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Supplementary Material for Learning Rate Adaptation by Line Search in Evolution Strategies with Recombination

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ABSTRACT

This is the supplementary material (proofs) for [1].

PROOF OF LEMMA 2.3 IN [1]. Let $t \in \mathbb{N}$. We know that $X_{t+1} =$ $X_{t} + \kappa_{t+1} \sum_{i=1}^{\mu} w_{i} U_{t+1}^{\varphi(i)}, \text{ with } \kappa_{t+1} = \bar{\kappa}(X_{t}, \alpha ||X_{t}|| \sum_{i=1}^{\mu} w_{i} U_{t+1}^{\varphi(i)}).$ Then, by (A1)

$$Z_t = \frac{\|X_{t+1}\|}{\|X_t\|} = \left\|\frac{X_t}{\|X_t\|} + \alpha \bar{\kappa} \left(\frac{X_t}{\|X_t\|}, \alpha S_{U_{t+1}}^{\varphi}\right) S_{U_{t+1}}^{\varphi}\right\|.$$

But then, according to (A2), we have that Z_t has the same distribution than $||e_1 + \alpha \bar{\kappa}(e_1, \alpha S_V^{\varphi}) S_V^{\varphi}||$. Thus, all the $Z_t, t \in \mathbb{N}$ are identically distributed.

Let $s \in \mathbb{R}$, and denote $\psi^s : \mathbb{R}^n \to \mathbb{C}$ the measurable function such that $\psi^{s}(X_{t}) = \mathbb{E}[\exp isZ_{t} \mid X_{t}]$. Then, for $x \in \mathbb{R}$, according to (A2) applied to X = x

$$\psi^{s}(x) = \mathbb{E} \exp is \left\| \frac{x}{\|x\|} + \alpha \bar{\kappa} \left(\frac{x}{\|x\|}, \alpha S^{\varphi}_{U_{t+1}} \right) S^{\varphi}_{U_{t+1}} \right\|$$
$$= \psi^{s}(e_{1}).$$

Thus, $\mathbb{E}[\exp isZ_t \mid X_t] = \psi^s(X_t) = \psi^s(e_1)$. Define \mathcal{F}_t the filtration induced by X_0, \ldots, X_t . We get then

$$\begin{split} & \mathbb{E}[\exp i(s_0 Z_0 + s_1 Z_1 + \dots + s_t Z_t)] \\ & = \mathbb{E}[\mathbb{E}[\exp i(s_0 Z_0 + s_1 Z_1 + \dots + s_t Z_t) \mid \mathcal{F}_t]] \\ & = \mathbb{E}[\exp i(s_0 Z_0 + s_1 Z_1 + \dots + s_{t-1} Z_{t-1}) \times \mathbb{E}[\exp i s_t Z_t \mid X_t]] \\ & = \mathbb{E}[\exp i(s_0 Z_0 + s_1 Z_1 + \dots + s_{t-1} Z_{t-1})] \times \psi^{s_t}(e_1) \\ & = \mathbb{E}[\exp i(s_0 Z_0 + s_1 Z_1 + \dots + s_{t-1} Z_{t-1})] \times \mathbb{E}[\exp i s_t Z_t] \end{split}$$

Hence, Z_t is independent from $(Z_0, Z_1, \ldots, Z_{t-1})$. By induction, we get that (Z_0, Z_1, \ldots, Z_t) is mutually independent. П

PROOF OF LEMMA 3.2 IN [1]. Assumption (A1) holds by [1, Lemma 2.1].

Let *R* be a rotation matrix. Then $\bar{\kappa}_{PLS}(Rx, Rv) = \arg \min_{\kappa \ge 0} ||Rx+$ $\kappa Rv\| = \arg\min_{\kappa \ge 0} \|R(x + \kappa v)\| = \arg\min_{\kappa \ge 0} \|x + \kappa v\| = \bar{\kappa}_{\text{PLS}}(x, v).$ Hence $\bar{\kappa}_{PLS}$ is rotation-invariant and [1, Lemma 2.2] implies that (A2) holds.

Now, let us show that (A4) is satisfied. Let $U^1, \ldots, U^{\lambda} \in \mathbb{R}^{\mathbb{N}}$ be i.i.d. infinite dimensional standard Gaussian vectors. Then, when $n \to \infty$, the following limit holds a.s. by the strong LLN

$$\frac{1}{n} \left\| \sum_{i=1}^{\mu} w_i[U^i]_{\leq n} \right\|^2 = \frac{1}{n} \sum_{k=1}^{n} \left| \sum_{i=1}^{\mu} w_i[U^i]_k \right|^2 \to \mu_{\rm w}^{-1},$$

thus by [1, Lemma 3.1]

$$\bar{\kappa}_{\text{PLS}}\left(e_{1}, \frac{\alpha}{n} \sum_{i=1}^{\mu} w_{i}[U^{i}]_{\leq n}\right) = -\mathbf{1}_{\sum_{i=1}^{\mu} w_{i}[U^{i}]_{1} < 0} \frac{\sum_{i=1}^{\mu} w_{i}[U^{i}]_{1}}{\frac{\alpha}{n} \|\sum_{i=1}^{\mu} w_{i}[U^{i}]_{\leq n} \|^{2}}$$
$$\to -\mathbf{1}_{\sum_{i=1}^{\mu} w_{i}[U^{i}]_{1} < 0} \alpha^{-1} \mu_{w} \sum_{i=1}^{\mu} w_{i}[U^{i}]_{1}.$$

Thus (A4) holds.

