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# ANALYSIS OF THE IMPLICIT EULER TIME-DISCRETIZATION OF SEMI-EXPLICIT DIFFERENTIAL-ALGEBRAIC LINEAR COMPLEMENTARITY SYSTEMS* 

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#### Abstract

This article is largely concerned with the time-discretization of differential-algebraic equations (DAE) with complementarity constraints, which we name differential algebraic linear complementarity systems (DALCS). Specifically, the Euler implicit discretization of DALCS is analysed: the one-step non-smooth problem (OSNSP), that is a generalized equation, is shown to be wellposed under some conditions, then the convergence of the discretized solutions is studied, and the existence of solutions to the continuous-time system is shown as a consequence. Passivity of some operators is pivotal to the analysis. Examples from circuits, mechanics and switching DAE illustrate the applicability of the developments.


Key words. differential-algebraic system, descriptor-variable system, linear complementarity system, recession function, discrete-time, well-posedness, passive system, Lur'e equations.

AMS subject classifications. $65 \mathrm{~K} 15, ~ 90 \mathrm{C} 33, ~ 90 \mathrm{C} 90, ~ 93 \mathrm{~B} 52$, 93D30, 93C30.

1. Introduction. The analysis of non-smooth dynamical systems with multivalued right-hand sides satisfying maximal monotone properties has witnessed a large number of contributions [19]. Within this class one finds linear and nonlinear complementarity dynamical systems, projected systems, differential variational inequalities, differential inclusions with maximal monotone right-hand sides, Moreau's sweeping process, some switching dynamics, etc. In this article we focus on singular, or differential-algebraic linear complementarity systems (DALCS), which may be viewed either as an extension of classical differential-algebraic equations (DAE), or of "classical" linear complementarity systems. As will become clear later, DALCS represent an extension of the set-valued Lur'e systems studied, e.g., in [39, 40]. The associated canonical form is motivated by applications in chemistry [53, 52, 62], switching DAE analysis [48], circuits with nonsmooth electronic components [2, section 3.5, chapter 4] [22], optimization-constrained differential equations [19, Example 2] with applications in atmospheric aerosol particles [38], and Lagrangian systems with both bilateral and unilateral constraints [17] (in spite of the fact that the results presented in this article do not apply to all of these systems).
The study of singular nonsmooth dynamical systems has not received a lot of attention yet. Singularly perturbed differential inclusions have been tackled in [26, 56, 32, 58]. The conditions on the set-valued mappings that are imposed (compactness [56, 58], boundedness and Hausdorff Lipschitz continuity [32], one-sided Lipschitz condition [26]) are not satisfied by complementarity conditions (or by normal cones to convex sets). Hence these results do not apply to DALCS. The well-posedness of DAE in semiexplicit form with nonsmooth constraints (which are related to the class of systems studied in this article, see Remark 2.1 below) has been analysed in [53, 54, 52]. Passive singular set-valued systems are studied in [22], who derived well-posedness relying on a special Weierstrass' representation. However the time-discretisation is not tackled

[^0]in the above references.
Our goal is to analyse the well-posedness of the one-step nonsmooth problem (OSNSP) obtained after an implict (backward) Euler discretization, and the convergence of the discrete-time solutions towards a continuous-time limit. The second step is useful not only for proving the existence of solutions to the continuous-time system, but also, perhaps most importantly, to show that the proposed time-discretization makes sense for numerical simulations. Passivity properties of suitable operators are pivotal throughout the article. It is noteworthy that implicit Euler methods are well-known in DAE [30, 14, 33, 42, 12, 35, 61, 7], differential variational inequalities [46, 24], and complementarity systems fields [41, 21, 59, 34], as well as in the field of maximal monotone differential inclusions $[8,47]$ (implicit Euler method is often called proximal algorithm, while Euler method refers to the explicit algorithm [47]), and Moreau's sweeping processes with the catching-up algorithm [19, 44, 45]. However their application to DALCS has not been studied yet.
The article is organised as follows: Section 2 introduces the general dynamics of the nonsmooth singular systems, section 3 presents their time-discretization, section 4 studies the well-posedness of the OSNSP, section 5 is dedicated to convergence analyses: results in section 5.1 are based on the properties of the numerical scheme, while section 5.2 uses properties of the system. Section 6 is dedicated to illustrating examples and simulations, from circuits, mechanics and switching DAE. Conclusions are drawn in section 7. The appendix is dedicated to recall various mathematical tools.

Notation and definitions: for any vector $x \in \mathbb{R}^{n}$ and any matrix $M \in \mathbb{R}^{m \times n}$, $\|x\|$ is the Euclidean norm and $\|M\|$ is the Frobenius norm, which are compatible norms [10, Proposition 9.3.5], i.e., $\|M x\| \leq\|M\|\|x\|$. Let $M \in \mathbb{R}^{n \times m}$, then $\operatorname{Im}(M)$ is its range, $\operatorname{Ker}(M)$ is its null space. We use $\langle x, y\rangle=x^{\top} y$, so $\langle x, x\rangle=\|x\|^{2}$. Positive definite matrix: $M \succ 0$ if $x^{\top} M x>0$ for all $x \neq 0$, positive semidefinite matrix: $M \succcurlyeq 0$ if $x^{\top} M x \geq 0$ for all $x$ (such $M$ is not necessarily symmetric). The maximum singular value is denoted as $\sigma_{\max }(M)$, and the minimum eigenvalue as $\lambda_{\min }(M)$. The $i$ th row of $M$ is denoted as $M_{i \bullet}$. The $n \times n$ identity matrix is denoted $I_{n}$. A set-valued mapping $A: \mathbb{R}^{n} \rightrightarrows \mathbb{R}^{n}$ is said monotone if for all $x_{1}, x_{2}, y_{1} \in A\left(x_{1}\right), y_{2} \in A\left(x_{2}\right)$, one has $\left\langle x_{1}-x_{2}, y_{1}-y_{2}\right\rangle \geq 0$. It is maximal monotone if its graph cannot be enlarged without destroying monotonicity. It is $\eta$-strongly monotone if there exists $\eta>0$ such that $\left\langle x_{1}-x_{2}, y_{1}-y_{2}\right\rangle \geq \eta\left\|x_{1}-x_{2}\right\|^{2}$. See also Appendix A for further definitions.
2. The class of nonsmooth singular dynamical systems. Let us consider the following differential algebraic linear complementarity system (DALCS):

$$
\left\{\begin{array}{l}
\text { (a) } P_{I} \dot{x}(t)=A_{I} x(t)+B_{I} z(t)+\bar{B}_{I} \lambda(t)+E_{I}(t)  \tag{2.1}\\
\text { (b) } 0=A_{I I} x(t)+B_{I I} z(t)+\bar{B}_{I I} \lambda(t)+E_{I I}(t) \\
(c) K^{\star} \ni \lambda(t) \perp w(t)=C_{I} x(t)+C_{I I} z(t)+D \lambda(t)+F(t) \in K,
\end{array}\right.
$$

with $x(t) \in \mathbb{R}^{n_{1}}, z(t) \in \mathbb{R}^{n_{2}}, \lambda(t) \in \mathbb{R}^{m}, w(t) \in \mathbb{R}^{m}, P_{I} \in \mathbb{R}^{n_{1} \times n_{1}}$ is full-rank, $K \subseteq \mathbb{R}^{m}$ is a closed nonempty convex cone, $K^{\star}$ is its dual cone, $\bar{B}_{I I} \in \mathbb{R}^{p \times m}$, $A_{I I} \in \mathbb{R}^{p \times n_{1}}, B_{I I} \in \mathbb{R}^{p \times n_{2}}, C_{I} \in \mathbb{R}^{m \times n_{1}}, C_{I I} \in \mathbb{R}^{m \times n_{2}}, D \in \mathbb{R}^{m \times m}, B_{I} \in$ $\mathbb{R}^{n_{1} \times n_{2}}, A_{I} \in \mathbb{R}^{n_{1} \times n_{1}}, \bar{B}_{I} \in \mathbb{R}^{n_{1} \times m}$, and $E_{I}(\cdot), E_{I I}(\cdot), F(\cdot)$ are bounded, Lipschitz continuous functions of time. The form in (2.1) is that of a semi-explicit DALCS. It corresponds to a kind of extension of the impulse-free Weierstrass form [20], where the transformed state matrix has non zero off-diagonal terms $A_{I I}$ and $B_{I}$.

Remark 2.1. Assume that $K=\mathbb{R}_{+}^{m}$. The complementarity constraint (2.1) (c) is equivalently rewritten as $\phi(\lambda, w)=0$ where $\phi(\cdot)$ is a $C$-function [3, 29]. Therefore (2.1) (b) (c) can be rewritten equivalently as $\Phi(x, z, \lambda, t)=0$ for some function $\Phi(\cdot)$, usually nondifferentiable. The system (2.1) thus possesses the form of a semi-explicit DAE as studied in [53, 52], assuming piecewise differentiable functions $\Phi(\cdot)$, with $x$ the differential state, $z$ and $\lambda$ the algebraic state variables. The so-called regularity conditions (which are an extension of index 1 conditions for DAE) used in [53, 54, 52, 51] may hold in particular cases for (2.1). Examples in section 6.3 illustrate the links betwen both approaches in $[53,54,52,51]$ and in this article.
Consider (2.1), then the initial data $x(0)=x_{0}$ and $z(0)=z_{0}$ are compatible with the system's constraints if and only if:

$$
\begin{gather*}
0 \in A_{I I} x_{0}+B_{I I} z_{0}+E_{I I}(0)-\bar{B}_{I I}\left(D+\mathcal{N}_{K^{\star}}\right)^{-1}\left(C_{I} x_{0}+C_{I I} z_{0}+F(0)\right)  \tag{2.2}\\
\lambda(0) \in-\left(D+\mathcal{N}_{K^{\star}}\right)^{-1}\left(C_{I} x_{0}+C_{I I} z_{0}+F(0)\right) .
\end{gather*}
$$

This implies that $C_{I} x_{0}+C_{I I} z_{0}+F(0) \in \operatorname{Im}\left(D+\mathcal{N}_{K^{\star}}\right)$ and $A_{I I} x_{0}+B_{I I} z_{0}+E_{I I}(0) \in$ $\bar{B}_{I I} \operatorname{Dom}\left(D+\mathcal{N}_{K^{\star}}\right)$.
3. Time-discretization of DALCS with implicit Euler methods. Let us consider the time interval $[0, T], T>0, h=\frac{T}{N}, N \in N_{+}, t_{k}=k h, k \in\{0,1, \ldots, N\}$, $t_{0}=0, t_{N}=T$. For a continuous function $f(\cdot), f_{k} \triangleq f\left(t_{k}\right)$. The implicit Euler discretization of (2.1) reads as:

$$
\begin{align*}
& \text { (a) } P_{I}\left(x_{k+1}-x_{k}\right)=h A_{I} x_{k+1}+h B_{I} z_{k+1}+h \bar{B}_{I} \lambda_{k+1}+h E_{I, k} \\
& \text { (b) } A_{I I} x_{k+1}+B_{I I} z_{k+1}+\bar{B}_{I I} \lambda_{k+1}+E_{I I, k}=0  \tag{3.1}\\
& \text { (c) } K^{\star} \ni \lambda_{k+1} \perp C_{I} x_{k+1}+C_{I I} z_{k+1}+D \lambda_{k+1}+F_{k} \in K .
\end{align*}
$$

The problem in (3.1) (b) (c) is a mixed LCP (MLCP). Let us now state a basic assumption.

Assumption 1. The matrix $P_{I}^{h} \triangleq P_{I}-h A_{I}$ is full-rank.
In case $P_{I} \succ 0$, [25, Theorem 2.11] can be used to set conditions on $h A_{I}$ so that $P_{I}^{h} \succ 0$ : there always exists $h_{\text {max }}$ such that the assumption holds for all $h \in\left[0, h_{\max }\right)$ (with $h_{\text {max }}=+\infty$ if $A_{I} \preccurlyeq 0$ ). Let us define the following matrices and vectors:

1. $\tilde{A}=h A_{I I}\left(P_{I}^{h}\right)^{-1} B_{I}+B_{I I}$,
2. $\tilde{\tilde{N}}=h A_{I I}\left(P_{I}^{h}\right)^{-1} \bar{B}_{I}+\bar{B}_{I I}$,
3. $\tilde{E}_{k}=E_{I I, k}+h A_{I I}\left(P_{I}^{h}\right)^{-1} E_{I, k}+A_{I I}\left(P_{I}^{h}\right)^{-1} P_{I} x_{k}$,
4. $\tilde{\tilde{C}}=h C_{I}\left(P_{I}^{h}\right)^{-1} B_{I}+C_{I I}$,
5. $\tilde{D}=h C_{I}\left(P_{I}^{h}\right)^{-1} \bar{B}_{I}+D$,
6. $\tilde{F}_{k}=F_{k}+h C_{I}\left(P_{I}^{h}\right)^{-1} E_{I, k}+C_{I}\left(P_{I}^{h}\right)^{-1} P_{I} x_{k}$.

Then we deduce from (3.1) the difference equations:

$$
\left\{\begin{array}{l}
\text { (a) } x_{k+1}=\left(P_{I}^{h}\right)^{-1}\left(P_{I} x_{k}+h B_{I} z_{k+1}+h \bar{B}_{I} \lambda_{k+1}+h E_{I, k}\right)  \tag{3.2}\\
(b) \tilde{A} z_{k+1}+\tilde{B} \lambda_{k+1}+\tilde{E} \tilde{E}_{k}=0 \\
(c)
\end{array} K^{\star} \ni \lambda_{k+1} \perp \tilde{C} z_{k+1}+\tilde{D} \lambda_{k+1}+\tilde{F}_{k} \in K,, ~ \$\right.
$$

where the last two lines of (3.2) are a MLCP with unknowns $z_{k+1}$ and $\lambda_{k+1}$. It is therefore possible to study the well-posedness of this MLCP independently of the first line in (3.2). Letting $\zeta_{k+1} \triangleq-z_{k+1}$, this MLCP can be rewritten equivalently as
follows:

$$
\left\{\begin{array}{l}
\lambda_{k+1} \in\left(\mathcal{N}_{K^{\star}}+\tilde{D}\right)^{-1}\left(-\tilde{C} z_{k+1}-\tilde{F}_{k}\right),  \tag{3.3}\\
-\tilde{A} \zeta_{k+1}+\tilde{E}_{k} \in-\tilde{B}\left(\mathcal{N}_{K^{\star}}+\tilde{D}\right)^{-1}\left(\tilde{C} \zeta_{k+1}-\tilde{F}_{k}\right), \\
x_{k+1}=\left(P_{I}^{h}\right)^{-1}\left(P_{I} x_{k}+h B_{I} z_{k+1}+h \bar{B}_{I} \lambda_{k+1}+h E_{I, k}\right)
\end{array}\right.
$$

Also the following calculations stem from the complementarity conditions in (3.2):

$$
\begin{array}{lll} 
& \lambda_{k+1} \in-\mathcal{N}_{K}\left(\tilde{C} z_{k+1}+\tilde{D} \lambda_{k+1}+\tilde{F}_{k}\right) & \Leftrightarrow-\lambda_{k+1} \in \mathcal{N}_{\Gamma_{k}}\left(\tilde{C} z_{k+1}+\tilde{D} \lambda_{k+1}\right)  \tag{3.4}\\
\Leftrightarrow & \partial \sigma_{\Gamma_{k}}\left(-\lambda_{k+1}\right) \ni \tilde{C} z_{k+1}+\tilde{D} \lambda_{k+1} & \Leftrightarrow\left(\partial \sigma_{\Gamma_{k}}+\tilde{D}\right)\left(-\lambda_{k+1}\right) \ni \tilde{C} z_{k+1} \\
\Leftrightarrow & \lambda_{k+1} \in-\left(\partial \sigma_{\Gamma_{k}}+\tilde{D}\right)^{-1}\left(\tilde{C} z_{k+1}\right) . &
\end{array}
$$

where the inversion of set-valued mappings is used in the second equivalence, and where $\sigma_{\Gamma_{k}}(\cdot)$ is the support function of the set $\Gamma_{k}\left(t_{k}, x_{k}\right) \triangleq K-\tilde{F}_{k}\left(t_{k}, x_{k}\right)$ (which is a closed convex set, but not a cone as long as $\tilde{F}_{k} \neq 0$ which is the generic case). Therefore equivalently we can rewrite (3.3) as:

$$
\left\{\begin{array}{l}
(\text { a }) \lambda_{k+1} \in-\left(\partial \sigma_{\Gamma_{k}}+\tilde{D}\right)^{-1}\left(\tilde{C} z_{k+1}\right)  \tag{3.5}\\
(b)-\tilde{A} z_{k+1}+\tilde{B}\left(\partial \sigma_{\Gamma_{k}}+\tilde{D}\right)^{-1}\left(\tilde{C} z_{k+1}\right) \ni \tilde{E}_{k} \\
(c) \quad x_{k+1}=\left(P_{I}^{h}\right)^{-1}\left(P_{I} x_{k}+h B_{I} z_{k+1}+h \bar{B}_{I} \lambda_{k+1}+h E_{I, k}\right)
\end{array}\right.
$$

The second line in (3.3) is a generalized equation (GE) with unknown $\zeta_{k+1}$. Its wellposedness depends on the properties of the operator $\mathcal{F}: \zeta \mapsto-\tilde{A} \zeta+\tilde{B}\left(\mathcal{N}_{K^{\star}}+\right.$ $\tilde{D})^{-1}\left(\tilde{C} \zeta-\tilde{F}_{k}\right)$. Similarly for (3.5) with the operator $\mathcal{G}: \zeta \mapsto-\tilde{A} \zeta+\tilde{B}\left(\partial \sigma_{\Gamma_{k}}+\right.$ $\tilde{D})^{-1}(\tilde{C} \zeta)$, and (3.5) (b) is: $\mathcal{G}\left(z_{k+1}\right) \ni \tilde{E}_{k}$. Various results have been published to characterize such operators $[16,23,6]$ and to guarantee that they are single-valued Lipschitz continuous [16, Propositions 1, 2, 3, Corollary 1], or maximal monotone (possibly set-valued) [23, 6]. The difficulty in the analysis of DALCS is that they involve the composition of two operators as seen in the OSNSP for advancing the scheme from step $k$ to step $k+1$ :

$$
\begin{equation*}
P_{I}^{h} x_{k+1} \in P_{I} x_{k}+h B_{I} \mathcal{G}^{-1}\left(\tilde{E}_{k}\left(x_{k}\right)\right)-h \bar{B}_{I}\left(\partial \sigma_{\Gamma_{k}}+\tilde{D}\right)^{-1}\left(\tilde{C} \mathcal{G}^{-1}\left(\tilde{E}_{k}\left(x_{k}\right)\right)\right)+h E_{I, k} \tag{3.6}
\end{equation*}
$$

Clearly the scheme can be advanced with a unique solution if both $\lambda_{k+1}$ and $z_{k+1}$ are uniquely defined (the "if and only if" holds with $\bar{B}_{I} \lambda_{k+1}$ and $B_{I} z_{k+1}$ uniquely defined). Thus next we are going to characterize the GEs in (3.3) and (3.5) in order to analyse the scheme (3.6).
Two main classes of results are provided in the sequel of the article: first the wellposedness (existence and uniqueness of solutions) for the OSNSP associated with the time-discretizations in (3.1) (Propositions 4.3 and 4.4, Corollary 4.6) and (3.2) (Propositions 4.7 and 4.8); second, the convergence of the discrete-time solutions is analysed, and the limit functions are shown to be solutions of the DALCS (Propositions 5.2 and 5.9).
4. OSNSP well-posedness. The approach chosen in this article consists of considering the system as an $\operatorname{ODE}$ (2.1) (a) with a nonsmooth constraint (2.1) (b) (c): either (3.1) (b) (c) that is a mixed LCP with unknowns $z_{k+1}, \lambda_{k+1}$ and parameter $x_{k+1}$ (section 4.2), or (3.2) (b) (c) (equivalently (3.5) (a) (b)) that is a mixed LCP with $z_{k+1}, \lambda_{k+1}$ (section 4.1). As will become clear later, the results rely heavily on the positive (semi) definiteness of $B_{I I}$, and on specific passivity couplings between $z$ in the complementarity constraint (matrix $C_{I I}$ ) and $\lambda$ in the algebraic constraint (matrix $\bar{B}_{I I}$ ). When $\tilde{D}=0$ in (3.2) (c), let us state the following:

Assumption 2. There exists a $p \times p$ matrix $\tilde{X}=\tilde{X}^{\top} \succ 0$ such that $\tilde{X} \tilde{B}=\tilde{C}^{\top}$. The alternative option when $D=0$ in (3.1) (c):

Assumption 3. There exists a $p \times p$ matrix $X=X^{\top} \succ 0$ such that $X \bar{B}_{I I}=C_{I I}^{\top}$.

Let us notice that Assumption 2 implies that $\tilde{X} \bar{B}_{I I}=C_{I I}^{\top}+\mathcal{O}(h)$, while Assumption 3 implies that $X \tilde{B}=\tilde{C}^{\top}+\mathcal{O}(h)$. The influence of the $\mathcal{O}(h)$ terms may not be easy to characterize precisely for finite $h>0$. Assumption 3 is a model assumption, while Assumption 2 is a property of the numerical method. One sees that Assumption 3 corresponds to a passivity constraint on the triple $\left(B_{I I}, \bar{B}_{I I}, C_{I I}\right)$ [18]. Assumption 2 imposes an input/output passivity-like constraint on the triple $(\tilde{A}, \tilde{B}, \tilde{C})$, which is an $\mathcal{O}(h)$ approximation of $\left(B_{I I}, \bar{B}_{I I}, C_{I I}\right)$. When $\tilde{D} \neq 0$ and $D \neq 0$, extensions of Assumptions 2 and 3 are as follows:

Assumption 4. The quadruple $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is passive.
and the counterpart of Assumption 3:
Assumption 5. The quadruple $\left(B_{I I}, \bar{B}_{I I}, C_{I I}, D\right)$ is passive.
Passivity is defined in Appendix C. Notice that the above assumptions imply that $n_{2}=p$. Assumption 4 refers to the numerical method while Assumption 5 is a model property. Assumptions 2 and 4 refer to (3.2) (b) (c) or (3.5) (a) (b), while Assumptions 3 and 5 refer to (3.1) (b) (c). Let us present a result about the relationships between Assumptions 4 and 5, i.e., when does the plant's passivity guarantees the numerical scheme's passivity.

Lemma 4.1. The following statements are true:

1. Let $\left(B_{I I}, \bar{B}_{I I}, C_{I I}, D\right)$ be strictly passive. Then $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is strictly passive for small enough $h>0$.
2. Let $\left(B_{I I}, \bar{B}_{I I}, C_{I I}, D\right)$ be strictly state passive. Then $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is strictly state passive if: a) $h>0$ is small enough, b) $D+D^{\top}=0$ and $C_{I}\left(P_{I}^{h}\right)^{-1} \bar{B}_{I}+$ $\left(C_{I}\left(P_{I}^{h}\right)^{-1} \bar{B}_{I}\right)^{\top} \succcurlyeq 0$, c) $X A_{I I}\left(P_{I}^{h}\right)^{-1} \bar{B}_{I}-B_{I}^{\top}\left(P_{I}^{h}\right)^{-\top} C_{I}^{\top}=0$, where $X$ is a solution of the passivity LMI for $\left(B_{I I}, \bar{B}_{I I}, C_{I I}, D\right)$.
3. Let $h>0$ and $\left(B_{I I}, \bar{B}_{I I}, C_{I I}, D\right)$ be passive. Then $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is passive if: b) and c) in item 2 hold, and a) $X A_{I I}\left(P_{I}^{h}\right)^{-1} B_{I}+B_{I}^{\top}\left(P_{I}^{h}\right)^{-\top} A_{I I}^{\top} X \preccurlyeq 0$, where $X$ is a solution of the passivity LMI for $\left(B_{I I}, \bar{B}_{I I}, C_{I I}, D\right)$.

## Proof:

1. The strict passivity of $\left(B_{I I}, \bar{B}_{I I}, C_{I I}, D\right)$ means that there exists $X=X^{\top} \succ 0$ such that:

$$
M \triangleq\left(\begin{array}{cc}
-X B_{I I}-B_{I I}^{\top} X & -X \bar{B}_{I I}+C_{I I}^{\top}  \tag{4.1}\\
-\bar{B}_{I I}^{\top} X+C_{I I} & D+D^{\top}
\end{array}\right) \succ 0 .
$$

In particular this implies that $D+D^{\top} \succ 0$ [18, Theorem A.65] and the associated transfer function is strongly SPR [18, Remark 3.19, Section 3.12.2]. Now we have $\tilde{A}=B_{I I}+\mathcal{O}(h), \tilde{B}=\bar{B}_{I I}+\mathcal{O}(h), \tilde{C}=C_{I I}+\mathcal{O}(h), \tilde{D}=$ $D+\mathcal{O}(h)$. Therefore

$$
M=\underbrace{\left(\begin{array}{cc}
-X \tilde{A}-\tilde{A}^{\top} X & -X \tilde{B}+\tilde{C}^{\top}  \tag{4.2}\\
-\tilde{B}^{\top} X+\tilde{C}^{\top} & \tilde{D}+\tilde{D}^{\top}
\end{array}\right)}_{\triangleq \tilde{M}}+N(h)
$$

where $N(h)$ stands for a matrix with suitable dimensions and which vanishes as $h \rightarrow 0$ (in fact $N(h)$ is $\mathcal{O}(h)$ ). Thus $\tilde{M}=M-N(h)$. Applying [25, Theorem 2.11], it follows that $\tilde{M} \succ 0$ provided that $\sigma_{\max }(N(h))<\lambda_{\min }^{2}(M)$, an inequality which can always be verified if $h>0$ is small enough. Under this condition, it follows that the strict passivity of $\left(B_{I I}, \bar{B}_{I I}, C_{I I}, D\right)$ implies the strict passivity of $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})(\Rightarrow \tilde{D} \succ 0)$.
2. The result is proved by developing $\tilde{M}$ in (4.2), then using [10, Proposition 8.2.4, Fact 8.21.22], or [18, Lemma A.69], as well as the fact that $-X B_{I I}-$ $B_{I I}^{\top} X \succ 0$ from the strict state passivity.
3. The proof is similar to that of item 2.

