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On the Nash number and the diminishing Grundy number of a graph

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Abstract

A *Nash k -colouring* is a k -colouring (S_1, \dots, S_k) such that every vertex of S_i is adjacent to a vertex in S_j , whenever $|S_j| \geq |S_i|$. The *Nash Number* of G , denoted by $\text{Nn}(G)$, is the largest k such that G admits a Nash k -colouring. A *diminishing greedy k -colouring* is a k -colouring (S_1, \dots, S_k) such that, for all $1 \leq j < i \leq k$, $|S_i| \leq |S_j|$ and every vertex of S_i is adjacent to a vertex in S_j . The *diminishing Grundy Number* of G , denoted by $\Gamma^\downarrow(G)$, is the largest k such that G admits a diminishing greedy k -colouring. In this paper, we prove some properties of Nn and Γ^\downarrow . We mainly study the relations between them and other graph parameters such as the clique number ω , the chromatic number χ , the Grundy number Γ , and the maximum degree Δ . In particular we study the chain of inequalities $\omega(G) \leq \chi(G) \leq \text{Nn}(G) \leq \Gamma^\downarrow(G) \leq \Gamma(G) \leq \Delta(G) + 1$. We show each inequality $\gamma_1(G) \leq \gamma_2(G)$ of this chain is loose, that is that there is no function f such that $\gamma_2(G) \leq f(\gamma_1(G))$. We also prove the existence or non-existence of Reed's like inequality who proved that there exists $\varepsilon > 0$ such that $\chi(G) \leq \varepsilon\omega(G) + (1 - \varepsilon)(\Delta(G) + 1)$.

We then study the Nash number and the diminishing Grundy number of trees and forests, and prove that $\Gamma(F) - 1 \leq \text{Nn}(F) \leq \Gamma^\downarrow(F) \leq \Gamma(F)$.

Finally we study the complexity of related problems. We show that computing the Nash number or the diminishing Grundy number is NP-hard even when the input graph is bipartite or chordal. We also show that deciding whether a graph satisfies $\gamma_1(G) = \gamma_2(G)$ is NP-hard for every pair (γ_1, γ_2) with $\gamma_1 \in \{\text{Nn}, \Gamma^\downarrow\}$ and $\gamma_2 \in \{\omega, \chi, \Gamma, \Delta + 1\}$.

1 Introduction

Graphs considered in this paper are undirected, finite and contain neither loops nor multiple edges (unless stated otherwise). The definitions and notations used in this paper are standard and may be found in any textbook on graph theory. See [3] for example. We use $n(G)$ to denote the number of vertices of a graph G , and $m(G)$ to denote its number of edges.

A **(proper) k -colouring** of a graph $G = (V, E)$ is a surjective mapping $\phi : V \rightarrow \{1, \dots, k\}$, such that for any edge $uv \in E(G)$, $\phi(u) \neq \phi(v)$. A **k -colouring** may also be seen as a partition of the vertex set of G into k disjoint non-empty **stable sets** (i.e. sets of pairwise non-adjacent vertices) $S_i = \{v \mid \phi(v) = i\}$ for $1 \leq i \leq k$. For convenience (and with a slight abuse of terminology), by k -colouring we mean both the mapping ϕ or the partition (S_1, \dots, S_k) . The elements of $\{1, \dots, k\}$ are called **colours**, and the S_i , $1 \leq i \leq k$, are the **colour classes**. A graph is **k -colourable** if it has a k -colouring. The **chromatic number** $\chi(G)$ is the least k such that G is k -colourable.

The problem of determining the chromatic number is one of Karp's twenty one NP-hard problems [10]. It is already NP-complete to decide whether a given planar 4-regular graph admits a 3-colouring [4]. Moreover, assuming $P \neq NP$, there is no algorithm that approximates the chromatic number within a factor of $n^{1-\varepsilon}$, for all $\varepsilon > 0$ [16].

To deal with this hard problem and its several applications, colouring heuristics have been designed. The most basic and most widespread one is the greedy algorithm. A **greedy colouring** relative to an ordering

$\sigma = v_1 < v_2 < \dots < v_n$ of $V(G)$ is obtained by colouring the vertices in the order σ , assigning to v_i the smallest positive integer not already used on its lower-indexed neighbours. Denoting by S_i the stable set of vertices coloured i , a greedy colouring has the following property:

(*)
For every $j < i$, every vertex in S_i has a neighbour in S_j .

Otherwise the vertex in S_i would have been coloured by an integer not greater than j . Conversely, a colouring satisfying Property (*) is a greedy colouring relative to any vertex ordering in which the vertices of S_i precede those of S_j when $i < j$. We call a vertex satisfying Property (*) a **greedy vertex**. The **Grundy number** $\Gamma(G)$ is the largest k such that G has a greedy k -colouring. It can be seen as a measure of the worst-case behaviour of the greedy algorithm.

Another colouring algorithm was proposed by Panagopoulou and Spirakis [13] based on a colouring game on a graph G . Each vertex is a player with action set corresponding to the set of colours $\{1, \dots, |V(G)|\}$. A configuration \mathbf{c} is the combination of the actions of each vertex. Given a configuration, a player's payoff is 0 if he selects the same colour as one of his neighbours, otherwise it is the number of vertices that selected the same colour. An elementary improvement occurs when a player unilaterally deviates to other colour and increases its payoff. A **pure Nash equilibrium** (PNE) is a configuration in which no vertex can do elementary improvements. It can be seen as a state of the game that is **sustainable**. Note that pure Nash equilibria are colourings of G , because a vertex having a neighbour with its colour may increase its payoff by choosing any unused colour.

Panagopoulou and Spirakis [13] studied the set $\text{PNE}(G)$ of pure Nash equilibria of G and proved that, starting with a proper colouring, any sequence of elementary improvements reaches a pure Nash equilibrium in $O(n \cdot \alpha(G))$ moves, where $\alpha(G)$ is the maximum size of a stable set in G . They also considered the social cost $\text{SC}(G, \sigma)$ associated with a graph G and a configuration $\sigma \in \text{PNE}(G)$, that is defined as the number of distinct colours selected by the players. They proved that:

$$\text{SC}(G, \sigma) \leq \min \left\{ \Delta_2(G) + 1, \frac{n(G) + \omega(G)}{2}, \frac{1 + \sqrt{1 + 8m(G)}}{2}, n(G) - \alpha(G) + 1 \right\}$$

where $\Delta_2(G) = \max_{v \in V} \max\{d_G(u) \mid u \in N_G(v) \text{ and } d_G(u) \leq d_G(v)\}$. They considered the approximation ratio $R(G) = \max_{\sigma \in \text{PNE}(G)} \frac{\text{SC}(G, \sigma)}{\chi(G)}$, and proved that there is a subclass of bipartite graphs satisfying $R(G) = \frac{n+2}{4}$. For the same subclass of bipartite graphs, they proved that by using a randomized sequence of elementary improvements one may attain a pure Nash equilibrium with two colours with high probability.

Escoffier et al. [6] continued their study and improved the bounds on $\text{SC}(G, \sigma)$. They also study the set of Pure Strong Equilibria, denoted $\text{PSE}(G)$. A **pure strong equilibrium** is a configuration where no unilateral deviation of a non-empty coalition of players is profitable to all of its members. They proved that a pure strong equilibrium always exists but it is NP-hard to compute one. They also provided bounds for $\text{SC}(G, \sigma)$ when $\sigma \in \text{PSE}(G)$.

1.1 The Nash Number and colouring heuristics

In this paper, we study Nash colourings, that correspond to pure Nash equilibria in Panagopoulou and Spirakis's game, and the Nash number, a parameter that measure the worst-case behaviour of their algorithm.

A **Nash k -colouring** is a k -colouring (S_1, \dots, S_k) such that for all $i \in [k]$, every vertex v in S_i has a neighbour in each S_j , $j \neq i$, such that $|S_j| \geq |S_i|$. Hence it corresponds to a pure Nash equilibrium σ such that $\text{SC}(G, \sigma) = k$. As observed in [13], we know that every graph admits a Nash colouring with $\chi(G)$ colours. The **Nash number** $\text{Nn}(G)$ is the largest k such that G has a Nash k -colouring. Hence $\text{Nn}(G) = \max\{\text{SC}(G, \sigma) \mid \sigma \in \text{PNE}(G)\}$. Therefore the above inequality of Panagopoulou and Spirakis implies

$$\text{Nn}(G) \leq \min \left\{ \Delta_2(G) + 1, \frac{n(G) + \omega(G)}{2}, \frac{1 + \sqrt{1 + 8m(G)}}{2}, n(G) - \alpha(G) + 1 \right\} \quad (1)$$

The starting point of our contributions is to link Nash colourings with the well-studied greedy colourings. To see this, let a colouring (S_1, \dots, S_k) be **diminishing** if $|S_i| \leq |S_{i-1}|$ for all $1 < i \leq k$.

Proposition 1. *Let G be a graph. If (S_1, \dots, S_k) is a Nash colouring of G , then there is a permutation σ over $[k]$ such that $(S_{\sigma(1)}, \dots, S_{\sigma(k)})$ is a diminishing greedy colouring.*

Proof. The result follows by considering a permutation σ satisfying $|S_{\sigma(i)}| \geq |S_{\sigma(j)}|$, for $i < j$. □

Henceforth, to each Nash k -colouring we can associate a **diminishing Nash k -colouring**. It follows directly from the definition that a diminishing Nash k -colouring is a diminishing greedy colouring. The **diminishing Grundy number** $\Gamma^\downarrow(G)$ is the largest k such that G has a diminishing greedy k -colouring. Thus, we have $\text{Nn}(G) \leq \Gamma^\downarrow(G) \leq \Gamma(G)$ for all graph G . Moreover, as we will show in Section 3, Panagopoulou and Spirakis's upper bound on $\text{Nn}(G)$ given at Equation (1) is in fact an upper bound on $\Gamma(G)$. So

$$\Gamma(G) \leq \min \left\{ \Delta_2(G) + 1, \frac{n(G) + \omega(G)}{2}, \frac{1 + \sqrt{1 + 8m(G)}}{2}, n(G) - \alpha(G) + 1 \right\} \quad (2)$$

We thus have the following chain of inequalities.

$$\omega(G) \leq \chi(G) \leq \text{Nn}(G) \leq \Gamma^\downarrow(G) \leq \Gamma(G) \leq \Delta_2(G) + 1 \leq \Delta(G) + 1 \leq n(G) \quad (3)$$

where $\omega(G)$ denotes the clique number and $\Delta(G)$ the maximum degree of G .

Note that all these inequalities are tight for complete graphs. Indeed $\Delta_2(K_n) + 1 = \frac{n(K_n) + \omega(K_n)}{2} = \frac{1 + \sqrt{1 + 8m(K_n)}}{2} = n(K_n) - \alpha(K_n) + 1 = n$.

1.2 Our results

Our results are organized as follows. In Section 2, we show that many useful properties of the chromatic and Grundy numbers are unfortunately not satisfied by the Nash number and the diminishing Grundy number. The chromatic number and the Grundy number are **monotone** (if H is an induced subgraph of G , then $\chi(H) \leq \chi(G)$ and $\gamma(H) \leq \gamma(G)$) and removing a vertex from a graph can only decrease each of those two parameters by 1. In contrast, the Nash number and the diminishing Grundy number are not monotone and removing a vertex from a graph may decrease or increase each of those numbers arbitrarily. See Propositions 11 and 12.

The chromatic number (resp. Grundy number) of a graph is well-known to be the maximum of the chromatic numbers (resp. Grundy numbers) of its connected components. We prove in Proposition 13 that the Nash number (resp. diminishing Grundy number) of a graph is greater or equal to the maximum of the diminishing Grundy numbers (resp. Nash numbers) of its components, but that it can also be arbitrarily larger (Proposition 14).

We also show that while the set of values k such that a graph G admits a greedy k -colouring is continuous (that is the set of all integers between $\chi(G)$ and $\Gamma(G)$) (Proposition 15), the same does not hold for Nash colouring nor diminishing greedy colouring (Proposition 16).

In Section 3, we first prove Equation (2). We then consider the chain of inequalities (3). In Subsection 3.1, we observe that each inequality $\gamma_1(G) \leq \gamma_2(G)$ of this chain is **loose**, that is that there is no function f such that $\gamma_2(G) \leq f(\gamma_1(G))$.

