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1 ANALYSIS OF THE IMPLICIT EULER TIME-DISCRETIZATION OF
2 SEMI-EXPLICIT DIFFERENTIAL-ALGEBRAIC LINEAR
3 COMPLEMENTARITY SYSTEMS*

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5 **Abstract.** This article is largely concerned with the time-discretization of differential-algebraic
6 equations (DAE) with complementarity constraints, which we name differential algebraic linear com-
7 plementarity systems (DALCS). Specifically, the Euler implicit discretization of DALCS is analysed:
8 the one-step non-smooth problem (OSNSP), that is a generalized equation, is shown to be well-
9 posed under some conditions, then the convergence of the discretized solutions is studied, and the
10 existence of solutions to the continuous-time system is shown as a consequence. Passivity of some
11 operators is pivotal to the analysis. Examples from circuits, mechanics and switching DAE illustrate
12 the applicability of the developments.

13 **Key words.** differential-algebraic system, descriptor-variable system, linear complementarity
14 system, recession function, discrete-time, well-posedness, passive system, Lur'e equations.

15 **AMS subject classifications.** 65K15, 90C33, 90C90, 93B52, 93D30, 93C30.

16 **1. Introduction.** The analysis of non-smooth dynamical systems with multi-
17 valued right-hand sides satisfying maximal monotone properties has witnessed a large
18 number of contributions [19]. Within this class one finds linear and nonlinear com-
19 plementarity dynamical systems, projected systems, differential variational inequali-
20 ties, differential inclusions with maximal monotone right-hand sides, Moreau's sweep-
21 ing process, some switching dynamics, *etc.* In this article we focus on singular, or
22 differential-algebraic linear complementarity systems (DALCS), which may be viewed
23 either as an extension of classical differential-algebraic equations (DAE), or of "clas-
24 sical" linear complementarity systems. As will become clear later, DALCS represent
25 an extension of the set-valued Lur'e systems studied, *e.g.*, in [39, 40]. The associated
26 canonical form is motivated by applications in chemistry [53, 52, 62], switching DAE
27 analysis [48], circuits with nonsmooth electronic components [2, section 3.5, chapter 4]
28 [22], optimization-constrained differential equations [19, Example 2] with applications
29 in atmospheric aerosol particles [38], and Lagrangian systems with both bilateral and
30 unilateral constraints [17] (in spite of the fact that the results presented in this article
31 do not apply to all of these systems).

32 The study of singular nonsmooth dynamical systems has not received a lot of attention
33 yet. Singularly perturbed differential inclusions have been tackled in [26, 56, 32, 58].
34 The conditions on the set-valued mappings that are imposed (compactness [56, 58],
35 boundedness and Hausdorff Lipschitz continuity [32], one-sided Lipschitz condition
36 [26]) are not satisfied by complementarity conditions (or by normal cones to convex
37 sets). Hence these results do not apply to DALCS. The well-posedness of DAE in semi-
38 explicit form with nonsmooth constraints (which are related to the class of systems
39 studied in this article, see Remark 2.1 below) has been analysed in [53, 54, 52]. Passive
40 singular set-valued systems are studied in [22], who derived well-posedness relying on
41 a special Weierstrass' representation. However the time-discretisation is not tackled

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42 in the above references.

43 Our goal is to analyse the well-posedness of the one-step nonsmooth problem (OS-
 44 NSP) obtained after an implicit (backward) Euler discretization, and the convergence
 45 of the discrete-time solutions towards a continuous-time limit. The second step is use-
 46 ful not only for proving the existence of solutions to the continuous-time system, but
 47 also, perhaps most importantly, to show that the proposed time-discretization makes
 48 sense for numerical simulations. Passivity properties of suitable operators are pivotal
 49 throughout the article. It is noteworthy that implicit Euler methods are well-known
 50 in DAE [30, 14, 33, 42, 12, 35, 61, 7], differential variational inequalities [46, 24],
 51 and complementarity systems fields [41, 21, 59, 34], as well as in the field of max-
 52 imal monotone differential inclusions [8, 47] (implicit Euler method is often called
 53 proximal algorithm, while Euler method refers to the explicit algorithm [47]), and
 54 Moreau's sweeping processes with the catching-up algorithm [19, 44, 45]. However
 55 their application to DALCS has not been studied yet.

56 The article is organised as follows: Section 2 introduces the general dynamics of
 57 the nonsmooth singular systems, section 3 presents their time-discretization, section
 58 4 studies the well-posedness of the OSNSP, section 5 is dedicated to convergence
 59 analyses: results in section 5.1 are based on the properties of the numerical scheme,
 60 while section 5.2 uses properties of the system. Section 6 is dedicated to illustrating
 61 examples and simulations, from circuits, mechanics and switching DAE. Conclusions
 62 are drawn in section 7. The appendix is dedicated to recall various mathematical
 63 tools.

64 **Notation and definitions:** for any vector $x \in \mathbb{R}^n$ and any matrix $M \in \mathbb{R}^{m \times n}$,
 65 $\|x\|$ is the Euclidean norm and $\|M\|$ is the Frobenius norm, which are compatible
 66 norms [10, Proposition 9.3.5], *i.e.*, $\|Mx\| \leq \|M\| \|x\|$. Let $M \in \mathbb{R}^{n \times m}$, then $\text{Im}(M)$
 67 is its range, $\text{Ker}(M)$ is its null space. We use $\langle x, y \rangle = x^\top y$, so $\langle x, x \rangle = \|x\|^2$. Positive
 68 definite matrix: $M \succ 0$ if $x^\top M x > 0$ for all $x \neq 0$, positive semidefinite matrix:
 69 $M \succcurlyeq 0$ if $x^\top M x \geq 0$ for all x (such M is not necessarily symmetric). The maximum
 70 singular value is denoted as $\sigma_{\max}(M)$, and the minimum eigenvalue as $\lambda_{\min}(M)$. The
 71 i th row of M is denoted as $M_{i\bullet}$. The $n \times n$ identity matrix is denoted I_n . A set-valued
 72 mapping $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said monotone if for all $x_1, x_2, y_1 \in A(x_1), y_2 \in A(x_2)$, one
 73 has $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$. It is maximal monotone if its graph cannot be enlarged
 74 without destroying monotonicity. It is η -strongly monotone if there exists $\eta > 0$ such
 75 that $\langle x_1 - x_2, y_1 - y_2 \rangle \geq \eta \|x_1 - x_2\|^2$. See also Appendix A for further definitions.

76 **2. The class of nonsmooth singular dynamical systems.** Let us consider
 77 the following differential algebraic linear complementarity system (DALCS):

$$78 \quad (2.1) \quad \begin{cases} (a) & P_I \dot{x}(t) = A_I x(t) + B_I z(t) + \bar{B}_I \lambda(t) + E_I(t) \\ (b) & 0 = A_{II} x(t) + B_{II} z(t) + \bar{B}_{II} \lambda(t) + E_{II}(t) \\ (c) & K^* \ni \lambda(t) \perp w(t) = C_I x(t) + C_{II} z(t) + D \lambda(t) + F(t) \in K, \end{cases}$$

79 with $x(t) \in \mathbb{R}^{n_1}$, $z(t) \in \mathbb{R}^{n_2}$, $\lambda(t) \in \mathbb{R}^m$, $w(t) \in \mathbb{R}^m$, $P_I \in \mathbb{R}^{n_1 \times n_1}$ is full-rank,
 80 $K \subseteq \mathbb{R}^m$ is a closed nonempty convex cone, K^* is its dual cone, $\bar{B}_{II} \in \mathbb{R}^{p \times m}$,
 81 $A_{II} \in \mathbb{R}^{p \times n_1}$, $B_{II} \in \mathbb{R}^{p \times n_2}$, $C_I \in \mathbb{R}^{m \times n_1}$, $C_{II} \in \mathbb{R}^{m \times n_2}$, $D \in \mathbb{R}^{m \times m}$, $B_I \in$
 82 $\mathbb{R}^{n_1 \times n_2}$, $A_I \in \mathbb{R}^{n_1 \times n_1}$, $\bar{B}_I \in \mathbb{R}^{n_1 \times m}$, and $E_I(\cdot)$, $E_{II}(\cdot)$, $F(\cdot)$ are bounded, Lipschitz
 83 continuous functions of time. The form in (2.1) is that of a semi-explicit DALCS. It
 84 corresponds to a kind of extension of the impulse-free Weierstrass form [20], where
 85 the transformed state matrix has non zero off-diagonal terms A_{II} and B_I .

86 *Remark 2.1.* Assume that $K = \mathbb{R}_{\neq}^m$. The complementarity constraint (2.1) (c)
 87 is equivalently rewritten as $\phi(\lambda, w) = 0$ where $\phi(\cdot)$ is a C -function [3, 29]. Therefore
 88 (2.1) (b) (c) can be rewritten equivalently as $\Phi(x, z, \lambda, t) = 0$ for some function $\Phi(\cdot)$,
 89 usually nondifferentiable. The system (2.1) thus possesses the form of a semi-explicit
 90 DAE as studied in [53, 52], assuming piecewise differentiable functions $\Phi(\cdot)$, with x
 91 the differential state, z and λ the algebraic state variables. The so-called regularity
 92 conditions (which are an extension of index 1 conditions for DAE) used in [53, 54, 52,
 93 51] may hold in particular cases for (2.1). Examples in section 6.3 illustrate the links
 94 between both approaches in [53, 54, 52, 51] and in this article.

95 Consider (2.1), then the initial data $x(0) = x_0$ and $z(0) = z_0$ are compatible with the
 96 system's constraints if and only if:

$$97 \quad (2.2) \quad \begin{aligned} 0 &\in A_{II}x_0 + B_{II}z_0 + E_{II}(0) - \bar{B}_{II}(D + \mathcal{N}_{K^*})^{-1}(C_Ix_0 + C_{II}z_0 + F(0)) \\ \lambda(0) &\in -(D + \mathcal{N}_{K^*})^{-1}(C_Ix_0 + C_{II}z_0 + F(0)). \end{aligned}$$

98 This implies that $C_Ix_0 + C_{II}z_0 + F(0) \in \text{Im}(D + \mathcal{N}_{K^*})$ and $A_{II}x_0 + B_{II}z_0 + E_{II}(0) \in$
 99 $\bar{B}_{II}\text{Dom}(D + \mathcal{N}_{K^*})$.

100 **3. Time-discretization of DALCS with implicit Euler methods.** Let us
 101 consider the time interval $[0, T]$, $T > 0$, $h = \frac{T}{N}$, $N \in \mathbb{N}_+$, $t_k = kh$, $k \in \{0, 1, \dots, N\}$,
 102 $t_0 = 0$, $t_N = T$. For a continuous function $f(\cdot)$, $f_k \triangleq f(t_k)$. The implicit Euler
 103 discretization of (2.1) reads as:

$$104 \quad (3.1) \quad \begin{cases} (a) & P_I(x_{k+1} - x_k) = hA_Ix_{k+1} + hB_Iz_{k+1} + h\bar{B}_I\lambda_{k+1} + hE_{I,k} \\ (b) & A_{II}x_{k+1} + B_{II}z_{k+1} + \bar{B}_{II}\lambda_{k+1} + E_{II,k} = 0 \\ (c) & K^* \ni \lambda_{k+1} \perp C_Ix_{k+1} + C_{II}z_{k+1} + D\lambda_{k+1} + F_k \in K. \end{cases}$$

105 The problem in (3.1) (b) (c) is a mixed LCP (MLCP). Let us now state a basic
 106 assumption.

107 **ASSUMPTION 1.** *The matrix $P_I^h \triangleq P_I - hA_I$ is full-rank.*

108 In case $P_I \succ 0$, [25, Theorem 2.11] can be used to set conditions on hA_I so that
 109 $P_I^h \succ 0$: there always exists h_{\max} such that the assumption holds for all $h \in [0, h_{\max})$
 110 (with $h_{\max} = +\infty$ if $A_I \preceq 0$). Let us define the following matrices and vectors:

- 111 1. $\tilde{A} = hA_{II}(P_I^h)^{-1}B_I + B_{II}$,
- 112 2. $\tilde{B} = hA_{II}(P_I^h)^{-1}\bar{B}_I + \bar{B}_{II}$,
- 113 3. $\tilde{E}_k = E_{II,k} + hA_{II}(P_I^h)^{-1}E_{I,k} + A_{II}(P_I^h)^{-1}P_Ix_k$,
- 114 4. $\tilde{C} = hC_I(P_I^h)^{-1}B_I + C_{II}$,
- 115 5. $\tilde{D} = hC_I(P_I^h)^{-1}\bar{B}_I + D$,
- 116 6. $\tilde{F}_k = F_k + hC_I(P_I^h)^{-1}E_{I,k} + C_I(P_I^h)^{-1}P_Ix_k$.

117 Then we deduce from (3.1) the difference equations:

$$118 \quad (3.2) \quad \begin{cases} (a) & x_{k+1} = (P_I^h)^{-1}(P_Ix_k + hB_Iz_{k+1} + h\bar{B}_I\lambda_{k+1} + hE_{I,k}) \\ (b) & \tilde{A}z_{k+1} + \tilde{B}\lambda_{k+1} + \tilde{E}_k = 0 \\ (c) & K^* \ni \lambda_{k+1} \perp \tilde{C}z_{k+1} + \tilde{D}\lambda_{k+1} + \tilde{F}_k \in K, \end{cases}$$

119 where the last two lines of (3.2) are a MLCP with unknowns z_{k+1} and λ_{k+1} . It is
 120 therefore possible to study the well-posedness of this MLCP independently of the
 121 first line in (3.2). Letting $\zeta_{k+1} \triangleq -z_{k+1}$, this MLCP can be rewritten equivalently as

122 follows:

$$123 \quad (3.3) \quad \begin{cases} \lambda_{k+1} \in (\mathcal{N}_{K^*} + \tilde{D})^{-1}(-\tilde{C}z_{k+1} - \tilde{F}_k), \\ -\tilde{A}\zeta_{k+1} + \tilde{E}_k \in -\tilde{B}(\mathcal{N}_{K^*} + \tilde{D})^{-1}(\tilde{C}\zeta_{k+1} - \tilde{F}_k), \\ x_{k+1} = (P_I^h)^{-1}(P_I x_k + hB_I z_{k+1} + h\tilde{B}_I \lambda_{k+1} + hE_{I,k}). \end{cases}$$

124 Also the following calculations stem from the complementarity conditions in (3.2):
 (3.4)

$$125 \quad \begin{aligned} & \lambda_{k+1} \in -\mathcal{N}_K(\tilde{C}z_{k+1} + \tilde{D}\lambda_{k+1} + \tilde{F}_k) \Leftrightarrow -\lambda_{k+1} \in \mathcal{N}_{\Gamma_k}(\tilde{C}z_{k+1} + \tilde{D}\lambda_{k+1}) \\ \Leftrightarrow & \partial\sigma_{\Gamma_k}(-\lambda_{k+1}) \ni \tilde{C}z_{k+1} + \tilde{D}\lambda_{k+1} \quad \Leftrightarrow (\partial\sigma_{\Gamma_k} + \tilde{D})(-\lambda_{k+1}) \ni \tilde{C}z_{k+1} \\ \Leftrightarrow & \lambda_{k+1} \in -(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}z_{k+1}). \end{aligned}$$

126 where the inversion of set-valued mappings is used in the second equivalence, and
 127 where $\sigma_{\Gamma_k}(\cdot)$ is the support function of the set $\Gamma_k(t_k, x_k) \triangleq K - \tilde{F}_k(t_k, x_k)$ (which
 128 is a closed convex set, but not a cone as long as $\tilde{F}_k \neq 0$ which is the generic case).
 129 Therefore equivalently we can rewrite (3.3) as:

$$130 \quad (3.5) \quad \begin{cases} (a) \lambda_{k+1} \in -(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}z_{k+1}) \\ (b) -\tilde{A}z_{k+1} + \tilde{B}(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}z_{k+1}) \ni \tilde{E}_k \\ (c) x_{k+1} = (P_I^h)^{-1}(P_I x_k + hB_I z_{k+1} + h\tilde{B}_I \lambda_{k+1} + hE_{I,k}). \end{cases}$$

131 The second line in (3.3) is a generalized equation (GE) with unknown ζ_{k+1} . Its well-
 132 posedness depends on the properties of the operator $\mathcal{F} : \zeta \mapsto -\tilde{A}\zeta + \tilde{B}(\mathcal{N}_{K^*} +$
 133 $\tilde{D})^{-1}(\tilde{C}\zeta - \tilde{F}_k)$. Similarly for (3.5) with the operator $\mathcal{G} : \zeta \mapsto -\tilde{A}\zeta + \tilde{B}(\partial\sigma_{\Gamma_k} +$
 134 $\tilde{D})^{-1}(\tilde{C}\zeta)$, and (3.5) (b) is: $\mathcal{G}(z_{k+1}) \ni \tilde{E}_k$. Various results have been published to
 135 characterize such operators [16, 23, 6] and to guarantee that they are single-valued
 136 Lipschitz continuous [16, Propositions 1, 2, 3, Corollary 1], or maximal monotone
 137 (possibly set-valued) [23, 6]. The difficulty in the analysis of DALCS is that they
 138 involve the composition of two operators as seen in the OSNSP for advancing the
 139 scheme from step k to step $k+1$:

$$140 \quad (3.6) \quad P_I^h x_{k+1} \in P_I x_k + hB_I \mathcal{G}^{-1}(\tilde{E}_k(x_k)) - h\tilde{B}_I(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}\mathcal{G}^{-1}(\tilde{E}_k(x_k))) + hE_{I,k}.$$

141 Clearly the scheme can be advanced with a unique solution if both λ_{k+1} and z_{k+1}
 142 are uniquely defined (the “if and only if” holds with $\tilde{B}_I \lambda_{k+1}$ and $B_I z_{k+1}$ uniquely
 143 defined). Thus next we are going to characterize the GEs in (3.3) and (3.5) in order
 144 to analyse the scheme (3.6).

145 Two main classes of results are provided in the sequel of the article: first the well-
 146 posedness (existence and uniqueness of solutions) for the OSNSP associated with the
 147 time-discretizations in (3.1) (Propositions 4.3 and 4.4, Corollary 4.6) and (3.2) (Propo-
 148 sitions 4.7 and 4.8); second, the convergence of the discrete-time solutions is analysed,
 149 and the limit functions are shown to be solutions of the DALCS (Propositions 5.2 and
 150 5.9).

151 **4. OSNSP well-posedness.** The approach chosen in this article consists of
 152 considering the system as an ODE (2.1) (a) with a nonsmooth constraint (2.1) (b)
 153 (c): either (3.1) (b) (c) that is a mixed LCP with unknowns z_{k+1} , λ_{k+1} and parameter
 154 x_{k+1} (section 4.2), or (3.2) (b) (c) (equivalently (3.5) (a) (b)) that is a mixed LCP
 155 with z_{k+1} , λ_{k+1} (section 4.1). As will become clear later, the results rely heavily on
 156 the positive (semi) definiteness of B_{II} , and on specific passivity couplings between
 157 z in the complementarity constraint (matrix C_{II}) and λ in the algebraic constraint
 158 (matrix \tilde{B}_{II}). When $\tilde{D} = 0$ in (3.2) (c), let us state the following:

159 ASSUMPTION 2. *There exists a $p \times p$ matrix $\tilde{X} = \tilde{X}^\top \succ 0$ such that $\tilde{X}\tilde{B} = \tilde{C}^\top$.*

160 The alternative option when $D = 0$ in (3.1) (c):

161 ASSUMPTION 3. *There exists a $p \times p$ matrix $X = X^\top \succ 0$ such that $X\bar{B}_{II} = C_{II}^\top$.*

162

163 Let us notice that Assumption 2 implies that $\tilde{X}\bar{B}_{II} = C_{II}^\top + \mathcal{O}(h)$, while Assumption
 164 3 implies that $X\tilde{B} = \tilde{C}^\top + \mathcal{O}(h)$. The influence of the $\mathcal{O}(h)$ terms may not be easy
 165 to characterize precisely for finite $h > 0$. Assumption 3 is a model assumption, while
 166 Assumption 2 is a property of the numerical method. One sees that Assumption 3
 167 corresponds to a passivity constraint on the triple $(B_{II}, \bar{B}_{II}, C_{II})$ [18]. Assumption
 168 2 imposes an input/output passivity-like constraint on the triple $(\tilde{A}, \tilde{B}, \tilde{C})$, which is
 169 an $\mathcal{O}(h)$ approximation of $(B_{II}, \bar{B}_{II}, C_{II})$. When $\tilde{D} \neq 0$ and $D \neq 0$, extensions of
 170 Assumptions 2 and 3 are as follows:

171 ASSUMPTION 4. *The quadruple $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is passive.*

172 and the counterpart of Assumption 3:

173 ASSUMPTION 5. *The quadruple $(B_{II}, \bar{B}_{II}, C_{II}, D)$ is passive.*

174 Passivity is defined in Appendix C. Notice that the above assumptions imply that
 175 $n_2 = p$. Assumption 4 refers to the numerical method while Assumption 5 is a model
 176 property. Assumptions 2 and 4 refer to (3.2) (b) (c) or (3.5) (a) (b), while Assumptions
 177 3 and 5 refer to (3.1) (b) (c). Let us present a result about the relationships between
 178 Assumptions 4 and 5, *i.e.*, when does the plant's passivity guarantees the numerical
 179 scheme's passivity.

