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# Rough differential equations with affine boundary conditions

Antoine Lejay\*      Renaud Marty†

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## Abstract

In this paper we consider rough differential equations with affine boundary conditions. Using rough paths and fixed-point techniques, we study properties of the solutions. We also prove a convergence theorem and apply our results to differential equations driven by Brownian motions and fractional Brownian motions.

**Keywords:** Rough path theory, differential equations, affine boundary conditions.

**MSC (2020):** 60H10, 34K10, 60L90

## 1 Introduction

In this paper we study rough differential equations with affine boundary conditions. If  $U$  and  $V$  are two Banach spaces, we consider the problem

$$\begin{cases} dY(t) = f(Y(t)) dW(t) \text{ for all } t \in [0, 1] \\ H_0 Y(0) + H_1 Y(1) = v \end{cases} \quad (1)$$

where  $W$  is a continuous function or a stochastic process with values in  $U$ ,  $f : V \rightarrow L(U, V)$  is a smooth function,  $v$  is a vector in  $V$ ,  $H_0$  and  $H_1$  are two operators on  $V$ , and  $Y$  is unknown. In the case where  $W$  is a finite-dimensional Brownian motion and  $dW$  stands for the Stratonovich integration, the existence, the uniqueness and other properties of the solutions of such problems (with possibly other forms of two-point boundary conditions) have been analyzed under various assumptions and with different methods. Let us mention a few examples. Linear equations are studied in [23], where anticipating stochastic calculus (see [21]) is used because the solutions are not

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adapted to  $W$ . Nonlinear equations with additive noises and more general two-point boundary conditions are studied in [22] with a pathwise approach. In [11] invariant imbedding (see [3]) is used for the analysis of nonlinear equations. In addition to the works mentioned above, we may also cite for instance [2, 7–9, 24] where various stochastic differential equations with two-point boundary conditions are studied in other frameworks or with other methods.

In our paper we address the problem (1) in a setting where  $W$  is a more general continuous process (such as a fractional Brownian motion for instance). We use the theory of rough paths (see for instance [10, 18]). This enable us to avoid any difficulties related to the anticipating stochastic calculus. We use an enhanced Brownian motion or any other suitable rough path  $\mathbf{W}$  in place of  $W$  in order to define rough differential equations  $Z(t) = Z(0) + \int_0^t f(Z(s)) d\mathbf{W}(s)$ . For every  $a \in V$ , if  $f$  is regular enough then there exists a unique solution  $Z$  such that  $Z(0) = a$ . Then the Itô map  $I$  which assigns  $Z$  to  $(a, f, \mathbf{W})$  can be defined and is regular with respect to all its variables.

In our work, setting  $\tau \in [0, T]$  we relate (1) to

$$\begin{cases} Y(t) = Y(0) + \int_0^t f(Y(s)) d\mathbf{W}(s) \text{ for } t \in [0, T], \\ H_0 Y(0) + H_1 Y(\tau) = v. \end{cases} \quad (2)$$

and study various properties. Equation (2) links the values of  $Y(0)$  and  $Y(\tau)$  for a given value of  $\tau \in [0, T]$ , while the path  $\{Y(t)\}_{t \in [0, T]}$  is defined on the whole interval  $[0, T]$ .

Using the Itô map and a fixed-point method, we prove that there exists a unique solution to (2) if  $\tau < C$  where  $C$  is a constant that depends on  $T$ ,  $f$ ,  $\mathbf{W}$ ,  $H_0$ , and  $H_1$  (we note that possibly  $C \geq T$ ). In this setting we also establish regularity results, which enable us to prove limit theorems.

For  $\tau \in [C, T]$  and when  $V$  is finite-dimensional, using the Brouwer degree we show that there exists a solution. While the uniqueness is not ensured, the number of solutions is finite for almost every  $v$ . The situation is summarized in Figure 1.

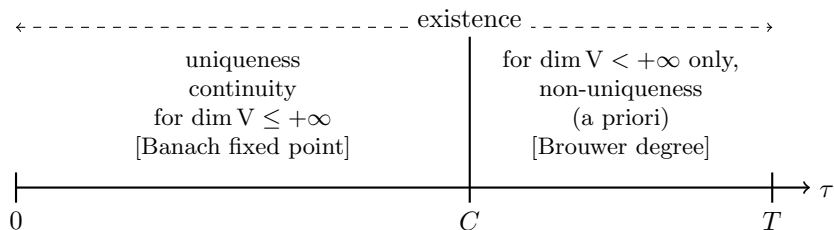


Figure 1: Summary of the results on existence and uniqueness depending on  $\tau$  for fixed time  $T$ .

**Outline.** In Section 2 we recall results on rough paths theory. In Section 3 we prove the existence and uniqueness of a solution for a small time  $\tau$ . In Section 4 we study the regularity of the solution with respect to the driving rough noise. In Sections 5 and 6 we establish more results on regularity and convergence. In Section 7 we study the existence of solutions for any time  $\tau$ . Finally, in Section 8 we consider the case of a driving rough noise with Hölder regularity greater than  $1/2$ . In this context, the rough integrals coincide with the Young integrals.

## 2 Reminders about rough paths

We consider in this section the case of rough differential equations driven by paths of Hölder regularity  $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ .

For two Banach space  $U$  and  $V$  whose norms are written  $|\cdot|$ , we denote by  $L(U, V)$  the space of bounded, linear operators from  $U$  to  $V$ . Here,  $U$  and  $V$  are finite or infinite dimensional.

We denote by  $C^\alpha(W, V)$  the space of  $\alpha$ -Hölder continuous maps from  $W$  to  $V$  for  $W \subset U$ . The space is equipped with the  $\alpha$ -Hölder semi-norm  $\|f\|_\alpha := \sup_{x \neq y} |f(x) - f(y)|/|x - y|^\alpha$ . When  $W = [0, T] \subset \mathbb{R}$ , we write  $\|\cdot\|_{\alpha, T}$  instead of  $\|\cdot\|_\alpha$ .

We denote by  $C^k(U, V)$  for some integer  $k$  the space of continuous functions which are differentiable up to order  $k$  with continuous  $k$ -th derivatives. We denote by  $C_b^k(U, V)$  the space of functions in  $C^k(U, V)$  with bounded  $j$ -th derivatives for all  $0 \leq j \leq k$ .

We denote by  $\mathcal{C}^\alpha([0, T], U)$  the space of  $\alpha$ -Hölder rough paths  $\mathbf{X} = (X, \mathbb{X})$  with values in  $U \oplus (U \otimes U)$ . The elements of  $\mathcal{C}^\alpha([0, T], U)$  satisfy the Chen relation

$$X(r, s) + X(s, t) = X(r, t), \quad \mathbb{X}(r, s) + \mathbb{X}(s, t) + X(r, s) \otimes X(s, t) = \mathbb{X}(r, t)$$

for any  $0 \leq r \leq s \leq t \leq T$ . Besides

$$\|X\|_{\alpha, T} = \sup_{0 \leq s \neq t \leq T} \frac{|X(s, t)|}{|t - s|^\alpha} < \infty \quad \text{and} \quad \|\mathbb{X}\|_{2\alpha, T} = \sup_{0 \leq s \neq t \leq T} \frac{|\mathbb{X}(s, t)|}{|t - s|^{2\alpha}} < \infty.$$

The metric on  $\mathcal{C}^\alpha([0, T], U)$  is denoted by  $\varrho_{\alpha, T}$  and defined by

$$\varrho_{\alpha, T}(\mathbf{X}_1, \mathbf{X}_2) = \sup_{0 \leq s \neq t \leq T} \frac{|X_1(s, t) - X_2(s, t)|}{|t - s|^\alpha} + \sup_{0 \leq s \neq t \leq T} \frac{|\mathbb{X}_1(s, t) - \mathbb{X}_2(s, t)|}{|t - s|^{2\alpha}}$$

for all  $\mathbf{X}_1$  and  $\mathbf{X}_2$  in  $\mathcal{C}^\alpha([0, T], U)$ .