Reed [14] proved that there exists $\varepsilon > 0$ such that $\chi(G) \leq \varepsilon\omega(G) + (1 - \varepsilon)(\Delta(G) + 1)$ and conjecture that it holds for $\varepsilon = \frac{1}{2}$. The blow-up of a 5-cycle, i.e. the Cartesian product of a complete graph and a 5-cycle, demonstrates that it is not true for $\varepsilon > \frac{1}{2}$. King and Reed [11] showed that it holds for $\varepsilon \leq 1/130000$. It was improved by Bonamy et al. [2] to $\varepsilon \leq 1/26$, and subsequently further improved by Delcourt and Postle [5] to $1/13$. In Subsection 3.2, we show that, for some triples of parameters appearing in the chain of inequalities (3), an analogue of Reed's conjecture holds (Theorem 20), while for some other triples, there is no analogue of Reed's result and conjecture (Theorem 22). For triples of the form (ω, χ, γ) for $\gamma \in \{\text{Nn}, \Gamma^\downarrow, \Gamma, \Delta_2 + 1\}$, we leave the existence of such an analogue as an open problem.

Problem 2. For $\gamma \in \{\text{Nn}, \Gamma^\downarrow, \Gamma, \Delta_2 + 1\}$, does there exists $\varepsilon > 0$ such that $\chi(G) \leq \varepsilon\omega(G) + (1 - \varepsilon)\gamma(G)$?

In Section 4, we study the Nash number and the diminishing Grundy number of trees and forests. The chromatic number of such graphs is 2, and the Grundy number can be arbitrarily large but is well-understood via binomial trees. The **binomial trees** \mathcal{B}_k , $k \in \mathbb{N}$, are defined inductively as follows. \mathcal{B}_1 is the graph with one vertex and no edges; its root is its vertex. For every $k > 1$, \mathcal{B}_k is obtained from the disjoint union of two copies of \mathcal{B}_{k-1} by adding an edge between the roots of these copies; the root of \mathcal{B}_k is one of the roots of the copies. Note that $|V(\mathcal{B}_k)| = 2^{k-1}$. For a graph G , we denote by $\text{bin}(G)$ the largest k such that G has an induced subgraph isomorphic to the binomial tree \mathcal{B}_k . Hedetniemi, Hedetniemi, and Beyer [8] showed that $\Gamma(F) = \text{bin}(F)$ for every forest F , and used it to design a linear-time algorithm for computing the Grundy number of a forest. We first show in Corollary 27 that a forest F satisfies $\text{Nn}(F) = 2$ if and only if $\Gamma^\downarrow(F) = 2$ if and only if $\Gamma(F) = 2$.

Unfortunately such an equivalence does not extend for larger value of k . For any $k \geq 4$, we show a tree T_k such that $\text{Nn}(T_k) = \Gamma^\downarrow(T_k) = k - 1$ and $\Gamma(T_k) = k$, a tree T'_k such that $\text{Nn}(T'_k) = k - 1$ and $\Gamma^\downarrow(T'_k) = \Gamma(T'_k) = k$. Nevertheless, we prove in Corollary 31 that the Nash number and the diminishing Grundy number of a forest are very close to its Grundy number:

$$\Gamma(F) - 1 \leq \text{Nn}(F) \leq \Gamma^\downarrow(F) \leq \Gamma(F)$$

However, the complexity of determining the Nash number or the the diminishing Grundy number of a tree (and more generally a forest) is still unknown.

Problem 3. Can we compute the Nash number (resp. the diminishing Grundy number) of a tree in polynomial time ?

This is equivalent to solving items (i) and (ii) of the following problem.

Problem 4. Can we decide in polynomial time whether a given tree T satisfies

(i) $\text{Nn}(T) = \Gamma(T)$?

(ii) $\Gamma^\downarrow(T) = \Gamma(T)$?

(iii) $\text{Nn}(T) = \Gamma^\downarrow(T)$?

The same questions may be asked for graphs in general. What is the complexity of deciding whether a given graph G satisfies $\text{Nn}(G) = \Gamma(G)$ (resp. $\Gamma^\downarrow(G) = \Gamma(G)$, $\text{Nn}(G) = \Gamma^\downarrow(G)$) ? In Section 5, we answer the two first questions by showing that those two problems are NP-complete even when restricted to bipartite graphs (Theorem 42) or chordal graphs (Theorem 35). A graph is **chordal** if every cycle of length at least 4 has a chord, and a graph is bipartite and chordal if and only if it is a forest. However the third question remains open for general graphs.

Problem 5. What is the complexity of deciding whether a given graph G satisfies $\text{Nn}(G) = \Gamma^\downarrow(G)$?

Zaker [15] showed that for any fixed $k \geq 0$, given a graph G it is co-NP-Complete to decide whether $\Gamma(G) \leq \chi(G) + k$. He also showed that, given a graph G which is the complement of bipartite graph, it is co-NP-Complete to decide if $\Gamma(G) = \chi(G)$. This implies that it is NP-hard to decide if $\Gamma(G) = \omega(G)$. Indeed, if G is the complement of a bipartite graph, then it is perfect, so $\chi(G) = \omega(G)$. We generalize Zaker's result to Nn and Γ^\downarrow . We show in Theorem 43 that for any $\gamma_1 \in \{\omega, \chi\}$ and $\gamma_2 \in \{\text{Nn}, \Gamma^\downarrow, \Gamma\}$, deciding whether a given graph G satisfies $\gamma_2(G) \leq \gamma_1(G) + k$ is NP-hard for all fixed k .

In Section 5, we also show that, for any fixed non-negative integer k , parameter $\gamma_1 \in \{\text{Nn}, \Gamma^\downarrow, \Gamma\}$ and parameter $\gamma_2 \in \{\Delta_2 + 1, \Delta + 1\}$, it is NP-complete to decide whether $\gamma_1(G) \geq \gamma_2(G) - k$ for a given graph G . NP-complete when restricted to chordal graphs (Theorem 34) or bipartite graphs (Theorem 40).

This implies that given a graph G and an integer, it is NP-complete to decide whether $\text{Nn}(G) \geq k$ (resp. $\Gamma^\downarrow(G) \geq k$). However, the complexity for any fixed k remains open even for chordal or bipartite graphs.

Problem 6. For any fixed integer k , what is the complexity of deciding whether a graph G has Nash number (resp. diminishing Grundy number) at least k ?

For any fixed k , it is trivial to decide whether $\omega(G) \geq k$, it is co-NP-complete to decide whether $\chi(G) \geq k$ for all $k \geq 3$, and Zaker [15] proved that, for any fixed k , one can decide in polynomial time whether $\Gamma(G) \geq k$.

Since computing the Nash number and the diminishing Grundy number is NP-hard, it is natural to ask for which classes of graphs it is polynomial-time solvable. Problem 3 asks whether it is the case for trees. We show in Corollary 38 that it is the case for split graphs, which is a subclass of chordal graphs. A **split graph** is a graph G such that its vertex set can be partitioned $V(G) = (X, Y)$ such that X is a clique and Y a stable set.

2 Properties of Nn and Γ^\downarrow .

Let $\phi = (S_1, \dots, S_k)$ be a k -colouring. If there exist S_i and S_j and a vertex $x \in S_j$ such that $|S_i| \geq |S_j|$ and $S_i \cup \{x\}$ is stable, then ϕ is not a Nash colouring. Moreover, the colouring $\phi' = (S'_1, \dots, S'_k)$ where $S'_i = S_i \cup \{x\}$, $S'_j = S'_j \setminus \{x\}$, and $S'_\ell = S_\ell$ is the **elementary improvement** obtained by **moving** x from S_j to S_i . A colouring ψ is an **improvement** of the colouring ϕ if it can be obtained from ϕ by a sequence of elementary improvements.

Lemma 7. Let G be a graph, $\phi = (S_1, \dots, S_k)$ a greedy colouring of G , and I a stable set in G . If $S_i \subseteq I$ and $S_j \subseteq I$, then $i = j$.

Proof. Assume for a contradiction that for some $i \neq j$, $S_i \subseteq I$ and $S_j \subseteq I$. Without loss of generality, $|S_i| \geq |S_j|$. But now no vertex of S_j has a neighbour in S_i , contradicting the fact that ϕ is a greedy colouring. \square

Two vertices u and v are **twins** in a graph if $N(u) = N(v)$. Note that it necessarily implies that u and v are not adjacent.

Lemma 8. Let G be a graph and ϕ a greedy colouring of G . If u and v are twins, then $\phi(u) = \phi(v)$.

Proof. Let $\phi = (S_1, \dots, S_k)$. Let $i = \phi(u)$ and $j = \phi(v)$. Without loss of generality, we have $i \leq j$. But now since S_i is a stable set and u and v are twins, v is adjacent to no vertex in S_i . Hence $i = j$ for otherwise ϕ would not be a greedy colouring. \square

We shall need the following propositions, which are easy and well-known. (E.g. see [1] for a proof of Proposition 9 (b).)

Proposition 9. If H is an induced subgraph of G , then

- (a) $\chi(H) \leq \chi(G)$, and
- (b) $\Gamma(H) \leq \Gamma(G)$.

Proposition 10. Let G be a graph and v a vertex of G , then

- (a) $\chi(G - v) \leq \chi(G) \leq \chi(G - v) + 1$, and
- (b) $\Gamma(G - v) \leq \Gamma(G) \leq \Gamma(G - v) + 1$.

We shall now prove that there is no analogue of Propositions 9 and 10 for the Nash number or the diminishing Grundy number. In order to do so, for $k \geq 0$, let M_k be graph obtained from the bipartite complete graph with bipartition (A, B) where $A = \{u_1, \dots, u_k\}$ and $B = \{v_1, \dots, v_k\}$ by deleting the edges $u_i v_i$ for all $i \in \{1, \dots, k\}$. Let G_k be the graph obtained from M_{k+1} by deleting the vertices u_{k+1} and u_k , and let H_k be the graph obtained from M_k by deleting the vertex u_k .

Proposition 11. Let $k \geq 2$ be an integer.

- (i) $\text{Nn}(G_k - v_{k+1}) \geq k$ and $\text{Nn}(G_k) = 2$.
- (ii) $\text{Nn}(H_k - u_{k-1}) = 2$ and $\text{Nn}(H_k) \geq k$.

Proof. Observe that $G_k - v_{k+1} = H_k$, and $H_k - u_{k-1} = G_{k-1}$. Hence to prove (i) and (ii) it suffices to prove that $\text{Nn}(G_k) = 2$ and $\text{Nn}(H_k) \geq k$.

Let us first prove that $\text{Nn}(G_k) = 2$. Let (S_1, \dots, S_p) be a Nash colouring of G_k . First note that v_k and v_{k+1} are twins, so by Lemma 8, they are in the same colour class, say S_α .

Suppose for a contradiction that for some $\ell \in [k-1]$, $v_\ell \in S_\beta$ with $\beta \neq \alpha$.

If $S_\beta \cap A \neq \emptyset$, then $S_\beta = \{u_\ell, v_\ell\}$, because $N(v_\ell) = A \setminus \{u_\ell\}$ and $N(u_\ell) = B \setminus \{v_\ell\}$. But then, moving v_ℓ to S_α would be an elementary improvement, a contradiction. Henceforth we can suppose $S_\beta \subseteq B$. But $S_\alpha \subseteq B$, because $N(v_{k+1}) = A$. This contradicts Lemma 7. Therefore $S_\alpha = B$.

Now the colour classes distinct from S_β are all included in A . Since A is a stable set, by Lemma 7, there is only one such colour class, so $p = 2$.

Let us now prove $\text{Nn}(H_k) \geq k$. Set $S_i = \{v_i, u_i\}$ for all $i \in [k-1]$ and set $S_k = \{v_k\}$. It is easy to verify that (S_1, \dots, S_k) is a Nash colouring of H_k . \square

Let G'_k be the graph obtained from M_{k+2} by deleting the vertices u_{k+2} , u_{k+1} and u_k . Let H'_k be the graph obtained from G'_{k-1} by adding a vertex x adjacent to all vertices of G'_{k-1} except v_{k+1} .

Proposition 12. Let $k \geq 2$ be an integer.

- (i) $\Gamma^\downarrow(G'_k) = 2$ and $\Gamma^\downarrow(G - v_{k+2}) \geq k$.

(ii) $\Gamma^\downarrow(H'_k - x) = 2$ and $\Gamma^\downarrow(H'_k) \geq k$.

Proof. (i) Let $U = \{u_i \mid i \in [k-1]\}$, $V = \{v_i \mid i \in [k-1]\}$, and $W = \{v_k, v_{k+1}, v_{k+2}\}$.

Observe that $(\{u_1, v_1\}, \dots, \{u_{k-1}, v_{k-1}\}, \{v_k, v_{k+1}\})$ is a diminishing greedy colouring of $G'_k - v_{k+2}$, so $\Gamma^\downarrow(G'_k - v_{k+2}) \geq k$.

Consider now a diminishing greedy colouring (S_1, \dots, S_q) of G'_k . Since the vertices of W are twins, by Lemma 8, there exists p such that $W \subset S_p$.

