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# Rough differential equations with affine boundary conditions 

Antoine Lejay* Renaud Marty ${ }^{\dagger}$

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#### Abstract

In this paper we consider rough differential equations with affine boundary conditions. Using rough paths and fixed-point techniques, we study properties of the solutions. We also prove a convergence theorem and apply our results to differential equations driven by Brownian motions and fractional Brownian motions.


Keywords: Rough path theory, differential equations, affine boundary conditions.
MSC (2020): 60H10, 34K10, 60L90

## 1 Introduction

In this paper we study rough differential equations with affine boundary conditions. If U and V are two Banach spaces, we consider the problem

$$
\left\{\begin{array}{l}
d Y(t)=f(Y(t)) \mathrm{d} W(t) \text { for all } t \in[0,1]  \tag{1}\\
H_{0} Y(0)+H_{1} Y(1)=v
\end{array}\right.
$$

where $W$ is a continuous function or a stochastic process with values in U , $f: \mathrm{V} \rightarrow \mathrm{L}(\mathrm{U}, \mathrm{V})$ is a smooth function, $v$ is a vector in $\mathrm{V}, H_{0}$ and $H_{1}$ are two operators on V , and $Y$ is unknown. In the case where $W$ is a finite-dimensional Brownian motion and d $W$ stands for the Stratonovich integration, the existence, the uniqueness and other properties of the solutions of such problems (with possibly other forms of two-point boundary conditions) have been analyzed under various assumptions and with different methods. Let us mention a few examples. Linear equations are studied in [23], where anticipating stochastic calculus (see [21]) is used because the solutions are not

[^0]adapted to $W$. Nonlinear equations with additive noises and more general two-point boundary conditions are studied in [22] with a pathwise approach. In [11] invariant imbedding (see [3]) is used for the analysis of nonlinear equations. In addition to the works mentioned above, we may also cite for instance $[2,7,9,24]$ where various stochastic differential equations with two-point boundary conditions are studied in other frameworks or with other methods.

In our paper we address the problem (1) in a setting where $W$ is a more general continuous process (such as a fractional Brownian motion for instance). We use the theory of rough paths (see for instance [10, 18). This enable us to avoid any difficulties related to the anticipating stochastic calculus. We use an enhanced Brownian motion or any other suitable rough path $\mathbf{W}$ in place of $W$ in order to define rough differential equations $Z(t)=Z(0)+\int_{0}^{t} f(Z(s)) \mathrm{d} \mathbf{W}(s)$. For every $a \in \mathrm{~V}$, if $f$ is regular enough then there exists a unique solution $Z$ such that $Z(0)=a$. Then the Itô map $I$ which assigns $Z$ to $(a, f, \mathbf{W})$ can be defined and is regular with respect to all its variables.

In our work, setting $\tau \in[0, T]$ we relate (1) to

$$
\left\{\begin{array}{l}
Y(t)=Y(0)+\int_{0}^{t} f(Y(s)) \mathrm{d} \mathbf{W}(s) \text { for } t \in[0, T]  \tag{2}\\
H_{0} Y(0)+H_{1} Y(\tau)=v
\end{array}\right.
$$

and study various properties. Equation (2) links the values of $Y(0)$ and $Y(\tau)$ for a given value of $\tau \in[0, T]$, while the path $\{Y(t)\}_{t \in[0, T]}$ is defined on the whole interval $[0, T]$.

Using the Itô map and a fixed-point method, we prove that there exits a unique solution to (2) if $\tau<C$ where $C$ is a constant that depends on $T$, $f, \mathbf{W}, H_{0}$, and $H_{1}$ (we note that possibly $C \geqslant T$ ). In this setting we also establish regularity results, which enable us to prove limit theorems.

For $\tau \in[C, T]$ and when V is finite-dimensional, using the Brouwer degree we show that there exists a solution. While the uniqueness is not ensured, the number of solutions is finite for almost every $v$. The situation is summarized in Figure 1.


Figure 1: Summary of the results on existence and uniqueness depending on $\tau$ for fixed time $T$.

Outline. In Section 2 we recall results on rough paths theory. In Section 3 we prove the existence and uniqueness of a solution for a small time $\tau$. In Section 4 we study the regularity of the solution with respect to the driving rough noise. In Sections 5 and 6 we establish more results on regularity and convergence. In Section 7 we study the existence of solutions for any time $\tau$. Finally, in Section 8 we consider the case of a driving rough noise with Hölder regularity greater than $1 / 2$. In this context, the rough integrals coincide with the Young integrals.

## 2 Reminders about rough paths

We consider in this section the case of rough differential equations driven by paths of Hölder regularity $\alpha \in\left(\frac{1}{3}, \frac{1}{2}\right]$.

For two Banach space U and V whose norms are written $|\cdot|$, we denote by $L(\mathrm{U}, \mathrm{V})$ the space of bounded, linear operators from U to V . Here, U and V are finite or infinite dimensional.

We denote by $C^{\alpha}(\mathrm{W}, \mathrm{V})$ the space of $\alpha$-Hölder continuous maps from W to V for $\mathrm{W} \subset \mathrm{U}$. The space is equipped with the $\alpha$-Hölder semi-norm $\|f\|_{\alpha}:=\sup _{x \neq y}|f(x)-f(y)| /|x-y|^{\alpha}$. When $\mathrm{W}=[0, T] \subset \mathbb{R}$, we write $\|\cdot\|_{\alpha, T}$ instead of $\|\cdot\|_{\alpha}$.

We denote by $C^{k}(\mathrm{U}, \mathrm{V})$ for some integer $k$ the space of continuous functions which are differentiable up to order $k$ with continuous $k$-th derivatives. We denote by $C_{b}^{k}(\mathrm{U}, \mathrm{V})$ the space of functions in $C^{k}(\mathrm{U}, \mathrm{V})$ with bounded $j$-th derivatives for all $0 \leqslant j \leqslant k$.

We denote by $\mathcal{C}^{\alpha}([0, T], \mathrm{U})$ the space of $\alpha$-Hölder rough paths $\mathbf{X}=(X, \mathbb{X})$ with values in $\mathrm{U} \oplus(\mathrm{U} \otimes \mathrm{U})$. The elements of $\mathcal{C}^{\alpha}([0, T], \mathrm{U})$ satisfy the Chen relation

$$
X(r, s)+X(s, t)=X(r, t), \mathbb{X}(r, s)+\mathbb{X}(s, t)+X(r, s) \otimes X(s, t)=\mathbb{X}(r, t)
$$

for any $0 \leqslant r \leqslant s \leqslant t \leqslant T$. Besides

$$
\|X\|_{\alpha, T}=\sup _{0 \leqslant s \neq t \leqslant T} \frac{|X(s, t)|}{|t-s|^{\alpha}}<\infty \text { and }\|\mathbb{X}\|_{2 \alpha, T}=\sup _{0 \leqslant s \neq t \leqslant T} \frac{|\mathbb{X}(s, t)|}{|t-s|^{2 \alpha}}<\infty
$$