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PROOF OF LEMMA 4.2 IN [1]. First, let us prove the scaling-invariance condition (A1) for $\bar{\kappa}_{\text{DLS}}^{\varepsilon,\beta}$. Consider $\bar{\kappa}_{\text{DLS}}^{\varepsilon,\beta}(x/r,v/r,\kappa_{\text{init}})$. Then, as in line 4 of [1, Algorithm 2], the condition $f(x + \kappa^0 v) < f(x + \kappa^1 v)$ is equivalent to $f(x/r + \kappa^0 v/r) < f(x/r + \kappa^1 v/r)$, as f is the sphere function is scaling-invariant, then $\bar{\kappa}_{\text{DLS}}^{\epsilon,\beta}(x/r,v/r,\kappa_{\text{init}}) =$ $\bar{\kappa}_{\text{DLS}}^{\varepsilon,\beta}(x,v,\kappa_{\text{init}})$. Thus, (A1) holds.

The function $\bar{\kappa}_{\text{DLS}}^{\epsilon,\beta}$ is rotation-invariant. Indeed if *R* is a rotation matrix, then, as in line 4 of [1, Algorithm 2], the condition f(x + $\kappa^0 v$) < $f(x + \kappa^1 v)$ is equivalent to $f(Rx + \kappa^0 Rv) < f(Rx + \kappa^1 Rv)$, as f is the sphere function is invariant by rotation, then this implies that $\bar{\kappa}_{\text{DLS}}^{\varepsilon,\beta}(Rx, Rv, \kappa_{\text{init}}) = \bar{\kappa}_{\text{DLS}}^{\varepsilon,\beta}(x, v, \kappa_{\text{init}})$. Thus, by [1, Lemma 2.2], (A2) holds.

We prove now that (A4) is satisfied. Consider $U^1, \ldots, U^{\mu} \in \mathbb{R}^{\mathbb{N}}$ μ i.i.d. infinite dimensional standard Gaussian vectors, and denote $S_U^n = \sum_{i=1}^{\mu} w_i [U^i]_{\leq n}.$

Consider the line search obtained with $\bar{\kappa}_{\text{DLS}}^{\varepsilon,\beta}\left(e_1,\frac{\alpha}{n}S_U^n,\kappa_{\text{init}}\right)$. We denote $(\kappa^{i,0}, \kappa^{i,1})_{i=0,\cdots,C(\beta,\varepsilon)-2}$ the value of κ^1 and κ^0 over the iterations of this line search.

We prove now by induction that for all i = 0, 1, ..., the following limits when $n \to \infty$ hold a.s. $\kappa^{i,0} \to \kappa^{\infty,i,0}$ and $\kappa^{i,1} \to \kappa^{\infty,i,1}$, where $\kappa^{\infty,i,0}, \kappa^{\infty,i,1}$ are given in the line 5 and line 7 of [1, Algorithm 2] over iteration initialized with parameters $X = \alpha^{-1} \mu_{w} S_{U}^{1} \in \mathbb{R}^{1}, v =$ $1 \in \mathbb{R}^1$, and given $\kappa_{\text{init}}, \varepsilon, \beta$. At iteration i = 0, this is trivially true as $\kappa^{i,0} = \kappa^{\infty,i,0} = \kappa_{\text{init}}/2$, and $\kappa^{i,1} = \kappa^{\infty,i,1} = 2\kappa_{\text{init}}$

Now consider an arbitrary step i of the line search, and assume that $\kappa^{i,0} \to \kappa^{\infty,i,0}$ and $\kappa^{i,1} \to \kappa^{\infty,i,1}$. Then, $\kappa^{i+1,0} = \kappa^{i,0} + (1 - 1)^{-1}$ $\beta) \left(\kappa^{i,1} - \kappa^{i,0} \right) \mathbf{1}_{\{h_{\alpha/n} \left(\kappa^{i,1} S_U^n \right) < h_{\alpha/n} \left(\kappa^{i,0} S_U^n \right) \}}, \text{ where }$

$$h_{\alpha/n}(\kappa^{i,j}S_U^n) = 2 \underbrace{\kappa^{i,j}}_{\to \kappa^{\infty,i,j}} S_U^1 + \alpha \left(\kappa^{i,j}\right)^2 \underbrace{\frac{\|S_U^n\|^2}{n}}_{\to \sum_{i=1}^{\mu} w}$$