Assume for a contradiction that $p > 1$. Let $i \in [p-1]$. Any element of W has a neighbour in S_i , which is in U . So $U \cap S_i \neq \emptyset$. Thus $|V \cap S_i| \leq 1$ since every vertex of U is adjacent to all vertices of V but one. Since the colouring is diminishing, $|S_i| \geq 3$, so $|S_i \cap U| \geq 2$. Consequently, $S_i \cap V = \emptyset$, so $S_i \subseteq U$. Thus, by Lemma 7, $p-1 = 1$ and $S_1 \subseteq U$. Hence $S_1 = U$. It follows that $S_2 = V \cup W$, and $|S_2| > |S_1|$, a contradiction.

Thus $p = 1$. Hence $V \subseteq S_1$, since $V \cup W$ is a stable set and U is complete to W . Therefore $U = S_2$, and $q = 2$. Hence $\Gamma^\downarrow(G'_k) = 2$. This proves (i).

$(\{x, v_{k+1}\}, \{u_1, v_1\}, \dots, \{u_{k-2}, v_{k-2}\}, \{v_{k-1}, v_k\})$ is a diminishing greedy colouring of H'_k , so $\Gamma^\downarrow(H'_k) \geq k$. And $H'_k - x = G'_{k-1}$, so $\Gamma^\downarrow(H'_k - x) = 2$. This proves (ii). \square

It is well-known that the chromatic number (resp. Grundy number) of a graph is the maximum of the chromatic numbers (resp. Grundy numbers) of its connected components. We shall now prove that the diminishing Grundy number (resp. Nash number) of a graph is greater or equal to the maximum of the diminishing Grundy numbers (resp. Nash numbers) of its components (Proposition 13), but, in contrast to the chromatic and Grundy numbers, the difference between those two numbers may be arbitrarily large (Proposition 14).

Proposition 13. *Let G be a graph.*

(i) $\Gamma^\downarrow(G) \geq \max\{\Gamma^\downarrow(C) \mid C \text{ component of } G\}$.

(ii) $\text{Nn}(G) \geq \max\{\text{Nn}(C) \mid C \text{ component of } G\}$.

Proof. (i) Set $k = \max\{\Gamma^\downarrow(C) \mid C \text{ component of } G\}$, and let C_1, \dots, C_p be the components of G . For every $i \in [p]$ of G , let $k_i = \Gamma^\downarrow(C_i)$, let $(S_1^i, \dots, S_{k_i}^i)$ be a diminishing greedy colouring of C_i , and set $S_j^i = \emptyset$ for all $k_i < j \leq k$.

For every $j \in [k]$, let $S_j = \bigcup_{i \in [p]} S_j^i$. One easily checks that (S_1, \dots, S_k) is a diminishing greedy colouring of G .

(ii) Let $k = \max\{\text{Nn}(C) \mid C \text{ component of } G\}$. Let C_1, \dots, C_p be the components of G . Without loss of generality, suppose $\text{Nn}(C_1) = k$. Let (S_1^1, \dots, S_k^1) be a diminishing Nash colouring of C_1 and for each $i \in \{2, \dots, p\}$ let $(S_1^i, \dots, S_{\chi(C_i)}^i)$ be a diminishing Nash colouring of C_i and set $S_j^i = \emptyset$ for all $j > \chi(C_i)$.

For each $j \in \{1, \dots, k\}$ set $S_j = \bigcup_{i=1}^p S_j^i$. We claim that (S_1, \dots, S_k) is a Nash colouring. First of all, note that for every $j \in \{1, \dots, k\}$ and $i \in \{1, \dots, p\}$ we have $|S_j^i| \geq |S_{j+1}^i|$. Therefore for each $j \in \{1, \dots, k-1\}$ the following inequality holds:

$$|S_j| = \sum_{i=1}^p |S_j^i| \geq \sum_{i=1}^p |S_{j+1}^i| = |S_{j+1}| \quad (4)$$

with equality occurring if and only if $|S_j^i| = |S_{j+1}^i|$ for every $i \in \{1, \dots, p\}$.

Suppose for a contradiction that moving some vertex w from S_j to S_l is an elementary improvement. Then we have $|S_l| \geq |S_j|$ and w has no neighbour in S_l . Let C_a the component containing w . Since $(S_1^a, \dots, S_{\chi(C_a)}^a)$ is a greedy colouring, w has a neighbour in S_l^a and so in S_l for all $l < j$. Therefore $l > j$. But then by Inequality (4), $|S_j^a| = |S_l^a|$. Thus, since $(S_1^a, \dots, S_{\chi(C_a)}^a)$ is a Nash colouring, w has a neighbour in S_l^a and so in S_l , a contradiction. \square

In contrast to the Grundy number of a graph, which is the maximum Grundy number of its connected components, the Nash number (resp. diminishing Grundy number) of a graph cannot be bounded by a function of the maximum Nash number (resp. diminishing Grundy number) of its connected components.

Proposition 14. *Let k be a positive integer. There exists a graph G such that $\max\{\Gamma^\downarrow(C) \mid C \text{ component of } G\} = 2$ and $\text{Nn}(G) = k$.*

Proof. For any $r \in [k]$, let A_r be the graph constructed from the disjoint union of the binomial tree \mathcal{B}_r with bipartition (X_r, Y_r) and a stable set U_r of 2^k new vertices by adding all edges between U_r and Y_r ; formally, $V(A_r) = V(\mathcal{B}_r) \cup U_r$ and $E(A_r) = E(\mathcal{B}_r) \cup \{uv \mid u \in Y_r \text{ and } v \in U_r\}$.

Claim 14.1. $\text{Nn}(A_r) = \Gamma^\downarrow(A_r) = 2$.

Proof of claim. Since A_r has an edge, we have $\Gamma^\downarrow(A_r) \geq \text{Nn}(A_r) \geq 2$. We shall now prove that $\Gamma^\downarrow(A_r) = 2$.

Consider a diminishing greedy colouring $\phi = (S_1, \dots, S_p)$ of A_r . Let j be the smallest index such that $S_j \cap U_r \neq \emptyset$. Observe that $S_j \subseteq U_r \cup X_r$. Therefore, for every $i > j$, no vertex of U_r is in S_i for otherwise it will not satisfy the greedy property. Thus $U_r \subseteq S_j$. Now $|S_j| > |V(A_r - S_j)|$, so $|S_j| > |S_i|$ for all $i \neq j$. Thus $j = 1$ because ϕ is diminishing. Now, for every $i > j$, no vertex of X_r is in S_i for otherwise it will not satisfy the greedy property. Hence $S_1 = U_r \cup X_r$. But then $V(A_r) \setminus S_1 = Y_r$ is a stable set, so $\phi = (U_r \cup X_r, Y_r)$ and $\Gamma^\downarrow(G) = 2$. \diamond

Let G be the disjoint union of the A_r for $r \in [k]$. By Claim 14.1, $\max\{\Gamma^\downarrow(C) \mid C \text{ is component of } G\} = 2$.

For $1 \leq r \leq k$ take a greedy r -colouring $(S_1^r, \dots, S_{r'}^r)$ of \mathcal{B}_r such that the unique vertex of $S_{r'}^r$ is in X_r . Note that for $1 \leq i \leq r-1$, $|S_i^r| = 2^{r-i-1}$. Clearly, $\phi_r = (S_1^r, \dots, S_{r'}^r \cup U_r)$ is a greedy r -colouring of A_r . For $i = 1$ to k , let $S_i = U_i \cup \bigcup_{r \geq i} S_i^r$. By construction, we have $|S_1| > |S_2| > \dots > |S_k|$. Moreover, since each ϕ_r was a greedy colouring, for all $1 \leq i < j \leq k$, every vertex of S_j has a neighbour in S_i . Therefore (S_1, \dots, S_k) is a Nash k -colouring of G . \square

Let $\text{Nash}(G)$ (resp. $\text{Dimin}(G)$, $\text{Greedy}(G)$) be the set of integers k such that G admits a Nash k -colouring (resp. diminishing greedy k -colouring, greedy k -colouring). Since every Nash k -colouring corresponds to a diminishing greedy k -colouring, we have $\text{Nash}(G) \subseteq \text{Dimin}(G) \subseteq \text{Greedy}(G)$. A graph G is **Nash-continuous** if $\text{Nash}(G) = \{\chi(G), \dots, \text{Nn}(G)\}$; it is **diminishing-continuous** if $\text{Dimin}(G) = \{\chi(G), \dots, \Gamma^\downarrow(G)\}$; it is **greedy-continuous** if $\text{Greedy}(G) = \{\chi(G), \dots, \Gamma(G)\}$.

Proposition 15. *Every graph is greedy-continuous.*

Proof. We proceed by induction on the number of vertices, the result holding trivially for edgeless graphs.

Let G be a graph which has at least one edge. Then $\chi(G) > 1$. Let $\chi(G) \leq k \leq \Gamma(G)$. We shall prove that G has a greedy k -colouring.

If $k = \chi(G)$, then there is a greedy k -colouring of G . Henceforth we assume that $k \geq \chi(G) + 1$.

Let (S_1, \dots, S_p) be a greedy colouring of G with $p = \Gamma(G)$. Then S_1 is a dominating stable set in G . Let us consider $G' = G - S_1$. Then $\Gamma(G') = \Gamma(G) - 1$ and $\chi(G') \leq \chi(G)$. So $\chi(G') \leq k - 1 \leq \Gamma(G')$. By the induction hypothesis, G' admits a greedy $(k-1)$ -colouring (S_2', \dots, S_k') . Thus (S_1, S_2', \dots, S_k') is a greedy k -colouring of G . \square

Since every graph has a Nash colouring with $\chi(G)$ colours, $\chi(G) \in \text{Nash}(G) \subseteq \text{Dimin}(G)$. We now prove that some graphs are not Nash-continuous (resp. diminishing continuous) and that $\text{Nash}(G)$ (resp. $\text{Dimin}(G)$) may only contain two far apart values, namely $\chi(G)$ and $\text{Nn}(G)$ (resp. $\Gamma^\downarrow(G)$). This is in particular the case for M_k . (Recall that M_k is obtained from the complete bipartite graph $K_{k,k}$ by removing a perfect matching.)

Proposition 16. *For every integer $k \geq 2$, $\text{Nash}(M_k) = \text{Dimin}(M_k) = \{2, k\}$.*

Proof. The bipartition (A, B) of M_k clearly is a Nash 2-colouring of M_k . Thus $2 \in \text{Nash}(M_k)$. Now it is easy to verify that $(\{u_1, v_1\}, \dots, \{u_k, v_k\})$ is a Nash k -colouring. Hence $\{2, k\} \subseteq \text{Nash}(M_k) \subseteq \text{Dimin}(M_k)$.

Assume that (S_1, \dots, S_q) is a diminishing greedy colouring of M_k with $q < k$. We have $|S_1| \geq |M_k|/q = 2k/q > 2$. So $|S_1| \geq 3$. Without loss of generality, we may assume that $|S_1 \cap A| \geq 2$. Since every vertex of B is adjacent to all vertices of A but one, we have $S_1 \cap B = \emptyset$. Hence, since the colouring is greedy and A is a stable set, $S_1 = A$. Therefore $S_2 = b$ and $q = 2$.

Consequently, $\text{Nash}(M_k) = \text{Dimin}(M_k) = \{2, k\}$. \square

Problem 17. What classes of graphs are Nash-continuous? Can we characterize the Nash-continuous graphs?

What classes of graphs are diminishing-continuous? Can we characterize the diminishing-continuous graphs?

As we shall see in Section 4, trees are an example of Nash-continuous and diminishing-continuous graphs.

We end this section by showing that the family of graphs with Nash number (resp. diminishing Grundy number) at most k cannot be characterized by forbidden induced subgraphs.

Proposition 18. *For every graph G , there is a graph A_G containing G as an induced subgraph such that $\Gamma^\downarrow(A_G) = \text{Nn}(A_G) = \chi(A_G) = \chi(G)$.*

Proof. Let G be a graph and (S_1, \dots, S_p) be a colouring of G with $p = \chi(G)$ colours.

Let A_G be the graph obtained from the disjoint union of G and a complete p -partite graph with parts V_1, \dots, V_p with $|V_i| = \left(\sum_{j=0}^{i-1} |V_j|\right) + 1$ where $V_0 = V(G)$, by adding all edges between V_i and $V(G) \setminus S_i$ for all $i \in [p]$.

Assume now that $\phi = (T_1, \dots, T_q)$ is a diminishing greedy colouring of A_G . By Lemma 8, V_p is contained in one of the T_j . By construction, $|V_p| > |V(A_G) \setminus T_p|$, so necessarily $V_p \subseteq T_1$. Since every vertex of $V(G) \setminus S_p$ is adjacent to all vertices of V_p , $T_1 \subseteq V_p \cup S_p$. Since $V_p \cup S_p$ is a stable set and ϕ is a greedy colouring, necessarily $T_1 = V_p \cup S_p$. Using the same reasoning, one shows by induction on i that $T_i = V_{p+1-i} \cup S_{p+1-i}$ for all $i \in [p]$. Hence $q = p$. So $\Gamma^\downarrow(A_G) \leq p$.

