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# Invariant imbedding and rough differential equations with affine boundary conditions

Antoine Lejay\*      Renaud Marty†

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## Abstract

In this paper we consider rough differential equations with affine boundary conditions. Using invariant imbedding techniques, we study qualitative properties of the solutions. We also prove a Wonk-Zakai theorem to apply our results to differential equations driven by Brownian motions and fractional Brownian motions.

**Keywords:** Invariant imbedding, Rough paths theory, Ordinary differential equations, Two-point affine boundary value problem.

**MSC (2020):** 60H10, 34K10, 60L90

## 1 Introduction

The *invariant imbedding* is a generic method to deal with two-point boundary problems for ordinary differential equations (ODE), partial differential equations (PDE) and integro-differential equations. The idea is to convert the problem to one on the initial condition. Used first by V.A. Ambarzumian [1] and by S. Chandrasekhar [4] in transport problems, this method was soon extended in many directions such that wave propagation, quantum mechanics,

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\*Université de Lorraine, CNRS, IECL, Inria, F-54000 Nancy, France; ORCID: 0000-0003-0406-9550; [Antoine.Lejay@univ-lorraine.fr](mailto:Antoine.Lejay@univ-lorraine.fr)

†Université de Lorraine, CNRS, IECL, F-54000 Nancy, France; [Renaud.Marty@univ-lorraine.fr](mailto:Renaud.Marty@univ-lorraine.fr)

scattering, etc. The book [2] by R. Bellman and G.M. Wing contains an excellent introduction to the subject as well as its many applications.

A typical affine two-point boundary value problem consists in relating both the initial and the terminal conditions of an ODE by an equation. The kind of problem we wish to consider here is the following one: let  $H_0$  and  $H_1$  be two bounded operators on a (finite-dimensional) vector space  $V$  and  $f : V \rightarrow V$  of class  $C^1$ . We are then interested in finding a path  $y : [0, \tau] \rightarrow V$  satisfying

$$\begin{cases} H_0 y(0; \tau, v) + H_1 y(\tau; \tau, v) = v, \\ \dot{y}(s; \tau, v) = f(y(s; \tau, v)) \text{ for } s \in [0, \tau] \end{cases} \quad (1)$$

for  $v$  and  $\tau \geq 0$  given. The main idea of the invariant imbedding consists in considering the length  $\tau$  of the domain as an independent variable and to define the initial condition  $a(\tau, v) := y(0; \tau, v)$  as the solution of a new problem. For  $\delta$  small enough,

$$y(\tau + \delta; \tau + \delta, v) = y(\tau; \tau + \delta, v) + \delta \cdot f(y(\tau; \tau + \delta, v)) + O(\delta^2)$$

so that

$$H_0 a(\tau + \delta, v) + H_1 y(\tau; \tau + \delta, v) = v - \delta \cdot H_1 f(y(\tau; \tau + \delta, v)) + O(\delta^2).$$

Hence,  $z := s \in [0, \tau] \mapsto y(s; \tau + \delta, v)$  is solution to the two-point non-linear boundary value problem

$$H_0 z(0) + H_1 z(\tau) = v - \delta \cdot H_1 f(z(\tau)) + O(\delta^2) \text{ and } \dot{z} = f(z) \text{ on } [0, \tau]. \quad (2)$$

Since  $f$  is  $C^1$ , the ODE  $\dot{z}(t, a) = f(z(t, a))$ ,  $z(0, a) = a$  defines a flow  $I : (t, a) \mapsto z(t, a)$  which is also  $C^1$  in space.

We now admit the uniqueness of the solution of (1) for any  $\tau$  and any  $v$  as well as the regularity of the solution with respect to  $(\tau, v)$ .

Eq. (2) means that  $a(\tau + \delta, v) \approx a(\tau, v - \delta \cdot H_1 f(z(\tau)))$ . Passing to the limit, we obtain the non-linear equation on  $a$ :

$$D_\tau a(\tau, v) + D_v a(\tau, v) \cdot H_1 f(I(\tau, a(\tau, v))) = 0,$$

where  $D_x$  is the Fréchet derivative in the  $x$ -variable,  $x = \tau, v$ . This equation is naturally solved through the method of characteristic: with

$$x(\tau, v) = v + H_1 \int_0^\tau f(I(\sigma, a(\sigma, v))) d\sigma,$$

we obtain that  $\sigma \in [0, \tau] \mapsto a(\sigma, x(\tau - \sigma, v))$  is constant and then

$$\begin{aligned} a(\tau, v) = a(0, x(\tau, v)) &= (H_0 + H_1)^{-1} \left( v + H_1 \int_0^\tau f(I(\sigma, a(\tau, \sigma))) d\sigma \right) \\ &= (H_0 + H_1)^{-1} (v + H_1 I(\tau, a(\tau, v))) \end{aligned} \quad (3)$$

assuming that  $H_0 + H_1$  is invertible, since  $(H_0 + H_1)a(0, v) = v$ .

The affine two-point boundary value problem (1) is then transformed to a fixed-point problem on the initial condition (3). Besides, we have identified the various technical conditions which are required: existence, uniqueness and regularity of the flow, *a priori* uniqueness and regularity of the solution to the two-point boundary value problem, as well as various differentiability conditions to perform the computations.

Now, let  $U$  be another finite-dimensional vector space and let  $W$  be a Brownian motion with values in  $U$ . We consider the problem

$$\begin{cases} H_0 Y(0) + H_1 Y(1) = v, \\ dY(t) = f(Y(t)) \circ dW(t) \text{ for } t \in [0, 1], \end{cases} \quad (4)$$

where  $Y : [0, 1] \rightarrow \mathbb{R}^d$  is an unknown function,  $f$  is a vector field in  $C_b^2(V, L(U, V))$ , and  $v$  is a vector in  $V$ . Here, the integral is understood in the Stratonovich form.

Eq. (4) has been studied since it has applications in physics, in particular in the study of waves in random media [9]. In particular, existence and uniqueness of a solution  $Y$  has been analyzed, see for instance [6, 9, 18, 19]. Moreover, [7, 9] deal with approximation-diffusion theorems where the limit is solution of an equation with the form of (4).

Passing from an ordinary to stochastic differential equations (SDE) implies to adapt the above proof. There are two apparent difficulties: (i) The solution of a SDE is not differentiable in time. (ii) The solution implies its terminal values. Therefore, it is anticipative while Itô or Stratonovich calculus requires non-anticipative solutions.

Problem (i) is bypassed by starting from (3) using the properties of the stochastic flow to get a solution to (4). Reversing the argument does not require to use differentiability in time, but just to compute the appropriate (random) characteristics. Problem (ii) is resolved by developing a proper theory of anticipating calculus, at the price of high technical difficulties.

To circumvent Problem (ii) and to consider other noises in place of the Brownian motion such as fractional Brownian motions, we use the theory of

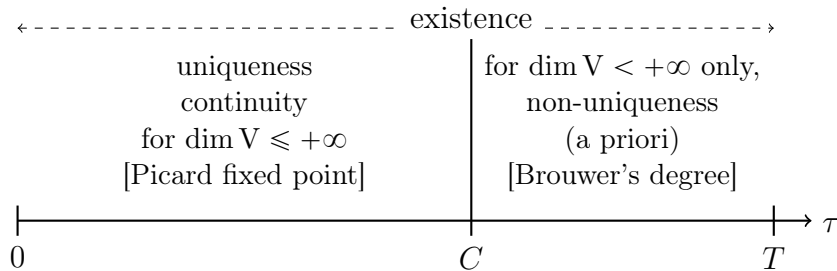


Figure 1: Summary of the results on uniqueness in function of  $\tau$  for fixed time horizon  $T$ .

rough paths [8, 15]. As this theory is actually a deterministic one, we avoid any difficulties related to the use of anticipating calculus. By using a so-called *enhanced Brownian motion*  $\mathbf{W}$  in place of  $W$  or any other suitable rough path, we define the Itô map  $I$  which transforms the driving signal  $\mathbf{W}$  to the solution  $Y$  of  $Y(t) = a + \int_0^t f(Y(s)) d\mathbf{W}(s)$  so that  $I$  is regular with respect to starting point, as well as to the driving signal or the vector field  $f$ . Hence, we relate (3) to

$$\begin{cases} H_0 Y(0) + H_1 Y(\tau) = v, \\ Y(t) = Y(0) + \int_0^t f(Y(s)) d\mathbf{W}(s) \text{ for } t \in [0, T]. \end{cases} \quad (5)$$

and study its various properties.