Hence, a.s. $h_{\alpha/n}(\kappa^{i,j}S_U^n)$ tends to $2\kappa^{\infty,i,j}S_U^1 + \alpha (\kappa^{\infty,i,j})^2 \mu_w^{-1}$ when $n \to \infty$, so that

$$\begin{split} \lim_{n \to \infty} \mathbf{1}_{\{h_{\alpha/n}(\kappa^{i,1}S_U^n) < h_{\alpha/n}(\kappa^{i,0}S_U^n)\}} \\ &= \mathbf{1}_{\{(\kappa^{\infty,i,1} - \kappa^{\infty,i,0}) \left(2S_U^n + (\kappa^{\infty,i,0} + \kappa^{\infty,i,1})\alpha \sum_{i=1}^{\mu} w_i^2\right) > 0\}}, \end{split}$$

hence $\kappa^{i+1,0} \to \kappa^{\infty,i+1,0}$. Similarly, $\lim_{k \to \infty} \kappa^{i+1,1} = \kappa^{\infty,i+1,1}$. In the end, by induction, we get that

 $\lim_{n\to\infty}\bar{\kappa}\left(e_{1},\frac{\alpha}{n}S_{U}^{n},\kappa_{\mathrm{init}}\right)=\bar{\kappa}^{\infty}\left(\left(\left[U^{i}\right]_{1}\right)_{i=1,\ldots,\lambda},\kappa_{\mathrm{init}}\right).$

where $\bar{\kappa}^{\infty}$ is defined in [1, Lemma 4.2]. Hence (A4) holds.

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LEMMA A.1. For $n \in \mathbb{N}^*$, let U_n^1, \ldots, U_n^{μ} be i.i.d. standard multivariate normal distribution of dimension n. Suppose that the functions $\bar{\kappa}_n \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ are either all equal to $\bar{\kappa}_{PLS}$ or all upperbounded by a positive constant κ^M . Then,

$$\left(n\ln\left\|e_{1}+\bar{\kappa}_{n}\left(e_{1},\alpha/n\sum_{i=1}^{\mu}w_{i}U_{n}^{i}\right)\alpha/n\sum_{i=1}^{n}w_{i}U_{n}^{i}\right\|^{2}\right)_{n\in\mathbb{N}}$$

is uniformly integrable.

PROOF. Suppose first that for all $n \in \mathbb{N}^*$, $\bar{\kappa}_n = \bar{\kappa}_{\text{PLS}}$. Note then that, as in the proof of [1, Theorem 3.3], we have

$$n \ln \left\| e_1 + \bar{\kappa}_n \left(e_1, \alpha/n \sum_{i=1}^{\mu} w_i U_n^i \right) \frac{\alpha}{n} \sum_{i=1}^{n} w_i U_n^i \right\|^2$$
$$= -8 \ln \left[\left(1 - \frac{\left[\sum_{i=1}^{\mu} w_i U_n^i \right]_1^2}{\left\| \sum_{i=1}^{\mu} w_i U_n^i \right\|^2} \right)^{-n/8} \right] \mathbf{1}_{\left[\sum_{i=1}^{\mu} w_i U_n^i \right]_1 < 0}.$$

But it is proven in [2, Proof of Proposition 4, page 23] that

$$\left(\ln\left|\left(1-\frac{\left[\sum_{i=1}^{\mu}w_{i}U_{n}^{i}\right]_{1}^{2}}{\left\|\sum_{i=1}^{\mu}w_{i}U_{n}^{i}\right\|^{2}}\right)^{-n/8}\right]\mathbf{1}_{\left[\sum_{i=1}^{\mu}w_{i}U_{n}^{i}\right]_{1}<0}\right)_{n\in\mathbb{N}}$$

is uniformly integrable.

Suppose now that there exists $\kappa^M > 0$ such that for all $n \in \mathbb{N}^*$, $\bar{\kappa}_n \leq \kappa^M$. We will prove the uniform integrability of the positive part and of the negative part of

$$\left(n\ln\left\|e_1+\bar{\kappa}_n\left(e_1,\alpha/n\sum_{i=1}^{\mu}w_iU_n^i\right)\alpha/n\sum_{i=1}^{n}w_iU_n^i\right\|^2\right)_{n\in\mathbb{N}^*}$$