Let  $U$  and  $V$  be two Banach spaces,  $\mathbf{X} \in \mathcal{C}^\alpha([0, T], U)$ ,  $f \in C_b^2(V, L(U, V))$  and  $a \in V$ . We consider the following differential equation

$$Y(t) = a + \int_0^t f(Y(s)) d\mathbf{X}(s) \quad \text{for all } t \in [0, T] \quad (3)$$

with unknown  $Y \in C^\alpha([0, T], V)$ . Following [6, 10], a solution  $Y$  of (9) can be defined as a path such that  $Y(0) = a$  and

$$Y(t) - Y(s) = f(Y(s))X(s, t) + Df(Y(s))f(Y(s))\mathbb{X}(s, t) + O(|t - s|^\theta) \quad (4)$$

for some  $\theta > 1$ .

Following [10, Theorem 8.3], if  $f \in C_b^3(V, L(U, V))$ , then there exists a unique solution  $Y \in C^\alpha([0, T], V)$  to Equation (4) (or (3)). In order to take account of the dependence on  $(a, \mathbf{X})$ , we set  $I(a, \mathbf{X}) := Y$ . This defines the *Itô map*  $I$  from  $V \times C^\alpha([0, T], U)$  to  $C^\alpha([0, T], V)$ . The next lemma establishes the boundedness of this solution (see for instance [10, Proposition 8.2]).

**Lemma 1.** *There exists a positive constant  $K = K(\alpha)$  depending only on  $\alpha$  such that for all  $a \in V$ ,*

$$\|I(a, \mathbf{X})\|_{\alpha, T} \leq K(\alpha) \max \left( \|f\|_{C^2} \|\mathbf{X}\|_{\alpha, T}, (\|f\|_{C^2} \|\mathbf{X}\|_{\alpha, T})^{1/\alpha} \right), \quad (5)$$

where

$$\|\mathbf{X}\|_{\alpha, T} := \|X\|_{\alpha, T} + \sqrt{\|\mathbb{X}\|_{2\alpha, T}}.$$

By T. Lyons' theorem [17] the Itô map  $I$  is continuous. Moreover it is locally Lipschitz continuous, as recalled in the following lemma (see for instance [5, 10, 15]). Notice that some extended statements consider also variations of the vector field  $f$ .

**Lemma 2.** *We fix  $M > 0$  and consider*

$$\mathcal{B}_{M, T} = \{\mathbf{X} \in C^\alpha([0, T], U) : \|\mathbf{X}\|_{\alpha, T} \leq M\}, \quad (6)$$

where  $\|\mathbf{X}\|_{\alpha, T}$  is defined in Lemma 1. There exists a positive constant  $C_{\alpha, f, M}$  depending only on  $\alpha$ ,  $f$  and  $M$  such that for all  $a$  and  $b \in V$ , and for all  $\mathbf{X}_1$  and  $\mathbf{X}_2 \in \mathcal{B}_{M, T}$ ,

$$\|I(a, \mathbf{X}_1) - I(b, \mathbf{X}_2)\|_{\alpha, T} \leq C_{\alpha, f, M} (|a - b| + \varrho_{\alpha, T}(\mathbf{X}_1, \mathbf{X}_2)). \quad (7)$$

In particular, for every  $\mathbf{X} \in C^\alpha([0, T], U)$  there exists a positive constant  $C_{\alpha, f, \mathbf{X}}$  depending only on  $\alpha$ ,  $f$  and  $\|\mathbf{X}\|_{\alpha, T}$  such that for all  $a$  and  $b \in V$ ,

$$\|I(a, \mathbf{X}) - I(b, \mathbf{X})\|_{\alpha, T} \leq C_{\alpha, f, \mathbf{X}} |a - b|. \quad (8)$$

For the sake of simplicity, we do not consider optimal hypotheses on the vector field. The assumptions on the regularity of  $f$  can be weakened (for example when  $f \in C_b^{2+\gamma}$  with  $\alpha(2 + \gamma) > 1$ ) but to get a global Lipschitz condition with respect to the starting point, we highlight that  $f$  shall be bounded. This latter hypothesis is used in several proofs.

### 3 Existence and uniqueness

Let  $v \in V$  and  $H_0$  and  $H_1$  be two bounded operators on  $V$  (finite or infinite dimensional). We set  $\tau$  and  $T$  such that  $0 < \tau \leq T$ .

Throughout the paper and unless otherwise mentioned, we assume that  $f \in C_b^3(V, L(U, V))$ .

In this section we fix  $\mathbf{X} \in C^\alpha([0, T], U)$  and we wish to find a continuous function  $Y$  such that

$$\begin{cases} Y(t) = Y(0) + \int_0^t f(Y(s)) d\mathbf{X}(s) \text{ for all } t \in [0, T], \\ H_0 Y(0) + H_1 Y(\tau) = v. \end{cases} \quad (9)$$

Then we consider the following problem. We look for an element  $a \in V$  such that

$$H_0 a + H_1 I(a)(\tau) = v, \quad (10)$$

where  $I$  is the Itô map and  $I(a, \mathbf{X})$  is denoted by  $I(a)$  since  $\mathbf{X}$  is fixed. The following lemma establishes the relation between (10) and (9).

**Lemma 3.** *If there exists  $a \in V$  such that (10) holds, then  $Y = I(a)$  satisfies (9). Moreover, if there exists a unique  $a \in V$  such that (10) holds, then  $Y = I(a)$  is the unique function in  $C^\alpha([0, T], V)$  satisfying (9).*

*Proof.* The first part of the lemma is obvious. Assume there exists a unique  $a \in V$  such that (10) and set  $Y = I(a)$ . Consider another  $Z \in C^\alpha([0, T], V)$  such that

$$\begin{cases} Z(t) = Z(0) + \int_0^t f(Z(s)) d\mathbf{X}(s) \text{ for all } t \in [0, T], \\ H_0 Z(0) + H_1 Z(\tau) = v. \end{cases}$$

By the definition of the Itô map,  $Z = I(Z(0))$ , so that  $H_0 Z(0) + H_1 I(Z(0))(\tau) = v$ . This implies that  $Z(0) = a$  (by uniqueness of  $a$ ), and then  $Z = I(Z(0)) = I(a) = Y$ .  $\square$

The following theorem establishes the existence of a unique solution of the problem (9) for small values of  $\tau$ . It is proved with the use of (10), which is solved with the Banach fixed-point theorem.