The solution to (5) links the values of  $Y(0)$  and  $Y(\tau)$  for a given value of  $\tau \in [0, T]$ , while the path  $\{Y(t)\}_{t \in [0, T]}$  that lives on the whole interval  $[0, T]$ . The solution to (12) will be unique for  $\tau < C$  for a constant  $C$  that depends on  $T$  the norms of  $f$ ,  $\mathbf{X}$ ,  $H_0$  and  $H_1$  (we note that possible,  $C \geq T$ ). We also study its regularity with respect to the driving signal. This enables us to give new proofs for results such as those proved in [9] and present new ones. In particular, we generalize limit theorems of [16] to a two-point boundary condition setting.

For  $\tau \in [C, T]$ , when  $V$  is finite-dimensional, we show existence for almost every  $v$ . For this, we use the degree theory. While uniqueness is not guaranteed, the number of solutions is finite. At the best of our knowledge, even in the classical cases of invariant imbedding, it seems that the degree theory has not been used yet in the field of invariance imbedding. The situation is summarized in Figure 1.

We also give several examples of applications to this problem to random media.

**Outline.** In Section 2, we recall elementary results on rough paths theory. In Section 3, we study the problem of existence for a small time horizon. In Section 4, we study the regularity of the solution with respect to the driving rough noise. In Section 5, we give a result on convergence in the probabilistic context. Section 6 is devoted to study the existence for any time horizon using the degree theory. Finally, Section 7 consider the case of a driving signal with Hölder regularity greater than  $1/2$ . As in this context, rough integrals coincide to Young integrals, we refer to this case as the Young case.

## 2 Reminders about rough paths with $\alpha \in (\frac{1}{3}, \frac{1}{2})$

We consider in this section the case of rough differential equations driven by paths of Hölder regularity  $\alpha \in (\frac{1}{3}, \frac{1}{2})$ . We mainly use the notation of [8] and recall some results.

For two Banach space  $U$  and  $V$  whose norms are written  $|\cdot|$ , we denote by  $L(U, V)$  the space of bounded, linear operators from  $U$  to  $V$ . Here,  $U$  and  $V$  are finite or infinite dimensional.

We denote by  $C^\alpha(W, V)$  the space of  $\alpha$ -Hölder continuous maps from  $W$  to  $V$  for  $W$  a subset of  $U$ . The space is equipped with the  $\alpha$ -Hölder semi-norm  $\|f\|_\alpha := \sup_{x \neq y} |f(x) - f(y)|/|x - y|^\alpha$ . When  $W = [0, T] \subset \mathbb{R}$ , we write  $\|\cdot\|_{\alpha, T}$  instead of  $\|\cdot\|_\alpha$  to keep the dependence in  $T$ .

We denote by  $C^k(U, V)$  for some integer  $k$  the space of continuous functions which are differentiable up to order  $k$  with continuous  $k$ -th derivatives.

We denote by  $\mathcal{C}^\alpha([0, T], U)$  the space of  $\alpha$ -Hölder rough paths  $\mathbf{X} = (X, \mathbb{X})$  with values in  $U \oplus (U \otimes U)$ . The elements of  $\mathcal{C}^\alpha([0, T], U)$  satisfy the Chen relation

$$X(r, s) + X(s, t) = X(r, t), \quad \mathbb{X}(r, s) + \mathbb{X}(s, t) + X(r, s) \otimes X(s, t) = \mathbb{X}(r, t)$$

for any  $0 \leq r \leq s \leq t \leq T$ . Besides

$$\|X\|_{\alpha, T} = \sup_{0 \leq s \neq t \leq T} \frac{|X(s, t)|}{|t - s|^\alpha} < \infty \quad \text{and} \quad \|\mathbb{X}\|_{2\alpha, T} = \sup_{0 \leq s \neq t \leq T} \frac{|\mathbb{X}(s, t)|}{|t - s|^{2\alpha}} < \infty.$$

The pseudo-metric on  $\mathcal{C}^\alpha([0, T], U)$  is denoted by  $\varrho_{\alpha, T}$  and defined by

$$\varrho_{\alpha, T}(\mathbf{X}_1, \mathbf{X}_2) = \sup_{0 \leq s \neq t \leq T} \frac{|X_1(s, t) - X_2(s, t)|}{|t - s|^\alpha} + \sup_{0 \leq s \neq t \leq T} \frac{|\mathbb{X}_1(s, t) - \mathbb{X}_2(s, t)|}{|t - s|^{2\alpha}}$$

for all  $\mathbf{X}_1$  and  $\mathbf{X}_2$  in  $\mathcal{C}^\alpha([0, T], U)$ .

Let  $U$  and  $V$  be two Banach spaces,  $\mathbf{X} \in \mathcal{C}^\alpha([0, T], U)$ ,  $f \in C_b^2(V, L(U, V))$  and  $a \in V$ . We consider the following differential equation

$$Y(t) = a + \int_0^t f(Y(s)) d\mathbf{X}(s) \text{ for all } t \in [0, T] \quad (6)$$

with unknown  $Y \in C^\alpha([0, T], V)$ . Following Davie's formulation, a solution  $Y$  of (12) is defined as a path such that  $Y(0) = a$  and

$$Y(t) - Y(s) = f(Y(s))X(s, t) + \nabla f(Y(s))f(Y(s))\mathbb{X}(s, t) + O(|t - s|^\theta) \quad (7)$$

for some  $\theta > 1$ .

Following [8, Theorem 8.3], if  $f \in C_b^3(V, L(U, V))$ , then there exists a unique solution  $Y \in C^\alpha([0, T], V)$  to Equation (7) (or (6)). In order to take account of the dependence on  $(a, \mathbf{X})$ , we set  $I(a, \mathbf{X}) := Y$ . This defines the *Itô map*  $I$  from  $V \times \mathcal{C}^\alpha([0, T], U)$  to  $C^\alpha([0, T], V)$ . The next lemma establishes the boundedness of this solution (see for instance [8, Proposition 8.3]).

**Lemma 1.** *There exists a positive constant  $K = K(\alpha)$  depending only on  $\alpha$  such that for all  $a \in V$ ,*

$$\|I(a, \mathbf{X})\|_{\alpha, T} \leq K(\alpha) \max(\|f\|_{C^2} \|\mathbf{X}\|_{\alpha, T}, (\|f\|_{C^2} \|\mathbf{X}\|_{\alpha, T})^{1/\alpha}), \quad (8)$$

where

$$\|\mathbf{X}\|_{\alpha, T} := \|X\|_{\alpha, T} + \sqrt{\|\mathbb{X}\|_{2\alpha, T}}.$$

(Notice that  $\|\cdot\|_{\alpha, T}$  is not a norm.)

T. Lyons' theorem establishes that  $I$  is continuous in  $(a, \mathbf{X})$ . Moreover it is locally Lipschitz as recalled in the following lemma (see for instance [5, 8, 13] for an extended statement incorporating variations of the vector field  $f$ ). For the sake of simplicity, we do not consider optimal hypotheses on the vector field. The assumptions on the regularity of  $f$  can be weakened (for example when  $f \in C_b^{2+\gamma}$  with  $\gamma(1 + \alpha) > p$  when  $2 \leq p < 3$ ) but to get a global Lipschitz condition with respect to the starting point, we highlight that  $f$  shall be bounded. This latter hypothesis is fundamental and appears in several proofs.

