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Rough differential equations with affine boundary conditions

Antoine Lejay* Renaud Marty†

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Abstract

In this paper we consider rough differential equations with affine boundary conditions. Using rough paths and fixed-point techniques, we study properties of the solutions. We also prove a convergence theorem and apply our results to differential equations driven by Brownian motions and fractional Brownian motions.

Keywords: Rough path theory, differential equations, affine boundary conditions.

MSC (2020): 60H10, 34K10, 60L90

1 Introduction

In this paper we study rough differential equations with affine boundary conditions. If U and V are two Banach spaces, we consider the problem

$$\begin{cases} dY(t) = f(Y(t)) dW(t) \text{ for all } t \in [0, 1] \\ H_0 Y(0) + H_1 Y(1) = v \end{cases} \quad (1)$$

where W is a continuous function or a stochastic process with values in U , $f : V \rightarrow L(U, V)$ is a smooth function, v is a vector in V , H_0 and H_1 are two operators on V , and Y is unknown. In the case where W is a finite-dimensional Brownian motion and dW stands for the Stratonovich integration, the existence, the uniqueness and other properties of the solutions of such problems (with possibly other forms of two-point boundary conditions) have been analyzed under various assumptions and with different methods. Let us mention a few examples. Linear equations are studied in [23], where anticipating stochastic calculus (see [21]) is used because the solutions are not

*Université de Lorraine, CNRS, IECL, Inria, F-54000 Nancy, France; ORCID: 0000-0003-0406-9550; Antoine.Lejay@univ-lorraine.fr

†Université de Lorraine, CNRS, IECL, F-54000 Nancy, France; Renaud.Marty@univ-lorraine.fr

adapted to W . Nonlinear equations with additive noises and more general two-point boundary conditions are studied in [22] with a pathwise approach. In [11] invariant imbedding (see [3]) is used for the analysis of nonlinear equations. In addition to the works mentioned above, we may also cite for instance [2, 7–9, 24] where various stochastic differential equations with two-point boundary conditions are studied in other frameworks or with other methods.

In our paper we address the problem (1) in a setting where W is a more general continuous process (such as a fractional Brownian motion for instance). We use the theory of rough paths (see for instance [10, 18]). This enable us to avoid any difficulties related to the anticipating stochastic calculus. We use an enhanced Brownian motion or any other suitable rough path \mathbf{W} in place of W in order to define rough differential equations $Z(t) = Z(0) + \int_0^t f(Z(s)) d\mathbf{W}(s)$. For every $a \in V$, if f is regular enough then there exists a unique solution Z such that $Z(0) = a$. Then the Itô map I which assigns Z to (a, f, \mathbf{W}) can be defined and is regular with respect to all its variables.

In our work, setting $\tau \in [0, T]$ we relate (1) to

$$\begin{cases} Y(t) = Y(0) + \int_0^t f(Y(s)) d\mathbf{W}(s) \text{ for } t \in [0, T], \\ H_0 Y(0) + H_1 Y(\tau) = v. \end{cases} \quad (2)$$

and study various properties. Equation (2) links the values of $Y(0)$ and $Y(\tau)$ for a given value of $\tau \in [0, T]$, while the path $\{Y(t)\}_{t \in [0, T]}$ is defined on the whole interval $[0, T]$.

Using the Itô map and a fixed-point method, we prove that there exists a unique solution to (2) if $\tau < C$ where C is a constant that depends on T , f , \mathbf{W} , H_0 , and H_1 (we note that possibly $C \geq T$). In this setting we also establish regularity results, which enable us to prove limit theorems.

For $\tau \in [C, T]$ and when V is finite-dimensional, using the Brouwer degree we show that there exists a solution. While the uniqueness is not ensured, the number of solutions is finite for almost every v . The situation is summarized in Figure 1.

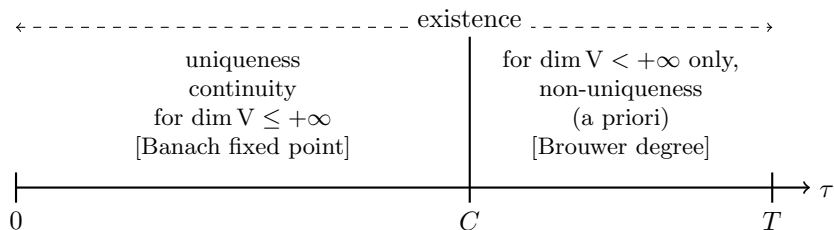


Figure 1: Summary of the results on existence and uniqueness depending on τ for fixed time T .

Outline. In Section 2 we recall results on rough paths theory. In Section 3 we prove the existence and uniqueness of a solution for a small time τ . In Section 4 we study the regularity of the solution with respect to the driving rough noise. In Sections 5 and 6 we establish more results on regularity and convergence. In Section 7 we study the existence of solutions for any time τ . Finally, in Section 8 we consider the case of a driving rough noise with Hölder regularity greater than $1/2$. In this context, the rough integrals coincide with the Young integrals.

2 Reminders about rough paths

We consider in this section the case of rough differential equations driven by paths of Hölder regularity $\alpha \in (\frac{1}{3}, \frac{1}{2}]$.

For two Banach space U and V whose norms are written $|\cdot|$, we denote by $L(U, V)$ the space of bounded, linear operators from U to V . Here, U and V are finite or infinite dimensional.

We denote by $C^\alpha(W, V)$ the space of α -Hölder continuous maps from W to V for $W \subset U$. The space is equipped with the α -Hölder semi-norm $\|f\|_\alpha := \sup_{x \neq y} |f(x) - f(y)|/|x - y|^\alpha$. When $W = [0, T] \subset \mathbb{R}$, we write $\|\cdot\|_{\alpha, T}$ instead of $\|\cdot\|_\alpha$.

We denote by $C^k(U, V)$ for some integer k the space of continuous functions which are differentiable up to order k with continuous k -th derivatives. We denote by $C_b^k(U, V)$ the space of functions in $C^k(U, V)$ with bounded j -th derivatives for all $0 \leq j \leq k$.