For the negative part, note that, by definition of $\bar{\kappa}_{PLS}$, then

$$\left\| e_1 + \bar{\kappa}_n \left(e_1, \alpha/n \sum_{i=1}^{\mu} w_i U_n^i \right) \alpha/n \sum_{i=1}^{n} w_i U_n^i \right\|^2$$

$$\geq \left\| e_1 + \bar{\kappa}_{\text{PLS}} \left(e_1, \alpha/n \sum_{i=1}^{\mu} w_i U_n^i \right) \alpha/n \sum_{i=1}^{n} w_i U_n^i \right\|^2.$$

In addition, the negative part of the logarithm ln⁻ is a decreasing function, thus

$$n \ln^{-} \left\| e_{1} + \bar{\kappa}_{n} \left(e_{1}, \alpha/n \sum_{i=1}^{\mu} w_{i} U_{n}^{i} \right) \alpha/n \sum_{i=1}^{n} w_{i} U_{n}^{i} \right\|^{2}$$
$$\leq n \ln^{-} \left\| e_{1} + \bar{\kappa}_{\text{PLS}} \left(e_{1}, \alpha/n \sum_{i=1}^{\mu} w_{i} U_{n}^{i} \right) \alpha/n \sum_{i=1}^{n} w_{i} U_{n}^{i} \right\|^{2}.$$
(1)

Note that, as $\|e_1 + \bar{\kappa}_{\text{PLS}}(e_1, \alpha/n \sum_{i=1}^{\mu} w_i U_n^i) \alpha/n \sum_{i=1}^{n} w_i U_n^i\| \le \|e_1 + 0\| = 1$, then $\ln \|e_1 + \bar{\kappa}_{\text{PLS}}(e_1, \alpha/n \sum_{i=1}^{\mu} w_i U_n^i) \alpha/n \sum_{i=1}^{n} w_i U_n^i\| \le 1$ $||e_1 + 0|| \leq 0$, thus is equal in absolute value to its negative part. However, as

$$\left(n\ln^{-}\left\|e_{1}+\bar{\kappa}_{\mathrm{PLS}}\left(e_{1},\alpha/n\sum_{i=1}^{\mu}w_{i}U_{n}^{i}\right)\alpha/n\sum_{i=1}^{n}w_{i}U_{n}^{i}\right\|^{2}\right)_{n\in\mathbb{N}^{*}}$$

is uniformly integrable (since it is equal to the sequence with ln instead of ln⁻ which is uniformly integrable as seen above), then according to Eq. (1) so is

$$\left(n\ln^{-}\left\|e_{1}+\bar{\kappa}_{n}\left(e_{1},\alpha/n\sum_{i=1}^{\mu}w_{i}U_{n}^{i}\right)\alpha/n\sum_{i=1}^{n}w_{i}U_{n}^{i}\right\|^{2}\right)_{n\in\mathbb{N}^{*}}$$

We claim that for all $x \in \mathbb{R}^n$, and $0 \le \kappa \le \kappa^M$, then

 $\ln^{+} \|e_{1} + \kappa x\| \leq \ln^{+} \|e_{1} + \kappa^{M} x\|.$

Indeed, the above equation is equivalent to

$$||e_1 + \kappa x||^2 \mathbf{1}_{||e_1 + \kappa x|| \ge 1} \le ||e_1 + \kappa^M x||^2 \mathbf{1}_{||e_1 + \kappa^M x|| \ge 1}.$$

However, the derivative of the function $\kappa \mapsto ||e_1 + \kappa x||^2$ is equal to $\kappa \mapsto 2[x]_1 + \kappa \langle x, x \rangle$, hence is nonnegative for any $\kappa \ge 0$ that also satisfies that $||e_1 + \kappa x||^2 = 1 + \kappa (2[x]_1 + \kappa \langle x, x \rangle)$ is greater than or equal to 1. Thus, the above condition is satisfied.

For the positive part, we have then

$$n \ln^{+} \left\| e_{1} + \bar{\kappa}_{n} \left(e_{1}, \frac{\alpha}{n} \sum_{i=1}^{\mu} w_{i} U_{n}^{i} \right) \frac{\alpha}{n} \sum_{i=1}^{\mu} w_{i} U_{n}^{i} \right\|^{2}$$

$$\leq n \ln^{+} \left\| e_{1} + \frac{\kappa^{\mathcal{M}} \alpha}{n} \sum_{i=1}^{\mu} w_{i} U_{n}^{i} \right\|^{2}.$$

But it is proven in $[2]^1$ that the RHS of the above equation is uniformly integrable. All in all, we get that

$$\left(n\ln\left\|e_{1}+\bar{\kappa}_{n}\left(e_{1},\alpha/n\sum_{i=1}^{\mu}w_{i}U_{n}^{i}\right)\alpha/n\sum_{i=1}^{n}w_{i}U_{n}^{i}\right\|^{2}\right)_{n\in\mathbb{N}^{*}}$$
formly integrable.

is uniformly integrable.

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¹replace σ^* by $\alpha \kappa^M$ in the proof of Proposition 4 of [2]