Now $p = \chi(G) \leq \chi(A_G) \leq \text{Nn}(A_G) \leq \Gamma^\downarrow(A_G) \leq p$. □

3 Relations between the parameters

We first show that the upper bound of Panagopoulou and Spirakis on the Nash number is in fact an upper bound on the Grundy number.

Theorem 19. *Let G be a graph.*

$$\Gamma(G) \leq \min \left\{ \Delta_2(G) + 1, \frac{n(G) + \omega(G)}{2}, \frac{1 + \sqrt{1 + 8m(G)}}{2}, n(G) - \alpha(G) + 1 \right\}.$$

Proof. Set $p = \Gamma(G)$ and let (S_1, \dots, S_p) be a greedy p -colouring of G .

Running the greedy algorithm on any ordering of $V(G)$, one easily checks that the colour assigned to every vertex v is at most $\Delta_2(G) + 1$.

Let $I = \{i \in [p] \mid |S_i| = 1\}$ and $J = [p] \setminus I$. For every $i \in I$, let x_i be the unique vertex in S_i . Note that the x_i , $i \in I$, form a clique in G . Now $n(G) = \sum_{i \in [p]} |S_i| \geq 2|J| + |I|$ and so $\Gamma(G) = |I| + |J| \leq |I| + \frac{n(G) - |I|}{2} = \frac{n(G) + |I|}{2} \leq \frac{n(G) + \omega(G)}{2}$.

For $i \in [p]$, let x_i be a vertex in S_i . By (\star) , x_i has a neighbour in each S_j for $1 \leq j < i$. Hence $m(G) \geq \sum_{i=1}^p (i-1) = \frac{1}{2}p(p-1)$. Hence $\Gamma(G) = p \leq \frac{1 + \sqrt{1 + 8m(G)}}{2}$.

Let T be a stable set of cardinality $\alpha(G)$. Let $I_T = \{i \in [p] \mid S_i \cap T \neq \emptyset\}$, and let $i_T = \max I_T$ and x_T be a vertex of S_{i_T} . Observe that for each $i \in I_T \setminus \{i_T\}$, x_T is adjacent to a vertex of S_i which is not in T . Furthermore, by definition, any vertex of S_j for $j \notin I_T$ is not in T . Consequently, for all $i \neq i_T$, there is a vertex in $S_i \setminus T$. Thus $n(G) \geq |T| + p - 1$, so $\Gamma(G) = p \geq n(G) - \alpha(G) + 1$. □

3.1 On the chain of inequalities (3)

In this subsection, we consider the chain of inequalities (3). As observed in the introduction, all these inequalities are tight for complete graphs. We shall now prove that they are loose. Recall that an inequality $\gamma_1(G) \leq \gamma_2(G)$ is **loose** if there is no function f such that $\gamma_2(G) \leq f(\gamma_1(G))$.

The inequality $\omega(G) \leq \chi(G)$ is loose, because there are many construction of graphs G with $\omega(G) = 2$ (triangle-free graphs) and $\chi(G) = k$ for all positive integer k , as the one due to Mycielski [12].

The inequality $\chi(G) \leq \text{Nn}(G)$ is also loose. Indeed by Proposition 16, $\chi(M_k) = 2$ and $\text{Nn}(M_k) = k$ for all positive integer k .

The inequality $\text{Nn}(G) \leq \Gamma^\downarrow(G)$ is also loose. Indeed, $\text{Nn}(G_k) = 2$ by Proposition 11, and $\Gamma^\downarrow(G_k) \geq k$ because $(\{u_1, v_1\}, \dots, \{u_{k-1}, v_{k-1}\}, \{v_k, v_{k+1}\})$ is a diminishing greedy k -colouring of G_k .

The inequality $\Gamma^\downarrow(G) \leq \Gamma(G)$ is also loose. Indeed, by Proposition 12 and 10, $\Gamma^\downarrow(G'_k) = 2$ and $\Gamma(G'_k) \geq \Gamma(G'_k - v_{k+1}) \geq \Gamma^\downarrow(G'_k - v_{k+1}) \geq k$.

The inequality $\Gamma(G) \leq \Delta_2(G) + 1$ is also loose. Consider for example the complete bipartite graph $K_{k,k}$. We have $\Delta_2(K_{k,k}) = k$ and $\Gamma(K_{k,k}) = 2$.

3.2 Reed's like inequalities

Reed [14] proved that there exists ε such that $\chi(G) \leq \varepsilon\omega(G) + (1 - \varepsilon)(\Delta(G) + 1)$ and conjecture that it holds for $\varepsilon = \frac{1}{2}$. A natural question is to investigate analogous bounds for the parameters in Equation (3).

An **increasing triple** is a triple $\tau = (\gamma_1, \gamma_2, \gamma_3)$ of distinct parameters in $\{\omega, \chi, \text{Nn}, \Gamma^\downarrow, \Gamma, \Delta_2 + 1, \Delta + 1, n\}$ such that $\gamma_1(G) \leq \gamma_2(G) \leq \gamma_3(G)$ for every graph G .

Let $\tau = (\gamma_1, \gamma_2, \gamma_3)$ be an increasing triple. It is ε -**convex** if $\gamma_2(G) \leq \varepsilon\gamma_1(G) + (1 - \varepsilon)\gamma_3(G)$ for all graph G . By definition, the increasing triple τ is 0-convex. It is **convex** if there exists $\varepsilon > 0$ such that it is ε -convex. The **convexity** of τ is $\text{conv}(\tau) = \sup\{\varepsilon \mid \tau \text{ is } \varepsilon\text{-convex}\}$. Note that τ is convex if and only if its convexity is greater than 0. Reed's theorem states that $(\omega, \chi, \Delta + 1)$ is convex and Reed's conjecture states that $\text{conv}(\omega, \chi, \Delta + 1) = \frac{1}{2}$.

It is then natural to ask which increasing triples are convex, and to determine their convexity.

Theorem 20. (i) $\text{conv}(\omega, \Gamma, n) = \text{conv}(\chi, \Gamma, n) = \text{conv}(\text{Nn}, \Gamma, n) = \text{conv}(\Gamma^\downarrow, \Gamma, n) = \frac{1}{2}$.

(ii) $\frac{1}{2} = \text{conv}(\omega, \Gamma, n) = \text{conv}(\omega, \Gamma^\downarrow, n) \leq \text{conv}(\omega, \text{Nn}, n) = \frac{1}{2}$.

Proof. Equation 2 (Theorem 19) shows that (ω, Γ, n) is $\frac{1}{2}$ -convex. Therefore

$$\frac{1}{2} \leq \text{conv}(\omega, \Gamma, n) \leq \text{conv}(\chi, \Gamma, n) \leq \text{conv}(\text{Nn}, \Gamma, n) \leq \text{conv}(\Gamma^\downarrow, \Gamma, n) \quad (5)$$

and

$$\frac{1}{2} \leq \text{conv}(\omega, \Gamma, n) \leq \text{conv}(\omega, \Gamma^\downarrow, n) \leq \text{conv}(\omega, \text{Nn}, n) \quad (6)$$

Now $n(G'_k) = 2k + 1$, and, by Proposition 12 and 10, $\Gamma^\downarrow(G'_k) = 2$ and $\Gamma(G'_k) \geq \Gamma(G'_k - v_{k+1}) \geq \Gamma^\downarrow(G'_k - v_{k+1}) \geq k$. Hence, for any $\varepsilon > 0$, taking $k > \frac{3}{4\varepsilon} + \frac{1}{2}$, we have

$$\Gamma(G'_k) = k > \left(\frac{1}{2} + \varepsilon\right)2 + \left(\frac{1}{2} - \varepsilon\right)(2k + 1) = \left(\frac{1}{2} + \varepsilon\right)\Gamma^\downarrow(G'_k) + \left(\frac{1}{2} - \varepsilon\right)n(G'_k).$$

Hence $\text{conv}(\Gamma^\downarrow, \Gamma, n) \leq \frac{1}{2}$. With Equation (5), this yields (i).

$n(M_k) = 2k$, $\omega(M_k) = 2$, and, by Proposition 16, $\text{Nn}(M_k) = k$. Hence, for any $\varepsilon > 0$, taking $k > \frac{1}{2\varepsilon} + 1$, we have

$$\text{Nn}(M_k) = k > \left(\frac{1}{2} + \varepsilon\right)2 + \left(\frac{1}{2} - \varepsilon\right)(2k) = \left(\frac{1}{2} + \varepsilon\right)\omega(M_k) + \left(\frac{1}{2} - \varepsilon\right)n(M_k).$$

Hence $\text{conv}(\omega, \text{Nn}, n) \leq \frac{1}{2}$. With Equation (6), this yields (ii). \square

Problem 21. What is $\text{conv}(\omega, \chi, n)$?

$\text{conv}(\omega, \chi, n) \geq \text{conv}(\omega, \Gamma, n) \geq \frac{1}{2}$. Because of the 5-cycle, $\text{conv}(\omega, \chi, n) \leq \frac{2}{3}$. Is this $2/3$ the right value ?

In the following theorem, we show that many increasing triples are not convex.

Theorem 22. Let $\tau = (\gamma_1, \gamma_2, \gamma_3)$ be an increasing triple.

- (i) If $\gamma_1 \in \{\omega, \chi\}$ and $\gamma_2, \gamma_3 \in \{\text{Nn}, \Gamma^\downarrow, \Gamma, \Delta_2 + 1, \Delta + 1\}$, then τ is not convex.
- (ii) If $\gamma_1 \in \{\omega, \chi, \text{Nn}\}$, and $\gamma_2, \gamma_3 \in \{\Gamma^\downarrow, \Gamma, \Delta_2 + 1, \Delta + 1\}$, then τ is not convex.
- (iii) If $\gamma_1 = \{\omega, \chi, \text{Nn}, \Gamma^\downarrow\}$, $\gamma_2, \gamma_3 \in \{\Gamma, \Delta_2 + 1, \Delta + 1\}$, then τ is not convex.
- (iv) If $\gamma_1 \in \{\omega, \chi, \text{Nn}, \Gamma^\downarrow, \Gamma\}$, $\gamma_2 = \Delta_2 + 1$, and $\gamma_3 = \Delta + 1$, then τ is not convex.

Proof. (i) Consider the graph M_p for $p \geq 3$. We have $\omega(M_p) = \chi(M_p) = 2$ and $\text{Nn}(M_p) = \Gamma^\downarrow(M_p) = \Gamma(M_p) = \Delta_2(M_p) + 1 = \Delta(M_p) + 1 = p$. Thus if $\gamma_1 \in \{\omega, \chi\}$ and $\gamma_2, \gamma_3 \in \{\text{Nn}, \Gamma^\downarrow, \Gamma, \Delta_2 + 1, \Delta + 1\}$, we have $\gamma_2(M_p) = p > p - (p-2)\varepsilon = \varepsilon\gamma_1(M_p) + (1-\varepsilon)\gamma_3(M_p)$ for all $\varepsilon > 0$. Hence $\text{conv}(\gamma_1, \gamma_2, \gamma_3) = 0$.

(ii) Consider the graph G_k . By Proposition 11, $\omega(G_k) = \chi(G_k) = \text{Nn}(G_k) = 2$. As observed above, $\Gamma^\downarrow(G_k) \geq k$, and one easily checks that $\Delta(G_k) = k$. Thus if $\gamma_1 \in \{\omega, \chi, \text{Nn}\}$, $\gamma_2, \gamma_3 \in \{\Gamma^\downarrow, \Gamma, \Delta_2 + 1, \Delta + 1\}$, we have $\gamma_2(G_k) \geq k > 2\varepsilon + (1-\varepsilon)(k+1) \geq \varepsilon\gamma_1(G_k) + (1-\varepsilon)\gamma_3(G_k)$ for all $k > \varepsilon^{-1} + 1$. Hence $\text{conv}(\gamma_1, \gamma_2, \gamma_3) = 0$.

(iii) Consider the graph G'_k . As observed above, by Proposition 12 and 10, $\omega(G'_k) = \chi(G'_k) = \text{Nn}(G'_k) = \Gamma^\downarrow(G'_k) = 2$ and $\Gamma(G'_k) \geq k$. Moreover, $\Delta(G'_k) = k + 1$.

Thus if $\gamma_1 \in \{\omega, \chi, \text{Nn}, \Gamma^\downarrow\}$, $\gamma_2, \gamma_3 \in \{\Gamma, \Delta_2 + 1, \Delta + 1\}$, we have $\gamma_2(G'_k) \geq k > 2\varepsilon + (1-\varepsilon)(k+2) \geq \varepsilon\gamma_1(G'_k) + (1-\varepsilon)\gamma_3(G'_k)$ for all $k > 2\varepsilon^{-1}$. Hence $\text{conv}(\gamma_1, \gamma_2, \gamma_3) = 0$.

