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# Supplementary Material for Learning Rate Adaptation by Line Search in Evolution Strategies with Recombination

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## ABSTRACT

This is the supplementary material (proofs) for [1].

PROOF OF LEMMA 2.3 IN [1]. Let  $t \in \mathbb{N}$ . We know that  $X_{t+1} = X_t + \kappa_{t+1} \sum_{i=1}^{\mu} w_i U_{t+1}^{\varphi(i)}$ , with  $\kappa_{t+1} = \bar{\kappa}(X_t, \alpha \|X_t\| \sum_{i=1}^{\mu} w_i U_{t+1}^{\varphi(i)})$ . Then, by (A1)

$$Z_t = \frac{\|X_{t+1}\|}{\|X_t\|} = \left\| \frac{X_t}{\|X_t\|} + \alpha \bar{\kappa} \left( \frac{X_t}{\|X_t\|}, \alpha S_{U_{t+1}}^{\varphi} \right) S_{U_{t+1}}^{\varphi} \right\|.$$

But then, according to (A2), we have that  $Z_t$  has the same distribution than  $\|e_1 + \alpha \bar{\kappa}(e_1, \alpha S_V^{\varphi}) S_V^{\varphi}\|$ . Thus, all the  $Z_t, t \in \mathbb{N}$  are identically distributed.

Let  $s \in \mathbb{R}$ , and denote  $\psi^s: \mathbb{R}^n \rightarrow \mathbb{C}$  the measurable function such that  $\psi^s(X_t) = \mathbb{E}[\exp is Z_t \mid X_t]$ . Then, for  $x \in \mathbb{R}$ , according to (A2) applied to  $X = x$

$$\begin{aligned} \psi^s(x) &= \mathbb{E} \exp is \left\| \frac{x}{\|x\|} + \alpha \bar{\kappa} \left( \frac{x}{\|x\|}, \alpha S_{U_{t+1}}^{\varphi} \right) S_{U_{t+1}}^{\varphi} \right\| \\ &= \psi^s(e_1). \end{aligned}$$

Thus,  $\mathbb{E}[\exp is Z_t \mid X_t] = \psi^s(X_t) = \psi^s(e_1)$ . Define  $\mathcal{F}_t$  the filtration induced by  $X_0, \dots, X_t$ . We get then

$$\begin{aligned} &\mathbb{E}[\exp i(s_0 Z_0 + s_1 Z_1 + \dots + s_t Z_t)] \\ &= \mathbb{E}[\mathbb{E}[\exp i(s_0 Z_0 + s_1 Z_1 + \dots + s_t Z_t) \mid \mathcal{F}_t]] \\ &= \mathbb{E}[\exp is(s_0 Z_0 + s_1 Z_1 + \dots + s_{t-1} Z_{t-1}) \times \mathbb{E}[\exp is_t Z_t \mid X_t]] \\ &= \mathbb{E}[\exp i(s_0 Z_0 + s_1 Z_1 + \dots + s_{t-1} Z_{t-1})] \times \psi^{s_t}(e_1) \\ &= \mathbb{E}[\exp i(s_0 Z_0 + s_1 Z_1 + \dots + s_{t-1} Z_{t-1})] \times \mathbb{E}[\exp is_t Z_t] \end{aligned}$$

Hence,  $Z_t$  is independent from  $(Z_0, Z_1, \dots, Z_{t-1})$ . By induction, we get that  $(Z_0, Z_1, \dots, Z_t)$  is mutually independent.  $\square$

PROOF OF LEMMA 3.2 IN [1]. Assumption (A1) holds by [1, Lemma 2.1].

Let  $R$  be a rotation matrix. Then  $\bar{\kappa}_{\text{PLS}}(Rx, Rv) = \arg \min_{\kappa \geq 0} \|Rx + \kappa Rv\| = \arg \min_{\kappa \geq 0} \|R(x + \kappa v)\| = \arg \min_{\kappa \geq 0} \|x + \kappa v\| = \bar{\kappa}_{\text{PLS}}(x, v)$ . Hence  $\bar{\kappa}_{\text{PLS}}$  is rotation-invariant and [1, Lemma 2.2] implies that (A2) holds.