**Lemma 2.** *We fix  $M > 0$  and consider*

$$\mathcal{B}_{M, T} = \{\mathbf{X} \in \mathcal{C}^\alpha([0, T], U) : \|\mathbf{X}\|_{\alpha, T} \leq M\}, \quad (9)$$

where  $\|\mathbf{X}\|_{\alpha,T}$  is defined in Lemma 1. There exists a positive constant  $C_{\alpha,f,M}$  depending only on  $\alpha$ ,  $f$  and  $M$  such that for all  $a$  and  $b \in V$ , and for all  $\mathbf{X}_1$  and  $\mathbf{X}_2 \in \mathcal{B}_{M,T}$ ,

$$\|I(a, \mathbf{X}_1) - I(b, \mathbf{X}_2)\|_{\alpha,T} \leq C_{\alpha,f,M} (|a - b| + \varrho_{\alpha,T}(\mathbf{X}_1, \mathbf{X}_2)). \quad (10)$$

In particular, for all  $\mathbf{X} \in \mathcal{C}^\alpha([0, T], U)$  there exists a positive constant  $C_{\alpha,f,\mathbf{X}}$  depending only on  $\alpha$ ,  $f$  and  $\|\mathbf{X}\|_{\alpha,T}$  such that for all  $a$  and  $b \in V$ ,

$$\|I(a, \mathbf{X}) - I(b, \mathbf{X})\|_{\alpha,T} \leq C_{\alpha,f,\mathbf{X}} |a - b|. \quad (11)$$

Estimates (10) and (11) are very useful in the sequel of the paper.

### 3 Existence and uniqueness

Let  $v \in V$  and  $H_0$  and  $H_1$  be two bounded operators on  $V$  (finite or infinite dimensional). We set  $\tau$  and  $T$  such that  $0 < \tau \leq T$ .

Throughout the paper and unless otherwise mentioned, we assume that  $f \in C_b^3(V, L(U, V))$ .

In this section fix  $\mathbf{X} \in \mathcal{C}^\alpha([0, T], U)$  and we wish to find a continuous function  $Y$  such that

$$\begin{cases} Y(t) = Y(0) + \int_0^t f(Y(s)) d\mathbf{X}(s) \text{ for all } t \in [0, T], \\ H_0 Y(0) + H_1 Y(\tau) = v. \end{cases} \quad (12)$$

For this goal, we consider the following problem. We look for an element  $a \in V$  such that

$$H_0 a + H_1 I(a)(\tau) = v, \quad (13)$$

where  $I$  is the Itô map and  $I(a, \mathbf{X})$  is denoted by  $I(a)$  since  $\mathbf{X}$  is fixed. The following lemma establishes the relation between (13) and (12).

**Lemma 3.** *If there exists  $a \in V$  such that (13) holds, then  $Y = I(a)$  satisfies (12). Moreover, if there exists a unique  $a \in V$  such that (13) holds, then  $Y = I(a)$  is the unique function in  $C^\alpha([0, T], V)$  satisfying (12).*

*Proof.* The first part of the lemma is obvious. Assume there exists a unique  $a \in V$  such that (13) and set  $Y = I(a)$ . Consider another  $Z \in C^\alpha([0, T], V)$  such that

$$\begin{cases} Z(t) = Z(0) + \int_0^t f(Z(s)) d\mathbf{X}(s) \text{ for all } t \in [0, T], \\ H_0 Z(0) + H_1 Z(\tau) = v. \end{cases}$$



By the definition of the Itô map,  $Z = I(Z(0))$ , so that  $H_0Z(0) + H_1I(Z(0))(\tau) = v$ . This implies that  $Z(0) = a$  (by uniqueness of  $a$ ), and then  $Z = I(Z(0)) = I(a) = Y$ .  $\square$

The following theorem establishes the existence of a unique solution of the problem (12) for small values of  $\tau$ . It is proved with the use of (13) which is solved with the Banach fixed-point theorem.

For  $a \in V$  and  $\tau \in [0, T]$ , we define

$$L(a)_\tau := H^{-1}v - H^{-1}H_1(I(a)(\tau) - a) \quad (14)$$

when  $H := H_0 + H_1$  is invertible.

*Remark 1.* The condition that  $H_0 + H_1$  is invertible means that our results cannot be applied to deal with a periodic solution.

**Theorem 1.** *Let  $C_{\alpha, f, \mathbf{X}}$  be the constant of (11). If  $H := H_0 + H_1$  is invertible and  $\tau$  satisfies*

$$\tau^\alpha |H^{-1}H_1| C_{\alpha, f, \mathbf{X}} < 1, \quad (15)$$

*then there exists a unique solution to (12).*

*Proof.* Because of Lemma 3, it suffices to prove that there exists a unique solution  $a \in V$  to (13). Since  $H := H_0 + H_1$  is invertible, we rewrite (13) as

$$H_0a + H_1I(a)(\tau) = v \text{ if and only if } (H_0 + H_1)a + H_1(I(a)(\tau) - a) = v.$$

Because  $H = H_0 + H_1$  is invertible,

$$H_0a + H_1I(a)(\tau) = v \text{ if and only if } a = H^{-1}v - H^{-1}H_1(I(a)(\tau) - a).$$

With (14), (13) can be reformulated as the fixed-point problem

$$a = L(a)_\tau. \quad (16)$$

By the Banach fixed-point theorem, it suffices to prove that  $L(\cdot)_\tau$  is a contraction. For all  $a, b \in V$ ,

$$\begin{aligned} |L(a)_\tau - L(b)_\tau| &\leq |H^{-1}H_1| \cdot |I(a)(\tau) - a - I(b)(\tau) + b| \\ &\leq |H^{-1}H_1| \cdot (|I(a)_0 - a - I(b)_0 + b| + \|I(a) - a - I(b) + b\|_\alpha \tau^\alpha) \\ &= |H^{-1}H_1| \cdot \|I(a) - a - I(b) + b\|_\alpha \tau^\alpha \\ &\leq |H^{-1}H_1| \cdot (\|a - b\| + \|I(a) - I(b)\|_\alpha) \tau^\alpha \\ &= |H^{-1}H_1| \cdot \|I(a) - I(b)\|_\alpha \tau^\alpha. \end{aligned}$$

Following (11),

$$\|I(a) - I(b)\|_\alpha \leq C_{\alpha, f, \mathbf{x}} |a - b|.$$

As a consequence of above,

$$|L(a)_\tau - L(b)_\tau| \leq \tau^\alpha |H^{-1} H_1| C_{\alpha, f, \mathbf{x}} |a - b|.$$

Then for all  $\tau$  satisfying (15), the function  $L(\cdot)_\tau$  is a contraction. Then by the Banach fixed-point theorem, there exists a unique fixed point  $a \in V$  such that (16), and then (13) are satisfied.  $\square$

As an application of Theorem 1, fix  $\tau \in [0, T]$  and consider that  $W = (W_1, \dots, W_d)$  is a Brownian motion with values in  $U = \mathbb{R}^d$  which is enhanced as  $\mathbf{W} = (W, \mathbb{W}) \in \mathcal{C}^\alpha([0, T], \mathbb{R}^d)$  with for all  $1 \leq j, k \leq d$ ,

$$\mathbb{W}_{j,k}(s, t) = \int_s^t W_j(r) \circ dW_k(r) \text{ for any } 0 \leq s \leq t \leq T. \quad (17)$$

Then, Theorem 1 establishes that for all  $\omega \in \Omega$  such that  $\tau^\alpha |H^{-1} H_1| C_{\alpha, f, \mathbf{W}(\omega)} < 1$ , there exists a unique solution  $Y(\omega) \in \mathcal{C}^\alpha([0, T], V)$  to

$$\begin{cases} Y(\omega)(t) = Y(\omega)(0) + \int_0^t \sum_{j=1}^d f_j(Y(\omega)(s)) \circ dW_j(\omega)(s), & t \in [0, T], \\ H_0 Y(\omega)(0) + H_1 Y(\omega)(\tau) = v. \end{cases} \quad (18)$$

Notice that  $\int_0^t \sum_{j=1}^d f_j(Y(\omega)(s)) \circ dW_j(\omega)(s)$  stands for  $\int_0^t f(Y(\omega)(s)) d\mathbf{W}(\omega)(s)$  because of the Stratonovich integration in (17) and both rough and Stratonovich integrals coincide. This result follows from Theorem 4 below which is the rough counterpart of the stochastic Wong-Zakai theorem, and the construction of [9] based on constructing characteristics as solutions to some Stratonovich SDE following the method proposed in the introduction.