(iv) Consider the graph $K_{p,p}$ for $p \geq 2$. We have $\omega(K_{p,p}) = \chi(K_{p,p}) = \text{Nn}(K_{p,p}) = \Gamma^\downarrow(K_{p,p}) = \Gamma(K_{p,p}) = 2$ and $\Delta_2(K_{p,p}) + 1 = \Delta(K_{p,p}) + 1 = p + 1$. Thus if $\gamma_1 \in \{\omega, \chi, \text{Nn}, \Gamma^\downarrow, \Gamma\}$, we have $\Delta_2(K_{p,p}) + 1 = p + 1 > p + 1 - (p-1)\varepsilon = \varepsilon\gamma_1(G) + (1-\varepsilon)(\Delta(G) + 1)$. Hence $\text{conv}(\gamma_1, \Delta_2 + 1, \Delta + 1) = 0$. □

Problem 23. For $\gamma_3 \in \{\text{Nn}, \Gamma^\downarrow, \Gamma, \Delta_2 + 1\}$, what is the convexity of (ω, χ, γ_3) ?

4 Trees

Binomial trees have been widely studied and many of their properties are known. The following proposition is well-known and easy to prove.

Proposition 24. For $i \in [k]$, let W_i be the set of vertices of degree i in \mathcal{B}_k .

- (i) $|W_i| = 2^{k-1-i}$ for all $i \in [k-2]$, and $|W_{k-1}| = 2$.
- (ii) Every vertex of W_i has exactly one neighbour in W_j for all $j \in [i-1]$.
- (iii) There are only two greedy k -colourings of \mathcal{B}_k , $(W_1, \dots, W_{k-2}, \{v_2\}, \{v_1\})$ and $(W_1, \dots, W_{k-2}, \{v_1\}, \{v_2\})$, where $\{v_1, v_2\} = W_{k-1}$.

Recall that, for a graph G , we denote by $\text{bin}(G)$ the largest k such that G has an induced subgraph isomorphic to the binomial tree \mathcal{B}_k . Hedetniemi, Hedetniemi, and Beyer [8] proved that $\Gamma(T) = \text{bin}(T)$ for every tree T . This equality directly implies that $\text{Nn}(T) \leq \Gamma^\downarrow(T) \leq \Gamma(T) \leq \log(n) + 1$ for every tree of order n , and thus the result of [6] stating that $\frac{\text{Nn}(T)}{\chi(T)} \leq \frac{\log(n)+1}{2}$.

Moreover, the equality $\Gamma(T) = \text{bin}(T)$ implies that one can compute the Grundy number of a tree in polynomial time. A natural question is the following.

Problem 25. Can we compute the Nash number (resp. the diminishing Grundy number) of a tree in polynomial time ?

Theorem 26. Let T be a tree. The following assertions are equivalent:

- (i) $\text{Nn}(T) = 2$;
- (ii) $\Gamma^\downarrow(T) = 2$;
- (iii) $\Gamma(T) = 2$.

Proof. Let T be a tree. If it has no edge, then $\text{Nn}(T) = \Gamma^\downarrow(T) = \Gamma(T) = 1$, so we may assume that G has an edge, and so $2 = \chi(T) \leq \text{Nn}(T) \leq \Gamma^\downarrow(T) \leq \Gamma(T)$

If $\Gamma(T) = 2$, then T does not contain \mathcal{B}_3 . Thus it is a star with centre v . Now all the vertices of $V(T) \setminus \{v\}$ are twins, and thus have the same colour in any greedy colouring of T , by Lemma 8. Therefore $\text{Nn}(T) = \Gamma^\downarrow(T) = 2$.

Assume now that $\Gamma(T) > 2$. Then T contains \mathcal{B}_3 . In particular, it has diameter at least 3. Let us prove that T has a Nash 3-colouring. Consider a longest path P of T , and let a, b, c, d be the four first vertices on P in this order. Set $L = N(b) \setminus \{c\}$. Since P is a longest path of T , all vertices in L are leaves. The tree $T' = T - (L \cup \{b\})$ has a bipartition (C, D) such that $c \in C$ and $d \in D$.

If $|D| + |L| > |C|$, then set $S_1 = D \cup L$, $S_2 = C$, and $S_3 = \{b\}$. We have $|S_1| > |S_2| \geq |S_3|$. Moreover every vertex of $S_2 = C$ has a neighbour in D because T' is a tree of order at least 2, and b , the vertex of S_3 , has a neighbour, namely c , in S_2 . Hence (S_1, S_2, S_3) is a Nash colouring of T .

If $|D| + |L| \leq |C|$, then at least one neighbour of c in T' is not a leaf. Without loss of generality, we may assume that this is d . Let I be set of leaves adjacent to c , and set $S_1 = (C \setminus \{c\}) \cup I \cup \{b\}$, $S_2 = (D \cup L) \setminus I$, and $S_3 = \{c\}$. Observe that $|S_1| = |C| + |I| \geq |S_2| + |I|$, and $|S_2| \geq |S_3|$ because $d \in S_2$. Moreover $|S_1| > |S_3|$ because $|T| \geq 4$. Let us prove that (S_1, S_2, S_3) is a Nash colouring of T . First, by definition, the vertices of S_2 have a neighbour in S_1 , and c , the vertex of S_3 , has a neighbour, namely d , in S_2 . Now, if $|S_2| = |S_3|$, then $S_2 = \{d\}$, and d has a neighbour, namely c , in S_3 . If $|S_1| = |S_2|$, then $I = \emptyset$, so every neighbour c in T' is not a leaf, so every vertex of D has a neighbour in $C \setminus \{c\} \subseteq S_1$, and every vertex of L is adjacent to $b \in S_1$. Thus, in all cases, (S_1, S_2, S_3) is a Nash colouring of T . \square

Corollary 27. *Let F be a forest. The following assertions are equivalent:*

- (i) $\text{Nn}(F) = 2$;
- (ii) $\Gamma^\downarrow(F) = 2$;
- (iii) $\Gamma(F) = 2$.

Proof. If $\Gamma(F) = 2$, the F has an edge and does not contain \mathcal{B}_3 , that is F is the disjoint union of stars, so $2 = \chi(F) \leq \text{Nn}(F) \leq \Gamma^\downarrow(F) \leq \Gamma(F) = 2$.

If $\Gamma(F) > 2$, then F has a component with Grundy number at least 3. Thus, by Proposition 13 and Theorem 26, we have $\Gamma^\downarrow(F) \geq \text{Nn}(F) \geq \text{Nn}(T) \geq 3$. \square

Corollary 28. *Deciding whether a forest has Nash number (resp. diminishing Grundy number) 2 is polynomial-time solvable.*

We shall now see that for $k \geq 4$, containing the binomial tree of order k does not necessarily imply that a tree has Nash number at least k .

Proposition 29. *Let k be an integer such that $k \geq 4$.*

- (i) *There is a tree T_k with $\Gamma(T_k) = k$ and $\text{Nn}(T_k) \leq \Gamma^\downarrow(T_k) \leq k - 1$.*
- (ii) *There is a tree T'_k with $\Gamma^\downarrow(T'_k) = \Gamma(T'_k) = k$ and $\text{Nn}(T'_k) \leq k - 1$.*
- (iii) *There is a forest F_k such that $\text{Nn}(F_k) = \Gamma^\downarrow(F_k) = k$ and $\max\{\text{Nn}(T) \mid T \text{ component of } F_k\} \leq \max\{\Gamma^\downarrow(T) \mid T \text{ component of } F_k\} \leq k - 1$.*

Proof. (i) Let u_1, u_2 be the two vertices of degree $k - 1$ in \mathcal{B}_k . Let v be the vertex of degree 2 in $N(u_1)$. By the binomial tree structure, $N(v) = \{u_1, w\}$ with w a leaf. Now consider the tree T_k such that $V(T_k) = V(\mathcal{B}_k) \cup S$ where S is a stable set of order 2^k and $E(T_k) = E(\mathcal{B}_k) \cup \{xv \mid x \in S\}$. By definition, \mathcal{B}_k is a subtree of T_k and T_k does not contain \mathcal{B}_{k+1} because all its vertices except w have degree less than k . Hence $\Gamma(T_k) = \text{bin}(T_k) = k$.

Suppose for a contradiction that T_k admits a diminishing greedy k -colouring (S_1, \dots, S_k) . All vertices in S are twins, so, by Lemma 8, they are all in the same colour class. Now since $|S| > |V(T_k)|$ we have $S \subseteq S_1$. Let s_k be an element of S_k has degree $k - 1$ because it has a neighbour in each S_i , $i \in [k - 1]$. Let s_{k-1} be one of its neighbour in S_{k-1} . It also has degree $k - 1$ because it has a neighbour in each S_i , $i \in [k - 2]$ and it is adjacent to s_k . Moreover, the neighbours of w are all in S_1 except possibly v . But in T_k the only two adjacent vertices of degree k are u_1 and u_2 because $k \geq 4$. Hence, either $S_k = \{u_1\}$ or $S_k = \{u_2\}$.

Suppose now that $S_k = \{u_1\}$. Since $d(u_1) = k - 1$, vertex u_1 has exactly one neighbour in each S_i , $1 \leq i \leq k - 1$. But since $S \subseteq S_1$, we have $w \notin S_1$ and so $v \in S_1$ because the colouring is greedy. Moreover, the leaf in the neighbourhood of u_1 is also in S_1 , a contradiction.

Suppose first that $S_k = \{u_2\}$. Since the colouring is greedy, for each $i \in \{1, \dots, k - 1\}$ if $x \in S_i \cap N(u_2)$, then $d(x) \geq i$. Because the only vertex with degree at least $k - 1$ in the neighbourhood of u_2 is u_1 , we have $u_1 \in S_{k-1}$. Then u_1 has exactly one neighbour in each S_i for $i \in \{1, \dots, k - 2\} \cup \{k\}$. We then obtain a contradiction, in the same way as in the above paragraph.

Therefore T_k has no diminishing greedy k -colouring, and so $\text{Nn}(T_k) \leq \Gamma^\downarrow(T_k) \leq k - 1$.

(ii) Let u_1, u_2 be the two vertices of degree $k-1$ in \mathcal{B}_k . Let v be the vertex of degree 2 in $N(u_1)$. The neighbour of v distinct from u_1 is a leaf, say w .

Let T'_k be the graph obtained from \mathcal{B}_k by adding a stable set S' of 2^{k-3} vertices adjacent to w . Formally, $V(T'_k) = V(\mathcal{B}_k) \cup S'$, and $E(T'_k) = E(\mathcal{B}_k) \cup \{xw \mid x \in V'\}$.

Clearly, $\Gamma(T'_k) = \text{bin}(T'_k) = k$, so $\text{Nn}(T'_k) \leq \Gamma^\downarrow(T'_k) \leq k$.

Let us first prove that $\Gamma^\downarrow(T'_k) = k$. We shall construct a diminishing greedy k -colouring of T'_k . First take the greedy colouring $(W_1, \dots, W_{k-2}, u_2, u_1)$ of \mathcal{B}_k . In this colouring, every leaf is in W_1 and $|W_1| = 2^{k-2} = 2|W_2| = 2^2|W_3| = \dots = 2^{k-2}|W_{k-1}| = 2^{k-2}|W_{k-1}|$. Hence $(W_1, W_2 \cup V', W_3, \dots, W_k)$ is a clearly diminishing colouring of T'_k because, by Proposition 24 (i), $|W_1| = 2^{k-2} = |W_2 \cup V'|$ and $|W_i| = 2^{k-1-i}$ for all $3 \leq i \leq k-2$. Moreover it is a greedy colouring because every vertex in V' is neighbour of w and $w \in W_1$ as it is a leaf.