The converse implication in item 1 does not hold, since one may have $\tilde{M} \succ 0$ for some $h>0$, but $M \nsucc 0$. It is possible to state necessary and sufficient conditions, relying on [10, Proposition 8.2.4, Fact 8.21.22] or [18, Lemma A.70]. However the obtained conditions are cumbersome and hardly usable. Lemma 4.1 can be used to determine the conditions under which Assumption 4 is verified in the results presented in the next section. The conditions b) and c) in Lemma 4.1 item 2 hold if: b) $C_{I}=0$ or $\bar{B}_{I}=0$, c) $\left(A_{I I}=0\right.$ or $\left.\bar{B}_{I}=0\right)$ and ( $B_{I}=0$ or $\left.C_{I}=0\right)$. Condition a) in Lemma 4.1 item 3 holds if $A_{I I}=0$ or $B_{I}=0$. Let us remind that the solution $X=X^{\top} \succ 0$ of the Lyapunov equation $X B_{I I}+B_{I I}^{\top} X=-Q, Q=Q^{\top} \succ 0$, can be written $X=\left(-\frac{1}{2} Q+F\right) B_{I I}^{-1}$, where $F=-F^{\top}$ is unique [60, Theorems 1 and 2]. Let us denote the symmetric part of a square matrix $M$ as $\operatorname{sym}(M)$ and its skew-symmetric part as $\operatorname{skw}(M)$. This allows us to state the following necessary conditions.

Corollary 4.2. 1) Let $\left(B_{I I}, \bar{B}_{I I}, C_{I I}, D\right)$ be strictly state passive with $X B_{I I}+$ $B_{I I}^{\top} X=-Q, Q=Q^{\top} \succ 0$, and assume that $A_{I I}\left(P_{I}^{h}\right)^{-1} \bar{B}_{I}$ is right-invertible $(\Rightarrow p \leq$ $m)$. Then $X=\left(-\frac{1}{2} Q+F\right) B_{I I}^{-1}$, for some $F=-F^{\top}$ uniquely defined, and condition c) item 2 in Lemma 4.1 holds only if

$$
\begin{equation*}
\operatorname{sym}\left(\frac{1}{2} Q B_{I I}^{-1} A_{I I}\left(P_{I}^{h}\right)^{-1} \bar{B}_{I}+B_{I}^{\top}\left(P_{I}^{h}\right)^{-\top} C_{I}^{\top}\right)=F \operatorname{skw}\left(B_{I I}^{-1} A_{I I}\left(P_{I}^{h}\right)^{-1} \bar{B}_{I}\right) F, \tag{4.3}
\end{equation*}
$$

for any pair $(Q, F)$, and

$$
\begin{equation*}
\operatorname{Ker}\left(C_{I}\left(P_{I}^{h}\right)^{-1} B_{I}\right)=\{0\} . \tag{4.4}
\end{equation*}
$$

2) Let $\left(B_{I I}, \bar{B}_{I I}, C_{I I}, D\right)$ be passive with $X B_{I I}+B_{I I}^{\top} X=-Q, Q=Q^{\top} \succcurlyeq 0$ and $B_{I I}$ invertible. Then $X=\left(-\frac{1}{2} Q+F\right) B_{I I}^{-1}$, for some $F=-F^{\top}$, and condition a) item 3 in Lemma 4.1 holds only if

$$
\begin{equation*}
\operatorname{sym}\left(-Q B_{I I}^{-1} A_{I I}\left(P_{I}^{h}\right)^{-1} B_{I}\right)+2 F \operatorname{skw}\left(B_{I I}^{-1} A_{I I}\left(P_{I}^{h}\right)^{-1} B_{I}\right) F \preccurlyeq 0 \tag{4.5}
\end{equation*}
$$

for any pair $(Q, F)$.
Proof: 1) Condition c) yields $\left(-\frac{1}{2} Q+F\right) B_{I I}^{-1} A_{I I}\left(P_{I}^{h}\right)^{-1} \bar{B}_{I}=B_{I}^{\top}\left(P_{I}^{h}\right)^{-\top} C_{I}^{\top}$, then (4.3) follows using $F=-F^{\top}$. Using the right-invertibility, (4.4) follows from [23, equ. (70)] and the fact that $X$ is full rank. 2) From passivity any solution must be symmetric. The Lyapunov equation and the skew-symmetry of $F$ yield (4.5). $\boxtimes$
We see that the right-invertibility condition (resp. condition (4.4)) in item 1 of the lemma, characterizes the couplings between $x$ and $\lambda$ in the index reduction (resp. between $x$ and $z$ in the relative degree reduction (i.e., in $\dot{w})$ ).
4.1. Analysis of the OSNSP for (3.2). The next proposition analyses the OSNSP well-posedness, and applies to two main cases: $\tilde{D}=0$, and $\tilde{D} \succ 0$. Let us remind that $p=n_{2}$ as a consequence of Assumptions 2, 3, 4 and 5, so that $\tilde{A}$ and $B_{I I}$ are $p \times p$ matrices.

Proposition 4.3. Let $h>0, k \geq 0$ be given.

1. Let Assumption 4 hold. Suppose that either $\operatorname{Im}(\tilde{C}) \cap \operatorname{int}\left(\operatorname{Im}\left(\partial \sigma_{\Gamma_{k}}+\tilde{D}\right)\right) \neq \emptyset$ or $\tilde{B}$ is square full rank. Suppose also that there is $\zeta_{0}$ such that $\tilde{C} \zeta_{0} \in$ $\operatorname{Im}\left(\partial \sigma_{\Gamma_{k}}+\tilde{D}\right)$. Then the operator $\mathcal{G}: \zeta \mapsto-\tilde{A} \zeta+\tilde{B}\left(\partial \sigma_{\Gamma_{k}}+\tilde{D}\right)^{-1}(\tilde{C} \zeta)$ is maximal monotone with non empty domain.
2. Let $\tilde{D} \succ 0$, then both mappings $\left(\mathcal{N}_{K^{\star}}+\tilde{D}\right)^{-1}(\cdot)$ and $\left(\partial \sigma_{\Gamma_{k}}+\tilde{D}\right)^{-1}(\cdot)$ are well-defined, single-valued Lipschitz continuous and maximal monotone for each $h \geq 0, k \geq 0$ (maximal monotonicity holds for $\tilde{D} \succcurlyeq 0$ ). The operator $\mathcal{G}(\cdot)$ is single-valued and Lipschitz continuous also.
3. Let $\tilde{D}=0$, Assumption 2 hold true, $\tilde{R}^{2}=\tilde{X}, \tilde{R}=\tilde{R}^{\top} \succ 0, \operatorname{Im}(\tilde{C}) \cap \Gamma_{k} \neq \emptyset$ and $K=P(G, 0)$ (see Proposition A.1 e)) for some matrix $G \in \mathbb{R}^{m \times m}$. Let us assume that $\tilde{R} \tilde{A} \tilde{R}^{-1} \preccurlyeq 0$. Let us consider the set:

$$
\begin{align*}
\tilde{\mathcal{S}} \triangleq & P\left(G \tilde{C} \tilde{R}^{-1}, 0\right) \\
& \cap\left\{\xi \in \mathbb{R ^ { p }} \mid \tilde{R} \tilde{A} \tilde{R}^{-1} \xi=-\sum_{i=1}^{m} \lambda_{i}\left(G \tilde{C} \tilde{R}^{-1}\right)_{i \bullet}^{\top}, \lambda_{i} \geq 0\right\}  \tag{4.6}\\
& \cap \operatorname{Ker}\left(\tilde{R} \tilde{A} \tilde{R}^{-1}+\tilde{R}^{-1} \tilde{A}^{\top} \tilde{R}\right)
\end{align*}
$$

Define $\tilde{q}_{k}=-\tilde{R} \tilde{E}_{k}, \tilde{f}(\cdot)=\psi_{\Gamma_{k}} \circ \tilde{C} \tilde{R}^{-1}(\cdot), \Gamma_{k}=K-\tilde{F}_{k}$, and consider the $\operatorname{VI}\left(-\tilde{R} \tilde{A} \tilde{R}^{-1}, \tilde{q}_{k}, \tilde{f}\right):$ Find $\xi_{k+1}=\tilde{R} z_{k+1}$ such that
(4.7) $\left\langle-\tilde{R} \tilde{A} \tilde{R}^{-1} \xi_{k+1}+\tilde{q}_{k}, v-\xi_{k+1}\right\rangle+\tilde{f}(v)-\tilde{f}\left(\xi_{k+1}\right) \geq 0$, for all $v \in \operatorname{dom}(\tilde{f})$.
(a) If $\tilde{\mathcal{S}}=\{0\}$, then the $\operatorname{VI}\left(-\tilde{R} \tilde{A} \tilde{R}^{-1}, \tilde{q}_{k}, \tilde{f}\right)$ has at least one solution.
(b) If $\tilde{\mathcal{S}} \neq\{0\}$, and if there exists $\xi_{0} \in \tilde{\Gamma}_{k}$ such that

$$
\begin{equation*}
\left\langle\tilde{q}_{k}-\left(-\tilde{R} \tilde{A} \tilde{R}^{-1}\right)^{\top} \xi_{0}, v\right\rangle>0 \text { for all } v \in \tilde{\mathcal{S}}, v \neq 0 \tag{4.8}
\end{equation*}
$$

where $\tilde{\Gamma}_{k}=\left\{\xi \in \mathbb{R}^{n_{2}} \mid G \tilde{C} \tilde{R}^{-1} \xi+G \tilde{F}_{k} \geq 0\right\}=\operatorname{dom}(\tilde{f})$, then the $V I\left(-\tilde{R} \tilde{A} \tilde{R}^{-1}, \tilde{q}_{k}, \tilde{f}\right)$ has at least one solution.
(c) If $\xi_{k+1}^{1}$ and $\xi_{k+1}^{2}$ are two solutions of the $\operatorname{VI}\left(-\tilde{R} \tilde{A} \tilde{R}^{-1}, \tilde{q}_{k}, \tilde{f}\right)$, then $\xi_{k+1}^{1}-$ $\xi_{k+1}^{2} \in \operatorname{Ker}\left(\tilde{R} \tilde{A} \tilde{R}^{-1}+\tilde{R}^{-1} \tilde{A}^{\top} \tilde{R}\right)$.
(d) Let $\tilde{R} \tilde{A} \tilde{R}^{-1}$ be symmetric. If $\xi_{k+1}^{1}$ and $\xi_{k+1}^{2}$ are two solutions of the $V I\left(-\tilde{R} \tilde{A} \tilde{R}^{-1}, \tilde{q}_{k}, \tilde{f}\right)$, then $\left\langle\tilde{q}_{k}, \xi_{k+1}^{1}-\xi_{k+1}^{2}\right\rangle=0$.
(e) Let $\tilde{R} \tilde{A} \tilde{R}^{-1}$ be symmetric. Then any solution of the $V I\left(-\tilde{R} \tilde{A} \tilde{R}^{-1}, \tilde{q}_{k}, \tilde{f}\right)$ is also a solution to the problem: $\min _{\xi \in \tilde{\Gamma}_{k}}-\frac{1}{2} \xi^{\top} \tilde{R} \tilde{A} \tilde{R}^{-1} \xi+\left\langle\tilde{q}_{k}, \xi\right\rangle$.
(f) Let $-\tilde{R} \tilde{A} \tilde{R}^{-1} \succ 0$ and be symmetric. Then $\xi_{k+1}=\operatorname{proj}_{-\tilde{R} \tilde{A} \tilde{R}^{-1}}\left[\tilde{\Gamma}_{k} ;-\tilde{q}_{k}\right]$.

## Proof:

1. Follows from [6, Theorem 1], which is itself inspired from [23] and [16, Theorem 2], and which states that the negative feedback interconnection of a passive system with a maximal monotone operator, defines another maximal monotone operator (under the stated basic conditions).
2. Follows from [16, Proposition 1] (see Proposition B.1) and from [55, Lemma 1] (or [9, Corollary 24.4 (ii)]) since $\operatorname{Im}(\tilde{D})=\mathbb{R}^{m}$ and $\operatorname{int}\left(\operatorname{Dom}\left(\mathcal{N}_{K^{\star}}\right)\right)=$ $\operatorname{int}\left(K^{\star}\right) \neq \emptyset$, and $\operatorname{int}\left(\operatorname{Dom}\left(\partial \sigma_{\Gamma_{k}}\right)\right) \neq \emptyset$ because $\Gamma_{k}$ is convex and $\operatorname{int}\left(\Gamma_{k}\right) \neq \emptyset$.
3. The proof follows from Propositions A. 1 and A.2. First notice that the MLCP $\tilde{A} z_{k+1}+\tilde{B} \lambda_{k+1}+\tilde{E}_{k}=0, K^{\star} \ni \lambda_{k+1} \perp \tilde{C} z_{k+1}+\tilde{D} \lambda_{k+1}+\tilde{F}_{k} \in K$, can be equivalently rewritten as the $\operatorname{VI}\left(-\tilde{R} \tilde{A} \tilde{R}^{-1}, \tilde{q}_{k}, \tilde{f}\right)$ in (4.7). Indeed one obtains $-\tilde{R} \tilde{A} \tilde{R}^{-1} \xi_{k+1}-\tilde{R} \tilde{E}_{k} \in-\tilde{R} \tilde{B} \partial \psi_{\Gamma_{k}}\left(\tilde{C} \tilde{R}^{-1} \xi_{k+1}\right)$. Using the chain rule of Convex Analysis [49, Theorem 23.9] and the fact that $\Gamma_{k}$ is polyhedral, the MLCP can written equivalently as in (4.7). Let us now compute the set $\mathcal{S}$ in (4.6). Here $\operatorname{dom}(\varphi)=\operatorname{dom}(\tilde{f})=\left\{z \in \mathbb{R}^{p} \mid \tilde{C} \tilde{R}^{-1} z \in \Gamma_{k}\right\}=$ $\left\{z \in \mathbb{R}^{p} \mid \tilde{C} \tilde{R}^{-1} z+\tilde{F}_{k} \in K\right\}=\left\{z \in \mathbb{R}^{p} \mid G \tilde{C} \tilde{R}^{-1} z+G \tilde{F}_{k} \geq 0\right\}$. Hence $(\operatorname{dom}(\varphi))_{\infty}=P\left(G \tilde{C} \tilde{R}^{-1}, 0\right)$ from Proposition A. 1 e). Now we have $\tilde{f}(\xi)=$ $\psi_{\Gamma_{k}}\left(\tilde{C} \tilde{R}^{-1} \xi\right)=\psi_{\tilde{\Gamma}_{k}}(\xi)$ with $\tilde{\Gamma}_{k}=\left\{\xi \in \mathbb{R}^{p} \mid \tilde{C} \tilde{R}^{-1} \xi \in K-\tilde{F}_{k}\right\}=\{\xi \in$ $\left.\mathbb{R}^{p} \mid G \tilde{C} \tilde{R}^{-1} \xi+G \tilde{F}_{k} \geq 0\right\}=P\left(G \tilde{C} \tilde{R}^{-1},-G \tilde{F}_{k}\right)$. Thus $\tilde{f}_{\infty}(\cdot)=\psi_{\left(\tilde{\Gamma}_{k}\right)_{\infty}}(\cdot)$ and $\operatorname{dom}\left(\tilde{f}_{\infty}\right)=P\left(G \tilde{C} \tilde{R}^{-1}, 0\right)$. Now we have to calculate the dual cone $P\left(G \tilde{C} \tilde{R}^{-1}, 0\right)^{\star}$. Using [31, Example 15], one finds that $P\left(G \tilde{C} \tilde{R}^{-1}, 0\right)^{\star}=\{w \in$ $\left.\mathbb{R}^{n_{2}} \mid w=\sum_{i=1}^{m} \lambda_{i}\left(G \tilde{C} \tilde{R}^{-1}\right)_{i \bullet}^{\top}, \lambda_{i} \geq 0\right\}$, where $\left(G \tilde{C} \tilde{R}^{-1}\right)_{i \bullet}$ denotes the $i$ th row of the matrix $G \tilde{C} \tilde{R}^{-1} \in \mathbb{R}^{m \times n_{2}}$. Hence it follows that $\mathcal{K}\left(-\tilde{R} \tilde{A} \tilde{R}^{-1}, \tilde{f}\right)=$ $\left\{\xi \in \mathbb{R}^{n_{2}} \mid \tilde{R} \tilde{A} \tilde{R}^{-1} \xi=-\sum_{i=1}^{m} \lambda_{i}\left(G \tilde{C} \tilde{R}^{-1}\right)_{i \bullet}^{\top}, \lambda_{i} \geq 0\right\}$. The third set in $\tilde{\mathcal{S}}$ is equal to $\operatorname{Ker}\left(\tilde{R} \tilde{A} \tilde{R}^{-1}+\tilde{R}^{-1} \tilde{A}^{\top} \tilde{R}\right)$, from $\operatorname{Ker}\left(\mathbf{M}+\mathbf{M}^{\top}\right)$ in Proposition A. 2 a). Then the results in items (a) to (f) are a consequence of Proposition A.2, where $\varphi_{\infty}(v)=0$ in item (b). Item (f) follows from item e) in Proposition A. 2 .
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Item 1 encompasses cases with $\tilde{D} \succcurlyeq 0$. If item 2 holds, then $\lambda_{k+1}$ is uniquely defined, provided that $z_{k+1}$ exists (which is guaranteed if conditions of item 1 hold). From item 3 (c), it follows that $z_{k+1}$ is unique if $\tilde{A}$ has full $\operatorname{rank}$ (recall that $B_{I I}$ and $\tilde{A}$ are square $p \times p$ matrices). However from (3.2) and (3.3), $x_{k+1}$ is uniquely defined if and only if $\bar{B}_{I}\left(\mathcal{N}_{K^{\star}}\right)^{-1}\left(-\tilde{C} z_{k+1}-\tilde{F}_{k}\right)$ is unique. This happens trivially when $\bar{B}_{I}=0$ (if one thinks of (2.1) as the limit of a singularly perturbed system, this simply means that the multiplier enters only the $z$-dynamics). Let us now deal with the case $\tilde{D} \succcurlyeq 0$.

Assumption 6. Let $h>0$, and $\tilde{D}=\left(\begin{array}{cc}\tilde{D}_{11} & 0 \\ 0 & 0\end{array}\right)$ with $0 \prec \tilde{D}_{11}=\tilde{D}_{11}^{\top} \in \mathbb{R}^{m_{1} \times m_{1}}$.

Such specific structure occurs for instance when $D$ has the same blockdiagonal structure and $\bar{B}_{I}=0$, i.e., the multiplier does not intervene in the $x$-dynamics. Let us split the multiplier as $\lambda=\left(\lambda_{1}^{\top}, \lambda_{2}^{\top}\right)^{\top}, \lambda_{1} \in \mathbb{R}^{m_{1}}, \lambda_{2} \in \mathbb{R}^{m_{2}}, m_{1}+m_{2}=m$, and $K=K_{1} \times K_{2} \subset \mathbb{R}^{m_{1}} \times \mathbb{R}^{m_{2}}$ (hence $\left.K^{\star}=K_{1}^{\star} \times K_{2}^{\star}\right), \tilde{B}=\left(\begin{array}{cc}\tilde{B}_{1} & \tilde{B}_{2}\end{array}\right), \tilde{B}_{1} \in \mathbb{R}^{p \times m_{1}}$, $\tilde{B}_{2} \in \mathbb{R}^{p \times m_{2}}, \tilde{C}=\binom{\tilde{C}_{1}}{\tilde{C}_{2}}, \tilde{C}_{1} \in \mathbb{R}^{m_{1} \times p}, \tilde{C}_{2} \in \mathbb{R}^{m_{2} \times p}$. If $\tilde{D}_{11}=0$ then $\lambda=\lambda_{2}$, $\tilde{B}=\tilde{B}_{2}, \tilde{C}=\tilde{C}_{2}$. If $\tilde{D}=\tilde{D}_{11} \succ 0$ then item 2 in Proposition 4.3 applies.

Assumption 7. Let $h>0$. There exists a $(p \times p)$-matrix $\tilde{X}_{2}=\tilde{X}_{2}^{\top} \succ 0$ such that $\tilde{X}_{2} \tilde{B}_{2}=\tilde{C}_{2}^{\top}$.

This assumption implies that there are couplings between the multiplier $\lambda_{k+1}$ in the algebraic equation, and the algebraic variable $z_{k+1}$ in the complementarity conditions, see (3.2). Define $\tilde{R}_{2}=\tilde{R}_{2}^{\top} \succ 0, \tilde{R}_{2}^{2}=\tilde{X}_{2}$, and $\xi_{k+1}=\tilde{R}_{2} z_{k+1}$. The next result extends Proposition 4.3 to the positive semidefinite case with blockdiagonal structure.

Proposition 4.4. Let Assumptions 6 and 7 hold true, $h>0$ and $k \geq 0$ be given.

Define $\Gamma_{2, k}=K_{2}-\tilde{F}_{2, k}$. Consider the VI: find $\xi_{k+1} \in \operatorname{dom}\left(\tilde{f}_{2}\right)$ such that (4.9)

$$
-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1} \xi_{k+1}-\tilde{R}_{2} \tilde{E}_{k}+\underbrace{\tilde{R}_{2} \tilde{B}_{1}\left(\tilde{D}_{11}+\partial \psi_{K_{1}^{\star}}\right)^{-1}\left(\tilde{C}_{1} \tilde{R}_{2}^{-1} \xi_{k+1}+\tilde{F}_{1, k}\right)}_{\triangleq \tilde{G}_{2}\left(t_{k}, \xi_{k+1}\right)} \in-\partial \tilde{f}_{2}\left(\xi_{k+1}\right)
$$

where $\tilde{f}_{2}(\cdot)=\psi_{\Gamma_{2, k}} \circ \tilde{C}_{2} \tilde{R}_{2}^{-1}(\cdot)$.

1. Suppose that $\operatorname{Im}\left(\tilde{C}_{2}\right) \cap \Gamma_{2, k} \neq \emptyset$. The MLCP $\tilde{A} z_{k+1}+\tilde{B} \lambda_{k+1}+\tilde{E}_{k}=0$, $K^{\star} \ni \lambda_{k+1} \perp \tilde{C} z_{k+1}+\tilde{D} \lambda_{k+1}+\tilde{F}_{k} \in K$, can be equivalently rewritten as the VI in (4.9).
2. Assume that the set $\tilde{\Gamma}_{2, k}=\left\{\xi \in \mathbb{R}^{p} \mid \tilde{C}_{2} \tilde{R}_{2}^{-1} \xi \in \Gamma_{2, k}\right\}$ is compact. Then the set of solutions to the VI in (4.9) is compact and nonempty.
3. Assume that for each $t_{k}$, the following well-defined, single-valued and Lipschitz continuous mapping, is maximal monotone:

$$
\tilde{G}_{2}: \xi_{k+1} \mapsto \tilde{R}_{2} \tilde{B}_{1}\left(\tilde{D}_{11}+\partial \psi_{K_{1}^{\star}}\right)^{-1}\left(\tilde{C}_{1} \tilde{R}_{2}^{-1} \xi_{k+1}-\tilde{F}_{1, k}\right)
$$

If $-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1} \succcurlyeq 0, \tilde{\Gamma}_{2, k} \cap \operatorname{Int}\left(\operatorname{Dom}\left(\tilde{G}_{2}\right)\right) \neq \emptyset$ or $\operatorname{Int}\left(\tilde{\Gamma}_{2, k}\right) \cap \operatorname{Dom}\left(\tilde{G}_{2}\right) \neq \emptyset$, then the set of solutions to the VI in (4.9) is closed and convex. If $-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1} \succ 0$, the VI in (4.9) has a unique solution.