For  $a \in V$  and  $\tau \in [0, T]$ , we define

$$L(a)_\tau := H^{-1}v - H^{-1}H_1(I(a)(\tau) - a) \quad (11)$$

when  $H := H_0 + H_1$  is invertible.

*Remark 1.* The condition that  $H_0 + H_1$  is invertible means that our results cannot be applied to deal with a periodic solution.

**Theorem 1.** Let  $C_{\alpha,f,\mathbf{X}}$  be the constant of (8). If  $H := H_0 + H_1$  is invertible and  $\tau$  satisfies

$$\tau^\alpha |H^{-1}H_1| C_{\alpha,f,\mathbf{X}} < 1, \quad (12)$$

then there exists a unique solution to (9) in  $C^\alpha([0, T], \mathbb{V})$ .

*Proof.* By Lemma 3, it suffices to prove that there exists a unique solution  $a \in \mathbb{V}$  to (10). We rewrite (10) as

$$(H_0 + H_1)a + H_1(I(a)(\tau) - a) = v.$$

Then, since  $H = H_0 + H_1$  is invertible,

$$H_0a + H_1I(a)(\tau) = v \text{ if and only if } a = H^{-1}v - H^{-1}H_1(I(a)(\tau) - a).$$

With (11), (10) can be reformulated as the fixed-point problem

$$a = L(a)_\tau. \quad (13)$$

By the Banach fixed-point theorem, it suffices to prove that  $L(\cdot)_\tau$  is a contraction. For all  $a, b \in \mathbb{V}$ ,

$$\begin{aligned} |L(a)_\tau - L(b)_\tau| &\leq |H^{-1}H_1| \cdot |I(a)(\tau) - a - I(b)(\tau) + b| \\ &\leq |H^{-1}H_1| \cdot (|I(a)_0 - a - I(b)_0 + b| + \tau^\alpha \|I(a) - a - I(b) + b\|_{\alpha,T}) \\ &= \tau^\alpha |H^{-1}H_1| \cdot \|I(a) - a - I(b) + b\|_{\alpha,T} \\ &= \tau^\alpha |H^{-1}H_1| \cdot \|I(a) - I(b)\|_{\alpha,T}. \end{aligned}$$

Following (8),

$$\|I(a) - I(b)\|_{\alpha,T} \leq C_{\alpha,f,\mathbf{X}} |a - b|.$$

As a consequence of above,

$$|L(a)_\tau - L(b)_\tau| \leq \tau^\alpha |H^{-1}H_1| C_{\alpha,f,\mathbf{X}} |a - b|.$$

Then for every  $\tau$  satisfying (12), the function  $L(\cdot)_\tau$  is a contraction. Then by the Banach fixed-point theorem, there exists a unique fixed point  $a \in \mathbb{V}$  such that (13), and then (10) are satisfied.  $\square$

As an application of Theorem 1, fix  $\tau \in [0, T]$  and consider that  $W = (W_1, \dots, W_d)$  is a Brownian motion with values in  $\mathbb{U} = \mathbb{R}^d$  which is enhanced with  $\mathbf{W} = (W, \mathbb{W}) \in \mathcal{C}^\alpha([0, T], \mathbb{R}^d)$  with for all  $1 \leq j, k \leq d$ ,

$$\mathbb{W}_{j,k}(s, t) = \int_s^t (W_j(r) - W_j(s)) \circ dW_k(r) \text{ for any } 0 \leq s \leq t \leq T. \quad (14)$$

Then, Theorem 1 establishes that for every  $\omega \in \Omega$  such that  $\tau^\alpha |H^{-1}H_1|C_{\alpha,f,\mathbf{W}(\omega)} < 1$ , there exists a unique solution  $Y(\omega) \in C^\alpha([0, T], V)$  to

$$\begin{cases} Y(\omega)(t) = Y(\omega)(0) + \int_0^t \sum_{j=1}^d f_j(Y(\omega)(s)) \circ dW_j(\omega)(s), & t \in [0, T], \\ H_0 Y(\omega)(0) + H_1 Y(\omega)(\tau) = v. \end{cases} \quad (15)$$

Notice that  $\int_0^t \sum_{j=1}^d f_j(Y(\omega)(s)) \circ dW_j(\omega)(s)$  stands for  $\int_0^t f(Y(\omega)(s)) d\mathbf{W}(\omega)(s)$  because of the Stratonovich integration in (14) and both rough and Stratonovich integrals coincide. This result follows from Theorem 4 below which is the rough counterpart of the stochastic Wong-Zakai theorem, and the construction of [11] based on characteristics as solutions to some Stratonovich SDE.

This problem can also be solved in an infinite-dimensional setting. Suppose that  $U$  is a separable Banach space and consider a  $U$ -valued Brownian motion  $W$  on the Wiener space  $(U, H, \mu)$  where  $\mu$  is a zero-mean Gaussian measure on  $U$  and  $H$  is the Cameron-Martin space of  $\mu$ . Following [14], under suitable conditions, there exists an enhanced Brownian motion  $\mathbf{W} = (W, \mathbb{W}) \in C^\alpha([0, T], U)$  obtained as follows. We consider the sequence  $(\mathbf{W}(m))_{m \geq 1} = (W(m), \mathbb{W}(m))_{m \geq 1}$  where  $(W(m))_{m \geq 1}$  is the dyadic piecewise linear approximation of  $W$ , and  $(\mathbb{W}(m))_{m \geq 1}$  is defined by

$$\mathbb{W}(m)(s, t) = \int_s^t (W(m)(r) - W(m)(s)) \otimes dW(m)(r)$$

for all  $s, t$ , and  $m$ . Assume that the tensor norm  $\|\cdot\|_{U \otimes U}$  on  $U \otimes U$  is exact for  $\mu$ , which means that there exist two constants  $\gamma \in [1/2, 1)$  and  $C > 0$  such that for all independent sequences  $(G_n)_{n \geq 1}$  and  $(\tilde{G}_n)_{n \geq 1}$  of independent  $U$ -valued random variables with distribution  $\mu$ , and for every  $N \geq 0$ ,

$$\mathbb{E} \left( \left\| \sum_{n=1}^N G_n \otimes \tilde{G}_n \right\|_{U \otimes U} \right) \leq CN^\gamma.$$

By [14], under this assumption, as  $m \rightarrow \infty$  the sequence  $\mathbf{W}(m)$  converges in  $C^\alpha([0, T], U)$  almost surely. Its limit defines an enhanced Brownian motion. Then by Theorem 1, for every  $\omega \in \Omega$  such that  $\tau^\alpha |H^{-1}H_1|C_{\alpha,f,\mathbf{W}(\omega)} < 1$ , there exists a unique solution  $Y(\omega) \in C^\alpha([0, T], V)$  to

$$\begin{cases} Y(\omega)(t) = Y(\omega)(0) + \int_0^t f(Y(\omega)(s)) d\mathbf{W}(\omega)(s) \text{ for all } t \in [0, T], \\ H_0 Y(\omega)(0) + H_1 Y(\omega)(\tau) = v. \end{cases}$$

## 4 Continuity of the solution with respect to the path

In the previous section the driving path  $\mathbf{X}$  is fixed. In this section we consider  $\mathbf{X}$  as a variable and we establish the continuity of the solution of (9) with respect to  $\mathbf{X}$ .