This problem can also be solved in an infinite-dimensional setting. Suppose that  $U$  is a separable Banach space and consider a  $U$ -valued Brownian motion  $W$  on the Wiener space  $(U, H, \mu)$  where  $\mu$  is a zero-mean Gaussian measure on  $U$  and  $H$  is the Cameron-Martin space of  $\mu$ . Following [12], under suitable conditions, there exists an enhanced Brownian motion  $\mathbf{W} = (W, \mathbb{W}) \in \mathcal{C}^\alpha([0, T], U)$  obtained as follows. We consider the sequence  $(\mathbf{W}(m))_{m \geq 1} = (W(m), \mathbb{W}(m))_{m \geq 1}$  where  $(W(m))_{m \geq 1}$  is the dyadic piecewise linear approximation of  $W$ , and  $(\mathbb{W}(m))_{m \geq 1}$  is defined by

$$\mathbb{W}(m)(s, t) = \int_s^t W(m) \otimes dW(m)$$

for all  $s, t$ , and  $m$ . Assume that the tensor norm  $\|\cdot\|_{\mathbb{U} \otimes \mathbb{U}}$  on  $\mathbb{U} \otimes \mathbb{U}$  is exact for  $\mu$ , which means that there exists two constants  $\gamma \in [1/2, 1)$  and  $C > 0$  such that for all independent sequences  $(G_n)_{n \geq 1}$  and  $(\tilde{G}_n)_{n \geq 1}$  of independent  $\mathbb{U}$ -valued random variables with distribution  $\mu$ , and for every  $N \geq 0$ ,

$$\mathbb{E} \left( \left\| \sum_{n=1}^N G_n \otimes \tilde{G}_n \right\|_{\mathbb{U} \otimes \mathbb{U}} \right) \leq CN^\gamma.$$

Following [12], under this assumption, as  $m \rightarrow \infty$ , the sequence  $\mathbf{W}(m)$  converges in  $\mathcal{C}^\alpha([0, T], \mathbb{U})$  almost surely to a limit of the form  $\mathbf{W} = (W, \mathbb{W}) \in \mathcal{C}^\alpha([0, T], \mathbb{U})$  which defines an enhanced Brownian motion. Then by Theorem 1, for all  $\omega \in \Omega$  such that  $\tau^\alpha |H^{-1}H_1| C_{\alpha, f, \mathbf{W}(\omega)} < 1$ , there exists a unique solution  $Y(\omega) \in \mathcal{C}^\alpha([0, T], \mathbb{V})$  to

$$\begin{cases} Y(\omega)(t) = Y(\omega)(0) + \int_0^t f(Y(\omega)(s)) d\mathbf{W}(\omega)(s) \text{ for all } t \in [0, T], \\ H_0 Y(\omega)(0) + H_1 Y(\omega)(\tau) = v. \end{cases} \quad (19)$$

## 4 Continuity of the solution with respect to the noise

In the previous section the driving path  $\mathbf{X}$  is fixed. In this section we consider  $\mathbf{X}$  as a variable and we establish the continuity of the solution of (12) with respect to it.

We fix  $M > 0$ , and consider  $\mathcal{B}_{M, T}$  defined by (9) and  $\tau$  satisfying

$$\tau^\alpha |H^{-1}H_1| C_{\alpha, f, M} < 1. \quad (20)$$

The condition (20) implies (15) when  $\mathbf{X} \in \mathcal{B}_{M, T}$ .

We define the function

$$\bar{I} : \mathcal{B}_{M, T} \rightarrow C^\alpha([0, T], \mathbb{V})$$

such that for all  $\mathbf{X} \in \mathcal{B}_{M, T}$ ,  $\bar{I}(\mathbf{X})$  is the unique solution  $Y$  to

$$\begin{cases} Y(t) = Y(0) + \int_0^t f(Y(s)) d\mathbf{X}(s) \text{ for all } t \in [0, T], \\ H_0 Y(0) + H_1 Y(\tau) = v. \end{cases}$$

The main theorem of this subsection establishes the continuity of  $\bar{I}$ .

**Theorem 2.** Under Condition (20), the map  $\bar{I}$  is Lipschitz continuous with respect to  $\varrho_{\alpha,T}$  on  $\mathcal{B}_{M,T}$ , with constant

$$C_{\alpha,f,M} + \frac{|H^{-1}H_1|C_{\alpha,f,M}\tau^\alpha}{1 - |H^{-1}H_1|C_{\alpha,f,M}\tau^\alpha}.$$

*Proof.* Previously we proved that for all  $\mathbf{X}$ ,  $\bar{I}(\mathbf{X})$  can be written as  $\bar{I}(\mathbf{X}) = I(a_{\mathbf{X}}, \mathbf{X})$  where  $I$  is the Itô map and  $a_{\mathbf{X}}$  is the fixed point of the function  $L(\cdot, \mathbf{X})_\tau : a \mapsto L(a, \mathbf{X})_\tau = H^{-1}v - H^{-1}H_1(I(a, \mathbf{X})(\tau) - a)$ .

Since (10), for all  $\mathbf{X}$  and  $\mathbf{Y} \in \mathcal{B}_{M,T}$ ,

$$\begin{aligned} \|\bar{I}(\mathbf{X}) - \bar{I}(\mathbf{Y})\|_{\alpha,T} &= \|I(a_{\mathbf{X}}, \mathbf{X}) - I(a_{\mathbf{Y}}, \mathbf{Y})\|_{\alpha,T} \\ &\leq C_{\alpha,f,M}(|a_{\mathbf{X}} - a_{\mathbf{Y}}| + \varrho_{\alpha,T}(\mathbf{X}, \mathbf{Y})). \end{aligned} \quad (21)$$

Moreover,

$$\begin{aligned} |a_{\mathbf{X}} - a_{\mathbf{Y}}| &= |L(a_{\mathbf{X}}, \mathbf{X})_\tau - L(a_{\mathbf{Y}}, \mathbf{Y})_\tau| \\ &\leq |L(a_{\mathbf{X}}, \mathbf{X})_\tau - L(a_{\mathbf{X}}, \mathbf{Y})_\tau| + |L(a_{\mathbf{X}}, \mathbf{Y})_\tau - L(a_{\mathbf{Y}}, \mathbf{Y})_\tau| \\ &= |H^{-1}H_1| \cdot |I(a_{\mathbf{X}}, \mathbf{X})_\tau - I(a_{\mathbf{X}}, \mathbf{Y})_\tau| \\ &\quad + |H^{-1}H_1| \cdot |I(a_{\mathbf{X}}, \mathbf{Y})_\tau - I(a_{\mathbf{Y}}, \mathbf{Y})_\tau - a_{\mathbf{X}} + a_{\mathbf{Y}}| \\ &\leq |H^{-1}H_1| \cdot \|I(a_{\mathbf{X}}, \mathbf{X}) - I(a_{\mathbf{X}}, \mathbf{Y})\|_{\alpha,T}\tau^\alpha \\ &\quad + |H^{-1}H_1| \cdot \|I(a_{\mathbf{X}}, \mathbf{Y}) - I(a_{\mathbf{Y}}, \mathbf{Y})\|_{\alpha,T}\tau^\alpha \\ &\leq |H^{-1}H_1|C_{\alpha,f,M}\varrho_{\alpha,T}(\mathbf{X}, \mathbf{Y})\tau^\alpha \\ &\quad + |H^{-1}H_1|C_{\alpha,f,M}|a_{\mathbf{X}} - a_{\mathbf{Y}}|\tau^\alpha. \end{aligned}$$

As a consequence, for all  $\mathbf{X}$  and  $\mathbf{Y} \in \mathcal{B}_{M,T}$ ,

$$|a_{\mathbf{X}} - a_{\mathbf{Y}}| \leq \frac{|H^{-1}H_1|C_{\alpha,f,M}\tau^\alpha}{1 - |H^{-1}H_1|C_{\alpha,f,M}\tau^\alpha} \varrho_{\alpha,T}(\mathbf{X}, \mathbf{Y})$$

and, due to (21),

$$\|\bar{I}(\mathbf{X}) - \bar{I}(\mathbf{Y})\|_{\alpha,T} \leq \left( C_{\alpha,f,M} + \frac{|H^{-1}H_1|C_{\alpha,f,M}\tau^\alpha}{1 - |H^{-1}H_1|C_{\alpha,f,M}\tau^\alpha} \right) \varrho_{\alpha,T}(\mathbf{X}, \mathbf{Y}).$$

This completes the proof.  $\square$

## 5 Convergence results

### 5.1 Regularity of the solution and convergence of numerical schemes

The Itô map  $I$  is actually regular with respect to all its variables: the starting point  $a$ , the driving path  $\mathbf{X}$  and the driving vector field  $f$ .