Now assume for a contradiction that T'_k has a Nash k -colouring $\sigma = (S_1, \dots, S_k)$. Every vertex $a \in S_k$ has degree at least $k-1$ and has a neighbour in S_i , $i \in [k-1]$ of degree at least i . In particular, its neighbour b in S_{k-1} has degree at least $k-1$. Since $k \geq 4$, u_1 and u_2 are the only adjacent vertices of degree at least $k-1$, so either $S_k = \{u_1\}$ and $u_2 \in S_{k-1}$ or $S_k = \{u_2\}$ and $u_1 \in S_{k-1}$. In both cases, for all $i \in [k-2]$ let s_i be the neighbour of u_1 in S_i . Since the colouring is greedy, s_i has degree at least i , and $s_i \neq u_2$ since $u_2 \in S_{k-1} \cup S_k$. But for every $i \in [k-2]$, v_1 has exactly one neighbour of degree i by Proposition 24 (ii). Hence s_i is the unique neighbour of u_1 with degree i . In particular, $v = s_2$, that is $v \in S_2$. Hence v has a neighbour in S_1 , which must be w because $u_1 \in S_{k-1} \cup S_k$. Now every vertex of V' is adjacent to w and only w , so $V' \subseteq S_2$. Hence $\sigma = (S_1, S_2 \setminus V', \dots, S_k)$ is a greedy colouring of \mathcal{B}_k . Thus, by Proposition 24 (iii), $S_1 = W_1$ and $S_2 = W_2 \cup V'$. In particular, $|S_1| = |S_2|$. Let s_1 be the leaf adjacent to v_1 . It is not adjacent to any element of $W_2 \cup V'$ so moving s_1 to S_2 would be an elementary improvement of σ , a contradiction.

(iii) Let F_k be the disjoint union of two copies T_1 and T_2 of T_k . By (i), we have $\max\{\text{Nn}(T) \mid T \text{ component of } F_k\} \leq \max\{\Gamma^\downarrow(T) \mid T \text{ component of } F_k\} \leq k-1$. Now let $\phi_1 = (S_1, \dots, S_k)$ be the greedy k -colouring of T_1 (it coincides with the greedy k -colouring of \mathcal{B}_k , and all vertices of S are coloured 2) and $\phi_2 = (T_1, T_2)$ be the 2-colouring of T_2 such that $|T_1| \geq |T_2|$. One easily checks that the union of ϕ_1 and ϕ_2 is a Nash k -colouring of F_k . Thus $\text{Nn}(F_k) = \Gamma^\downarrow(F_k) = k$. \square

However, as we now show the Nash number of a tree T cannot be much smaller than $\text{bin}(T)$.

Lemma 30. *Let $k \geq 3$ be an integer. Let T be a tree. If T contains \mathcal{B}_k , then T has a Nash $(k-1)$ -colouring.*

Proof. Let T be a tree containing a copy B of \mathcal{B}_k and let (U_1, \dots, U_k) be a greedy k -colouring of B . Note that $|U_i| = 2^{k-1-i}$ for $1 \leq i \leq k-1$ and $|U_k| = 1$. Let W_2 (resp. W_3) be the set of vertices of U_1 that are adjacent to a vertex in U_2 (resp. U_3) and let $U'_1 = U_1 \setminus (W_2 \cup W_3)$. Note that (U'_1, W_2, W_3) is a partition of U_1 . For each component C of $T - B$, there is a unique vertex $v \in V(B)$ that has a neighbour in C . We say that C is **attached** to v . Let F be the forest induced by the vertices in $U_2 \cup U_3 \cup W_2 \cup W_3$ and in the components of $T - B$ attached to a vertex in this set, let F' be the forest induced by the vertices in U'_1 and in the components of $T - B$ attached to a vertex in U'_1 , and let F'' be the forest which is the union of the components of $T - B$ attached to a vertex in $\bigcup_{i=3}^k U_i$.

Forest F has a bipartition (X, Y) such that $U_2 \cup W_3 \subseteq X$ and $U_3 \cup W_2 \subseteq Y$. Let (X', Y') be an arbitrary partition of F' , and let (X'', Y'') be an arbitrary partition of F'' .

Set $S_1 = X \cup X' \cup X''$, $S_2 = Y \cup Y' \cup Y''$, and $S_i = U_{i+1}$ for $3 \leq i \leq k-1$. The $(k-1)$ -colouring $\phi = (S_1, \dots, S_{k-1})$ may not be yet a Nash colouring. But it has the following properties:

(P1) $\min\{|S_1|, |S_2|\} > |S_3| > |S_4| > \dots > |S_{k-2}| = |S_{k-1}| = 1$;

(P2) for every $i \geq 3$, every vertex of S_i has a neighbour in S_j for all $j \in [i-1]$;

(P3) every vertex in $U_3 \cup W_2$ has a neighbour in $U_2 \cup W_3$, and every vertex in $U_2 \cup W_3$ has a neighbour in $U_3 \cup W_2$.

Those properties implies that one can only improve ϕ by either moving a vertex of $S_1 \setminus (U_2 \cup W_3)$ to S_2 or moving a vertex of $S_2 \setminus (U_3 \cup W_2)$ to S_1 . But then the resulting colouring will also have Properties (P1) – (P3). Hence, by induction, any improvement of ϕ will have those properties. In particular, any Nash colouring which is an improvement of ϕ is a Nash $(k-1)$ -colouring. \square

Lemma 30, Proposition 13 and the fact that $\Gamma(F) = \text{bin}(F)$ for every forest immediately imply the following.

Corollary 31. *For every forest F , $\Gamma(F) - 1 \leq \text{Nn}(F) \leq \Gamma^\downarrow(F) \leq \Gamma(F)$.*

This immediately implies the following.

Corollary 32. *Every forest is Nash-continuous.*

5 Complexity

In this section, we prove some complexity results about the Nash number and the diminishing Grundy number. All the NP-completeness proofs are reduction from 3-REGULAR 3-EDGE-COLOURABILITY, which is well-known to be NP-complete [9]. In this problem, we are given a 3-regular graph and want to assign colours in $\{1, 2, 3\}$ to the edges of the graph in a way that edges sharing one endpoint are to be assigned distinct colours.

5.1 Chordal graphs

Let G be a 3-regular graph with $n \geq 5$ vertices, where $V(G) = \{v_1, v_2, \dots, v_n\}$ and $E(G) = \{e_1, \dots, e_m\}$. Let I_G be the vertex-edge incidence graph of G , that is the bipartite graph with vertex set $V(I_G) = V(G) \cup E(G)$ in which $e_i \in E(G)$ is adjacent to $v_j \in V(G)$ if and only if vertex v_j is an end-vertex of e_i . We construct from I_G a new graph H_G as follows. Start with I_G . Add an edge between every pair of vertices of $V(G)$, thus making $V(G)$ a clique in H_G . Finally, for every vertex $e \in E(G)$, add three new vertices x_e, y_e, z_e and the three new edges $x_e y_e, y_e z_e, z_e x_e$ so that the subgraph induced by $\{e, x_e, y_e, z_e\}$ is a copy $\mathcal{B}_3(e)$ of \mathcal{B}_3 .

Observe that $|H_G| = 4|E(G)| + |V(G)| = 7n$ and $\Delta_2(H_G) = \Delta(H_G) = n + 2$. In H_G , $V(G)$ is a clique and $E(G)$ is a stable set, so $H_G[V(G) \cup E(G)]$ is a split graph, and so chordal. The vertices that are not in $V(G) \cup E(G)$ belong to one of the binomial trees $\mathcal{B}_3(e)$, so no induced cycle pass through one of these vertices and therefore H_G is chordal.

The key ingredient in the NP-completeness proofs is the following lemma.

Lemma 33. *Let G be a 3-regular graph with $n \geq 5$ vertices. The following are equivalent :*

- (i) G admits a 3-edge-colouring ;
- (ii) $\Gamma(H_G) = n + 3$;
- (iii) $\Gamma^\downarrow(H_G) = n + 3$;
- (iv) $\text{Nn}(H_G) = n + 3$.

Proof. We have $\text{Nn}(H_G) \leq dG(H_G) \leq \Gamma(H_G) \leq \Delta_2(H_G) + 1 = \Delta(H_G) + 1 = n + 3$.

Assume G admits a 3-edge-colouring c of which in H_G corresponds to a 3-colouring (S'_1, S'_2, S'_3) of the vertices in $E(G)$ that uses colours $\{1, 2, 3\}$. We may assume, without loss of generality, that $|S'_1| \geq |S'_2| \geq |S'_3|$. Let ϕ be the $(n+3)$ -colouring ϕ of H_G defined as follows: $\phi(e) = c(e)$, for $e \in E$, and $\phi(v_i) = i + 3$, for $1 \leq i \leq n$, $v_i \in V(G)$; if $c(e) = 1$ then $\phi(y_e) = \phi(z_e) = 2$ and $\phi(x_e) = 1$, if $c(e) = 2$ then $\phi(y_e) = \phi(z_e) = 1$ and $\phi(x_e) = 2$, and if $c(e) = 3$ then $\phi(x_e) = \phi(z_e) = 1$ and $\phi(y_e) = 2$; $\phi(a) = 2$ and $\phi(b_j) = 1$ for all $j \in n + 2 + k$. It is easy to check that ϕ is a greedy colouring. Moreover, setting $S_i = \phi^{-1}(i)$, $|S_1| > |S_2| > |S_3|$ (because each colour is used in c) and $|S_i| = 1$ for $4 \leq i \leq n + 3$. Therefore ϕ is a diminishing Nash $(n+3)$ -colouring of H_G . Hence $\text{Nn}(H_G) \geq n + 3$, and so $\Gamma(H_G) = \Gamma^\downarrow(H_G) = \text{Nn}(H_G) = n + 3$.

Conversely, assume $\Gamma(H_G) = n + 3$. Let ϕ be a greedy $(n+3)$ -colouring of H_G . Then every vertex coloured in $\{6, \dots, n+3\}$ has degree at least 5 and so must be in $V(G)$. So every vertex coloured in $\{6, \dots, n+3\}$ must be in $V(G)$. Without loss of generality, we may assume that $\phi(v_i) = i + 3$, for $3 \leq i \leq n$.

Claim 33.1. $\{\phi(v_1), \phi(v_2)\} = \{4, 5\}$.

Proof of claim. Suppose for a contradiction that $5 \notin \{\phi(v_1), \phi(v_2)\}$. Observe that vertex v_i , $3 \leq i \leq n$, has neighbours coloured 4 and 5, as it is coloured $i+3$. Therefore vertex v_n should be adjacent to a vertex $e \in E(G)$, such that $\phi(e) = 5$. But now e has at most 3 neighbours with colour in $\{1, \dots, 4\}$, a contradiction to $\phi(e) = 5$. Therefore, $5 \in \{\phi(v_1), \phi(v_2)\}$.

Without loss of generality we may assume $\phi(v_2) = 5$. Suppose for a contradiction that $\phi(v_1) \neq 4$. Every vertex in $v \in \{v_2, \dots, v_n\}$ should be adjacent to a vertex in $e \in E(G)$ that is coloured 4. In turn, e is adjacent to a vertex with colour 3. This vertex can only be v_1 , because every neighbour of e in $\mathcal{B}_3(e) - e$ is coloured in $\{1, 2\}$ since e is coloured 4. Hence e is adjacent to both v and v_1 , so vv_1 was an edge of G . Hence, in G , v_1 is adjacent to all vertices of $V \setminus \{v_1\}$. This is a contradiction as G is cubic and has at least 5 vertices. \diamond

Claim 33.1 implies that every vertex $v \in V(G)$ is coloured in $\{4, \dots, n+3\}$. Therefore the three neighbours of v in $E(G)$ must be coloured with distinct colours in $\{1, 2, 3\}$. Therefore ϕ corresponds to a proper 3-edge-colouring of G . This completes the proof. \square

Since H_G is chordal, Lemma 33 directly implies that it is NP-complete to decide whether a chordal graph satisfies $\gamma(G) = \gamma_2(G)$ for every parameters γ_1 in $\{\text{Nn}, \Gamma^\downarrow, \Gamma\}$ and γ_2 in $\{\Delta_2 + 1, \Delta + 1\}$. But we can obtain the following slightly more general statement.

Theorem 34. *Let k be a fixed non-negative integer, let γ_1 be a parameter in $\{\text{Nn}, \Gamma^\downarrow, \Gamma\}$ and γ_2 a parameter in $\{\Delta_2 + 1, \Delta + 1\}$.*

Given a chordal graph G , deciding whether $\gamma_1(G) \geq \gamma_2(G) - k$ is NP-complete.

Proof. Clearly, one can verify in polynomial time if a colouring is a Nash colouring, a diminishing greedy colouring, or a greedy colouring, and so the problem is in NP. To show that it is also NP-complete, we present a reduction from 3-REGULAR 3-EDGE-COLOURABILITY.

Let G be a 3-regular graph with $n \geq 5$ vertices. Let $D_{n,k}$ be the graph defined by

$$\begin{aligned} V(D_{n,k}) &= \{a, b\} \cup \{a_i \mid i \in [n+2+k]\} \cup \{b_i \mid i \in [n+2+k]\} \text{ and} \\ E(D_{n,k}) &= \{ab\} \cup \{aa_i \mid i \in [n+2+k]\} \cup \{bb_i \mid i \in [n+2+k]\}. \end{aligned}$$

Let H_k be the disjoint union of H_G and $D_{n,k}$. This graph is chordal and $\Delta_2(H_k) = \Delta(H_k) = n+2+k$.