The metric on $\mathcal{C}^{\alpha}([0, T], \mathrm{U})$ is denoted by $\varrho_{\alpha, T}$ and defined by

$$
\varrho_{\alpha, T}\left(\mathbf{X}_{1}, \mathbf{X}_{2}\right)=\sup _{0 \leqslant s \neq t \leqslant T} \frac{\left|X_{1}(s, t)-X_{2}(s, t)\right|}{|t-s|^{\alpha}}+\sup _{0 \leqslant s \neq t \leqslant T} \frac{\left|\mathbb{X}_{1}(s, t)-\mathbb{X}_{2}(s, t)\right|}{|t-s|^{2 \alpha}}
$$

for all $\mathbf{X}_{1}$ and $\mathbf{X}_{2}$ in $\mathcal{C}^{\alpha}([0, T], \mathrm{U})$.
Let U and V be two Banach spaces, $\mathbf{X} \in \mathcal{C}^{\alpha}([0, T], \mathrm{U}), f \in C_{b}^{2}(\mathrm{~V}, \mathrm{~L}(\mathrm{U}, \mathrm{V}))$ and $a \in \mathrm{~V}$. We consider the following differential equation

$$
\begin{equation*}
Y(t)=a+\int_{0}^{t} f(Y(s)) \mathrm{d} \mathbf{X}(s) \text { for all } t \in[0, T] \tag{3}
\end{equation*}
$$

with unknown $Y \in C^{\alpha}([0, T], \mathrm{V})$. Following [6, 10, a solution $Y$ of (9) can be defined as a path such that $Y(0)=a$ and

$$
\begin{equation*}
Y(t)-Y(s)=f(Y(s)) X(s, t)+D f(Y(s)) f(Y(s)) \mathbb{X}(s, t)+\mathrm{O}\left(|t-s|^{\theta}\right) \tag{4}
\end{equation*}
$$

for some $\theta>1$.
Following [10, Theorem 8.3], if $f \in C_{b}^{3}(\mathrm{~V}, \mathrm{~L}(\mathrm{U}, \mathrm{V}))$, then there exists a unique solution $Y \in C^{\alpha}([0, T], \mathrm{V})$ to Equation (4) (or (3)). In order to take account of the dependence on $(a, \mathbf{X})$, we set $I(a, \mathbf{X}):=Y$. This defines the Itô map $I$ from $V \times \mathcal{C}^{\alpha}([0, T], \mathrm{U})$ to $C^{\alpha}([0, T], \mathrm{V})$. The next lemma establishes the boundedness of this solution (see for instance 10, Proposition 8.2]).

Lemma 1. There exists a positive constant $K=K(\alpha)$ depending only on $\alpha$ such that for all $a \in \mathrm{~V}$,

$$
\begin{equation*}
\|I(a, \mathbf{X})\|_{\alpha, T} \leqslant K(\alpha) \max \left(\|f\|_{C^{2}}\|\mathbf{X}\|_{\alpha, T},\left(\|f\|_{C^{2}}\|\mathbf{X}\|_{\alpha, T}\right)^{1 / \alpha}\right) \tag{5}
\end{equation*}
$$

where

$$
\|\mathbf{X}\|_{\alpha, T}:=\|X\|_{\alpha, T}+\sqrt{\|\mathbb{X}\|_{2 \alpha, T}}
$$

By T. Lyons' theorem [17] the Itô map $I$ is continuous. Moreover it is locally Lipschitz continuous, as recalled in the following lemma (see for instance $[5,10,15]$ ). Notice that some extended statements consider also variations of the vector field $f$.

Lemma 2. We fix $M>0$ and consider

$$
\begin{equation*}
\mathcal{B}_{M, T}=\left\{\mathbf{X} \in \mathcal{C}^{\alpha}([0, T], \mathrm{U}):\|\mathbf{X}\|_{\alpha, T} \leqslant M\right\} \tag{6}
\end{equation*}
$$

where $\|\mathbf{X}\|_{\alpha, T}$ is defined in Lemma 1. There exists a positive constant $C_{\alpha, f, M}$ depending only on $\alpha, f$ and $M$ such that for all a and $b \in V$, and for all $\mathbf{X}_{\mathbf{1}}$ and $\mathbf{X}_{\mathbf{2}} \in \mathcal{B}_{M, T}$,

$$
\begin{equation*}
\left\|I\left(a, \mathbf{X}_{1}\right)-I\left(b, \mathbf{X}_{\mathbf{2}}\right)\right\|_{\alpha, T} \leqslant C_{\alpha, f, M}\left(|a-b|+\varrho_{\alpha, T}\left(\mathbf{X}_{\mathbf{1}}, \mathbf{X}_{\mathbf{2}}\right)\right) . \tag{7}
\end{equation*}
$$

In particular, for every $\mathbf{X} \in \mathcal{C}^{\alpha}([0, T], \mathrm{U})$ there exists a positive constant $C_{\alpha, f, \mathbf{X}}$ depending only on $\alpha, f$ and $\|\mathbb{X}\|_{\alpha, T}$ such that for all a and $b \in \mathrm{~V}$,

$$
\begin{equation*}
\|I(a, \mathbf{X})-I(b, \mathbf{X})\|_{\alpha, T} \leqslant C_{\alpha, f, \mathbf{X}}|a-b| \tag{8}
\end{equation*}
$$

For the sake of simplicity, we do not consider optimal hypotheses on the vector field. The assumptions on the regularity of $f$ can be weakened (for example when $f \in C_{\mathrm{b}}^{2+\gamma}$ with $\alpha(2+\gamma)>1$ ) but to get a global Lipschitz condition with respect to the starting point, we highlight that $f$ shall be bounded. This latter hypothesis is used in several proofs.

## 3 Existence and uniqueness

Let $v \in \mathrm{~V}$ and $H_{0}$ and $H_{1}$ be two bounded operators on V (finite or infinite dimensional). We set $\tau$ and $T$ such that $0<\tau \leqslant T$.

Throughout the paper and unless otherwise mentioned, we assume that $f \in C_{b}^{3}(\mathrm{~V}, \mathrm{~L}(\mathrm{U}, \mathrm{V}))$.

In this section we fix $\mathbf{X} \in \mathcal{C}^{\alpha}([0, T], \mathrm{U})$ and we wish to find a continuous function $Y$ such that

$$
\left\{\begin{array}{l}
Y(t)=Y(0)+\int_{0}^{t} f(Y(s)) \mathrm{d} \mathbf{X}(s) \text { for all } t \in[0, T]  \tag{9}\\
H_{0} Y(0)+H_{1} Y(\tau)=v
\end{array}\right.
$$

Then we consider the following problem. We look for an element $a \in \mathrm{~V}$ such that

$$
\begin{equation*}
H_{0} a+H_{1} I(a)(\tau)=v \tag{10}
\end{equation*}
$$

where $I$ is the Itô map and $I(a, \mathbf{X})$ is denoted by $I(a)$ since $\mathbf{X}$ is fixed. The following lemma establishes the relation between 10 and (9).

Lemma 3. If there exists $a \in \mathrm{~V}$ such that (10) holds, then $Y=I(a)$ satisfies (9). Moreover, if there exists a unique $a \in \mathrm{~V}$ such that (10) holds, then $Y=I(a)$ is the unique function in $C^{\alpha}([0, T], \mathrm{V})$ satisfying (9).

Proof. The first part of the lemma is obvious. Assume there exists a unique $a \in \mathrm{~V}$ such that 10 ) and set $Y=I(a)$. Consider another $Z \in C^{\alpha}([0, T], \mathrm{V})$ such that

$$
\left\{\begin{array}{l}
Z(t)=Z(0)+\int_{0}^{t} f(Z(s)) \mathrm{d} \mathbf{X}(s) \text { for all } t \in[0, T] \\
H_{0} Z(0)+H_{1} Z(\tau)=v
\end{array}\right.
$$

By the definition of the Itô map, $Z=I(Z(0))$, so that $H_{0} Z(0)+H_{1} I(Z(0))(\tau)=$ $v$. This implies that $Z(0)=a$ (by uniqueness of $a$ ), and then $Z=I(Z(0))=$ $I(a)=Y$.

The following theorem establishes the existence of a unique solution of the problem (9) for small values of $\tau$. It is proved with the use of 10 , which is solved with the Banach fixed-point theorem.

For $a \in \mathrm{~V}$ and $\tau \in[0, T]$, we define

$$
\begin{equation*}
L(a)_{\tau}:=H^{-1} v-H^{-1} H_{1}(I(a)(\tau)-a) \tag{11}
\end{equation*}
$$

when $H:=H_{0}+H_{1}$ is invertible.
Remark 1. The condition that $H_{0}+H_{1}$ is invertible means that our results cannot be applied to deal with a periodic solution.