We denote by $\mathcal{C}^\alpha([0, T], U)$ the space of α -Hölder rough paths $\mathbf{X} = (X, \mathbb{X})$ with values in $U \oplus (U \otimes U)$. The elements of $\mathcal{C}^\alpha([0, T], U)$ satisfy the Chen relation

$$X(r, s) + X(s, t) = X(r, t), \quad \mathbb{X}(r, s) + \mathbb{X}(s, t) + X(r, s) \otimes X(s, t) = \mathbb{X}(r, t)$$

for any $0 \leq r \leq s \leq t \leq T$. Besides

$$\|X\|_{\alpha, T} = \sup_{0 \leq s \neq t \leq T} \frac{|X(s, t)|}{|t - s|^\alpha} < \infty \quad \text{and} \quad \|\mathbb{X}\|_{2\alpha, T} = \sup_{0 \leq s \neq t \leq T} \frac{|\mathbb{X}(s, t)|}{|t - s|^{2\alpha}} < \infty.$$

The metric on $\mathcal{C}^\alpha([0, T], U)$ is denoted by $\varrho_{\alpha, T}$ and defined by

$$\varrho_{\alpha, T}(\mathbf{X}_1, \mathbf{X}_2) = \sup_{0 \leq s \neq t \leq T} \frac{|X_1(s, t) - X_2(s, t)|}{|t - s|^\alpha} + \sup_{0 \leq s \neq t \leq T} \frac{|\mathbb{X}_1(s, t) - \mathbb{X}_2(s, t)|}{|t - s|^{2\alpha}}$$

for all \mathbf{X}_1 and \mathbf{X}_2 in $\mathcal{C}^\alpha([0, T], U)$.

Let U and V be two Banach spaces, $\mathbf{X} \in \mathcal{C}^\alpha([0, T], U)$, $f \in C_b^2(V, L(U, V))$ and $a \in V$. We consider the following differential equation

$$Y(t) = a + \int_0^t f(Y(s)) d\mathbf{X}(s) \quad \text{for all } t \in [0, T] \quad (3)$$

with unknown $Y \in C^\alpha([0, T], V)$. Following [6, 10], a solution Y of (9) can be defined as a path such that $Y(0) = a$ and

$$Y(t) - Y(s) = f(Y(s))X(s, t) + Df(Y(s))f(Y(s))\mathbb{X}(s, t) + O(|t - s|^\theta) \quad (4)$$

for some $\theta > 1$.

Following [10, Theorem 8.3], if $f \in C_b^3(V, L(U, V))$, then there exists a unique solution $Y \in C^\alpha([0, T], V)$ to Equation (4) (or (3)). In order to take account of the dependence on (a, \mathbf{X}) , we set $I(a, \mathbf{X}) := Y$. This defines the *Itô map* I from $V \times C^\alpha([0, T], U)$ to $C^\alpha([0, T], V)$. The next lemma establishes the boundedness of this solution (see for instance [10, Proposition 8.2]).

Lemma 1. *There exists a positive constant $K = K(\alpha)$ depending only on α such that for all $a \in V$,*

$$\|I(a, \mathbf{X})\|_{\alpha, T} \leq K(\alpha) \max \left(\|f\|_{C^2} \|\mathbf{X}\|_{\alpha, T}, (\|f\|_{C^2} \|\mathbf{X}\|_{\alpha, T})^{1/\alpha} \right), \quad (5)$$

where

$$\|\mathbf{X}\|_{\alpha, T} := \|X\|_{\alpha, T} + \sqrt{\|\mathbb{X}\|_{2\alpha, T}}.$$

By T. Lyons' theorem [17] the Itô map I is continuous. Moreover it is locally Lipschitz continuous, as recalled in the following lemma (see for instance [5, 10, 15]). Notice that some extended statements consider also variations of the vector field f .

Lemma 2. *We fix $M > 0$ and consider*

$$\mathcal{B}_{M, T} = \{\mathbf{X} \in C^\alpha([0, T], U) : \|\mathbf{X}\|_{\alpha, T} \leq M\}, \quad (6)$$

where $\|\mathbf{X}\|_{\alpha, T}$ is defined in Lemma 1. There exists a positive constant $C_{\alpha, f, M}$ depending only on α , f and M such that for all a and $b \in V$, and for all \mathbf{X}_1 and $\mathbf{X}_2 \in \mathcal{B}_{M, T}$,

$$\|I(a, \mathbf{X}_1) - I(b, \mathbf{X}_2)\|_{\alpha, T} \leq C_{\alpha, f, M} (|a - b| + \varrho_{\alpha, T}(\mathbf{X}_1, \mathbf{X}_2)). \quad (7)$$

In particular, for every $\mathbf{X} \in C^\alpha([0, T], U)$ there exists a positive constant $C_{\alpha, f, \mathbf{X}}$ depending only on α , f and $\|\mathbf{X}\|_{\alpha, T}$ such that for all a and $b \in V$,

$$\|I(a, \mathbf{X}) - I(b, \mathbf{X})\|_{\alpha, T} \leq C_{\alpha, f, \mathbf{X}} |a - b|. \quad (8)$$

For the sake of simplicity, we do not consider optimal hypotheses on the vector field. The assumptions on the regularity of f can be weakened (for example when $f \in C_b^{2+\gamma}$ with $\alpha(2 + \gamma) > 1$) but to get a global Lipschitz condition with respect to the starting point, we highlight that f shall be bounded. This latter hypothesis is used in several proofs.

3 Existence and uniqueness

Let $v \in V$ and H_0 and H_1 be two bounded operators on V (finite or infinite dimensional). We set τ and T such that $0 < \tau \leq T$.

Throughout the paper and unless otherwise mentioned, we assume that $f \in C_b^3(V, L(U, V))$.

In this section we fix $\mathbf{X} \in C^\alpha([0, T], U)$ and we wish to find a continuous function Y such that

$$\begin{cases} Y(t) = Y(0) + \int_0^t f(Y(s)) d\mathbf{X}(s) \text{ for all } t \in [0, T], \\ H_0 Y(0) + H_1 Y(\tau) = v. \end{cases} \quad (9)$$

Then we consider the following problem. We look for an element $a \in V$ such that

$$H_0 a + H_1 I(a)(\tau) = v, \quad (10)$$

where I is the Itô map and $I(a, \mathbf{X})$ is denoted by $I(a)$ since \mathbf{X} is fixed. The following lemma establishes the relation between (10) and (9).

Lemma 3. *If there exists $a \in V$ such that (10) holds, then $Y = I(a)$ satisfies (9). Moreover, if there exists a unique $a \in V$ such that (10) holds, then $Y = I(a)$ is the unique function in $C^\alpha([0, T], V)$ satisfying (9).*

Proof. The first part of the lemma is obvious. Assume there exists a unique $a \in V$ such that (10) and set $Y = I(a)$. Consider another $Z \in C^\alpha([0, T], V)$ such that

$$\begin{cases} Z(t) = Z(0) + \int_0^t f(Z(s)) d\mathbf{X}(s) \text{ for all } t \in [0, T], \\ H_0 Z(0) + H_1 Z(\tau) = v. \end{cases}$$

By the definition of the Itô map, $Z = I(Z(0))$, so that $H_0 Z(0) + H_1 I(Z(0))(\tau) = v$. This implies that $Z(0) = a$ (by uniqueness of a), and then $Z = I(Z(0)) = I(a) = Y$. \square

The following theorem establishes the existence of a unique solution of the problem (9) for small values of τ . It is proved with the use of (10), which is solved with the Banach fixed-point theorem.