180 LEMMA 4.1. *The following statements are true:*

- 181 1. *Let $(B_{II}, \bar{B}_{II}, C_{II}, D)$ be strictly passive. Then $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is strictly passive*
 182 *for small enough $h > 0$.*
- 183 2. *Let $(B_{II}, \bar{B}_{II}, C_{II}, D)$ be strictly state passive. Then $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is strictly*
 184 *state passive if: a) $h > 0$ is small enough, b) $D + D^\top = 0$ and $C_I(P_I^h)^{-1}\bar{B}_I +$
 185 $(C_I(P_I^h)^{-1}\bar{B}_I)^\top \succcurlyeq 0$, c) $XA_{II}(P_I^h)^{-1}\bar{B}_I - B_I^\top(P_I^h)^{-\top}C_I^\top = 0$, where X is a*
 186 *solution of the passivity LMI for $(B_{II}, \bar{B}_{II}, C_{II}, D)$.*
- 187 3. *Let $h > 0$ and $(B_{II}, \bar{B}_{II}, C_{II}, D)$ be passive. Then $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is passive if:*
 188 *b) and c) in item 2 hold, and a) $XA_{II}(P_I^h)^{-1}\bar{B}_I + B_I^\top(P_I^h)^{-\top}A_{II}^\top X \preccurlyeq 0$,*
 189 *where X is a solution of the passivity LMI for $(B_{II}, \bar{B}_{II}, C_{II}, D)$.*

190 **Proof:**

- 191 1. The strict passivity of $(B_{II}, \bar{B}_{II}, C_{II}, D)$ means that there exists $X = X^\top \succ 0$
 192 such that:

$$193 \quad (4.1) \quad M \triangleq \begin{pmatrix} -XB_{II} - B_{II}^\top X & -X\bar{B}_{II} + C_{II}^\top \\ -\bar{B}_{II}^\top X + C_{II} & D + D^\top \end{pmatrix} \succ 0.$$

194 In particular this implies that $D + D^\top \succ 0$ [18, Theorem A.65] and the
 195 associated transfer function is strongly SPR [18, Remark 3.19, Section 3.12.2].
 196 Now we have $\tilde{A} = B_{II} + \mathcal{O}(h)$, $\tilde{B} = \bar{B}_{II} + \mathcal{O}(h)$, $\tilde{C} = C_{II} + \mathcal{O}(h)$, $\tilde{D} =$
 197 $D + \mathcal{O}(h)$. Therefore

$$198 \quad (4.2) \quad M = \underbrace{\begin{pmatrix} -X\tilde{A} - \tilde{A}^\top X & -X\tilde{B} + \tilde{C}^\top \\ -\tilde{B}^\top X + \tilde{C}^\top & \tilde{D} + \tilde{D}^\top \end{pmatrix}}_{\triangleq \tilde{M}} + N(h)$$

where $N(h)$ stands for a matrix with suitable dimensions and which vanishes as $h \rightarrow 0$ (in fact $N(h)$ is $\mathcal{O}(h)$). Thus $\tilde{M} = M - N(h)$. Applying [25, Theorem 2.11], it follows that $\tilde{M} \succ 0$ provided that $\sigma_{\max}(N(h)) < \lambda_{\min}^2(M)$, an inequality which can always be verified if $h > 0$ is small enough. Under this condition, it follows that the strict passivity of $(B_{II}, \bar{B}_{II}, C_{II}, D)$ implies the strict passivity of $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ ($\Rightarrow \tilde{D} \succ 0$).

2. The result is proved by developing \tilde{M} in (4.2), then using [10, Proposition 8.2.4, Fact 8.21.22], or [18, Lemma A.69], as well as the fact that $-XB_{II} - B_{II}^\top X \succ 0$ from the strict state passivity.
3. The proof is similar to that of item 2.

□

The converse implication in item 1 does not hold, since one may have $\tilde{M} \succ 0$ for some $h > 0$, but $M \not\succeq 0$. It is possible to state necessary and sufficient conditions, relying on [10, Proposition 8.2.4, Fact 8.21.22] or [18, Lemma A.70]. However the obtained conditions are cumbersome and hardly usable. Lemma 4.1 can be used to determine the conditions under which Assumption 4 is verified in the results presented in the next section. The conditions b) and c) in Lemma 4.1 item 2 hold if: b) $C_I = 0$ or $\bar{B}_I = 0$, c) ($A_{II} = 0$ or $\bar{B}_I = 0$) and ($B_I = 0$ or $C_I = 0$). Condition a) in Lemma 4.1 item 3 holds if $A_{II} = 0$ or $B_I = 0$. Let us remind that the solution $X = X^\top \succ 0$ of the Lyapunov equation $XB_{II} + B_{II}^\top X = -Q$, $Q = Q^\top \succ 0$, can be written $X = (-\frac{1}{2}Q + F)B_{II}^{-1}$, where $F = -F^\top$ is unique [60, Theorems 1 and 2]. Let us denote the symmetric part of a square matrix M as $\text{sym}(M)$ and its skew-symmetric part as $\text{skw}(M)$. This allows us to state the following necessary conditions.

COROLLARY 4.2. 1) Let $(B_{II}, \bar{B}_{II}, C_{II}, D)$ be strictly state passive with $XB_{II} + B_{II}^\top X = -Q$, $Q = Q^\top \succ 0$, and assume that $A_{II}(P_I^h)^{-1}\bar{B}_I$ is right-invertible ($\Rightarrow p \leq m$). Then $X = (-\frac{1}{2}Q + F)B_{II}^{-1}$, for some $F = -F^\top$ uniquely defined, and condition c) item 2 in Lemma 4.1 holds only if

$$(4.3) \quad \text{sym}\left(\frac{1}{2}QB_{II}^{-1}A_{II}(P_I^h)^{-1}\bar{B}_I + B_I^\top(P_I^h)^{-\top}C_I^\top\right) = F\text{skw}(B_{II}^{-1}A_{II}(P_I^h)^{-1}\bar{B}_I)F,$$

for any pair (Q, F) , and

$$(4.4) \quad \text{Ker}(C_I(P_I^h)^{-1}B_I) = \{0\}.$$

2) Let $(B_{II}, \bar{B}_{II}, C_{II}, D)$ be passive with $XB_{II} + B_{II}^\top X = -Q$, $Q = Q^\top \succcurlyeq 0$ and B_{II} invertible. Then $X = (-\frac{1}{2}Q + F)B_{II}^{-1}$, for some $F = -F^\top$, and condition a) item 3 in Lemma 4.1 holds only if

$$(4.5) \quad \text{sym}(-QB_{II}^{-1}A_{II}(P_I^h)^{-1}B_I) + 2F\text{skw}(B_{II}^{-1}A_{II}(P_I^h)^{-1}B_I)F \preccurlyeq 0,$$

for any pair (Q, F) .

Proof: 1) Condition c) yields $(-\frac{1}{2}Q + F)B_{II}^{-1}A_{II}(P_I^h)^{-1}\bar{B}_I = B_I^\top(P_I^h)^{-\top}C_I^\top$, then (4.3) follows using $F = -F^\top$. Using the right-invertibility, (4.4) follows from [23, equ. (70)] and the fact that X is full rank. 2) From passivity any solution must be symmetric. The Lyapunov equation and the skew-symmetry of F yield (4.5). □

We see that the right-invertibility condition (resp. condition (4.4)) in item 1 of the lemma, characterizes the couplings between x and λ in the index reduction (resp. between x and z in the relative degree reduction (*i.e.*, in \dot{w})).

241 **4.1. Analysis of the OSNSP for (3.2).** The next proposition analyses the
 242 OSNSP well-posedness, and applies to two main cases: $\tilde{D} = 0$, and $\tilde{D} \succ 0$. Let us
 243 remind that $p = n_2$ as a consequence of Assumptions 2, 3, 4 and 5, so that \tilde{A} and B_{II}
 244 are $p \times p$ matrices.

245 **PROPOSITION 4.3.** *Let $h > 0$, $k \geq 0$ be given.*

- 246 1. *Let Assumption 4 hold. Suppose that either $\text{Im}(\tilde{C}) \cap \text{int}(\text{Im}(\partial\sigma_{\Gamma_k} + \tilde{D})) \neq \emptyset$
 247 or \tilde{B} is square full rank. Suppose also that there is ζ_0 such that $\tilde{C}\zeta_0 \in$
 248 $\text{Im}(\partial\sigma_{\Gamma_k} + \tilde{D})$. Then the operator $\mathcal{G} : \zeta \mapsto -\tilde{A}\zeta + \tilde{B}(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}\zeta)$ is
 249 maximal monotone with non empty domain.*
- 250 2. *Let $\tilde{D} \succ 0$, then both mappings $(\mathcal{N}_{K^*} + \tilde{D})^{-1}(\cdot)$ and $(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\cdot)$ are
 251 well-defined, single-valued Lipschitz continuous and maximal monotone for
 252 each $h \geq 0$, $k \geq 0$ (maximal monotonicity holds for $\tilde{D} \succ 0$). The operator
 253 $\mathcal{G}(\cdot)$ is single-valued and Lipschitz continuous also.*
- 254 3. *Let $\tilde{D} = 0$, Assumption 2 hold true, $\tilde{R}^2 = \tilde{X}$, $\tilde{R} = \tilde{R}^\top \succ 0$, $\text{Im}(\tilde{C}) \cap \Gamma_k \neq \emptyset$
 255 and $K = P(G, 0)$ (see Proposition A.1 e)) for some matrix $G \in \mathbb{R}^{m \times m}$. Let
 256 us assume that $\tilde{R}\tilde{A}\tilde{R}^{-1} \preceq 0$. Let us consider the set:*

$$257 \quad (4.6) \quad \begin{aligned} \tilde{\mathcal{S}} &\triangleq P(G\tilde{C}\tilde{R}^{-1}, 0) \\ &\cap \{ \xi \in \mathbb{R}^p \mid \tilde{R}\tilde{A}\tilde{R}^{-1}\xi = -\sum_{i=1}^m \lambda_i (G\tilde{C}\tilde{R}^{-1})_{i\bullet}^\top, \lambda_i \geq 0 \} \\ &\cap \text{Ker}(\tilde{R}\tilde{A}\tilde{R}^{-1} + \tilde{R}^{-1}\tilde{A}^\top\tilde{R}). \end{aligned}$$

258 Define $\tilde{q}_k = -\tilde{R}\tilde{E}_k$, $\tilde{f}(\cdot) = \psi_{\Gamma_k} \circ \tilde{C}\tilde{R}^{-1}(\cdot)$, $\Gamma_k = K - \tilde{F}_k$, and consider the
 259 $VI(-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, \tilde{f})$: Find $\xi_{k+1} = \tilde{R}z_{k+1}$ such that

$$260 \quad (4.7) \quad \langle -\tilde{R}\tilde{A}\tilde{R}^{-1}\xi_{k+1} + \tilde{q}_k, v - \xi_{k+1} \rangle + \tilde{f}(v) - \tilde{f}(\xi_{k+1}) \geq 0, \quad \text{for all } v \in \text{dom}(\tilde{f}).$$

- 261 (a) *If $\tilde{\mathcal{S}} = \{0\}$, then the $VI(-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, \tilde{f})$ has at least one solution.*
- 262 (b) *If $\tilde{\mathcal{S}} \neq \{0\}$, and if there exists $\xi_0 \in \tilde{\Gamma}_k$ such that*

$$263 \quad (4.8) \quad \langle \tilde{q}_k - (-\tilde{R}\tilde{A}\tilde{R}^{-1})^\top \xi_0, v \rangle > 0 \quad \text{for all } v \in \tilde{\mathcal{S}}, v \neq 0,$$

264 where $\tilde{\Gamma}_k = \{ \xi \in \mathbb{R}^{n_2} \mid G\tilde{C}\tilde{R}^{-1}\xi + G\tilde{F}_k \geq 0 \} = \text{dom}(\tilde{f})$, then the
 265 $VI(-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, \tilde{f})$ has at least one solution.

- 266 (c) *If ξ_{k+1}^1 and ξ_{k+1}^2 are two solutions of the $VI(-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, \tilde{f})$, then $\xi_{k+1}^1 -$
 267 $\xi_{k+1}^2 \in \text{Ker}(\tilde{R}\tilde{A}\tilde{R}^{-1} + \tilde{R}^{-1}\tilde{A}^\top\tilde{R})$.*
- 268 (d) *Let $\tilde{R}\tilde{A}\tilde{R}^{-1}$ be symmetric. If ξ_{k+1}^1 and ξ_{k+1}^2 are two solutions of the
 269 $VI(-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, \tilde{f})$, then $\langle \tilde{q}_k, \xi_{k+1}^1 - \xi_{k+1}^2 \rangle = 0$.*
- 270 (e) *Let $\tilde{R}\tilde{A}\tilde{R}^{-1}$ be symmetric. Then any solution of the $VI(-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, \tilde{f})$
 271 is also a solution to the problem: $\min_{\xi \in \tilde{\Gamma}_k} -\frac{1}{2}\xi^\top \tilde{R}\tilde{A}\tilde{R}^{-1}\xi + \langle \tilde{q}_k, \xi \rangle$.*
- 272 (f) *Let $-\tilde{R}\tilde{A}\tilde{R}^{-1} \succ 0$ and be symmetric. Then $\xi_{k+1} = \text{proj}_{-\tilde{R}\tilde{A}\tilde{R}^{-1}}[\tilde{\Gamma}_k; -\tilde{q}_k]$.*

273 **Proof:**

- 274 1. Follows from [6, Theorem 1], which is itself inspired from [23] and [16, The-
 275 orem 2], and which states that the negative feedback interconnection of a
 276 passive system with a maximal monotone operator, defines another maximal
 277 monotone operator (under the stated basic conditions).
- 278 2. Follows from [16, Proposition 1] (see Proposition B.1) and from [55, Lemma
 279 1] (or [9, Corollary 24.4 (ii)]) since $\text{Im}(\tilde{D}) = \mathbb{R}^m$ and $\text{int}(\text{Dom}(\mathcal{N}_{K^*})) =$
 280 $\text{int}(K^*) \neq \emptyset$, and $\text{int}(\text{Dom}(\partial\sigma_{\Gamma_k})) \neq \emptyset$ because Γ_k is convex and $\text{int}(\Gamma_k) \neq \emptyset$.

281 3. The proof follows from Propositions A.1 and A.2. First notice that the MLCP
 282 $\tilde{A}z_{k+1} + \tilde{B}\lambda_{k+1} + \tilde{E}k = 0$, $K^* \ni \lambda_{k+1} \perp \tilde{C}z_{k+1} + \tilde{D}\lambda_{k+1} + \tilde{F}k \in K$, can
 283 be equivalently rewritten as the VI($-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, \tilde{f}$) in (4.7). Indeed one
 284 obtains $-\tilde{R}\tilde{A}\tilde{R}^{-1}\xi_{k+1} - \tilde{R}\tilde{E}k \in -\tilde{R}\tilde{B} \partial\psi_{\Gamma_k}(\tilde{C}\tilde{R}^{-1}\xi_{k+1})$. Using the chain rule
 285 of Convex Analysis [49, Theorem 23.9] and the fact that Γ_k is polyhedral,
 286 the MLCP can be written equivalently as in (4.7). Let us now compute the
 287 set \mathcal{S} in (4.6). Here $\text{dom}(\varphi) = \text{dom}(\tilde{f}) = \{z \in \mathbb{R}^p \mid \tilde{C}\tilde{R}^{-1}z \in \Gamma_k\} =$
 288 $\{z \in \mathbb{R}^p \mid \tilde{C}\tilde{R}^{-1}z + \tilde{F}k \in K\} = \{z \in \mathbb{R}^p \mid G\tilde{C}\tilde{R}^{-1}z + G\tilde{F}k \geq 0\}$. Hence
 289 $(\text{dom}(\varphi))_\infty = P(G\tilde{C}\tilde{R}^{-1}, 0)$ from Proposition A.1 e). Now we have $\tilde{f}(\xi) =$
 290 $\psi_{\Gamma_k}(\tilde{C}\tilde{R}^{-1}\xi) = \psi_{\tilde{\Gamma}_k}(\xi)$ with $\tilde{\Gamma}_k = \{\xi \in \mathbb{R}^p \mid \tilde{C}\tilde{R}^{-1}\xi \in K - \tilde{F}k\} = \{\xi \in$
 291 $\mathbb{R}^p \mid G\tilde{C}\tilde{R}^{-1}\xi + G\tilde{F}k \geq 0\} = P(G\tilde{C}\tilde{R}^{-1}, -G\tilde{F}k)$. Thus $\tilde{f}_\infty(\cdot) = \psi_{(\tilde{\Gamma}_k)_\infty}(\cdot)$
 292 and $\text{dom}(\tilde{f}_\infty) = P(G\tilde{C}\tilde{R}^{-1}, 0)$. Now we have to calculate the dual cone
 293 $P(G\tilde{C}\tilde{R}^{-1}, 0)^*$. Using [31, Example 15], one finds that $P(G\tilde{C}\tilde{R}^{-1}, 0)^* = \{w \in$
 294 $\mathbb{R}^{n_2} \mid w = \sum_{i=1}^m \lambda_i (G\tilde{C}\tilde{R}^{-1})_{i\bullet}^\top, \lambda_i \geq 0\}$, where $(G\tilde{C}\tilde{R}^{-1})_{i\bullet}$ denotes the i th
 295 row of the matrix $G\tilde{C}\tilde{R}^{-1} \in \mathbb{R}^{m \times n_2}$. Hence it follows that $\mathcal{K}(-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{f}) =$
 296 $\{\xi \in \mathbb{R}^{n_2} \mid \tilde{R}\tilde{A}\tilde{R}^{-1}\xi = -\sum_{i=1}^m \lambda_i (G\tilde{C}\tilde{R}^{-1})_{i\bullet}^\top, \lambda_i \geq 0\}$. The third set in $\tilde{\mathcal{S}}$
 297 is equal to $\text{Ker}(\tilde{R}\tilde{A}\tilde{R}^{-1} + \tilde{R}^{-1}A^\top\tilde{R})$, from $\text{Ker}(\mathbf{M} + \mathbf{M}^\top)$ in Proposition A.2
 298 a). Then the results in items (a) to (f) are a consequence of Proposition A.2,
 299 where $\varphi_\infty(v) = 0$ in item (b). Item (f) follows from item e) in Proposition
 300 A.2.

301 \square

302 Item 1 encompasses cases with $\tilde{D} \succ 0$. If item 2 holds, then λ_{k+1} is uniquely defined,
 303 provided that z_{k+1} exists (which is guaranteed if conditions of item 1 hold). From
 304 item 3 (c), it follows that z_{k+1} is unique if \tilde{A} has full rank (recall that B_{II} and \tilde{A} are
 305 square $p \times p$ matrices). However from (3.2) and (3.3), x_{k+1} is uniquely defined if and
 306 only if $\tilde{B}_I(\mathcal{N}_{K^*})^{-1}(-\tilde{C}z_{k+1} - \tilde{F}k)$ is unique. This happens trivially when $\tilde{B}_I = 0$ (if
 307 one thinks of (2.1) as the limit of a singularly perturbed system, this simply means
 308 that the multiplier enters only the z -dynamics). Let us now deal with the case $\tilde{D} \succ 0$.

309 ASSUMPTION 6. Let $h > 0$, and $\tilde{D} = \begin{pmatrix} \tilde{D}_{11} & 0 \\ 0 & 0 \end{pmatrix}$ with $0 \prec \tilde{D}_{11} = \tilde{D}_{11}^\top \in \mathbb{R}^{m_1 \times m_1}$.

311 Such specific structure occurs for instance when D has the same blockdiagonal struc-
 312 ture and $\tilde{B}_I = 0$, i.e., the multiplier does not intervene in the x -dynamics. Let us
 313 split the multiplier as $\lambda = (\lambda_1^\top, \lambda_2^\top)^\top$, $\lambda_1 \in \mathbb{R}^{m_1}$, $\lambda_2 \in \mathbb{R}^{m_2}$, $m_1 + m_2 = m$, and
 314 $K = K_1 \times K_2 \subset \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ (hence $K^* = K_1^* \times K_2^*$), $\tilde{B} = (\tilde{B}_1 \quad \tilde{B}_2)$, $\tilde{B}_1 \in \mathbb{R}^{p \times m_1}$,
 315 $\tilde{B}_2 \in \mathbb{R}^{p \times m_2}$, $\tilde{C} = \begin{pmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{pmatrix}$, $\tilde{C}_1 \in \mathbb{R}^{m_1 \times p}$, $\tilde{C}_2 \in \mathbb{R}^{m_2 \times p}$. If $\tilde{D}_{11} = 0$ then $\lambda = \lambda_2$,
 316 $\tilde{B} = \tilde{B}_2$, $\tilde{C} = \tilde{C}_2$. If $\tilde{D} = \tilde{D}_{11} \succ 0$ then item 2 in Proposition 4.3 applies.

317 ASSUMPTION 7. Let $h > 0$. There exists a $(p \times p)$ -matrix $\tilde{X}_2 = \tilde{X}_2^\top \succ 0$ such that
 318 $\tilde{X}_2\tilde{B}_2 = \tilde{C}_2^\top$.

319 This assumption implies that there are couplings between the multiplier λ_{k+1} in the
 320 algebraic equation, and the algebraic variable z_{k+1} in the complementarity conditions,
 321 see (3.2). Define $\tilde{R}_2 = \tilde{R}_2^\top \succ 0$, $\tilde{R}_2^2 = \tilde{X}_2$, and $\xi_{k+1} = \tilde{R}_2 z_{k+1}$. The next result extends
 322 Proposition 4.3 to the positive semidefinite case with blockdiagonal structure.