Observe that $\Gamma(D_{n,k}) = 2$. Thus

$$\gamma_1(H_k) \leq \Gamma(H_k) = \max\{\Gamma(H_G), \Gamma(D_{n,k})\} = \Gamma(H_G) \leq \Delta(H_G) + 1 = n+3 = \gamma_2(H_k) - k.$$

Moreover, by Proposition 13, we have $\gamma_1(H_k) \geq \gamma(H_G)$. Hence, by Lemma 33, G admits a 3-edge-colouring if and only if $\gamma_1(H_k) = \gamma_1(H_G) = n+3 = \gamma_2(H_k) - k$. \square

Theorem 35. *Let γ be a parameter in $\{\text{Nn}, \Gamma^\downarrow\}$. Given a chordal graph G , deciding whether $\gamma(G) = \Gamma(G)$ is NP-complete.*

Proof. We give a reduction from 3-REGULAR 3-EDGE-COLOURABILITY.

Let G be a 3-regular graph with n vertices. Let $s = |H_G|$.

Let N be the graph obtained from the disjoint union of the complete graphs K_i^j for $i \in [n+1]$ and $j \in [2]$ where K_i^j has order i . For $i \in [n+1]$ and $j \in [2]$, pick a vertex v_i^j of K_i^j . Add two new vertices u_1 and u_2 , connect them with an edge, and for $j \in [2]$ add edges between u_j and all v_i^j , $i \in [n+1]$. Finally, add a stable set S of size $s + (n+1)(n+2) + 3$ and connect all its vertices to w , the vertex of K_2^1 distinct from v_2^1 . This graph is clearly chordal.

By Lemma 8, in any greedy colouring of N the vertices of S are coloured the same. Hence at most $n+2$ colours appear in the neighbourhood of each vertex. Thus $\Gamma(N) \leq n+3$. Now taking an i -colouring of each K_i^j such that v_i^j is coloured i , colouring the vertices of S by 2, and colouring u_2 with $n+2$ and u_1 with $n+3$, we obtain a greedy $(n+3)$ -colouring of N . Thus $\Gamma(N) = n+3$.

Let L be the disjoint union of N and H_G . It is chordal because N and H_G are chordal. Moreover $\Gamma(L) = \max\{\Gamma(H), \Gamma(H_G)\} = n+3$ because $\Gamma(H_G) \leq \Delta(H_G) + 1 = n+3$. We now prove that G admits a 3-edge-colouring if and only if $\gamma(L) = \Gamma(L)$.

Assume first that $\gamma(L) = \Gamma(L)$. Then L admits a diminishing greedy $(n+3)$ -colouring $\phi = (S_1, \dots, S_{n+3})$. By Lemma 8, S is included in one of the S_j . Since $|S| = s + (n+1)(n+2) + 3 > |V(L) \setminus S|$, necessarily $S \subseteq S_1$. Thus $w \notin S_1$, because w is adjacent to the vertices of S .

Let s_{n+3} be a vertex in S_{n+3} . It has degree at least $n+2$ because it has a neighbour in each S_i , $i \in [n+2]$. Let s_{n+2} be one of its neighbour in S_{n+2} . It also has degree $n+2$ because it has a neighbour in each S_i , $i \in [n+1]$ and it is adjacent to s_{n+3} .

Assume for a contradiction that s_{n+3} is in N . Since u_1 and u_2 is the only pair of adjacent vertices of degree at least $n+2$, we necessarily have $\{s_{n+3}, s_{n+2}\} = \{u_1, u_2\}$. Now, because these vertices have degree exactly $n+2$, they have exactly one neighbour in each S_i for all $i \in [n+1]$. But $v_1^1 \in S_1$ because its unique neighbour u_1 is in $S_{n+3} \cup S_{n+2}$ and v_1^2 is also in S_1 because its two neighbours w and u_1 are not in S_1 . This is a contradiction.

Therefore s_{n+3} is in H_G , and thus the restriction of ϕ to H_G is a greedy $(n+3)$ -colouring. Hence, by Lemma 33, G admits a 3-edge-colouring.

Conversely, assume that G admits a 3-edge-colouring. By Lemma 33, $\gamma(H_G) = n+3$. Thus, by Proposition 13, $\gamma(L) \geq \gamma(H_G) = n+3$, and so $\gamma(L) = n+3 = \Gamma(L)$. \square

5.2 Split graphs

Recall that a **split graph** is a graph G such that its vertex set can be partitioned $V(G) = (X, Y)$ such that X is a clique and Y a stable set. From [7], we know that $\Gamma(G) \leq \omega(G) + 1$, whenever G is a split graph.

Proposition 36. *Let G be a split graph with bipartition (K, S) with K a clique of size $\omega(G)$ and S a stable set. Then the following assertions are equivalent:*

- (i) $\text{Nn}(G) = \omega(G) + 1$.
- (ii) $\Gamma^\downarrow(G) = \omega(G) + 1$.
- (iii) $\Gamma(G) = \omega(G) + 1$.
- (iv) *Every vertex of K has a neighbour in S .*

Proof. As $\text{Nn}(G) \leq \Gamma^\downarrow(G) \leq \Gamma(G) \leq \omega(G) + 1$, (i) \Rightarrow (ii) \Rightarrow (iii).

Assume that every vertex of K has a neighbour in S . Then $|S| \geq 2$, because $K \cup \{s\}$ is not a clique since $|K| = \omega(G)$. Set $p = \omega(G)$ and let $K = \{v_1, \dots, v_p\}$. Then $(S, \{v_1\}, \dots, \{v_p\})$ is a diminishing Nash $(p+1)$ -colouring of G . Hence (iv) \Rightarrow (i).

Assume now that G has a greedy $(\omega(G) + 1)$ -colouring ϕ . Let k be the colour of $[\omega(G) + 1]$ such that no vertex of K is coloured k . Let $Y \subseteq S$ be the set of vertices coloured k . Let x be a vertex of K . If $\phi(x) > k$, then x has a neighbour in Y , so a neighbour in S . If $\phi(x) < k$, then x is the unique vertex of K coloured $\phi(x)$. Now consider a vertex $y \in Y$. It has a neighbour coloured $\phi(x)$ which is in K , and thus is necessarily x . Hence x has a neighbour in Y , and so in S . Therefore (iii) \Rightarrow (iv). \square

Proposition 36 directly implies the following two corollaries

Corollary 37. *If G is a split graph, then $\text{Nn}(G) = \Gamma^\downarrow(G) = \Gamma(G)$.*

Corollary 38. *The Nash number (resp. diminishing Grundy number, Grundy number) of a split graph can be computed in polynomial time.*

5.3 Bipartite Graphs

We now prove the NP-hardness of computing the Nash number of a bipartite graph in a similar way to chordal graphs. In order to do so, we shall construct a bipartite graph J_G which plays a similar role to H_G .

Recall that M_p is graph obtained from the bipartite complete graph (A, B) such that $A = \{u_1, \dots, u_p\}$ and $B = \{v_1, \dots, v_p\}$ by deleting the edges $u_i v_i$ for all $i \in \{1, \dots, p\}$. The **canonical colouring** of M_p is the colouring c such that $c(u_i) = c(v_i) = i$ for all $i \in [p]$. This is clearly a Nash colouring of M_p .

Let G be a 3-regular graph with $n-3$ vertices. Set $V(G) = \{v_4, v_5, \dots, v_n\}$ and $E(G) = \{e_1, \dots, e_m\}$. Let I_G be the vertex-edge incidence graph of G , that is the bipartite graph with vertex set $V(I) = V(G) \cup E(G)$ in which an edge of G is adjacent to its two endvertices.

We construct from I_G a new bipartite graph J_G as follows. For each vertex $e_i \in E(G)$, we add a copy $M_3(e_i)$ of M_3 and identify one of its vertices with e_i . We add a new vertex w adjacent to all the vertices of $V(G)$. We add copies $M_1^w, M_2^w, M_3^w, M_{n+1, n+1}^w$ of K_1, K_2, M_3, M_{n+1} and we choose arbitrary vertices v_1, v_2, v_3, v_{n+1} respectively from each copy and add the edges $v_1 w, v_2 w, v_3 w, v_{n+1} w$. Finally, for every $5 \leq i \leq n$, we do the

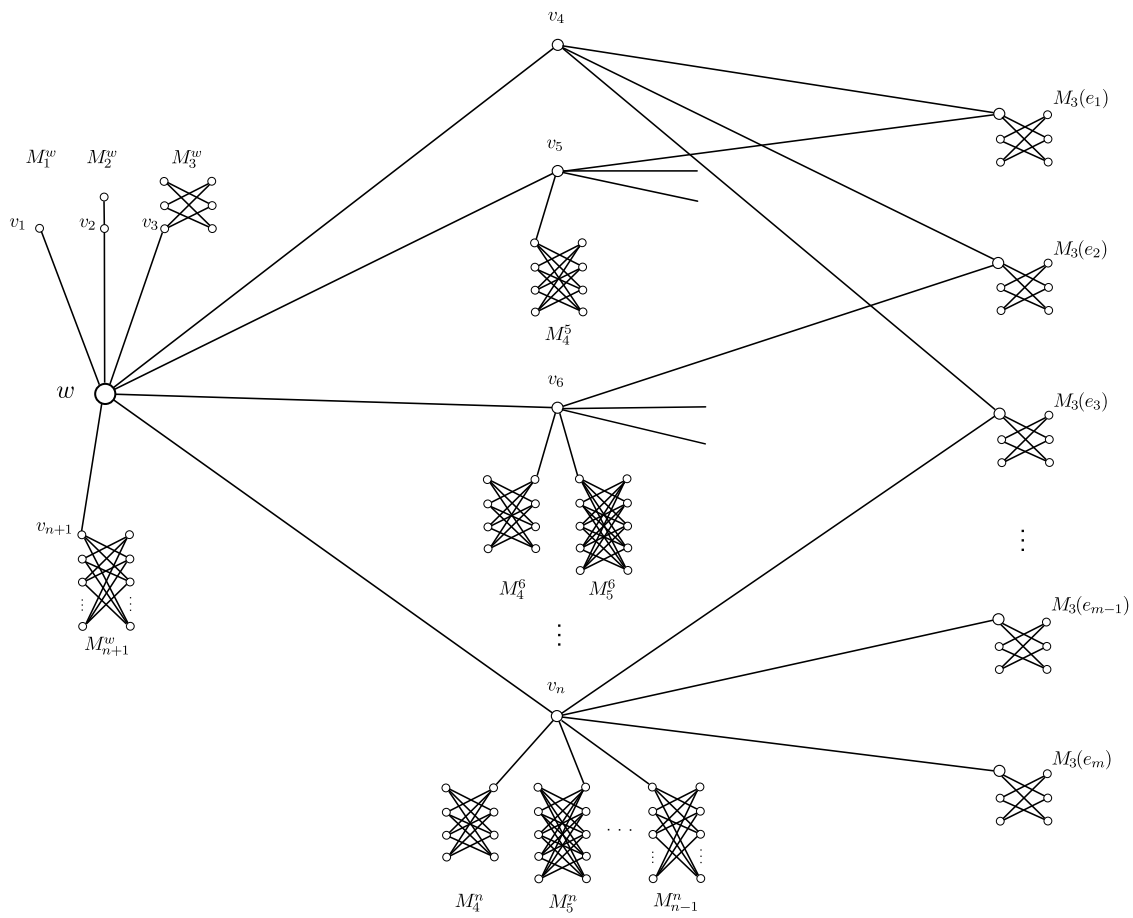


Figure 1: The graph J_G of the reduction in Theorem 40

following: for every $4 \leq j \leq i-1$, we add a copy M_j^i of M_j , choose an arbitrary vertex v_j^i of it and add the edge $v_i v_j^i$. See Figure 1.

Observe that:

- (i) $d_{J_G}(w) = n+1$,
- (ii) $d_{J_G}(v_i) = 1 + (i-1) = i$, for $4 \leq i \leq n+1$.
- (iii) $d_{J_G}(e_j) = 4$, for $1 \leq j \leq m$, since e_i has two neighbours in I and two in $M_3(e_i)$.
- (iv) $d_{J_G}(v_j^i) = j$, for $5 \leq i \leq n$ and $4 \leq j < i$, since a vertex in M_j^i has degree $j-1$ and v_j^i is adjacent to v_i .
- (v) $\Delta_2(J_G) = \Delta(J_G) = n+1$ and the only vertices with degree $n+1$ are w and v_{n+1} .

We shall now prove a Lemma which is an analogue to Lemma 33.

Lemma 39. *Let G be a 3-regular graph with n vertices. The following are equivalent :*

- (i) G admits a 3-edge-colouring ;
- (ii) $\Gamma(J_G) = n+2$;
- (iii) $\Gamma^\downarrow(J_G) = n+2$;
- (iv) $\text{Nn}(J_G) = n+2$.