## Proof:

1. The complementarity conditions in (3.2) (c) are rewritten as:

$$
\begin{equation*}
\binom{K_{1}^{\star}}{K_{2}^{\star}} \ni\binom{\lambda_{1, k+1}}{\lambda_{2, k+1}} \perp\binom{\tilde{C}_{1}}{\tilde{C}_{2}} z_{k+1}+\binom{\tilde{D}_{11} \lambda_{1, k+1}}{0}+\binom{\tilde{F}_{1, k}}{\tilde{F}_{2, k}} \in\binom{K_{1}}{K_{2}} \tag{4.10}
\end{equation*}
$$

while the equality constraint in (3.2) (b) is rewritten as:

$$
\begin{equation*}
\tilde{A} z_{k+1}+\tilde{B}_{1} \lambda_{1, k+1}+\tilde{B}_{2} \lambda_{2, k+1}+\tilde{E}_{k}=0 \tag{4.11}
\end{equation*}
$$

From (4.10) it is deduced that:

$$
\left\{\begin{array}{l}
\lambda_{1, k+1}=-\left(\mathcal{N}_{K_{1}^{\star}}+\tilde{D}_{11}\right)^{-1}\left(\tilde{C}_{1} z_{k+1}+\tilde{F}_{1, k}\right)  \tag{4.12}\\
\lambda_{2, k+1} \in-\mathcal{N}_{K_{2}-\tilde{F}_{2, k}}\left(\tilde{C}_{2} z_{k+1}\right) .
\end{array}\right.
$$

Using (4.11) it is inferred:

$$
\begin{gather*}
\tilde{A} z_{k+1}-\tilde{B}_{1}\left(\mathcal{N}_{K_{1}}+\tilde{D}_{11}\right)^{-1}\left(\tilde{C}_{1} z_{k+1}+\tilde{F}_{1, k}\right) \\
-\tilde{B}_{2} \mathcal{N}_{K_{2}-}-\tilde{F}_{2, k}\left(\tilde{C}_{2} z_{k+1}\right)+\tilde{E}_{k}=0  \tag{4.13}\\
\tilde{\Downarrow} \\
-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1} \xi_{k+1}+\tilde{R}_{2} \tilde{B}_{1}\left(\mathcal{N}_{K_{1}^{\star}}+\tilde{D}_{11}\right)^{-1}\left(\tilde{C}_{1} z_{k+1}+\tilde{F}_{1, k}\right)-\tilde{R}_{2} \tilde{E}_{k} \\
\in-\tilde{R}_{2} \tilde{B}_{2} \mathcal{N}_{K_{2}-\tilde{F}_{2, k}}\left(\tilde{C}_{2} \tilde{R}_{2}^{-1} \xi_{k+1}\right),
\end{gather*}
$$

which is (4.9) since $\tilde{R}_{2} \tilde{B}_{2}=\tilde{R}_{2}^{-1} \tilde{X}_{2} \tilde{B}_{2}=\tilde{R}_{2}^{-1} \tilde{C}_{2}^{\top}$ and applying the chain rule of Convex Analysis for polyhedral functions.
2. This follows from [29, Corollary 2.2.5] and since the map $\left(\tilde{D}_{11}+\partial \psi_{K_{1}^{\star}}\right)^{-1}(\cdot)$ is single-valued, well defined and Lipschitz continuous (see Proposition B.1).
3. The first assertion is a consequence of [9, Corollary 24.4] (for the maximal monotonicity of the mapping $\left(-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1}+\tilde{G}_{2}+\partial \tilde{f}_{2}\right)(\cdot)$, and of $[9$, Proposition 23.39]. The second assertion follows from [29, Theorem 2.3.3], noting that the
mapping in the left-hand side of (4.9) is maximal monotone [9, Corollary 24.4 (ii)] and strongly monotone since $-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1} \succ 0$. This can also be proved using [9, Corollary 23.37]. Single-valuedness and Lipschitz continuity stem from $\tilde{D}_{11} \succ 0$.

Further features of the solutions to the VI in (4.9) will be studied in section 5.1.3. The next lemma states a classical condition that guarantees the maximal monotonicity of Proposition 4.4 item 3. The first condition in item 1 is a constraint qualification which guarantees that the function $\tilde{f}_{2}(\cdot)$ has a nonempty domain. It is usually seen as a technical condition, however in some applications it may not be satisfied. Item 1 states conditions under which the VI in (4.9) can be constructed. Item 3 uses some results about the zeroes of maximal monotone mappings, or of the sum of these.

Lemma 4.5. Let $\tilde{X}_{2} \tilde{B}_{1}=\tilde{C}_{1}^{\top}$ for $\tilde{X}_{2}$ in Assumption 7 , then for each fixed $t_{k}$, the mapping $\tilde{G}_{2}: \xi_{k+1} \mapsto \tilde{R}_{2} \tilde{B}_{1}\left(\tilde{D}_{11}+\partial \psi_{K_{1}^{\star}}\right)^{-1}\left(\tilde{C}_{1} \tilde{R}_{2}^{-1} \xi_{k+1}+\tilde{F}_{1, k}\right)$ is maximal monotone.

Proof: Indeed $\tilde{X}_{2} \tilde{B}_{1}=\tilde{C}_{1}^{\top} \Rightarrow \tilde{R}_{2} \tilde{B}_{1}=\tilde{R}_{2}^{-1} \tilde{C}_{1}^{\top}=\left(\tilde{C}_{1} \tilde{R}_{2}^{-1}\right)^{\top}$. The mapping $\left(\tilde{D}_{11}+\right.$ $\left.\partial \psi_{K_{1}^{\star}}\right)^{-1}(\cdot)$ is maximal monotone, and the following operation: $\left(\tilde{C}_{1} \tilde{R}_{2}^{-1}\right)^{\top}\left(\tilde{D}_{11}+\right.$ $\left.\partial \psi_{K_{1}^{\star}}\right)^{-1}\left(\tilde{C}_{1} \tilde{R}_{2}^{-1} \cdot+\tilde{F}_{1, k}\right)$ preserves the maximal monotonicity [50, Exercise 12.4]. $\boxtimes$

Propositions 4.3 and 4.4 provide some informations about the existence and uniqueness of the unknown $z_{k+1}$, and characterize some mappings. It is noteworthy that other well-posedness results for VIs are available in the literature. For instance if the set $\tilde{\Gamma}_{k}$ is compact, [29, Corollary 2.2.5] guarantees existence of a solution to $\mathrm{VI}\left(-\tilde{R} \tilde{A} \tilde{R}^{-1}, \tilde{q}_{k}, \tilde{f}\right)$ with a compact set of solutions. Also when $\tilde{\Gamma}_{k}=\mathbb{R}_{+}^{m}$ one may rely on P-matrices and LCPs properties, when it is a closed convex cone one may use copositive matrices [29, Theorem 2.5.10].
The next corollary completes the existence and uniqueness results for the OSNSP.
Corollary 4.6. The next results hold true:

1. (set of solutions $z_{k+1}$ ) Let Assumption 4 and the conditions of item 1 in Proposition 4.3 hold. Then the set of solutions $z_{k+1}$ in (3.5) (or the set of solutions $\zeta_{k+1}$ in (3.3)), is closed and convex. If in addition the conditions of item 3 (a) or (b) in Proposition 4.3 hold, then the set of solutions is non empty. If in addition $\tilde{A} \prec 0$, then $z_{k+1}$ is unique.
2. (set of solutions $\left.\lambda_{k+1}\right)$ Let $\tilde{C} \operatorname{Dom}(\mathcal{G}) \subseteq \operatorname{Im}\left(\partial \sigma_{\Gamma_{k}}+\tilde{D}\right)$, and the conditions of item 1 in Proposition 4.3 hold true. Then the set of solutions $\lambda_{k+1}$ in (3.5) (a) is non empty for each $z_{k+1}$.
3. Let Assumption 4 hold with strict passivity, and the conditions of item 1 in Proposition 4.3 be true. Then the GE in (3.5) (or the GE in (3.3)) has a unique solution $\left(\lambda_{k+1}, z_{k+1}\right)$ for any $\tilde{E}_{k}$. Consequently $x_{k+1}$ in (3.2) is unique.
4. Let $\tilde{D}=0$ and $\mathcal{G}(\cdot)$ be maximal monotone. Then if the set $\tilde{\Gamma}_{k}=\{\zeta \in$ $\left.\mathbb{R}^{n_{2}} \mid G \tilde{C} \tilde{R}^{-1} \zeta+G \tilde{F}_{k} \geq 0\right\}$ is bounded, the $G E$ with unknown $z_{k+1}$ in (3.5) (or in (3.3)) has at least one solution and the set of solutions is compact.

Proof:

1. Follows from [9, Proposition 23.39] by the maximal monotonicity of $\mathcal{G}(\cdot)$. If $-\tilde{A} \succ 0$ then $\mathcal{G}(\cdot)$ is strongly monotone (however it may not be Lipschitz continuous and this property is used later in section 5.2 ), and so is $\mathcal{G}(\cdot)+\tilde{E}_{k}$. Hence the generalized equation $\mathcal{G}\left(z_{k+1}\right)+\tilde{E}_{k} \ni 0$ has a unique solution [9, Corollary 23.37].
2. Let us consider $\lambda_{k+1} \in-\left(\partial \sigma_{\Gamma_{k}}+\tilde{D}\right)^{-1}\left(\tilde{C} z_{k+1}\right)$. We know that any solution
$z_{\tilde{D}+1} \in \mathcal{G}^{-1}\left(\tilde{E}_{k}\right) \subseteq \operatorname{Im}\left(\mathcal{G}^{-1}\right)=\operatorname{Dom}(\mathcal{G})$. Since $\operatorname{Dom}\left(\partial \sigma_{\Gamma_{k}}+\tilde{D}\right)^{-1}=\operatorname{Im}\left(\partial \sigma_{\Gamma_{k}}+\right.$ $\tilde{D})$ the result follows.
3. Strict passivity implies that $\tilde{D} \succ 0$ and $\tilde{A}$ is a Hurwitz matrix, therefore from a slight modification of $[6$, Theorem 1] it follows that the operator $\mathcal{G}(\cdot)$ is strongly, hence strictly monotone (indeed there exists a real $\alpha>0$ small enough such that $\left(\tilde{A}+\alpha I_{n_{2}}, \tilde{B}, \tilde{C}, \tilde{D}\right)$ is passive). Using [9, Proposition 23.35] shows that $z_{k+1}$ is unique, and item 2 in Proposition 4.3 shows that $\lambda_{k+1}$ is unique. The conclusion follows.
4. Under these conditions $\operatorname{Dom}(\mathcal{G})$ is bounded (compact). Hence from [9, Proposition 23.36 (iii)] existence of a solution holds true. From [29, Corollary 2.2.5] the compactness of the set of solutions follows. This could also be inferred from item (e) in Proposition 4.3 in case $\tilde{R} \tilde{A} \tilde{R}^{-1}$ is symmetric.

Item 3 in Corollary 4.6 guarantees the uniqueness of solutions to the OSNSP in (3.6), in which case the $\in$ can be replaced by equality. Concerning item 2 in Corollary 4.6, an important step is to characterize both $\operatorname{Dom}(\mathcal{G})$ and $\operatorname{Im}\left(\partial \sigma_{\Gamma_{k}}+\tilde{D}\right)$. If $\tilde{D}=\tilde{D}^{\top} \succcurlyeq 0$, one can apply [15, Theorem 3], and infer that $\operatorname{Im}\left(\partial \sigma_{\Gamma_{k}}+\tilde{D}\right) \simeq \operatorname{Im}\left(\partial \sigma_{\Gamma_{k}}\right)+\operatorname{Im}(\tilde{D})=$ $\operatorname{Dom}\left(\partial \psi_{\Gamma_{k}}\right)+\operatorname{Im}(\tilde{D}) \subset \operatorname{dom}\left(\psi_{\Gamma_{k}}\right)+\operatorname{Im}(\tilde{D})=\Gamma_{k}+\operatorname{Im}(\tilde{D})$ (see [15] for the exact meaning of $\simeq$ in this context). Similar calculations may be done for in item 1 of Proposition 4.3. It is inferred that the basic condition in Proposition 4.3 item 1 , i.e., $\operatorname{Im}(\tilde{C}) \cap \operatorname{int}\left(\operatorname{Im}\left(\partial \sigma_{\Gamma_{k}}+\tilde{D}\right)\right) \neq \emptyset$, or in Corollary 4.6 item 2, i.e., $\tilde{C} \operatorname{Dom}(\mathcal{G}) \subseteq$ $\operatorname{Im}\left(\partial \sigma_{\Gamma_{k}}+\tilde{D}\right)$, always are satisfied when $\tilde{D} \succ 0$. For the sake of briefness the conditions for the existence and uniqueness of $\lambda_{k+1}$ in (4.12) are not stated, similar tools as for Corollary 4.6 can be used.
4.2. Analysis of the OSNSP for (3.1). It is also interesting to investigate the case when Assumption 3 (resp. Assumption 5) is satisfied instead of Assumption 2 (resp. Assumption 4). The operator $\mathcal{G}(\cdot)$ is now defined as $\mathcal{G}: \zeta \mapsto-B_{I I} \zeta+$ $\left(\partial \sigma_{\Gamma\left(t_{k}, x_{k+1}\right)}+D\right)^{-1}\left(C_{I I} \zeta\right)$, where $\Gamma\left(t_{k}, x_{k+1}\right)=\left\{\xi \in \mathbb{R}^{m} \mid \xi+C_{I} x_{k+1}+F_{k} \in\right.$ $K$ \}. Consequently the results of Proposition 4.3 item 1, Corollary 4.6 item 3, with Assumption 5 plus strict passivity, and Proposition 4.3 item 2 when $D \succ 0$, hold. For the sake of briefness we do not develop all the arguments once again.
Let us focus now on the case $D \succcurlyeq 0$. From Assumption $3, D=0$ and (2.1) one infers:

$$
\begin{equation*}
-R B_{I I} R^{-1} \xi(t)-R A_{I I} x(t)-R E_{I I}(t) \in-\partial \psi_{\Phi(t, x(t))}(\xi(t)) \tag{4.14}
\end{equation*}
$$

where $\xi=-R z, R^{2}=X, R=R^{\top} \succ 0, \Phi(t, x)=\left\{\xi \in \mathbb{R}^{p} \mid C_{I I} R^{-1} \xi+C_{I} x+F(t) \in\right.$ $K\}$, and the set-valued term is obtained using the chain rule, under the Convex Analysis chain rule qualification condition [49, Theorem 23.9]. The first step is to analyse the existence and uniqueness of solutions to the VI :

$$
\begin{equation*}
-R B_{I I} R^{-1} \xi_{k+1}-R A_{I I} x_{k+1}-R E_{I I, k} \in-\partial \psi_{\Phi\left(t_{k}, x_{k+1}\right)}\left(\xi_{k+1}\right) \tag{4.15}
\end{equation*}
$$

which is the counterpart of (4.7), relying on Proposition A.2. The unknown is $\xi_{k+1}$ and the other terms and variables are considered as data for this VI. Similar results to those in item 3 of Proposition 4.3 can be obtained.

Proposition 4.7. Let $D=0$, Assumption 3 be true, $h>0$ be given, assume that $\operatorname{Im}\left(C_{I I}\right) \cap\left(K-C_{I} x_{k+1}+F_{k}\right) \neq \emptyset$ and that $K$ is polyhedral. Consider the discrete-time system in (3.1), and the associated $V I\left(-R B_{I I} R^{-1},-R A_{I I} x_{k+1}-R E_{I I, k}, \psi_{\Phi\left(t_{k}, x_{k+1}\right)}\right)$
in (4.15). Assume that $-R B_{I I} R^{-1} \succcurlyeq 0$. Consider the set:

$$
\begin{equation*}
\mathcal{S} \triangleq P\left(G C_{I I} R^{-1}, 0\right) \cap\left\{\xi \in \mathbb{R}^{n_{2}} \mid R B_{I I} R^{-1} \xi=\sum_{i=1}^{m} \lambda_{i}\left(G C_{I I} R^{-1}\right)_{i \bullet}^{\top}, \lambda_{i} \geq 0\right\} \tag{4.16}
\end{equation*}
$$

$\cap \operatorname{Ker}\left(R B_{I I} R^{-1}+R^{-1} B_{I I}^{\top} R\right)$.
Then:

1. If $\mathcal{S}=\{0\}$ the $V I\left(-R B_{I I} R^{-1},-R A_{I I} x_{k+1}-R E_{I I, k}, \psi_{\Phi\left(t_{k}, x_{k+1}\right)}\right)$ has a unique solution.
2. If $\mathcal{S} \neq\{0\}$, and if there exists $\xi_{0} \in \Phi\left(t_{k}, x_{k+1}\right)$ such that $\left\langle-R A_{I I} x_{k+1}-\right.$ $\left.R E_{I I, k}+R B_{I I} R^{-1} \xi_{0}, v\right\rangle>0$ for all $v \neq 0$ and $v \in \mathcal{S}$, then it follows that the $V I\left(-R B_{I I} R^{-1},-R A_{I I} x_{k+1}-\tilde{R} E_{I I, k}, \psi_{\Phi\left(t_{k}, x_{k+1}\right)}\right)$ has a unique solution.
Proof: The set $\mathcal{S}$ in (4.16) is constructed as follows. The chain rule holds from [49, Theorem 23.9] and the fact that $\psi_{K-C_{I} x_{k+1}+F_{k}}(\cdot)$ is polyhedral. Hence the VI in (4.15) is equivalent to (3.1) (b) (c). Here $\varphi=\psi_{\Phi\left(t_{k}, x_{k+1}\right)}$, hence $\operatorname{dom}(\varphi)=$ $\Phi\left(t_{k}, x_{k+1}\right)$ and $(\operatorname{dom}(\varphi))_{\infty}$. Assume that $K=P(G, 0)$ (see Proposition A. 1 (e)), then $(\operatorname{dom}(\varphi))_{\infty}=\left\{\xi \in \mathbb{R}^{p} \mid G C_{I I} R^{-1} \xi \geq 0\right\}=P\left(G C_{I I} R^{-1}, 0\right)$. Further, $\varphi_{\infty}=$ $\psi_{\left(\Phi\left(t_{k}, x_{k+1}\right)\right)_{\infty}}=\psi_{P\left(G C_{I I} R^{-1}, 0\right)}$, hence $\operatorname{dom}\left(\varphi_{\infty}\right)=P\left(G C_{I I} R^{-1}, 0\right)$ and $\operatorname{dom}\left(\varphi_{\infty}\right)^{\star}=$ $\left(P\left(G C_{I I} R^{-1}, 0\right)\right)^{\star}=\left\{w \in \mathbb{R}^{n_{2}} \mid w=-\sum_{i=1}^{m} \lambda_{i}\left(G C_{I I} R^{-1}\right)_{i \bullet}^{\top}, \lambda_{i} \geq 0\right\}$, where [31, Example 15] was used as in the proof of Proposition 4.3. Thus $\mathcal{K}\left(-R B_{I I} R^{-1}, \varphi\right)=$ $\left\{\xi \in \mathbb{R}^{n_{2}} \mid-R B_{I I} R^{-1} \xi=-\sum_{i=1}^{m} \lambda_{i}\left(G C_{I I} R^{-1}\right)_{i \bullet}^{\top}, \lambda_{i} \geq 0\right\}$. Finally in Proposition A.2: $\operatorname{Ker}\left(\mathbf{M}+\mathbf{M}^{\top}\right)=\operatorname{Ker}\left(R B_{I I} R^{-1}+\left(R B_{I I} R^{-1}\right)^{\top}\right)$ and $R=R^{\top}$. Items 1 and 2 are a direct consequence of Proposition A. 2 (a) and (b). $\boxtimes$
For the sake of briefness we do not state similar results as in Proposition 4.3 (c) to (f). To enlarge the analysis let us now state the counterpart of Assumption 6:

$$
\text { Assumption 8. Let } D=\left(\begin{array}{cc}
D_{11} & 0 \\
0 & 0
\end{array}\right) \text { with } 0 \prec D_{11}=D_{11}^{\top} \in \mathbb{R}^{m_{1} \times m_{1}} \text {. }
$$

Similarly to Assumption 7, with the same splitting of the multiplier $\lambda$, the matrices $B_{I I}$ and $C_{I I}$, and of the cone $K$ (see the comments after Assumption 6, in particular if $D_{11}=0$ the case $D=0$ is recovered with $G_{2}(\cdot, \cdot)$ in (4.17) which vanishes, and the set $\Phi_{2}=\Phi$ ), let us state the following:

Assumption 9. There exists a $(p \times p)$-matrix $X_{2}=X_{2}^{\top} \succ 0$ such that $X_{2} \bar{B}_{I I}^{2}=$ $\left(C_{I I}^{2}\right)^{\top}$.
Let us recall that $p$ is the number of equality constraints in (2.1). Let us define $R_{2}=R_{2}^{\top} \succ 0, R_{2}^{2}=X_{2}, \xi_{k+1}=R_{2} z_{k+1}, \Phi_{2}\left(t_{k}, x_{k+1}\right)=\left\{\xi \in \mathbb{R}^{p} \mid C_{I I}^{2} R_{2}^{-1} \xi+\right.$ $\left.C_{I}^{2} x_{k+1}+F_{2, k} \in K_{2}\right\}$. The next proposition is the counterpart of Proposition 4.4. It states conditions that guarantee that $z_{k+1}$ can be calculated from (3.1) (b) (c).

Proposition 4.8. Let $h>0$, and Assumptions 8 and 9 hold true. Consider the VI: find $\xi_{k+1} \in \operatorname{dom}\left(f_{2}\right)$ such that

$$
\begin{align*}
& -R_{2} B_{I I} R_{2}^{-1} \xi_{k+1}-R_{2} A_{I I} x_{k+1}-R_{2} E_{I I, k} \\
& +\underbrace{R_{2} \bar{B}_{I I}^{1}\left(D_{11}+\partial \psi_{K_{1}^{\star}}\right)^{-1}\left(C_{I I}^{1} R_{2}^{-1} \xi_{k+1}-F_{1, k}\right)}_{\triangleq_{G_{2}\left(t_{k}, \xi_{k+1}\right)}} \in-\partial f_{2}\left(\xi_{k+1}\right) \tag{4.17}
\end{align*}
$$

with $f_{2}(\cdot)=\left(\psi_{K_{2}-C_{I}^{2} x_{k+1}-F_{2, k}} \circ C_{I I}^{2} R_{2}^{-1}\right)(\cdot)=\psi_{\Phi_{2}\left(t_{k}, x_{k+1}\right)}(\cdot)$.

1. Assume that $\operatorname{Im}\left(C_{I I}^{2}\right) \cap\left(K_{2}-C_{I}^{2} x_{k+1}-F_{2, k}\right) \neq \emptyset$, then the VI in (4.17) is equivalent to the MLCP in (3.1) (b) (c).
2. Suppose that the set $\Phi_{2}\left(t_{k}, x_{k+1}\right)$ is compact and nonempty, then the set of solutions to the VI in (4.17) is compact and nonempty.
3. Assume that for each fixed $t_{k}$, the mapping $G_{2}(\cdot)$ is maximal monotone, and that $-R_{2} B_{I I} R_{2}^{-1} \succ 0$, then the VI in (4.17) has a unique solution.
4. Assume that $-R_{2} B_{I I} R_{2}^{-1} \succcurlyeq 0, G_{2}(\cdot)$ is maximal monotone for each fixed $t_{k}$, and $\Phi_{2}\left(t_{k}, x_{k+1}\right) \cap \operatorname{Int}\left(\operatorname{Dom}\left(G_{2}\right)\right) \neq \emptyset$ or $\operatorname{Int}\left(\Phi_{2}\left(t_{k}, x_{k+1}\right)\right) \cap \operatorname{Dom}\left(G_{2}\right) \neq \emptyset$. Then the set of solutions to the VI in (4.17) is closed and convex.