We set  $M > 0$  and  $\tau$  satisfying

$$\tau^\alpha |H^{-1}H_1|C_{\alpha,f,M} < 1. \quad (16)$$

When  $\mathbf{X} \in \mathcal{B}_{M,T}$  (defined by (6)), (16) implies (12).

We define the function

$$\bar{I} : \mathcal{B}_{M,T} \rightarrow C^\alpha([0, T], \mathbf{V})$$

such that for every  $\mathbf{X} \in \mathcal{B}_{M,T}$ ,  $\bar{I}(\mathbf{X})$  is the unique solution  $Y$  to

$$\begin{cases} Y(t) = Y(0) + \int_0^t f(Y(s)) d\mathbf{X}(s) \text{ for all } t \in [0, T], \\ H_0 Y(0) + H_1 Y(\tau) = v. \end{cases}$$

The main theorem of this subsection establishes the continuity of  $\bar{I}$ .

**Theorem 2.** *The map  $\bar{I}$  is Lipschitz continuous with respect to  $\varrho_{\alpha,T}$  on  $\mathcal{B}_{M,T}$ , with constant*

$$C_{\alpha,f,M} + \frac{|H^{-1}H_1|C_{\alpha,f,M}\tau^\alpha}{1 - |H^{-1}H_1|C_{\alpha,f,M}\tau^\alpha}.$$

*Proof.* Previously we proved that for every  $\mathbf{X}$ ,  $\bar{I}(\mathbf{X})$  can be written as  $\bar{I}(\mathbf{X}) = I(a_{\mathbf{X}}, \mathbf{X})$  where  $I$  is the Itô map and  $a_{\mathbf{X}}$  is the fixed point of the function  $L(\cdot, \mathbf{X})_\tau : a \mapsto L(a, \mathbf{X})_\tau = H^{-1}v - H^{-1}H_1(I(a, \mathbf{X})(\tau) - a)$ .

By (7), for all  $\mathbf{X}$  and  $\mathbf{Y} \in \mathcal{B}_{M,T}$ ,

$$\begin{aligned} \|\bar{I}(\mathbf{X}) - \bar{I}(\mathbf{Y})\|_{\alpha,T} &= \|I(a_{\mathbf{X}}, \mathbf{X}) - I(a_{\mathbf{Y}}, \mathbf{Y})\|_{\alpha,T} \\ &\leq C_{\alpha,f,M}(|a_{\mathbf{X}} - a_{\mathbf{Y}}| + \varrho_{\alpha,T}(\mathbf{X}, \mathbf{Y})). \end{aligned} \quad (17)$$

Moreover,

$$\begin{aligned} |a_{\mathbf{X}} - a_{\mathbf{Y}}| &= |L(a_{\mathbf{X}}, \mathbf{X})_\tau - L(a_{\mathbf{Y}}, \mathbf{Y})_\tau| \\ &\leq |L(a_{\mathbf{X}}, \mathbf{X})_\tau - L(a_{\mathbf{X}}, \mathbf{Y})_\tau| + |L(a_{\mathbf{X}}, \mathbf{Y})_\tau - L(a_{\mathbf{Y}}, \mathbf{Y})_\tau| \\ &= |H^{-1}H_1| \cdot |I(a_{\mathbf{X}}, \mathbf{X})_\tau - I(a_{\mathbf{X}}, \mathbf{Y})_\tau| \\ &\quad + |H^{-1}H_1| \cdot |I(a_{\mathbf{X}}, \mathbf{Y})_\tau - I(a_{\mathbf{Y}}, \mathbf{Y})_\tau - a_{\mathbf{X}} + a_{\mathbf{Y}}| \\ &\leq |H^{-1}H_1| \cdot \|I(a_{\mathbf{X}}, \mathbf{X}) - I(a_{\mathbf{X}}, \mathbf{Y})\|_{\alpha,T}\tau^\alpha \\ &\quad + |H^{-1}H_1| \cdot \|I(a_{\mathbf{X}}, \mathbf{Y}) - I(a_{\mathbf{Y}}, \mathbf{Y})\|_{\alpha,T}\tau^\alpha \\ &\leq |H^{-1}H_1|C_{\alpha,f,M}\varrho_{\alpha,T}(\mathbf{X}, \mathbf{Y})\tau^\alpha \\ &\quad + |H^{-1}H_1|C_{\alpha,f,M}|a_{\mathbf{X}} - a_{\mathbf{Y}}|\tau^\alpha. \end{aligned}$$

As a consequence, for all  $\mathbf{X}$  and  $\mathbf{Y} \in \mathcal{B}_{M,T}$ ,

$$|a_{\mathbf{X}} - a_{\mathbf{Y}}| \leq \frac{|H^{-1}H_1|C_{\alpha,f,M}\tau^\alpha}{1 - |H^{-1}H_1|C_{\alpha,f,M}\tau^\alpha} \varrho_{\alpha,T}(\mathbf{X}, \mathbf{Y})$$

and, by (17),

$$\|\bar{I}(\mathbf{X}) - \bar{I}(\mathbf{Y})\|_{\alpha, T} \leq \left( C_{\alpha, f, M} + \frac{|H^{-1}H_1|C_{\alpha, f, M}\tau^\alpha}{1 - |H^{-1}H_1|C_{\alpha, f, M}\tau^\alpha} \right) \varrho_{\alpha, T}(\mathbf{X}, \mathbf{Y}).$$

This completes the proof.  $\square$

## 5 Regularity and numerical analysis

In Section 4 we proved the continuity of the solution with respect to the path  $\mathbf{X}$ . In this section, with the implicit function theorem, we consider the regularity with respect to the other variables. Then we briefly present an application to numerical analysis.

For  $\gamma \in (0, 1]$  and  $k \in \mathbb{N}^*$ , we denote by  $C^{k+\gamma}$  the class of functions  $f$  of class  $C^k$  such that  $D^k f$  is  $\gamma$ -Hölder continuous. We denote by  $C_b^{k+\gamma}$  the space of functions in  $C^{k+\gamma}$  with bounded  $j$ -th derivatives for all  $0 \leq j \leq k$ . This space is equipped with

$$\|f\|_{C^{k+\gamma}} := \max\{\|f\|_\infty, \dots, \|D^k f\|_\infty, \|D^k f\|_\gamma\}.$$

Let  $\mathbf{X} \in \mathcal{C}^\alpha([0, T], \mathbf{U})$ . We define

$$\begin{aligned} \mathcal{P} &:= \mathbf{V} \times C_b^{k+1+\gamma}(\mathbf{V}, L(\mathbf{U}, \mathbf{V})), \\ \text{and } \mathcal{Q} &:= C_b^{k+1+\gamma}(\mathbf{V}, L(\mathbf{U}, \mathbf{V})) \times \mathbf{V} \times L(\mathbf{V}, \mathbf{V}) \times L(\mathbf{V}, \mathbf{V}). \end{aligned}$$

**Proposition 1** ([5, Theorem 3]). *For any  $\kappa \in (0, 1)$  such that  $\alpha(2 + \kappa\gamma) > 1$ , the Itô map  $I : \mathcal{P} \rightarrow C^\alpha([0, T], \mathbf{V})$  is locally of class  $C^{k+(1-\kappa)\gamma}$ .*

Let us write

$$P(a, q)_\tau = H_0 a + H_1 I(a, f, \mathbf{X})(\tau) - v \text{ with } q = (f, v, H_0, H_1) \in \mathcal{Q}.$$

If  $H_0 + H_1$  is invertible, the problem  $L(a)_\tau = a$  is equivalent to  $P(a, q)_\tau = 0$ . Besides, by Proposition 1,  $P$  is locally of class  $C^{k+(1-\kappa)\gamma}$  on  $\mathcal{Q}$  when  $\alpha(2 + \kappa\gamma) > 1$ . Moreover,

$$D_a P(a, q)_\tau = H_0 + H_1 D_a I(a, f, \mathbf{X})(\tau), \quad (18)$$

where

$$D_a I(a, f, \mathbf{X})(t) = \text{Id} + \int_0^t Df(I(a, f, \mathbf{X})(s)) \cdot D_a I(a, f, \mathbf{X})(s) ds.$$

The regularity is established by the following result, which is a generalized implicit function theorem (see *e.g.*, [1]) applied to  $P$ .