Theorem 1. Let $C_{\alpha, f, \mathbf{X}}$ be the constant of (8). If $H:=H_{0}+H_{1}$ is invertible and $\tau$ satisfies

$$
\begin{equation*}
\tau^{\alpha}\left|H^{-1} H_{1}\right| C_{\alpha, f, \mathbf{X}}<1, \tag{12}
\end{equation*}
$$

then there exists a unique solution to (9) in $C^{\alpha}([0, T], \mathrm{V})$.
Proof. By Lemma 3, it suffices to prove that there exists a unique solution $a \in \mathrm{~V}$ to 10 . We rewrite (10) as

$$
\left(H_{0}+H_{1}\right) a+H_{1}(I(a)(\tau)-a)=v
$$

Then, since $H=H_{0}+H_{1}$ is invertible,

$$
H_{0} a+H_{1} I(a)(\tau)=v \text { if and only if } a=H^{-1} v-H^{-1} H_{1}(I(a)(\tau)-a)
$$

With (11), 10) can be reformulated as the fixed-point problem

$$
\begin{equation*}
a=L(a)_{\tau} \tag{13}
\end{equation*}
$$

By the Banach fixed-point theorem, it suffices to prove that $L(\cdot)_{\tau}$ is a contraction. For all $a, b \in \mathrm{~V}$,

$$
\begin{aligned}
\left|L(a)_{\tau}-L(b)_{\tau}\right| & \leqslant\left|H^{-1} H_{1}\right| \cdot|I(a)(\tau)-a-I(b)(\tau)+b| \\
& \leqslant\left|H^{-1} H_{1}\right| \cdot\left(\left|I(a)_{0}-a-I(b)_{0}+b\right|+\tau^{\alpha}\|I(a)-a-I(b)+b\|_{\alpha, T}\right) \\
& =\tau^{\alpha}\left|H^{-1} H_{1}\right| \cdot\|I(a)-a-I(b)+b\|_{\alpha, T} \\
& =\tau^{\alpha}\left|H^{-1} H_{1}\right| \cdot\|I(a)-I(b)\|_{\alpha, T}
\end{aligned}
$$

Following (8),

$$
\|I(a)-I(b)\|_{\alpha, T} \leqslant C_{\alpha, f, \mathbf{X}}|a-b|
$$

As a consequence of above,

$$
\left|L(a)_{\tau}-L(b)_{\tau}\right| \leqslant \tau^{\alpha}\left|H^{-1} H_{1}\right| C_{\alpha, f, \mathbf{X}}|a-b|
$$

Then for every $\tau$ satisfying (12), the function $L(\cdot)_{\tau}$ is a contraction. Then by the Banach fixed-point theorem, there exists a unique fixed point $a \in V$ such that $(13)$, and then 10 are satisfied.

As an application of Theorem 1, fix $\tau \in[0, T]$ and consider that $W=$ $\left(W_{1}, \ldots, W_{d}\right)$ is a Brownian motion with values in $\mathrm{U}=\mathbb{R}^{d}$ which is enhanced with $\mathbf{W}=(W, \mathbb{W}) \in \mathcal{C}^{\alpha}\left([0, T], \mathbb{R}^{d}\right)$ with for all $1 \leqslant j, k \leqslant d$,

$$
\begin{equation*}
\mathbb{W}_{j, k}(s, t)=\int_{s}^{t}\left(W_{j}(r)-W_{j}(s)\right) \circ \mathrm{d} W_{k}(r) \text { for any } 0 \leqslant s \leqslant t \leqslant T \tag{14}
\end{equation*}
$$

Then, Theorem 1 establishes that for every $\omega \in \Omega$ such that $\tau^{\alpha}\left|H^{-1} H_{1}\right| C_{\alpha, f, \mathbf{W}(\omega)}<$ 1, there exists a unique solution $Y(\omega) \in C^{\alpha}([0, T], \mathrm{V})$ to

$$
\left\{\begin{array}{l}
Y(\omega)(t)=Y(\omega)(0)+\int_{0}^{t} \sum_{j=1}^{d} f_{j}(Y(\omega)(s)) \circ \mathrm{d} W_{j}(\omega)(s), t \in[0, T]  \tag{15}\\
H_{0} Y(\omega)(0)+H_{1} Y(\omega)(\tau)=v
\end{array}\right.
$$

Notice that $\int_{0}^{t} \sum_{j=1}^{d} f_{j}(Y(\omega)(s)) \circ \mathrm{d} W_{j}(\omega)(s)$ stands for $\int_{0}^{t} f(Y(\omega)(s)) \mathrm{d} \mathbf{W}(\omega)(s)$ because of the Stratonovich integration in (14) and both rough and Stratonovich integrals coincide. This result follows from Theorem 4 below which is the rough counterpart of the stochastic Wong-Zakai theorem, and the construction of 11 based on characteristics as solutions to some Stratonovich SDE.

This problem can also be solved in an infinite-dimensional setting. Suppose that $U$ is a separable Banach space and consider a U-valued Brownian motion $W$ on the Wiener space ( $\mathrm{U}, \mathrm{H}, \mu$ ) where $\mu$ is a zero-mean Gaussian measure on U and H is the Cameron-Martin space of $\mu$. Following [14], under suitable conditions, there exists an enhanced Brownian motion $\mathbf{W}=(W, \mathbb{W}) \in \mathcal{C}^{\alpha}([0, T], \mathrm{U})$ obtained as follows. We consider the sequence $(\mathbf{W}(m))_{m \geqslant 1}=(W(m), \mathbb{W}(m))_{m \geqslant 1}$ where $(W(m))_{m \geqslant 1}$ is the dyadic piecewise linear approximation of $W$, and $(\mathbb{W}(m))_{m \geqslant 1}$ is defined by

$$
\mathbb{W}(m)(s, t)=\int_{s}^{t}(W(m)(r)-W(m)(s)) \otimes \mathrm{d} W(m)(r)
$$

for all $s, t$, and $m$. Assume that the tensor norm $\|\cdot\|_{\mathrm{U} \otimes \mathrm{U}}$ on $\mathrm{U} \otimes \mathrm{U}$ is exact for $\mu$, which means that there exist two constants $\gamma \in[1 / 2,1)$ and $C>0$ such that for all independent sequences $\left(G_{n}\right)_{n \geqslant 1}$ and $\left(\widetilde{G}_{n}\right)_{n \geqslant 1}$ of independent U-valued random variables with distribution $\mu$, and for every $N \geqslant 0$,

$$
\mathbb{E}\left(\left\|\sum_{n=1}^{N} G_{n} \otimes \widetilde{G}_{n}\right\|_{\mathrm{U} \otimes \mathrm{U}}\right) \leqslant C N^{\gamma}
$$

By [14], under this assumption, as $m \rightarrow \infty$ the sequence $\mathbf{W}(m)$ converges in $\mathcal{C}^{\alpha}([0, T], \mathrm{U})$ almost surely. Its limit defines an enhanced Brownian motion. Then by Theorem 1 , for every $\omega \in \Omega$ such that $\tau^{\alpha}\left|H^{-1} H_{1}\right| C_{\alpha, f, \mathbf{W}(\omega)}<1$, there exists a unique solution $Y(\omega) \in C^{\alpha}([0, T], \mathrm{V})$ to

$$
\left\{\begin{array}{l}
Y(\omega)(t)=Y(\omega)(0)+\int_{0}^{t} f(Y(\omega)(s)) \mathrm{d} \mathbf{W}(\omega)(s) \text { for all } t \in[0, T] \\
H_{0} Y(\omega)(0)+H_{1} Y(\omega)(\tau)=v
\end{array}\right.
$$

## 4 Continuity of the solution with respect to the path

In the previous section the driving path $\mathbf{X}$ is fixed. In this section we consider $\mathbf{X}$ as a variable and we establish the continuity of the solution of (9) with respect to $\mathbf{X}$.