For $a \in V$ and $\tau \in [0, T]$, we define

$$L(a)_\tau := H^{-1}v - H^{-1}H_1(I(a)(\tau) - a) \quad (11)$$

when $H := H_0 + H_1$ is invertible.

Remark 1. The condition that $H_0 + H_1$ is invertible means that our results cannot be applied to deal with a periodic solution.

Theorem 1. Let $C_{\alpha,f,\mathbf{X}}$ be the constant of (8). If $H := H_0 + H_1$ is invertible and τ satisfies

$$\tau^\alpha |H^{-1}H_1| C_{\alpha,f,\mathbf{X}} < 1, \quad (12)$$

then there exists a unique solution to (9) in $C^\alpha([0, T], \mathbb{V})$.

Proof. By Lemma 3, it suffices to prove that there exists a unique solution $a \in \mathbb{V}$ to (10). We rewrite (10) as

$$(H_0 + H_1)a + H_1(I(a)(\tau) - a) = v.$$

Then, since $H = H_0 + H_1$ is invertible,

$$H_0a + H_1I(a)(\tau) = v \text{ if and only if } a = H^{-1}v - H^{-1}H_1(I(a)(\tau) - a).$$

With (11), (10) can be reformulated as the fixed-point problem

$$a = L(a)_\tau. \quad (13)$$

By the Banach fixed-point theorem, it suffices to prove that $L(\cdot)_\tau$ is a contraction. For all $a, b \in \mathbb{V}$,

$$\begin{aligned} |L(a)_\tau - L(b)_\tau| &\leq |H^{-1}H_1| \cdot |I(a)(\tau) - a - I(b)(\tau) + b| \\ &\leq |H^{-1}H_1| \cdot (|I(a)_0 - a - I(b)_0 + b| + \tau^\alpha \|I(a) - a - I(b) + b\|_{\alpha,T}) \\ &= \tau^\alpha |H^{-1}H_1| \cdot \|I(a) - a - I(b) + b\|_{\alpha,T} \\ &= \tau^\alpha |H^{-1}H_1| \cdot \|I(a) - I(b)\|_{\alpha,T}. \end{aligned}$$

Following (8),

$$\|I(a) - I(b)\|_{\alpha,T} \leq C_{\alpha,f,\mathbf{X}} |a - b|.$$

As a consequence of above,

$$|L(a)_\tau - L(b)_\tau| \leq \tau^\alpha |H^{-1}H_1| C_{\alpha,f,\mathbf{X}} |a - b|.$$

Then for every τ satisfying (12), the function $L(\cdot)_\tau$ is a contraction. Then by the Banach fixed-point theorem, there exists a unique fixed point $a \in \mathbb{V}$ such that (13), and then (10) are satisfied. \square

As an application of Theorem 1, fix $\tau \in [0, T]$ and consider that $W = (W_1, \dots, W_d)$ is a Brownian motion with values in $\mathbb{U} = \mathbb{R}^d$ which is enhanced with $\mathbf{W} = (W, \mathbb{W}) \in \mathcal{C}^\alpha([0, T], \mathbb{R}^d)$ with for all $1 \leq j, k \leq d$,

$$\mathbb{W}_{j,k}(s, t) = \int_s^t (W_j(r) - W_j(s)) \circ dW_k(r) \text{ for any } 0 \leq s \leq t \leq T. \quad (14)$$

Then, Theorem 1 establishes that for every $\omega \in \Omega$ such that $\tau^\alpha |H^{-1}H_1|C_{\alpha,f,\mathbf{W}(\omega)} < 1$, there exists a unique solution $Y(\omega) \in C^\alpha([0, T], V)$ to

$$\begin{cases} Y(\omega)(t) = Y(\omega)(0) + \int_0^t \sum_{j=1}^d f_j(Y(\omega)(s)) \circ dW_j(\omega)(s), & t \in [0, T], \\ H_0 Y(\omega)(0) + H_1 Y(\omega)(\tau) = v. \end{cases} \quad (15)$$

Notice that $\int_0^t \sum_{j=1}^d f_j(Y(\omega)(s)) \circ dW_j(\omega)(s)$ stands for $\int_0^t f(Y(\omega)(s)) d\mathbf{W}(\omega)(s)$ because of the Stratonovich integration in (14) and both rough and Stratonovich integrals coincide. This result follows from Theorem 4 below which is the rough counterpart of the stochastic Wong-Zakai theorem, and the construction of [11] based on characteristics as solutions to some Stratonovich SDE.

This problem can also be solved in an infinite-dimensional setting. Suppose that U is a separable Banach space and consider a U -valued Brownian motion W on the Wiener space (U, H, μ) where μ is a zero-mean Gaussian measure on U and H is the Cameron-Martin space of μ . Following [14], under suitable conditions, there exists an enhanced Brownian motion $\mathbf{W} = (W, \mathbb{W}) \in C^\alpha([0, T], U)$ obtained as follows. We consider the sequence $(\mathbf{W}(m))_{m \geq 1} = (W(m), \mathbb{W}(m))_{m \geq 1}$ where $(W(m))_{m \geq 1}$ is the dyadic piecewise linear approximation of W , and $(\mathbb{W}(m))_{m \geq 1}$ is defined by

$$\mathbb{W}(m)(s, t) = \int_s^t (W(m)(r) - W(m)(s)) \otimes dW(m)(r)$$

for all s, t , and m . Assume that the tensor norm $\|\cdot\|_{U \otimes U}$ on $U \otimes U$ is exact for μ , which means that there exist two constants $\gamma \in [1/2, 1)$ and $C > 0$ such that for all independent sequences $(G_n)_{n \geq 1}$ and $(\tilde{G}_n)_{n \geq 1}$ of independent U -valued random variables with distribution μ , and for every $N \geq 0$,

$$\mathbb{E} \left(\left\| \sum_{n=1}^N G_n \otimes \tilde{G}_n \right\|_{U \otimes U} \right) \leq CN^\gamma.$$

By [14], under this assumption, as $m \rightarrow \infty$ the sequence $\mathbf{W}(m)$ converges in $C^\alpha([0, T], U)$ almost surely. Its limit defines an enhanced Brownian motion. Then by Theorem 1, for every $\omega \in \Omega$ such that $\tau^\alpha |H^{-1}H_1|C_{\alpha,f,\mathbf{W}(\omega)} < 1$, there exists a unique solution $Y(\omega) \in C^\alpha([0, T], V)$ to

$$\begin{cases} Y(\omega)(t) = Y(\omega)(0) + \int_0^t f(Y(\omega)(s)) d\mathbf{W}(\omega)(s) \text{ for all } t \in [0, T], \\ H_0 Y(\omega)(0) + H_1 Y(\omega)(\tau) = v. \end{cases}$$

4 Continuity of the solution with respect to the path

In the previous section the driving path \mathbf{X} is fixed. In this section we consider \mathbf{X} as a variable and we establish the continuity of the solution of (9) with respect to \mathbf{X} .