For  $\gamma \in (0, 1]$ , we denote by  $C^{k+\gamma}$  the class of functions  $f$  of class  $C^k$  such that  $D^k f$  is  $\gamma$ -Hölder continuous. This space is equipped with

$$\|f\|_{k+\gamma} := \max\{\|f\|_\infty, \dots, \|D^k f\|_\infty, \|D^k f\|_\alpha\}.$$

Let us define

$$\begin{aligned} \mathcal{P} &:= V \times C^{k+\gamma}(V, L(U, V)) \times \mathcal{C}^\alpha([0, T], U), \\ \text{and } \mathcal{Q} &:= C^{k+\gamma}(V, L(U, V)) \times \mathcal{C}^\alpha([0, T], U) \times V \times L(V, V) \times L(V, V). \end{aligned}$$

**Proposition 1** ([5, Theorem 3]). *For any  $\kappa$  such that  $\alpha(1 + \kappa\gamma) > 1$ , the Itô map  $I : \mathcal{P} \rightarrow C^\alpha([0, T], V)$  is locally of class  $C^{k+\kappa\gamma}$ .*

Let us write

$$P(a, q)_\tau = H_0 a + H_1 I(a, f, \mathbf{X})(\tau) - v \text{ with } q = (f, \mathbf{X}, v, H_0, H_1) \in \mathcal{Q}.$$

Provided that (20) holds, the problem  $L(a)_\tau = a$  is equivalent to  $P(a, q) = 0$ . Besides, since  $P$  is linear in  $H_0$ ,  $H_1$  and  $v$ , it is of class  $C^{k+\kappa\gamma}$  on  $\mathcal{Q}$  when  $f$  is of class  $C^{k+\gamma}$  with  $\alpha(1 + \kappa\gamma) > 1$ . Finally,

$$D_a P(a, q)_\tau = H_0 + H_1 D_a I(a, f, \mathbf{X})(\tau),$$

where

$$D_a I(a, f, \mathbf{X})(t) = \text{Id} + \int_0^t Df(I(a, f, \mathbf{X})(s)) \cdot D_a I(a, f, \mathbf{X})(s) ds.$$

The next result is a direct consequence of the Implicit Function Theorem [4].

**Theorem 3.** *Let us consider  $\tau \geq 0$  satisfying (20),  $a^\dagger \in V$  and  $q^\dagger \in \mathcal{Q}$  such that  $P(a^\dagger, q^\dagger)_\tau = 0$  and  $D_a P(a^\dagger, q^\dagger)_\tau \in L(V, V)$  is an isomorphism. Then there exist neighborhoods  $U^\dagger$  of  $q^\dagger$  and  $V^\dagger$  of  $a^\dagger$  as well as a function  $Q \in C^{k+\kappa\gamma}(U^\dagger, V^\dagger)$  with  $Q(q^\dagger) = a^\dagger$  such that  $P(Q(q), q) = 0$  for any  $q \in U^\dagger$ .*

*Remark 2.* At  $\tau = 0$ ,  $D_a I(a, f, \mathbf{X})(0) = \text{Id}$  and then  $D_a P(a, q)_0 = H_0 + H_1$ . To apply the Implicit Function Theorem,  $D_a P(a, q)_0$  has to be invertible. Hence,  $H_0 + H_1$  shall be invertible which is an hypothesis we used to get existence.

The next example shows that one cannot expect Theorem 3 to hold for any time.

*Example 1.* Let us consider the one-dimensional ODE  $Y(t) = a + \int_0^t f(Y(s)) ds$  for  $f \in C^1(\mathbb{R}_+, \mathbb{R})$ . Its derivative with respect to  $a \in V := \mathbb{R}$  is then

$$D_a Y(t) = \text{Id} + \int_0^t A(s) D_a Y(s) ds \text{ with } A(t) = f'(Y(s)).$$

Because this equation is linear and  $A(t) \in \mathbb{R}$ ,

$$D_a Y(t) = \exp\left(\int_0^t A(s) ds\right).$$

Therefore,  $D_a P(a, q)_\tau = H_0 + H_1 D_a Y(\tau)$ . Since we are in dimension one so that  $H_0 + H_1$  are scalar, we see that when  $D_a Y(\tau) = -H_0/H_1$ ,  $D_a P(a, q)$  is no longer invertible.

Let us consider for any starting point  $a$ , we have at our disposition a numerical scheme  $J(a)$  such that  $J(a) = I(a, f, \mathbf{X})(\tau) + R(a)$  for a Lipschitz continuous error  $a \mapsto R(a)$  with a Lipschitz norm  $\ell$  smaller than 1. Several numerical schemes exist for rough paths, see *e.g.* [11, 14, 17], for RDE driven by Brownian motion or fractional Brownian motion.

Besides, let us assume that we are able to solve the fixed-point problem  $H_0 a + H_1 J(a) - v = 0$  at precision  $\eta$ , that is we find  $a'$  such that

$$H_0 a' + H_1 J(a') = v + \eta.$$

This is the case for example when we use a finite number of iterations in the Picard approximation, or a Newton scheme. Then

$$H_0 a' + H_1 I(a', f, \mathbf{X})(\tau) = v + \eta + R(a').$$

It follows that there exists a constant  $K$ , depending on  $\tau, \ell, f, \mathbf{X}, H_0$  and  $H_1$  such that

$$|a' - a| \leq K|\eta| + K|R(a)|.$$

This implies that the numerical scheme converges.

## 5.2 A limit theorem for random driving noise

In this section we establish some limit theorems. We fix  $\tau$  and  $M$  such that (20), namely

$$\tau^\alpha |H^{-1}H_1| C_{\alpha,f,M} < 1.$$

For all  $\tau \in [0, T]$  we denote by  $\mathcal{C}^{0,\alpha}([0, \tau], U)$  the closure of smooth rough paths in  $\mathcal{C}^\alpha([0, \tau], U)$ . We assume that the Banach space  $U$  is separable and recall that as a consequence the space  $\mathcal{C}^{0,\alpha}([0, \tau], U)$  is Polish (see Exercise 2.8 of [8] for instance). The main result is the following.

**Theorem 4.** *Suppose (20) and let  $\mathbf{X}$  and  $(\mathbf{X}^n)_n$  be a random variable and a sequence of random variables with values in  $\mathcal{C}^{0,\alpha}([0, T], V)$  such that  $\mathbf{X}^n$  converges in distribution to  $\mathbf{X}$  in  $\mathcal{C}^{0,\alpha}([0, T], V)$  and  $\mathbb{P}(\|\mathbf{X}\|_{\alpha,T} = M) = 0$ . For all continuous bounded function  $h : C^\alpha([0, T], V) \rightarrow \mathbb{R}$ ,*

$$\lim_{n \rightarrow \infty} \mathbb{E}(h(\bar{I}(\mathbf{X}^n)) \mathbb{1}_{\|\mathbf{X}^n\|_{\alpha,T} \leq M}) = \mathbb{E}(h(\bar{I}(\mathbf{X})) \mathbb{1}_{\|\mathbf{X}\|_{\alpha,T} \leq M}).$$

Moreover, if  $\mathbb{P}(\|\mathbf{X}\|_{\alpha,T} \leq M) \neq 0$  and  $\mathbb{P}(\|\mathbf{X}^n\|_{\alpha,T} \leq M) \neq 0$  for all  $n$ , then

$$\lim_{n \rightarrow \infty} \mathbb{E}(h(\bar{I}(\mathbf{X}^n)) | \mathbf{X}^n \in \mathcal{B}_{M,T}) = \mathbb{E}(h(\bar{I}(\mathbf{X})) | \mathbf{X} \in \mathcal{B}_{M,T}).$$