**Theorem 3.** Let  $\tau \in [0, T]$ ,  $a^\dagger \in V$  and  $q^\dagger \in \mathcal{Q}$  such that  $P(a^\dagger, q^\dagger)_\tau = 0$  and  $D_a P(a^\dagger, q^\dagger)_\tau \in L(V, V)$  is an isomorphism. Then there exist neighborhoods  $U^\dagger$  of  $q^\dagger$  and  $V^\dagger$  of  $a^\dagger$  as well as a function  $A \in C_b^{k+(1-\kappa)\gamma}(U^\dagger, V^\dagger)$  with  $A(q^\dagger) = a^\dagger$  such that  $P(A(q), q) = 0$  for any  $q \in U^\dagger$ .

*Remark 2.* At  $\tau = 0$ ,  $D_a I(a, f, \mathbf{X})(0) = \text{Id}$  and then  $D_a P(a, q)_0 = H_0 + H_1$ . To apply the implicit function theorem,  $D_a P(a, q)_0$  has to be invertible. Hence,  $H_0 + H_1$  shall be invertible which is an hypothesis we used to get the existence and uniqueness of a solution.

*Remark 3.* If  $H_0 + H_1$  is invertible, then we can deduce from (18) that  $D_a P(a, q)_\tau$  is invertible when  $\tau$  is small enough. This not ensured for large  $\tau$ .

Theorem 3 implies that the solution  $a$  is locally Lipschitz continuous with respect to  $q = (f, v, H_0, H_1)$ . Notice that we can also establish that  $a$  is globally Lipschitz continuous with respect to  $v$ . More precisely, using (8) we get

$$|a(v_1) - a(v_2)| \leq (1 - \tau^\alpha |H^{-1} H_1| C_{\alpha, f, \mathbf{X}}) |v_1 - v_2| \quad (19)$$

for all  $v_1$  and  $v_2$ .

We now present an application to numerical analysis. Let us consider that for any starting point  $a$  we have a numerical scheme  $J(a)$  such that  $J(a) = I(a, f, \mathbf{X})(\tau) + R(a)$  for a Lipschitz continuous error  $a \mapsto R(a)$  with a Lipschitz norm  $\ell$  smaller than 1. Several numerical schemes exist for rough paths and for RDE driven by Brownian motion or fractional Brownian motion (see for instance [13, 16, 20]).

Besides, let us assume that we are able to solve the fixed-point problem  $H_0 a + H_1 J(a) - v = 0$  at precision  $\eta$ , that is we find  $a'$  such that

$$H_0 a' + H_1 J(a') = v + \eta.$$

This is the case for example when we use a finite number of iterations in the fixed-point iteration, or the Newton scheme. Then

$$H_0 a' + H_1 I(a', f, \mathbf{X})(\tau) = v + \eta - H_1 R(a').$$

It follows from Theorem 3 or (19) that there exists a constant  $K$ , depending on  $\tau, \ell, f, \mathbf{X}, H_0$  and  $H_1$  such that

$$|a' - a| \leq K|\eta| + K|R(a)|.$$

This implies that the numerical scheme converges.

## 6 Limit theorems

In this section we deal with limit theorems. We fix  $\tau$  and  $M$  such that (16), namely

$$\tau^\alpha |H^{-1} H_1| C_{\alpha, f, M} < 1.$$

For every  $T > 0$  we denote by  $\mathcal{C}^{0,\alpha}([0, T], \mathbb{U})$  the closure of smooth rough paths in  $\mathcal{C}^\alpha([0, T], \mathbb{U})$ . We assume that the Banach space  $\mathbb{U}$  is separable and recall that, as a consequence,  $\mathcal{C}^{0,\alpha}([0, T], \mathbb{U})$  is a Polish space (see Exercise 2.8 of [10] for instance). The main result is the following.

**Theorem 4.** *Suppose (16) and let  $\mathbf{X}$  and  $(\mathbf{X}^n)_n$  be a random variable and a sequence of random variables with values in  $\mathcal{C}^{0,\alpha}([0, T], \mathbb{U})$  such that  $\mathbf{X}^n$  converges in distribution to  $\mathbf{X}$  in  $\mathcal{C}^{0,\alpha}([0, T], \mathbb{U})$  and  $\mathbb{P}(\|\mathbf{X}\|_{\alpha,T} = M) = 0$ . For every continuous bounded function  $h : \mathcal{C}^\alpha([0, T], \mathbb{V}) \rightarrow \mathbb{R}$ ,*

$$\lim_{n \rightarrow \infty} \mathbb{E}(h(\bar{I}(\mathbf{X}^n))\mathbf{1}_{\|\mathbf{X}^n\|_{\alpha,T} \leq M}) = \mathbb{E}(h(\bar{I}(\mathbf{X}))\mathbf{1}_{\|\mathbf{X}\|_{\alpha,T} \leq M}).$$

*Proof.* Since  $\mathcal{C}^{0,\alpha}([0, T], \mathbb{U})$  is a Polish space, by the Skorokhod representation theorem there exists a probability space  $(\Omega^*, \mathcal{T}^*, \mathbb{P}^*)$  and random variables  $\mathbf{X}^{*,n}$  and  $\mathbf{X}^*$  from  $(\Omega^*, \mathcal{T}^*, \mathbb{P}^*)$  to  $\mathcal{C}^{0,\alpha}([0, T], \mathbb{U})$  such that  $\mathbf{X}^{*,n} \stackrel{\text{law}}{=} \mathbf{X}^n$ ,  $\mathbf{X}^* \stackrel{\text{law}}{=} \mathbf{X}$  and  $\mathbf{X}^{*,n}$  converges  $\mathbb{P}^*$ -almost surely to  $\mathbf{X}^*$ . Then  $\mathbb{P}^*$ -almost surely,  $h(\bar{I}(\mathbf{X}^{*,n}))\mathbf{1}_{\|\mathbf{X}^{*,n}\|_{\alpha,T} \leq M}$  converges to  $h(\bar{I}(\mathbf{X}^*))\mathbf{1}_{\|\mathbf{X}^*\|_{\alpha,T} \leq M}$  as  $n \rightarrow \infty$ . Moreover, by the bounded convergence theorem, since for all  $n$ ,

$$h(\bar{I}(\mathbf{X}^{*,n}))\mathbf{1}_{\|\mathbf{X}^{*,n}\|_{\alpha,T} \leq M} \leq \|h\|_\infty,$$

we deduce that

$$\lim_{n \rightarrow \infty} \mathbb{E}(h(\bar{I}(\mathbf{X}^{*,n}))\mathbf{1}_{\|\mathbf{X}^{*,n}\|_{\alpha,T} \leq M}) = \mathbb{E}(h(\bar{I}(\mathbf{X}^*))\mathbf{1}_{\|\mathbf{X}^*\|_{\alpha,T} \leq M}).$$