Now, let us show that (A4) is satisfied. Let  $U^1, \dots, U^{\lambda} \in \mathbb{R}^{\mathbb{N}}$  be i.i.d. infinite dimensional standard Gaussian vectors. Then, when  $n \rightarrow \infty$ , the following limit holds a.s. by the strong LLN

$$\frac{1}{n} \left\| \sum_{i=1}^{\mu} w_i [U^i]_{\leq n} \right\|^2 = \frac{1}{n} \sum_{k=1}^n \left| \sum_{i=1}^{\mu} w_i [U^i]_k \right|^2 \rightarrow \mu_w^{-1},$$

thus by [1, Lemma 3.1]

$$\begin{aligned} \bar{\kappa}_{\text{PLS}} \left( e_1, \frac{\alpha}{n} \sum_{i=1}^{\mu} w_i [U^i]_{\leq n} \right) &= -1_{\sum_{i=1}^{\mu} w_i [U^i]_1 < 0} \frac{\sum_{i=1}^{\mu} w_i [U^i]_1}{\frac{\alpha}{n} \left\| \sum_{i=1}^{\mu} w_i [U^i]_{\leq n} \right\|^2} \\ &\rightarrow -1_{\sum_{i=1}^{\mu} w_i [U^i]_1 < 0} \alpha^{-1} \mu_w \sum_{i=1}^{\mu} w_i [U^i]_1. \end{aligned}$$

Thus (A4) holds.  $\square$

PROOF OF LEMMA 4.2 IN [1]. First, let us prove the scaling-invariance condition (A1) for  $\bar{\kappa}_{\text{DLS}}^{\varepsilon, \beta}$ . Consider  $\bar{\kappa}_{\text{DLS}}^{\varepsilon, \beta}(x/r, v/r, \kappa_{\text{init}})$ . Then, as in line 4 of [1, Algorithm 2], the condition  $f(x + \kappa^0 v) < f(x + \kappa^1 v)$  is equivalent to  $f(x/r + \kappa^0 v/r) < f(x/r + \kappa^1 v/r)$ , as  $f$  is the sphere function is scaling-invariant, then  $\bar{\kappa}_{\text{DLS}}^{\varepsilon, \beta}(x/r, v/r, \kappa_{\text{init}}) = \bar{\kappa}_{\text{DLS}}^{\varepsilon, \beta}(x, v, \kappa_{\text{init}})$ . Thus, (A1) holds.

The function  $\bar{\kappa}_{\text{DLS}}^{\varepsilon, \beta}$  is rotation-invariant. Indeed if  $R$  is a rotation matrix, then, as in line 4 of [1, Algorithm 2], the condition  $f(x + \kappa^0 v) < f(x + \kappa^1 v)$  is equivalent to  $f(Rx + \kappa^0 Rv) < f(Rx + \kappa^1 Rv)$ , as  $f$  is the sphere function is invariant by rotation, then this implies that  $\bar{\kappa}_{\text{DLS}}^{\varepsilon, \beta}(Rx, Rv, \kappa_{\text{init}}) = \bar{\kappa}_{\text{DLS}}^{\varepsilon, \beta}(x, v, \kappa_{\text{init}})$ . Thus, by [1, Lemma 2.2], (A2) holds.

We prove now that (A4) is satisfied. Consider  $U^1, \dots, U^{\mu} \in \mathbb{R}^{\mathbb{N}}$   $\mu$  i.i.d. infinite dimensional standard Gaussian vectors, and denote  $S_U^n = \sum_{i=1}^{\mu} w_i [U^i]_{\leq n}$ .

Consider the line search obtained with  $\bar{\kappa}_{\text{DLS}}^{\varepsilon, \beta} \left( e_1, \frac{\alpha}{n} S_U^n, \kappa_{\text{init}} \right)$ . We denote  $(\kappa^{i,0}, \kappa^{i,1})_{i=0, \dots, C(\beta, \varepsilon) - 2}$  the value of  $\kappa^1$  and  $\kappa^0$  over the iterations of this line search.