*Proof.* Since  $\mathcal{C}^{0,\alpha}([0, T], V)$  is Polish, by the Skorokhod representation theorem there exists a probability space  $(\Omega^*, \mathcal{T}^*, \mathbb{P}^*)$  and random variables  $\mathbf{X}^{*,n}$  and  $\mathbf{X}^*$  from  $(\Omega^*, \mathcal{T}^*, \mathbb{P}^*)$  to  $\mathcal{C}^{0,\alpha}([0, T], V)$  such that  $\mathbf{X}^{*,n} \stackrel{\text{law}}{=} \mathbf{X}^n$ ,  $\mathbf{X}^* \stackrel{\text{law}}{=} \mathbf{X}$  and  $\mathbf{X}^{*,n}$  converges  $\mathbb{P}^*$ -almost surely to  $\mathbf{X}^*$ . Then  $\mathbb{P}^*$ -almost surely,  $h(\bar{I}(\mathbf{X}^{*,n})) \mathbb{1}_{\|\mathbf{X}^{*,n}\|_{\alpha,T} \leq M}$  converges to  $h(\bar{I}(\mathbf{X}^*)) \mathbb{1}_{\|\mathbf{X}^*\|_{\alpha,T} \leq M}$  as  $n \rightarrow \infty$ . Moreover, by the bounded convergence theorem, since for all  $n$ ,

$$h(\bar{I}(\mathbf{X}^{*,n})) \mathbb{1}_{\|\mathbf{X}^{*,n}\|_{\alpha,T} \leq M} \leq \|h\|_\infty,$$

we deduce that

$$\lim_{n \rightarrow \infty} \mathbb{E}(h(\bar{I}(\mathbf{X}^{*,n})) \mathbb{1}_{\|\mathbf{X}^{*,n}\|_{\alpha,T} \leq M}) = \mathbb{E}(h(\bar{I}(\mathbf{X}^*)) \mathbb{1}_{\|\mathbf{X}^*\|_{\alpha,T} \leq M}).$$

This completes the proof. □

As an application, we consider for all  $n \in \mathbb{N}^*$  the problem

$$\begin{cases} \frac{dY^n(t)}{dt} = \sqrt{n} \sum_{j=1}^d \xi_{j,[nt]} f_j(Y^n(t)) \text{ for all } t \in [0, T], \\ H_0 Y^n(0) + H_1 Y^n(\tau) = v, \end{cases} \quad (22)$$

where  $\xi := (\xi_{j,k})_{(j,k) \in \llbracket 1,d \rrbracket \times \mathbb{N}}$  is a discrete family of independent random variables with mean zero, variance 1, same distribution and finite moments. We define the sequence of  $\mathbb{R}^d$ -valued processes  $W^n = (W_1^n, \dots, W_d^n)$  such that for all  $j = 1, \dots, d$ ,

$$W_j^n(t) = \frac{1}{\sqrt{n}} \left( \sum_{k=0}^{n-1} \xi_{\cdot,k} + (nt - [nt])\xi_{\cdot,[nt]} \right).$$

Notice that  $W^n$  is Lipschitz-continuous. Therefore,  $W^n$  can be enhanced as  $\mathbf{W}^n = (W^n, \mathbb{W}^n) \in \mathcal{C}^\alpha([0, T], \mathbb{R}^d)$  with

$$\mathbb{W}^n(s, t) = \int_s^t W^n(r) \otimes dW^n(r).$$

The following theorem establishes the convergence in distribution of  $Y^n$  as  $n$  goes to infinity.

**Theorem 5.** *For all continuous bounded function  $h : C^\alpha([0, \tau], V) \rightarrow \mathbb{R}$ ,*

$$\lim_{n \rightarrow \infty} \mathbb{E}(h(Y^n) \mathbb{1}_{\mathbf{W}^n \in \mathcal{B}_{M,T}}) = \mathbb{E}(h(Y) \mathbb{1}_{\mathbf{W} \in \mathcal{B}_{M,T}}),$$

where  $\mathbf{W}$  is the enhanced Brownian motion defined by (17) and  $Y$  is the solution of

$$\begin{cases} dY(t) = \sum_{j=1}^d f_j(Y(t)) \circ dW_j(t) \text{ for all } t \in [0, T], \\ H_0 Y(0) + H_1 Y(\tau) = v. \end{cases}$$

This theorem is of the same type as [9, Theorem 7] with a different proof.

*Proof.* It is a direct consequence of [3, Theorem 1] and of Theorem 4.  $\square$

## 6 Global result on existence

In Section 3 we prove that (12) (and (13)) has a unique solution for all  $\tau$  small enough. In this section we use the Brouwer degree in order to prove that (12) has a solution (maybe more than one) for all  $\tau \in [0, T]$ , yet only when  $V$  is finite-dimensional.

Throughout the section, we assume that  $V = \mathbb{R}^N$  for some  $N \in \mathbb{N}^*$ . We equip  $V$  with the Lebesgue measure.



Before we establish the existence of solution for (12) in Theorem 6, we remind the definition and some properties of the Brouwer degree. For a more complete exposition on the degree theory we refer the reader for instance to [10].

Consider a bounded open subset  $\mathcal{O} \subset \mathbb{R}^N = V$ , a continuous function  $\Phi : \overline{\mathcal{O}} \rightarrow \mathbb{R}^N$  and  $y \in V$  such that  $y \notin \Phi(\partial\mathcal{O})$ . The Brouwer degree  $\deg(\Phi, \mathcal{O}, y)$  with  $\Phi \in C^1(\mathcal{O}, V)$  can be defined as follows (see for instance [10, Definition 13.2.9]).

**Definition 1.** *If  $y$  is a regular value of  $\Phi$  (namely such that  $\det J_\Phi(x) \neq 0$  for all  $x \in \Phi^{-1}(\{y\})$ ), then  $\deg(\Phi, \mathcal{O}, y)$  is defined by*

$$\deg(\Phi, \mathcal{O}, y) = \sum_{x \in \Phi^{-1}(\{y\})} \text{sign}(\det J_\Phi(x)).$$

*If  $y$  is a critical value of  $\Phi$  ( $y$  is not a regular value of  $\Phi$ ), then  $\deg(\Phi, \mathcal{O}, y)$  is defined by*

$$\deg(\Phi, \mathcal{O}, y) = \deg(\Phi, \mathcal{O}, y'),$$

*where  $y'$  is a regular value of  $\Phi$  such that  $\text{dist}(y, y') < \text{dist}(p, \Phi(\partial\mathcal{O}))$ .*

Notice that this definition of  $\deg(\Phi, \mathcal{O}, y)$  for a critical value  $y$  is independent of the choice of  $y'$ .

The Brouwer degree is a great tool for the study of existence of solution(s) to equations. It has the following property (see for instance [10, Theorem 13.2.11(iv)]).

**Lemma 4.** *Let  $y \in V$  such that  $y \notin \Phi(\partial\mathcal{O})$ . If  $\deg(\Phi, \mathcal{O}, y) \neq 0$  then there exists (at least) a solution  $x \in \mathcal{O}$  to the equation  $\Phi(x) = y$ .*

The invariance by homotopy is another important property of the Brouwer degree and can be stated as follows (see for instance [10, Theorem 13.2.11(ii)]).

**Lemma 5.** *We fix  $\tau > 0$  and consider a continuous function  $H : [0, \tau] \times \overline{\mathcal{O}} \rightarrow V$ . If we assume that  $y \notin H(t, \partial\mathcal{O})$  for all  $t \in [0, \tau]$ , then*

$$\deg(H(0, \cdot), \mathcal{O}, y) = \deg(H(\tau, \cdot), \mathcal{O}, y).$$

The existence of solution(s) of (12) can now be established in the following theorem.