323 PROPOSITION 4.4. Let Assumptions 6 and 7 hold true, $h > 0$ and $k \geq 0$ be given.

324 Define $\Gamma_{2,k} = K_2 - \tilde{F}_{2,k}$. Consider the VI: find $\xi_{k+1} \in \text{dom}(\tilde{f}_2)$ such that
 (4.9)

$$325 \quad -\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_{k+1} - \tilde{R}_2 \tilde{E}_k + \underbrace{\tilde{R}_2 \tilde{B}_1 (\tilde{D}_{11} + \partial \psi_{K_1^*})^{-1} (\tilde{C}_1 \tilde{R}_2^{-1} \xi_{k+1} + \tilde{F}_{1,k})}_{\triangleq \tilde{G}_2(t_k, \xi_{k+1})} \in -\partial \tilde{f}_2(\xi_{k+1})$$

326 where $\tilde{f}_2(\cdot) = \psi_{\Gamma_{2,k}} \circ \tilde{C}_2 \tilde{R}_2^{-1}(\cdot)$.

- 327 1. Suppose that $\text{Im}(\tilde{C}_2) \cap \Gamma_{2,k} \neq \emptyset$. The MLCP $\tilde{A}z_{k+1} + \tilde{B}\lambda_{k+1} + \tilde{E}_k = 0$,
- 328 $K^* \ni \lambda_{k+1} \perp \tilde{C}z_{k+1} + \tilde{D}\lambda_{k+1} + \tilde{F}_k \in K$, can be equivalently rewritten as the
- 329 VI in (4.9).
- 330 2. Assume that the set $\tilde{\Gamma}_{2,k} = \{\xi \in \mathbb{R}^p \mid \tilde{C}_2 \tilde{R}_2^{-1} \xi \in \Gamma_{2,k}\}$ is compact. Then the
- 331 set of solutions to the VI in (4.9) is compact and nonempty.
- 332 3. Assume that for each t_k , the following well-defined, single-valued and Lip-
- 333 schitz continuous mapping, is maximal monotone:

$$\tilde{G}_2 : \xi_{k+1} \mapsto \tilde{R}_2 \tilde{B}_1 (\tilde{D}_{11} + \partial \psi_{K_1^*})^{-1} (\tilde{C}_1 \tilde{R}_2^{-1} \xi_{k+1} - \tilde{F}_{1,k}).$$

332 If $-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \succ 0$, $\tilde{\Gamma}_{2,k} \cap \text{Int}(\text{Dom}(\tilde{G}_2)) \neq \emptyset$ or $\text{Int}(\tilde{\Gamma}_{2,k}) \cap \text{Dom}(\tilde{G}_2) \neq \emptyset$, then
 333 the set of solutions to the VI in (4.9) is closed and convex. If $-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \succ 0$,
 334 the VI in (4.9) has a unique solution.

335 **Proof:**

- 336 1. The complementarity conditions in (3.2) (c) are rewritten as:

$$337 \quad (4.10) \quad \begin{pmatrix} K_1^* \\ K_2^* \end{pmatrix} \ni \begin{pmatrix} \lambda_{1,k+1} \\ \lambda_{2,k+1} \end{pmatrix} \perp \begin{pmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{pmatrix} z_{k+1} + \begin{pmatrix} \tilde{D}_{11} \lambda_{1,k+1} \\ 0 \end{pmatrix} + \begin{pmatrix} \tilde{F}_{1,k} \\ \tilde{F}_{2,k} \end{pmatrix} \in \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$$

338 while the equality constraint in (3.2) (b) is rewritten as:

$$339 \quad (4.11) \quad \tilde{A}z_{k+1} + \tilde{B}_1 \lambda_{1,k+1} + \tilde{B}_2 \lambda_{2,k+1} + \tilde{E}_k = 0.$$

340 From (4.10) it is deduced that:

$$341 \quad (4.12) \quad \begin{cases} \lambda_{1,k+1} = -(\mathcal{N}_{K_1^*} + \tilde{D}_{11})^{-1} (\tilde{C}_1 z_{k+1} + \tilde{F}_{1,k}) \\ \lambda_{2,k+1} \in -\mathcal{N}_{K_2 - \tilde{F}_{2,k}}(\tilde{C}_2 z_{k+1}). \end{cases}$$

342 Using (4.11) it is inferred:

$$343 \quad (4.13) \quad \begin{aligned} & \tilde{A}z_{k+1} - \tilde{B}_1 (\mathcal{N}_{K_1} + \tilde{D}_{11})^{-1} (\tilde{C}_1 z_{k+1} + \tilde{F}_{1,k}) \\ & \quad - \tilde{B}_2 \mathcal{N}_{K_2 - \tilde{F}_{2,k}}(\tilde{C}_2 z_{k+1}) + \tilde{E}_k = 0 \\ & \quad \quad \quad \updownarrow \\ & -\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_{k+1} + \tilde{R}_2 \tilde{B}_1 (\mathcal{N}_{K_1^*} + \tilde{D}_{11})^{-1} (\tilde{C}_1 z_{k+1} + \tilde{F}_{1,k}) - \tilde{R}_2 \tilde{E}_k \\ & \quad \quad \quad \in -\tilde{R}_2 \tilde{B}_2 \mathcal{N}_{K_2 - \tilde{F}_{2,k}}(\tilde{C}_2 \tilde{R}_2^{-1} \xi_{k+1}), \end{aligned}$$

344 which is (4.9) since $\tilde{R}_2 \tilde{B}_2 = \tilde{R}_2^{-1} \tilde{X}_2 \tilde{B}_2 = \tilde{R}_2^{-1} \tilde{C}_2^\top$ and applying the chain rule
 345 of Convex Analysis for polyhedral functions.

- 346 2. This follows from [29, Corollary 2.2.5] and since the map $(\tilde{D}_{11} + \partial \psi_{K_1^*})^{-1}(\cdot)$
- 347 is single-valued, well defined and Lipschitz continuous (see Proposition B.1).
- 348 3. The first assertion is a consequence of [9, Corollary 24.4] (for the maximal
- 349 monotonicity of the mapping $(-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} + \tilde{G}_2 + \partial f_2)(\cdot)$, and of [9, Proposition
- 350 23.39]. The second assertion follows from [29, Theorem 2.3.3], noting that the

351 mapping in the left-hand side of (4.9) is maximal monotone [9, Corollary 24.4
 352 (ii)] and strongly monotone since $-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1} \succ 0$. This can also be proved
 353 using [9, Corollary 23.37]. Single-valuedness and Lipschitz continuity stem
 354 from $\tilde{D}_{11} \succ 0$. \square

355 Further features of the solutions to the VI in (4.9) will be studied in section 5.1.3. The
 356 next lemma states a classical condition that guarantees the maximal monotonicity of
 357 Proposition 4.4 item 3. The first condition in item 1 is a constraint qualification
 358 which guarantees that the function $\tilde{f}_2(\cdot)$ has a nonempty domain. It is usually seen
 359 as a technical condition, however in some applications it may not be satisfied. Item 1
 360 states conditions under which the VI in (4.9) can be constructed. Item 3 uses some
 361 results about the zeroes of maximal monotone mappings, or of the sum of these.

362 LEMMA 4.5. *Let $\tilde{X}_2\tilde{B}_1 = \tilde{C}_1^\top$ for \tilde{X}_2 in Assumption 7, then for each fixed t_k , the
 363 mapping $\tilde{G}_2 : \xi_{k+1} \mapsto \tilde{R}_2\tilde{B}_1(\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(\tilde{C}_1\tilde{R}_2^{-1}\xi_{k+1} + \tilde{F}_{1,k})$ is maximal monotone.*

364 **Proof:** Indeed $\tilde{X}_2\tilde{B}_1 = \tilde{C}_1^\top \Rightarrow \tilde{R}_2\tilde{B}_1 = \tilde{R}_2^{-1}\tilde{C}_1^\top = (\tilde{C}_1\tilde{R}_2^{-1})^\top$. The mapping $(\tilde{D}_{11} +$
 365 $\partial\psi_{K_1^*})^{-1}(\cdot)$ is maximal monotone, and the following operation: $(\tilde{C}_1\tilde{R}_2^{-1})^\top(\tilde{D}_{11} +$
 366 $\partial\psi_{K_1^*})^{-1}(\tilde{C}_1\tilde{R}_2^{-1} \cdot + \tilde{F}_{1,k})$ preserves the maximal monotonicity [50, Exercise 12.4]. \square

367 Propositions 4.3 and 4.4 provide some informations about the existence and unique-
 368 ness of the unknown z_{k+1} , and characterize some mappings. It is noteworthy that
 369 other well-posedness results for VIs are available in the literature. For instance if
 370 the set $\tilde{\Gamma}_k$ is compact, [29, Corollary 2.2.5] guarantees existence of a solution to
 371 VI($-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, f$) with a compact set of solutions. Also when $\tilde{\Gamma}_k = \mathbb{R}_+^m$ one may
 372 rely on P-matrices and LCPs properties, when it is a closed convex cone one may use
 373 copositive matrices [29, Theorem 2.5.10].

374 The next corollary completes the existence and uniqueness results for the OSNSP.

375 COROLLARY 4.6. *The next results hold true:*

- 376 1. (set of solutions z_{k+1}) *Let Assumption 4 and the conditions of item 1 in*
 377 *Proposition 4.3 hold. Then the set of solutions z_{k+1} in (3.5) (or the set of*
 378 *solutions ζ_{k+1} in (3.3)), is closed and convex. If in addition the conditions*
 379 *of item 3 (a) or (b) in Proposition 4.3 hold, then the set of solutions is non*
 380 *empty. If in addition $\tilde{A} \prec 0$, then z_{k+1} is unique.*
- 381 2. (set of solutions λ_{k+1}) *Let $\tilde{C} \text{ Dom}(\mathcal{G}) \subseteq \text{Im}(\partial\sigma_{\Gamma_k} + \tilde{D})$, and the conditions of*
 382 *item 1 in Proposition 4.3 hold true. Then the set of solutions λ_{k+1} in (3.5)*
 383 *(a) is non empty for each z_{k+1} .*
- 384 3. *Let Assumption 4 hold with strict passivity, and the conditions of item 1 in*
 385 *Proposition 4.3 be true. Then the GE in (3.5) (or the GE in (3.3)) has*
 386 *a unique solution (λ_{k+1}, z_{k+1}) for any \tilde{E}_k . Consequently x_{k+1} in (3.2) is*
 387 *unique.*
- 388 4. *Let $\tilde{D} = 0$ and $\mathcal{G}(\cdot)$ be maximal monotone. Then if the set $\tilde{\Gamma}_k = \{\zeta \in$
 389 $\mathbb{R}^{n_2} \mid G\tilde{C}\tilde{R}^{-1}\zeta + G\tilde{F}_k \geq 0\}$ is bounded, the GE with unknown z_{k+1} in (3.5)*
 390 *(or in (3.3)) has at least one solution and the set of solutions is compact.*

391 **Proof:**

- 392 1. Follows from [9, Proposition 23.39] by the maximal monotonicity of $\mathcal{G}(\cdot)$. If
 393 $-\tilde{A} \succ 0$ then $\mathcal{G}(\cdot)$ is strongly monotone (however it may not be Lipschitz
 394 continuous and this property is used later in section 5.2), and so is $\mathcal{G}(\cdot) + \tilde{E}_k$.
 395 Hence the generalized equation $\mathcal{G}(z_{k+1}) + \tilde{E}_k \ni 0$ has a unique solution [9,
 396 Corollary 23.37].
- 397 2. Let us consider $\lambda_{k+1} \in -(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}z_{k+1})$. We know that any solution

398 $z_{k+1} \in \mathcal{G}^{-1}(\tilde{E}_k) \subseteq \text{Im}(\mathcal{G}^{-1}) = \text{Dom}(\mathcal{G})$. Since $\text{Dom}(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1} = \text{Im}(\partial\sigma_{\Gamma_k} +$
 399 $\tilde{D})$ the result follows.
 400 3. Strict passivity implies that $\tilde{D} \succ 0$ and \tilde{A} is a Hurwitz matrix, therefore
 401 from a slight modification of [6, Theorem 1] it follows that the operator $\mathcal{G}(\cdot)$
 402 is strongly, hence strictly monotone (indeed there exists a real $\alpha > 0$ small
 403 enough such that $(\tilde{A} + \alpha I_{n_2}, \tilde{B}, \tilde{C}, \tilde{D})$ is passive). Using [9, Proposition 23.35]
 404 shows that z_{k+1} is unique, and item 2 in Proposition 4.3 shows that λ_{k+1} is
 405 unique. The conclusion follows.
 406 4. Under these conditions $\text{Dom}(\mathcal{G})$ is bounded (compact). Hence from [9, Propo-
 407 sition 23.36 (iii)] existence of a solution holds true. From [29, Corollary 2.2.5]
 408 the compactness of the set of solutions follows. This could also be inferred
 409 from item (e) in Proposition 4.3 in case $\tilde{R}\tilde{A}\tilde{R}^{-1}$ is symmetric.
 410 \square

411 Item 3 in Corollary 4.6 guarantees the uniqueness of solutions to the OSNSP in (3.6),
 412 in which case the \in can be replaced by equality. Concerning item 2 in Corollary 4.6,
 413 an important step is to characterize both $\text{Dom}(\mathcal{G})$ and $\text{Im}(\partial\sigma_{\Gamma_k} + \tilde{D})$. If $\tilde{D} = \tilde{D}^\top \succ 0$,
 414 one can apply [15, Theorem 3], and infer that $\text{Im}(\partial\sigma_{\Gamma_k} + \tilde{D}) \simeq \text{Im}(\partial\sigma_{\Gamma_k}) + \text{Im}(\tilde{D}) =$
 415 $\text{Dom}(\partial\psi_{\Gamma_k}) + \text{Im}(\tilde{D}) \subset \text{dom}(\psi_{\Gamma_k}) + \text{Im}(\tilde{D}) = \Gamma_k + \text{Im}(\tilde{D})$ (see [15] for the exact
 416 meaning of \simeq in this context). Similar calculations may be done for in item 1 of
 417 Proposition 4.3. It is inferred that the basic condition in Proposition 4.3 item 1,
 418 *i.e.*, $\text{Im}(\tilde{C}) \cap \text{int}(\text{Im}(\partial\sigma_{\Gamma_k} + \tilde{D})) \neq \emptyset$, or in Corollary 4.6 item 2, *i.e.*, $\tilde{C} \text{Dom}(\mathcal{G}) \subseteq$
 419 $\text{Im}(\partial\sigma_{\Gamma_k} + \tilde{D})$, always are satisfied when $\tilde{D} \succ 0$. For the sake of brevity the conditions
 420 for the existence and uniqueness of λ_{k+1} in (4.12) are not stated, similar tools as for
 421 Corollary 4.6 can be used.

422 **4.2. Analysis of the OSNSP for (3.1).** It is also interesting to investigate
 423 the case when Assumption 3 (resp. Assumption 5) is satisfied instead of Assumption
 424 2 (resp. Assumption 4). The operator $\mathcal{G}(\cdot)$ is now defined as $\mathcal{G} : \zeta \mapsto -B_{II}\zeta +$
 425 $(\partial\sigma_{\Gamma(t_k, x_{k+1})} + D)^{-1}(C_{II}\zeta)$, where $\Gamma(t_k, x_{k+1}) = \{\xi \in \mathbb{R}^m \mid \xi + C_I x_{k+1} + F_k \in$
 426 $K\}$. Consequently the results of Proposition 4.3 item 1, Corollary 4.6 item 3, with
 427 Assumption 5 plus strict passivity, and Proposition 4.3 item 2 when $D \succ 0$, hold. For
 428 the sake of brevity we do not develop all the arguments once again.
 429 Let us focus now on the case $D \succcurlyeq 0$. From Assumption 3, $D = 0$ and (2.1) one infers:

$$430 \quad (4.14) \quad -RB_{II}R^{-1}\xi(t) - RA_{II}x(t) - RE_{II}(t) \in -\partial\psi_{\Phi(t, x(t))}(\xi(t)),$$

431 where $\xi = -Rz$, $R^2 = X$, $R = R^\top \succ 0$, $\Phi(t, x) = \{\xi \in \mathbb{R}^p \mid C_{II}R^{-1}\xi + C_I x + F(t) \in$
 432 $K\}$, and the set-valued term is obtained using the chain rule, under the Convex
 433 Analysis chain rule qualification condition [49, Theorem 23.9]. The first step is to
 434 analyse the existence and uniqueness of solutions to the VI:

$$435 \quad (4.15) \quad -RB_{II}R^{-1}\xi_{k+1} - RA_{II}x_{k+1} - RE_{II,k} \in -\partial\psi_{\Phi(t_k, x_{k+1})}(\xi_{k+1}),$$

436 which is the counterpart of (4.7), relying on Proposition A.2. The unknown is ξ_{k+1}
 437 and the other terms and variables are considered as data for this VI. Similar results
 438 to those in item 3 of Proposition 4.3 can be obtained.

439 **PROPOSITION 4.7.** *Let $D = 0$, Assumption 3 be true, $h > 0$ be given, assume that*
 440 *$\text{Im}(C_{II}) \cap (K - C_I x_{k+1} + F_k) \neq \emptyset$ and that K is polyhedral. Consider the discrete-time*
 441 *system in (3.1), and the associated VI $(-RB_{II}R^{-1}, -RA_{II}x_{k+1} - RE_{II,k}, \psi_{\Phi(t_k, x_{k+1})})$*

442 in (4.15). Assume that $-RB_{II}R^{-1} \succ 0$. Consider the set:
 (4.16)

$$443 \quad \mathcal{S} \triangleq P(GC_{II}R^{-1}, 0) \cap \{\xi \in \mathbb{R}^{n_2} \mid RB_{II}R^{-1}\xi = \sum_{i=1}^m \lambda_i (GC_{II}R^{-1})_{i\bullet}^\top, \lambda_i \geq 0\} \\ \cap \text{Ker}(RB_{II}R^{-1} + R^{-1}B_{II}^\top R).$$

444 Then:

- 445 1. If $\mathcal{S} = \{0\}$ the VI $(-RB_{II}R^{-1}, -RA_{II}x_{k+1} - RE_{II,k}, \psi_{\Phi(t_k, x_{k+1})})$ has a unique
 446 solution.
- 447 2. If $\mathcal{S} \neq \{0\}$, and if there exists $\xi_0 \in \Phi(t_k, x_{k+1})$ such that $\langle -RA_{II}x_{k+1} -$
 448 $RE_{II,k} + RB_{II}R^{-1}\xi_0, v \rangle > 0$ for all $v \neq 0$ and $v \in \mathcal{S}$, then it follows that the
 449 VI $(-RB_{II}R^{-1}, -RA_{II}x_{k+1} - \tilde{R}E_{II,k}, \psi_{\Phi(t_k, x_{k+1})})$ has a unique solution.

450 **Proof:** The set \mathcal{S} in (4.16) is constructed as follows. The chain rule holds from
 451 [49, Theorem 23.9] and the fact that $\psi_{K-C_I x_{k+1} + F_k}(\cdot)$ is polyhedral. Hence the
 452 VI in (4.15) is equivalent to (3.1) (b) (c). Here $\varphi = \psi_{\Phi(t_k, x_{k+1})}$, hence $\text{dom}(\varphi) =$
 453 $\Phi(t_k, x_{k+1})$ and $(\text{dom}(\varphi))_\infty$. Assume that $K = P(G, 0)$ (see Proposition A.1 (e)),
 454 then $(\text{dom}(\varphi))_\infty = \{\xi \in \mathbb{R}^p \mid GC_{II}R^{-1}\xi \geq 0\} = P(GC_{II}R^{-1}, 0)$. Further, $\varphi_\infty =$
 455 $\psi_{(\Phi(t_k, x_{k+1}))_\infty} = \psi_{P(GC_{II}R^{-1}, 0)}$, hence $\text{dom}(\varphi_\infty) = P(GC_{II}R^{-1}, 0)$ and $\text{dom}(\varphi_\infty)^* =$
 456 $(P(GC_{II}R^{-1}, 0))^* = \{w \in \mathbb{R}^{n_2} \mid w = -\sum_{i=1}^m \lambda_i (GC_{II}R^{-1})_{i\bullet}^\top, \lambda_i \geq 0\}$, where [31,
 457 Example 15] was used as in the proof of Proposition 4.3. Thus $\mathcal{K}(-RB_{II}R^{-1}, \varphi) =$
 458 $\{\xi \in \mathbb{R}^{n_2} \mid -RB_{II}R^{-1}\xi = -\sum_{i=1}^m \lambda_i (GC_{II}R^{-1})_{i\bullet}^\top, \lambda_i \geq 0\}$. Finally in Proposition
 459 A.2: $\text{Ker}(\mathbf{M} + \mathbf{M}^\top) = \text{Ker}(RB_{II}R^{-1} + (RB_{II}R^{-1})^\top)$ and $R = R^\top$. Items 1 and 2
 460 are a direct consequence of Proposition A.2 (a) and (b). \square

461 For the sake of briefness we do not state similar results as in Proposition 4.3 (c) to
 462 (f). To enlarge the analysis let us now state the counterpart of Assumption 6:

463 ASSUMPTION 8. Let $D = \begin{pmatrix} D_{11} & 0 \\ 0 & 0 \end{pmatrix}$ with $0 \prec D_{11} = D_{11}^\top \in \mathbb{R}^{m_1 \times m_1}$.