Proof: The proof of the first three items follows closely that of Proposition 4.4 and it is therefore omitted. Item 4: the VI is rewritten as $\underbrace{\left(-R_{2} B_{I I} R_{2}^{-1}+G_{2}+\partial f_{2}\right)}_{\triangleq_{H_{2}}}\left(\xi_{k+1}\right)-$
$R_{2} A_{I I} x_{k+1}-R_{2} E_{I I, k} \ni 0$, and the assumption guarantees that $H_{2}(\cdot)$ is maximal monotone [9, Corollary 24.4]. The result follows from [9, Proposition 23.39]. $\boxtimes$
The same as Lemma 4.5 can be stated to guarantee the maximal monotonicity in items 3 and 4. It is noteworthy that the main discrepancy between Proposition 4.4 and Proposition 4.8, is that the solution $\xi_{k+1}$ depends on $x_{k+1}$ in the latter, while it depends on $x_{k}$ in the former. This is a difficulty with the application of Proposition 4.8 in some cases, for instance when $C_{I I}=0$ (examples in sections 6.2 and 6.4 will demonstrate that this can occur). Actually, the condition $\operatorname{Im}\left(C_{I I}^{2}\right) \cap\left(K_{2}-\right.$ $\left.C_{I}^{2} x_{k+1}-F_{2, k}\right) \neq \emptyset$ (that is a form of constraint qualification), or its counterparts in Propositions 4.3, 4.4, 4.7, happens to be crucial in the setting of this work. It states that the algebraic variable $z$ has to enter the complementarity constraints through a $C_{I I}$ matrix which has a big enough range. This may occur in some systems, see section 6.5.
5. Convergence results. In this section it is assumed that the conditions for existence of solutions in Propositions 4.3, 4.4, 4.7, or 4.8, are fulfilled, so that sequences can be calculated. The section is divided into two main parts: in sections 5.1.1, 5.1.2 and 5.1.3, convergence is analysed from the properties of the quadruple $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ which characterize the MLCP in (3.2) (b) (c). In section 5.2 , the analysis is led with the properties of $\left(B_{I I}, \bar{B}_{I I}, C_{I I}, D\right)$ which characterize the MLCP in (3.1) (b) (c). The relationships between both approaches are explained.
5.1. Convergence results from (3.2). In this section the numerical scheme in (3.2) and the results in section 4.1 are used. First let us notice that $z_{k+1}$ and $\lambda_{k+1}$ are solutions of the generalized equation in (3.5) (a) (b). Thus if existence of solutions hold, they both are functions of $\tilde{E}_{k}$ (in addition to being functions of $\tilde{F}_{k}$ through the set $\Gamma_{k}$ ). Under some conditions which are recalled below, like item 2 of Proposition 4.3 , they can be single-valued Lipschitz continuous functions of $\tilde{E}_{k}$. Then the right-hand side of the first line in (3.2) may be written compactly as

$$
\begin{equation*}
P_{I}^{h} x_{k+1}=P_{I} x_{k}+h \bar{E}_{k}\left(t_{k}, x_{k}\right) \tag{5.1}
\end{equation*}
$$

for some $\bar{E}_{k}\left(t_{k}, x_{k}\right)$ which is a linear function of $E_{I, k}, E_{I I, k}, \lambda_{k+1}$ and $z_{k+1}$. The time-dependence of $\bar{E}_{k}$ comes from $E_{I, k}, E_{I I, k}$ and $F_{k}$, while the state dependence comes from the third terms in $\tilde{E}_{k}$ and in $\tilde{F}_{k}$ (which enters $\Gamma_{k}\left(t_{k}, x_{k}\right)=K-\tilde{F}_{k}$ ). From (5.1) it can be calculated and checked by induction that:

$$
\begin{equation*}
x_{k}=\left(\left(P_{I}^{h}\right)^{-1} P_{I}\right)^{k} x_{0}+h \sum_{j=0}^{k-1}\left(\left(P_{I}^{h}\right)^{-1} P_{I}\right)^{k-1-j}\left(P_{I}^{h}\right)^{-1} \bar{E}_{j}\left(t_{j}, x_{j}\right), \quad k \geq 1 \tag{5.2}
\end{equation*}
$$

Since $P_{I}^{h}=P_{I}-h A_{I}$, it follows that for $h>0$ small enough, $\left(P_{I}^{h}\right)^{-1} P_{I}=I_{p}+\mathcal{O}(h)$ [10, Proposition 9.4.13] hence $\left(P_{I}^{h}\right)^{-1}=P_{I}^{-1}+\mathcal{O}(h)$. Thus

$$
\begin{align*}
x_{k} & =x_{0}+\mathcal{O}(h)+h \sum_{j=0}^{k-1}\left(I_{p}+\mathcal{O}(h)\right)^{k-1-j}\left(P_{I}^{-1}+\mathcal{O}(h)\right) \bar{E}_{j}\left(t_{j}, x_{j}\right) \\
& =x_{0}+\mathcal{O}(h)+h \sum_{j=0}^{k-1}\left(I_{p}+\mathcal{O}(h)\right)\left(P_{I}^{-1}+\mathcal{O}(h)\right) \bar{E}_{j}\left(t_{j}, x_{j}\right)  \tag{5.3}\\
& =x_{0}+\mathcal{O}(h)+h \sum_{j=0}^{k-1}\left(P_{I}^{-1}+\mathcal{O}(h)\right) \bar{E}_{j}\left(t_{j}, x_{j}\right) \\
& =x_{0}+\mathcal{O}(h)+h\left(P_{I}^{-1}+\mathcal{O}(h)\right) \sum_{j=0}^{k-1} \bar{E}_{j}\left(t_{j}, x_{j}\right),
\end{align*}
$$

for all $k \geq 1$. It is inferred that:

$$
\begin{align*}
\dot{x}_{k} \triangleq \frac{x_{k}-x_{k-1}}{h} & =\frac{\mathcal{O}(h)}{h}+\left(P_{I}^{-1}+\mathcal{O}(h)\right) \bar{E}\left(t_{k-1}, x_{k-1}\right)  \tag{5.4}\\
& =\mathcal{O}(1)+\left(P_{I}^{-1}+\mathcal{O}(h)\right) \bar{E}\left(t_{k-1}, x_{k-1}\right)
\end{align*}
$$

Our aim is now to prove that both $\left\{x_{k}\right\}$ and $\left\{\dot{x}_{k}\right\}$ are uniformly bounded sequences, which will allow us to conclude about the convergence of the sequence $\left\{x^{h}(\cdot)\right\}$ (see the definition in (5.7)). For this it is necessary and sufficient that $\bar{E}_{k}\left(t_{k}, x_{k}\right)$ be uniformly bounded for all $k \geq 0$, since $P_{I}$ is full rank. It follows from (3.5) (a) and (b), that there are two operators which are crucial to characterize the right-hand side in (3.6).
5.1.1. Analysis of the operator $\mathcal{G}(\cdot)$. Notice that the mapping $\mathcal{Z}: \zeta \mapsto \eta=$ $\left(\partial \sigma_{\Gamma_{k}}+\tilde{D}\right)^{-1}(\zeta)$ is under the conditions of Proposition 4.3 item 2, a well-defined $\mathbb{R}^{m} \rightarrow \mathbb{R}^{m}$ single-valued Lipschitz continuous mapping which is parameterized by $t_{k}$ and $x_{k}$ through the set $\Gamma_{k}\left(t_{k}, x_{k}\right)$. Assume that $x_{k}$ is bounded, then redoing the proof of Proposition B.1, it follows that the Lipschitz constant of $\mathcal{Z}(\cdot)$ does not depend on $\partial \sigma_{\Gamma_{k}}(\cdot)$ but depends only on $\tilde{D}$, being equal to $\frac{2}{\lambda_{\min }\left(\tilde{D}+\tilde{D}^{\top}\right)}$. In turn $\lambda_{\min }\left(\tilde{D}+\tilde{D}^{\top}\right)>0$ uniformly in $h>0$ only if $D \succ 0$. Thus the operator $\mathcal{G}$ : $\zeta \mapsto-\tilde{A} \zeta+\tilde{B}\left(\partial \sigma_{\Gamma_{k}}+\tilde{D}\right)^{-1}(\tilde{C} \zeta)$ is also single-valued Lipschitz continuous (being the composition of Lipschitz continuous mappings) with a Lipschitz constant that does not depend on $x_{k}$, and which is bounded for bounded $h$. Let us now assume the following which reinforces Assumption 4 and guarantees that $\tilde{D} \succ 0$ :

Assumption 10. The system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is strictly passive, uniformly in $h \geq 0$.

Then if the basic conditions stated in item 1 of Proposition 4.3 are verified in addition to Assumption 10, the operator $\mathcal{G}(\cdot)$ is strongly monotone for some constant $\epsilon>0$, and we know from Corollary 4.6 item 3, that $x_{k+1}$ is unique. As stated in Proposition 4.3 item 1, maximal monotonicity follows from [6, Theorem 1]. Strong monotonicity follows from the fact that under Assumption 10, the quadruple $\left(\tilde{A}+\epsilon I_{n_{2}}, \tilde{B}, \tilde{C}, \tilde{D}\right)$ is passive for a small enough $\epsilon>0$, so that $\left(\mathcal{G}-\epsilon I_{n_{2}}\right)(\cdot)$ is maximal monotone for such $\epsilon$. Thus if Assumption 10 and the basic conditions in item 1 of Proposition 4.3 are true, $\mathcal{G}: \mathbb{R}^{p} \rightarrow \mathbb{R}^{p}$ is well-defined and single-valued, Lipschitz continuous and strongly monotone. Consequently it is inferred from [50, Proposition 12.54] that under the above set of conditions, $\mathcal{G}^{-1}(\cdot)$ is single-valued, well defined and globally Lipschitz, and the solution $z_{k+1}$ of the generalized equation $\tilde{E}_{k}\left(t_{k}, x_{k}\right) \in \mathcal{G}\left(z_{k+1}\right) \Leftrightarrow$ $z_{k+1}=\mathcal{G}^{-1}\left(\tilde{E}_{k}\left(t_{k}, x_{k}\right)\right)$ in (3.5) is Lipschitz continuous in $x_{k}$ and $t_{k}$. Thus $\lambda_{k+1}$ is also Lipschitz continuous in $x_{k}$, and we conclude that the same applies to $x_{k+1}$ in (3.2).

Remark 5.1. Assumption 10 is stringent. Actually the strict state passivity of $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is sufficient to guarantee that $\mathcal{G}(\cdot)$ is strongly monotone. However in the body of sections 5.1.1 and 5.1.2 we need also that $\lambda_{k+1}$ be a Lipschitz continuous
function of its argument, a sufficient condition being that $\tilde{D} \succ 0$, which in turn implies that strict state passivity and strict passivity are identical. Thus strict passivity could be relaxed to strict state passivity if it is assured that $\bar{B}_{I}\left(\partial \sigma_{\Gamma_{k}}+\tilde{D}\right)^{-1}(\tilde{C} \cdot)$ is Lipschitz continuous. This may occur in certain cases (the trivial case is $\bar{B}_{I}=0$ ).
5.1.2. Boundedness and convergence analysis. Under the above conditions, the term $\bar{E}_{k}\left(t_{k}, x_{k}\right)$ and consequently $x_{k+1}\left(t_{k}, x_{k}\right)$ in (3.2) and $\dot{x}_{k}\left(t_{k}, x_{k}\right)$ in (5.4) are single-valued and Lipschitz continuous in $x_{k}$ and $t_{k}$ (provided $E_{I}(\cdot), E_{I I}(\cdot)$ and $F(\cdot)$ are Lipschitz continuous functions of $t$, which is a mild requirement on the data), since $\tilde{E}_{k}$ and $\tilde{F}_{k}$ are linear in the argument $x_{k}$. We have $\left\|\bar{E}\left(t_{k}^{1}, x_{k}^{1}\right)-\bar{E}\left(t_{k}^{2}, x_{k}^{2}\right)\right\| \leq$ $L_{t}\left|t_{k}^{1}-t_{k}^{2}\right|+L_{x}| | x_{k}^{1}-x_{k}^{2} \|$ for any $t_{k}^{1}, t_{k}^{2}, x_{k}^{1}, x_{k}^{2}$, and for some $L_{t}>0$ and $L_{x}>0$. Thus $\left\|\bar{E}\left(t_{k}, x_{k}^{1}\right)-\bar{E}\left(t_{k}, x_{k}^{2}\right)\right\| \leq L_{x}\left\|x_{k}^{1}-x_{k}^{2}\right\|$ and $\left\|\bar{E}\left(t_{k}, x_{k}\right)-\bar{E}\left(t_{k}, 0\right)\right\| \leq L_{x}\left\|x_{k}\right\|$. Notice that due to the Lipschitz continuity both $z_{k+1}$ and $\lambda_{k+1}$ are uniformly bounded in time for fixed $x_{k}$ (provided $E_{I}(\cdot), E_{I I}(\cdot)$ and $F(\cdot)$ are uniformly bounded, which is a mild requirement on the data). Therefore $\bar{E}\left(t_{k}, 0\right)$ is uniformly bounded also, and $\left\|\bar{E}\left(t_{k}, x_{k}\right)\right\| \leq\left\|\bar{E}\left(t_{k}, 0\right)\right\|+L_{x}\left\|x_{k}\right\| \leq m+L_{x}\left\|x_{k}\right\|$. Now using (5.2) we get:

$$
\begin{equation*}
\left\|x_{k}\right\| \leq \sigma_{\max }^{k}\left(\left(P_{I}^{h}\right)^{-1} P_{I}\right)\left\|x_{0}\right\|+h \sigma_{\min }\left(P_{I}^{h}\right) \sum_{j=1}^{k-1} \sigma_{\max }^{k-j-1}\left(P_{I}\left(P_{I}^{h}\right)^{-1}\right)\left(m+L_{x}\left\|x_{j-1}\right\|\right) \tag{5.5}
\end{equation*}
$$

where we used the Hölder induced matrix norm $\|\cdot\|_{2,2}=\sigma_{\max }(\cdot)$ and $[10$, Proposition 9.4.9, Corollary 9.4.12]. Let us make the following:

Assumption 11. One has $\sigma_{\max }\left(\left(P_{I}^{h}\right)^{-1} P_{I}\right) \leq 1$ for all $h \geq 0$.
Assumption 11 is a technical assumption which is satisfied if for instance the $n_{1} \times n_{1}$ matrix $A_{I} \prec 0$ and $P_{I} \succ 0$. Indeed this implies that $P_{I}^{h} \succ P_{I}$, thus $\left(P_{I}^{h}\right)^{-1} P_{I} \prec I_{n_{1}}$. Notice that under Assumption 11 one no longer needs $h>0$ small enough to obtain the developments in (5.4), because the spectral radius of $\left(P_{I}^{h}\right)^{-1} P_{I}$ is smaller than one. It is also noteworthy that changing $A_{I} x_{k+1}$ to $A_{I} x_{k}$ in the right-hand side of (3.1) (a), allows one to dispense with Assumption 11, with a slight modification of the matrices $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E}_{k}$ and $\tilde{F}_{k}$ (where $\left(P_{I}^{h}\right)^{-1}$ is replaced by $P_{I}^{-1}$ in all of them). Consequently from (5.5) one infers that:

$$
\begin{equation*}
\left\|x_{k}\right\| \leq\left\|x_{0}\right\|+h \sigma_{\min }\left(P_{I}^{h}\right) \sum_{j=1}^{k}\left(m+L_{x}\left\|x_{j-1}\right\|\right) \tag{5.6}
\end{equation*}
$$

Assume now that $x_{0}, x_{1}, \ldots, x_{k-1}$ are uniformly bounded by some $x_{\text {max }}$, then from (5.6) one has $\left\|x_{k}\right\| \leq x_{\max }+h \sigma_{\min }\left(P_{I}^{h}\right) k\left(m+L_{x} x_{\max }\right) \leq x_{\max }+h \sigma_{\min }\left(P_{I}^{h}\right) n\left(m+L_{x} x_{\max }\right)=$ $x_{\max }+h \sigma_{\min }\left(P_{I}^{h}\right) \frac{T}{h}\left(m+L_{x} x_{\max }\right)=x_{\max }+\sigma_{\min }\left(P_{I}^{h}\right) T\left(m+L_{x} x_{\max }\right)$. Therefore $x_{k}$ is also uniformly bounded (its upperbound depends only on $T$ which is fixed), and we conclude that all $x_{k}, 1 \leq k \leq n$, are bounded provided $x_{0}$ is bounded. Then it is inferred from (5.4) that $\dot{x}_{k}$ is also uniformly bounded. Let us define the sequences of piecewise-linear approximations $\left\{x^{h}(\cdot)\right\}$ and $\left\{\dot{x}^{h}(\cdot)\right\}$ as follows:

$$
\left\{\begin{array}{l}
x^{h}(t)=x_{k+1}+\frac{t_{k+1}-t}{h}\left(x_{k}-x_{k+1}\right) \quad t \in\left[t_{k}, t_{k+1}\right) .  \tag{5.7}\\
\dot{x}^{h}(t)=\frac{x_{k+1}-x_{k}}{h}
\end{array}\right.
$$

We have shown that $x^{h}(\cdot)$ and $\dot{x}^{h}(\cdot)$ are uniformly bounded on $[0, T]$ and for any $h>0$, hence $x^{h}(\cdot)$ has a Lipschitz constant that does not depend on $h$, consequently the family $\left\{x^{h}\right\}_{h>0}$ is equi-Lipschitz continuous. Thus from Arzela-Ascoli Theorem
$\left\{x^{h}(\cdot)\right\}$ stays in a compact subset of $\mathcal{C}^{0}\left([0, T] ; \mathbb{R}^{n_{1}}\right)$ and there exist subsequences which converge uniformly in $\mathcal{C}^{0}\left([0, T] ; \mathbb{R}^{n_{1}}\right)$ towards a Lipschitz continuous limit $x(\cdot)$. Let $\tilde{E}^{h}(\cdot)$ and $\tilde{F}^{h}(\cdot)$ be defined in a similar way as $x^{h}(\cdot)$ in (5.7). Due to the Lipschitz continuity of $E_{I}(\cdot), E_{I I}(\cdot)$, and $F(\cdot)$, the sequences of functions $\left\{\tilde{E}^{h}(\cdot)\right\}_{h \geq 0}$ and $\left\{\tilde{F}^{h}(\cdot)\right\}_{h \geq 0}$ also converge uniformy in $\mathcal{C}^{0}\left([0, T] ; \mathbb{R}^{p}\right)$ and $\mathcal{C}^{0}\left([0, T] ; \mathbb{R}^{m}\right)$ to limits $\tilde{E}(\cdot)=E_{I I}(\cdot)+A_{I I} x(\cdot)$ and $\tilde{F}(\cdot)=F(\cdot)+C_{I} x(\cdot)$ as $h \rightarrow 0$, respectively. From (5.4) it is deduced that $\dot{x}^{h}(\cdot)$ is uniformly bounded. Therefore from the Banach-Alaoglu Theorem, it follows that $\dot{x}^{h}(\cdot)$ converges weakly ${ }^{\star}$ to a limit $v(\cdot)$. Since the domain $[0, T]$ is bounded, one has $\mathcal{L}^{2}\left([0, T] ; \mathbb{R}^{n_{1}}\right) \subseteq \mathcal{L}^{1}\left([0, T] ; \mathbb{R}^{n_{1}}\right)[36$, Corollary 1, p.80], and consequently $\dot{x}^{h}(\cdot)$ converges weakly in $\mathcal{L}^{2}\left([0, T] ; \mathbb{R}^{n_{1}}\right)$. Since $x^{h}(t)=x^{h}(0)+\int_{0}^{t} \dot{x}^{h}(\tau) d \tau$, we infer that $v(\cdot)=\dot{x}(\cdot)$ almost everywhere.
Now notice that using (3.2) the following holds:

$$
\begin{align*}
& \text { (a) } P_{I} \dot{x}^{h}(t)=A_{I} x_{h}^{\star}(t)+B_{I} z_{h}^{\star}(t)+\bar{B}_{I} \lambda_{h}^{\star}(t)+E_{I}^{h}(t) \\
& \text { (b) } \tilde{A} z_{h}^{\star}(t)+\tilde{B} \lambda_{h}^{\star}(t)+\tilde{E}_{I I}^{h}(t)+A_{I I} x^{h}(t)=0  \tag{5.8}\\
& \text { (c) } 0 \leq \lambda_{h}^{*}(t) \perp \tilde{C} z_{h}^{\star}(t)+\tilde{D} \lambda_{h}^{\star}(t)+F^{h}(t)+C_{I} x^{h}(t) \geq 0,
\end{align*}
$$

for all $t \in\left[t_{k}, t_{k+1}\right), k \geq 0, x_{h}^{\star}(t)=x_{k+1}$ for all $t \in\left[t_{k}, t_{k+1}\right)$ and similarly for the other terms are step-functions. From the boundedness of $\left\|\dot{x}_{k}\right\| \triangleq\left\|\frac{x_{k+1}-x_{x}}{h}\right\| \leq C$ for some $C>0$, we infer that $x_{\star}^{h}(\cdot)$ converges strongly in $\mathcal{L}^{2}\left([0, T] ; \mathbb{R}^{n_{1}}\right)$ towards $x(\cdot)$. Indeed:

$$
\begin{align*}
\left\|x^{h}-x_{\star}^{h}\right\|_{\mathcal{L}_{2}\left([0, T] ; \mathbb{R}^{n_{1}}\right)}^{2} & \leq \sum_{k=0}^{n-1} \int_{t_{k}}^{t_{k+1}}\left(t_{k+1}-t_{k}\right)^{2}\left\|\dot{x}_{k}\right\|^{2} d t  \tag{5.9}\\
& \leq C^{2} \sum_{k=0}^{n-1}\left(t_{k+1}-t_{k}\right)^{3}=\frac{C^{2} n h^{3}}{3}=\frac{C^{2} T h^{2}}{3} \longrightarrow 0 \text { as } h \rightarrow 0 .
\end{align*}
$$

To conclude one notices that $\left\|x^{h}-x_{\star}^{h}\right\|_{\mathcal{L}_{2}\left([0, T] ; \mathbb{R}^{n_{1}}\right)}^{2}=\left\|x^{h} \pm x-x_{\star}^{h}\right\|_{\mathcal{L}_{2}\left([0, T] ; \mathbb{R}^{n_{1}}\right)}^{2}$. Now we have $z_{\star}^{h}(t) \in \mathcal{G}^{-1}\left(E_{I I, \star}^{h}(t)+A_{I I} x_{\star}^{h}(t)\right)$ for all $t \in\left[t_{k}, t_{k+1}\right), k \geq 0$. Since $E_{I}(\cdot)$, $E_{I I}(\cdot)$ are Lipschitz continuous, it follows that $\left\|\dot{E}_{k}\right\| \triangleq\left\|\frac{E_{k+1}-E_{k}}{h}\right\| \leq k_{E}$ where $k_{E}$ is its Lipschitz constant. We infer that $E_{I I, \star}^{h}(\cdot)$ converges strongly in $\mathcal{L}^{2}\left([0, T] ; \mathbb{R}^{p}\right)$ towards $E_{I I}(\cdot)$, and using the Lipschitz continuity of $\mathcal{G}^{-1}(\cdot)$ it follows that $z_{\star}^{h}(\cdot)$ converges strongly in $\mathcal{L}^{2}\left([0, T] ; \mathbb{R}^{p}\right)$ towards a limit $z(\cdot)=\mathcal{G}^{-1}\left(E_{I I}(\cdot)+A_{I I} x(\cdot)\right)$. Then from $(3.3) \lambda_{\star}^{h}(t) \in\left(\mathcal{N}_{\mathbb{R}_{+}^{m}}+\tilde{D}\right)^{-1}\left(-\tilde{C} z_{\star}^{h}(t)-F_{\star}^{h}(t)+C_{I} x_{\star}^{h}(t)\right)$, so that $\lambda_{\star}^{h}(\cdot)$ converges towards a limit $\lambda(\cdot)=\left(\mathcal{N}_{\mathbb{R}_{+}^{m}}+\tilde{D}\right)^{-1}\left(-\tilde{C} z(\cdot)-F(\cdot)+C_{I} x(\cdot)\right)$ strongly in $\mathcal{L}^{2}\left([0, T] ; \mathbb{R}^{m}\right)$, due to the single-valuedness, Lipschitzness and maximal monotonicity of $\left(\mathcal{N}_{\mathbb{R}_{+}^{m}}+\tilde{D}\right)^{-1}(\cdot)$. It is concluded that $P_{I} \dot{x}^{h}(t)=A_{I} x_{h}^{\star}(t)+B_{I} z_{h}^{\star}(t)+\bar{B}_{I} \lambda_{h}^{\star}(t)+$ $E_{I}^{h}(t) \longrightarrow P_{I} \dot{x}(t)=A_{I} x(t)+B_{I} z(t)+\bar{B}_{I} \lambda(t)+E_{I}(t)$ weakly in $\mathcal{L}^{2}\left([0, T] ; \mathbb{R}^{n_{1}}\right)$ as $h \rightarrow$ $0, \tilde{A} z_{h}^{\star}(t)+\tilde{B} \lambda_{h}^{\star}(t)+\tilde{E}_{I I}^{h}(t)+A_{I I} x^{h}(t) \rightarrow B_{I I} z(t)+\bar{B}_{I I} \lambda(t)+E_{I I}(t)+A_{I I} x(t)(=0)$, and $0 \leq \lambda_{h}^{\star}(t) \perp \tilde{C} z_{h}^{\star}(t)+\tilde{D} \lambda_{h}^{\star}(t)+F^{h}(t)+C_{I} x^{h}(t) \geq 0 \rightarrow 0 \leq \lambda(t) \perp C_{I I} z(t)+$ $D \lambda(t)+F(t)+C_{I} x(t) \geq 0$ as $h \rightarrow 0$ strongly in $\mathcal{L}^{2}\left([0, T] ; \mathbb{R}^{l}\right)$ where $l$ depends on the variables dimensions. Hence we have proved the existence of a continuous solution $(x, z, \lambda)$ to $(2.1)$ on $[0, T]$. Consequently the following result is proved:

Proposition 5.2. Let the conditions of item 2 in Corollary 4.6 hold true uniformly in $h(\Rightarrow D \succ 0)$. Assume that Assumptions 10 and 11 hold, $E(\cdot)$ and $F(\cdot)$ are bounded Lipschitz continuous functions, initial data are bounded. Then the DALCS in (2.1) possesses for each admissible bounded initial data, a continuous solution $(x, z, \lambda):[0, T] \rightarrow \mathbb{R}^{n_{1}+n_{2}+m}, T>0$. Moreover the Euler approximations converge to a solution as $h \rightarrow 0$.
5.1.3. The case blockdiagonal $\tilde{D}=\tilde{D}^{\top} \succcurlyeq 0$. This section is dedicated to analyse an alternative to the conditions stated in sections 5.1.1 and 5.1.2. Let us consider the case of Proposition 4.4 with Assumption 6. Then it follows from the complementarity conditions that $\lambda_{1, k+1}=\left(\tilde{D}_{11}+\partial \psi_{K_{1}^{\star}}\right)^{-1}\left(-\tilde{C}_{1} z_{k+1}-\tilde{F}_{1, k}\right)$, where the mapping $\left(\tilde{D}_{11}+\partial \psi_{K_{1}^{\star}}\right)^{-1}(\cdot)$ is maximal monotone (being the sum of two maximal monotone mappings satisfying [9, Corollary 24.4 (ii)]), single-valued, well-defined and Lipschitz continuous, see Proposition B.1. Thus if $\bar{B}_{I}=\left(\bar{B}_{I}^{1}, 0\right)$ with $\bar{B}_{I}^{1} \in \mathbb{R}^{n_{1} \times m_{1}}$ one obtains $\bar{B}_{I} \lambda_{k+1}=\bar{B}_{I}^{1} \lambda_{1, k+1}=\bar{B}_{I}^{1}\left(\tilde{D}_{11}+\partial \psi_{K_{1}^{\star}}\right)^{-1}\left(-\tilde{C}_{1} z_{k+1}-\tilde{F}_{1, k}\right)$ in the righthand side of (3.5) (c).
It may be less restrictive to compute upperbounds using the maximal monotonicity. To start with let us consider the VI in (4.9) and let us state several features of its solutions. Here Assumption 7 has to be reinforced a little:

Assumption 12. The conditions of Assumption 7 hold where $\tilde{X}_{2} \succ 0$ for any $h \geq 0$.
This secures that the variable change $\xi_{k+1}=-\tilde{R}_{2}^{-1} z_{k+1}$ remains well-defined as $h \rightarrow$ 0 . It is also important that item 1 in Proposition 4.4 holds, so that the VI in (4.9) is equivalent to the MLCP in (3.2) (b) (c).