We set $M>0$ and $\tau$ satisfying

$$
\begin{equation*}
\tau^{\alpha}\left|H^{-1} H_{1}\right| C_{\alpha, f, M}<1 \tag{16}
\end{equation*}
$$

When $\mathbf{X} \in \mathcal{B}_{M, T}$ (defined by (6)), (16) implies (12).
We define the function

$$
\bar{I}: \mathcal{B}_{M, T} \rightarrow C^{\alpha}([0, T], \mathrm{V})
$$

such that for every $\mathbf{X} \in \mathcal{B}_{M, T}, \bar{I}(\mathbf{X})$ is the unique solution $Y$ to

$$
\left\{\begin{array}{l}
Y(t)=Y(0)+\int_{0}^{t} f(Y(s)) \mathrm{d} \mathbf{X}(s) \text { for all } t \in[0, T] \\
H_{0} Y(0)+H_{1} Y(\tau)=v
\end{array}\right.
$$

The main theorem of this subsection establishes the continuity of $\bar{I}$.
Theorem 2. The map $\bar{I}$ is Lipschitz continuous with respect to $\varrho_{\alpha, T}$ on $\mathcal{B}_{M, T}$, with constant

$$
C_{\alpha, f, M}+\frac{\left|H^{-1} H_{1}\right| C_{\alpha, f, M} \tau^{\alpha}}{1-\left|H^{-1} H_{1}\right| C_{\alpha, f, M} \tau^{\alpha}}
$$

Proof. Previously we proved that for every $\mathbf{X}, \bar{I}(\mathbf{X})$ can be written as $\bar{I}(\mathbf{X})=I\left(a_{\mathbf{X}}, \mathbf{X}\right)$ where $I$ is the Itô map and $a_{\mathbf{X}}$ is the fixed point of the function $L(\cdot, \mathbf{X})_{\tau}: a \mapsto L(a, \mathbf{X})_{\tau}=H^{-1} v-H^{-1} H_{1}(I(a, \mathbf{X})(\tau)-a)$.

By (7), for all $\mathbf{X}$ and $\mathbf{Y} \in \mathcal{B}_{M, T}$,

$$
\begin{align*}
\|\bar{I}(\mathbf{X})-\bar{I}(\mathbf{Y})\|_{\alpha, T} & =\left\|I\left(a_{\mathbf{X}}, \mathbf{X}\right)-I\left(a_{\mathbf{Y}}, \mathbf{Y}\right)\right\|_{\alpha, T} \\
& \leqslant C_{\alpha, f, M}\left(\left|a_{\mathbf{X}}-a_{\mathbf{Y}}\right|+\varrho_{\alpha, T}(\mathbf{X}, \mathbf{Y})\right) . \tag{17}
\end{align*}
$$

Moreover,

$$
\begin{aligned}
\left|a_{\mathbf{X}}-a_{\mathbf{Y}}\right|= & \left|L\left(a_{\mathbf{X}}, \mathbf{X}\right)_{\tau}-L\left(a_{\mathbf{Y}}, \mathbf{Y}\right)_{\tau}\right| \\
\leqslant & \left|L\left(a_{\mathbf{X}}, \mathbf{X}\right)_{\tau}-L\left(a_{\mathbf{X}}, \mathbf{Y}\right)_{\tau}\right|+\left|L\left(a_{\mathbf{X}}, \mathbf{Y}\right)_{\tau}-L\left(a_{\mathbf{Y}}, \mathbf{Y}\right)_{\tau}\right| \\
= & \left|H^{-1} H_{1}\right| \cdot\left|I\left(a_{\mathbf{X}}, \mathbf{X}\right)_{\tau}-I\left(a_{\mathbf{X}}, \mathbf{Y}\right)_{\tau}\right| \\
& +\left|H^{-1} H_{1}\right| \cdot\left|I\left(a_{\mathbf{X}}, \mathbf{Y}\right)_{\tau}-I\left(a_{\mathbf{Y}}, \mathbf{Y}\right)_{\tau}-a_{\mathbf{X}}+a_{\mathbf{Y}}\right| \\
\leqslant & \left|H^{-1} H_{1}\right| \cdot\left\|I\left(a_{\mathbf{X}}, \mathbf{X}\right) .-I\left(a_{\mathbf{X}}, \mathbf{Y}\right) \cdot\right\|_{\alpha, T} \tau^{\alpha} \\
& +\left|H^{-1} H_{1}\right| \cdot\left\|I\left(a_{\mathbf{X}}, \mathbf{Y}\right) .-I\left(a_{\mathbf{Y}}, \mathbf{Y}\right) \cdot\right\|_{\alpha, T} \tau^{\alpha} \\
\leqslant & \left|H^{-1} H_{1}\right| C_{\alpha, f, M} \varrho_{\alpha, T}(\mathbf{X}, \mathbf{Y}) \tau^{\alpha} \\
& +\left|H^{-1} H_{1}\right| C_{\alpha, f, M}\left|a_{\mathbf{X}}-a_{\mathbf{Y}}\right| \tau^{\alpha} .
\end{aligned}
$$

As a consequence, for all $\mathbf{X}$ and $\mathbf{Y} \in \mathcal{B}_{M, T}$,

$$
\left|a_{\mathbf{X}}-a_{\mathbf{Y}}\right| \leqslant \frac{\left|H^{-1} H_{1}\right| C_{\alpha, f, M} \tau^{\alpha}}{1-\left|H^{-1} H_{1}\right| C_{\alpha, f, M} \tau^{\alpha}} \varrho_{\alpha, T}(\mathbf{X}, \mathbf{Y})
$$

and, by (17),

$$
\|\bar{I}(\mathbf{X})-\bar{I}(\mathbf{Y})\|_{\alpha, T} \leqslant\left(C_{\alpha, f, M}+\frac{\left|H^{-1} H_{1}\right| C_{\alpha, f, M} \tau^{\alpha}}{1-\left|H^{-1} H_{1}\right| C_{\alpha, f, M} \tau^{\alpha}}\right) \varrho_{\alpha, T}(\mathbf{X}, \mathbf{Y})
$$

This completes the proof.

## 5 Regularity and numerical analysis

In Section 4 we proved the continuity of the solution with respect to the path $\mathbf{X}$. In this section, with the implicit function theorem, we consider the regularity with respect to the other variables. Then we briefly present an application to numerical analysis.

For $\gamma \in(0,1]$ and $k \in \mathbb{N}^{*}$, we denote by $C^{k+\gamma}$ the class of functions $f$ of class $C^{k}$ such that $\mathrm{D}^{k} f$ is $\gamma$-Hölder continuous. We denote by $C_{b}^{k+\gamma}$ the space of functions in $C^{k+\gamma}$ with bounded $j$-th derivatives for all $0 \leqslant j \leqslant k$. This space is equipped with

$$
\|f\|_{C^{k+\gamma}}:=\max \left\{\|f\|_{\infty}, \ldots,\left\|\mathrm{D}^{k} f\right\|_{\infty},\left\|\mathrm{D}^{k} f\right\|_{\gamma}\right\} .
$$

Let $\mathbf{X} \in \mathcal{C}^{\alpha}([0, T], \mathrm{U})$. We define

$$
\begin{aligned}
\mathcal{P} & :=\mathrm{V} \times C_{b}^{k+1+\gamma}(\mathrm{V}, L(\mathrm{U}, \mathrm{~V})), \\
\text { and } \mathcal{Q} & :=C_{b}^{k+1+\gamma}(\mathrm{V}, L(\mathrm{U}, \mathrm{~V})) \times \mathrm{V} \times L(\mathrm{~V}, \mathrm{~V}) \times L(\mathrm{~V}, \mathrm{~V})
\end{aligned}
$$

Proposition 1 ( [5, Theorem 3]). For any $\kappa \in(0,1)$ such that $\alpha(2+\kappa \gamma)>1$, the Itô map $I: \mathcal{P} \rightarrow C^{\alpha}([0, T], \mathrm{V})$ is locally of class $C^{k+(1-\kappa) \gamma}$.