464 Similarly to Assumption 7, with the same splitting of the multiplier λ , the matrices
 465 B_{II} and C_{II} , and of the cone K (see the comments after Assumption 6, in particular
 466 if $D_{11} = 0$ the case $D = 0$ is recovered with $G_2(\cdot, \cdot)$ in (4.17) which vanishes, and the
 467 set $\Phi_2 = \Phi$), let us state the following:

468 ASSUMPTION 9. There exists a $(p \times p)$ -matrix $X_2 = X_2^\top \succ 0$ such that $X_2 \bar{B}_{II}^2 =$
 469 $(C_{II}^2)^\top$.

470 Let us recall that p is the number of equality constraints in (2.1). Let us define
 471 $R_2 = R_2^\top \succ 0$, $R_2^2 = X_2$, $\xi_{k+1} = R_2 z_{k+1}$, $\Phi_2(t_k, x_{k+1}) = \{\xi \in \mathbb{R}^p \mid C_{II}^2 R_2^{-1} \xi +$
 472 $C_{II}^2 x_{k+1} + F_{2,k} \in K_2\}$. The next proposition is the counterpart of Proposition 4.4. It
 473 states conditions that guarantee that z_{k+1} can be calculated from (3.1) (b) (c).

474 PROPOSITION 4.8. Let $h > 0$, and Assumptions 8 and 9 hold true. Consider the
 475 VI: find $\xi_{k+1} \in \text{dom}(f_2)$ such that

$$476 \quad (4.17) \quad \underbrace{-R_2 B_{II} R_2^{-1} \xi_{k+1} - R_2 A_{II} x_{k+1} - R_2 E_{II,k} + R_2 \bar{B}_{II}^1 (D_{11} + \partial \psi_{K_1^*})^{-1} (C_{II}^1 R_2^{-1} \xi_{k+1} - F_{1,k})}_{\triangleq G_2(t_k, \xi_{k+1})} \in -\partial f_2(\xi_{k+1})$$

477 with $f_2(\cdot) = (\psi_{K_2 - C_{II}^2 x_{k+1} - F_{2,k}} \circ C_{II}^2 R_2^{-1})(\cdot) = \psi_{\Phi_2(t_k, x_{k+1})}(\cdot)$.

- 478 1. Assume that $\text{Im}(C_{II}^2) \cap (K_2 - C_{II}^2 x_{k+1} - F_{2,k}) \neq \emptyset$, then the VI in (4.17) is
 479 equivalent to the MLCP in (3.1) (b) (c).

- 480 2. Suppose that the set $\Phi_2(t_k, x_{k+1})$ is compact and nonempty, then the set of
 481 solutions to the VI in (4.17) is compact and nonempty.
 482 3. Assume that for each fixed t_k , the mapping $G_2(\cdot)$ is maximal monotone, and
 483 that $-R_2 B_{II} R_2^{-1} \succ 0$, then the VI in (4.17) has a unique solution.
 484 4. Assume that $-R_2 B_{II} R_2^{-1} \succcurlyeq 0$, $G_2(\cdot)$ is maximal monotone for each fixed t_k ,
 485 and $\Phi_2(t_k, x_{k+1}) \cap \text{Int}(\text{Dom}(G_2)) \neq \emptyset$ or $\text{Int}(\Phi_2(t_k, x_{k+1})) \cap \text{Dom}(G_2) \neq \emptyset$.
 486 Then the set of solutions to the VI in (4.17) is closed and convex.

487 **Proof:** The proof of the first three items follows closely that of Proposition 4.4 and it
 488 is therefore omitted. Item 4: the VI is rewritten as $\underbrace{(-R_2 B_{II} R_2^{-1} + G_2 + \partial f_2)}_{\triangleq H_2}(\xi_{k+1}) -$

489 $R_2 A_{II} x_{k+1} - R_2 E_{II,k} \ni 0$, and the assumption guarantees that $H_2(\cdot)$ is maximal
 490 monotone [9, Corollary 24.4]. The result follows from [9, Proposition 23.39]. \square

491 The same as Lemma 4.5 can be stated to guarantee the maximal monotonicity in
 492 items 3 and 4. It is noteworthy that the main discrepancy between Proposition
 493 4.4 and Proposition 4.8, is that the solution ξ_{k+1} depends on x_{k+1} in the latter,
 494 while it depends on x_k in the former. This is a difficulty with the application of
 495 Proposition 4.8 in some cases, for instance when $C_{II} = 0$ (examples in sections 6.2
 496 and 6.4 will demonstrate that this can occur). Actually, the condition $\text{Im}(C_{II}^2) \cap (K_2 -$
 497 $C_{II}^2 x_{k+1} - F_{2,k}) \neq \emptyset$ (that is a form of constraint qualification), or its counterparts in
 498 Propositions 4.3, 4.4, 4.7, happens to be crucial in the setting of this work. It states
 499 that the algebraic variable z has to enter the complementarity constraints through
 500 a C_{II} matrix which has a big enough range. This may occur in some systems, see
 501 section 6.5.

502 **5. Convergence results.** In this section it is assumed that the conditions for
 503 existence of solutions in Propositions 4.3, 4.4, 4.7, or 4.8, are fulfilled, so that sequences
 504 can be calculated. The section is divided into two main parts: in sections 5.1.1, 5.1.2
 505 and 5.1.3, convergence is analysed from the properties of the quadruple $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$
 506 which characterize the MLCP in (3.2) (b) (c). In section 5.2, the analysis is led with
 507 the properties of $(B_{II}, \bar{B}_{II}, C_{II}, D)$ which characterize the MLCP in (3.1) (b) (c).
 508 The relationships between both approaches are explained.

509 **5.1. Convergence results from (3.2).** In this section the numerical scheme
 510 in (3.2) and the results in section 4.1 are used. First let us notice that z_{k+1} and
 511 λ_{k+1} are solutions of the generalized equation in (3.5) (a) (b). Thus if existence of
 512 solutions hold, they both are functions of \tilde{E}_k (in addition to being functions of \tilde{F}_k
 513 through the set Γ_k). Under some conditions which are recalled below, like item 2 of
 514 Proposition 4.3, they can be single-valued Lipschitz continuous functions of \tilde{E}_k . Then
 515 the right-hand side of the first line in (3.2) may be written compactly as

$$516 \quad (5.1) \quad P_I^h x_{k+1} = P_I x_k + h \bar{E}_k(t_k, x_k)$$

517 for some $\bar{E}_k(t_k, x_k)$ which is a linear function of $E_{I,k}$, $E_{II,k}$, λ_{k+1} and z_{k+1} . The
 518 time-dependence of \bar{E}_k comes from $E_{I,k}$, $E_{II,k}$ and F_k , while the state dependence
 519 comes from the third terms in \tilde{E}_k and in \tilde{F}_k (which enters $\Gamma_k(t_k, x_k) = K - \tilde{F}_k$). From
 520 (5.1) it can be calculated and checked by induction that:

$$521 \quad (5.2) \quad x_k = ((P_I^h)^{-1} P_I)^k x_0 + h \sum_{j=0}^{k-1} ((P_I^h)^{-1} P_I)^{k-1-j} (P_I^h)^{-1} \bar{E}_j(t_j, x_j), \quad k \geq 1,$$

522 Since $P_I^h = P_I - hA_I$, it follows that for $h > 0$ small enough, $(P_I^h)^{-1}P_I = I_p + \mathcal{O}(h)$
 523 [10, Proposition 9.4.13] hence $(P_I^h)^{-1} = P_I^{-1} + \mathcal{O}(h)$. Thus

$$\begin{aligned}
 x_k &= x_0 + \mathcal{O}(h) + h \sum_{j=0}^{k-1} (I_p + \mathcal{O}(h))^{k-1-j} (P_I^{-1} + \mathcal{O}(h)) \bar{E}_j(t_j, x_j) \\
 &= x_0 + \mathcal{O}(h) + h \sum_{j=0}^{k-1} (I_p + \mathcal{O}(h)) (P_I^{-1} + \mathcal{O}(h)) \bar{E}_j(t_j, x_j) \\
 524 \quad (5.3) \quad &= x_0 + \mathcal{O}(h) + h \sum_{j=0}^{k-1} (P_I^{-1} + \mathcal{O}(h)) \bar{E}_j(t_j, x_j) \\
 &= x_0 + \mathcal{O}(h) + h (P_I^{-1} + \mathcal{O}(h)) \sum_{j=0}^{k-1} \bar{E}_j(t_j, x_j),
 \end{aligned}$$

525 for all $k \geq 1$. It is inferred that:

$$\begin{aligned}
 526 \quad (5.4) \quad \dot{x}_k &\stackrel{\Delta}{=} \frac{x_k - x_{k-1}}{h} = \frac{\mathcal{O}(h)}{h} + (P_I^{-1} + \mathcal{O}(h)) \bar{E}(t_{k-1}, x_{k-1}) \\
 &= \mathcal{O}(1) + (P_I^{-1} + \mathcal{O}(h)) \bar{E}(t_{k-1}, x_{k-1}).
 \end{aligned}$$

527 Our aim is now to prove that both $\{x_k\}$ and $\{\dot{x}_k\}$ are uniformly bounded sequences,
 528 which will allow us to conclude about the convergence of the sequence $\{x^h(\cdot)\}$ (see the
 529 definition in (5.7)). For this it is necessary and sufficient that $\bar{E}_k(t_k, x_k)$ be uniformly
 530 bounded for all $k \geq 0$, since P_I is full rank. It follows from (3.5) (a) and (b), that
 531 there are two operators which are crucial to characterize the right-hand side in (3.6).

532 **5.1.1. Analysis of the operator $\mathcal{G}(\cdot)$.** Notice that the mapping $\mathcal{Z} : \zeta \mapsto \eta =$
 533 $(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\zeta)$ is under the conditions of Proposition 4.3 item 2, a well-defined
 534 $\mathbb{R}^m \rightarrow \mathbb{R}^m$ single-valued Lipschitz continuous mapping which is parameterized by
 535 t_k and x_k through the set $\Gamma_k(t_k, x_k)$. Assume that x_k is bounded, then redoing
 536 the proof of Proposition B.1, it follows that the Lipschitz constant of $\mathcal{Z}(\cdot)$ does not
 537 depend on $\partial\sigma_{\Gamma_k}(\cdot)$ but depends only on \tilde{D} , being equal to $\frac{2}{\lambda_{\min}(\tilde{D} + \tilde{D}^\top)}$. In turn
 538 $\lambda_{\min}(\tilde{D} + \tilde{D}^\top) > 0$ uniformly in $h > 0$ only if $D \succ 0$. Thus the operator $\mathcal{G} :$
 539 $\zeta \mapsto -\tilde{A}\zeta + \tilde{B}(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}\zeta)$ is also single-valued Lipschitz continuous (being the
 540 composition of Lipschitz continuous mappings) with a Lipschitz constant that does
 541 not depend on x_k , and which is bounded for bounded h . Let us now assume the
 542 following which reinforces Assumption 4 and guarantees that $\tilde{D} \succ 0$:

543 ASSUMPTION 10. *The system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is strictly passive, uniformly in $h \geq 0$.*
 544

545 Then if the basic conditions stated in item 1 of Proposition 4.3 are verified in addition
 546 to Assumption 10, the operator $\mathcal{G}(\cdot)$ is strongly monotone for some constant $\epsilon > 0$,
 547 and we know from Corollary 4.6 item 3, that x_{k+1} is unique. As stated in Proposition
 548 4.3 item 1, maximal monotonicity follows from [6, Theorem 1]. Strong monotonicity
 549 follows from the fact that under Assumption 10, the quadruple $(\tilde{A} + \epsilon I_{n_2}, \tilde{B}, \tilde{C}, \tilde{D})$
 550 is passive for a small enough $\epsilon > 0$, so that $(\mathcal{G} - \epsilon I_{n_2})(\cdot)$ is maximal monotone for
 551 such ϵ . Thus if Assumption 10 and the basic conditions in item 1 of Proposition
 552 4.3 are true, $\mathcal{G} : \mathbb{R}^p \rightarrow \mathbb{R}^p$ is well-defined and single-valued, Lipschitz continuous
 553 and strongly monotone. Consequently it is inferred from [50, Proposition 12.54] that
 554 under the above set of conditions, $\mathcal{G}^{-1}(\cdot)$ is single-valued, well defined and globally
 555 Lipschitz, and the solution z_{k+1} of the generalized equation $\bar{E}_k(t_k, x_k) \in \mathcal{G}(z_{k+1}) \Leftrightarrow$
 556 $z_{k+1} = \mathcal{G}^{-1}(\bar{E}_k(t_k, x_k))$ in (3.5) is Lipschitz continuous in x_k and t_k . Thus λ_{k+1} is
 557 also Lipschitz continuous in x_k , and we conclude that the same applies to x_{k+1} in
 558 (3.2).

559 *Remark 5.1.* Assumption 10 is stringent. Actually the strict state passivity of
 560 $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is sufficient to guarantee that $\mathcal{G}(\cdot)$ is strongly monotone. However in
 561 the body of sections 5.1.1 and 5.1.2 we need also that λ_{k+1} be a Lipschitz continuous

562 function of its argument, a sufficient condition being that $\tilde{D} \succ 0$, which in turn implies
 563 that strict state passivity and strict passivity are identical. Thus strict passivity could
 564 be relaxed to strict state passivity if it is assured that $\tilde{B}_I(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}\cdot)$ is Lipschitz
 565 continuous. This may occur in certain cases (the trivial case is $\tilde{B}_I = 0$).

566 **5.1.2. Boundedness and convergence analysis.** Under the above conditions,
 567 the term $\tilde{E}_k(t_k, x_k)$ and consequently $x_{k+1}(t_k, x_k)$ in (3.2) and $\dot{x}_k(t_k, x_k)$ in (5.4) are
 568 single-valued and Lipschitz continuous in x_k and t_k (provided $E_I(\cdot)$, $E_{II}(\cdot)$ and $F(\cdot)$
 569 are Lipschitz continuous functions of t , which is a mild requirement on the data),
 570 since \tilde{E}_k and \tilde{F}_k are linear in the argument x_k . We have $\|\tilde{E}(t_k^1, x_k^1) - \tilde{E}(t_k^2, x_k^2)\| \leq$
 571 $L_t|t_k^1 - t_k^2| + L_x\|x_k^1 - x_k^2\|$ for any $t_k^1, t_k^2, x_k^1, x_k^2$, and for some $L_t > 0$ and $L_x > 0$. Thus
 572 $\|\tilde{E}(t_k, x_k^1) - \tilde{E}(t_k, x_k^2)\| \leq L_x\|x_k^1 - x_k^2\|$ and $\|\tilde{E}(t_k, x_k) - \tilde{E}(t_k, 0)\| \leq L_x\|x_k\|$. Notice
 573 that due to the Lipschitz continuity both z_{k+1} and λ_{k+1} are uniformly bounded in
 574 time for fixed x_k (provided $E_I(\cdot)$, $E_{II}(\cdot)$ and $F(\cdot)$ are uniformly bounded, which is
 575 a mild requirement on the data). Therefore $\tilde{E}(t_k, 0)$ is uniformly bounded also, and
 576 $\|\tilde{E}(t_k, x_k)\| \leq \|\tilde{E}(t_k, 0)\| + L_x\|x_k\| \leq m + L_x\|x_k\|$. Now using (5.2) we get:
 (5.5)

$$577 \quad \|x_k\| \leq \sigma_{\max}^k((P_I^h)^{-1}P_I)\|x_0\| + h\sigma_{\min}(P_I^h) \sum_{j=1}^{k-1} \sigma_{\max}^{k-j-1}(P_I(P_I^h)^{-1})(m + L_x\|x_{j-1}\|),$$

578 where we used the Hölder induced matrix norm $\|\cdot\|_{2,2} = \sigma_{\max}(\cdot)$ and [10, Proposition
 579 9.4.9, Corollary 9.4.12]. Let us make the following:

580 **ASSUMPTION 11.** *One has $\sigma_{\max}((P_I^h)^{-1}P_I) \leq 1$ for all $h \geq 0$.*

581 Assumption 11 is a technical assumption which is satisfied if for instance the $n_1 \times n_1$
 582 matrix $A_I \prec 0$ and $P_I \succ 0$. Indeed this implies that $P_I^h \succ P_I$, thus $(P_I^h)^{-1}P_I \prec I_{n_1}$.
 583 Notice that under Assumption 11 one no longer needs $h > 0$ small enough to obtain
 584 the developments in (5.4), because the spectral radius of $(P_I^h)^{-1}P_I$ is smaller than
 585 one. It is also noteworthy that changing $A_I x_{k+1}$ to $A_I x_k$ in the right-hand side of
 586 (3.1) (a), allows one to dispense with Assumption 11, with a slight modification of the
 587 matrices \tilde{A} , \tilde{B} , \tilde{C} , \tilde{D} , \tilde{E}_k and \tilde{F}_k (where $(P_I^h)^{-1}$ is replaced by P_I^{-1} in all of them).
 588 Consequently from (5.5) one infers that:

$$589 \quad (5.6) \quad \|x_k\| \leq \|x_0\| + h\sigma_{\min}(P_I^h) \sum_{j=1}^k (m + L_x\|x_{j-1}\|),$$

590 Assume now that x_0, x_1, \dots, x_{k-1} are uniformly bounded by some x_{\max} , then from (5.6)
 591 one has $\|x_k\| \leq x_{\max} + h\sigma_{\min}(P_I^h)k(m + L_x x_{\max}) \leq x_{\max} + h\sigma_{\min}(P_I^h)n(m + L_x x_{\max}) =$
 592 $x_{\max} + h\sigma_{\min}(P_I^h)\frac{T}{h}(m + L_x x_{\max}) = x_{\max} + \sigma_{\min}(P_I^h)T(m + L_x x_{\max})$. Therefore x_k
 593 is also uniformly bounded (its upperbound depends only on T which is fixed), and
 594 we conclude that all x_k , $1 \leq k \leq n$, are bounded provided x_0 is bounded. Then it is
 595 inferred from (5.4) that \dot{x}_k is also uniformly bounded. Let us define the sequences of
 596 piecewise-linear approximations $\{x^h(\cdot)\}$ and $\{\dot{x}^h(\cdot)\}$ as follows:

$$597 \quad (5.7) \quad \begin{cases} x^h(t) = x_{k+1} + \frac{t_{k+1}-t}{h}(x_k - x_{k+1}) \\ \dot{x}^h(t) = \frac{x_{k+1}-x_k}{h} \end{cases} \quad t \in [t_k, t_{k+1}).$$

598 We have shown that $x^h(\cdot)$ and $\dot{x}^h(\cdot)$ are uniformly bounded on $[0, T]$ and for any
 599 $h > 0$, hence $x^h(\cdot)$ has a Lipschitz constant that does not depend on h , consequently
 600 the family $\{x^h\}_{h \geq 0}$ is equi-Lipschitz continuous. Thus from Arzela-Ascoli Theorem

601 $\{x^h(\cdot)\}$ stays in a compact subset of $\mathcal{C}^0([0, T]; \mathbb{R}^{n_1})$ and there exist subsequences
 602 which converge uniformly in $\mathcal{C}^0([0, T]; \mathbb{R}^{n_1})$ towards a Lipschitz continuous limit $x(\cdot)$.
 603 Let $\tilde{E}^h(\cdot)$ and $\tilde{F}^h(\cdot)$ be defined in a similar way as $x^h(\cdot)$ in (5.7). Due to the Lip-
 604 schitz continuity of $E_I(\cdot)$, $E_{II}(\cdot)$, and $F(\cdot)$, the sequences of functions $\{\tilde{E}^h(\cdot)\}_{h \geq 0}$
 605 and $\{\tilde{F}^h(\cdot)\}_{h \geq 0}$ also converge uniformly in $\mathcal{C}^0([0, T]; \mathbb{R}^p)$ and $\mathcal{C}^0([0, T]; \mathbb{R}^m)$ to limits
 606 $\tilde{E}(\cdot) = E_{II}(\cdot) + A_{II}x(\cdot)$ and $\tilde{F}(\cdot) = F(\cdot) + C_Ix(\cdot)$ as $h \rightarrow 0$, respectively. From (5.4) it
 607 is deduced that $\dot{x}^h(\cdot)$ is uniformly bounded. Therefore from the Banach-Alaoglu The-
 608 orem, it follows that $\dot{x}^h(\cdot)$ converges weakly* to a limit $v(\cdot)$. Since the domain $[0, T]$ is
 609 bounded, one has $\mathcal{L}^2([0, T]; \mathbb{R}^{n_1}) \subseteq \mathcal{L}^1([0, T]; \mathbb{R}^{n_1})$ [36, Corollary 1, p.80], and conse-
 610 quently $\dot{x}^h(\cdot)$ converges weakly in $\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})$. Since $x^h(t) = x^h(0) + \int_0^t \dot{x}^h(\tau) d\tau$,
 611 we infer that $v(\cdot) = \dot{x}(\cdot)$ almost everywhere.
 612 Now notice that using (3.2) the following holds:

$$613 \quad (5.8) \quad \begin{cases} (a) P_I \dot{x}^h(t) = A_I x_h^*(t) + B_I z_h^*(t) + \bar{B}_I \lambda_h^*(t) + E_I^h(t) \\ (b) \tilde{A} z_h^*(t) + \tilde{B} \lambda_h^*(t) + \tilde{E}_{II}^h(t) + A_{II} x^h(t) = 0 \\ (c) 0 \leq \lambda_h^*(t) \perp C z_h^*(t) + \tilde{D} \lambda_h^*(t) + F^h(t) + C_I x^h(t) \geq 0, \end{cases}$$