We set $M > 0$ and τ satisfying

$$\tau^\alpha |H^{-1}H_1|C_{\alpha,f,M} < 1. \quad (16)$$

When $\mathbf{X} \in \mathcal{B}_{M,T}$ (defined by (6)), (16) implies (12).

We define the function

$$\bar{I} : \mathcal{B}_{M,T} \rightarrow C^\alpha([0, T], \mathbf{V})$$

such that for every $\mathbf{X} \in \mathcal{B}_{M,T}$, $\bar{I}(\mathbf{X})$ is the unique solution Y to

$$\begin{cases} Y(t) = Y(0) + \int_0^t f(Y(s)) d\mathbf{X}(s) \text{ for all } t \in [0, T], \\ H_0 Y(0) + H_1 Y(\tau) = v. \end{cases}$$

The main theorem of this subsection establishes the continuity of \bar{I} .

Theorem 2. *The map \bar{I} is Lipschitz continuous with respect to $\varrho_{\alpha,T}$ on $\mathcal{B}_{M,T}$, with constant*

$$C_{\alpha,f,M} + \frac{|H^{-1}H_1|C_{\alpha,f,M}\tau^\alpha}{1 - |H^{-1}H_1|C_{\alpha,f,M}\tau^\alpha}.$$

Proof. Previously we proved that for every \mathbf{X} , $\bar{I}(\mathbf{X})$ can be written as $\bar{I}(\mathbf{X}) = I(a_{\mathbf{X}}, \mathbf{X})$ where I is the Itô map and $a_{\mathbf{X}}$ is the fixed point of the function $L(\cdot, \mathbf{X})_\tau : a \mapsto L(a, \mathbf{X})_\tau = H^{-1}v - H^{-1}H_1(I(a, \mathbf{X})(\tau) - a)$.

By (7), for all \mathbf{X} and $\mathbf{Y} \in \mathcal{B}_{M,T}$,

$$\begin{aligned} \|\bar{I}(\mathbf{X}) - \bar{I}(\mathbf{Y})\|_{\alpha,T} &= \|I(a_{\mathbf{X}}, \mathbf{X}) - I(a_{\mathbf{Y}}, \mathbf{Y})\|_{\alpha,T} \\ &\leq C_{\alpha,f,M}(|a_{\mathbf{X}} - a_{\mathbf{Y}}| + \varrho_{\alpha,T}(\mathbf{X}, \mathbf{Y})). \end{aligned} \quad (17)$$

Moreover,

$$\begin{aligned} |a_{\mathbf{X}} - a_{\mathbf{Y}}| &= |L(a_{\mathbf{X}}, \mathbf{X})_\tau - L(a_{\mathbf{Y}}, \mathbf{Y})_\tau| \\ &\leq |L(a_{\mathbf{X}}, \mathbf{X})_\tau - L(a_{\mathbf{X}}, \mathbf{Y})_\tau| + |L(a_{\mathbf{X}}, \mathbf{Y})_\tau - L(a_{\mathbf{Y}}, \mathbf{Y})_\tau| \\ &= |H^{-1}H_1| \cdot |I(a_{\mathbf{X}}, \mathbf{X})_\tau - I(a_{\mathbf{X}}, \mathbf{Y})_\tau| \\ &\quad + |H^{-1}H_1| \cdot |I(a_{\mathbf{X}}, \mathbf{Y})_\tau - I(a_{\mathbf{Y}}, \mathbf{Y})_\tau - a_{\mathbf{X}} + a_{\mathbf{Y}}| \\ &\leq |H^{-1}H_1| \cdot \|I(a_{\mathbf{X}}, \mathbf{X}) - I(a_{\mathbf{X}}, \mathbf{Y})\|_{\alpha,T}\tau^\alpha \\ &\quad + |H^{-1}H_1| \cdot \|I(a_{\mathbf{X}}, \mathbf{Y}) - I(a_{\mathbf{Y}}, \mathbf{Y})\|_{\alpha,T}\tau^\alpha \\ &\leq |H^{-1}H_1|C_{\alpha,f,M}\varrho_{\alpha,T}(\mathbf{X}, \mathbf{Y})\tau^\alpha \\ &\quad + |H^{-1}H_1|C_{\alpha,f,M}|a_{\mathbf{X}} - a_{\mathbf{Y}}|\tau^\alpha. \end{aligned}$$

As a consequence, for all \mathbf{X} and $\mathbf{Y} \in \mathcal{B}_{M,T}$,

$$|a_{\mathbf{X}} - a_{\mathbf{Y}}| \leq \frac{|H^{-1}H_1|C_{\alpha,f,M}\tau^\alpha}{1 - |H^{-1}H_1|C_{\alpha,f,M}\tau^\alpha} \varrho_{\alpha,T}(\mathbf{X}, \mathbf{Y})$$

and, by (17),

$$\|\bar{I}(\mathbf{X}) - \bar{I}(\mathbf{Y})\|_{\alpha, T} \leq \left(C_{\alpha, f, M} + \frac{|H^{-1}H_1|C_{\alpha, f, M}\tau^\alpha}{1 - |H^{-1}H_1|C_{\alpha, f, M}\tau^\alpha} \right) \varrho_{\alpha, T}(\mathbf{X}, \mathbf{Y}).$$

This completes the proof. \square

5 Regularity and numerical analysis

In Section 4 we proved the continuity of the solution with respect to the path \mathbf{X} . In this section, with the implicit function theorem, we consider the regularity with respect to the other variables. Then we briefly present an application to numerical analysis.

For $\gamma \in (0, 1]$ and $k \in \mathbb{N}^*$, we denote by $C^{k+\gamma}$ the class of functions f of class C^k such that $D^k f$ is γ -Hölder continuous. We denote by $C_b^{k+\gamma}$ the space of functions in $C^{k+\gamma}$ with bounded j -th derivatives for all $0 \leq j \leq k$. This space is equipped with

$$\|f\|_{C^{k+\gamma}} := \max\{\|f\|_\infty, \dots, \|D^k f\|_\infty, \|D^k f\|_\gamma\}.$$

Let $\mathbf{X} \in \mathcal{C}^\alpha([0, T], U)$. We define

$$\begin{aligned} \mathcal{P} &:= V \times C_b^{k+1+\gamma}(V, L(U, V)), \\ \text{and } \mathcal{Q} &:= C_b^{k+1+\gamma}(V, L(U, V)) \times V \times L(V, V) \times L(V, V). \end{aligned}$$

Proposition 1 ([5, Theorem 3]). *For any $\kappa \in (0, 1)$ such that $\alpha(2 + \kappa\gamma) > 1$, the Itô map $I : \mathcal{P} \rightarrow C^\alpha([0, T], V)$ is locally of class $C^{k+(1-\kappa)\gamma}$.*