**Theorem 6.** *We consider the above hypotheses on  $V$ . If  $H := H_0 + H_1$  is invertible, then for all  $\tau \in [0, T]$ , there exists  $Y \in C^\alpha([0, T], V)$  such that (12) is satisfied. Moreover, for almost every  $v \in V$  the number of solution(s) of (12) is finite.*

*Proof of Theorem 6.* As a consequence of Lemma 3 and the beginning of the proof of Theorem 1, it suffices to prove that there exists  $a \in V$  such that  $a = L(a)_\tau$ . With  $\mathcal{L}$  defined by

$$\mathcal{L}(\tau, a) = a + H^{-1}H_1(I(a)(\tau) - a) \text{ for } (\tau, a) \in [0, T] \times V,$$

the problem  $a = L(a)_\tau$  is equivalent to

$$\mathcal{L}(\tau, a) = \tilde{v} \text{ where } \tilde{v} = H^{-1}v. \quad (23)$$

Hence, we look for  $a \in V$  solving (23). In order to prove the existence of solutions it suffices to prove that for  $r > 0$  large enough and for all  $\tau \in [0, T]$  the Brouwer degree of  $\mathcal{L}(\tau, \cdot)$  exists and satisfies

$$\deg(\mathcal{L}(\tau, \cdot), B_r(\tilde{v}), \tilde{v}) \neq 0, \quad (24)$$

where  $B_r(\tilde{v})$  stands for the ball in  $V$  of center  $\tilde{v}$  and radius  $r$ .

For all  $r > 0$ ,

$$\deg(\mathcal{L}(0, \cdot), B_r(\tilde{v}), \tilde{v}) = \deg(\text{Id}, B_r(\tilde{v}), \tilde{v}) = 1.$$

For all  $(\tau, x) \in [0, T] \times V$  such that  $\mathcal{L}(\tau, x) = \tilde{v}$ , we have

$$\begin{aligned} |x - \tilde{v}| &= |H^{-1}H_1(I(x)(\tau) - x)| \\ &\leq |H^{-1}H_1| \cdot |I(x)(\tau) - I(x)(0)| \\ &\leq |H^{-1}H_1| \cdot \|I(x)\|_\alpha \tau^\alpha \\ &\leq |H^{-1}H_1| C_{\alpha, f, \mathbf{X}} T^\alpha. \end{aligned}$$

This implies that for all  $r > |H^{-1}H_1| C_{\alpha, f, \mathbf{X}} T^\alpha$  and all  $\tau \in [0, T]$ ,

$$\tilde{v} \notin \mathcal{L}(\tau, \partial B_r(\tilde{v}))$$

and then the degree  $\deg(\mathcal{L}(\tau, \cdot), B_r(\tilde{v}), (\tilde{v}))$  is well-defined. Moreover, by the continuity of  $I$ , the function  $\mathcal{L}$  is a homotopy. This implies that for all  $\tau \in [0, T]$ ,

$$\deg(\mathcal{L}(\tau, \cdot), B_r(\tilde{v}), \tilde{v}) = \deg(\mathcal{L}(0, \cdot), B_r(\tilde{v}), \tilde{v}) = 1$$

and then (24) and the existence of solution(s)  $a \in B_r(\tilde{v})$  to  $a = L(a)_\tau$ .

It remains to prove that for all  $\tau \in [0, T]$  the number of solution(s) of  $a = L(a)_\tau$  is finite for almost every  $v$ , or equivalently for almost every  $\tilde{v}$ . We fix  $r > |H^{-1}H_1| C_{\alpha, f, \mathbf{X}} T^\alpha$ . Since  $f \in C_b^3$  the function  $a \mapsto I(a)$  is  $C^1$ . Following the Sard theorem [10, Lemma 13.2.1, p. 331], the Lebesgue measure

of the set of critical values of  $\mathcal{L}(\tau, \cdot)$  is equal to 0. Consider a regular value  $\tilde{v}$  of  $\mathcal{L}(\tau, \cdot)$ . The set  $\overline{B_r(\tilde{v})} \cap \mathcal{L}(\tau, \cdot)^{-1}(\tilde{v})$  is closed and bounded, and consequently compact because  $V$  is finite-dimensional. Since  $\tilde{v}$  is a regular value,  $\mathcal{L}(\tau, \cdot)$  is locally invertible at a neighborhood of each point of  $\overline{B_r(\tilde{v})} \cap \mathcal{L}(\tau, \cdot)^{-1}(\tilde{v})$ . Then each point of  $\overline{B_r(\tilde{v})} \cap \mathcal{L}(\tau, \cdot)^{-1}(\tilde{v})$  is isolated. Since  $|L(\tau, \cdot)|$  is bounded (by  $|H^{-1}v| + |H^{-1}H_1|\tau^\alpha C_{\alpha, f, \mathbf{X}}$  for instance), we choose  $r$  large enough and deduce that  $\mathcal{L}(\tau, \cdot)^{-1}(\tilde{v})$  is finite. This completes the proof.  $\square$

As a consequence of Theorem 6, we prove the existence of solution(s) to (18) without the assumption  $\tau^\alpha |H^{-1}H_1| C_{\alpha, f, \mathbf{W}(\omega)} < 1$ .

*Remark 3.* If  $f$  is not bounded, we can find differential equations such that the global solution does not exist. For instance consider the real-valued differential equation

$$dy(t) = y(t) dw(t) \text{ with } y(0) = a$$

where  $w$  is a Hölder function and  $a \in \mathbb{R}$ . The solutions are written in the form

$$y(t) = a \exp(w(t) - w(0)).$$

where  $a$  is a constant. The condition

$$h_0 y(0) + h_1 y(T) = v$$

can be rewritten as

$$h_0 a + h_1 a \exp(w(T) - w(0)) = v$$

and then

$$a = \frac{v}{h_0 + h_1 \exp(w(T) - w(0))}$$

which is defined if and only if  $h_0 + h_1 \exp(w(T) - w(0)) \neq 0$ . The solution is then

$$y(t) = \frac{v \exp(w(t) - w(0))}{h_0 + h_1 \exp(w(T) - w(0))} = \frac{v \exp(w(t))}{h_0 \exp(w(0)) + h_1 \exp(w(T))}.$$

If  $h_0$  and  $h_1$  have different signs and are different from 0, then the condition  $h_0 + h_1 \exp(w(T) - w(0)) \neq 0$  is equivalent to

$$w(T) - w(0) \neq \ln \left( -\frac{h_0}{h_1} \right).$$

This is not ensured even when  $w$  is bounded.

## 7 The Young case

In this section we wish to extend previous results to a framework with long-range dependence. This generally involves more regular processes and the Young approach suffices. We consider in this section  $\alpha \in (1/2, 1)$ . If  $U$  is a Banach space we denote by  $\mathcal{C}^\alpha([0, T], U)$  the space of  $\alpha$ -Hölder rough paths  $X$  and define

$$\|X\|_{\alpha, T} = \sup_{0 \leq s \neq t \leq T} \frac{|X(s, t)|}{|t - s|^\alpha} < \infty.$$

The pseudo-metric on  $\mathcal{C}^\alpha([0, T], U)$  is then reduced to

$$\varrho_{\alpha, T}(X_1, X_2) = \sup_{0 \leq s \neq t \leq T} \frac{|X_1(s, t) - X_2(s, t)|}{|t - s|^\alpha}$$

for all  $X_1$  and  $X_2$  in  $\mathcal{C}^\alpha([0, T], U)$ . If  $V$  is another Banach space,  $f \in C_b^2(V, \mathcal{L}(U, V))$  and  $a \in V$  then the following differential equation

$$Y(t) = a + \int_0^t f(Y(s)) dX(s) \text{ for all } t \in [0, T]$$

(in the Young sense) has a unique solution  $Y \in C^\alpha([0, T], V)$ . If we denote this solution by  $I(X, a)$  to take account of the dependence on  $(X, a)$ , then it defines the Itô map  $I$  from  $V \times \mathcal{C}^\alpha([0, T], U)$  to  $C^\alpha([0, T], V)$ . Moreover it can be proved that, if we consider

$$\mathcal{B}_{M, T} = \{X \in \mathcal{C}^\alpha([0, T], U) \mid \|X\|_{\alpha, T} \leq M\},$$

that there exists a positive constant  $C_{\alpha, f, M}$  depending only on  $\alpha$ ,  $f$  and  $M$  such that for all  $a, b \in V$ , and for all  $X_1$  and  $X_2 \in \mathcal{B}_{M, T}$ ,

$$\|I(a, X_1) - I(b, X_2)\|_{\alpha, T} \leq C_{\alpha, f, M} (|a - b| + \varrho_{\alpha, T}(X_1, X_2)). \quad (25)$$

In particular, for all  $X \in \mathcal{C}^\alpha([0, T], U)$  there exists a positive constant  $C_{\alpha, f, X}$  depending only on  $\alpha$ ,  $f$  and  $\|X\|_{\alpha, T}$  such that for all  $a$  and  $b \in V$ ,

$$\|I(a, X) - I(b, X)\|_{\alpha, T} \leq C_{\alpha, f, X} |a - b|. \quad (26)$$

Due to (25), we can deduce the following result whose proof is similar to the one of Theorem 1.