This completes the proof.  $\square$

As an application, we consider for every  $n \in \mathbb{N}^*$  the problem

$$\begin{cases} \frac{dY^n(t)}{dt} = \sqrt{n} \sum_{j=1}^d \xi_{j,[nt]} f_j(Y^n(t)) \text{ for all } t \in [0, T], \\ H_0 Y^n(0) + H_1 Y^n(\tau) = v, \end{cases}$$

where  $\xi := (\xi_{j,k})_{(j,k) \in \{1, \dots, d\} \times \mathbb{N}}$  is a discrete family of independent random variables with mean zero, variance 1, same distribution and finite moments. We define the sequence of  $\mathbb{R}^d$ -valued processes  $W^n = (W_1^n, \dots, W_d^n)$  such that for all  $j = 1, \dots, d$ ,

$$W_j^n(t) = \frac{1}{\sqrt{n}} \left( \sum_{k=0}^{[nt]-1} \xi_{j,k} + (nt - [nt])\xi_{j,[nt]} \right).$$

Notice that  $W^n$  is Lipschitz-continuous. Therefore,  $W^n$  can be enhanced with  $\mathbf{W}^n = (W^n, \mathbb{W}^n) \in \mathcal{C}^\alpha([0, T], \mathbb{R}^d)$  where

$$\mathbb{W}^n(s, t) = \int_s^t (W^n(r) - W^n(s)) \otimes dW^n(r).$$

The following theorem establishes the convergence in distribution of  $Y^n$  as  $n$  goes to infinity.

**Theorem 5.** For every continuous bounded function  $h : C^\alpha([0, T], V) \rightarrow \mathbb{R}$ ,

$$\lim_{n \rightarrow \infty} \mathbb{E}(h(Y^n) \mathbf{1}_{\mathbf{W}^n \in \mathcal{B}_{M, T}}) = \mathbb{E}(h(Y) \mathbf{1}_{\mathbf{W} \in \mathcal{B}_{M, T}}),$$

where  $\mathbf{W}$  is the enhanced Brownian motion defined by (14) and  $Y$  is the solution of

$$\begin{cases} dY(t) = \sum_{j=1}^d f_j(Y(t)) \circ dW_j(t) \text{ for all } t \in [0, T], \\ H_0 Y(0) + H_1 Y(\tau) = v. \end{cases}$$

This theorem is similar to [11, Theorem 7] but its proof is different.

*Proof.* It is a direct consequence of [4, Theorem 1] and of Theorem 4.  $\square$

## 7 Global existence result

In Section 3 we prove that (9) (and (10)) has a unique solution for every  $\tau$  small enough. In this section we use the Brouwer degree in order to prove that (9) has a solution (maybe more than one) for every  $\tau \in [0, T]$ , yet only when  $V$  is finite-dimensional.

Throughout the section, we assume that  $V = \mathbb{R}^N$  for some  $N \in \mathbb{N}^*$ . We equip  $V$  with the Lebesgue measure.

Before we establish the existence of solution for (9) in Theorem 6, we remind the definition and some properties of the Brouwer degree. For a more complete exposition on the degree theory we refer the reader for instance to [12].

Consider a bounded open subset  $\mathcal{O} \subset \mathbb{R}^N = V$ , a continuous function  $\Phi : \overline{\mathcal{O}} \rightarrow \mathbb{R}^N$  and  $y \in V$  such that  $y \notin \Phi(\partial\mathcal{O})$ . The Brouwer degree  $\deg(\Phi, \mathcal{O}, y)$  with  $\Phi \in C^1(\mathcal{O}, V)$  can be defined as follows (see for instance [12, Definition 13.2.9]).

**Definition 1.** If  $y$  is a regular value of  $\Phi$  (namely such that  $\det J_\Phi(x) \neq 0$  for all  $x \in \Phi^{-1}(\{y\})$ ), then  $\deg(\Phi, \mathcal{O}, y)$  is defined by

$$\deg(\Phi, \mathcal{O}, y) = \sum_{x \in \Phi^{-1}(\{y\})} \text{sign}(\det J_\Phi(x)).$$

If  $y$  is a critical value of  $\Phi$  ( $y$  is not a regular value of  $\Phi$ ), then  $\deg(\Phi, \mathcal{O}, y)$  is defined by

$$\deg(\Phi, \mathcal{O}, y) = \deg(\Phi, \mathcal{O}, y'),$$

where  $y'$  is a regular value of  $\Phi$  such that  $|y - y'| < \text{dist}(y, \Phi(\partial\mathcal{O}))$ .

The Brouwer degree has the following property (see for instance [12, Theorem 13.2.11(iv)]).

**Lemma 4.** *Let  $y \in V$  such that  $y \notin \Phi(\partial\mathcal{O})$ . If  $\deg(\Phi, \mathcal{O}, y) \neq 0$  then there exists a solution  $x \in \mathcal{O}$  to the equation  $\Phi(x) = y$ .*

The invariance by homotopy is another important property of the Brouwer degree and can be stated as follows (see for instance [12, Theorem 13.2.11(ii)]).

**Lemma 5.** *We fix  $\tau > 0$  and consider a continuous function  $H : [0, \tau] \times \overline{\mathcal{O}} \rightarrow V$ . If we assume that  $y \notin H(t, \partial\mathcal{O})$  for all  $t \in [0, \tau]$ , then*

$$\deg(H(0, \cdot), \mathcal{O}, y) = \deg(H(\tau, \cdot), \mathcal{O}, y).$$

The existence of solutions of (9) can now be established in the following theorem.

**Theorem 6.** *If  $H := H_0 + H_1$  is invertible, then for every  $\tau \in [0, T]$ , there exists  $Y \in C^\alpha([0, T], V)$  such that (9) is satisfied. Moreover, for almost every  $v \in V$  the number of solutions of (9) is finite.*

*Proof.* As a consequence of Lemma 3 and the beginning of the proof of Theorem 1, it suffices to prove that there exists  $a \in V$  such that  $a = L(a)_\tau$ . With  $\mathcal{L}$  defined by

$$\mathcal{L}(\tau, a) = a + H^{-1}H_1(I(a)(\tau) - a) \text{ for } (\tau, a) \in [0, T] \times V,$$

the problem  $a = L(a)_\tau$  is equivalent to

$$\mathcal{L}(\tau, a) = \tilde{v} \text{ where } \tilde{v} = H^{-1}v. \quad (20)$$

Hence, we look for  $a \in V$  solving (20). In order to prove the existence of solutions it suffices to prove that for  $r > 0$  large enough and for all  $\tau \in [0, T]$  the Brouwer degree of  $\mathcal{L}(\tau, \cdot)$  exists and satisfies

$$\deg(\mathcal{L}(\tau, \cdot), B_r(\tilde{v}), \tilde{v}) \neq 0, \quad (21)$$

where  $B_r(\tilde{v})$  stands for the ball in  $V$  of center  $\tilde{v}$  and radius  $r$ .