614 for all $t \in [t_k, t_{k+1})$, $k \geq 0$, $x_h^*(t) = x_{k+1}$ for all $t \in [t_k, t_{k+1})$ and similarly for the
 615 other terms are step-functions. From the boundedness of $\|\dot{x}_k\| \triangleq \|\frac{x_{k+1} - x_k}{h}\| \leq C$ for
 616 some $C > 0$, we infer that $x_h^*(\cdot)$ converges strongly in $\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})$ towards $x(\cdot)$.
 617 Indeed:

$$618 \quad (5.9) \quad \begin{aligned} \|x^h - x_h^*\|_{\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})}^2 &\leq \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_{k+1} - t_k)^2 \|\dot{x}_k\|^2 dt \\ &\leq C^2 \sum_{k=0}^{n-1} (t_{k+1} - t_k)^3 = \frac{C^2 n h^3}{3} = \frac{C^2 T h^2}{3} \rightarrow 0 \text{ as } h \rightarrow 0. \end{aligned}$$

619 To conclude one notices that $\|x^h - x_h^*\|_{\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})}^2 = \|x^h \pm x - x_h^*\|_{\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})}^2$. Now
 620 we have $z_h^*(t) \in \mathcal{G}^{-1}(E_{II, \star}^h(t) + A_{II} x_h^*(t))$ for all $t \in [t_k, t_{k+1})$, $k \geq 0$. Since $E_I(\cdot)$,
 621 $E_{II}(\cdot)$ are Lipschitz continuous, it follows that $\|\dot{E}_k\| \triangleq \|\frac{E_{k+1} - E_k}{h}\| \leq k_E$ where k_E
 622 is its Lipschitz constant. We infer that $E_{II, \star}^h(\cdot)$ converges strongly in $\mathcal{L}^2([0, T]; \mathbb{R}^p)$
 623 towards $E_{II}(\cdot)$, and using the Lipschitz continuity of $\mathcal{G}^{-1}(\cdot)$ it follows that $z_h^*(\cdot)$
 624 converges strongly in $\mathcal{L}^2([0, T]; \mathbb{R}^p)$ towards a limit $z(\cdot) = \mathcal{G}^{-1}(E_{II}(\cdot) + A_{II}x(\cdot))$.
 625 Then from (3.3) $\lambda_h^*(t) \in (\mathcal{N}_{\mathbb{R}_+^m} + \tilde{D})^{-1}(-\tilde{C} z_h^*(t) - F_h^*(t) + C_I x_h^*(t))$, so that $\lambda_h^*(\cdot)$
 626 converges towards a limit $\lambda(\cdot) = (\mathcal{N}_{\mathbb{R}_+^m} + \tilde{D})^{-1}(-\tilde{C} z(\cdot) - F(\cdot) + C_I x(\cdot))$ strongly in
 627 $\mathcal{L}^2([0, T]; \mathbb{R}^m)$, due to the single-valuedness, Lipschitzness and maximal monotonicity
 628 of $(\mathcal{N}_{\mathbb{R}_+^m} + \tilde{D})^{-1}(\cdot)$. It is concluded that $P_I \dot{x}^h(t) = A_I x_h^*(t) + B_I z_h^*(t) + \bar{B}_I \lambda_h^*(t) +$
 629 $E_I^h(t) \rightarrow P_I \dot{x}(t) = A_I x(t) + B_I z(t) + \bar{B}_I \lambda(t) + E_I(t)$ weakly in $\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})$ as $h \rightarrow$
 630 0 , $\tilde{A} z_h^*(t) + \tilde{B} \lambda_h^*(t) + \tilde{E}_{II}^h(t) + A_{II} x^h(t) \rightarrow B_{II} z(t) + \bar{B}_{II} \lambda(t) + E_{II}(t) + A_{II} x(t) (= 0)$,
 631 and $0 \leq \lambda_h^*(t) \perp \tilde{C} z_h^*(t) + \tilde{D} \lambda_h^*(t) + F^h(t) + C_I x^h(t) \geq 0 \rightarrow 0 \leq \lambda(t) \perp C_{II} z(t) +$
 632 $D \lambda(t) + F(t) + C_I x(t) \geq 0$ as $h \rightarrow 0$ strongly in $\mathcal{L}^2([0, T]; \mathbb{R}^l)$ where l depends on the
 633 variables dimensions. Hence we have proved the existence of a continuous solution
 634 (x, z, λ) to (2.1) on $[0, T]$. Consequently the following result is proved:

635 **PROPOSITION 5.2.** *Let the conditions of item 2 in Corollary 4.6 hold true uni-*
 636 *formly in h ($\Rightarrow D \succ 0$). Assume that Assumptions 10 and 11 hold, $E(\cdot)$ and $F(\cdot)$ are*
 637 *bounded Lipschitz continuous functions, initial data are bounded. Then the DALCS*
 638 *in (2.1) possesses for each admissible bounded initial data, a continuous solution*
 639 *$(x, z, \lambda) : [0, T] \rightarrow \mathbb{R}^{n_1 + n_2 + m}$, $T > 0$. Moreover the Euler approximations converge*
 640 *to a solution as $h \rightarrow 0$.*

641 **5.1.3. The case blockdiagonal** $\tilde{D} = \tilde{D}^\top \succcurlyeq 0$. This section is dedicated to
 642 analyse an alternative to the conditions stated in sections 5.1.1 and 5.1.2. Let us
 643 consider the case of Proposition 4.4 with Assumption 6. Then it follows from the
 644 complementarity conditions that $\lambda_{1,k+1} = (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(-\tilde{C}_1 z_{k+1} - \tilde{F}_{1,k})$, where
 645 the mapping $(\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(\cdot)$ is maximal monotone (being the sum of two maximal
 646 monotone mappings satisfying [9, Corollary 24.4 (ii)]), single-valued, well-defined and
 647 Lipschitz continuous, see Proposition B.1. Thus if $\tilde{B}_I = (\tilde{B}_I^1, 0)$ with $\tilde{B}_I^1 \in \mathbb{R}^{n_1 \times m_1}$
 648 one obtains $\tilde{B}_I \lambda_{k+1} = \tilde{B}_I^1 \lambda_{1,k+1} = \tilde{B}_I^1 (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(-\tilde{C}_1 z_{k+1} - \tilde{F}_{1,k})$ in the right-
 649 hand side of (3.5) (c).

650 It may be less restrictive to compute upperbounds using the maximal monotonicity.
 651 To start with let us consider the VI in (4.9) and let us state several features of its
 652 solutions. Here Assumption 7 has to be reinforced a little:

653 ASSUMPTION 12. *The conditions of Assumption 7 hold where $\tilde{X}_2 \succ 0$ for any*
 654 *$h \geq 0$.*

655 This secures that the variable change $\xi_{k+1} = -\tilde{R}_2^{-1} z_{k+1}$ remains well-defined as $h \rightarrow$
 656 0 . It is also important that item 1 in Proposition 4.4 holds, so that the VI in (4.9) is
 657 equivalent to the MLCP in (3.2) (b) (c).

658 LEMMA 5.3. *Assume that $\lambda_{\min}(-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1}) (= \lambda_{\min}(-\tilde{A}))$ is large enough, then*
 659 *any solution $\xi_{k+1} = -\tilde{R}_2^{-1} z_{k+1}$ of the VI in (4.9) is upperbounded by an affine function*
 660 *of $\|x_k\|$.*

661 **Proof:** Let us manipulate the VI in (4.9) as follows:

$$(5.10) \quad \begin{aligned} & -\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_{k+1} + \tilde{R}_2 \tilde{E}_k + \tilde{G}_2(t_k, \xi_{k+1}) \in -\partial \tilde{f}_2(\xi_{k+1}) \\ \Leftrightarrow & -\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} (\xi_{k+1} - \xi_0) + \tilde{R}_2 \tilde{E}_k + \tilde{G}_2(t_k, \xi_{k+1}) + \zeta_0 - \tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_0 = -\zeta_{k+1} + \zeta_0 \\ & \text{with } \zeta_{k+1} \in \partial \tilde{f}_2(\xi_{k+1}), \zeta_0 \in \partial \tilde{f}_2(\xi_0) \\ \Rightarrow & \langle -\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} (\xi_{k+1} - \xi_0), \xi_{k+1} - \xi_0 \rangle + \langle \tilde{R}_2 \tilde{E}_k, \xi_{k+1} - \xi_0 \rangle \\ & + \langle \tilde{G}_2(t_k, \xi_{k+1}) - \tilde{G}_2(t_k, \xi_0), \xi_{k+1} - \xi_0 \rangle + \langle \zeta_0 - \tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_0, \xi_{k+1} - \xi_0 \rangle \\ & + \langle \tilde{G}_2(t_k, \xi_0), \xi_{k+1} - \xi_0 \rangle = -\langle \zeta_{k+1} - \zeta_0, \xi_{k+1} - \xi_0 \rangle \leq 0, \end{aligned}$$

663 where we have denoted $\tilde{G}_2(t_k, \xi_{k+1})$ to highlight the dependency of this mapping on t_k
 664 and x_k through \tilde{F}_k . We have $\|\tilde{R}_2 \tilde{B}_1 (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(z_1) - \tilde{R}_2 \tilde{B}_1 (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(z_2)\| \leq$
 665 $\eta \|z_1 - z_2\|$ for any z_1 and z_2 , so that $\|\tilde{G}_2(t_k, \xi_{k+1}) - \tilde{G}_2(t_k, \xi_0)\| \leq \eta \|\tilde{C}_1 \tilde{R}_2^{-1} (\xi_{k+1} -$
 666 $\xi_0)\| \leq \eta' \|\xi_{k+1} - \xi_0\|$, where η depends on \tilde{D}_{11} , \tilde{R}_2 and \tilde{B}_1 (see the proof of Propo-
 667 sition B.1 in [16]), and η' depends on \tilde{D}_{11} , \tilde{R}_2 , \tilde{B}_1 , and \tilde{C}_1 . Thus we obtain:

$$(5.11) \quad \begin{aligned} & \langle -\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} (\xi_{k+1} - \xi_0), \xi_{k+1} - \xi_0 \rangle \leq \langle -\tilde{R}_2 \tilde{E}_k, \xi_{k+1} - \xi_0 \rangle - \langle \tilde{G}_2(t_k, \xi_0), \xi_{k+1} - \xi_0 \rangle \\ & + \langle \zeta_0 + \tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_0, \xi_{k+1} - \xi_0 \rangle + \eta' \|\xi_{k+1} - \xi_0\|^2 \end{aligned}$$

669 which implies

$$(5.12) \quad \begin{aligned} & \langle (-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} - \eta' I_{n_2})(\xi_{k+1} - \xi_0), \xi_{k+1} - \xi_0 \rangle \leq \|\tilde{R}_2 \tilde{E}_k\| \|\xi_{k+1} - \xi_0\| \\ & + \|\tilde{G}_2(t_k, \xi_0) \pm \tilde{G}_2(t_0, \xi_0)\| \|\xi_{k+1} - \xi_0\| + \|\zeta_0 + \tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_0\| \|\xi_{k+1} - \xi_0\|. \end{aligned}$$

671 Now one has: $\|\tilde{E}_k\| \leq \|E_{II,k} + h A_{II} E_{I,k}\| + \|A_{II} (P_I^h)^{-1} P_I\| \|x_k\|$, $\tilde{G}_2(t_k, \xi_0) =$
 672 $\tilde{R}_2 \tilde{B}_1 (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1} (\tilde{C}_1 \tilde{R}_2^{-1} \xi_0 - \tilde{F}_{1,k})$, $\|\tilde{G}_2(t_k, \xi_0) - \tilde{G}_2(t_0, \xi_0)\| = \|\tilde{R}_2 \tilde{B}_1 (\tilde{D}_{11} +$
 673 $\partial\psi_{K_1^*})^{-1} (\tilde{C}_1 \tilde{R}_2^{-1} \xi_0 - \tilde{F}_{1,k}) - \tilde{R}_2 \tilde{B}_1 (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1} (\tilde{C}_1 \tilde{R}_2^{-1} \xi_0 - \tilde{F}_{1,0})\| \leq \eta \|\tilde{F}_{1,k} +$
 674 $\tilde{F}_{1,0}\| \leq \eta \|F_k + h C_I (P_I^h)^{-1} E_{I,k}\| + \eta \|C_I (P_I^h)^{-1} P_I\| \|x_k\|$. Let us denote $\alpha_k =$

675 $\|\tilde{R}_2\| \|E_{II,k} + hA_{II}E_{I,k}\| + \eta \|F_k + hC_I(P_I^h)^{-1}E_{I,k}\|$, $\beta_0 = \|\tilde{G}_2(t_0, \xi_0)\| + \|\zeta_0 +$
 676 $\tilde{R}_2\tilde{A}\tilde{R}_2^{-1}\xi_0\|$, and $\gamma = \eta \|C_I(P_I^h)^{-1}P_I\| + \|\tilde{R}_2\| \|A_{II}(P_I^h)^{-1}P_I\|$. Then:
 (5.13)

$$677 \quad \langle (-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1} - \eta'I_{n_2})(\xi_{k+1} - \xi_0), \xi_{k+1} - \xi_0 \rangle \leq (\alpha_k + \beta_0 + \gamma \|x_k\|) \|\xi_{k+1} - \xi_0\| \\ \leq \frac{1}{2}\alpha_k^2 + \frac{1}{2}\beta_0^2 + \frac{1}{2}\gamma^2 \|x_k\|^2 + \frac{1}{2}\|\xi_{k+1} - \xi_0\|^2$$

678 which implies:

$$(5.14)$$

$$679 \quad \langle (-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1} - \eta'I_{n_2} - \frac{1}{2}I_{n_2})(\xi_{k+1} - \xi_0), \xi_{k+1} - \xi_0 \rangle \leq \frac{1}{2}\alpha_k^2 + \frac{1}{2}\beta_0^2 + \frac{1}{2}\gamma^2 \|x_k\|^2.$$

680 Therefore if $-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1} - \eta'I_{n_2} - \frac{1}{2}I_{n_2} \succ 0$, one obtains

$$681 \quad (5.15) \quad \|\xi_{k+1} - \xi_0\| \leq \frac{1}{\sqrt{\lambda_{\min}(-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1} - \eta'I_{n_2} - \frac{1}{2}I_{n_2})}} \sqrt{\frac{1}{2}\alpha_k^2 + \frac{1}{2}\beta_0^2 + \frac{1}{2}\gamma^2 \|x_k\|^2} \\ \leq \frac{1}{\sqrt{2}\sqrt{\lambda_{\min}(-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1} - \eta'I_{n_2} - \frac{1}{2}I_{n_2})}} \left(\sqrt{\alpha_k^2 + \beta_0^2} + \gamma \|x_k\| \right)$$

682 which ends the proof. \square

683 Therefore the term $hB_I z_{k+1}$ in the right-hand side of (3.5) (c), admits also an affine
 684 upperbound in $\|x_k\|$. A further characterization of the iterates is given now.

685 LEMMA 5.4. Assume that $-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1} \succ \frac{3}{2}I_{n_2}$. Assume also that the mapping
 686 $\tilde{G}_2(\cdot)$ is monotone (see Lemma 4.5 for conditions), and that $E_I(\cdot)$, $E_{II}(\cdot)$ and $F(\cdot)$
 687 are uniformly bounded and Lipschitz continuous. Then for any k and $p \geq 0$ and any
 688 solution of the VI in (4.9), one has:

$$689 \quad (5.16) \quad \|\xi_k - \xi_p\| \leq \alpha + \beta \|x_{k-1} - x_{p-1}\|$$

690 for some bounded constants $\alpha \geq 0$ and $\beta \geq 0$, and α is proportional to $(t_{k-1} - t_{p-1})$.

691 **Proof:** From

$$692 \quad (5.17) \quad \begin{cases} -\tilde{R}_2\tilde{A}\tilde{R}_2^{-1}\xi_k + \tilde{R}_2\tilde{E}_{k-1} + \tilde{G}_2(t_{k-1}, \xi_k) = -\zeta_k \\ -\tilde{R}_2\tilde{A}\tilde{R}_2^{-1}\xi_p + \tilde{R}_2\tilde{E}_{p-1} + \tilde{G}_2(t_{p-1}, \xi_p) = -\zeta_p \end{cases}$$

693 with $\zeta_k \in \partial f_2(\xi_k)$, $\zeta_p \in \partial f_2(\xi_p)$, it follows that

$$694 \quad (5.18) \quad \langle -\tilde{R}_2\tilde{A}\tilde{R}_2^{-1}(\xi_k - \xi_p), \xi_k - \xi_p \rangle + \langle \tilde{R}_2(\tilde{E}_{k-1} - \tilde{E}_{p-1}), \xi_k - \xi_p \rangle \\ + \langle \tilde{G}_2(t_{k-1}, \xi_k) - \tilde{G}_2(t_{k-1}, \xi_p) + \tilde{G}_2(t_{k-1}, \xi_p) - \tilde{G}_2(t_{p-1}, \xi_p), \xi_k - \xi_p \rangle \\ = -\langle \zeta_k - \zeta_p, \xi_k - \xi_p \rangle$$

695 which implies

$$696 \quad (5.19) \quad \langle -\tilde{R}_2\tilde{A}\tilde{R}_2^{-1}(\xi_k - \xi_p), \xi_k - \xi_p \rangle + \langle \tilde{R}_2(\tilde{E}_{k-1} - \tilde{E}_{p-1}), \xi_k - \xi_p \rangle \\ + \langle \tilde{G}_2(t_{k-1}, \xi_p) - \tilde{G}_2(t_{p-1}, \xi_p), \xi_k - \xi_p \rangle \leq 0.$$

697 One has:

$$(5.20)$$

$$\begin{aligned} & \langle -\tilde{R}_2(\tilde{E}_{k-1} - \tilde{E}_{p-1}), \xi_k - \xi_p \rangle \\ & \leq \langle -\tilde{R}_2(E_{II,k-1} - E_{II,p-1}), \xi_k - \xi_p \rangle - \langle \tilde{R}_2(hA_{II}(E_{I,k-1} - E_{I,p-1})), \xi_k - \xi_p \rangle \\ & \quad - \langle A_{II}(P_I^h)^{-1}P_I(x_{k-1} - x_{p-1}), \xi_k - \xi_p \rangle \\ 698 & \leq \frac{1}{2}\|\tilde{R}_2((E_{II,k-1} - E_{II,p-1}) + hA_{II}(E_{I,k-1} - E_{I,p-1}))\|^2 + \frac{1}{2}\|\xi_k - \xi_p\|^2 \\ & \quad + \frac{1}{2}\|A_{II}(P_I^h)^{-1}P_I\|^2\|x_{k-1} - x_{p-1}\|^2 + \frac{1}{2}\|\xi_k - \xi_p\|^2 \\ & \leq \|\tilde{R}_2\|^2\|E_{II,k-1} - E_{II,p-1}\|^2 + \|\xi_k - \xi_p\|^2 + h\|A_{II}\|^2\|E_{I,k-1} - E_{I,p-1}\|^2 \\ & \quad + \frac{1}{2}\|A_{II}(P_I^h)^{-1}P_I\|^2\|x_{k-1} - x_{p-1}\|^2 \leq \alpha' + \beta'\|x_{k-1} - x_{p-1}\|^2 + \|\xi_k - \xi_p\|^2, \end{aligned}$$

699 for some $\alpha' \geq 0$ proportional to $(t_{k-1} - t_{p-1})^2$ since $E(\cdot)$ is Lipschitz continuous,
 700 $\beta' \geq 0$, and it was used that $x^\top y \leq \|x\| \|y\| \leq \frac{1}{2}(\|x\|^2 + \|y\|^2)$. Also:

$$\begin{aligned}
 & \langle -\tilde{G}_2(t_{k-1}, \xi_p) + \tilde{G}_2(t_{p-1}, \xi_p), \xi_k - \xi_p \rangle \\
 & \leq \|\tilde{G}_2(t_{k-1}, \xi_p) - \tilde{G}_2(t_{p-1}, \xi_p)\| \|\xi_k - \xi_p\| \\
 701 \quad (5.21) \quad & \leq \frac{1}{2} k \tilde{G}_2 \|\tilde{F}_{1,k-1} - \tilde{F}_{p-1}\|^2 + \frac{1}{2} \|\xi_k - \xi_p\|^2 \\
 & \leq k \tilde{G}_2 \|F_{k-1} - F_{p-1}\|^2 + h k \tilde{G}_2 \|C_I(P_I^h)^{-1}(E_{I,k-1} - E_{I,p-1})\|^2 \\
 & \quad + k \tilde{G}_2 \|C_I(P_I^h)^{-1}P_I(x_{k-1} - x_{p-1})\|^2 + \frac{1}{2} \|\xi_k - \xi_p\|^2 \\
 & \leq \alpha'' + \beta'' \|x_{k-1} - x_{p-1}\|^2 + \frac{1}{2} \|\xi_k - \xi_p\|^2,
 \end{aligned}$$

702 for some $\alpha'' \geq 0$ proportional to $(t_{k-1} - t_{p-1})^2$ (since $E(\cdot)$ and $F(\cdot)$ are Lipschitz
 703 continuous), and $\beta'' \geq 0$. Hence using (5.19):

$$\begin{aligned}
 704 \quad (5.22) \quad \|\xi_k - \xi_p\| & \leq \frac{1}{\sqrt{\lambda_{\min}(-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} - \frac{3}{2} I_p)}} \sqrt{\alpha' + \alpha'' + (\beta' + \beta'') \|x_{k-1} - x_{p-1}\|^2} \\
 & \leq \frac{1}{\sqrt{\lambda_{\min}(-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} - \frac{3}{2} I_p)}} (\sqrt{\alpha' + \alpha''} + \sqrt{\beta' + \beta''} \|x_{k-1} - x_{p-1}\|)
 \end{aligned}$$

705 and the lemma is proved. \square

706

707 **COROLLARY 5.5.** *Let $h > 0$. Assume that the conditions and assumptions of*
 708 *Lemma 5.4 are fulfilled. For all $k \geq 1$ and any solution of the VI in (4.9), one has:*

$$709 \quad (5.23) \quad \left\| \frac{\xi_{k+1} - \xi_k}{h} \right\| \leq \alpha + \beta \left\| \frac{x_k - x_{k-1}}{h} \right\|$$

710 for some bounded constants $\alpha \geq 0$ and $\beta \geq 0$.