LEMMA 5.3. Assume that $\lambda_{\min }\left(-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1}\right)\left(=\lambda_{\min }(-\tilde{A})\right)$ is large enough, then any solution $\xi_{k+1}=-\tilde{R}_{2}^{-1} z_{k+1}$ of the VI in (4.9) is upperbounded by an affine function of $\left\|x_{k}\right\|$.
Proof: Let us manipulate the VI in (4.9) as follows:

$$
\begin{align*}
&-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1} \xi_{k+1}+\tilde{R}_{2} \tilde{E}_{k}+\tilde{G}_{2}\left(t_{k}, \xi_{k+1}\right) \in-\partial \tilde{f}_{2}\left(\xi_{k+1}\right)  \tag{5.10}\\
& \Leftrightarrow \quad-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1}\left(\xi_{k+1}-\xi_{0}\right)+\tilde{R}_{2} \tilde{E}_{k}+\tilde{G}_{2}\left(t_{k}, \xi_{k+1}\right)+\zeta_{0}-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1} \xi_{0}=-\zeta_{k+1}+\zeta_{0} \\
& \quad \text { with } \zeta_{k+1} \in \partial \tilde{f}_{2}\left(\xi_{k+1}\right), \zeta_{0} \in \partial \tilde{f}_{2}\left(\xi_{0}\right) \\
& \Rightarrow \quad\left\langle-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1}\left(\xi_{k+1}-\xi_{0}\right), \xi_{k+1}-\xi_{0}\right\rangle+\left\langle\tilde{R}_{2} \tilde{E}_{k}, \xi_{k+1}-\xi_{0}\right\rangle \\
&+\left\langle\tilde{G}_{2}\left(t_{k}, \xi_{k+1}\right)-\tilde{G}_{2}\left(t_{k}, \xi_{0}\right), \xi_{k+1}-\xi_{0}\right\rangle+\left\langle\zeta_{0}-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1} \xi_{0}, \xi_{k+1}-\xi_{0}\right\rangle \\
&+\left\langle\tilde{G}_{2}\left(t_{k}, \xi_{0}\right), \xi_{k+1}-\xi_{0}\right\rangle=-\left\langle\zeta_{k+1}-\zeta_{0}, \xi_{k+1}-\xi_{0}\right\rangle \leq 0,
\end{align*}
$$

where we have denoted $\tilde{G}_{2}\left(t_{k}, \xi_{k+1}\right)$ to highlight the dependency of this mapping on $t_{k}$ and $x_{k}$ through $\tilde{F}_{k}$. We have $\left\|\tilde{R}_{2} \tilde{B}_{1}\left(\tilde{D}_{11}+\partial \psi_{K_{1}^{\star}}\right)^{-1}\left(z_{1}\right)-\tilde{R}_{2} \tilde{B}_{1}\left(\tilde{D}_{11}+\partial \psi_{K_{1}^{\star}}\right)^{-1}\left(z_{2}\right)\right\| \leq$ $\eta\left\|z_{1}-z_{2}\right\|$ for any $z_{1}$ and $z_{2}$, so that $\left\|\tilde{G}_{2}\left(t_{k}, \xi_{k+1}\right)-\tilde{G}_{2}\left(t_{k}, \xi_{0}\right)\right\| \leq \eta \| \tilde{C}_{1} \tilde{R}_{2}^{-1}\left(\xi_{k+1}-\right.$ $\left.\xi_{0}\right)\left\|\leq \eta^{\prime}\right\| \xi_{k+1}-\xi_{0} \|$, where $\eta$ depends on $\tilde{D}_{11}, \tilde{R}_{2}$ and $\tilde{B}_{1}$ (see the proof of Proposition B. 1 in [16]), and $\eta^{\prime}$ depends on $\tilde{D}_{11}, \tilde{R}_{2}, \tilde{B}_{1}$, and $\tilde{C}_{1}$. Thus we obtain:

$$
\begin{gather*}
\left\langle-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1}\left(\xi_{k+1}-\xi_{0}\right), \xi_{k+1}-\xi_{0}\right\rangle \leq\left\langle-\tilde{R}_{2} \tilde{E}_{k}, \xi_{k+1}-\xi_{0}\right\rangle-\left\langle\tilde{G}_{2}\left(t_{k}, \xi_{0}\right), \xi_{k+1}-\xi_{0}\right\rangle  \tag{5.11}\\
+\left\langle\zeta_{0}+\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1} \xi_{0}, \xi_{k+1}-\xi_{0}\right\rangle+\eta^{\prime}\left\|\xi_{k+1}-\xi_{0}\right\|^{2}
\end{gather*}
$$

which implies

$$
\begin{align*}
& \left\langle\left(-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1}-\eta^{\prime} I_{n_{2}}\right)\left(\xi_{k+1}-\xi_{0}\right), \xi_{k+1}-\xi_{0}\right\rangle \leq\left\|\tilde{R}_{2} \tilde{E}_{k}\right\|\left\|\xi_{k+1}-\xi_{0}\right\|  \tag{5.12}\\
& \quad+\left\|\tilde{G}_{2}\left(t_{k}, \xi_{0}\right) \pm \tilde{G}_{2}\left(t_{0}, \xi_{0}\right)\right\|\left\|\xi_{k+1}-\xi_{0}\right\|+\left\|\zeta_{0}+\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1} \xi_{0}\right\|\left\|\xi_{k+1}-\xi_{0}\right\|
\end{align*}
$$

Now one has: $\left\|\tilde{E}_{k}\right\| \leq\left\|E_{I I, k}+h A_{I I} E_{I, k}\right\|+\left\|A_{I I}\left(P_{I}^{h}\right)^{-1} P_{I}\right\|\left\|x_{k}\right\|, \tilde{G}_{2}\left(t_{k}, \xi_{0}\right)=$ $\tilde{R}_{2} \tilde{B}_{1}\left(\tilde{D}_{11}+\partial \psi_{K_{1}^{\star}}\right)^{-1}\left(\tilde{C}_{1} \tilde{R}_{2}^{-1} \xi_{0}-\tilde{F}_{1, k}\right),\left\|\tilde{G}_{2}\left(t_{k}, \xi_{0}\right)-\tilde{G}_{2}\left(t_{0}, \xi_{0}\right)\right\|=\| \tilde{R}_{2} \tilde{B}_{1}\left(\tilde{D}_{11}+\right.$ $\left.\partial \psi_{K_{1}^{\star}}\right)^{-1}\left(\tilde{C}_{1} \tilde{R}_{2}^{-1} \xi_{0}-\tilde{F}_{1, k}\right)-\tilde{R}_{2} \tilde{B}_{1}\left(\tilde{D}_{11}+\partial \psi_{K_{1}^{\star}}\right)^{-1}\left(\tilde{C}_{1} \tilde{R}_{2}^{-1} \xi_{0}-\tilde{F}_{1,0}\right)\|\leq \eta\|-\tilde{F}_{1, k}+$ $\tilde{F}_{1,0}\|\leq \eta\| F_{k}+h C_{I}\left(P_{I}^{h}\right)^{-1} E_{I, k}\|+\eta\| C_{I}\left(P_{I}^{h}\right)^{-1} P_{I}\| \| x_{k} \|$. Let us denote $\alpha_{k}=$
$\left\|\tilde{R}_{2}\right\|\left\|E_{I I, k}+h A_{I I} E_{I, k}\right\|+\eta\left\|F_{k}+h C_{I}\left(P_{I}^{h}\right)^{-1} E_{I, k}\right\|, \beta_{0}=\left\|\tilde{G}_{2}\left(t_{0}, \xi_{0}\right)\right\|+\| \zeta_{0}+$ $\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1} \xi_{0} \|$, and $\gamma=\eta\left\|C_{I}\left(P_{I}^{h}\right)^{-1} P_{I}\right\|+\left\|\tilde{R}_{2}\right\|\left\|A_{I I}\left(P_{I}^{h}\right)^{-1} P_{I}\right\|$. Then:

$$
\begin{gather*}
\left\langle\left(-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1}-\eta^{\prime} I_{n_{2}}\right)\left(\xi_{k+1}-\xi_{0}\right), \xi_{k+1}-\xi_{0}\right\rangle \leq\left(\alpha_{k}+\beta_{0}+\gamma\left\|x_{k}\right\|\right)\left\|\xi_{k+1}-\xi_{0}\right\|  \tag{5.13}\\
\leq \frac{1}{2} \alpha_{k}^{2}+\frac{1}{2} \beta_{0}^{2}+\frac{1}{2} \gamma^{2}\left\|x_{k}\right\|^{2}+\frac{1}{2}\left\|\xi_{k+1}-\xi_{0}\right\|^{2}
\end{gather*}
$$

which implies:

$$
\begin{equation*}
\left\langle\left(-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1}-\eta^{\prime} I_{n_{2}}-\frac{1}{2} I_{n_{2}}\right)\left(\xi_{k+1}-\xi_{0}\right), \xi_{k+1}-\xi_{0}\right\rangle \leq \frac{1}{2} \alpha_{k}^{2}+\frac{1}{2} \beta_{0}^{2}+\frac{1}{2} \gamma^{2}\left\|x_{k}\right\|^{2} . \tag{5.14}
\end{equation*}
$$

Therefore if $-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1}-\eta^{\prime} I_{n_{2}}-\frac{1}{2} I_{n_{2}} \succ 0$, one obtains

$$
\begin{align*}
\left\|\xi_{k+1}-\xi_{0}\right\| & \leq \frac{1}{\sqrt{\lambda_{\min }\left(-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1}-\eta^{\prime} I_{n_{2}}-\frac{1}{2} I_{n_{2}}\right)}} \sqrt{\frac{1}{2} \alpha_{k}^{2}+\frac{1}{2} \beta_{0}^{2}+\frac{1}{2} \gamma^{2}\left\|x_{k}\right\|^{2}}  \tag{5.15}\\
& \leq \frac{1}{\sqrt{2} \sqrt{\lambda_{\min }\left(-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1}-\eta^{\prime} I_{n_{2}}-\frac{1}{2} I_{n_{2}}\right)}}\left(\sqrt{\alpha_{k}^{2}+\beta_{0}^{2}}+\gamma\left\|x_{k}\right\|\right)
\end{align*}
$$

which ends the proof. $\boxtimes$
Therefore the term $h B_{I} z_{k+1}$ in the right-hand side of (3.5) (c), admits also an affine upperbound in $\left\|x_{k}\right\|$. A further characterization of the iterates is given now.

Lemma 5.4. Assume that $-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1} \succ \frac{3}{2} I_{n_{2}}$. Assume also that the mapping $\tilde{G}_{2}(\cdot)$ is monotone (see Lemma 4.5 for conditions), and that $E_{I}(\cdot), E_{I I}(\cdot)$ and $F(\cdot)$ are uniformly bounded and Lipschitz continuous. Then for any $k$ and $p \geq 0$ and any solution of the VI in (4.9), one has:

$$
\begin{equation*}
\left\|\xi_{k}-\xi_{p}\right\| \leq \alpha+\beta\left\|x_{k-1}-x_{p-1}\right\| \tag{5.16}
\end{equation*}
$$

for some bounded constants $\alpha \geq 0$ and $\beta \geq 0$, and $\alpha$ is proportional to $\left(t_{k-1}-t_{p-1}\right)$.
Proof: From

$$
\left\{\begin{array}{l}
-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1} \xi_{k}+\tilde{R}_{2} \tilde{E}_{k-1}+\tilde{G}_{2}\left(t_{k-1}, \xi_{k}\right)=-\zeta_{k}  \tag{5.17}\\
-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1} \xi_{p}+\tilde{R}_{2} \tilde{E}_{p-1}+\tilde{G}_{2}\left(t_{p-1}, \xi_{p}\right)=-\zeta_{p}
\end{array}\right.
$$

with $\zeta_{k} \in \partial f_{2}\left(\xi_{k}\right), \zeta_{p} \in \partial f_{2}\left(\xi_{p}\right)$, it follows that

$$
\begin{align*}
& \left\langle-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1}\left(\xi_{k}-\xi_{p}\right), \xi_{k}-\xi_{p}\right\rangle+\left\langle\tilde{R}_{2}\left(\tilde{E}_{k-1}-\tilde{E}_{p-1}\right), \xi_{k}-\xi_{p}\right\rangle \\
& +\left\langle\tilde{G}_{2}\left(t_{k-1}, \xi_{k}\right)-\tilde{G}_{2}\left(t_{k-1}, \xi_{p}\right)+\tilde{G}_{2}\left(t_{k-1}, \xi_{p}\right)-\tilde{G}_{2}\left(t_{p-1}, \xi_{p}\right), \xi_{k}-\xi_{p}\right\rangle  \tag{5.18}\\
& \quad=-\left\langle\zeta_{k}-\zeta_{p}, \xi_{k}-\xi_{p}\right\rangle
\end{align*}
$$

which implies

$$
\begin{align*}
& \left\langle-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1}\left(\xi_{k}-\xi_{p}\right), \xi_{k}-\xi_{p}\right\rangle+\left\langle\tilde{R}_{2}\left(\tilde{E}_{k-1}-\tilde{E}_{p-1}\right), \xi_{k}-\xi_{p}\right\rangle  \tag{5.19}\\
& \quad+\left\langle\tilde{G}_{2}\left(t_{k-1}, \xi_{p}\right)-\tilde{G}_{2}\left(t_{p-1}, \xi_{p}\right), \xi_{k}-\xi_{p}\right\rangle \leq 0 .
\end{align*}
$$

One has:

$$
\begin{align*}
& \left\langle-\tilde{R}_{2}\left(\tilde{E}_{k-1}-\tilde{E}_{p-1}\right), \xi_{k}-\xi_{p}\right\rangle  \tag{5.20}\\
& \leq\left\langle-\tilde{R}_{2}\left(E_{I I, k-1}-E_{I I, p-1}\right), \xi_{k}-\xi_{p}\right\rangle-\left\langle\tilde{R}_{2}\left(h A_{I I}\left(E_{I, k-1}-E_{I, p-1}\right)\right), \xi_{k}-\xi_{p}\right\rangle \\
& \quad-\left\langle A_{I I}\left(P_{I}^{h}\right)^{-1} P_{I}\left(x_{k-1}-x_{p-1}\right), \xi_{k}-\xi_{p}\right\rangle \\
& \left.\leq \frac{1}{2}\left\|\tilde{R}_{2}\left(\left(E_{I I, k-1}-E_{I I, p-1}\right)+h A_{I I}\left(E_{I, k-1}-E_{I, p-1}\right)\right)\right\|^{2}+\frac{1}{2} \right\rvert\, \xi_{k}-\xi_{p} \|^{2} \\
& \quad+\frac{1}{2}\left\|A_{I I}\left(P_{I}^{h}\right)^{-1} P_{I}\right\|^{2}\left\|x_{k-1}-x_{p-1}\right\|^{2}+\frac{1}{2}\left\|\xi_{k}-\xi_{p}\right\|^{2} \\
& \leq\left\|R_{2}\right\|^{2}\left\|E_{I I, k-1}-E_{I I, p-1}\right\|^{2}+\left\|\xi_{k}-\xi_{p}\right\|^{2}+h\left\|A_{I I}\right\|^{2}\left\|E_{I, k-1}-E_{I, p-1}\right\|^{2} \\
& \quad+\frac{1}{2}\left\|A_{I I}\left(P_{I}^{h}\right)^{-1} P_{I}\right\|^{2}\left\|x_{k-1}-x_{p-1}\right\|^{2} \leq \alpha^{\prime}+\beta^{\prime}\left\|x_{k-1}-x_{p-1}\right\|^{2}+\left\|\xi_{k}-\xi_{p}\right\|^{2},
\end{align*}
$$

for some $\alpha^{\prime} \geq 0$ proportional to $\left(t_{k-1}-t_{p-1}\right)^{2}$ since $E(\cdot)$ is Lipschitz continuous, $\beta^{\prime} \geq 0$, and it was used that $x^{\top} y \leq\|x\|\|y\| \leq \frac{1}{2}\left(\|x\|^{2}+\|y\|^{2}\right)$. Also:

$$
\begin{align*}
& \quad\left\langle-\tilde{G}_{2}\left(t_{k-1}, \xi_{p}\right)+\tilde{G}_{2}\left(t_{p-1}, \xi_{p}\right), \xi_{k}-\xi_{p}\right\rangle \\
& \leq\left\|\tilde{G}_{2}\left(t_{k-1}, \xi_{p}\right)-\tilde{G}_{2}\left(t_{p-1}, \xi_{p}\right)\right\|\left\|\xi_{k}-\xi_{p}\right\| \\
& \leq \frac{1}{2} k_{\tilde{G}_{2}}\left\|\tilde{F}_{1, k-1}-\tilde{F}_{p-1}\right\|^{2}+\frac{1}{2}\left\|\xi_{k}-\xi_{p}\right\|^{2}  \tag{5.21}\\
& \leq k_{\tilde{G}_{2}}\left\|F_{k-1}-F_{p-1}\right\|^{2}+h k_{\tilde{G}_{2}}\left\|C_{I}\left(P_{I}^{h}\right)^{-1}\left(E_{I, k-1}-E_{I, p-1}\right)\right\|^{2} \\
& \\
& \quad+k_{\tilde{G}_{2}}\left\|C_{I}\left(P_{I}^{h}\right)^{-1} P_{I}\left(x_{k-1}-x_{p-1}\right)\right\|^{2}+\frac{1}{2}\left\|\xi_{k}-\xi_{p}\right\|^{2} \\
& \leq \alpha^{\prime \prime}+\beta^{\prime \prime}\left\|x_{k-1}-x_{p-1}\right\|^{2}+\frac{1}{2}\left\|\xi_{k}-\xi_{p}\right\|^{2},
\end{align*}
$$

for some $\alpha^{\prime \prime} \geq 0$ proportional to $\left(t_{k-1}-t_{p-1}\right)^{2}$ (since $E(\cdot)$ and $F(\cdot)$ are Lipschitz continuous), and $\beta^{\prime \prime} \geq 0$. Hence using (5.19):

$$
\begin{align*}
\left\|\xi_{k}-\xi_{p}\right\| & \leq \frac{1}{\sqrt{\lambda_{\min }\left(-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1}-\frac{3}{2} I_{p}\right)}} \sqrt{\alpha^{\prime}+\alpha^{\prime \prime}+\left(\beta^{\prime}+\beta^{\prime \prime}\right)\left\|x_{k-1}-x_{p-1}\right\|^{2}}  \tag{5.22}\\
& \leq \frac{1}{\sqrt{\lambda_{\min }\left(-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1}-\frac{3}{2} I_{p}\right)}}\left(\sqrt{\alpha^{\prime}+\alpha^{\prime \prime}}+\sqrt{\beta^{\prime}+\beta^{\prime \prime}}\left\|x_{k-1}-x_{p-1}\right\|\right)
\end{align*}
$$

and the lemma is proved. $\boxtimes$

Corollary 5.5. Let $h>0$. Assume that the conditions and assumptions of Lemma 5.4 are fulfilled. For all $k \geq 1$ and any solution of the VI in (4.9), one has:

$$
\begin{equation*}
\left\|\frac{\xi_{k+1}-\xi_{k}}{h}\right\| \leq \alpha+\beta\left\|\frac{x_{k}-x_{k-1}}{h}\right\| \tag{5.23}
\end{equation*}
$$

for some bounded constants $\alpha \geq 0$ and $\beta \geq 0$.
Proof: Let us consider the proof of Lemma 5.4, replacing $k$ by $k+1$ and $p$ by $k-1$. Then $\alpha$ in (5.16) is proportional to $h$. Redoing the calculations in (5.20) and (5.22), one obtains (5.23). $\boxtimes$

Clearly for all $h>0$ (5.23) can be written equivalently as: $\left\|\xi_{k+1}-\xi_{k}\right\| \leq h \alpha+$ $\beta\left\|x_{k}-x_{k-1}\right\|$. The bounds calculated for the variable $\xi_{k}$ also hold for the variable $z_{k}=-\tilde{R}_{2} \xi_{k}$. Let us now analyse $\lambda_{k+1}$ in (3.5) (a). As seen above, $\lambda_{1, k+1}=\left(\tilde{D}_{11}+\right.$ $\left.\partial \psi_{K_{1}^{\star}}\right)^{-1}\left(-\tilde{C}_{1} z_{k+1}-\tilde{F}_{1, k}\right)$ and $\lambda_{2, k+1} \in-\mathcal{N}_{K_{2}}\left(\tilde{C}_{2} z_{k+1}+\tilde{F}_{2, k}\right)$. If the matrix $\bar{B}_{I}$ has the above structure (i.e., $\bar{B}_{I}=\left(\bar{B}_{I}^{1}, 0\right)$ ), then the term $\bar{B}_{I} \lambda_{k+1}=\bar{B}_{I}^{1}$ is Lipschitz continuous with respect to the variables $z_{k+1}, x_{k}$ and $F_{k}$. In a more general setting one obtains:

$$
\begin{align*}
P_{I}^{h} x_{k+1} \in & P_{I} x_{k}+h B_{I} z_{k+1}+h \bar{B}_{I}^{1}\left(\tilde{D}_{11}+\partial \psi_{K_{1}^{\star}}\right)^{-1}\left(-\tilde{C}_{1} z_{k+1}-\tilde{F}_{1, k}\right)  \tag{5.24}\\
& -h \bar{B}_{I}^{2} \mathcal{N}_{K_{2}}\left(\tilde{C}_{2} z_{k+1}+\tilde{F}_{2, k}\right) .
\end{align*}
$$

The difficulty is that the last term in the right-hand side of (5.24) is set-valued in general, and this is intrinsically due to the structure of the implicit scheme in (2.1). A solution is to consider the VI in (4.9). Its right-hand side matches with the last term in the right-hand side of $(5.24)$ and it is set-valued. However provided the VI is well-posed, we know that one element of this set is equal to the left-hand side of (4.9). We may assume that the scheme in (5.24) is implemented with this element. This element appears to define a Lipschitz continuous function of $\xi_{k+1}$ (let us recall that the Lipschitz constant of the third term is equal to $\frac{1}{\lambda_{\min }\left(\tilde{D}_{11}\right)}$, see Proposition B.1). If this element is chosen then the numerical scheme (5.24) becomes:

$$
\begin{align*}
P_{I}^{h} x_{k+1}= & P_{I} x_{k}+h B_{I} z_{k+1}+h \bar{B}_{I}^{1}\left(\tilde{D}_{11}+\partial \psi_{K_{1}^{\star}}\right)^{-1}\left(\tilde{C}_{1} z_{k+1}-\tilde{F}_{1, k}\right) \\
& +h \bar{B}_{I}^{2}\left(-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1} \xi_{k+1}-\tilde{R}_{2} \tilde{E}_{k}+\tilde{G}_{2}\left(t_{k}, \xi_{k+1}\right)\right) . \tag{5.25}
\end{align*}
$$

Let us now study (5.25), using Lemmata 5.3, 5.4, Corollary 5.5. It is noteworthy that the constant $\alpha$ in both Lemmata and in Corollary can be made arbitrarily small by increasing $-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1}$ arbitrarily (in the set of positive definite matrices).