Let us write

$$
P(a, q)_{\tau}=H_{0} a+H_{1} I(a, f, \mathbf{X})(\tau)-v \text { with } q=\left(f, v, H_{0}, H_{1}\right) \in \mathcal{Q} .
$$

If $H_{0}+H_{1}$ is invertible, the problem $L(a)_{\tau}=a$ is equivalent to $P(a, q)_{\tau}=$ 0 . Besides, by Proposition 1, $P$ is locally of class $C^{k+(1-\kappa) \gamma}$ on $\mathcal{Q}$ when $\alpha(2+\kappa \gamma)>1$. Moreover,

$$
\begin{equation*}
\mathrm{D}_{a} P(a, q)_{\tau}=H_{0}+H_{1} \mathrm{D}_{a} I(a, f, \mathbf{X})(\tau), \tag{18}
\end{equation*}
$$

where

$$
\mathrm{D}_{a} I(a, f, \mathbf{X})(t)=\mathrm{Id}+\int_{0}^{t} \mathrm{D} f(I(a, f, \mathbf{X})(s)) \cdot \mathrm{D}_{a} I(a, f, \mathbf{X})(s) \mathrm{d} s
$$

The regularity is established by the following result, which is a generalized implicit function theorem (see e.g., (1) applied to $P$.

Theorem 3. Let $\tau \in[0, T]$, $a^{\dagger} \in \mathrm{V}$ and $q^{\dagger} \in \mathcal{Q}$ such that $P\left(a^{\dagger}, q^{\dagger}\right)_{\tau}=0$ and $\mathrm{D}_{a} P\left(a^{\dagger}, q^{\dagger}\right)_{\tau} \in L(\mathrm{~V}, \mathrm{~V})$ is an isomorphism. Then there exist neighborhoods $U^{\dagger}$ of $q^{\dagger}$ and $V^{\dagger}$ of $a^{\dagger}$ as well as a function $A \in C_{b}^{k+(1-\kappa) \gamma}\left(U^{\dagger}, V^{\dagger}\right)$ with $A\left(q^{\dagger}\right)=a^{\dagger}$ such that $P(A(q), q)=0$ for any $q \in U^{\dagger}$.
Remark 2. At $\tau=0, \mathrm{D}_{a} I(a, f, \mathbf{X})(0)=\mathrm{Id}$ and then $\mathrm{D}_{a} P(a, q)_{0}=H_{0}+H_{1}$. To apply the implicit function theorem, $\mathrm{D}_{a} P(a, q)_{0}$ has to be invertible. Hence, $H_{0}+H_{1}$ shall be invertible which is an hypothesis we used to get the existence and uniqueness of a solution.
Remark 3. If $H_{0}+H_{1}$ is invertible, then we can deduce from (18) that $\mathrm{D}_{a} P(a, q)_{\tau}$ is invertible when $\tau$ is small enough. This not ensured for large $\tau$.

Theorem 3 implies that the solution $a$ is locally Lipschitz continuous with respect to $q=\left(f, v, H_{0}, H_{1}\right)$. Notice that we can also establish that $a$ is globally Lipschitz continuous with respect to $v$. More precisely, using (8) we get

$$
\begin{equation*}
\left|a\left(v_{1}\right)-a\left(v_{2}\right)\right| \leqslant\left(1-\tau^{\alpha}\left|H^{-1} H_{1}\right| C_{\alpha, f, \mathbf{X}}\right)\left|v_{1}-v_{2}\right| \tag{19}
\end{equation*}
$$

for all $v_{1}$ and $v_{2}$.
We now present an application to numerical analysis. Let us consider that for any starting point $a$ we have a numerical scheme $J(a)$ such that $J(a)=I(a, f, \mathbf{X})(\tau)+R(a)$ for a Lipschitz continuous error $a \mapsto R(a)$ with a Lipschitz norm $\ell$ smaller than 1. Several numerical schemes exist for rough paths and for RDE driven by Brownian motion or fractional Brownian motion (see for instance [13, 16, 20]).

Besides, let us assume that we are able to solve the fixed-point problem $H_{0} a+H_{1} J(a)-v=0$ at precision $\eta$, that is we find $a^{\prime}$ such that

$$
H_{0} a^{\prime}+H_{1} J\left(a^{\prime}\right)=v+\eta .
$$

This is the case for example when we use a finite number of iterations in the fixed-point iteration, or the Newton scheme. Then

$$
H_{0} a^{\prime}+H_{1} I\left(a^{\prime}, f, \mathbf{X}\right)(\tau)=v+\eta-H_{1} R\left(a^{\prime}\right) .
$$

It follows from Theorem 3 or 19 ) that there exists a constant $K$, depending on $\tau, \ell, f, \mathbf{X}, H_{0}$ and $H_{1}$ such that

$$
\left|a^{\prime}-a\right| \leqslant K|\eta|+K|R(a)| .
$$

This implies that the numerical scheme converges.

## 6 Limit theorems

In this section we deal with limit theorems. We fix $\tau$ and $M$ such that (16), namely

$$
\tau^{\alpha}\left|H^{-1} H_{1}\right| C_{\alpha, f, M}<1
$$

For every $T>0$ we denote by $\mathcal{C}^{0, \alpha}([0, T], \mathrm{U})$ the closure of smooth rough paths in $\mathcal{C}^{\alpha}([0, T], \mathrm{U})$. We assume that the Banach space U is separable and recall that, as a consequence, $\mathcal{C}^{0, \alpha}([0, T], \mathrm{U})$ is a Polish space (see Exercise 2.8 of $[10$ for instance). The main result is the following.
Theorem 4. Suppose (16) and let $\mathbf{X}$ and $\left(\mathbf{X}^{n}\right)_{n}$ be a random variable and a sequence of random variables with values in $\mathcal{C}^{0, \alpha}([0, T], \mathrm{U})$ such that $\mathbf{X}^{n}$ converges in distribution to $\mathbf{X}$ in $\mathcal{C}^{0, \alpha}([0, T], \mathrm{U})$ and $\mathbb{P}\left(\|\mathbf{X}\|_{\alpha, T}=M\right)=0$. For every continuous bounded function $h: C^{\alpha}([0, T], \mathrm{V}) \rightarrow \mathbb{R}$,

$$
\lim _{n \rightarrow \infty} \mathbb{E}\left(h\left(\bar{I}\left(\mathbf{X}^{n}\right)\right) \mathbf{1}_{\left\|\mathbf{X}^{n}\right\|_{\alpha, T} \leqslant M}\right)=\mathbb{E}\left(h(\bar{I}(\mathbf{X})) \mathbf{1}_{\|\mathbf{X}\|_{\alpha, T} \leqslant M}\right)
$$

Proof. Since $\mathcal{C}^{0, \alpha}([0, T], \mathrm{U})$ is a Polish space, by the Skorokhod representation theorem there exists a probability space $\left(\Omega^{*}, \mathcal{T}^{*}, \mathbb{P}^{*}\right)$ and random variables $\mathbf{X}^{*, n}$ and $\mathbf{X}^{*}$ from $\left(\Omega^{*}, \mathcal{T}^{*}, \mathbb{P}^{*}\right)$ to $\mathcal{C}^{0, \alpha}([0, T], \mathrm{U})$ such that $\mathbf{X}^{*, n} \stackrel{\text { law }}{=} \mathbf{X}^{n}$, $\mathbf{X}^{*} \stackrel{\text { law }}{=} \mathbf{X}$ and $\mathbf{X}^{*, n}$ converges $\mathbb{P}^{*}$-almost surely to $\mathbf{X}^{*}$. Then $\mathbb{P}^{*}$-almost surely, $h\left(\bar{I}\left(\mathbf{X}^{*, n}\right)\right) \mathbf{1}_{\left\|\mathbf{X}^{*, n}\right\|_{\alpha, T} \leqslant M}$ converges to $h\left(\bar{I}\left(\mathbf{X}^{*}\right)\right) \mathbf{1}_{\| \mathbf{X}} \|_{\|_{\alpha, T} \leqslant M}$ as $n \rightarrow \infty$. Moreover, by the bounded convergence theorem, since for all $n$,

$$
h\left(\bar{I}\left(\mathbf{X}^{*, n}\right)\right) \mathbf{1}_{\left\|\mathbf{X}^{*, n}\right\|_{\alpha, T} \leqslant M} \leqslant\|h\|_{\infty},
$$

we deduce that

$$
\lim _{n \rightarrow \infty} \mathbb{E}\left(h\left(\bar{I}\left(\mathbf{X}^{*, n}\right)\right) \mathbf{1}_{\left\|\mathbf{X}^{*, n}\right\|_{\alpha, T} \leqslant M}\right)=\mathbb{E}\left(h\left(\bar{I}\left(\mathbf{X}^{*}\right)\right) \mathbf{1}_{\left\|\mathbf{X}^{*}\right\|_{\alpha, T} \leqslant M}\right)
$$