Let us write

$$P(a, q)_\tau = H_0 a + H_1 I(a, f, \mathbf{X})(\tau) - v \text{ with } q = (f, v, H_0, H_1) \in \mathcal{Q}.$$

If $H_0 + H_1$ is invertible, the problem $L(a)_\tau = a$ is equivalent to $P(a, q)_\tau = 0$. Besides, by Proposition 1, P is locally of class $C^{k+(1-\kappa)\gamma}$ on \mathcal{Q} when $\alpha(2 + \kappa\gamma) > 1$. Moreover,

$$D_a P(a, q)_\tau = H_0 + H_1 D_a I(a, f, \mathbf{X})(\tau), \quad (18)$$

where

$$D_a I(a, f, \mathbf{X})(t) = \text{Id} + \int_0^t Df(I(a, f, \mathbf{X})(s)) \cdot D_a I(a, f, \mathbf{X})(s) ds.$$

The regularity is established by the following result, which is a generalized implicit function theorem (see *e.g.*, [1]) applied to P .

Theorem 3. Let $\tau \in [0, T]$, $a^\dagger \in V$ and $q^\dagger \in \mathcal{Q}$ such that $P(a^\dagger, q^\dagger)_\tau = 0$ and $D_a P(a^\dagger, q^\dagger)_\tau \in L(V, V)$ is an isomorphism. Then there exist neighborhoods U^\dagger of q^\dagger and V^\dagger of a^\dagger as well as a function $A \in C_b^{k+(1-\kappa)\gamma}(U^\dagger, V^\dagger)$ with $A(q^\dagger) = a^\dagger$ such that $P(A(q), q) = 0$ for any $q \in U^\dagger$.

Remark 2. At $\tau = 0$, $D_a I(a, f, \mathbf{X})(0) = \text{Id}$ and then $D_a P(a, q)_0 = H_0 + H_1$. To apply the implicit function theorem, $D_a P(a, q)_0$ has to be invertible. Hence, $H_0 + H_1$ shall be invertible which is an hypothesis we used to get the existence and uniqueness of a solution.

Remark 3. If $H_0 + H_1$ is invertible, then we can deduce from (18) that $D_a P(a, q)_\tau$ is invertible when τ is small enough. This not ensured for large τ .

Theorem 3 implies that the solution a is locally Lipschitz continuous with respect to $q = (f, v, H_0, H_1)$. Notice that we can also establish that a is globally Lipschitz continuous with respect to v . More precisely, using (8) we get

$$|a(v_1) - a(v_2)| \leq (1 - \tau^\alpha |H^{-1} H_1| C_{\alpha, f, \mathbf{X}}) |v_1 - v_2| \quad (19)$$

for all v_1 and v_2 .

We now present an application to numerical analysis. Let us consider that for any starting point a we have a numerical scheme $J(a)$ such that $J(a) = I(a, f, \mathbf{X})(\tau) + R(a)$ for a Lipschitz continuous error $a \mapsto R(a)$ with a Lipschitz norm ℓ smaller than 1. Several numerical schemes exist for rough paths and for RDE driven by Brownian motion or fractional Brownian motion (see for instance [13, 16, 20]).

Besides, let us assume that we are able to solve the fixed-point problem $H_0 a + H_1 J(a) - v = 0$ at precision η , that is we find a' such that

$$H_0 a' + H_1 J(a') = v + \eta.$$

This is the case for example when we use a finite number of iterations in the fixed-point iteration, or the Newton scheme. Then

$$H_0 a' + H_1 I(a', f, \mathbf{X})(\tau) = v + \eta - H_1 R(a').$$

It follows from Theorem 3 or (19) that there exists a constant K , depending on $\tau, \ell, f, \mathbf{X}, H_0$ and H_1 such that

$$|a' - a| \leq K|\eta| + K|R(a)|.$$

This implies that the numerical scheme converges.

6 Limit theorems

In this section we deal with limit theorems. We fix τ and M such that (16), namely

$$\tau^\alpha |H^{-1} H_1| C_{\alpha, f, M} < 1.$$

For every $T > 0$ we denote by $\mathcal{C}^{0,\alpha}([0, T], \mathbb{U})$ the closure of smooth rough paths in $\mathcal{C}^\alpha([0, T], \mathbb{U})$. We assume that the Banach space \mathbb{U} is separable and recall that, as a consequence, $\mathcal{C}^{0,\alpha}([0, T], \mathbb{U})$ is a Polish space (see Exercise 2.8 of [10] for instance). The main result is the following.

Theorem 4. *Suppose (16) and let \mathbf{X} and $(\mathbf{X}^n)_n$ be a random variable and a sequence of random variables with values in $\mathcal{C}^{0,\alpha}([0, T], \mathbb{U})$ such that \mathbf{X}^n converges in distribution to \mathbf{X} in $\mathcal{C}^{0,\alpha}([0, T], \mathbb{U})$ and $\mathbb{P}(\|\mathbf{X}\|_{\alpha,T} = M) = 0$. For every continuous bounded function $h : \mathcal{C}^\alpha([0, T], \mathbb{V}) \rightarrow \mathbb{R}$,*

$$\lim_{n \rightarrow \infty} \mathbb{E}(h(\bar{I}(\mathbf{X}^n))\mathbf{1}_{\|\mathbf{X}^n\|_{\alpha,T} \leq M}) = \mathbb{E}(h(\bar{I}(\mathbf{X}))\mathbf{1}_{\|\mathbf{X}\|_{\alpha,T} \leq M}).$$

Proof. Since $\mathcal{C}^{0,\alpha}([0, T], \mathbb{U})$ is a Polish space, by the Skorokhod representation theorem there exists a probability space $(\Omega^*, \mathcal{T}^*, \mathbb{P}^*)$ and random variables $\mathbf{X}^{*,n}$ and \mathbf{X}^* from $(\Omega^*, \mathcal{T}^*, \mathbb{P}^*)$ to $\mathcal{C}^{0,\alpha}([0, T], \mathbb{U})$ such that $\mathbf{X}^{*,n} \stackrel{\text{law}}{=} \mathbf{X}^n$, $\mathbf{X}^* \stackrel{\text{law}}{=} \mathbf{X}$ and $\mathbf{X}^{*,n}$ converges \mathbb{P}^* -almost surely to \mathbf{X}^* . Then \mathbb{P}^* -almost surely, $h(\bar{I}(\mathbf{X}^{*,n}))\mathbf{1}_{\|\mathbf{X}^{*,n}\|_{\alpha,T} \leq M}$ converges to $h(\bar{I}(\mathbf{X}^*))\mathbf{1}_{\|\mathbf{X}^*\|_{\alpha,T} \leq M}$ as $n \rightarrow \infty$. Moreover, by the bounded convergence theorem, since for all n ,

$$h(\bar{I}(\mathbf{X}^{*,n}))\mathbf{1}_{\|\mathbf{X}^{*,n}\|_{\alpha,T} \leq M} \leq \|h\|_\infty,$$

we deduce that

$$\lim_{n \rightarrow \infty} \mathbb{E}(h(\bar{I}(\mathbf{X}^{*,n}))\mathbf{1}_{\|\mathbf{X}^{*,n}\|_{\alpha,T} \leq M}) = \mathbb{E}(h(\bar{I}(\mathbf{X}^*))\mathbf{1}_{\|\mathbf{X}^*\|_{\alpha,T} \leq M}).$$