711 **Proof:** Let us consider the proof of Lemma 5.4, replacing k by $k+1$ and p by $k-1$.
 712 Then α in (5.16) is proportional to h . Redoing the calculations in (5.20) and (5.22),
 713 one obtains (5.23). \square

714 Clearly for all $h > 0$ (5.23) can be written equivalently as: $\|\xi_{k+1} - \xi_k\| \leq h\alpha +$
 715 $\beta \|x_k - x_{k-1}\|$. The bounds calculated for the variable ξ_k also hold for the variable
 716 $z_k = -\tilde{R}_2 \xi_k$. Let us now analyse λ_{k+1} in (3.5) (a). As seen above, $\lambda_{1,k+1} = (\tilde{D}_{11} +$
 717 $\partial\psi_{K_1^*})^{-1}(-\tilde{C}_1 z_{k+1} - \tilde{F}_{1,k})$ and $\lambda_{2,k+1} \in -\mathcal{N}_{K_2}(\tilde{C}_2 z_{k+1} + \tilde{F}_{2,k})$. If the matrix \tilde{B}_I has
 718 the above structure (i.e., $\tilde{B}_I = (\tilde{B}_I^1, 0)$), then the term $\tilde{B}_I \lambda_{k+1} = \tilde{B}_I^1$ is Lipschitz
 719 continuous with respect to the variables z_{k+1} , x_k and F_k . In a more general setting
 720 one obtains:

$$721 \quad (5.24) \quad P_I^h x_{k+1} \in P_I x_k + h B_I z_{k+1} + h \tilde{B}_I^1 (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1} (-\tilde{C}_1 z_{k+1} - \tilde{F}_{1,k}) \\
 \quad \quad \quad - h \tilde{B}_I^2 \mathcal{N}_{K_2}(\tilde{C}_2 z_{k+1} + \tilde{F}_{2,k}).$$

722 The difficulty is that the last term in the right-hand side of (5.24) is set-valued in
 723 general, and this is intrinsically due to the structure of the implicit scheme in (2.1).
 724 A solution is to consider the VI in (4.9). Its right-hand side matches with the last
 725 term in the right-hand side of (5.24) and it is set-valued. However provided the VI
 726 is well-posed, we know that one element of this set is equal to the left-hand side of
 727 (4.9). We may assume that the scheme in (5.24) is implemented with this element.
 728 This element appears to define a Lipschitz continuous function of ξ_{k+1} (let us recall
 729 that the Lipschitz constant of the third term is equal to $\frac{1}{\lambda_{\min}(\tilde{D}_{11})}$, see Proposition
 730 B.1). If this element is chosen then the numerical scheme (5.24) becomes:

$$731 \quad (5.25) \quad P_I^h x_{k+1} = P_I x_k + h B_I z_{k+1} + h \tilde{B}_I^1 (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1} (\tilde{C}_1 z_{k+1} - \tilde{F}_{1,k}) \\
 \quad \quad \quad + h \tilde{B}_I^2 (-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_{k+1} - \tilde{R}_2 \tilde{E}_k + \tilde{G}_2(t_k, \xi_{k+1})).$$

732 Let us now study (5.25), using Lemmata 5.3, 5.4, Corollary 5.5. It is noteworthy that
 733 the constant α in both Lemmata and in Corollary can be made arbitrarily small by
 734 increasing $-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1}$ arbitrarily (in the set of positive definite matrices).

735 **PROPOSITION 5.6.** *Assume that the conditions of Lemmata 5.3, 5.4 and of Corol-*
 736 *lary 5.5 are verified, $E_I(\cdot)$, $E_{II}(\cdot)$ and $F(\cdot)$ are bounded Lipschitz continuous func-*
 737 *tions of time, and Assumption 13 below is satisfied. Then the sequences $\{x^h\}_{h \geq 0}$ and*
 738 *$\{\dot{x}^h\}_{h \geq 0}$ in (5.7) are uniformly bounded in h on $[0, T]$. Thus $\{x^h\}_{h \geq 0}$ is equi-Lipschitz*
 739 *continuous and there is a subsequence of it that converges uniformly in $\mathcal{C}^0([0, T]; \mathbb{R}^{n_1})$*
 740 *towards a Lipschitz continuous limit denoted as $x(\cdot)$.*

741 **Proof:** In the proof, use is made of positive bounded constants denoted generically as
 742 k_i . Consider (5.25). First recall that the operator $\zeta \mapsto (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(\zeta)$ is Lipschitz
 743 continuous with Lipschitz constant $\frac{1}{\lambda_{\min}(\tilde{D}_{11})}$. Hence we obtain:

$$\begin{aligned}
 & \| (P_I^h)^{-1} \tilde{B}_I^1 (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1} (-\tilde{C}_1 z_{k+1} - \tilde{F}_{1,k}) \\
 & \quad - (P_I^h)^{-1} \tilde{B}_I^1 (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1} (-\tilde{C}_1 z_{p+1} - \tilde{F}_{1,p}) \| \\
 744 \quad (5.26) \quad & \leq k_1 |\tilde{C}_1(z_{k+1} - z_{p+1} + \tilde{F}_{1,k} - \tilde{F}_{1,p})| \leq k_2 \|z_{k+1} - z_{p+1}\| + k_3 \|\tilde{F}_{1,k} - \tilde{F}_{1,p}\| \\
 & \leq k_4 \|\xi_{k+1} - \xi_{p+1}\| + k_4 \|F_{1,k} - F_{1,p}\| + k_5 \|E_{I,k} - E_{I,p}\| + k_6 \|x_k - x_p\| \\
 & \leq k_4(\alpha + \beta \|x_k - x_p\|) + k_7 |t_k - t_p| + k_6 \|x_k - x_p\| \leq k_8 + k_9 \|x_k - x_p\| \\
 & \quad + k_7 |t_k - t_p|,
 \end{aligned}$$

745 for any k and $p \geq 0$, where we used Lemma 5.4 and the Lipschitzness property. In a
 746 quite similar way one has $\| (P_I^h)^{-1} \tilde{B}_I^2 (\tilde{G}_2(\xi_{k+1}) - \tilde{G}_2(\xi_{p+1})) \| \leq k_8 + k_9 \|x_k - x_p\| +$
 747 $k_7 |t_k - t_p|$, where k_7, k_8 and k_9 depend on the Lipschitz constants of $\tilde{G}_2(\cdot, \cdot)$, $F(\cdot)$, and
 748 on $\| (P_I^h)^{-1} \tilde{B}_I^2 \|$. One also has $\| (P_I^h)^{-1} (B_I + \tilde{B}_I^2 (\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1})(z_{k+1} - z_{p+1})) \| \leq k_{10}(\alpha +$
 749 $\beta \|x_k - x_p\|) \leq k_{11} + k_{12} \|x_k - x_p\|$, and $\| \tilde{B}_I^2 \tilde{R}_2 (\tilde{E}_k - \tilde{E}_p) \| \leq k_{13} |t_k - t_p| + k_{14} \|x_k - x_p\|$.
 750 It is deduced that $\|x_{k+1} - x_{p+1}\| \leq (\| (P_I^h)^{-1} P_I \| + hk_{15}) \|x_k - x_p\| + hk_{16} |t_k - t_p| + hk_{17}$,
 751 where the term k_{17} stems from $k_8 + k_{11} = \alpha(k_4 + k_{10})$. Here we are led to make the
 752 following which is more stringent than Assumption 11:

753 **ASSUMPTION 13.** *There exists $h_{\max} > 0$ such that $\| (P_I^h)^{-1} P_I \| + hk_{15} < 1$ for all*
 754 $0 < h \leq h_{\max}$.

755 Obviously for $h = 0$ and the Hölder-induced norm one has $\| (P_I^h)^{-1} P_I \|_{2,2} + hk_{15} = 1$.
 756 Let us choose $p = 0$, it follows that $\|x_{k+1} - x_1\| \leq (\| (P_I^h)^{-1} P_I \| + hk_{15}) \|x_k -$
 757 $x_0\| + hk_{16}T + hk_{17}$, hence $\|x_{k+1}\| \leq (\| (P_I^h)^{-1} P_I \| + hk_{15}) \|x_k\| + hk_{16}T + hk_{17} +$
 758 $\|x_1\| + (\| (P_I^h)^{-1} P_I \| + hk_{15}) \|x_0\|$. Now it has to be proved that $\|x_1\|$ is bounded.
 759 From Lemma 5.3 and (5.15), $\|\xi_1\|$ and $\|z_1\|$ are upperbounded by an affine function of
 760 initial data $\|x_0\|$ and $\|z_0\|$. The operator $(\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(\cdot)$ is well-defined and single-
 761 valued. Taking $k = 0$ in (5.25) one infers that indeed $\|x_1\|$ is bounded for bounded
 762 initial data. Consequently it follows that $\|x_{k+1}\| \leq (\| (P_I^h)^{-1} P_I \| + hk_{15}) \|x_k\| +$
 763 $hk_{16}T + hk_{17} + hf(x_0, z_0) + \| (P_I^h)^{-1} P_I \| \|x_0\|$ for some function $f(\cdot)$ that is bounded
 764 for bounded initial data. To simplify the notations let us set $h_1 \triangleq \| (P_I^h)^{-1} P_I \| + hk_{15}$
 765 and $h_2 \triangleq k_{16}T + k_{17} + f(x_0, z_0)$. We obtain for all $k \geq 1$:

$$\begin{aligned}
 766 \quad (5.27) \quad \|x_k\| & \leq h_1^k \|x_0\| + \sum_{j=0}^{k-1} h_1^j (hh_2 + \| (P_I^h)^{-1} P_I \| \|x_0\|) \\
 & \leq h_1^k \|x_0\| + (hh_2 + \| (P_I^h)^{-1} P_I \| \|x_0\|) \frac{1 - h_1^{k+1}}{1 - h_1}.
 \end{aligned}$$

767 Thus in view of Assumption 13 the sequence $\{x_k\}_{k \geq 0}$ generated by (5.25) is uniformly
 768 bounded for bounded initial data.

769 Let us now reconsider the inequality $\|x_{k+1} - x_{p+1}\| \leq (\|(P_I^h)^{-1}P_I\| + hk_{15}) \|x_k -$
 770 $x_p\| + hk_{16}|t_k - t_p| + hk_{17}$ obtained above. Setting $p = k - 1$ yields $\|x_{k+1} - x_k\| \leq$
 771 $(\|(P_I^h)^{-1}P_I\| + hk_{15}) \|x_k - x_{k-1}\| + h^2k_{16} + hk_{17} \Rightarrow \frac{\|x_{k+1} - x_k\|}{h} \leq (\|(P_I^h)^{-1}P_I\| +$
 772 $hk_{15}) \frac{\|x_k - x_{k-1}\|}{h} + hk_{16} + k_{17}$. Now using Lemma 5.4 and the Lipschitzness it follows
 773 that α , hence k_{17} , are proportional to $k_E^2 h^2$. We may therefore write the inequality
 774 as $\frac{\|x_{k+1} - x_k\|}{h} \leq (\|(P_I^h)^{-1}P_I\| + hk_{15}) \frac{\|x_k - x_{k-1}\|}{h} + hk_{16} + k_{18}h^2$ for some $k_{18} \geq 0$.
 775 Proceeding as for (5.27) one infers that if $\frac{\|x_1 - x_0\|}{h}$ is bounded then $\frac{\|x_{k+1} - x_k\|}{h}$ is
 776 bounded for all $k \geq 1$. Evaluating (5.25) at $k = 0$ and using the boundedness of x_1
 777 as shown above, proves that this is indeed the case. The proposition is proved using
 778 the Arzela-Ascoli Theorem. \square

779 Clearly in practice one needs to compute the constants k_i , for instance k_{15} is needed
 780 to determine h_{\max} in Assumption 13.

781 **COROLLARY 5.7.** *Under the conditions of Proposition 5.6, the sequences $\{z^h\}_{h \geq 0}$*
 782 *and $\{z^h\}_{h \geq 0}$ (respectively $\{\xi^h\}_{h \geq 0}$ and $\{\xi^h\}_{h \geq 0}$) are uniformly bounded in h on $[0, T]$,*
 783 *$0 < T < +\infty$. Thus $\{z^h\}_{h \geq 0}$ (respectively $\{\xi^h\}_{h \geq 0}$) is equi-Lipschitz continuous and*
 784 *there is a subsequence of it that converges uniformly in $C^0([0, T]; \mathbb{R}^{n_2})$ towards a*
 785 *Lipschitz continuous limit denoted as $z(\cdot)$ (respectively $\xi(\cdot)$).*

786 **Proof:** This follows from Proposition 5.6 and using Lemmata 5.3, 5.4, Corollary 5.5.
 787 \square

788 Let us now pass to the analysis of the multiplier convergence. Recall that $\lambda_{1,k+1} =$
 789 $(\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(-\tilde{C}_1 z_{k+1} - \tilde{F}_{1,k})$ and $\lambda_{2,k+1} = -\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_{k+1} + \tilde{R}_2 \tilde{E}_k + \tilde{G}_2(\xi_{k+1}) \in$
 790 $-\mathcal{N}_{K_2}(\tilde{C}_2 z_{k+1} + \tilde{F}_{2,k})$, therefore for all $t \geq 0$, $\lambda_{1,\star}^h(t) = (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(-\tilde{C}_1 z_\star^h(t) -$
 791 $\tilde{F}_{1,\star}^h(t))$ and $\lambda_{2,\star}^h(t) = -\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_\star^h(t) + \tilde{R}_2 \tilde{E}_\star^h(t) + \tilde{G}_2(\xi_\star^h(t)) \in -\mathcal{N}_{K_2}(\tilde{C}_2 z_\star^h(t) +$
 792 $\tilde{F}_{2,\star}^h(t))$, where we recall that the step-functions are defined as $f_\star^h(t) = f_{k+1}$ for
 793 $t \in [t_k, t_{k+1})$. We know that $\dot{\xi}_k \triangleq \frac{\xi_{k+1} - \xi_k}{h} \leq C$ for some $C > 0$, hence:

$$\begin{aligned}
 794 \quad (5.28) \quad \|\xi^h - \xi_\star^h\|_{\mathcal{L}_2([0, T]; \mathbb{R}^{n_2})}^2 &\leq \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_{k+1} - t_k)^2 \|\dot{\xi}_k\|^2 dt \\
 &\leq C^2 \sum_{k=0}^{n-1} (t_{k+1} - t_k)^3 = \frac{C^2 n h^3}{3} = \frac{C^2 T h^2}{3}.
 \end{aligned}$$

795 Consequently $\|\xi^h - \xi_\star^h\|_{\mathcal{L}_2([0, T]; \mathbb{R}^{n_2})} \rightarrow 0$ as $h \rightarrow 0$. Since $\|\xi^h - \xi_\star^h\|_{\mathcal{L}_2([0, T]; \mathbb{R}^{n_2})} =$
 796 $\|\xi^h \pm \xi - \xi_\star^h\|_{\mathcal{L}_2([0, T]; \mathbb{R}^{n_2})} \rightarrow \|\xi - \xi_\star^h\|_{\mathcal{L}_2([0, T]; \mathbb{R}^{n_2})} \rightarrow 0$, we conclude that the sequence
 797 $\{\xi_\star^h\}_{h \geq 0}$ converges strongly in $\mathcal{L}_2([0, T]; \mathbb{R}^{n_2})$ towards the limit $\xi(\cdot)$. The same result
 798 holds for the sequence $\{x_\star^h\}_{h \geq 0}$ with the limit $x(\cdot)$. Therefore the sequence $\{-\tilde{C}_1 z_\star^h -$
 799 $\tilde{F}_{1,\star}^h\}_{h \geq 0}$ also converges strongly in \mathcal{L}_2 towards a limit, and we infer that the same
 800 conclusion holds for both $\{\lambda_{1,\star}^h(\cdot)\}_{h \geq 0}$ and $\{\lambda_{2,\star}^h(\cdot)\}_{h \geq 0}$. Since $\mathcal{N}_{K_2}(\cdot)$ is maximal
 801 monotone, it follows from [9, Proposition 20.33] that the limit $\lambda_2(\cdot)$ satisfies $\lambda_2(t) \in$
 802 $-\mathcal{N}_{K_2}(C_{II,2} z(t) + F_2(t) + C_{I,2} x(t))$. Thus we proved the following result.

803 **LEMMA 5.8.** *Let the conditions of Lemmata 5.3, 5.4, Corollary 5.5 be satisfied for*
 804 *all $h > 0$. The sequences $\{\xi_\star^h\}_{h \geq 0}$, $\{x_\star^h\}_{h \geq 0}$, $\{\lambda_{1,\star}^h(\cdot)\}_{h \geq 0}$ and $\{\lambda_{2,\star}^h(\cdot)\}_{h \geq 0}$ converge*
 805 *strongly in $\mathcal{L}_2([0, T]; \mathbb{R}^{n_2})$ towards the limits $\xi(\cdot)$, $x(\cdot)$, $\lambda_1(\cdot)$ and $\lambda_2(\cdot)$, respectively.*

806 To finish this section, one can consider again (5.8) and the analysis which follows it,
 807 to obtain a proposition similar to Proposition 5.2, relying on Lemma 5.8, Proposition
 808 5.6, and on Corollary 5.5. It is noteworthy that the three fundamental assumptions
 809 in this section, are that $-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1}$ be sufficiently large as a positive definite matrix,
 810 that a suitable selection is chosen for the multiplier $\lambda_{2,k}$, and that a coupling as stated
 811 in Assumptions 2 or 7 exists.

812 PROPOSITION 5.9. *Assume that the conditions of Proposition 5.6, Corollary 5.7*
 813 *and Lemma 5.8 are verified. Then the limit functions $x(\cdot)$, $z(\cdot)$, $\lambda(\cdot)$, are solutions of*
 814 *the DALCS (2.1).*

815 *Proof.* Let us consider (5.8). From Lemma 5.8, $\{x_\star^h(\cdot)\}$, $\{z_\star^h(\cdot)\}$, $\{\lambda_\star^h(\cdot)\}$ converge
 816 strongly in $\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})$ towards limits $x(\cdot)$, $z(\cdot)$ and $\lambda(\cdot)$. Using the same argu-
 817 ments as those developed before Proposition 5.2, it is inferred from (5.8) (b) (c) that
 818 the limits verify (2.1) (b) (c). Using the same arguments as those developed just above
 819 (5.8), it is deduced that $\{\dot{x}^h(\cdot)\}$ converges weakly* to a limit $v(\cdot)$ where $v(\cdot) = \dot{x}(\cdot)$
 820 almost everywhere. Consequently $P_I \dot{x}^h(t) = A_I x_\star^h(t) + B_I z_\star^h(t) + \bar{B}_I \lambda_\star^h(t) + E_I^h(t) \rightarrow$
 821 $P_I \dot{x}(t) = A_I x(t) + B_I z(t) + \bar{B}_I \lambda(t) + E_I(t)$ weakly in $\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})$ as $h \rightarrow 0$. \square

822 **5.2. Convergence results from (3.1).** In this section the properties of the
 823 plant's model (2.1) are used, thus the numerical scheme in (3.1) and the results in
 824 section 4.2 are analysed. This section is devoted to the case $D \succcurlyeq 0$ as in Proposition
 825 4.8. Indeed, since the strict passivity of $(B_{II}, \bar{B}_{II}, C_{II}, D)$ implies Assumption 10 for
 826 small enough $h > 0$, it is unnecessary to repeat the developments in sections 5.1.1 and
 827 5.1.2. However, it is not possible in general to determine the implications between
 828 Assumption 7 and Assumption 9, nor the relations between $\lambda_{\min}(\hat{A})$ and $\lambda_{\min}(B_{II})$
 829 for $h > 0$. Therefore let us deal with the VI in (4.17) where x_{k+1} is replaced by
 830 $x_{k+\theta} = \theta x_k + (1 - \theta)x_{k+1}$, $\theta \in [0, 1]$. The following holds, which is the counterpart of
 831 Lemma 5.3.

832 LEMMA 5.10. *Assume that $\lambda_{\min}(-R_2 B_{II} R_2^{-1}) (= \lambda_{\min}(-B_{II}))$ is large enough,*
 833 *$E_I(\cdot)$, $E_{II}(\cdot)$ and $F(\cdot)$ are uniformly bounded and Lipschitz continuous functions of*
 834 *time. Then any solution $\xi_{k+1} = R_2^{-1} z_{k+1}$ of the VI in (4.17) is upperbounded by an*
 835 *affine function of $\|x_{k+\theta}\|$.*

836 The proof follows the same steps as the proof of Lemma 5.3 and it is therefore omitted.
 837 Lemma 5.10 implies that the term $B_I z_{k+1}$ in the right-hand side of (3.1) (a), is
 838 upperbounded by an affine function of $\|x_{k+\theta}\|$. The coefficients of the affine function
 839 depend only on the system's data (constant matrices, Lipschitz constants) and on the
 840 initial conditions. Let us now propose the counterpart of Lemma 5.4.