This completes the proof.
As an application, we consider for every $n \in \mathbb{N}^{*}$ the problem

$$
\left\{\begin{array}{l}
\frac{\mathrm{d} Y^{n}(t)}{\mathrm{d} t}=\sqrt{n} \sum_{j=1}^{d} \xi_{j,[n t]} f_{j}\left(Y^{n}(t)\right) \text { for all } t \in[0, T] \\
H_{0} Y^{n}(0)+H_{1} Y^{n}(\tau)=v,
\end{array}\right.
$$

where $\xi:=\left(\xi_{j, k}\right)_{(j, k) \in\{1, \cdots, d\} \times \mathbb{N}}$ is a discrete family of independent random variables with mean zero, variance 1 , same distribution and finite moments. We define the sequence of $\mathbb{R}^{d}$-valued processes $W^{n}=\left(W_{1}^{n}, \ldots, W_{d}^{n}\right)$ such that for all $j=1, \ldots, d$,

$$
W_{j}^{n}(t)=\frac{1}{\sqrt{n}}\left(\sum_{k=0}^{[n t]-1} \xi_{j, k}+(n t-[n t]) \xi_{j,[n t]}\right)
$$

Notice that $W^{n}$ is Lipschitz-continuous. Therefore, $W^{n}$ can be enhanced with $\mathbf{W}^{n}=\left(W^{n}, \mathbb{W}^{n}\right) \in \mathcal{C}^{\alpha}\left([0, T], \mathbb{R}^{d}\right)$ where

$$
\mathbb{W}^{n}(s, t)=\int_{s}^{t}\left(W^{n}(r)-W^{n}(s)\right) \otimes \mathrm{d} W^{n}(r)
$$

The following theorem establishes the convergence in distribution of $Y^{n}$ as $n$ goes to infinity.

Theorem 5. For every continuous bounded function $h: C^{\alpha}([0, T], \mathrm{V}) \rightarrow \mathbb{R}$,

$$
\lim _{n \rightarrow \infty} \mathbb{E}\left(h\left(Y^{n}\right) \mathbf{1}_{\mathbf{W}^{n} \in \mathcal{B}_{M, T}}\right)=\mathbb{E}\left(h(Y) \mathbf{1}_{\mathbf{W} \in \mathcal{B}_{M, T}}\right),
$$

where $\mathbf{W}$ is the enhanced Brownian motion defined by (14) and $Y$ is the solution of

$$
\left\{\begin{array}{l}
\mathrm{d} Y(t)=\sum_{j=1}^{d} f_{j}(Y(t)) \circ \mathrm{d} W_{j}(t) \text { for all } t \in[0, T] \\
H_{0} Y(0)+H_{1} Y(\tau)=v .
\end{array}\right.
$$

This theorem is similar to [11, Theorem 7] but its proof is different.
Proof. It is a direct consequence of [4, Theorem 1] and of Theorem 4

## 7 Global existence result

In Section 3 we prove that (9) (and (10) has a unique solution for every $\tau$ small enough. In this section we use the Brouwer degree in order to prove that (9) has a solution (maybe more than one) for every $\tau \in[0, T]$, yet only when V is finite-dimensional.

Throughout the section, we assume that $\mathrm{V}=\mathbb{R}^{N}$ for some $N \in \mathbb{N}^{*}$. We equip V with the Lebesgue measure.

Before we establish the existence of solution for (9) in Theorem 6, we remind the definition and some properties of the Brouwer degree. For a more complete exposition on the degree theory we refer the reader for instance to 12 .

Consider a bounded open subset $\mathcal{O} \subset \mathbb{R}^{N}=\mathrm{V}$, a continuous function $\Phi: \overline{\mathcal{O}} \rightarrow \mathbb{R}^{N}$ and $y \in V$ such that $y \notin \Phi(\partial \mathcal{O})$. The Brouwer degree $\operatorname{deg}(\Phi, \mathcal{O}, y)$ with $\Phi \in C^{1}(\mathcal{O}, \mathrm{~V})$ can be defined as follows (see for instance 12 , Definition 13.2.9]).
Definition 1. If $y$ is a regular value of $\Phi$ (namely such that $\operatorname{det} J_{\Phi}(x) \neq 0$ for all $\left.x \in \Phi^{-1}(\{y\})\right)$, then $\operatorname{deg}(\Phi, \mathcal{O}, y)$ is defined by

$$
\operatorname{deg}(\Phi, \mathcal{O}, y)=\sum_{x \in \Phi^{-1}(\{y\})} \operatorname{sign}\left(\operatorname{det} J_{\Phi}(x)\right) .
$$

If $y$ is a critical value of $\Phi$ ( $y$ is not a regular value of $\Phi$ ), then $\operatorname{deg}(\Phi, \mathcal{O}, y)$ is defined by

$$
\operatorname{deg}(\Phi, \mathcal{O}, y)=\operatorname{deg}\left(\Phi, \mathcal{O}, y^{\prime}\right)
$$

where $y^{\prime}$ is a regular value of $\Phi$ such that $\left|y-y^{\prime}\right|<\operatorname{dist}(y, \Phi(\partial \mathcal{O}))$.

The Brouwer degree has the following property (see for instance 12 , Theorem 13.2.11(iv)]).

Lemma 4. Let $y \in \mathrm{~V}$ such that $y \notin \Phi(\partial \mathcal{O})$. If $\operatorname{deg}(\Phi, \mathcal{O}, y) \neq 0$ then there exists a solution $x \in \mathcal{O}$ to the equation $\Phi(x)=y$.

The invariance by homotopy is another important property of the Brouwer degree and can be stated as follows (see for instance [12, Theorem 13.2.11(ii)]).
Lemma 5. We fix $\tau>0$ and consider a continuous function $H:[0, \tau] \times \overline{\mathcal{O}} \rightarrow$ V . If we assume that $y \notin H(t, \partial \mathcal{O})$ for all $t \in[0, \tau]$, then

$$
\operatorname{deg}(H(0, \cdot), \mathcal{O}, y)=\operatorname{deg}(H(\tau, \cdot), \mathcal{O}, y)
$$

The existence of solutions of (9) can now be established in the following theorem.

Theorem 6. If $H:=H_{0}+H_{1}$ is invertible, then for every $\tau \in[0, T]$, there exists $Y \in C^{\alpha}([0, T], V)$ such that (9) is satisfied. Moreover, for almost every $v \in \mathrm{~V}$ the number of solutions of (9) is finite.

Proof. As a consequence of Lemma 3 and the beginning of the proof of Theorem 1, it suffices to prove that there exists $a \in \mathrm{~V}$ such that $a=L(a)_{\tau}$. With $\mathcal{L}$ defined by

$$
\mathcal{L}(\tau, a)=a+H^{-1} H_{1}(I(a)(\tau)-a) \text { for }(\tau, a) \in[0, T] \times \mathrm{V},
$$

the problem $a=L(a)_{\tau}$ is equivalent to

$$
\begin{equation*}
\mathcal{L}(\tau, a)=\tilde{v} \text { where } \tilde{v}=H^{-1} v \tag{20}
\end{equation*}
$$

Hence, we look for $a \in \mathrm{~V}$ solving 20). In order to prove the existence of solutions it suffices to prove that for $r>0$ large enough and for all $\tau \in[0, T]$ the Brouwer degree of $\mathcal{L}(\tau, \cdot)$ exists and satisfies

$$
\begin{equation*}
\operatorname{deg}\left(\mathcal{L}(\tau, \cdot), B_{r}(\tilde{v}), \tilde{v}\right) \neq 0 \tag{21}
\end{equation*}
$$

where $B_{r}(\tilde{v})$ stands for the ball in $V$ of center $\tilde{v}$ and radius $r$.
For all $r>0$,

$$
\operatorname{deg}\left(\mathcal{L}(0, \cdot), B_{r}(\tilde{v}), \tilde{v}\right)=\operatorname{deg}\left(\operatorname{Id}, B_{r}(\tilde{v}), \tilde{v}\right)=1
$$