Proposition 5.6. Assume that the conditions of Lemmata 5.3, 5.4 and of Corollary 5.5 are verified, $E_{I}(\cdot), E_{I I}(\cdot)$ and $F(\cdot)$ are bounded Lipschitz continuous functions of time, and Assumption 13 below is satisfied. Then the sequences $\left\{x^{h}\right\}_{h \geq 0}$ and $\left\{\dot{x}^{h}\right\}_{h \geq 0}$ in (5.7) are uniformly bounded in $h$ on $[0, T]$. Thus $\left\{x^{h}\right\}_{h \geq 0}$ is equi-Lipschitz continuous and there is a subsequence of it that converges uniformly in $\mathcal{C}^{0}\left([0, T] ; \mathbb{R}^{n_{1}}\right)$ towards a Lipschitz continuous limit denoted as $x(\cdot)$.
Proof: In the proof, use is made of positive bounded constants denoted generically as $k_{i}$. Consider (5.25). First recall that the operator $\zeta \mapsto\left(\tilde{D}_{11}+\partial \psi_{K_{1}^{\star}}\right)^{-1}(\zeta)$ is Lipschitz continuous with Lipschitz constant $\frac{1}{\lambda_{\min }\left(\tilde{D}_{11}\right)}$. Hence we obtain:

$$
\begin{align*}
& \|\left(P_{I}^{h}\right)^{-1} \bar{B}_{I}^{1}\left(\tilde{D}_{11}+\partial \psi_{K_{1}^{\star}}\right)^{-1}\left(-\tilde{C}_{1} z_{k+1}-\tilde{F}_{1, k}\right) \\
& \quad-\left(P_{I}^{h}\right)^{-1} \bar{B}_{I}^{1}\left(\tilde{D}_{11}+\partial \psi_{K_{1}^{\star}}\right)^{-1}\left(-\tilde{C}_{1} z_{p+1}-\tilde{F}_{1, p}\right) \| \\
& \leq k_{1} \| \tilde{C}_{1}\left(z_{k+1}-z_{p+1}+\tilde{F}_{1, k}-\tilde{F}_{1, p}\left\|\leq k_{2}\right\| z_{k+1}-z_{p+1}\left\|+k_{3}\right\| \tilde{F}_{1, k}-\tilde{F}_{1, p} \|\right.  \tag{5.26}\\
& \leq k_{4}\left\|\xi_{k+1}-\xi_{p+1}\right\|+k_{4}\left\|F_{1, k}-F_{1, p}\right\|+k_{5}\left\|E_{I, k}-E_{I, p}\right\|+k_{6}\left\|x_{k}-x_{p}\right\| \\
& \leq k_{4}\left(\alpha+\beta\left\|x_{k}-x_{p}\right\|\right)+k_{7}\left|t_{k}-t_{p}\right|+k_{6}\left\|x_{k}-x_{p}\right\| \leq k_{8}+k_{9}\left\|x_{k}-x_{p}\right\| \\
& \quad+k_{7}\left|t_{k}-t_{p}\right|,
\end{align*}
$$

for any $k$ and $p \geq 0$, where we used Lemma 5.4 and the Lipschitzness property. In a quite similar way one has $\left\|\left(P_{I}^{h}\right)^{-1} \bar{B}_{I}^{2}\left(\tilde{G}_{2}\left(\xi_{k+1}\right)-\tilde{G}_{2}\left(\xi_{p+1}\right)\right)\right\| \leq k_{8}+k_{9}\left\|x_{k}-x_{p}\right\|+$ $k_{7}\left|t_{k}-t_{p}\right|$, where $k_{7}, k_{8}$ and $k_{9}$ depend on the Lipschitz constants of $\tilde{G}_{2}(\cdot, \cdot), F(\cdot)$, and on $\left\|\left(P_{I}^{h}\right)^{-1} \bar{B}_{I}^{2}\right\|$. One also has $\|\left(P_{I}^{h}\right)^{-1}\left(B_{I}+\bar{B}_{I}^{2}\left(\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1}\right)\left(z_{k+1}-z_{p+1}\right) \| \leq k_{10}(\alpha+\right.$ $\left.\beta\left\|x_{k}-x_{p}\right\|\right) \leq k_{11}+k_{12}\left\|x_{k}-x_{p}\right\|$, and $\left\|\bar{B}_{I}^{2} \tilde{R}_{2}\left(\tilde{E}_{k}-\tilde{E}_{p}\right)\right\| \leq k_{13}\left|t_{k}-t_{p}\right|+k_{14}\left\|x_{k}-x_{p}\right\|$. It is deduced that $\left\|x_{k+1}-x_{p+1}\right\| \leq\left(\left\|\left(P_{I}^{h}\right)^{-1} P_{I}\right\|+h k_{15}\right)\left\|x_{k}-x_{p}\right\|+h k_{16}\left|t_{k}-t_{p}\right|+h k_{17}$, where the term $k_{17}$ stems from $k_{8}+k_{11}=\alpha\left(k_{4}+k_{10}\right)$. Here we are led to make the following which is more stringent that Assumption 11:

Assumption 13. There exists $h_{\max }>0$ such that $\left\|\left(P_{I}^{h}\right)^{-1} P_{I}\right\|+h k_{15}<1$ for all $0<h \leq h_{\text {max }}$.
Obviously for $h=0$ and the Hölder-induced norm one has $\left\|\left(P_{I}^{h}\right)^{-1} P_{I}\right\|_{2,2}+h k_{15}=1$. Let us choose $p=0$, it follows that $\left\|x_{k+1}-x_{1}\right\| \leq\left(\left\|\left(P_{I}^{h}\right)^{-1} P_{I}\right\|+h k_{15}\right) \| x_{k}-$ $x_{0} \|+h k_{16} T+h k_{17}$, hence $\left\|x_{k+1}\right\| \leq\left(\left\|\left(P_{I}^{h}\right)^{-1} P_{I}\right\|+h k_{15}\right)\left\|x_{k}\right\|+h k_{16} T+h k_{17}+$ $\left\|x_{1}\right\|+\left(\left\|\left(P_{I}^{h}\right)^{-1} P_{I}\right\|+h k_{15}\right)\left\|x_{0}\right\|$. Now it has to be proved that $\left\|x_{1}\right\|$ is bounded. From Lemma 5.3 and (5.15), $\left\|\xi_{1}\right\|$ and $\left\|z_{1}\right\|$ are upperbounded by an affine function of initial data $\left\|x_{0}\right\|$ and $\left\|z_{0}\right\|$. The operator $\left(\tilde{D}_{11}+\partial \psi_{K_{1}^{\star}}\right)^{-1}(\cdot)$ is well-defined and singlevalued. Taking $k=0$ in (5.25) one infers that indeed $\left\|x_{1}\right\|$ is bounded for bounded initial data. Consequently it follows that $\left\|x_{k+1}\right\| \leq\left(\left\|\left(P_{I}^{h}\right)^{-1} P_{I}\right\|+h k_{15}\right)\left\|x_{k}\right\|+$ $h k_{16} T+h k_{17}+h f\left(x_{0}, z_{0}\right)+\left\|\left(P_{I}^{h}\right)^{-1} P_{I}\right\|\left\|x_{0}\right\|$ for some function $f(\cdot)$ that is bounded for bounded initial data. To simplify the notations let us set $h_{1} \triangleq\left\|\left(P_{I}^{h}\right)^{-1} P_{I}\right\|+h k_{15}$ and $h_{2} \triangleq k_{16} T+k_{17}+f\left(x_{0}, z_{0}\right)$. We obtain for all $k \geq 1$ :

$$
\begin{align*}
\left\|x_{k}\right\| & \leq h_{1}^{k}\left\|x_{0}\right\|+\sum_{j=0}^{k-1} h_{1}^{j}\left(h h_{2}+\left\|\left(P_{I}^{h}\right)^{-1} P_{I}\right\|\left\|x_{0}\right\|\right)  \tag{5.27}\\
& \leq h_{1}^{k}\left\|x_{0}\right\|+\left(h h_{2}+\left\|\left(P_{I}^{h}\right)^{-1} P_{I}\right\|\left\|x_{0}\right\|\right) \frac{1-h_{1}^{k+1}}{1-h_{1}} .
\end{align*}
$$

Thus in view of Assumption 13 the sequence $\left\{x_{k}\right\}_{k \geq 0}$ generated by (5.25) is uniformly bounded for bounded initial data.

Let us now reconsider the inequality $\left\|x_{k+1}-x_{p+1}\right\| \leq\left(\left\|\left(P_{I}^{h}\right)^{-1} P_{I}\right\|+h k_{15}\right) \| x_{k}-$ $x_{p} \|+h k_{16}\left|t_{k}-t_{p}\right|+h k_{17}$ obtained above. Setting $p=k-1$ yields $\left\|x_{k+1}-x_{k}\right\| \leq$ $\left(\left\|\left(P_{I}^{h}\right)^{-1} P_{I}\right\|+h k_{15}\right)\left\|x_{k}-x_{k-1}\right\|+h^{2} k_{16}+h k_{17} \Rightarrow \frac{\left\|x_{k+1}-x_{k}\right\|}{h} \leq\left(\left\|\left(P_{I}^{h}\right)^{-1} P_{I}\right\|+\right.$ $\left.h k_{15}\right) \frac{\left\|x_{k}-x_{k-1}\right\|}{h}+h k_{16}+k_{17}$. Now using Lemma 5.4 and the Lipschitzness it follows that $\alpha$, hence $k_{17}$, are proportional to $k_{E}^{2} h^{2}$. We may therefore write the inequality as $\frac{\left\|x_{k+1}-x_{k}\right\|}{h} \leq\left(\left\|\left(P_{I}^{h}\right)^{-1} P_{I}\right\|+h k_{15}\right) \frac{\left\|x_{k}-x_{k-1}\right\|}{h}+h k_{16}+k_{18} h^{2}$ for some $k_{18} \geq 0$. Proceeding as for (5.27) one infers that if $\frac{\left\|x_{1}-x_{0}\right\|}{h}$ is bounded then $\frac{\left\|x_{k+1}-x_{k}\right\|}{h}$ is bounded for all $k \geq 1$. Evaluating (5.25) at $k=0$ and using the boundedness of $x_{1}$ as shown above, proves that this is indeed the case. The proposition is proved using the Arzela-Ascoli Theorem.

Clearly in practice one needs to compute the constants $k_{i}$, for instance $k_{15}$ is needed to determine $h_{\max }$ in Assumption 13.

Corollary 5.7. Under the conditions of Proposition 5.6, the sequences $\left\{z^{h}\right\}_{h \geq 0}$ and $\left\{\dot{z}^{h}\right\}_{h \geq 0}$ (respectively $\left\{\xi^{h}\right\}_{h \geq 0}$ and $\left\{\dot{\xi}^{h}\right\}_{h \geq 0}$ ) are uniformly bounded in $h$ on $[0, T]$, $0<T<+\infty$. Thus $\left\{z^{h}\right\}_{h \geq 0}$ (respectively $\left\{\xi^{\bar{h}}\right\}_{h \geq 0}$ ) is equi-Lipschitz continuous and there is a subsequence of it that converges uniformly in $\mathcal{C}^{0}\left([0, T] ; \mathbb{R}^{n_{2}}\right)$ towards a Lipschitz continuous limit denoted as $z(\cdot)$ (respectively $\xi(\cdot)$ ).
Proof: This follows from Proposition 5.6 and using Lemmata 5.3, 5.4, Corollary 5.5. $\boxtimes$

Let us now pass to the analysis of the multiplier convergence. Recall that $\lambda_{1, k+1}=$ $\left(\tilde{D}_{11}+\partial \psi_{K_{1}^{\star}}\right)^{-1}\left(-\tilde{C}_{1} z_{k+1}-\tilde{F}_{1, k}\right)$ and $\lambda_{2, k+1}=-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1} \xi_{k+1}+\tilde{R}_{2} \tilde{E}_{k}+\tilde{G}_{2}\left(\xi_{k+1}\right) \in$ $-\mathcal{N}_{K_{2}}\left(\tilde{C}_{2} z_{k+1}+\tilde{F}_{2, k}\right)$, therefore for all $t \geq 0, \lambda_{1, \star}^{h}(t)=\left(\tilde{D}_{11}+\partial \psi_{K_{1}^{\star}}\right)^{-1}\left(-\tilde{C}_{1} z_{\star}^{h}(t)-\right.$ $\left.\tilde{F}_{1, \star}^{h}(t)\right)$ and $\lambda_{2, \star}^{h}(t)=-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1} \xi_{\star}^{h}(t)+\tilde{R}_{2} \tilde{E}_{\star}^{h}(t)+\tilde{G}_{2}\left(\xi_{\star}^{h}(t)\right) \in-\mathcal{N}_{K_{2}}\left(\tilde{C}_{2} z_{\star}^{h}(t)+\right.$ $\left.\tilde{F}_{2, \star}^{h}(t)\right)$, where we recall that the step-functions are defined as $f_{\star}^{h}(t)=f_{k+1}$ for $t \in\left[t_{k}, t_{k+1}\right)$. We know that $\dot{\xi}_{k} \triangleq\left\|\frac{\xi_{k+1}-\xi_{k}}{h}\right\| \leq C$ for some $C>0$, hence:

$$
\begin{align*}
\left\|\xi^{h}-\xi_{\star}^{h}\right\|_{\mathcal{L}_{2}\left([0, T] ; \mathbb{R}^{n_{2}}\right)}^{2} & \leq \sum_{k=0}^{n-1} \int_{t_{k}}^{t_{k+1}}\left(t_{k+1}-t_{k}\right)^{2}\left\|\dot{\xi}_{k}\right\|^{2} d t  \tag{5.28}\\
& \leq C^{2} \sum_{k=0}^{n-1}\left(t_{k+1}-t_{k}\right)^{3}=\frac{C^{2} n h^{3}}{3}=\frac{C^{2} T h^{2}}{3}
\end{align*}
$$

Consequently $\left\|\xi^{h}-\xi_{\star}^{h}\right\|_{\mathcal{L}_{2}\left([0, T] ; \mathbb{R}^{n_{2}}\right)} \rightarrow 0$ as $h \rightarrow 0$. Since $\left\|\xi^{h}-\xi_{\star}^{h}\right\|_{\mathcal{L}_{2}\left([0, T] ; \mathbb{R}^{n_{2}}\right)}=$ $\left\|\xi^{h} \pm \xi-\xi_{\star}^{h}\right\|_{\mathcal{L}_{2}\left([0, T] ; \mathbb{R}^{n_{2}}\right)} \rightarrow\left\|\xi-\xi_{\star}^{h}\right\|_{\mathcal{L}_{2}\left([0, T] ; \mathbb{R}^{n_{2}}\right)} \rightarrow 0$, we conclude that the sequence $\left\{\xi_{\star}^{h}\right\}_{h \geq 0}$ converges strongly in $\mathcal{L}_{2}\left([0, T] ; \mathbb{R}^{n_{2}}\right)$ towards the limit $\xi(\cdot)$. The same result holds for the sequence $\left\{x_{\star}^{h}\right\}_{h \geq 0}$ with the limit $x(\cdot)$. Therefore the sequence $\left\{-\tilde{C}_{1} z_{\star}^{h}-\right.$ $\left.\tilde{F}_{1, \star}^{h}\right\}_{h \geq 0}$ also converges strongly in $\mathcal{L}_{2}$ towards a limit, and we infer that the same conclusion holds for both $\left\{\lambda_{1, \star}^{h}(\cdot)\right\}_{h \geq 0}$ and $\left\{\lambda_{2, \star}^{h}(\cdot)\right\}_{h \geq 0}$. Since $\mathcal{N}_{K_{2}}(\cdot)$ is maximal monotone, it follows from [9, Proposition 20.33] that the limit $\lambda_{2}(\cdot)$ satisfies $\lambda_{2}(t) \in$ $-\mathcal{N}_{K_{2}}\left(C_{I I, 2} z(t)+F_{2}(t)+C_{I, 2} x(t)\right)$. Thus we proved the following result.

Lemma 5.8. Let the conditions of Lemmata 5.3, 5.4, Corollary 5.5 be satisfied for all $h>0$. The sequences $\left\{\xi_{\star}^{h}\right\}_{h \geq 0},\left\{x_{\star}^{h}\right\}_{h \geq 0},\left\{\lambda_{1, \star}^{h}(\cdot)\right\}_{h \geq 0}$ and $\left\{\lambda_{2, \star}^{h}(\cdot)\right\}_{h \geq 0}$ converge strongly in $\mathcal{L}_{2}\left([0, T] ; \mathbb{R}^{n_{2}}\right)$ towards the limits $\xi(\cdot), x(\cdot), \bar{\lambda}_{1}(\cdot)$ and $\lambda_{2}(\cdot)$, respectively.

To finish this section, one can consider again (5.8) and the analysis which follows it, to obtain a proposition similar to Proposition 5.2, relying on Lemma 5.8, Proposition 5.6, and on Corollary 5.5. It is noteworthy that the three fundamental assumptions in this section, are that $-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1}$ be sufficiently large as a positive definite matrix, that a suitable selection is chosen for the multiplier $\lambda_{2, k}$, and that a coupling as stated in Assumptions 2 or 7 exists.

Proposition 5.9. Assume that the conditions of Proposition 5.6, Corollary 5.7 and Lemma 5.8 are verified. Then the limit functions $x(\cdot), z(\cdot), \lambda(\cdot)$, are solutions of the DALCS (2.1).

Proof. Let us consider (5.8). From Lemma 5.8, $\left\{x_{\star}^{h}(\cdot)\right\},\left\{z_{\star}^{h}(\cdot)\right\},\left\{\lambda_{\star}^{h}(\cdot)\right\}$ converge strongly in $\mathcal{L}^{2}\left([0, T] ; \mathbb{R}^{n_{1}}\right)$ towards limits $x(\cdot), z(\cdot)$ and $\lambda(\cdot)$. Using the same arguments as those developed before Proposition 5.2, it is inferred from (5.8) (b) (c) that the limits verify (2.1) (b) (c). Using the same arguments as those developed just above (5.8), it is deduced that $\left\{\dot{x}^{h}(\cdot)\right\}$ converges weakly ${ }^{\star}$ to a limit $v(\cdot)$ where $v(\cdot)=\dot{x}(\cdot)$ almost everywhere. Consequently $P_{I} \dot{x}^{h}(t)=A_{I} x_{h}^{\star}(t)+B_{I} z_{h}^{\star}(t)+\bar{B}_{I} \lambda_{h}^{\star}(t)+E_{I}^{h}(t) \longrightarrow$ $P_{I} \dot{x}(t)=A_{I} x(t)+B_{I} z(t)+\bar{B}_{I} \lambda(t)+E_{I}(t)$ weakly in $\mathcal{L}^{2}\left([0, T] ; \mathbb{R}^{n_{1}}\right)$ as $h \rightarrow 0$.
5.2. Convergence results from (3.1). In this section the properties of the plant's model (2.1) are used, thus the numerical scheme in (3.1) and the results in section 4.2 are analysed. This section is devoted to the case $D \succcurlyeq 0$ as in Proposition 4.8. Indeed, since the strict passivity of $\left(B_{I I}, \bar{B}_{I I}, C_{I I}, D\right)$ implies Assumption 10 for small enough $h>0$, it is unnecessary to repeat the developments in sections 5.1.1 and 5.1.2. However, it is not possible in general to determine the implications between Assumption 7 and Assumption 9, nor the relations between $\lambda_{\min }(\tilde{A})$ and $\lambda_{\min }\left(B_{I I}\right)$ for $h>0$. Therefore let us deal with the VI in (4.17) where $x_{k+1}$ is replaced by $x_{k+\theta}=\theta x_{k}+(1-\theta) x_{k+1}, \theta \in[0,1]$. The following holds, which is the counterpart of Lemma 5.3.

Lemma 5.10. Assume that $\lambda_{\min }\left(-R_{2} B_{I I} R_{2}^{-1}\right)\left(=\lambda_{\min }\left(-B_{I I}\right)\right)$ is large enough, $E_{I}(\cdot), E_{I I}(\cdot)$ and $F(\cdot)$ are uniformly bounded and Lipschitz continuous functions of time. Then any solution $\xi_{k+1}=R_{2}^{-1} z_{k+1}$ of the VI in (4.17) is upperbounded by an affine function of $\left\|x_{k+\theta}\right\|$.

The proof follows the same steps as the proof of Lemma 5.3 and it is therefore omitted. Lemma 5.10 implies that the term $B_{I} z_{k+1}$ in the right-hand side of (3.1) (a), is upperbounded by an affine function of $\left\|x_{k+\theta}\right\|$. The coefficients of the affine function depend only on the system's data (constant matrices, Lipschitz constants) and on the initial conditions. Let us now propose the counterpart of Lemma 5.4.

LEMMA 5.11. Assume that $-R_{2} B_{I I} R_{2}^{-1} \succ I_{n_{2}}$. Assume also that the well-defined, single-valued and Lipschitz continuous mapping $G_{2}(\cdot)$ is monotone, and that $E_{I}(\cdot)$, $E_{I I}(\cdot)$ and $F(\cdot)$ are Lispchitz continuous functions of time. Then for any $k$ and $p \geq 0$ and any solution of the VI in (4.17), one has:

$$
\begin{equation*}
\left\|\xi_{k}-\xi_{p}\right\| \leq \alpha+\beta\left\|x_{k-1+\theta}-x_{p-1+\theta}\right\| \tag{5.29}
\end{equation*}
$$

for all $k \geq 1$ and some bounded constants $\alpha \geq 0$ (proportional to $\left|t_{k-1}-t_{p-1}\right|$ ) and $\beta \geq 0$.

Proof: The proof is similar to the proof of Lemma 5.4, since both VIs in (4.17) and (4.9) possess similar structures. Using (4.17) one finds:

$$
\left\{\begin{array}{l}
-R_{2} B_{I I} R_{2}^{-1} \xi_{k+1}-R_{2} A_{I I} x_{k+\theta}-R_{2} E_{I I, k}+G_{2}\left(t_{k}, \xi_{k+1}\right)=-\zeta_{k+1}  \tag{5.30}\\
-R_{2} B_{I I} R_{2}^{-1} \xi_{p+1}-R_{2} A_{I I} x_{p+\theta}-R_{2} E_{I I, p}+G_{2}\left(t_{p}, \xi_{p+1}\right)=-\zeta_{p+1}
\end{array}\right.
$$

with $\zeta_{k} \in \partial f_{2}\left(\xi_{k}\right), k \geq 0$. Using the monotonicity of $G_{2}\left(t_{k}, \cdot\right)$ and of $\partial f_{2}(\cdot)$ it follows
that:

$$
\begin{align*}
\langle- & \left.R_{2} B_{I I} R_{2}^{-1}\left(\xi_{k+1}-\xi_{p+1}\right), \xi_{k+1}-\xi_{p+1}\right\rangle \\
\leq & \left\langle R_{2} A_{I I}\left(x_{k+\theta}-x_{p+\theta}\right)+R_{2}\left(E_{I I, k}-E_{I I, p}\right), \xi_{k+1}-\xi_{p+1}\right\rangle \\
& -\left\langle G_{2}\left(t_{k}, \xi_{k+1}\right)-G_{2}\left(t_{k}, \xi_{p+1}\right), \xi_{k+1}-\xi_{p+1}\right\rangle \\
& +\left\langle G_{2}\left(t_{k}, \xi_{p+1}\right)-G_{2}\left(t_{p}, \xi_{p+1}\right), \xi_{k+1}-\xi_{p+1}\right\rangle \\
\leq & \left\|R_{2} A_{I I}\right\|^{2}\left\|x_{k+\theta}-x_{p+\theta}\right\|^{2}+\left\|R_{2}\right\|^{2} k_{E}^{2}\left(t_{k}-t_{p}\right)^{2}  \tag{5.31}\\
& +\frac{1}{2} k_{G_{2}}^{2}\left\|F_{k}-F_{p}\right\|^{2}+\left\|\xi_{k+1}-\xi_{p+1}\right\|^{2} \\
\leq & \left\|R_{2} A_{I I}\right\|^{2}\left\|x_{k+\theta}-x_{p+\theta}\right\|^{2}+\left\|R_{2}\right\|^{2} k_{E}^{2}\left(t_{k}-t_{p}\right)^{2}+\frac{1}{2} k_{G_{2}}^{2} k_{F}^{2}\left(t_{k}-t_{p}\right)^{2} \\
& +\left\|\xi_{k+1}-\xi_{p+1}\right\|^{2} .
\end{align*}
$$

The lemma is proved. $\boxtimes$
Thus Corollary 5.5 extends as well, and in particular one has $\left\|\xi_{k+1}-\xi_{k}\right\| \leq h \alpha+$ $\beta\left\|x_{k+\theta}-x_{k-1+\theta}\right\|$ for all $k \geq 1$, for bounded constants $\alpha \geq 0$ and $\beta \geq 0$. The scheme in (5.25) is now replaced by:

$$
\begin{align*}
P_{I}^{h} x_{k+1}= & P_{I} x_{k}+h B_{I} z_{k+1}+h \bar{B}_{I}^{1}\left(D_{11}+\partial \psi_{K_{1}^{\star}}\right)^{-1}\left(C_{I I}^{1} R_{2}^{-1} \xi_{k+1}+F_{1, k}\right)  \tag{5.32}\\
& +h \bar{B}_{I}^{2}\left(-R_{2} B_{I I} R_{2}^{-1} \xi_{k+1}-R_{2} A_{I I} x_{k+\theta}-R_{2} E_{I I, k}+G_{2}\left(t_{k}, \xi_{k+1}\right)\right),
\end{align*}
$$

equivalently:
(5.33)

$$
\begin{aligned}
\left(P_{I}^{h}+h(1\right. & \left.-\theta) \bar{B}_{I}^{2} R_{2} A_{I I}\right) x_{k+1}=P_{I} x_{k}+h B_{I} z_{k+1} \\
& +h \bar{B}_{I}^{1}\left(D_{11}+\partial \psi_{K_{1}^{\star}}\right)^{-1}\left(C_{I I}^{1} R_{2}^{-1} \xi_{k+1}+F_{1, k}\right)+h \bar{B}_{I}^{2}\left(-R_{2} B_{I I} R_{2}^{-1} \xi_{k+1}\right. \\
& \left.-\theta R_{2} A_{I I} x_{k}-R_{2} E_{I I, k}+G_{2}\left(t_{k}, \xi_{k+1}\right)\right)
\end{aligned}
$$

and it appears that both schemes in (5.25) and in (5.33) possess the same structure. Consequently the results of convergence obtained for the foregoing implicit scheme (Proposition 5.6, Corollary 5.7, Lemma 5.8) extend to the scheme in this section, for suitably chosen numerical and system parameters. For the sake of briefness we do not repeat the developments.