We prove now by induction that for all  $i = 0, 1, \dots$ , the following limits when  $n \rightarrow \infty$  hold a.s.  $\kappa^{i,0} \rightarrow \kappa^{\infty, i, 0}$  and  $\kappa^{i,1} \rightarrow \kappa^{\infty, i, 1}$ , where  $\kappa^{\infty, i, 0}, \kappa^{\infty, i, 1}$  are given in the line 5 and line 7 of [1, Algorithm 2] over iteration initialized with parameters  $X = \alpha^{-1} \mu_w S_U^1 \in \mathbb{R}^1, v = 1 \in \mathbb{R}^1$ , and given  $\kappa_{\text{init}}, \varepsilon, \beta$ . At iteration  $i = 0$ , this is trivially true as  $\kappa^{i,0} = \kappa^{\infty, i, 0} = \kappa_{\text{init}}/2$ , and  $\kappa^{i,1} = \kappa^{\infty, i, 1} = 2\kappa_{\text{init}}$ .

Now consider an arbitrary step  $i$  of the line search, and assume that  $\kappa^{i,0} \rightarrow \kappa^{\infty, i, 0}$  and  $\kappa^{i,1} \rightarrow \kappa^{\infty, i, 1}$ . Then,  $\kappa^{i+1,0} = \kappa^{i,0} + (1 - \beta) (\kappa^{i,1} - \kappa^{i,0}) \mathbf{1}_{\{h_{\alpha/n}(\kappa^{i,1} S_U^n) < h_{\alpha/n}(\kappa^{i,0} S_U^n)\}}$ , where

$$h_{\alpha/n}(\kappa^{i,j} S_U^n) = 2 \underbrace{\kappa^{i,j}}_{\rightarrow \kappa^{\infty, i, j}} S_U^1 + \alpha \left( \kappa^{i,j} \right)^2 \underbrace{\frac{\|S_U^n\|^2}{n}}_{\rightarrow \sum_{i=1}^{\mu} w_i^2}$$

Hence, a.s.  $h_{\alpha/n}(\kappa^{i,j} S_U^n)$  tends to  $2\kappa^{\infty,i,j} S_U^1 + \alpha (\kappa^{\infty,i,j})^2 \mu_w^{-1}$  when  $n \rightarrow \infty$ , so that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbf{1}_{\{h_{\alpha/n}(\kappa^{i,1} S_U^n) < h_{\alpha/n}(\kappa^{i,0} S_U^n)\}} \\ &= \mathbf{1}_{\{(\kappa^{\infty,i,1} - \kappa^{\infty,i,0}) (2S_U^n + (\kappa^{\infty,i,0} + \kappa^{\infty,i,1}) \alpha \sum_{i=1}^{\mu} w_i^2) > 0\}}, \end{aligned}$$

hence  $\kappa^{i+1,0} \rightarrow \kappa^{\infty,i+1,0}$ . Similarly,  $\lim_{n \rightarrow \infty} \kappa^{i+1,1} = \kappa^{\infty,i+1,1}$ .

In the end, by induction, we get that

$$\lim_{n \rightarrow \infty} \bar{\kappa} \left( e_1, \frac{\alpha}{n} S_U^n, \kappa_{\text{init}} \right) = \bar{\kappa}^{\infty} \left( \left( [U^i]_1 \right)_{i=1, \dots, \lambda}, \kappa_{\text{init}} \right).$$

where  $\bar{\kappa}^{\infty}$  is defined in [1, Lemma 4.2]. Hence (A4) holds.  $\square$

**LEMMA A.1.** For  $n \in \mathbb{N}^*$ , let  $U_1^n, \dots, U_n^n$  be i.i.d. standard multi-variate normal distribution of dimension  $n$ . Suppose that the functions  $\bar{\kappa}_n: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_+$  are either all equal to  $\bar{\kappa}_{\text{PLS}}$  or all upperbounded by a positive constant  $\kappa^M$ . Then,

$$\left( n \ln \left\| e_1 + \bar{\kappa}_n \left( e_1, \alpha/n \sum_{i=1}^{\mu} w_i U_n^i \right) \alpha/n \sum_{i=1}^n w_i U_n^i \right\|^2 \right)_{n \in \mathbb{N}^*}$$

is uniformly integrable.