Proof. Assume first G has a 3-edge-colouring ϕ . For every $1 \leq i \leq m$, we colour $M_3(e_i)$ with the canonical colouring such that e_i is coloured $\phi(e_i)$. Then in I_G every vertex in $V(G)$ has one neighbour of each colour in $\{1, 2, 3\}$.

For each $4 \leq j < i \leq n$, we colour M_j^i with the canonical colouring such that v_j^i is coloured j . We extend the union of these colourings to v_i , $4 \leq i \leq n$, so that v_i is coloured i .

Next, we colour v_1 with 1, v_2 with 2 and its neighbour in M_2^v with 1, and for $j \in \{3, n+1\}$, we colour M_j^w with the canonical colouring such that v_j is coloured j . We colour w with $n+2$.

It is simple matter to check that this colouring is a greedy colouring of J_G . Moreover, since canonical colourings of M_k contains exactly 2 vertices coloured i for all $1 \leq i \leq k$, and there are more copies of M_i than M_j when $i < j$, we clearly have $|S_i| < |S_j|$ for all $1 \leq j \leq n+2$. Hence this is a Nash colouring with $n+2$ colours, so $\text{Nn}(J_G) \geq n+2$. Thus $\text{Nn}(J_G) = \Gamma^\downarrow(J_G) = \Gamma(J_G) = n+2$ because $\text{Nn}(J_G) \leq \Gamma^\downarrow(J_G) \leq \Gamma(J_G) \leq \Delta(J_G) + 1 = n+2$.

Assume now that J_G admits a greedy $(n+2)$ -colouring c .

Claim 39.1. $\{c(w), c(v_{n+1})\} = \{n+1, n+2\}$

Proof of claim. Let x be a vertex such that $c(x) = n+2$. Vertex x must have one neighbour coloured with each of the other $n+1$ colours and so $d(x) \geq n+1$. Its neighbour y coloured $n+1$ must also have degree at least $n+1$: a neighbour of each colour in $[n]$ and x .

Hence, by Observation (v), $\{x, y\} = \{w, v_{n+1}\}$. ◇

Claim 39.2. For $1 \leq i \leq n$, $c(v_i) = i$.

Proof of claim. By Claim 39.1, $\{c(w), c(v_{n+1})\} = \{n+1, n+2\}$. Since $d_H(w) = n+1$, w has one neighbour coloured i , for each $1 \leq i \leq n$. A neighbour of w which is coloured n must have degree at least n . So, by Observation (ii), it must be v_n . And so on, by decreasing induction, we show that $c(v_i) = i$, for $1 \leq i \leq n$. ◇

We now prove that c induces a proper 3-edge-colouring of G .

Consider vertex v_i , $4 \leq i \leq n$. By Claim 39.2, it is coloured i , and by Observation (ii) it has degree i . Since it is adjacent to w , which by Claim 39.1 has a colour greater than i , there are only $i-1$ vertices remaining for the other $i-1$ colours. So, v_i has exactly one neighbour coloured j , for each $1 \leq j \leq i-1$ and therefore the three edges incident to v_i in G , which are adjacent to v_i in I , have different colours. Furthermore, for $1 \leq i \leq m$, the edge vertex e_i has at most two neighbours in J_G with a colour at most 3. Thus, $c(e_i) \in \{1, 2, 3\}$. □

Theorem 40. *Let k be a fixed non-negative integer, let γ_1 be a parameter in $\{\text{Nn}, \Gamma^\perp, \Gamma\}$ and γ_2 a parameter in $\{\Delta_2 + 1, \Delta + 1\}$.*

Given a bipartite graph G , deciding whether $\gamma_1(G) \geq \gamma_2(G) - k$ is NP-complete.

Proof. The problem is in NP because a Nash colouring using $\Delta(G) + 1 - k$ colours is a certificate. To show that it is also NP-complete, we present a reduction from 3-REGULAR 3-EDGE-COLOURABILITY, which is NP-complete [9].

Let J_k be the disjoint union of J_G and the graph $D_{n-1,k}$ defined in the proof of Theorem 34. This graph is bipartite and $\Delta_2(J_k) = \Delta(J_k) = n + 2 + k$.

Since $\Gamma(D_{n-1,k}) = 2$, we have

$$\gamma_1(J_k) \leq \Gamma(J_k) = \max\{\Gamma(J_G), \Gamma(D_{n-1,k})\} = \Gamma(J_G) \leq \Delta(J_G) + 1 = n + 2 = \gamma_2(J_k) - k$$

. Moreover, by Proposition 13, we have $\gamma_1(J_k) \geq \gamma(J_G)$. Hence, by Lemma 39, G admits a 3-edge-colouring if and only if $\gamma_1(J_k) = \gamma_1(J_G) = n + 2 = \gamma_2(J_k) - k$. \square

As a corollary to Theorem 40, it is NP-hard to compute the Nash number of a bipartite graph.

Corollary 41. *Given a bipartite graph G and an integer k , it is NP-complete to decide whether $\text{Nn}(G) \geq k$.*

Theorem 42. *Let γ be a parameter in $\{\text{Nn}, \Gamma^\perp\}$. Given a bipartite graph G , deciding whether $\gamma(G) = \Gamma(G)$ is NP-complete.*

Proof. We give a reduction from 3-REGULAR 3-EDGE-COLOURABILITY.

Let G be a 3-regular graph with n vertices. Let $s = |J_G|$.

Let F be the graph obtained from the complete bipartite graph $K_{s+3n+4, n+1}$ with bipartition (A, B) where $A = \{a_i \mid i \in [s+3n+4]\}$ and $B = \{b_i \mid i \in [n+1]\}$ by removing the edge $a_i b_i$ for all $i \in [n+1]$. Let $A' = \{a_i \mid i \in [n+1]\}$ and $A^* = A \setminus A'$.

Let L be the disjoint union of F and J_G .

By Lemma 8, in any greedy colouring the vertices of A^* are coloured the same. Hence at most $n + 1$ colours appear in the neighbourhood of each vertex. Thus $\Gamma(F) \leq n + 2$. Now $(\{a_1, b_1\}, \dots, \{a_{n+1}, b_{n+1}\}, A^*)$ is a greedy $(n + 2)$ -colouring. So $\Gamma(F) = n + 2$. Thus $\Gamma(L) = \max\{\Gamma(F), \Gamma(J_G)\} = n + 2$ because $\Gamma(J_G) \leq \Delta(J_G) + 1 = n + 2$.

We now prove that G admits a 3-edge-colouring if and only if $\gamma(L) = \Gamma(L)$.

Assume first that $\gamma(L) = \Gamma(L)$. Then L admits a diminishing greedy $(n + 2)$ -colouring (S_1, \dots, S_{n+2}) . By Lemma 8, A^* is included in one of the S_j . Since $|A^*| = s + 2n + 3 > s + 2n + 2 = |V(L) \setminus A^*|$, necessarily $A^* \subseteq S_1$. Therefore, no vertex of B is in S_1 , and thus $A \subseteq S_1$ because $N(a) \subseteq B$ for every $a \in A$. Now every vertex b of B must be in S_2 because $N(b) \subseteq A \subseteq S_1$. It follows that $(S_1 \setminus A, S_2 \setminus B, S_3, \dots, S_{n+2})$ is a greedy $(n + 2)$ -colouring of J_G . Hence, by Lemma 39, G admits a 3-edge-colouring.

Conversely, assume that G admits a 3-edge-colouring. By Lemma 39, $\gamma(J_G) = n + 2$. Thus, by Proposition 13, $\gamma(L) \geq \gamma(J_G) = n + 2$, and so $\gamma(L) = n + 2 = \Gamma(L)$. \square

5.4 Testing upper bounds based on ω and χ in general graphs

We end this section by showing the following NP-hardness result.

Theorem 43. *Let k be a fixed non-negative integer, let $\gamma_1 \in \{\omega, \chi\}$ and let $\gamma_2 \in \{\text{Nn}, \Gamma^\perp, \Gamma\}$. Deciding whether a given graph G satisfies $\gamma_2(G) \leq \gamma_1(G) + k$ is NP-hard.*

Proof. To show that it is NP-hard, we present a reduction from 3-REGULAR 3-EDGE-COLOURABILITY, which is NP-complete [9].

Let J'_k be the disjoint union of J_G and the complete graph K_{n-k+2} on $n - k + 2$ vertices. We have $\gamma_1(J'_k) = \max\{\gamma_1(J_G), \gamma_1(K_{n-k+2})\} = \max\{2, n - k + 2\} = n - k + 2$.

Since $\Gamma(K_{n-k+2}) = n - k + 2$, we have

$$\gamma_2(J'_k) \leq \Gamma(J'_k) = \max\{\Gamma(J_G), \Gamma(K_{n-k+2})\} \leq \max\{\Delta(J_G) + 1, n - k + 2\} = n + 2 = \gamma_1(J'_k) - k.$$

Moreover, by Proposition 13, we have $\gamma_2(J'_k) \geq \gamma_2(J_G)$. Hence, by Lemma 39, G admits a 3-edge-colouring if and only if $\gamma_2(J'_k) = \gamma_2(J_G) = n + 2 = \gamma_1(J'_k) - k$. \square

Remark 44. If $\gamma_1 = \chi$, then deciding whether a given graph G satisfies $\gamma_2(G) \leq \gamma_1(G) - k$ is co-NP-complete. Indeed a k_1 -colouring of G and a Nash k_2 -colouring (resp. a diminishing greedy k_2 -colouring) if $\gamma_2 = \text{Nn}$ (resp. $\gamma_2 = \Gamma^\downarrow$), with $k_2 > k_1 + k$ is a certificate for $\gamma_2(G) > \chi(G) + k$. So the problem is in co-NP.

References

- [1] Marie Asté, Frédéric Havet, and Cláudia Linhares Sales. Grundy number and products of graphs. *Discrete Mathematics*, 310(9):1482–1490, 2010.
- [2] Marthe Bonamy, Thomas Perrett, and Luke Postle. Colouring graphs with sparse neighbourhoods: Bounds and applications, 2018.
- [3] J.A. Bondy and U.S.R Murty. *Graph Theory*. Springer Publishing Company, Incorporated, 1st edition, 2008.
- [4] David P. Dailey. Uniqueness of colorability and colorability of planar 4-regular graphs are np-complete. *Discrete Mathematics*, 30(3):289–293, 1980.
- [5] Michelle Delcourt and Luke Postle. On the list coloring version of reed’s conjecture. *Electronic Notes in Discrete Mathematics*, 61:343 – 349, 2017. The European Conference on Combinatorics, Graph Theory and Applications (EUROCOMB’17).
- [6] Bruno Escoffier, Laurent Gourvès, and Jérôme Monnot. Strategic coloring of a graph. In Tiziana Calamoneri and Josep Diaz, editors, *Algorithms and Complexity*, pages 155–166, Berlin, Heidelberg, 2010. Springer Berlin Heidelberg.
- [7] A. Gyárfás and J. Lehel. On-line and first fit colorings of graphs. *Journal of Graph Theory*, 12(2):217–227, 1988.
- [8] S. M. Hedetniemi, S. T. Hedetniemi, and T. Beyer. A linear algorithm for the Grundy number of a tree. In *Proceedings of the Thirteenth Southeastern Conference on Combinatorics, Graph Theory and Computing*, number 36 in Congressus Numerantium, pages 351–363, 1982.
- [9] I. Holyer. The NP-completeness of edge-coloring. *SIAM J. Comput.*, 10(4):718–720, 1981.
- [10] R. M. Karp. Reducibility among combinatorial problems. In R. E. Miller and J. W. Thatcher, editors, *Complexity of Computer Computations*, pages 85–103. Plenum Press, 1972.
- [11] Andrew D. King and Bruce A. Reed. A short proof that χ can be bounded ε away from $\delta + 1$ toward ω . *Journal of Graph Theory*, 81(1):30–34, 2016.
- [12] Jan Mycielski. Sur le coloriage des graphes. *Colloquium Mathematicae*, 3(2):161–162, 1955.
- [13] Panagiota N. Panagopoulou and Paul G. Spirakis. A game theoretic approach for efficient graph coloring. In Seok-Hee Hong, Hiroshi Nagamochi, and Takuro Fukunaga, editors, *Algorithms and Computation*, pages 183–195. Springer Berlin Heidelberg, 2008.
- [14] B. Reed. ω , δ , and χ . *Journal of Graph Theory*, 27(4):177–212, 1998.
- [15] Manouchehr Zaker. Results on the Grundy chromatic number of graphs. *Discrete Mathematics*, 306(23):3166 – 3173, 2006. International Workshop on Combinatorics, Linear Algebra, and Graph Coloring.
- [16] David Zuckerman. Linear degree extractors and the inapproximability of max clique and chromatic number. *Theory of Computing*, 3(6), 2007.