Remark 5.12. Using Lemma 4.1 item 1, Assumption 10 could be replaced by assuming the strict passivity of the quadruple $\left(B_{I I}, \bar{B}_{I I}, C_{I I}, D\right)$ (a system property) and $h>0$ small enough.

Remark 5.13. Obviously when $B_{I I}$ is invertible it is possible to recast (2.1) into an LCS by solving the equality constraint for $z$. This yields the LCS quadruple $\left(P_{I}^{-1}\left(A_{I}-B_{I} B_{I I}^{-1} A_{I I}\right), P_{I}^{-1}\left(\bar{B}_{I}-B_{I} B_{I I}^{-1} \bar{B}_{I I}\right), C_{I}-C_{I I} B_{I I}^{-1} A_{I I}, D-C_{I I} B_{I I}^{-1} \bar{B}_{I I}\right)$. It is unclear how to relate in general the passivity of this quadruple, to the above assumptions which focus on the MLCP part (2.1) (b) (c) of the system. Especially no specific assumptions are made on the matrices $A_{I}, B_{I}, \bar{B}_{I}, C_{I}$ in the above analyses. For instance Assumption 3 implies that the feedthrough matrix of the LCS is $D-C_{I I} B_{I I}^{-1} X^{-1} C_{I I}^{\top}$, which may be indefinite in general. It is also noteworthy that transforming the MLCP into an LCP may be undesirable [59].
5.3. Recapitulation. Let us recapitulate the various assumptions upon which the results in this article rely. Assumptions 2, 4, 7, 10, 12 (respectively Assumptions $3,5,9)$, characterize the passivity properties of an auxiliary operator defined by the DALCS constraints in (3.2) (b) (c) (respectively in (2.1) (b) (c)). These passivity properties constrain the relative degree between the "input" $\lambda_{k+1}$ and the "output" $w_{k+1}=\tilde{C} z_{k+1}+\tilde{D} \lambda_{k+1}$ (respectively $\lambda$ and $\left.w=C_{I I} z+D \lambda\right)$. Assumptions 6 and 8 have practical importance (because some systems possess such $D$ matrix), they are
also related to passivity in the sense that passive systems possess positive semidefinite throughout matrices. Thus all these assumptions are fundamental ones for the developments in this article, which basically deals with the analysis of two types of generalised equations (GE): $0 \in A(u)+B(u)$ in (4.7) and (4.15), and $0 \in A(u)+B(u)+C(u)$ in (4.9) and (4.17), where the mappings $A(\cdot), B(\cdot)$ and $C(\cdot)$ are maximal monotone (notice in passing that due to time-dependency, these GEs do not fit with those in [28, Chapter 2], so that in particular [28, Theorem 2F.6] cannot be applied in general). Most importantly, the convergence results (Proposition 5.2, Lemmata 5.3 and 5.4, Corollary 5.5, Proposition 5.6, Corollary 5.7, Lemmata 5.8 and 5.10) rely upon the positive definiteness of $-R_{2} B_{I I} R_{2}^{-1}$ and $-\tilde{R}_{2} \tilde{A} \tilde{R}_{2}^{-1}$, which is an index one condition and guarantees strong monotonicity properties, allowing us to characterize the properties of the solutions to the GEs. It would be worth relaxing it in future studies. One path could be to use least norm elements [34]. Finally, it is worth noting that many practical systems like circuits with nonlinear resistors and inductances contain nonlinearities [19, section 4.1]. An extension of the above developments, consists of replacing the linear term $A_{I} x$ in (2.1), by a nonlinear term $A_{I}(x)$, where $A_{I}(\cdot)$ is globally Lipschitz continuous, and setting $A_{I}\left(x_{k}\right)$ in (3.1). The matrices $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ keep the same structure with $P_{I}^{-1}$ instead of $\left(P_{I}^{h}\right)^{-1}$, while both $\tilde{E}_{k}$ and $\tilde{F}_{k}$ contain $A_{I}\left(x_{k}\right)$. Nonlinearities in the equality and in the complementarity constraints should also be considered, where the results in [27] could be useful.
6. Examples and Numerical Simulations. The next examples are taken from circuits, mechanics and state-dependent-switch DAE. They aim at illustrating the above developments, and the limitations imposed by some assumptions.
6.1. Example 1 (circuit). Let us consider the circuit in Figure 1b. Using Kirchhoff laws its dynamics are given by:

$$
\left\{\begin{array}{l}
\dot{x}_{1}(t)=\frac{-2}{R \mathbf{C}} x_{1}(t)+\frac{1}{R \mathbf{C}} x_{2}(t)+\frac{1}{R} u_{D_{1}}(t)  \tag{6.1}\\
\dot{x}_{2}(t)=\frac{-1}{R \mathbf{C}} x_{2}(t)+\frac{1}{2 R \mathbf{C}} x_{1}(t)+\frac{1}{2 R} u_{D_{2}}(t) \\
0=\frac{3}{2} u_{D_{2}}(t)-R x_{3}(t)-\frac{1}{\mathbf{C}} x_{2}(t)+\frac{1}{2 \mathbf{C}} x_{1}(t) \\
0 \leq u_{D_{1}}(t) \perp \frac{-2}{R \mathbf{C}} x_{1}(t)+\frac{1}{R \mathbf{C}} x_{2}(t)+\frac{1}{R} u_{D_{1}}(t) \geq 0 \\
0 \leq u_{D_{2}}(t) \perp x_{3}(t) \geq 0,
\end{array}\right.
$$

with $x_{1}(t)=\int_{0}^{t} i_{1}(s) d s, x_{2}(t)=\int_{0}^{t} i_{2}(s) d s, x_{3}=i_{3}=z, \lambda=\left(\begin{array}{ll}u_{D_{1}} & u_{D_{2}}\end{array}\right)^{\top}$ (the voltages across the diodes), $K=K^{\star}=\mathbb{R}_{+}^{2}, n_{1}=2, p=n_{2}=1, m=2$. One has $D=\left(\begin{array}{cc}\frac{1}{R} & 0 \\ 0 & 0\end{array}\right), P_{I}=I_{2}, A_{I}=\left(\begin{array}{cc}\frac{-2}{R \mathbf{C}} & \frac{1}{R \mathbf{C}} \\ \frac{1}{2 R \mathbf{C}} & \frac{-1}{R \mathbf{C}}\end{array}\right), B_{I}=\binom{0}{0}, \bar{B}_{I}=\left(\begin{array}{cc}\frac{1}{R} & 0 \\ 0 & \frac{1}{2 R}\end{array}\right)$, $A_{I I}=\left(\begin{array}{ll}\frac{1}{2 \mathbf{C}} & -\frac{1}{\mathbf{C}}\end{array}\right), B_{I I}=-R, \bar{B}_{I I}=\left(\begin{array}{ll}0 & \frac{3}{2}\end{array}\right), C_{I}=\left(\begin{array}{cc}\frac{-2}{R \mathbf{C}} & \frac{1}{R \mathbf{C}} \\ 0 & 0\end{array}\right), C_{I I}=\binom{0}{1}$. Assumption 8 holds with $D_{11}=\frac{1}{R}$. Thus $\bar{B}_{I I}^{2}=\frac{3}{2}, C_{I I}^{2}=1, X_{2}=\frac{2}{3}$ and Assumption 9 is verified. Item 1 in Proposition 4.8 applies since $\operatorname{Im}\left(C_{I I}^{2}\right)=\mathbb{R}$. Since $\bar{B}_{I I}^{1}=0$ the mapping $G_{2}(\cdot)$ is zero, and item 3 in Proposition 4.8 applies. The quadruple $\left(B_{I I}, \bar{B}_{I I}, C_{I I}, D\right)$ is passive because the LMI (C.1) is verified with $X=\frac{2}{3}$, hence Assumption 5 is satisfied, but it is not strictly passive since $D \succcurlyeq 0$. It can be checked that the transfer matrix $H(s)=D+C_{I I}\left(s-B_{I I}\right)^{-1} \bar{B}_{I I}=\left(\begin{array}{cc}\frac{1}{R} & 0 \\ 0 & \frac{3}{2(s+R)}\end{array}\right)$ is positive real [18, Theorem 2.48]. The conditions of Lemmata 5.10 and 5.11 are verified by taking $R$ large enough (notice that all other parameters depend on $\frac{1}{R}$ ). Here $\tilde{A}=$ $B_{I I}=-R$, but Assumptions 2 and 6 do not hold, hence the material in Proposition 4.4 cannot be used.


Fig. 1: Two circuits with ideal diodes.
6.2. Example 2 (circuit). Let us consider now the circuit in Figure 1a. Using Kirchhoff laws its dynamics may be written as:

$$
\left\{\begin{array}{l}
\dot{x}_{1}(t)=-\frac{R_{1}}{L} x_{1}(t)-\frac{R_{3}}{L} z(t)+\frac{1}{L} u(t)+\frac{1}{L} u_{D}(t)  \tag{6.2}\\
\dot{x}_{2}(t)=\frac{R_{3}}{R_{2}} z(t)-\frac{1}{R_{2} \mathbf{C}} x_{2}(t) \\
0=x_{1}(t)+\frac{1}{R_{2} \mathbf{C}} x_{2}(t)-\left(1+\frac{R_{3}}{R_{2}}\right) z(t) \\
0 \leq u_{D}(t) \perp x_{1}(t) \geq 0
\end{array}\right.
$$

where $x_{1}=i_{1}, x_{2}(t)=\int_{0}^{t} i_{2}(s) d s, z=i_{3}, \lambda=u_{D}$ (the voltage across the diode), $K=K^{\star}=\mathbb{R}_{+}, n_{1}=2, p=n_{2}=1, m=1$. Let apply the feedback control $u=2 R_{3} z$, which modifies $B_{I}$. We get: $A_{I}=\left(\begin{array}{cc}-\frac{R_{1}}{L} & 0 \\ 0 & -\frac{1}{R_{2} \mathbf{C}}\end{array}\right), B_{I}=\binom{\frac{R_{3}}{L}}{\frac{R_{3}}{R_{2}}}, \bar{B}_{I}=\binom{\frac{1}{L}}{0}$, $A_{I I}=\left(\begin{array}{ll}1 & \frac{1}{R_{2} \mathbf{C}}\end{array}\right), B_{I I}=-\left(1+\frac{R_{3}}{R_{2}}\right), \bar{B}_{I I}=0, C_{I}=\left(\begin{array}{ll}1 & 0\end{array}\right), C_{I I}=0, D=0, E_{I}=\binom{\frac{1}{L} u}{0}$, $E_{I I}=0, F(t)=0, P_{I}=I_{2}, P_{I}^{h}=\left(\begin{array}{cc}1+h \frac{R_{1}}{L} & 0 \\ 0 & 1+h \frac{1}{R_{2} \mathbf{C}}\end{array}\right) \succ 0$ for all $h \geq 0$. Thus $\tilde{B}=\frac{h}{h R_{1}+L}, \tilde{C}=\frac{h R_{3}}{L+h R_{1}}, \tilde{D}=\frac{h}{L+h R_{1}}, \tilde{A}=-\frac{R_{2}+R_{3}}{R_{2}}+h R_{3}\left(\frac{1}{L+h R_{1}}+\frac{1}{R_{2}\left(h+R_{2} \mathbf{C}\right)}\right)$. Assumption 2 holds with $\tilde{X}=R_{3}, \tilde{D}>0$ for all $h>0$, and Assumption 4 holds for $h>0$ small enough such that $\tilde{A}<0$. For $h>0$ small enough the set $\tilde{\mathcal{S}}=\{0\}$ in (4.6) since $\tilde{R} \tilde{A} \tilde{R}^{-1}=\tilde{A}<0$, hence the VI in (4.7) always has a solution $z_{k+1}$ by Proposition 4.3 item 3 (a). By item (c) (or (f) ) of the same proposition, the solution is unique. When $\tilde{A}<0$, the quadruple $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is strictly passive for all $h>0$, hence items 1 and 3 of Corollary 4.6 apply. However Assumption 10 does not hold since $\tilde{D} \rightarrow 0$ as $h \rightarrow 0$. Consider now the quadruple $\left(B_{I I}, \bar{B}_{I I}, C_{I I}, D\right)=\left(B_{I I}, 0,0,0\right)$. It is passive since (C.1) is verified, it is even strictly state passive but not strictly passive since $D=0$. Assumptions 3 and 9 are trivially satisfied since both sides of the passivity equality are null. Provided that $\{0\} \in \mathbb{R}_{+}-x_{1, k+1} \Leftrightarrow x_{1, k+1} \geq 0$, item 1 in Proposition 4.7 applies and the OSNSP for (3.1) has a unique solution (but clearly there is no guarantee that this inequality is satisfied along the integration process, rendering Proposition 4.7 unapplicable). Proposition 4.8 item 3 also applies here for the VI in (4.17), however one has to check the equivalence stated in item 1 of the same proposition.
As noted above, the matrix $A_{I}$ intervenes only in Assumption 1, hence its modification does not alter the results. Setting $u=2 R_{3} z+R_{1} x_{1}+a L x_{1}+v(t)$ yields the modified $A_{I}=\left(\begin{array}{cc}a & 0 \\ 0 & -\frac{1}{R_{2} \mathbf{C}}\end{array}\right)$, and $B_{I}$ as above. This yields (see Remark 5.13): $P_{I}^{-1}\left(A_{I}-\right.$
$\left.B_{I} B_{I I}^{-1} A_{I I}\right)=\left(\begin{array}{cc}a+\frac{R_{2} R_{3}}{L\left(R_{2}+R_{3}\right)} & \frac{R_{3}}{L \mathbf{C}\left(R_{2}+R_{3}\right)} \\ \frac{R_{3}}{R_{2}+R_{3}} & -\frac{1}{\left(R_{2}+R_{3}\right) \mathbf{C}}\end{array}\right)$, which is unstable if $a+\frac{R_{2} R_{3}}{\left(R_{2}+R_{3}\right) L}>$ $\frac{1}{\left(R_{2}+R_{3}\right) \mathbf{C}}$. The other elements of the reduced LCS quadruple are given by $P_{I}^{-1}\left(\bar{B}_{I}-\right.$ $\left.B_{I} B_{I I}^{-1} \bar{B}_{I I}\right)=\binom{\frac{1}{L}}{0}, C_{I}-C_{I I} B_{I I}^{-1} A_{I I}=\left(\begin{array}{ll}1 & 0\end{array}\right), D-C_{I I} B_{I I}^{-1} \bar{B}_{I I}=0$. Consequently the reduced LCS is not passive with the above feedback controller. Also $P_{I}^{h}=\operatorname{diag}(1-$ $\left.h a, 1+\frac{h}{R_{2} \mathbf{C}}\right) \succ 0$ for all $h<\frac{1}{a}$.
To simplify the calculations let us set $R_{2}=R_{3}=1 \Omega, L=1 \mathrm{H}, \mathbf{C}=1 \mathrm{~F}, a=1$. Then the discretization (3.1) of (6.2) writes as:

$$
\left\{\begin{array}{l}
x_{1, k+1}-x_{1, k}=h x_{1, k+1}+h z_{k+1}+h \lambda_{k+1}+h v_{k}  \tag{6.3}\\
x_{2, k+1}-x_{2, k}=-h x_{2, k+1}+h z_{k+1} \\
0=x_{1, k+1}+x_{2, k+1}-2 z_{k+1} \\
0 \leq \lambda_{k+1} \perp x_{1, k+1} \geq 0
\end{array}\right.
$$

Equivalently the discretization in (3.2) can be used with the above matrices $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$.
6.3. Example 3 (switching DAE). Let us consider the following system, which is a nonsmooth DAE (similar to those studied in [53, 54, 52, 51], and which can be interpreted as a DAE with state-dependent switches):

$$
\left\{\begin{array}{l}
\text { (a) } \dot{x}(t)=A_{I} x(t)+B_{I} z(t)+E_{I}(t)  \tag{6.4}\\
(b) 0=a|x(t)|+b|z(t)|-c z(t)+E_{I I}(t)
\end{array}\right.
$$

with $x(t) \in \mathbb{R}, z(t) \in \mathbb{R}, a, b, c \in \mathbb{R}, c>0$. The constraint in (6.4) (b) is equivalently rewritten as:

$$
\begin{align*}
& 0=a \lambda_{1}(t)+a \lambda_{2}(t)+b \lambda_{3}(t)+b \lambda_{4}(t)-c z(t)+E_{I I}(t) \\
& 0 \leq\left(\begin{array}{l}
\lambda_{1}(t) \\
\lambda_{2}(t) \\
\lambda_{3}(t) \\
\lambda_{4}(t)
\end{array}\right) \perp\left(\begin{array}{l}
\lambda_{1}(t) \\
\lambda_{2}(t) \\
\lambda_{3}(t) \\
\lambda_{4}(t)
\end{array}\right)+\left(\begin{array}{c}
-x(t) \\
x(t) \\
0 \\
0
\end{array}\right)+\left(\begin{array}{c}
0 \\
0 \\
-z(t) \\
z(t)
\end{array}\right) \geq 0 . \tag{6.5}
\end{align*}
$$

Therefore (6.4) (a) (6.5) fits with (2.1) with $A_{I I}=0, B_{I I}=-c, \bar{B}_{I I}=\left(\begin{array}{lll}a & a & b\end{array}\right)$, $\left.D=I_{4}, C_{I}=\left(\begin{array}{lll}-1 & 1 & 0\end{array}\right)^{\top}, C_{I I}=\left(\begin{array}{ll}0 & 0\end{array}\right)_{1} 1\right)^{\top}, F(t)=0, \bar{B}_{I}=0$. Let us check the strict passivity of the quadruple $\left(B_{I I}, \bar{B}_{I I}, C_{I I}, D\right)$. This boils down to checking that (see (C.1)):

$$
M \triangleq\left(\begin{array}{ccccc}
2 c X & -a X & -a X & -b X-1 & -b X+1  \tag{6.6}\\
-a X & 2 & 0 & 0 & \\
-a X & 0 & 2 & 0 & 0 \\
-b X-1 & 0 & 0 & 2 & 0 \\
-b X+1 & 0 & 0 & 0 & 2
\end{array}\right) \succ 0
$$

for some $X>0$. Let us denote $M=P+N$ with $P=\operatorname{diag}\left(2 c X, 2 I_{4}\right) \succ 0$, and $N_{00}$ the matrix obtained for $a=b=0$. Applying [25, Theorem 2.11] or [10, Fact 8.19.2], it follows that $P+N_{00} \succ 0$ if $\sigma_{\max }^{2}\left(N_{00}\right)=2<\lambda_{\min }(2 c X) \lambda_{\min }\left(2 I_{4}\right)=4 c X$. Using continuity of the eigenvalues [10, Fact 10.11.8], it is deduced that given $c>0$, $X>0$, there exists $a$ and $b$ small enough such that (6.6) holds (or, given $a, b$ and $X>0$, there exists $c>0$ large enough). Thus, item 1 in Lemma 4.1 holds, item 2 in Proposition 4.3 holds for $h>0$ small enough (guaranteeing $\tilde{D} \succ 0$ ), Assumption

10 holds true for small enough $h>0$ (see the proof of item 1 in Lemma 4.1), and the arguments stated at the beginning of section 5.2 about the convergence analysis led in sections 5.1.1 and 5.1.2, hold. We have: $\tilde{A}=-c, \tilde{B}=\left(\begin{array}{lll}a & a & b\end{array}\right), \tilde{C}=$ $\left(\begin{array}{llll}0 & 0 & -1 & 1\end{array}\right)^{\top}+h\left(\begin{array}{llll}-1 & 1 & 0 & 0\end{array}\right)^{\top}\left(1-h A_{I}\right)^{-1} B_{I}=\left(\begin{array}{llll}\frac{-h B_{I}}{1-h A_{I}} & \frac{h B_{I}}{1-h A_{I}} & 0 & 0\end{array}\right)^{\top}, ~ \tilde{D}=D$. Integration can proceed using (3.2).
Let us now pass to a second example of state-dependent-switch DAE:

$$
\left\{\begin{array}{l}
(a) \dot{x}_{1}(t)=a_{1} x_{1}(t)+b_{1} x_{2}(t)+c_{1} z(t)+e_{1}(t)  \tag{6.7}\\
(b) \dot{x}_{2}(t)=a_{2} x_{1}(t)+b_{2} x_{2}(t)+c_{2} z(t)+e_{2}(t) \\
(c) 0=a_{3} x_{1}(t)+b_{3} x_{2}(t)+c_{3} z(t)+e_{3}(t) \text { if } x_{1}(t)<0 \\
(d) 0=a_{4} x_{1}(t)+b_{3} x_{2}(t)+c_{3} z(t)+e_{3}(t) \text { if } x_{1}(t)>0
\end{array}\right.
$$

where the switching conditions in (c) and (d) are equivalently rewritten as:

$$
\left\{\begin{array}{l}
0=-a_{3} \lambda_{1}(t)+a_{4} \lambda_{2}(t)+b_{3} x_{2}(t)+c_{3} z(t)+e_{3}(t)  \tag{6.8}\\
0 \leq \lambda_{1}(t) \perp \lambda_{1}(t)+x_{1}(t) \geq 0 \\
0 \leq \lambda_{2}(t) \perp \lambda_{2}(t)-x_{1}(t) \geq 0
\end{array}\right.
$$

Thus we have: $A_{I}=\left(\begin{array}{ll}a_{1} & b_{1} \\ a_{2} & b_{2}\end{array}\right), B_{I}=\left(\begin{array}{ll}c_{1} & c_{2}\end{array}\right)^{\top}, \bar{B}_{I}=\left(\begin{array}{ll}0 & 0\end{array}\right)^{\top}, E_{I}=\left(\begin{array}{ll}e_{1} & e_{2}\end{array}\right)^{\top}, \bar{B}_{I I}=$ $\left(\begin{array}{ll}-a_{3} & a_{4}\end{array}\right)^{\top}, A_{I I}=\left(\begin{array}{ll}0 & b_{3}\end{array}\right), B_{I I}=c_{3}, E_{I I}=e_{3}, C_{I}=\left(\begin{array}{cc}1 & 0 \\ -1 & 0\end{array}\right), C_{I I}=\left(\begin{array}{ll}0 & 0\end{array}\right)^{\top}, D=I_{2}$, $F(t)=\left(\begin{array}{ll}0 & 0\end{array}\right)^{\top}$. Also $\tilde{D}=D=I_{2}, \tilde{A}=c_{3}+h\left(\begin{array}{ll}0 & b_{3}\end{array}\right)\left(I_{2}-h A_{I}\right)^{-1}\binom{c_{1}}{c_{2}}=c_{3}+\mathcal{O}(h)$, $\tilde{B}=\bar{B}_{I I}=\left(\begin{array}{ll}-a_{3} & a_{4}\end{array}\right)^{\top}, \tilde{C}=h\left(\begin{array}{cc}1 & 0 \\ -1 & 0\end{array}\right)\left(I_{2}-h A_{I}\right)^{-1}\binom{c_{1}}{c_{2}}=\mathcal{O}(h)$. It can be checked that Proposition 4.3 items 1 and 2, Corollary 4.6 items 1,2 and 3 (if the passivity LMI holds strictly) apply. The passivity LMI $\left(\begin{array}{cc}-2 X \tilde{A} & -X\left(a_{3} a_{4}\right)+\tilde{C} \\ \left(-X\left(a_{3} a_{4}\right)+\tilde{C}\right)^{\top} & 2 I_{2}\end{array}\right) \succ 0$ is verified for $a_{3}, a_{4}$ and $h>0$ small enough, given $c_{3}<0, X>0$, implying the strict passivity of the quadruple $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$, uniformly as $h \rightarrow 0$ (Assumption 10 holds true). It is of interest to investigate what happens if $c_{3}=0$, which makes the "index" increase. Let us assume for simplicity that $A_{I}=0$, the matrix of the passivity LMI becomes $\left(\begin{array}{cc}-2 h X b_{3} c_{2} & -X\left(-a_{3} a_{4}\right)+h\left(c_{1}-c_{1}\right) \\ \left(-X\left(-a_{3} a_{4}\right)+h\left(c_{1}-c_{1}\right)\right)^{\top} & 2 I_{2}\end{array}\right)$. For given $X>0$ and $h>0$, there always exists $b_{3} c_{2}<0$ such that this matrix is $\succ 0$, showing that there exists conditions under which the OSNP is still well-posed when $c_{3}=0$. It is interesting to note that the term $b_{3} c_{2} z(t)$ appears when an index reduction is applied to the equality in (6.8) by differentiating it once. Simulations are shown in Figures 2 and 3. They are performed with the INRIA software package sICONOs ${ }^{1}$ [1]. The MLCP is solved with an enumeration method and the differential variables $x_{1}(\cdot)$ and $x_{2}(\cdot)$ are calculated in a second step (hence we follow the philosophy of the foregoing sections). The parameters are chosen as follows: $a_{1}=a_{2}=b_{1}=b_{2}=a_{3}=$ $a_{4}=b_{3}=c_{3}=1, c_{1}=10, c_{2}=20, e_{1}(t)=40 \sin (25 t), e_{2}(t)=\sin (t), e_{3}(t)=\sin (10 t)$, $h=2.10^{-3} \mathrm{~s}, x_{1}(0)=0, x_{2}(0)=0.1$. Simulations are done on $[0,2] \mathrm{s}$. The errors are calculated as $\max _{t \in[0,2]}\left\|x_{i}(t)-x_{i}^{r e f}(t)\right\|, i=1,2$, with 14 different values of step size $h$ between $10^{-1} \mathrm{~s}$ and $10^{-4} \mathrm{~s}$, where $x_{i}^{\text {ref }}(\cdot)$ is calculated with $h=10^{-4} \mathrm{~s}$. It is observed in Figure 2 that the order is approximately equal to one for small enough

[^1]step size. Figure 3 depicts the time evolution of various variables, demonstrating that the complementarity constraints are respected. Clearly the multipliers are continuous but non differentiable when the complementarity switches from one mode to another.