841 LEMMA 5.11. *Assume that $-R_2 B_{II} R_2^{-1} \succcurlyeq I_{n_2}$. Assume also that the well-defined,*
 842 *single-valued and Lipschitz continuous mapping $G_2(\cdot)$ is monotone, and that $E_I(\cdot)$,*
 843 *$E_{II}(\cdot)$ and $F(\cdot)$ are Lipschitz continuous functions of time. Then for any k and $p \geq 0$*
 844 *and any solution of the VI in (4.17), one has:*

$$845 \quad (5.29) \quad \|\xi_k - \xi_p\| \leq \alpha + \beta \|x_{k-1+\theta} - x_{p-1+\theta}\|$$

846 *for all $k \geq 1$ and some bounded constants $\alpha \geq 0$ (proportional to $|t_{k-1} - t_{p-1}|$) and*
 847 *$\beta \geq 0$.*

848 **Proof:** The proof is similar to the proof of Lemma 5.4, since both VIs in (4.17) and
 849 (4.9) possess similar structures. Using (4.17) one finds:

$$850 \quad (5.30) \quad \begin{cases} -R_2 B_{II} R_2^{-1} \xi_{k+1} - R_2 A_{II} x_{k+\theta} - R_2 E_{II,k} + G_2(t_k, \xi_{k+1}) = -\zeta_{k+1} \\ -R_2 B_{II} R_2^{-1} \xi_{p+1} - R_2 A_{II} x_{p+\theta} - R_2 E_{II,p} + G_2(t_p, \xi_{p+1}) = -\zeta_{p+1} \end{cases}$$

851 with $\zeta_k \in \partial f_2(\xi_k)$, $k \geq 0$. Using the monotonicity of $G_2(t_k, \cdot)$ and of $\partial f_2(\cdot)$ it follows

852 that:

$$\begin{aligned}
 & \langle -R_2 B_{II} R_2^{-1} (\xi_{k+1} - \xi_{p+1}), \xi_{k+1} - \xi_{p+1} \rangle \\
 & \leq \langle R_2 A_{II} (x_{k+\theta} - x_{p+\theta}) + R_2 (E_{II,k} - E_{II,p}), \xi_{k+1} - \xi_{p+1} \rangle \\
 & \quad - \langle G_2(t_k, \xi_{k+1}) - G_2(t_k, \xi_{p+1}), \xi_{k+1} - \xi_{p+1} \rangle \\
 853 \quad (5.31) \quad & \quad + \langle G_2(t_k, \xi_{p+1}) - G_2(t_p, \xi_{p+1}), \xi_{k+1} - \xi_{p+1} \rangle \\
 & \leq \|R_2 A_{II}\|^2 \|x_{k+\theta} - x_{p+\theta}\|^2 + \|R_2\|^2 k_E^2 (t_k - t_p)^2 \\
 & \quad + \frac{1}{2} k_{G_2}^2 \|F_k - F_p\|^2 + \|\xi_{k+1} - \xi_{p+1}\|^2 \\
 & \leq \|R_2 A_{II}\|^2 \|x_{k+\theta} - x_{p+\theta}\|^2 + \|R_2\|^2 k_E^2 (t_k - t_p)^2 + \frac{1}{2} k_{G_2}^2 k_F^2 (t_k - t_p)^2 \\
 & \quad + \|\xi_{k+1} - \xi_{p+1}\|^2.
 \end{aligned}$$

854 The lemma is proved. \square

855 Thus Corollary 5.5 extends as well, and in particular one has $\|\xi_{k+1} - \xi_k\| \leq h\alpha +$
 856 $\beta\|x_{k+\theta} - x_{k-1+\theta}\|$ for all $k \geq 1$, for bounded constants $\alpha \geq 0$ and $\beta \geq 0$. The scheme
 857 in (5.25) is now replaced by:

$$\begin{aligned}
 858 \quad (5.32) \quad P_I^h x_{k+1} = & P_I x_k + h B_I z_{k+1} + h \bar{B}_I^1 (D_{11} + \partial\psi_{K_1^*})^{-1} (C_{II}^1 R_2^{-1} \xi_{k+1} + F_{1,k}) \\
 & + h \bar{B}_I^2 (-R_2 B_{II} R_2^{-1} \xi_{k+1} - R_2 A_{II} x_{k+\theta} - R_2 E_{II,k} + G_2(t_k, \xi_{k+1})),
 \end{aligned}$$

859 equivalently:

$$\begin{aligned}
 860 \quad (5.33) \quad (P_I^h + h(1 - \theta) \bar{B}_I^2 R_2 A_{II}) x_{k+1} = & P_I x_k + h B_I z_{k+1} \\
 & + h \bar{B}_I^1 (D_{11} + \partial\psi_{K_1^*})^{-1} (C_{II}^1 R_2^{-1} \xi_{k+1} + F_{1,k}) + h \bar{B}_I^2 (-R_2 B_{II} R_2^{-1} \xi_{k+1} \\
 & - \theta R_2 A_{II} x_k - R_2 E_{II,k} + G_2(t_k, \xi_{k+1})),
 \end{aligned}$$

861 and it appears that both schemes in (5.25) and in (5.33) possess the same structure.
 862 Consequently the results of convergence obtained for the foregoing implicit scheme
 863 (Proposition 5.6, Corollary 5.7, Lemma 5.8) extend to the scheme in this section, for
 864 suitably chosen numerical and system parameters. For the sake of brevity we do not
 865 repeat the developments.

866 *Remark 5.12.* Using Lemma 4.1 item 1, Assumption 10 could be replaced by
 867 assuming the strict passivity of the quadruple $(B_{II}, \bar{B}_{II}, C_{II}, D)$ (a system property)
 868 and $h > 0$ small enough.

869 *Remark 5.13.* Obviously when B_{II} is invertible it is possible to recast (2.1) into
 870 an LCS by solving the equality constraint for z . This yields the LCS quadruple
 871 $(P_I^{-1} (A_I - B_I B_{II}^{-1} A_{II}), P_I^{-1} (\bar{B}_I - B_I B_{II}^{-1} \bar{B}_{II}), C_I - C_{II} B_{II}^{-1} A_{II}, D - C_{II} B_{II}^{-1} \bar{B}_{II})$.
 872 It is unclear how to relate in general the passivity of this quadruple, to the above
 873 assumptions which focus on the MLCP part (2.1) (b) (c) of the system. Especially
 874 no specific assumptions are made on the matrices A_I, B_I, \bar{B}_I, C_I in the above analy-
 875 ses. For instance Assumption 3 implies that the feedthrough matrix of the LCS is
 876 $D - C_{II} B_{II}^{-1} X^{-1} C_{II}^T$, which may be indefinite in general. It is also noteworthy that
 877 transforming the MLCP into an LCP may be undesirable [59].

878 **5.3. Recapitulation.** Let us recapitulate the various assumptions upon which
 879 the results in this article rely. Assumptions 2, 4, 7, 10, 12 (respectively Assumptions
 880 3, 5, 9), characterize the passivity properties of an auxiliary operator defined by the
 881 DALCS constraints in (3.2) (b) (c) (respectively in (2.1) (b) (c)). These passivity
 882 properties constrain the relative degree between the ‘‘input’’ λ_{k+1} and the ‘‘output’’
 883 $w_{k+1} = \tilde{C} z_{k+1} + \tilde{D} \lambda_{k+1}$ (respectively λ and $w = C_{II} z + D \lambda$). Assumptions 6 and 8
 884 have practical importance (because some systems possess such D matrix), they are

885 also related to passivity in the sense that passive systems possess positive semidefinite
 886 throughout matrices. Thus all these assumptions are fundamental ones for the devel-
 887 opments in this article, which basically deals with the analysis of two types of gener-
 888 alised equations (GE): $0 \in A(u)+B(u)$ in (4.7) and (4.15), and $0 \in A(u)+B(u)+C(u)$
 889 in (4.9) and (4.17), where the mappings $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ are maximal monotone
 890 (notice in passing that due to time-dependency, these GEs do not fit with those in [28,
 891 Chapter 2], so that in particular [28, Theorem 2F.6] cannot be applied in general).
 892 Most importantly, the convergence results (Proposition 5.2, Lemmata 5.3 and 5.4,
 893 Corollary 5.5, Proposition 5.6, Corollary 5.7, Lemmata 5.8 and 5.10) rely upon the
 894 positive definiteness of $-R_2 B_{II} R_2^{-1}$ and $-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1}$, which is an index one condition
 895 and guarantees strong monotonicity properties, allowing us to characterize the prop-
 896 erties of the solutions to the GEs. It would be worth relaxing it in future studies.
 897 One path could be to use least norm elements [34]. Finally, it is worth noting that
 898 many practical systems like circuits with nonlinear resistors and inductances contain
 899 nonlinearities [19, section 4.1]. An extension of the above developments, consists of
 900 replacing the linear term $A_I x$ in (2.1), by a nonlinear term $A_I(x)$, where $A_I(\cdot)$ is
 901 globally Lipschitz continuous, and setting $A_I(x_k)$ in (3.1). The matrices $\tilde{A}_2, \tilde{B}, \tilde{C}, \tilde{D}$
 902 keep the same structure with P_I^{-1} instead of $(P_I^h)^{-1}$, while both \tilde{E}_k and \tilde{F}_k contain
 903 $A_I(x_k)$. Nonlinearities in the equality and in the complementarity constraints should
 904 also be considered, where the results in [27] could be useful.

905 **6. Examples and Numerical Simulations.** The next examples are taken
 906 from circuits, mechanics and state-dependent-switch DAE. They aim at illustrating
 907 the above developments, and the limitations imposed by some assumptions.

908 **6.1. Example 1 (circuit).** Let us consider the circuit in Figure 1b. Using
 909 Kirchoff laws its dynamics are given by:

$$910 \quad (6.1) \quad \begin{cases} \dot{x}_1(t) = \frac{-2}{RC} x_1(t) + \frac{1}{RC} x_2(t) + \frac{1}{R} u_{D_1}(t) \\ \dot{x}_2(t) = \frac{1}{RC} x_2(t) + \frac{1}{2RC} x_1(t) + \frac{1}{2R} u_{D_2}(t) \\ 0 = \frac{3}{2} u_{D_2}(t) - R x_3(t) - \frac{1}{C} x_2(t) + \frac{1}{2C} x_1(t) \\ 0 \leq u_{D_1}(t) \perp \frac{-2}{RC} x_1(t) + \frac{1}{RC} x_2(t) + \frac{1}{R} u_{D_1}(t) \geq 0 \\ 0 \leq u_{D_2}(t) \perp x_3(t) \geq 0, \end{cases}$$

911 with $x_1(t) = \int_0^t i_1(s) ds$, $x_2(t) = \int_0^t i_2(s) ds$, $x_3 = i_3 = z$, $\lambda = (u_{D_1} \ u_{D_2})^\top$ (the
 912 voltages across the diodes), $K = K^* = \mathbb{R}^2_+$, $n_1 = 2$, $p = n_2 = 1$, $m = 2$. One
 913 has $D = \begin{pmatrix} \frac{1}{R} & 0 \\ 0 & 0 \end{pmatrix}$, $P_I = I_2$, $A_I = \begin{pmatrix} \frac{-2}{RC} & \frac{1}{RC} \\ \frac{1}{2RC} & \frac{1}{RC} \end{pmatrix}$, $B_I = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\bar{B}_I = \begin{pmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{2R} \end{pmatrix}$,
 914 $A_{II} = \begin{pmatrix} \frac{1}{2C} & -\frac{1}{C} \end{pmatrix}$, $B_{II} = -R$, $\bar{B}_{II} = \begin{pmatrix} 0 & \frac{3}{2} \end{pmatrix}$, $C_I = \begin{pmatrix} \frac{-2}{RC} & \frac{1}{RC} \\ 0 & 0 \end{pmatrix}$, $C_{II} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
 915 Assumption 8 holds with $D_{11} = \frac{1}{R}$. Thus $\bar{B}_{II}^2 = \frac{3}{2}$, $C_{II}^2 = 1$, $X_2 = \frac{2}{3}$ and Assumption
 916 9 is verified. Item 1 in Proposition 4.8 applies since $\text{Im}(C_{II}^2) = \mathbb{R}$. Since $\bar{B}_{II}^1 = 0$
 917 the mapping $G_2(\cdot)$ is zero, and item 3 in Proposition 4.8 applies. The quadruple
 918 $(B_{II}, \bar{B}_{II}, C_{II}, D)$ is passive because the LMI (C.1) is verified with $X = \frac{2}{3}$, hence
 919 Assumption 5 is satisfied, but it is not strictly passive since $D \not\succ 0$. It can be checked
 920 that the transfer matrix $H(s) = D + C_{II}(s - B_{II})^{-1} \bar{B}_{II} = \begin{pmatrix} \frac{1}{R} & 0 \\ 0 & \frac{3}{2(s+R)} \end{pmatrix}$ is positive
 921 real [18, Theorem 2.48]. The conditions of Lemmata 5.10 and 5.11 are verified by
 922 taking R large enough (notice that all other parameters depend on $\frac{1}{R}$). Here $\tilde{A} =$
 923 $B_{II} = -R$, but Assumptions 2 and 6 do not hold, hence the material in Proposition
 924 4.4 cannot be used.

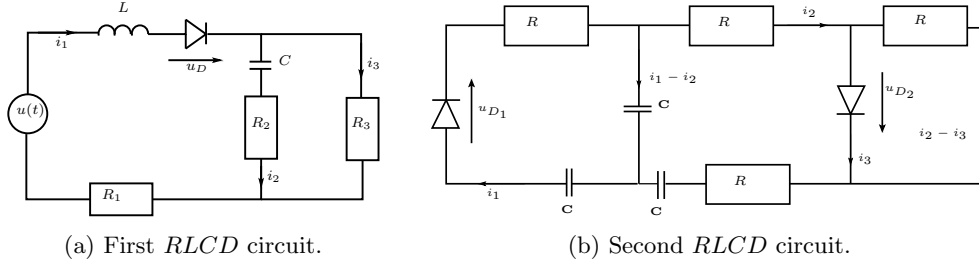


Fig. 1: Two circuits with ideal diodes.

925 **6.2. Example 2 (circuit).** Let us consider now the circuit in Figure 1a. Using
 926 Kirchhoff laws its dynamics may be written as:

$$927 \quad (6.2) \quad \begin{cases} \dot{x}_1(t) = -\frac{R_1}{L}x_1(t) - \frac{R_3}{L}z(t) + \frac{1}{L}u(t) + \frac{1}{L}u_D(t) \\ \dot{x}_2(t) = \frac{R_3}{R_2}z(t) - \frac{1}{R_2C}x_2(t) \\ 0 = x_1(t) + \frac{1}{R_2C}x_2(t) - (1 + \frac{R_3}{R_2})z(t) \\ 0 \leq u_D(t) \perp x_1(t) \geq 0, \end{cases}$$

928 where $x_1 = i_1$, $x_2(t) = \int_0^t i_2(s)ds$, $z = i_3$, $\lambda = u_D$ (the voltage across the diode),
 929 $K = K^* = \mathbb{R}_+$, $n_1 = 2$, $p = n_2 = 1$, $m = 1$. Let apply the feedback control $u = 2R_3z$,

930 which modifies B_I . We get: $A_I = \begin{pmatrix} -\frac{R_1}{L} & 0 \\ 0 & -\frac{1}{R_2C} \end{pmatrix}$, $B_I = \begin{pmatrix} \frac{R_3}{L} \\ \frac{R_3}{R_2} \end{pmatrix}$, $\bar{B}_I = \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix}$,

931 $A_{II} = (1 - \frac{1}{R_2C})$, $B_{II} = -(1 + \frac{R_3}{R_2})$, $\bar{B}_{II} = 0$, $C_I = (1 \ 0)$, $C_{II} = 0$, $D = 0$, $E_I = \begin{pmatrix} \frac{1}{L}u \\ 0 \end{pmatrix}$,

932 $E_{II} = 0$, $F(t) = 0$, $P_I = I_2$, $P_I^h = \begin{pmatrix} 1 + h\frac{R_1}{L} & 0 \\ 0 & 1 + h\frac{1}{R_2C} \end{pmatrix} \succ 0$ for all $h \geq 0$. Thus

933 $\tilde{B} = \frac{h}{hR_1 + L}$, $\tilde{C} = \frac{hR_3}{L + hR_1}$, $\tilde{D} = \frac{h}{L + hR_1}$, $\tilde{A} = -\frac{R_2 + R_3}{R_2} + hR_3 \left(\frac{1}{L + hR_1} + \frac{1}{R_2(h + R_2C)} \right)$.

934 Assumption 2 holds with $\tilde{X} = R_3$, $\tilde{D} > 0$ for all $h > 0$, and Assumption 4 holds
 935 for $h > 0$ small enough such that $\tilde{A} < 0$. For $h > 0$ small enough the set $\tilde{S} = \{0\}$

936 in (4.6) since $\tilde{R}\tilde{A}\tilde{R}^{-1} = \tilde{A} < 0$, hence the VI in (4.7) always has a solution z_{k+1} by
 937 Proposition 4.3 item 3 (a). By item (c) (or (f)) of the same proposition, the solution is

938 unique. When $\tilde{A} < 0$, the quadruple $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is strictly passive for all $h > 0$, hence
 939 items 1 and 3 of Corollary 4.6 apply. However Assumption 10 does not hold since

940 $\tilde{D} \rightarrow 0$ as $h \rightarrow 0$. Consider now the quadruple $(B_{II}, \bar{B}_{II}, C_{II}, D) = (B_{II}, 0, 0, 0)$.

941 It is passive since (C.1) is verified, it is even strictly state passive but not strictly
 942 passive since $D = 0$. Assumptions 3 and 9 are trivially satisfied since both sides of

943 the passivity equality are null. Provided that $\{0\} \in \mathbb{R}_+ - x_{1,k+1} \Leftrightarrow x_{1,k+1} \geq 0$, item 1
 944 in Proposition 4.7 applies and the OSNSP for (3.1) has a unique solution (but clearly

945 there is no guarantee that this inequality is satisfied along the integration process,
 946 rendering Proposition 4.7 unapplicable). Proposition 4.8 item 3 also applies here for

947 the VI in (4.17), however one has to check the equivalence stated in item 1 of the
 948 same proposition.

949 As noted above, the matrix A_I intervenes only in Assumption 1, hence its modification
 950 does not alter the results. Setting $u = 2R_3z + R_1x_1 + aLx_1 + v(t)$ yields the modified

951 $A_I = \begin{pmatrix} a & 0 \\ 0 & -\frac{1}{R_2C} \end{pmatrix}$, and B_I as above. This yields (see Remark 5.13): $P_I^{-1}(A_I -$

952 $B_I B_{II}^{-1} A_{II} = \begin{pmatrix} a + \frac{R_2 R_3}{L(R_2 + R_3)} & \frac{R_3}{L\mathbf{C}(R_2 + R_3)} \\ \frac{R_3}{R_2 + R_3} & -\frac{1}{(R_2 + R_3)\mathbf{C}} \end{pmatrix}$, which is unstable if $a + \frac{R_2 R_3}{(R_2 + R_3)L} >$
953 $\frac{1}{(R_2 + R_3)\mathbf{C}}$. The other elements of the reduced LCS quadruple are given by $P_I^{-1}(\bar{B}_I -$
954 $B_I B_{II}^{-1} \bar{B}_{II}) = \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix}$, $C_I - C_{II} B_{II}^{-1} A_{II} = (1 \ 0)$, $D - C_{II} B_{II}^{-1} \bar{B}_{II} = 0$. Consequently
955 the reduced LCS is not passive with the above feedback controller. Also $P_I^h = \text{diag}(1 -$
956 $ha, 1 + \frac{h}{R_2 \mathbf{C}}) \succ 0$ for all $h < \frac{1}{a}$.
957 To simplify the calculations let us set $R_2 = R_3 = 1\Omega$, $L = 1\text{H}$, $\mathbf{C} = 1\text{F}$, $a = 1$. Then
958 the discretization (3.1) of (6.2) writes as:

$$959 \quad (6.3) \quad \begin{cases} x_{1,k+1} - x_{1,k} = hx_{1,k+1} + hz_{k+1} + h\lambda_{k+1} + hv_k \\ x_{2,k+1} - x_{2,k} = -hx_{2,k+1} + hz_{k+1} \\ 0 = x_{1,k+1} + x_{2,k+1} - 2z_{k+1} \\ 0 \leq \lambda_{k+1} \perp x_{1,k+1} \geq 0. \end{cases}$$

960 Equivalently the discretization in (3.2) can be used with the above matrices $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$.