This completes the proof. \square

As an application, we consider for every $n \in \mathbb{N}^*$ the problem

$$\begin{cases} \frac{dY^n(t)}{dt} = \sqrt{n} \sum_{j=1}^d \xi_{j,[nt]} f_j(Y^n(t)) \text{ for all } t \in [0, T], \\ H_0 Y^n(0) + H_1 Y^n(\tau) = v, \end{cases}$$

where $\xi := (\xi_{j,k})_{(j,k) \in \{1, \dots, d\} \times \mathbb{N}}$ is a discrete family of independent random variables with mean zero, variance 1, same distribution and finite moments. We define the sequence of \mathbb{R}^d -valued processes $W^n = (W_1^n, \dots, W_d^n)$ such that for all $j = 1, \dots, d$,

$$W_j^n(t) = \frac{1}{\sqrt{n}} \left(\sum_{k=0}^{[nt]-1} \xi_{j,k} + (nt - [nt])\xi_{j,[nt]} \right).$$

Notice that W^n is Lipschitz-continuous. Therefore, W^n can be enhanced with $\mathbf{W}^n = (W^n, \mathbb{W}^n) \in \mathcal{C}^\alpha([0, T], \mathbb{R}^d)$ where

$$\mathbb{W}^n(s, t) = \int_s^t (W^n(r) - W^n(s)) \otimes dW^n(r).$$

The following theorem establishes the convergence in distribution of Y^n as n goes to infinity.

Theorem 5. For every continuous bounded function $h : C^\alpha([0, T], V) \rightarrow \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \mathbb{E}(h(Y^n) \mathbf{1}_{\mathbf{W}^n \in \mathcal{B}_{M, T}}) = \mathbb{E}(h(Y) \mathbf{1}_{\mathbf{W} \in \mathcal{B}_{M, T}}),$$

where \mathbf{W} is the enhanced Brownian motion defined by (14) and Y is the solution of

$$\begin{cases} dY(t) = \sum_{j=1}^d f_j(Y(t)) \circ dW_j(t) \text{ for all } t \in [0, T], \\ H_0 Y(0) + H_1 Y(\tau) = v. \end{cases}$$

This theorem is similar to [11, Theorem 7] but its proof is different.

Proof. It is a direct consequence of [4, Theorem 1] and of Theorem 4. \square

7 Global existence result

In Section 3 we prove that (9) (and (10)) has a unique solution for every τ small enough. In this section we use the Brouwer degree in order to prove that (9) has a solution (maybe more than one) for every $\tau \in [0, T]$, yet only when V is finite-dimensional.

Throughout the section, we assume that $V = \mathbb{R}^N$ for some $N \in \mathbb{N}^*$. We equip V with the Lebesgue measure.

Before we establish the existence of solution for (9) in Theorem 6, we remind the definition and some properties of the Brouwer degree. For a more complete exposition on the degree theory we refer the reader for instance to [12].

Consider a bounded open subset $\mathcal{O} \subset \mathbb{R}^N = V$, a continuous function $\Phi : \overline{\mathcal{O}} \rightarrow \mathbb{R}^N$ and $y \in V$ such that $y \notin \Phi(\partial\mathcal{O})$. The Brouwer degree $\deg(\Phi, \mathcal{O}, y)$ with $\Phi \in C^1(\mathcal{O}, V)$ can be defined as follows (see for instance [12, Definition 13.2.9]).

Definition 1. If y is a regular value of Φ (namely such that $\det J_\Phi(x) \neq 0$ for all $x \in \Phi^{-1}(\{y\})$), then $\deg(\Phi, \mathcal{O}, y)$ is defined by

$$\deg(\Phi, \mathcal{O}, y) = \sum_{x \in \Phi^{-1}(\{y\})} \text{sign}(\det J_\Phi(x)).$$

If y is a critical value of Φ (y is not a regular value of Φ), then $\deg(\Phi, \mathcal{O}, y)$ is defined by

$$\deg(\Phi, \mathcal{O}, y) = \deg(\Phi, \mathcal{O}, y'),$$

where y' is a regular value of Φ such that $|y - y'| < \text{dist}(y, \Phi(\partial\mathcal{O}))$.

The Brouwer degree has the following property (see for instance [12, Theorem 13.2.11(iv)]).

Lemma 4. *Let $y \in V$ such that $y \notin \Phi(\partial\mathcal{O})$. If $\deg(\Phi, \mathcal{O}, y) \neq 0$ then there exists a solution $x \in \mathcal{O}$ to the equation $\Phi(x) = y$.*

The invariance by homotopy is another important property of the Brouwer degree and can be stated as follows (see for instance [12, Theorem 13.2.11(ii)]).

Lemma 5. *We fix $\tau > 0$ and consider a continuous function $H : [0, \tau] \times \overline{\mathcal{O}} \rightarrow V$. If we assume that $y \notin H(t, \partial\mathcal{O})$ for all $t \in [0, \tau]$, then*

$$\deg(H(0, \cdot), \mathcal{O}, y) = \deg(H(\tau, \cdot), \mathcal{O}, y).$$

The existence of solutions of (9) can now be established in the following theorem.

Theorem 6. *If $H := H_0 + H_1$ is invertible, then for every $\tau \in [0, T]$, there exists $Y \in C^\alpha([0, T], V)$ such that (9) is satisfied. Moreover, for almost every $v \in V$ the number of solutions of (9) is finite.*

Proof. As a consequence of Lemma 3 and the beginning of the proof of Theorem 1, it suffices to prove that there exists $a \in V$ such that $a = L(a)_\tau$. With \mathcal{L} defined by

$$\mathcal{L}(\tau, a) = a + H^{-1}H_1(I(a)(\tau) - a) \text{ for } (\tau, a) \in [0, T] \times V,$$

the problem $a = L(a)_\tau$ is equivalent to

$$\mathcal{L}(\tau, a) = \tilde{v} \text{ where } \tilde{v} = H^{-1}v. \quad (20)$$

Hence, we look for $a \in V$ solving (20). In order to prove the existence of solutions it suffices to prove that for $r > 0$ large enough and for all $\tau \in [0, T]$ the Brouwer degree of $\mathcal{L}(\tau, \cdot)$ exists and satisfies

$$\deg(\mathcal{L}(\tau, \cdot), B_r(\tilde{v}), \tilde{v}) \neq 0, \quad (21)$$

where $B_r(\tilde{v})$ stands for the ball in V of center \tilde{v} and radius r .