By Lemma 11, for all $(\tau, x) \in[0, T] \times \mathrm{V}$ such that $\mathcal{L}(\tau, x)=\tilde{v}$, we have

$$
\begin{aligned}
|x-\tilde{v}|= & \left|H^{-1} H_{1}(I(x)(\tau)-x)\right| \\
& \leqslant\left|H^{-1} H_{1}\right| \cdot|I(x)(\tau)-I(x)(0)| \\
& \leqslant\left|H^{-1} H_{1}\right| \cdot\|I(x)\|_{\alpha} \tau^{\alpha} \\
& \leqslant\left|H^{-1} H_{1}\right| c_{\alpha, f, \mathbf{X}} T^{\alpha} .
\end{aligned}
$$

This implies that for all $r>\left|H^{-1} H_{1}\right| c_{\alpha, f, \mathbf{X}} T^{\alpha}$ and all $\tau \in[0, T]$,

$$
\tilde{v} \notin \mathcal{L}\left(\tau, \partial B_{r}(\tilde{v})\right)
$$

and then the $\operatorname{degree} \operatorname{deg}\left(\mathcal{L}(\tau, \cdot), B_{r}(\tilde{v}),(\tilde{v})\right)$ is well-defined. Moreover, by the continuity of $I$, the function $\mathcal{L}$ is a homotopy. This implies that for all $\tau \in[0, T]$,

$$
\operatorname{deg}\left(\mathcal{L}(\tau, \cdot), B_{r}(\tilde{v}), \tilde{v}\right)=\operatorname{deg}\left(\mathcal{L}(0, \cdot), B_{r}(\tilde{v}), \tilde{v}\right)=1
$$

and then (21) and the existence of a solution $a \in B_{r}(\tilde{v})$ to $a=L(a)_{\tau}$.
It remains to prove that for every $\tau \in[0, T]$ the number of solutions of $a=L(a)_{\tau}$ is finite for almost every $v$, or equivalently for almost every $\tilde{v}$. We fix $r>\left|H^{-1} H_{1}\right| c_{\alpha, f, \mathbf{X}} T^{\alpha}$. Since $f \in C_{b}^{3}$ the function $a \mapsto I(a)$ is $C^{1}$. Following the Sard theorem [12, Lemma 13.2.1, p. 331], the Lebesgue measure of the set of critical values of $\mathcal{L}(\tau, \cdot)$ is equal to 0 . Consider a regular value $\tilde{v}$ of $\mathcal{L}(\tau, \cdot)$. The set $\overline{B_{r}(\tilde{v})} \cap \mathcal{L}(\tau, \cdot)^{-1}(\tilde{v})$ is closed and bounded, and consequently compact because V is finite-dimensional. Since $\tilde{v}$ is a regular value, $\mathcal{L}(\tau, \cdot)$ is locally invertible on a neighborhood of each point of $\overline{B_{r}(\tilde{v})} \cap \mathcal{L}(\tau, \cdot)^{-1}(\tilde{v})$. Then each point of $\overline{B_{r}(\tilde{v})} \cap \mathcal{L}(\tau, \cdot)^{-1}(\tilde{v})$ is isolated. Since $|L(\tau, \cdot)|$ is bounded (by $\left|H^{-1} v\right|+\left|H^{-1} H_{1}\right| \tau^{\alpha} c_{\alpha, f, \mathbf{X}}$ for instance), we choose $r$ large enough and deduce that $\mathcal{L}(\tau, \cdot)^{-1}(\tilde{v})$ is finite. This completes the proof.

Remark 4. If $f$ is not bounded, the existence of solutions is not ensured for every $\tau$. Consider for instance the real-valued differential equation

$$
\mathrm{d} Y(t)=Y(t) \mathrm{d} W(t)
$$

where $W$ is a Hölder function and $a \in \mathbb{R}$. The solutions are written as

$$
Y(t)=a \exp (W(t)-W(0))
$$

where $a$ is a constant. The condition

$$
H_{0} Y(0)+H_{1} Y(T)=v
$$

can be rewritten as

$$
H_{0} a+H_{1} a \exp (W(T)-W(0))=v
$$

and then

$$
a=\frac{v}{H_{0}+H_{1} \exp (W(T)-W(0))}
$$

which is defined if and only if $H_{0}+H_{1} \exp (W(T)-W(0)) \neq 0$. The solution is then

$$
Y(t)=\frac{v \exp (W(t)-W(0))}{H_{0}+H_{1} \exp (W(T)-W(0))}=\frac{v \exp (W(t))}{H_{0} \exp (W(0))+H_{1} \exp (W(T))} .
$$

If $H_{0}$ and $H_{1}$ have different signs and are different from 0 , then the condition $H_{0}+H_{1} \exp (W(T)-W(0)) \neq 0$ is equivalent to

$$
W(T)-W(0) \neq \ln \left(-\frac{H_{0}}{H_{1}}\right)
$$

This is not ensured even when $W$ is bounded.

## 8 Young differential equations

In this section we wish to extend previous results to a framework with long-range dependence. This generally involves more regular processes and the Young integration suffices. We consider in this section $\alpha \in\left(\frac{1}{2}, 1\right]$. If U is a Banach space we denote by $\mathcal{C}^{\alpha}([0, T], \mathrm{U})$ the space of $\alpha$-Hölder rough paths $X$ and define

$$
\|X\|_{\alpha, T}=\sup _{0 \leqslant s \neq t \leqslant T} \frac{|X(s, t)|}{|t-s|^{\alpha}}<\infty .
$$

The metric on $\mathcal{C}^{\alpha}([0, T], \mathrm{U})$ is defined by

$$
\varrho_{\alpha, T}\left(X_{1}, X_{2}\right)=\sup _{0 \leqslant s \neq t \leqslant T} \frac{\left|X_{1}(s, t)-X_{2}(s, t)\right|}{|t-s|^{\alpha}}
$$

for all $X_{1}$ and $X_{2}$ in $\mathcal{C}^{\alpha}([0, T], \mathrm{U})$. If V is another Banach space, $f \in$ $C_{b}^{2}(\mathrm{~V}, \mathrm{~L}(\mathrm{U}, \mathrm{V}))$ and $a \in \mathrm{~V}$ then the following differential equation

$$
Y(t)=a+\int_{0}^{t} f(Y(s)) \mathrm{d} X(s) \text { for all } t \in[0, T]
$$

(in the Young sense) has a unique solution $Y \in C^{\alpha}([0, T], \mathrm{V})$. If we denote this solution by $I(a, X)$ to take account of the dependence on $(a, X)$, then it defines the Itô map $I$ from $\mathrm{V} \times \mathcal{C}^{\alpha}([0, T], \mathrm{U})$ to $C^{\alpha}([0, T], \mathrm{V})$. Moreover it can be proved that, if we set

$$
\mathcal{B}_{M, T}=\left\{X \in \mathcal{C}^{\alpha}([0, T], \mathrm{U}) \mid\|X\|_{\alpha, T} \leqslant M\right\}
$$

that there exists a positive constant $C_{\alpha, f, M}$ depending only on $\alpha, f$ and $M$ such that for all $a, b \in \mathrm{~V}$, and for all $X_{1}$ and $X_{2} \in \mathcal{B}_{M, T}$,

$$
\begin{equation*}
\left\|I\left(a, X_{1}\right)-I\left(b, X_{2}\right)\right\|_{\alpha, T} \leqslant C_{\alpha, f, M}\left(|a-b|+\varrho_{\alpha, T}\left(X_{1}, X_{2}\right)\right) \tag{22}
\end{equation*}
$$