**PROOF.** Suppose first that for all  $n \in \mathbb{N}^*$ ,  $\bar{\kappa}_n = \bar{\kappa}_{\text{PLS}}$ . Note then that, as in the proof of [1, Theorem 3.3], we have

$$\begin{aligned} & n \ln \left\| e_1 + \bar{\kappa}_n \left( e_1, \alpha/n \sum_{i=1}^{\mu} w_i U_n^i \right) \frac{\alpha}{n} \sum_{i=1}^n w_i U_n^i \right\|^2 \\ &= -8 \ln \left[ \left( 1 - \frac{[\sum_{i=1}^{\mu} w_i U_n^i]_1^2}{\|\sum_{i=1}^{\mu} w_i U_n^i\|^2} \right)^{-n/8} \right] \mathbf{1}_{[\sum_{i=1}^{\mu} w_i U_n^i]_1 < 0}. \end{aligned}$$

But it is proven in [2, Proof of Proposition 4, page 23] that

$$\left( \ln \left[ \left( 1 - \frac{[\sum_{i=1}^{\mu} w_i U_n^i]_1^2}{\|\sum_{i=1}^{\mu} w_i U_n^i\|^2} \right)^{-n/8} \right] \mathbf{1}_{[\sum_{i=1}^{\mu} w_i U_n^i]_1 < 0} \right)_{n \in \mathbb{N}^*}$$

is uniformly integrable.

Suppose now that there exists  $\kappa^M > 0$  such that for all  $n \in \mathbb{N}^*$ ,  $\bar{\kappa}_n \leq \kappa^M$ . We will prove the uniform integrability of the positive part and of the negative part of

$$\left( n \ln \left\| e_1 + \bar{\kappa}_n \left( e_1, \alpha/n \sum_{i=1}^{\mu} w_i U_n^i \right) \alpha/n \sum_{i=1}^n w_i U_n^i \right\|^2 \right)_{n \in \mathbb{N}^*}.$$

For the negative part, note that, by definition of  $\bar{\kappa}_{\text{PLS}}$ , then

$$\begin{aligned} & \left\| e_1 + \bar{\kappa}_n \left( e_1, \alpha/n \sum_{i=1}^{\mu} w_i U_n^i \right) \alpha/n \sum_{i=1}^n w_i U_n^i \right\|^2 \\ & \geq \left\| e_1 + \bar{\kappa}_{\text{PLS}} \left( e_1, \alpha/n \sum_{i=1}^{\mu} w_i U_n^i \right) \alpha/n \sum_{i=1}^n w_i U_n^i \right\|^2. \end{aligned}$$

In addition, the negative part of the logarithm  $\ln^-$  is a decreasing function, thus

$$\begin{aligned} & n \ln^- \left\| e_1 + \bar{\kappa}_n \left( e_1, \alpha/n \sum_{i=1}^{\mu} w_i U_n^i \right) \alpha/n \sum_{i=1}^n w_i U_n^i \right\|^2 \\ & \leq n \ln^- \left\| e_1 + \bar{\kappa}_{\text{PLS}} \left( e_1, \alpha/n \sum_{i=1}^{\mu} w_i U_n^i \right) \alpha/n \sum_{i=1}^n w_i U_n^i \right\|^2. \end{aligned} \quad (1)$$

Note that, as  $\|e_1 + \bar{\kappa}_{\text{PLS}}(e_1, \alpha/n \sum_{i=1}^{\mu} w_i U_n^i) \alpha/n \sum_{i=1}^n w_i U_n^i\| \leq \|e_1 + 0\| = 1$ , then  $\ln \|e_1 + \bar{\kappa}_{\text{PLS}}(e_1, \alpha/n \sum_{i=1}^{\mu} w_i U_n^i) \alpha/n \sum_{i=1}^n w_i U_n^i\| \leq \|e_1 + 0\| \leq 0$ , thus is equal in absolute value to its negative part.