For all $r > 0$,

$$\deg(\mathcal{L}(0, \cdot), B_r(\tilde{v}), \tilde{v}) = \deg(\text{Id}, B_r(\tilde{v}), \tilde{v}) = 1.$$

By Lemma 1, for all $(\tau, x) \in [0, T] \times V$ such that $\mathcal{L}(\tau, x) = \tilde{v}$, we have

$$\begin{aligned} |x - \tilde{v}| &= |H^{-1}H_1(I(x)(\tau) - x)| \\ &\leq |H^{-1}H_1| \cdot |I(x)(\tau) - I(x)(0)| \\ &\leq |H^{-1}H_1| \cdot \|I(x)\|_\alpha \tau^\alpha \\ &\leq |H^{-1}H_1| c_{\alpha, f, \mathbf{x}} T^\alpha. \end{aligned}$$

This implies that for all $r > |H^{-1}H_1|c_{\alpha,f,\mathbf{X}}T^\alpha$ and all $\tau \in [0, T]$,

$$\tilde{v} \notin \mathcal{L}(\tau, \partial B_r(\tilde{v}))$$

and then the degree $\deg(\mathcal{L}(\tau, \cdot), B_r(\tilde{v}), (\tilde{v}))$ is well-defined. Moreover, by the continuity of I , the function \mathcal{L} is a homotopy. This implies that for all $\tau \in [0, T]$,

$$\deg(\mathcal{L}(\tau, \cdot), B_r(\tilde{v}), \tilde{v}) = \deg(\mathcal{L}(0, \cdot), B_r(\tilde{v}), \tilde{v}) = 1$$

and then (21) and the existence of a solution $a \in B_r(\tilde{v})$ to $a = L(a)_\tau$.

It remains to prove that for every $\tau \in [0, T]$ the number of solutions of $a = L(a)_\tau$ is finite for almost every v , or equivalently for almost every \tilde{v} . We fix $r > |H^{-1}H_1|c_{\alpha,f,\mathbf{X}}T^\alpha$. Since $f \in C_b^3$ the function $a \mapsto I(a)$ is C^1 . Following the Sard theorem [12, Lemma 13.2.1, p. 331], the Lebesgue measure of the set of critical values of $\mathcal{L}(\tau, \cdot)$ is equal to 0. Consider a regular value \tilde{v} of $\mathcal{L}(\tau, \cdot)$. The set $\overline{B_r(\tilde{v})} \cap \mathcal{L}(\tau, \cdot)^{-1}(\tilde{v})$ is closed and bounded, and consequently compact because V is finite-dimensional. Since \tilde{v} is a regular value, $\mathcal{L}(\tau, \cdot)$ is locally invertible on a neighborhood of each point of $\overline{B_r(\tilde{v})} \cap \mathcal{L}(\tau, \cdot)^{-1}(\tilde{v})$. Then each point of $\overline{B_r(\tilde{v})} \cap \mathcal{L}(\tau, \cdot)^{-1}(\tilde{v})$ is isolated. Since $|L(\tau, \cdot)|$ is bounded (by $|H^{-1}v| + |H^{-1}H_1|\tau^\alpha c_{\alpha,f,\mathbf{X}}$ for instance), we choose r large enough and deduce that $\mathcal{L}(\tau, \cdot)^{-1}(\tilde{v})$ is finite. This completes the proof. \square

Remark 4. If f is not bounded, the existence of solutions is not ensured for every τ . Consider for instance the real-valued differential equation

$$dY(t) = Y(t) dW(t)$$

where W is a Hölder function and $a \in \mathbb{R}$. The solutions are written as

$$Y(t) = a \exp(W(t) - W(0)).$$

where a is a constant. The condition

$$H_0Y(0) + H_1Y(T) = v$$

can be rewritten as

$$H_0a + H_1a \exp(W(T) - W(0)) = v$$

and then

$$a = \frac{v}{H_0 + H_1 \exp(W(T) - W(0))}$$

which is defined if and only if $H_0 + H_1 \exp(W(T) - W(0)) \neq 0$. The solution is then

$$Y(t) = \frac{v \exp(W(t) - W(0))}{H_0 + H_1 \exp(W(T) - W(0))} = \frac{v \exp(W(t))}{H_0 \exp(W(0)) + H_1 \exp(W(T))}.$$

If H_0 and H_1 have different signs and are different from 0, then the condition $H_0 + H_1 \exp(W(T) - W(0)) \neq 0$ is equivalent to

$$W(T) - W(0) \neq \ln \left(-\frac{H_0}{H_1} \right).$$

This is not ensured even when W is bounded.

8 Young differential equations

In this section we wish to extend previous results to a framework with long-range dependence. This generally involves more regular processes and the Young integration suffices. We consider in this section $\alpha \in (\frac{1}{2}, 1]$. If U is a Banach space we denote by $\mathcal{C}^\alpha([0, T], U)$ the space of α -Hölder rough paths X and define

$$\|X\|_{\alpha, T} = \sup_{0 \leq s \neq t \leq T} \frac{|X(s, t)|}{|t - s|^\alpha} < \infty.$$

The metric on $\mathcal{C}^\alpha([0, T], U)$ is defined by

$$\varrho_{\alpha, T}(X_1, X_2) = \sup_{0 \leq s \neq t \leq T} \frac{|X_1(s, t) - X_2(s, t)|}{|t - s|^\alpha}$$

for all X_1 and X_2 in $\mathcal{C}^\alpha([0, T], U)$. If V is another Banach space, $f \in C_b^2(V, L(U, V))$ and $a \in V$ then the following differential equation

$$Y(t) = a + \int_0^t f(Y(s)) dX(s) \text{ for all } t \in [0, T]$$

(in the Young sense) has a unique solution $Y \in C^\alpha([0, T], V)$. If we denote this solution by $I(a, X)$ to take account of the dependence on (a, X) , then it defines the Itô map I from $V \times \mathcal{C}^\alpha([0, T], U)$ to $C^\alpha([0, T], V)$. Moreover it can be proved that, if we set

$$\mathcal{B}_{M, T} = \{X \in \mathcal{C}^\alpha([0, T], U) \mid \|X\|_{\alpha, T} \leq M\},$$

that there exists a positive constant $C_{\alpha, f, M}$ depending only on α , f and M such that for all $a, b \in V$, and for all X_1 and $X_2 \in \mathcal{B}_{M, T}$,

$$\|I(a, X_1) - I(b, X_2)\|_{\alpha, T} \leq C_{\alpha, f, M} (|a - b| + \varrho_{\alpha, T}(X_1, X_2)). \quad (22)$$

In particular, for every $X \in \mathcal{C}^\alpha([0, T], U)$ there exists a positive constant $C_{\alpha, f, X}$ depending only on α , f and $\|X\|_{\alpha, T}$ such that for all a and $b \in V$,

$$\|I(a, X) - I(b, X)\|_{\alpha, T} \leq C_{\alpha, f, X} |a - b|. \quad (23)$$

From (22), we can deduce the following result whose proof is similar to that of Theorem 1.