For all  $r > 0$ ,

$$\deg(\mathcal{L}(0, \cdot), B_r(\tilde{v}), \tilde{v}) = \deg(\text{Id}, B_r(\tilde{v}), \tilde{v}) = 1.$$

By Lemma 1, for all  $(\tau, x) \in [0, T] \times V$  such that  $\mathcal{L}(\tau, x) = \tilde{v}$ , we have

$$\begin{aligned} |x - \tilde{v}| &= |H^{-1}H_1(I(x)(\tau) - x)| \\ &\leq |H^{-1}H_1| \cdot |I(x)(\tau) - I(x)(0)| \\ &\leq |H^{-1}H_1| \cdot \|I(x)\|_\alpha \tau^\alpha \\ &\leq |H^{-1}H_1| c_{\alpha, f, \mathbf{x}} T^\alpha. \end{aligned}$$

This implies that for all  $r > |H^{-1}H_1|c_{\alpha,f,\mathbf{X}}T^\alpha$  and all  $\tau \in [0, T]$ ,

$$\tilde{v} \notin \mathcal{L}(\tau, \partial B_r(\tilde{v}))$$

and then the degree  $\deg(\mathcal{L}(\tau, \cdot), B_r(\tilde{v}), (\tilde{v}))$  is well-defined. Moreover, by the continuity of  $I$ , the function  $\mathcal{L}$  is a homotopy. This implies that for all  $\tau \in [0, T]$ ,

$$\deg(\mathcal{L}(\tau, \cdot), B_r(\tilde{v}), \tilde{v}) = \deg(\mathcal{L}(0, \cdot), B_r(\tilde{v}), \tilde{v}) = 1$$

and then (21) and the existence of a solution  $a \in B_r(\tilde{v})$  to  $a = L(a)_\tau$ .

It remains to prove that for every  $\tau \in [0, T]$  the number of solutions of  $a = L(a)_\tau$  is finite for almost every  $v$ , or equivalently for almost every  $\tilde{v}$ . We fix  $r > |H^{-1}H_1|c_{\alpha,f,\mathbf{X}}T^\alpha$ . Since  $f \in C_b^3$  the function  $a \mapsto I(a)$  is  $C^1$ . Following the Sard theorem [12, Lemma 13.2.1, p. 331], the Lebesgue measure of the set of critical values of  $\mathcal{L}(\tau, \cdot)$  is equal to 0. Consider a regular value  $\tilde{v}$  of  $\mathcal{L}(\tau, \cdot)$ . The set  $\overline{B_r(\tilde{v})} \cap \mathcal{L}(\tau, \cdot)^{-1}(\tilde{v})$  is closed and bounded, and consequently compact because  $V$  is finite-dimensional. Since  $\tilde{v}$  is a regular value,  $\mathcal{L}(\tau, \cdot)$  is locally invertible on a neighborhood of each point of  $\overline{B_r(\tilde{v})} \cap \mathcal{L}(\tau, \cdot)^{-1}(\tilde{v})$ . Then each point of  $\overline{B_r(\tilde{v})} \cap \mathcal{L}(\tau, \cdot)^{-1}(\tilde{v})$  is isolated. Since  $|L(\tau, \cdot)|$  is bounded (by  $|H^{-1}v| + |H^{-1}H_1|\tau^\alpha c_{\alpha,f,\mathbf{X}}$  for instance), we choose  $r$  large enough and deduce that  $\mathcal{L}(\tau, \cdot)^{-1}(\tilde{v})$  is finite. This completes the proof.  $\square$

*Remark 4.* If  $f$  is not bounded, the existence of solutions is not ensured for every  $\tau$ . Consider for instance the real-valued differential equation

$$dY(t) = Y(t) dW(t)$$

where  $W$  is a Hölder function and  $a \in \mathbb{R}$ . The solutions are written as

$$Y(t) = a \exp(W(t) - W(0)).$$

where  $a$  is a constant. The condition

$$H_0Y(0) + H_1Y(T) = v$$

can be rewritten as

$$H_0a + H_1a \exp(W(T) - W(0)) = v$$

and then

$$a = \frac{v}{H_0 + H_1 \exp(W(T) - W(0))}$$

which is defined if and only if  $H_0 + H_1 \exp(W(T) - W(0)) \neq 0$ . The solution is then

$$Y(t) = \frac{v \exp(W(t) - W(0))}{H_0 + H_1 \exp(W(T) - W(0))} = \frac{v \exp(W(t))}{H_0 \exp(W(0)) + H_1 \exp(W(T))}.$$

If  $H_0$  and  $H_1$  have different signs and are different from 0, then the condition  $H_0 + H_1 \exp(W(T) - W(0)) \neq 0$  is equivalent to

$$W(T) - W(0) \neq \ln \left( -\frac{H_0}{H_1} \right).$$

This is not ensured even when  $W$  is bounded.

## 8 Young differential equations

In this section we wish to extend previous results to a framework with long-range dependence. This generally involves more regular processes and the Young integration suffices. We consider in this section  $\alpha \in (\frac{1}{2}, 1]$ . If  $U$  is a Banach space we denote by  $\mathcal{C}^\alpha([0, T], U)$  the space of  $\alpha$ -Hölder rough paths  $X$  and define

$$\|X\|_{\alpha, T} = \sup_{0 \leq s \neq t \leq T} \frac{|X(s, t)|}{|t - s|^\alpha} < \infty.$$

The metric on  $\mathcal{C}^\alpha([0, T], U)$  is defined by

$$\varrho_{\alpha, T}(X_1, X_2) = \sup_{0 \leq s \neq t \leq T} \frac{|X_1(s, t) - X_2(s, t)|}{|t - s|^\alpha}$$

for all  $X_1$  and  $X_2$  in  $\mathcal{C}^\alpha([0, T], U)$ . If  $V$  is another Banach space,  $f \in C_b^2(V, L(U, V))$  and  $a \in V$  then the following differential equation

$$Y(t) = a + \int_0^t f(Y(s)) dX(s) \text{ for all } t \in [0, T]$$

(in the Young sense) has a unique solution  $Y \in C^\alpha([0, T], V)$ . If we denote this solution by  $I(a, X)$  to take account of the dependence on  $(a, X)$ , then it defines the Itô map  $I$  from  $V \times \mathcal{C}^\alpha([0, T], U)$  to  $C^\alpha([0, T], V)$ . Moreover it can be proved that, if we set

$$\mathcal{B}_{M, T} = \{X \in \mathcal{C}^\alpha([0, T], U) \mid \|X\|_{\alpha, T} \leq M\},$$

that there exists a positive constant  $C_{\alpha, f, M}$  depending only on  $\alpha$ ,  $f$  and  $M$  such that for all  $a, b \in V$ , and for all  $X_1$  and  $X_2 \in \mathcal{B}_{M, T}$ ,

$$\|I(a, X_1) - I(b, X_2)\|_{\alpha, T} \leq C_{\alpha, f, M} (|a - b| + \varrho_{\alpha, T}(X_1, X_2)). \quad (22)$$

In particular, for every  $X \in \mathcal{C}^\alpha([0, T], U)$  there exists a positive constant  $C_{\alpha, f, X}$  depending only on  $\alpha$ ,  $f$  and  $\|X\|_{\alpha, T}$  such that for all  $a$  and  $b \in V$ ,

$$\|I(a, X) - I(b, X)\|_{\alpha, T} \leq C_{\alpha, f, X} |a - b|. \quad (23)$$

From (22), we can deduce the following result whose proof is similar to that of Theorem 1.