However, as

$$\left( n \ln^- \left\| e_1 + \bar{\kappa}_{\text{PLS}} \left( e_1, \alpha/n \sum_{i=1}^{\mu} w_i U_n^i \right) \alpha/n \sum_{i=1}^n w_i U_n^i \right\|^2 \right)_{n \in \mathbb{N}^*}$$

is uniformly integrable (since it is equal to the sequence with  $\ln$  instead of  $\ln^-$  which is uniformly integrable as seen above), then according to Eq. (1) so is

$$\left( n \ln^- \left\| e_1 + \bar{\kappa}_n \left( e_1, \alpha/n \sum_{i=1}^{\mu} w_i U_n^i \right) \alpha/n \sum_{i=1}^n w_i U_n^i \right\|^2 \right)_{n \in \mathbb{N}^*}.$$

We claim that for all  $x \in \mathbb{R}^n$ , and  $0 \leq \kappa \leq \kappa^M$ , then

$$\ln^+ \|e_1 + \kappa x\| \leq \ln^+ \|e_1 + \kappa^M x\|.$$

Indeed, the above equation is equivalent to

$$\|e_1 + \kappa x\|^2 \mathbf{1}_{\|e_1 + \kappa x\| \geq 1} \leq \|e_1 + \kappa^M x\|^2 \mathbf{1}_{\|e_1 + \kappa^M x\| \geq 1}.$$

However, the derivative of the function  $\kappa \mapsto \|e_1 + \kappa x\|^2$  is equal to  $\kappa \mapsto 2[x]_1 + \kappa \langle x, x \rangle$ , hence is nonnegative for any  $\kappa \geq 0$  that also satisfies that  $\|e_1 + \kappa x\|^2 = 1 + \kappa(2[x]_1 + \kappa \langle x, x \rangle)$  is greater than or equal to 1. Thus, the above condition is satisfied.

For the positive part, we have then

$$\begin{aligned} & n \ln^+ \left\| e_1 + \bar{\kappa}_n \left( e_1, \frac{\alpha}{n} \sum_{i=1}^{\mu} w_i U_n^i \right) \frac{\alpha}{n} \sum_{i=1}^n w_i U_n^i \right\|^2 \\ & \leq n \ln^+ \left\| e_1 + \frac{\kappa^M \alpha}{n} \sum_{i=1}^{\mu} w_i U_n^i \right\|^2. \end{aligned}$$

But it is proven in [2]<sup>1</sup> that the RHS of the above equation is uniformly integrable. All in all, we get that

$$\left( n \ln \left\| e_1 + \bar{\kappa}_n \left( e_1, \alpha/n \sum_{i=1}^{\mu} w_i U_n^i \right) \alpha/n \sum_{i=1}^n w_i U_n^i \right\|^2 \right)_{n \in \mathbb{N}^*}$$

is uniformly integrable.  $\square$

## REFERENCES

- [1] Armand Gissler, Anne Auger, and Nikolaus Hansen. Learning rate adaptation by line search in evolution strategies with recombination. In *Genetic and Evolutionary Computation Conference (GECCO 2022)*, Boston, United States, July 2022.
- [2] Mohamed Jebalia and Anne Auger. Log-linear convergence of the scale-invariant  $(\mu/\mu_w, \lambda)$ -ES and optimal  $\mu$  for intermediate recombination for large population sizes. In *International Conference on Parallel Problem Solving from Nature*, pages 52–62. Springer, 2010.

<sup>1</sup>replace  $\sigma^*$  by  $\alpha \kappa^M$  in the proof of Proposition 4 of [2]