Theorem 7. *Let $\tau > 0$ such that*

$$\tau < \frac{1}{(|H^{-1}H_1|C_{\alpha,f,X})^{1/\alpha}}. \quad (24)$$

If $H := H_0 + H_1$ is invertible, then there exists a unique solution $Y \in C^\alpha([0, T], V)$ to

$$\begin{cases} dY(t) = f(Y(t)) dX(t) \text{ for all } t \in [0, T], \\ H_0Y(0) + H_1Y(\tau) = v. \end{cases} \quad (25)$$

We fix $M > 0$ and $\tau > 0$ such that

$$\tau < \frac{1}{(|H^{-1}H_1|C_{\alpha,f,M})^{1/\alpha}}. \quad (26)$$

Theorem 7 enables us to define the function $\bar{I} : \mathcal{B}_{M,T} \rightarrow C^\alpha([0, T], V)$ such that for every $X \in C^\alpha([0, T], U)$, $\bar{I}(X)$ is the unique solution of (25). The following theorem states that \bar{I} is continuous. Its proof is similar to that of Theorem 2.

Theorem 8. *The map \bar{I} is Lipschitz continuous with respect to $\varrho_{\alpha,T}$ on $\mathcal{B}_{M,T}$, with constant*

$$C_{\alpha,f,M} + \frac{|H^{-1}H_1|C_{\alpha,f,M}\tau^\alpha}{1 - |H^{-1}H_1|C_{\alpha,f,M}\tau^\alpha}.$$

We can deduce the following limit theorem.

Theorem 9. *We assume that U is separable and that (26) is satisfied. Let X and $(X^n)_n$ be a random variable and a sequence of random variables with values in $C^{0,\alpha}([0, T], U)$ such that X^n converges in distribution to X in $C^{0,\alpha}([0, T], U)$ and $\mathbb{P}(\|X\|_{\alpha,T} = M) = 0$. For every continuous bounded function $h : C^\alpha([0, T], V) \rightarrow \mathbb{R}$,*

$$\lim_{n \rightarrow \infty} \mathbb{E}(h(\bar{I}(X^n))\mathbf{1}_{X^n \in \mathcal{B}_{M,T}}) = \mathbb{E}(h(\bar{I}(X))\mathbf{1}_{X \in \mathcal{B}_{M,T}}).$$

Now we give an application of the above result. We consider for every $\varepsilon \in (0, 1]$ the problem

$$\begin{cases} \frac{dY^\varepsilon(t)}{dt} = \sum_{j=1}^d \frac{1}{\varepsilon^{\gamma_j}} m_j \left(\frac{t}{\varepsilon} \right) f_j(Y^\varepsilon(t)) \text{ for all } t \in [0, T], \\ H_0Y^\varepsilon(0) + H_1Y^\varepsilon(\tau) = v, \end{cases} \quad (27)$$

where for every j , $\gamma_j \in (0, 1)$ and m_j is a continuous stationary Gaussian process with mean zero and such that there exists c_j such that the covariance satisfies as $t \rightarrow \infty$,

$$\mathbb{E}(m_j(0)m_j(t)) \sim \frac{c_j}{t^{\gamma_j}}.$$

Moreover we assume that m_1, m_2, \dots, m_d are independent, and without loss of generality, that $\gamma_1 \geq \dots \geq \gamma_d$.

We define for every ε the Lipschitz continuous process w^ε such that for all (s, t) ,

$$w^\varepsilon(s, t) = (w_1^\varepsilon(s, t), \dots, w_d^\varepsilon(s, t))$$

where for all j ,

$$w_j^\varepsilon(s, t) = \int_s^t \frac{1}{\varepsilon^{\gamma_j}} m_j \left(\frac{r}{\varepsilon^2} \right) dr.$$

The following result deals with the asymptotic behavior of (27).

Theorem 10. *Consider $\alpha \in (\frac{1}{2}, \frac{2-\gamma_1}{2})$. For every continuous bounded function $h : C^\alpha([0, T], V) \rightarrow \mathbb{R}$,*

$$\lim_{n \rightarrow \infty} \mathbb{E}(h(Y^\varepsilon) \mathbf{1}_{w^\varepsilon \in \mathcal{B}_{M,T}}) = \mathbb{E}(h(Y) \mathbf{1}_{W \in \mathcal{B}_{M,T}}), \quad (28)$$

where Y is the solution of

$$\begin{cases} dY(t) = \sum_{j=1}^d f_j(Y(t)) dW_j(t) \text{ for all } t \in [0, T], \\ H_0 Y(0) + H_1 Y(\tau) = v \end{cases}$$

and $W = (W_1, \dots, W_d)$ is a multidimensional fractional Brownian motion such that for all (j, k) ,

$$\mathbb{E}(W_j(t)W_k(s)) = \delta_{j=k} \frac{\rho_j}{2} (|t|^{2H_j} + |s|^{2H_j} - |t-s|^{2H_j})$$

with $H_j = (2 - \gamma_j)/2$ and $\rho_j = H_j^{-1}(2H_j - 1)^{-1}c_j$.

Proof. Since $\alpha \in (\frac{1}{2}, \frac{2-\gamma_1}{2})$ and by [19, Theorem 5], w^ε converges in distribution to W in $C^{0,\alpha}([0, T], V)$ as $\varepsilon \rightarrow 0$. By Theorem 9, the proof is completed. \square

9 Conclusion

We reduced the affine boundary value problem (9) to a fixed-point problem. This approach may be used whenever the Itô map is well defined. This is especially suitable for a large range of driving rough signals (semi-martingales, fractional Brownian motion and many more) and enable us to avoid any technical difficulties related to anticipating stochastic calculus.

For every time τ small enough, using properties of the Itô map we established the existence and uniqueness of a solution, regularity results, and limit theorems.

Using the Brouwer degree we proved a global existence result. The number of solutions is finite for almost every v . We also gave counterexample showing that, if f is not bounded, then the existence of a solution for every time τ is not ensured. This is related to the behavior of the Itô map and to the operators H_0 and H_1 .

Further studies could concern sharper hypotheses, counterexamples for the existence or the uniqueness of a solution, as well as asymptotic results and their applications in multiscale modeling.

Finally, let us mention that the use of regularity structures [10] could lead to similar results where the ordinary/rough differential equations are replaced by (rough) partial differential equations.

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