In particular, for every $X \in \mathcal{C}^{\alpha}([0, T], \mathrm{U})$ there exists a positive constant $C_{\alpha, f, X}$ depending only on $\alpha, f$ and $\|X\|_{\alpha, T}$ such that for all $a$ and $b \in \mathrm{~V}$,

$$
\begin{equation*}
\|I(a, X)-I(b, X)\|_{\alpha, T} \leqslant C_{\alpha, f, X}|a-b| \tag{23}
\end{equation*}
$$

From $(22)$, we can deduce the following result whose proof is similar to that of Theorem 1

Theorem 7. Let $\tau>0$ such that

$$
\begin{equation*}
\tau<\frac{1}{\left(\left|H^{-1} H_{1}\right| C_{\alpha, f, X}\right)^{1 / \alpha}} \tag{24}
\end{equation*}
$$

If $H:=H_{0}+H_{1}$ is invertible, then there exists a unique solution $Y \in$ $C^{\alpha}([0, T], \mathrm{V})$ to

$$
\left\{\begin{array}{l}
\mathrm{d} Y(t)=f(Y(t)) \mathrm{d} X(t) \text { for all } t \in[0, T]  \tag{25}\\
H_{0} Y(0)+H_{1} Y(\tau)=v
\end{array}\right.
$$

We fix $M>0$ and $\tau>0$ such that

$$
\begin{equation*}
\tau<\frac{1}{\left(\left|H^{-1} H_{1}\right| C_{\alpha, f, M}\right)^{1 / \alpha}} \tag{26}
\end{equation*}
$$

Theorem 7 enables us to define the function $\bar{I}: \mathcal{B}_{M, T} \rightarrow C^{\alpha}([0, T], \mathrm{V})$ such that for every $X \in \mathcal{C}^{\alpha}([0, T], \mathrm{U}), \bar{I}(X)$ is the unique solution of 25 . The following theorem states that $\bar{I}$ is continuous. Its proof is similar to that of Theorem 2.

Theorem 8. The map $\bar{I}$ is Lipschitz continuous with respect to $\varrho_{\alpha, T}$ on $\mathcal{B}_{M, T}$, with constant

$$
C_{\alpha, f, M}+\frac{\left|H^{-1} H_{1}\right| C_{\alpha, f, M} \tau^{\alpha}}{1-\left|H^{-1} H_{1}\right| C_{\alpha, f, M} \tau^{\alpha}}
$$

We can deduce the following limit theorem.
Theorem 9. We assume that U is separable and that (26) is satisfied. Let $X$ and $\left(X^{n}\right)_{n}$ be a random variable and a sequence of random variables with values in $\mathcal{C}^{0, \alpha}([0, T], \mathrm{U})$ such that $X^{n}$ converges in distribution to $X$ in $\mathcal{C}^{0, \alpha}([0, T], \mathrm{U})$ and $\mathbb{P}\left(\|X\|_{\alpha, T}=M\right)=0$. For every continuous bounded function $h: C^{\alpha}([0, T], \mathrm{V}) \rightarrow \mathbb{R}$,

$$
\lim _{n \rightarrow \infty} \mathbb{E}\left(h\left(\bar{I}\left(X^{n}\right)\right) \mathbf{1}_{X^{n} \in \mathcal{B}_{M, T}}\right)=\mathbb{E}\left(h(\bar{I}(X)) \mathbf{1}_{X \in \mathcal{B}_{M, T}}\right)
$$

Now we give an application of the above result. We consider for every $\varepsilon \in(0,1]$ the problem

$$
\left\{\begin{array}{l}
\frac{\mathrm{d} Y^{\varepsilon}(t)}{\mathrm{d} t}=\sum_{j=1}^{d} \frac{1}{\varepsilon^{\gamma_{j}}} m_{j}\left(\frac{t}{\varepsilon^{2}}\right) f_{j}\left(Y^{\varepsilon}(t)\right) \text { for all } t \in[0, T]  \tag{27}\\
H_{0} Y^{\varepsilon}(0)+H_{1} Y^{\varepsilon}(\tau)=v
\end{array}\right.
$$

where for every $j, \gamma_{j} \in(0,1)$ and $m_{j}$ is a continuous stationary Gaussian process with mean zero and such that there exists $c_{j}$ such that the covariance satisfies as $t \rightarrow \infty$,

$$
\mathbb{E}\left(m_{j}(0) m_{j}(t)\right) \sim \frac{c_{j}}{t^{\gamma_{j}}}
$$

Moreover we assume that $m_{1}, m_{2}, \ldots, m_{d}$ are independent, and without loss of generality, that $\gamma_{1} \geqslant \cdots \geqslant \gamma_{d}$.

We define for every $\varepsilon$ the Lipschitz continuous process $w^{\varepsilon}$ such that for all $(s, t)$,

$$
w^{\varepsilon}(s, t)=\left(w_{1}^{\varepsilon}(s, t), \ldots, w_{d}^{\varepsilon}(s, t)\right)
$$

where for all $j$,

$$
w_{j}^{\varepsilon}(s, t)=\int_{s}^{t} \frac{1}{\varepsilon^{\gamma_{j}}} m_{j}\left(\frac{r}{\varepsilon^{2}}\right) \mathrm{d} r
$$

The following result deals with the asymptotic behavior of (27).
Theorem 10. Consider $\alpha \in\left(\frac{1}{2}, \frac{2-\gamma_{1}}{2}\right)$. For every continuous bounded function $h: C^{\alpha}([0, T], \mathrm{V}) \rightarrow \mathbb{R}$,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mathbb{E}\left(h\left(Y^{\varepsilon}\right) \mathbf{1}_{w^{\varepsilon} \in \mathcal{B}_{M, T}}\right)=\mathbb{E}\left(h(Y) \mathbf{1}_{W \in \mathcal{B}_{M, T}}\right) \tag{28}
\end{equation*}
$$

where $Y$ is the solution of

$$
\left\{\begin{array}{l}
\mathrm{d} Y(t)=\sum_{j=1}^{d} f_{j}(Y(t)) \mathrm{d} W_{j}(t) \text { for all } t \in[0, T] \\
H_{0} Y(0)+H_{1} Y(\tau)=v
\end{array}\right.
$$

and $W=\left(W_{1}, \ldots, W_{d}\right)$ is a multidimensional fractional Brownian motion such that for all $(j, k)$,

$$
\mathbb{E}\left(W_{j}(t) W_{k}(s)\right)=\delta_{j=k} \frac{\rho_{j}}{2}\left(|t|^{2 H_{j}}+|s|^{2 H_{j}}-|t-s|^{2 H_{j}}\right)
$$

with $H_{j}=\left(2-\gamma_{j}\right) / 2$ and $\rho_{j}=H_{j}^{-1}\left(2 H_{j}-1\right)^{-1} c_{j}$.
Proof. Since $\alpha \in\left(\frac{1}{2}, \frac{2-\gamma_{1}}{2}\right)$ and by 19, Theorem 5], $w^{\varepsilon}$ converges in distribution to $W$ in $\mathcal{C}^{0, \alpha}([0, T], \mathrm{V})$ as $\varepsilon \rightarrow 0$. By Theorem 9 , the proof is completed.

## 9 Conclusion

We reduced the affine boundary value problem (9) to a fixed-point problem. This approach may be used whenever the Itô map is well defined. This is especially suitable for a large range of driving rough signals (semi-martingales, fractional Brownian motion and many more) and enable us to avoid any technical difficulties related to anticipating stochastic calculus.

For every time $\tau$ small enough, using properties of the Itô map we established the existence and uniqueness of a solution, regularity results, and limit theorems.

Using the Brouwer degree we proved a global existence result. The number of solutions is finite for almost every $v$. We also gave counterexample showing that, if $f$ is not bounded, then the existence of a solution for every time $\tau$ is not ensured. This is related to the behavior of the Itô map and to the operators $H_{0}$ and $H_{1}$.

Further studies could concern sharper hypotheses, counterexamples for the existence or the uniqueness of a solution, as well as asymptotic results and their applications in multiscale modeling.

Finally, let us mention that the use of regularity structures [10] could lead to similar results where the ordinary/rough differential equations are replaced by (rough) partial differential equations.

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