**Theorem 7.** *Let  $\tau > 0$  such that*

$$\tau < \frac{1}{(|H^{-1}H_1|C_{\alpha,f,X})^{1/\alpha}}. \quad (24)$$

*If  $H := H_0 + H_1$  is invertible, then there exists a unique solution  $Y \in C^\alpha([0, T], V)$  to*

$$\begin{cases} dY(t) = f(Y(t)) dX(t) \text{ for all } t \in [0, T], \\ H_0Y(0) + H_1Y(\tau) = v. \end{cases} \quad (25)$$

We fix  $M > 0$  and  $\tau > 0$  such that

$$\tau < \frac{1}{(|H^{-1}H_1|C_{\alpha,f,M})^{1/\alpha}}. \quad (26)$$

Theorem 7 enables us to define the function  $\bar{I} : \mathcal{B}_{M,T} \rightarrow C^\alpha([0, T], V)$  such that for every  $X \in C^\alpha([0, T], U)$ ,  $\bar{I}(X)$  is the unique solution of (25). The following theorem states that  $\bar{I}$  is continuous. Its proof is similar to that of Theorem 2.

**Theorem 8.** *The map  $\bar{I}$  is Lipschitz continuous with respect to  $\varrho_{\alpha,T}$  on  $\mathcal{B}_{M,T}$ , with constant*

$$C_{\alpha,f,M} + \frac{|H^{-1}H_1|C_{\alpha,f,M}\tau^\alpha}{1 - |H^{-1}H_1|C_{\alpha,f,M}\tau^\alpha}.$$

We can deduce the following limit theorem.

**Theorem 9.** *We assume that  $U$  is separable and that (26) is satisfied. Let  $X$  and  $(X^n)_n$  be a random variable and a sequence of random variables with values in  $C^{0,\alpha}([0, T], U)$  such that  $X^n$  converges in distribution to  $X$  in  $C^{0,\alpha}([0, T], U)$  and  $\mathbb{P}(\|X\|_{\alpha,T} = M) = 0$ . For every continuous bounded function  $h : C^\alpha([0, T], V) \rightarrow \mathbb{R}$ ,*

$$\lim_{n \rightarrow \infty} \mathbb{E}(h(\bar{I}(X^n))\mathbf{1}_{X^n \in \mathcal{B}_{M,T}}) = \mathbb{E}(h(\bar{I}(X))\mathbf{1}_{X \in \mathcal{B}_{M,T}}).$$

Now we give an application of the above result. We consider for every  $\varepsilon \in (0, 1]$  the problem

$$\begin{cases} \frac{dY^\varepsilon(t)}{dt} = \sum_{j=1}^d \frac{1}{\varepsilon^{\gamma_j}} m_j \left( \frac{t}{\varepsilon} \right) f_j(Y^\varepsilon(t)) \text{ for all } t \in [0, T], \\ H_0Y^\varepsilon(0) + H_1Y^\varepsilon(\tau) = v, \end{cases} \quad (27)$$

where for every  $j$ ,  $\gamma_j \in (0, 1)$  and  $m_j$  is a continuous stationary Gaussian process with mean zero and such that there exists  $c_j$  such that the covariance satisfies as  $t \rightarrow \infty$ ,

$$\mathbb{E}(m_j(0)m_j(t)) \sim \frac{c_j}{t^{\gamma_j}}.$$

Moreover we assume that  $m_1, m_2, \dots, m_d$  are independent, and without loss of generality, that  $\gamma_1 \geq \dots \geq \gamma_d$ .

We define for every  $\varepsilon$  the Lipschitz continuous process  $w^\varepsilon$  such that for all  $(s, t)$ ,

$$w^\varepsilon(s, t) = (w_1^\varepsilon(s, t), \dots, w_d^\varepsilon(s, t))$$

where for all  $j$ ,

$$w_j^\varepsilon(s, t) = \int_s^t \frac{1}{\varepsilon^{\gamma_j}} m_j \left( \frac{r}{\varepsilon^2} \right) dr.$$

The following result deals with the asymptotic behavior of (27).

**Theorem 10.** *Consider  $\alpha \in (\frac{1}{2}, \frac{2-\gamma_1}{2})$ . For every continuous bounded function  $h : C^\alpha([0, T], V) \rightarrow \mathbb{R}$ ,*

$$\lim_{n \rightarrow \infty} \mathbb{E}(h(Y^\varepsilon) \mathbf{1}_{w^\varepsilon \in \mathcal{B}_{M,T}}) = \mathbb{E}(h(Y) \mathbf{1}_{W \in \mathcal{B}_{M,T}}), \quad (28)$$

where  $Y$  is the solution of

$$\begin{cases} dY(t) = \sum_{j=1}^d f_j(Y(t)) dW_j(t) \text{ for all } t \in [0, T], \\ H_0 Y(0) + H_1 Y(\tau) = v \end{cases}$$

and  $W = (W_1, \dots, W_d)$  is a multidimensional fractional Brownian motion such that for all  $(j, k)$ ,

$$\mathbb{E}(W_j(t) W_k(s)) = \delta_{j=k} \frac{\rho_j}{2} (|t|^{2H_j} + |s|^{2H_j} - |t-s|^{2H_j})$$

with  $H_j = (2 - \gamma_j)/2$  and  $\rho_j = H_j^{-1}(2H_j - 1)^{-1} c_j$ .

*Proof.* Since  $\alpha \in (\frac{1}{2}, \frac{2-\gamma_1}{2})$  and by [19, Theorem 5],  $w^\varepsilon$  converges in distribution to  $W$  in  $C^{0,\alpha}([0, T], V)$  as  $\varepsilon \rightarrow 0$ . By Theorem 9, the proof is completed.  $\square$

## 9 Conclusion

We reduced the affine boundary value problem (9) to a fixed-point problem. This approach may be used whenever the Itô map is well defined. This is especially suitable for a large range of driving rough signals (semi-martingales, fractional Brownian motion and many more) and enable us to avoid any technical difficulties related to anticipating stochastic calculus.

For every time  $\tau$  small enough, using properties of the Itô map we established the existence and uniqueness of a solution, regularity results, and limit theorems.

Using the Brouwer degree we proved a global existence result. The number of solutions is finite for almost every  $v$ . We also gave counterexample showing that, if  $f$  is not bounded, then the existence of a solution for every time  $\tau$  is not ensured. This is related to the behavior of the Itô map and to the operators  $H_0$  and  $H_1$ .

Further studies could concern sharper hypotheses, counterexamples for the existence or the uniqueness of a solution, as well as asymptotic results and their applications in multiscale modeling.

Finally, let us mention that the use of regularity structures [10] could lead to similar results where the ordinary/rough differential equations are replaced by (rough) partial differential equations.

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