Fig. 2: Log-log error diagram: $x_{1}$ (left) and $x_{2}$ (right).
6.4. Example 4 (mechanical system). As an extension of the systems studied in [37], let us consider a linear mechanical system with both unilateral springs and bilateral constraints, whose dynamics are:

$$
\left\{\begin{array}{l}
(a) M \ddot{q}(t)+K_{v} \dot{q}(t)+K_{p} q(t)=\tau(t)+H^{\top} \lambda_{u}(t)+G^{\top} \lambda_{b}(t)  \tag{6.9}\\
\text { (b) } 0=G q(t)-f(t) \\
(c) 0 \leq \lambda_{u}(t) \perp D \lambda_{u}(t)+C_{I} x(t)+F(t) \geq 0,
\end{array}\right.
$$

where $M=M^{\top} \succ 0, D \succ 0, x=\left(q^{\top} \dot{q}^{\top}\right)^{\top}, q(t) \in \mathbb{R}^{\frac{n_{1}}{2}}, z=\lambda_{b} \in \mathbb{R}^{n_{2}}, \lambda=\lambda_{u} \in$ $\mathbb{R}^{m}, G \in \mathbb{R}^{n_{2} \times \frac{n_{1}}{2}}, f: \mathbb{R}^{+} \rightarrow \mathbb{R}^{n_{2}}$ is twice differentiable, $\tau(\cdot)$ is an exogenous force. Here an index reduction is possible, in order to replace (6.9) (b) by:
(6.10) $0=G \ddot{q}(t)-\ddot{f}(t)=G M^{-1} H^{\top} \lambda(t)+G M^{-1} G^{\top} z(t)-G M^{-1}\left(K_{p} K_{v}\right) x(t)-\ddot{f}(t)$, and $G q(0)-f(0)=0, G \dot{q}(0)-\dot{f}(0)=0$. Thus $B_{I I}=-G M^{-1} G^{\top}(\prec 0$ if bilateral constraints are independent, i.e., $\left.\operatorname{rank}(G)=p=n_{2} \Rightarrow n_{1} \geq 2 n_{2}\right), A_{I I}=G M^{-1}\left(K_{p} K_{v}\right)$, $\bar{B}_{I I}=-G M^{-1} H^{\top}, B_{I}=\binom{0}{-M^{-1} G^{\top}}$, and $C_{I I}=0$ (similarly to the example in section 6.2, one major difference being that here $D \succ 0$ ). Let us examine the OSNSP for (6.9) on a particular example as depicted in Figure 4, whose dynamics are given by:

$$
\left\{\begin{array}{l}
m \ddot{q}_{1}(t)=\tau(t)+\lambda_{1}(t)-\lambda_{2}(t)  \tag{6.11}\\
m \ddot{q}_{2}(t)=z(t) \\
0=q_{2}(t)-f(t) \Leftrightarrow \frac{-z(t)}{m}+\ddot{f}(t)=0, q_{2}(0)-f(0)=0, \dot{q}_{2}(0)-\dot{f}(0)=0 \\
0 \leq \lambda_{1}(t) \perp \lambda_{1}(t)+k\left(q_{1}(t)-L-\alpha(t)-l\right) \geq 0 \\
0 \leq \lambda_{2}(t) \perp \lambda_{2}(t)-k\left(q_{1}(t)+L-\beta(t)+l\right) \geq 0 .
\end{array}\right.
$$

where $q=\left(\begin{array}{ll}q_{1} & q_{2}\end{array}\right)^{\top}$ are the mass coordinates, $k>0$ is a stiffness constant, $\alpha(\cdot)$ and $\beta(\cdot)$ are the massless springs' varying positions, $l>0$ is the spring's free length,


Fig. 3: (a) $x_{1}(t)$ (dashed) and $x_{2}(t)$, (b) $z(t)$ (dashed) and $\lambda_{1}(t)$, (c) $\lambda_{2}(t)-x_{1}(t)$ and $\lambda_{2}(t)$ (dashed), (d) $\lambda_{1}(t)+x_{1}(t)$ and $\lambda_{1}(t)$ (dashed).
$\beta(t)>\alpha(t)+2 L+2 l$ for all $t \geq 0$. We have: $x=\left(\begin{array}{lll}q_{1} & \dot{q}_{1} & q_{2} \\ \dot{q}_{2}\end{array}\right)^{\top}, G=(01), H=$ $\left(\begin{array}{cc}1 & 0 \\ -1 & 0\end{array}\right), M=\operatorname{diag}(m, m), D=I_{2}, A_{I I}=0, B_{I I}=\frac{-1}{m}, \bar{B}_{I I}=0, E_{I I}(t)=\ddot{f}(t)$, $C_{I}=\left(\begin{array}{cccc}k & 0 & 0 & 0 \\ -k & 0 & 0 & 0\end{array}\right), F(t)=\binom{k(-L-\alpha(t)-l)}{-k(L-\beta(t)+l)}, C_{I I}=0, B_{I}=\left(\begin{array}{llll}0 & 0 & 0 & \frac{1}{m}\end{array}\right)^{\top}$, $P_{I}=I_{4}\left(\right.$ and Assumption 1 holds for $h>0$ small enough), $\bar{B}_{I}=\left(\begin{array}{cccc}0 & \frac{1}{m} & 0 & 0 \\ 0 & \frac{-1}{m} & 0 & 0\end{array}\right)^{\top}$, $\tilde{A}=\frac{-1}{m}, \tilde{B}=0, \tilde{C}=h C_{I}\left(P_{I}^{h}\right)^{-1} B_{I}, \tilde{D}=I_{2}+h C_{I}\left(P_{I}^{h}\right)^{-1} \bar{B}_{I} \succ 0$ if $h>0$ is small enough. Let us check Assumption 4. The passivity LMI is $\left(\begin{array}{cc}\frac{2 X}{m} & \tilde{C} \\ C^{\top} & \tilde{D}+\tilde{D}^{\top}\end{array}\right) \succ 0$, $X>0$, which is verified for $h>0$ small enough and $X>0$ large enough. Hence under these conditions the quadruple $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is strictly passive. It can be checked that items 1 and 2 in Proposition 4.3 hold, as well as items 1,2 and 3 in Corollary 4.6.

Therefore the dynamics can be integrated using (3.2) with $q_{2,0}+f_{0}=0, \dot{q}_{2,0}+\dot{f}_{0}=0$. We note also that the quadruple $\left(B_{I I}, \bar{B}_{I I}, C_{I I}, D\right)=\left(\frac{-1}{m}, 0,0, I_{2}\right)$ is strictly passive (Assumption 5 is verified), hence item 1 in Lemma 4.1 holds. Assumption 10 is also verified (for $h>0$ small enough but uniformly in $h \geq 0$ ). Let us analyse the convergence as in section 5.1. Assumption 11 may not be verified in this case, however as noted after this assumption it is easy to modify slightly the numerical scheme so that it is satisfied with $P_{I}=I_{4}$. Hence it is inferred that Proposition 5.2 applies.


Fig. 4: A mechanical DALCS.


Fig. 5: An $R L C D$ passive circuit.
6.5. Example 5 (circuit). Let us consider the circuit in Figure 5 , with $R_{1}=$ $R_{3}=0 \Omega$. Let $x_{1}=i_{2}, x_{2}=\int_{0}^{t} i_{3}(t) d t, z=i_{1}, \lambda=u_{D}$ is the voltage across the diode, $u_{1}(\cdot)$ and $u_{2}(\cdot)$ are two voltage sources. One way to write its dynamics is:

$$
\left\{\begin{array}{l}
\dot{x}_{1}(t)=-\frac{R_{2}}{L} x_{1}(t)+\frac{1}{L} \lambda(t)+\frac{1}{L} u_{1}(t)  \tag{6.12}\\
\dot{x}_{2}(t)=x_{1}(t)-z(t) \\
0=\frac{1}{\mathbf{C}} x_{2}(t)+\lambda(t)+u_{2}(t) \\
0 \leq \lambda(t) \perp z(t) \geq 0
\end{array}\right.
$$

Assume that the feedback $u_{2}=-a z+u_{3}$ is applied, so that the equality constraint becomes $0=\frac{1}{\mathbf{C}} x_{2}(t)+u_{D}(t)-a z(t)+u_{3}(t)$. We have: $A_{I I}=\left(\begin{array}{ll}0 & \frac{1}{\mathbf{C}}\end{array}\right), C_{I}=0$, $C_{I I}=1, B_{I I}=-a, \bar{B}_{I I}=1, D=0, \bar{B}_{I}=\left(\begin{array}{cc}\frac{1}{L} & 0\end{array}\right)^{\top}, B_{I}=\left(\begin{array}{ll}0 & -1\end{array}\right)^{\top}$, and $\tilde{D}=D=0$, $\tilde{C}=C_{I I}=1, \tilde{A}=-a+h A_{I I}\left(P_{I}^{h}\right)^{-1} B_{I}, \tilde{B}=1+h A_{I I}\left(P_{I}^{h}\right)^{-1} \bar{B}_{I}$. The condition $\operatorname{Im}\left(C_{I I}\right) \cap\left(K-C_{I} x_{k+1}+F_{k}\right)=\mathbb{R} \cap \mathbb{R}_{+}-\frac{1}{\mathbf{C}} x_{2, k+1} \neq \emptyset$ is verified always. The set $\mathcal{S}$ in (4.16) is equal to $\{0\}$ hence Proposition 4.7 item 1 applies and the VI in (4.15) always has a solution. Proposition 4.8 items 1, 2, 3 apply, with $D_{11}=0$ in Assumption 8.

The system $\left(B_{I I}, \bar{B}_{I I}, C_{I I}\right)=(-a, 1,1)$ is passive and Assumptions 3 and 5 hold true. Also the system $(\tilde{A}, \tilde{B}, \tilde{C})$ is passive for $h>0$ small enough and Assumptions 2 and 4 hold true. The set $\mathcal{S}$ in (4.6) is equal to $\{0\}$ so Proposition 4.3 item 3 (a) applies. For $h>0$ small enough, $\tilde{A}<0$ so Corollary 4.6 items 1 and 2 apply. Assumption 12 is verified for $h \geq 0$ small enough with $\tilde{X}=\tilde{B}^{-1}$. It is inferred that if $a>0$ is large enough, the convergence results in sections 5.1.3 and 5.2 apply.
7. Conclusion. This article is devoted to analysing the implicit Euler discretisation of a class of nonsmooth nonlinear systems with equality constraints, named differential-algebraic linear complementarity systems. The well-posedness of the onestep nonsmooth problem is studied first, then the convergence of the discrete-time solutions is analysed. This article only scratches the surface of singular nonsmooth systems analysis. The above results could be extended in various directions. Most importantly, it would be worth extending the results to more general indices and relative degrees, to deeper analyse how the algebraic and the complementarity constraints are coupled, and to consider state discontinuities and solutions of bounded variation.

Appendix A. Well-posedness of Variational Inequalities. Some basic notions of Convex Analysis and Complementarity Theory are briefly introduce now [11, 29, 49, 31]. Let $f: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ be a proper convex and lower semicontinuous (lsc) function, we denote by $\operatorname{dom}(f) \triangleq\left\{x \in \mathbb{R}^{n} \mid f(x)<+\infty\right\}$ the domain of the function $f(\cdot)$. The Fenchel transform $f^{\star}(\cdot)$ of $f(\cdot)$ is the proper, convex and lsc function defined by: for all $z \in \mathbb{R}^{n}, f^{\star}(z)=\sup _{x \in \operatorname{dom}(f)}\{\langle x, z\rangle-f(x)\}$. The subdifferential $\partial f(x)$ of $f(\cdot)$ at $x \in \mathbb{R}^{n}$ is defined by: $\partial f(x)=\left\{\omega \in \mathbb{R}^{n} \mid f(v)-f(x) \geq\right.$ $\left.\langle\omega, v-x\rangle, \forall v \in \mathbb{R}^{n}\right\}$. We denote by $\operatorname{Dom}(\partial f) \triangleq\left\{x \in \mathbb{R}^{n} \mid \partial f(x) \neq \emptyset\right\}$ the domain of the subdifferential operator $\partial f: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$.
Let $x_{0}$ be any element in the domain $\operatorname{dom}(f)$, the recession function $f_{\infty}(\cdot)$ of $f(\cdot)$ is defined by: for all $x \in \mathbb{R}^{n}: f_{\infty}(x)=\lim _{\lambda \rightarrow+\infty} \frac{1}{\lambda} f\left(x_{0}+\lambda x\right)$. The function $f_{\infty}$ : $\mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ is a proper convex and lsc function which describes the asymptotic behavior of $f(\cdot)$. Let $K \subset \mathbb{R}^{n}$ be a nonempty closed convex set. Let $x_{0}$ be any element in $K$. The recession cone of $K$ is defined by [49]:

$$
K_{\infty}=\bigcap_{\lambda>0} \frac{1}{\lambda}\left(K-x_{0}\right)=\left\{u \in \mathbb{R}^{n} \mid x+\lambda u \in K, \forall \lambda \geq 0, \forall x \in K\right\}
$$

The set $K_{\infty}$ is a nonempty closed convex cone that is described in terms of the directions which recede from $K$. The indicator function of a set $K \subseteq \mathbb{R}^{n}$ is $\Psi_{K}(x)=0$ if $x \in K, \Psi_{K}(x)=+\infty$ if $x \notin K$. If $K$ is closed non empty convex, we have $\partial \Psi_{K}(x)=\mathcal{N}_{K}(x)$, the so-called normal cone to $K$ at $x$, defined as $\mathcal{N}_{K}(x)=\{v \in$ $\mathbb{R}^{n} \mid v^{\top}(s-x) \leq 0$ for all $\left.s \in K\right\}$.
Important properties of the recession function and recession cone are recalled now:
Proposition A.1. [11, Proposition 1.4.8] The following statements hold:
a) Let $f_{1}: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ and $f_{2}: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ be two proper, convex and lsc functions. Suppose that $f_{1}+f_{2}$ is proper. Then for all $x \in \mathbb{R}^{n}:\left(f_{1}+f_{2}\right)_{\infty}(x)=$ $\left(f_{1}\right)_{\infty}(x)+\left(f_{2}\right)_{\infty}(x)$.
b) Let $f: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ be a proper, convex and lsc function, and let $K$ be a nonempty closed convex set, such that $f+\Psi_{K}$ is proper (equivalently $\operatorname{dom}(f) \cap K$ is non empty). Then for all $x \in \mathbb{R}^{n}:\left(f+\Psi_{K}\right)_{\infty}(x)=f_{\infty}(x)+\left(\Psi_{K}\right)_{\infty}(x)$.
c) Let $K \subset \mathbb{R}^{n}$ be a nonempty, closed and convex set. Then for all $x \in \mathbb{R}^{n}$ : $\left(\Psi_{K}\right)_{\infty}(x)=\Psi_{K_{\infty}}(x)$. Moreover for all $x \in K$ and $e \in K_{\infty}: x+e \in K$.
d) If $K \subseteq \mathbb{R}^{n}$ is a nonempty closed and convex cone, then $K_{\infty}=K$.
e) Let $K=P(A, b) \triangleq\left\{x \in \mathbb{R}^{n} \mid A x \geq b\right\}$ for $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^{m}$. If $K \neq \emptyset$ then $K_{\infty}=P(A, 0)=\left\{x \in \mathbb{R}^{n} \mid A x \geq 0\right\}$.
Let us now concatenate [4, Theorem 3, Corollaries 3 and 4]. They concern variational inequalities (VIs) of the form: Find $u \in \mathbb{R}^{n}$ such that
$\langle\mathbf{M} u+\mathbf{q}, v-u\rangle+\varphi(v)-\varphi(u) \geq 0$, for all $v \in \mathbb{R}^{n}$
where $\mathbf{M} \in \mathbb{R}^{n \times n}$ is a real matrix, $\mathbf{q} \in \mathbb{R}^{n}$ a vector and $\varphi: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ a proper convex and lsc function. The VI in (A.1) is equivalent to the inclusion $\mathbf{M} u+\mathbf{q} \in-\partial \varphi(u)$, which is a generalized equation (GE) of the form: $0 \in A(u)+B(u)$ where $A(\cdot)$ and $B(\cdot)$ are two maximal monotone operators (since it will be assumed $\mathbf{M} \succcurlyeq 0$ ). In fact the problems encountered in this paper (see (4.7), (4.9), (4.14), (4.15), (4.17)) all belong to this class of GEs. Numerical schemes that compute solutions of such GEs have been studied for a long time in the mathematical literature, see [57] and references therein. The next proposition states conditions about existence and uniqueness of solutions.
The problem in (A.1) is denoted as $\operatorname{VI}(\mathbf{M}, \mathbf{q}, \varphi)$, and we set:

$$
\begin{equation*}
\mathcal{K}(\mathbf{M}, \varphi)=\left\{x \in \mathbb{R}^{n} \mid \mathbf{M} x \in\left(\operatorname{dom}\left(\varphi_{\infty}\right)\right)^{\star}\right\} \tag{A.2}
\end{equation*}
$$

Note that $\left(\operatorname{dom}\left(\varphi_{\infty}\right)\right)^{\star}$ is the dual cone of the domain of the recession function $\varphi_{\infty}$ while $(\operatorname{dom}(\varphi))_{\infty}$ (that we may denote also as $\left.\operatorname{dom}(\varphi)_{\infty}\right)$ is the recession cone of $\operatorname{dom}(\varphi)$.

Proposition A.2. [4] Let $\varphi: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ be a proper, convex and lsc function with closed domain, $\mathbf{M} \in \mathbb{R}^{n \times n}$, and suppose that $\mathbf{M} \succcurlyeq 0$.
a) If $(\operatorname{dom}(\varphi))_{\infty} \cap \operatorname{Ker}\left\{\mathbf{M}+\mathbf{M}^{\top}\right\} \cap \mathcal{K}(\mathbf{M}, \varphi)=\{0\}$ then for each $\mathbf{q} \in \mathbb{R}^{n}$, problem $\operatorname{VI}(\mathbf{M}, \mathbf{q}, \varphi)$ has at least one solution.
b) Suppose that $(\operatorname{dom}(\varphi))_{\infty} \cap \operatorname{Ker}\left\{\mathbf{M}+\mathbf{M}^{\top}\right\} \cap \mathcal{K}(\mathbf{M}, \varphi) \neq\{0\}$. If there exists $x_{0} \in$ $\operatorname{dom}(\varphi)$ such that
(A.3)
$\left\langle\mathbf{q}-\mathbf{M}^{\top} x_{0}, v\right\rangle+\varphi_{\infty}(v)>0, \forall v \in(\operatorname{dom}(\varphi))_{\infty} \cap \operatorname{Ker}\left\{\mathbf{M}+\mathbf{M}^{\top}\right\} \cap \mathcal{K}(\mathbf{M}, \varphi), v \neq 0$,
then problem $\operatorname{VI}(\mathbf{M}, \mathbf{q}, \varphi)$ has at least one solution.
c) If $u_{1}$ and $u_{2}$ are two solutions of problem $\operatorname{VI}(\mathbf{M}, \mathbf{q}, \varphi)$ then $u_{1}-u_{2} \in \operatorname{Ker}\{\mathbf{M}+$ $\left.\mathbf{M}^{\top}\right\}$.
d) If $\mathbf{M}=\mathbf{M}^{\top}$ and $u_{1}$ and $u_{2}$ denote two solutions of problem $\operatorname{VI}(\mathbf{M}, \mathbf{q}, \varphi)$, then $\left\langle\mathbf{q}, u_{1}-u_{2}\right\rangle=\varphi\left(u_{2}\right)-\varphi\left(u_{1}\right)$.
e) If $\mathbf{M}=\mathbf{M}^{\top}$, then $u$ is a solution of $\operatorname{VI}(\mathbf{M}, \mathbf{q}, \varphi)$ if and only if it is a solution of the optimization problem $\min _{x \in \mathbb{R}^{n}} \frac{1}{2} x^{\top} \mathbf{M} x+\langle\mathbf{q}, x\rangle+\varphi(x)$.

Appendix B. Some Convex Analysis and Complementarity Theory tools. If $K \subseteq \mathbb{R}^{n}$ is a set, then $K^{\star}=\left\{z \in \mathbb{R}^{n} \mid\langle z, x\rangle \geq 0\right.$ for all $\left.x \in K\right\}$ is its dual cone. Let $K$ be a nonempty closed convex cone, then:

$$
\begin{equation*}
K^{\star} \ni x \perp y \in K \Leftrightarrow x \in-\mathcal{N}_{K}(y) \Longleftrightarrow y \in-\mathcal{N}_{K^{\star}}(x) . \tag{B.1}
\end{equation*}
$$

Let $M=M^{\top} \succ 0, x$ and $y$ two vectors, then [29]

$$
\begin{equation*}
M(x-y) \in-\mathcal{N}_{K}(x) \Leftrightarrow x=\operatorname{proj}_{M}[K ; y] \Leftrightarrow x=\min _{z \in K} \frac{1}{2}(z-y)^{\top} M(z-y) \tag{B.2}
\end{equation*}
$$

We note that this is a particular case of (A.1), so that Proposition A. 2 can be considered as the characterization of a generalized projection operator $V I(\mathbf{M}, \mathbf{q}, \varphi)$. The next proposition is used several times in the article:

Proposition B.1. Suppose that the $m \times m$ matrix $D \succ 0$, and $\mathcal{M}: \mathbb{R}^{m} \rightrightarrows \mathbb{R}^{m}$ is a maximal monotone operator. Then the operator $(D+\mathcal{M})^{-1}(\cdot)$ is well-defined (i.e., for each $y \in \mathbb{R}^{m}$ there is an $x \in \mathbb{R}^{m}$ such that $\left.y \in D x+\mathcal{M}(x)\right)$, single-valued and Lipschitz continuous with modulus $L=\frac{2}{\lambda_{\min }\left(D+D^{\top}\right)}$.
It was proposed in [16, Proposition 1] when $\mathcal{M}=\partial \phi$ for some convex proper lsc $\phi(\cdot)$, extended to maximal monotone $\mathcal{M}(\cdot)$ in [5].

Appendix C. Passive Systems. A quadruple $(A, B, C, D)$ is said passive if:

$$
\left(\begin{array}{cc}
-X A-A^{\top} X & -X B+C^{\top}  \tag{C.1}\\
-B^{\top} X+C & D+D^{\top}
\end{array}\right) \succcurlyeq 0, \quad X=X^{\top} \succ 0 .
$$

for some $X$. It is said strictly passive if the first matrix inequality is satisfied with $\succ 0$ (this implies that $D \succ 0$ ). This is not to be confused with the strict state passivity [18, Definition 4.54] (strict passivity implies strict state passivity). In fact strict passivity is directy related to strongly SPR transfer matricess [18, p. 62 and section 3.12.2], while SPR transfer matrices with minimal realization are strictly state passive [18, Theorem 4.73] [43]. Strict state passivity holds when $-X A-A^{\top} X \succ 0$ in (C.1), with $D+D^{\top} \succcurlyeq 0$. The two inequalities in (C.1) make a linear matrix inequality (LMI) with unknown $X$, equivalently a Riccati inequality using the Schur complement Theorem [18, Theorem A.65]. Numerical algorithms exist that efficiently solve LMIs, see [13], and can be used to test whether or not a quadruple is passive.

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[^1]:    ${ }^{1}$ https://nonsmooth.gricad-pages.univ-grenoble-alpes.fr/siconos/index.html