**Theorem 7.** *Let  $\tau > 0$  be such that*

$$\tau < \frac{1}{(|H^{-1}H_1|C_{\alpha,f,X})^{1/\alpha}}. \quad (27)$$

*If  $H := H_0 + H_1$  is invertible, then there exists a unique solution  $Y \in C^\alpha([0, T], U)$  to*

$$\begin{cases} dY(t) = f(Y(t)) dX(t) \text{ for all } t \in [0, T], \\ H_0Y(0) + H_1Y(\tau) = v. \end{cases} \quad (28)$$

Theorem 7 enables us to define the function  $\bar{I} : \mathcal{B}_{M,T} \rightarrow \mathcal{C}^\alpha([0, T], V)$  such that for all  $X \in \mathcal{C}^\alpha([0, T], U)$ ,  $\bar{I}(X)$  is the unique solution of (28). The following theorem states that  $\bar{I}$  is continuous. Its proof is similar to the one of Theorem 2.

**Theorem 8.** *Under Condition (27), the map  $\bar{I}$  is Lipschitz continuous with respect to  $\varrho_{\alpha,T}$  on  $\mathcal{B}_{M,T}$ , with constant*

$$C_{\alpha,f,M} + \frac{|H^{-1}H_1|C_{\alpha,f,M}\tau^\alpha}{1 - |H^{-1}H_1|C_{\alpha,f,M}\tau^\alpha}.$$

We can deduce the following limit theorem.

**Theorem 9.** *We assume that  $V$  is separable and that (20) is satisfied. Let  $X$  and  $(X^n)_n$  be a random variable and a sequence of random variables with values in  $\mathcal{C}^{0,\alpha}([0, T], V)$  such that  $X^n$  converges in distribution to  $X$  in  $\mathcal{C}^{0,\alpha}([0, T], V)$  and  $\mathbb{P}(\|X\|_{\alpha,T} = M) = 0$ . For all continuous bounded functions  $h : C^\alpha([0, \tau], V) \rightarrow \mathbb{R}$ ,*

$$\lim_{n \rightarrow \infty} \mathbb{E}(h(\bar{I}(X^n))\mathbb{1}_{X^n \in \mathcal{B}_{M,T}}) = \mathbb{E}(h(\bar{I}(X))\mathbb{1}_{X \in \mathcal{B}_{M,T}}).$$

*Moreover, if  $\mathbb{P}(\|X\|_{\alpha,T} \leq M) \neq 0$  and  $\mathbb{P}(\|X^n\|_{\alpha,T} \leq M) \neq 0$  for all  $n$ , then*

$$\lim_{n \rightarrow \infty} \mathbb{E}(h(\bar{I}(X^n))|X^n \in \mathcal{B}_{M,T}) = \mathbb{E}(h(\bar{I}(X))|X \in \mathcal{B}_{M,T}).$$

As an application we consider for all  $\varepsilon \in (0, 1]$  the problem

$$\begin{cases} \frac{dY^\varepsilon(t)}{dt} = \sum_{j=1}^d \frac{1}{\varepsilon^{\gamma_j}} m_j \left( \frac{t}{\varepsilon^2} \right) f_j(Y^\varepsilon(t)) \text{ for all } t \in [0, T], \\ H_0Y^\varepsilon(0) + H_1Y^\varepsilon(\tau) = v, \end{cases} \quad (29)$$

where for all  $j$ ,  $\gamma_j \in (0, 1)$  and  $m_j$  is a continuous stationary Gaussian process with zero mean and such that there exists  $c_j$  such that the covariance satisfies as  $t \rightarrow \infty$ ,

$$\mathbb{E}(m_j(0)m_j(t)) \sim \frac{c_j}{t^{\gamma_j}}.$$

Moreover we assume that  $m_1, m_2, \dots, m_d$  are independent, and without loss of generality, that  $\gamma_1 \geq \dots \geq \gamma_d$ .

We define for all  $\varepsilon$  the Lipschitz continuous process  $w$  such that for all  $(s, t)$ ,

$$w^\varepsilon(s, t) = (w_1^\varepsilon(s, t), \dots, w_d^\varepsilon(s, t))$$

where for all  $j$ ,

$$w_j^\varepsilon(s, t) = \int_s^t \frac{1}{\varepsilon^{\gamma_j}} m_j\left(\frac{r}{\varepsilon^2}\right) dr.$$

The following result deals with the asymptotic behavior of (29).

**Theorem 10.** *Let  $\alpha \in (\frac{1}{2}, \frac{2-\gamma_1}{2})$ . For all continuous bounded functions  $h : C^\alpha([0, T], V) \rightarrow \mathbb{R}$ ,*

$$\lim_{n \rightarrow \infty} \mathbb{E}(h(Y^\varepsilon) \mathbb{1}_{w^\varepsilon \in \mathcal{B}_{M,T}}) = \mathbb{E}(h(Y) \mathbb{1}_{W \in \mathcal{B}_{M,T}}), \quad (30)$$

where  $Y$  is the solution of

$$\begin{cases} dY(t) = \sum_{j=1}^d f_j(Y(t)) dW_j(t) \text{ for all } t \in [0, T], \\ H_0 Y(0) + H_1 Y(\tau) = v \end{cases}$$

and  $W = (W_1, \dots, W_d)$  is a multidimensional fractional Brownian motion such that for all  $(j, k)$ ,

$$\mathbb{E}(W_j(t)W_k(s)) = \delta_{j=k} \frac{\rho_j}{2} (|t|^{2H_j} + |s|^{2H_j} - |t-s|^{2H_j})$$

with  $H_j = (2 - \gamma_j)/2$  and  $\rho_j = H_j^{-1}(2H_j - 1)^{-1}c_j$ .

This theorem is a fractional extension of [9, Theorem 7].

*Proof.* Since  $\alpha \in (\frac{1}{2}, \frac{2-\gamma_1}{2})$  and by [16, Theorem 5],  $w^\varepsilon$  converges in distribution in  $\mathcal{C}^{0,\alpha}([0, T], V)$  as  $\varepsilon \rightarrow 0$ . By Theorem 9, the proof is completed.  $\square$

## 8 Conclusion

The principle of invariant imbedding reduces the problem of a two-point boundary value problem to solving a non-linear fixed-point theorem. The features of the Itô map, mainly its regularity with all the parameters (driving noise, starting point, vector field, ...) allows one to extend the existence, uniqueness and regularity problems to a wide range of irregular, driving signals. Our approach may be used whenever the Itô map is well defined. This is especially suitable for a large range of driving noises (semi-martingales, fractional Brownian motion and many more). In addition, the continuity of the Itô map is relevant to apply limit theorems (diffusion principle) as well as approximation by smooth signals. In particular, we circumvented any technical difficulties related to anticipative calculus.

In the context of random signals, one difficulties is that existence and uniqueness hold only for short time, in a way which depends on pathwise bounds on the signal.

We have however proved existence for larger time by using the degree theory. This seems to be new in this context. Uniqueness is however lost. We gave one example showing that for long time, there could exist critical points to the non-linear functional to invert. Therefore, one cannot expect to get existence and uniqueness for any time. This is related to the behavior of the flow of the Itô map and the choice of the operators  $H_0$  and  $H_1$ .

Further studies could concern the obtention of sharper hypotheses, to give more complex counter-example to existence, as well as asymptotic results and their applications in multi-scale modeling.

Finally, let us mention that using the theory of regularity structures [8] or other alternative theory could lead to similar results where the ordinary/rough differential equations are replaced by (rough) partial differential equations.

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