961 **6.3. Example 3 (switching DAE).** Let us consider the following system,
962 which is a nonsmooth DAE (similar to those studied in [53, 54, 52, 51], and which
963 can be interpreted as a DAE with state-dependent switches):

$$964 \quad (6.4) \quad \begin{cases} (a) \dot{x}(t) = A_I x(t) + B_I z(t) + E_I(t) \\ (b) 0 = a|x(t)| + b|z(t)| - cz(t) + E_{II}(t), \end{cases}$$

965 with $x(t) \in \mathbb{R}$, $z(t) \in \mathbb{R}$, $a, b, c \in \mathbb{R}$, $c > 0$. The constraint in (6.4) (b) is equivalently
966 rewritten as:

$$967 \quad (6.5) \quad \begin{aligned} 0 &= a\lambda_1(t) + a\lambda_2(t) + b\lambda_3(t) + b\lambda_4(t) - cz(t) + E_{II}(t) \\ 0 &\leq \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \lambda_3(t) \\ \lambda_4(t) \end{pmatrix} \perp \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \lambda_3(t) \\ \lambda_4(t) \end{pmatrix} + \begin{pmatrix} -x(t) \\ x(t) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -z(t) \\ z(t) \end{pmatrix} \geq 0. \end{aligned}$$

968 Therefore (6.4) (a) (6.5) fits with (2.1) with $A_{II} = 0$, $B_{II} = -c$, $\bar{B}_{II} = (a \ a \ b \ b)$,
969 $D = I_4$, $C_I = (-1 \ 1 \ 0 \ 0)^\top$, $C_{II} = (0 \ 0 \ -1 \ 1)^\top$, $F(t) = 0$, $\bar{B}_I = 0$. Let us check the
970 strict passivity of the quadruple $(B_{II}, \bar{B}_{II}, C_{II}, D)$. This boils down to checking that
971 (see (C.1)):

$$972 \quad (6.6) \quad M \triangleq \begin{pmatrix} 2cX & -aX & -aX & -bX - 1 & -bX + 1 \\ -aX & 2 & 0 & 0 & 0 \\ -aX & 0 & 2 & 0 & 0 \\ -bX - 1 & 0 & 0 & 2 & 0 \\ -bX + 1 & 0 & 0 & 0 & 2 \end{pmatrix} \succ 0,$$

973 for some $X > 0$. Let us denote $M = P + N$ with $P = \text{diag}(2cX, 2I_4) \succ 0$, and
974 N_{00} the matrix obtained for $a = b = 0$. Applying [25, Theorem 2.11] or [10, Fact
975 8.19.2], it follows that $P + N_{00} \succ 0$ if $\sigma_{\max}^2(N_{00}) = 2 < \lambda_{\min}(2cX)\lambda_{\min}(2I_4) = 4cX$.
976 Using continuity of the eigenvalues [10, Fact 10.11.8], it is deduced that given $c > 0$,
977 $X > 0$, there exists a and b small enough such that (6.6) holds (or, given a, b and
978 $X > 0$, there exists $c > 0$ large enough). Thus, item 1 in Lemma 4.1 holds, item 2
979 in Proposition 4.3 holds for $h > 0$ small enough (guaranteeing $\tilde{D} \succ 0$), Assumption

980 10 holds true for small enough $h > 0$ (see the proof of item 1 in Lemma 4.1), and
 981 the arguments stated at the beginning of section 5.2 about the convergence analysis
 982 led in sections 5.1.1 and 5.1.2, hold. We have: $\tilde{A} = -c$, $\tilde{B} = (a \ a \ b \ b)$, $\tilde{C} =$
 983 $(0 \ 0 \ -1 \ 1)^\top + h(-1 \ 1 \ 0 \ 0)^\top(1 - hA_I)^{-1}B_I = \begin{pmatrix} -hB_I & hB_I & 0 & 0 \\ 1-hA_I & 1-hA_I & 0 & 0 \end{pmatrix}^\top$, $\tilde{D} = D$.
 984 Integration can proceed using (3.2).
 985 Let us now pass to a second example of state-dependent-switch DAE:

$$986 \quad (6.7) \quad \begin{cases} (a) \dot{x}_1(t) = a_1x_1(t) + b_1x_2(t) + c_1z(t) + e_1(t) \\ (b) \dot{x}_2(t) = a_2x_1(t) + b_2x_2(t) + c_2z(t) + e_2(t) \\ (c) 0 = a_3x_1(t) + b_3x_2(t) + c_3z(t) + e_3(t) \text{ if } x_1(t) < 0 \\ (d) 0 = a_4x_1(t) + b_3x_2(t) + c_3z(t) + e_3(t) \text{ if } x_1(t) > 0, \end{cases}$$

987 where the switching conditions in (c) and (d) are equivalently rewritten as:

$$988 \quad (6.8) \quad \begin{cases} 0 = -a_3\lambda_1(t) + a_4\lambda_2(t) + b_3x_2(t) + c_3z(t) + e_3(t) \\ 0 \leq \lambda_1(t) \perp \lambda_1(t) + x_1(t) \geq 0 \\ 0 \leq \lambda_2(t) \perp \lambda_2(t) - x_1(t) \geq 0. \end{cases}$$

989 Thus we have: $A_I = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$, $B_I = (c_1 \ c_2)^\top$, $\bar{B}_I = (0 \ 0)^\top$, $E_I = (e_1 \ e_2)^\top$, $\bar{B}_{II} =$
 990 $(-a_3 \ a_4)^\top$, $A_{II} = (0 \ b_3)$, $B_{II} = c_3$, $E_{II} = e_3$, $C_I = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$, $C_{II} = (0 \ 0)^\top$, $D = I_2$,
 991 $F(t) = (0 \ 0)^\top$. Also $\tilde{D} = D = I_2$, $\tilde{A} = c_3 + h(0 \ b_3)(I_2 - hA_I)^{-1} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_3 + \mathcal{O}(h)$,

992 $\tilde{B} = \bar{B}_{II} = (-a_3 \ a_4)^\top$, $\tilde{C} = h \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} (I_2 - hA_I)^{-1} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathcal{O}(h)$. It can be checked

993 that Proposition 4.3 items 1 and 2, Corollary 4.6 items 1, 2 and 3 (if the passivity LMI
 994 holds strictly) apply. The passivity LMI $\begin{pmatrix} -2X\tilde{A} & -X(a_3 \ a_4) + \tilde{C} \\ -X(a_3 \ a_4) + \tilde{C}^\top & 2I_2 \end{pmatrix} \succ 0$

995 is verified for a_3 , a_4 and $h > 0$ small enough, given $c_3 < 0$, $X > 0$, implying the
 996 strict passivity of the quadruple $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$, uniformly as $h \rightarrow 0$ (Assumption 10
 997 holds true). It is of interest to investigate what happens if $c_3 = 0$, which makes
 998 the “index” increase. Let us assume for simplicity that $A_I = 0$, the matrix of the

999 passivity LMI becomes $\begin{pmatrix} -2hXb_3c_2 & -X(-a_3 \ a_4) + h(c_1 \ -c_1) \\ (-X(-a_3 \ a_4) + h(c_1 \ -c_1))^\top & 2I_2 \end{pmatrix}$.

1000 For given $X > 0$ and $h > 0$, there always exists $b_3c_2 < 0$ such that this matrix is $\succ 0$,
 1001 showing that there exists conditions under which the OSNP is still well-posed when
 1002 $c_3 = 0$. It is interesting to note that the term $b_3c_2z(t)$ appears when an index reduction
 1003 is applied to the equality in (6.8) by differentiating it once. Simulations are shown
 1004 in Figures 2 and 3. They are performed with the INRIA software package SICONOS¹
 1005 [1]. The MLCP is solved with an enumeration method and the differential variables
 1006 $x_1(\cdot)$ and $x_2(\cdot)$ are calculated in a second step (hence we follow the philosophy of the
 1007 foregoing sections). The parameters are chosen as follows: $a_1 = a_2 = b_1 = b_2 = a_3 =$
 1008 $a_4 = b_3 = c_3 = 1$, $c_1 = 10$, $c_2 = 20$, $e_1(t) = 40 \sin(25t)$, $e_2(t) = \sin(t)$, $e_3(t) = \sin(10t)$,
 1009 $h = 2 \cdot 10^{-3}$ s, $x_1(0) = 0$, $x_2(0) = 0.1$. Simulations are done on $[0, 2]$ s. The errors are
 1010 calculated as $\max_{t \in [0, 2]} \|x_i(t) - x_i^{ref}(t)\|$, $i = 1, 2$, with 14 different values of step
 1011 size h between 10^{-1} s and 10^{-4} s, where $x_i^{ref}(\cdot)$ is calculated with $h = 10^{-4}$ s. It is
 1012 observed in Figure 2 that the order is approximately equal to one for small enough

¹<https://nonsmooth.gricad-pages.univ-grenoble-alpes.fr/siconos/index.html>

1013 step size. Figure 3 depicts the time evolution of various variables, demonstrating that
 1014 the complementarity constraints are respected. Clearly the multipliers are continuous
 1015 but non differentiable when the complementarity switches from one mode to another.

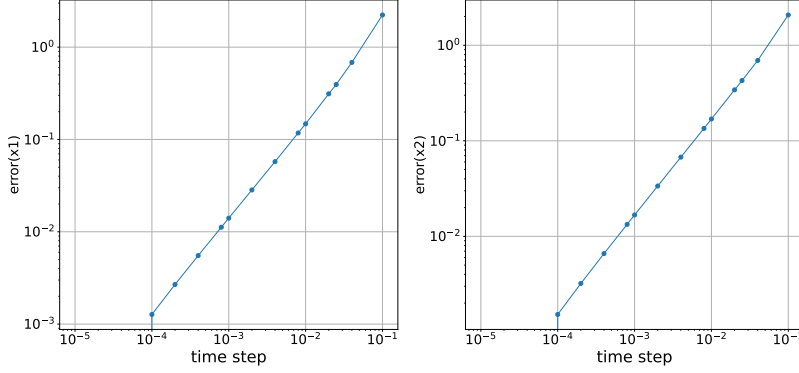


Fig. 2: Log-log error diagram: x_1 (left) and x_2 (right).

1016

1017 **6.4. Example 4 (mechanical system).** As an extension of the systems studied
 1018 in [37], let us consider a linear mechanical system with both unilateral springs and
 1019 bilateral constraints, whose dynamics are:

$$1020 \quad (6.9) \quad \begin{cases} (a) M\ddot{q}(t) + K_v\dot{q}(t) + K_pq(t) = \tau(t) + H^\top \lambda_u(t) + G^\top \lambda_b(t) \\ (b) 0 = Gq(t) - f(t) \\ (c) 0 \leq \lambda_u(t) \perp D\lambda_u(t) + C_Ix(t) + F(t) \geq 0, \end{cases}$$

1021 where $M = M^\top \succ 0$, $D \succ 0$, $x = (q^\top \dot{q}^\top)^\top$, $q(t) \in \mathbb{R}^{\frac{n_1}{2}}$, $z = \lambda_b \in \mathbb{R}^{n_2}$, $\lambda = \lambda_u \in$
 1022 \mathbb{R}^m , $G \in \mathbb{R}^{n_2 \times \frac{n_1}{2}}$, $f: \mathbb{R}^+ \rightarrow \mathbb{R}^{n_2}$ is twice differentiable, $\tau(\cdot)$ is an exogenous force.
 1023 Here an index reduction is possible, in order to replace (6.9) (b) by:

$$1024 \quad (6.10) \quad 0 = G\ddot{q}(t) - \ddot{f}(t) = GM^{-1}H^\top \lambda(t) + GM^{-1}G^\top z(t) - GM^{-1}(K_p K_v)x(t) - \ddot{f}(t),$$

1025 and $Gq(0) - f(0) = 0$, $G\dot{q}(0) - \dot{f}(0) = 0$. Thus $B_{II} = -GM^{-1}G^\top$ ($\prec 0$ if bilateral con-
 1026 straints are independent, *i.e.*, $\text{rank}(G) = p = n_2 \Rightarrow n_1 \geq 2n_2$), $A_{II} = GM^{-1}(K_p K_v)$,

1027 $\bar{B}_{II} = -GM^{-1}H^\top$, $B_I = \begin{pmatrix} 0 \\ -M^{-1}G^\top \end{pmatrix}$, and $C_{II} = 0$ (similarly to the example in
 1028 section 6.2, one major difference being that here $D \succ 0$). Let us examine the OSNSP
 1029 for (6.9) on a particular example as depicted in Figure 4, whose dynamics are given
 1030 by:

$$1031 \quad (6.11) \quad \begin{cases} m\ddot{q}_1(t) = \tau(t) + \lambda_1(t) - \lambda_2(t) \\ m\ddot{q}_2(t) = z(t) \\ 0 = q_2(t) - f(t) \Leftrightarrow \frac{-z(t)}{m} + \ddot{f}(t) = 0, q_2(0) - f(0) = 0, \dot{q}_2(0) - \dot{f}(0) = 0 \\ 0 \leq \lambda_1(t) \perp \lambda_1(t) + k(q_1(t) - L - \alpha(t) - l) \geq 0 \\ 0 \leq \lambda_2(t) \perp \lambda_2(t) - k(q_1(t) + L - \beta(t) + l) \geq 0. \end{cases}$$

1032 where $q = (q_1 \ q_2)^\top$ are the mass coordinates, $k > 0$ is a stiffness constant, $\alpha(\cdot)$
 1033 and $\beta(\cdot)$ are the massless springs' varying positions, $l > 0$ is the spring's free length,

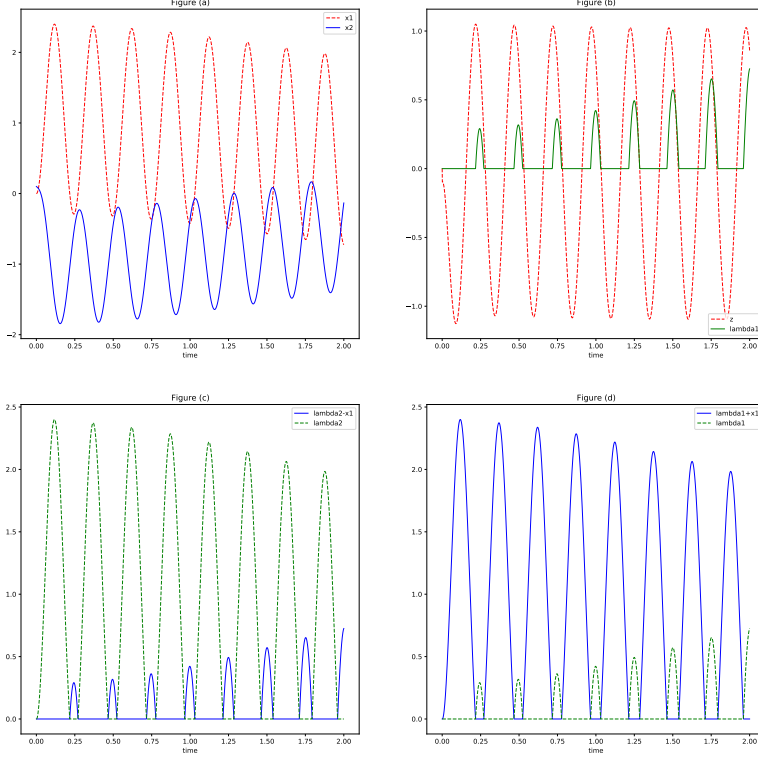


Fig. 3: (a) $x_1(t)$ (dashed) and $x_2(t)$, (b) $z(t)$ (dashed) and $\lambda_1(t)$, (c) $\lambda_2(t) - x_1(t)$ and $\lambda_2(t)$ (dashed), (d) $\lambda_1(t) + x_1(t)$ and $\lambda_1(t)$ (dashed).

1034 $\beta(t) > \alpha(t) + 2L + 2l$ for all $t \geq 0$. We have: $x = (q_1 \dot{q}_1 \ q_2 \ \dot{q}_2)^\top$, $G = (0 \ 1)$, $H =$
 1035 $\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$, $M = \text{diag}(m, m)$, $D = I_2$, $A_{II} = 0$, $B_{II} = \frac{-1}{m}$, $\bar{B}_{II} = 0$, $E_{II}(t) = \ddot{f}(t)$,
 1036 $C_I = \begin{pmatrix} k & 0 & 0 & 0 \\ -k & 0 & 0 & 0 \end{pmatrix}$, $F(t) = \begin{pmatrix} k(-L - \alpha(t) - l) \\ -k(L - \beta(t) + l) \end{pmatrix}$, $C_{II} = 0$, $B_I = (0 \ 0 \ 0 \ \frac{1}{m})^\top$,
 1037 $P_I = I_4$ (and Assumption 1 holds for $h > 0$ small enough), $\bar{B}_I = \begin{pmatrix} 0 & \frac{1}{m} & 0 & 0 \\ 0 & \frac{m}{1} & 0 & 0 \end{pmatrix}^\top$,
 1038 $\tilde{A} = \frac{-1}{m}$, $\tilde{B} = 0$, $\tilde{C} = hC_I(P_I^h)^{-1}B_I$, $\tilde{D} = I_2 + hC_I(P_I^h)^{-1}\bar{B}_I \succ 0$ if $h > 0$ is small
 1039 enough. Let us check Assumption 4. The passivity LMI is $\begin{pmatrix} \frac{2X}{C^\top} & \tilde{C} \\ \tilde{C}^\top & \tilde{D} + \tilde{D}^\top \end{pmatrix} \succ 0$,
 1040 $X > 0$, which is verified for $h > 0$ small enough and $X > 0$ large enough. Hence under
 1041 these conditions the quadruple $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is strictly passive. It can be checked that
 1042 items 1 and 2 in Proposition 4.3 hold, as well as items 1, 2 and 3 in Corollary 4.6.

1043 Therefore the dynamics can be integrated using (3.2) with $q_{2,0} + f_0 = 0$, $\dot{q}_{2,0} + \dot{f}_0 = 0$.
 1044 We note also that the quadruple $(B_{II}, \bar{B}_{II}, C_{II}, D) = (\frac{-1}{m}, 0, 0, I_2)$ is strictly passive
 1045 (Assumption 5 is verified), hence item 1 in Lemma 4.1 holds. Assumption 10 is
 1046 also verified (for $h > 0$ small enough but uniformly in $h \geq 0$). Let us analyse the
 1047 convergence as in section 5.1. Assumption 11 may not be verified in this case, however
 1048 as noted after this assumption it is easy to modify slightly the numerical scheme so
 1049 that it is satisfied with $P_I = I_4$. Hence it is inferred that Proposition 5.2 applies.

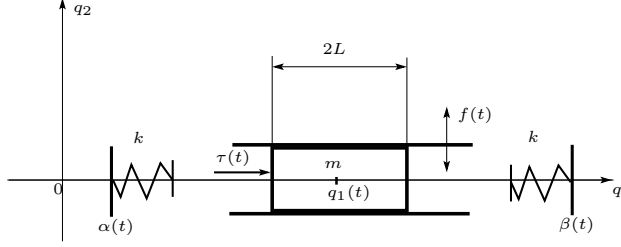


Fig. 4: A mechanical DALCS.

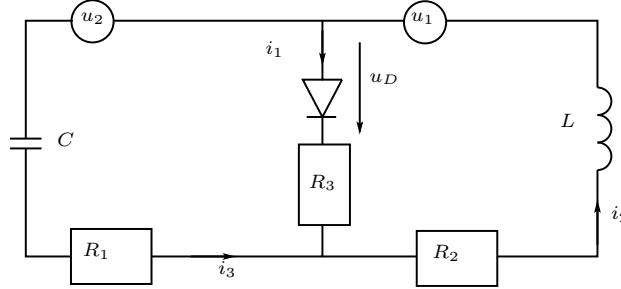


Fig. 5: An $RLCD$ passive circuit.

1050 **6.5. Example 5 (circuit).** Let us consider the circuit in Figure 5, with $R_1 =$
 1051 $R_3 = 0 \Omega$. Let $x_1 = i_2$, $x_2 = \int_0^t i_3(t)dt$, $z = i_1$, $\lambda = u_D$ is the voltage across the
 1052 diode, $u_1(\cdot)$ and $u_2(\cdot)$ are two voltage sources. One way to write its dynamics is:

$$1053 \quad (6.12) \quad \begin{cases} \dot{x}_1(t) = -\frac{R_2}{L}x_1(t) + \frac{1}{L}\lambda(t) + \frac{1}{L}u_1(t) \\ \dot{x}_2(t) = x_1(t) - z(t) \\ 0 = \frac{1}{C}x_2(t) + \lambda(t) + u_2(t) \\ 0 \leq \lambda(t) \perp z(t) \geq 0. \end{cases}$$

1054 Assume that the feedback $u_2 = -az + u_3$ is applied, so that the equality constraint
 1055 becomes $0 = \frac{1}{C}x_2(t) + u_D(t) - az(t) + u_3(t)$. We have: $A_{II} = (0 \quad \frac{1}{C})$, $C_I = 0$,
 1056 $C_{II} = 1$, $B_{II} = -a$, $\bar{B}_{II} = 1$, $D = 0$, $\bar{B}_I = (\frac{1}{L} \quad 0)^\top$, $B_I = (0 \quad -1)^\top$, and $\bar{D} = D = 0$,
 1057 $\tilde{C} = C_{II} = 1$, $\tilde{A} = -a + hA_{II}(P_I^h)^{-1}B_I$, $\tilde{B} = 1 + hA_{II}(P_I^h)^{-1}\bar{B}_I$. The condition
 1058 $\text{Im}(C_{II}) \cap (K - C_I x_{k+1} + F_k) = \mathbb{R} \cap \mathbb{R}_+ - \frac{1}{C}x_{2,k+1} \neq \emptyset$ is verified always. The set \mathcal{S} in
 1059 (4.16) is equal to $\{0\}$ hence Proposition 4.7 item 1 applies and the VI in (4.15) always
 1060 has a solution. Proposition 4.8 items 1, 2, 3 apply, with $D_{11} = 0$ in Assumption 8.

1061 The system $(B_{II}, \bar{B}_{II}, C_{II}) = (-a, 1, 1)$ is passive and Assumptions 3 and 5 hold true.
 1062 Also the system $(\bar{A}, \bar{B}, \bar{C})$ is passive for $h > 0$ small enough and Assumptions 2 and
 1063 4 hold true. The set \mathcal{S} in (4.6) is equal to $\{0\}$ so Proposition 4.3 item 3 (a) applies.
 1064 For $h > 0$ small enough, $\bar{A} < 0$ so Corollary 4.6 items 1 and 2 apply. Assumption 12
 1065 is verified for $h \geq 0$ small enough with $\bar{X} = \bar{B}^{-1}$. It is inferred that if $a > 0$ is large
 1066 enough, the convergence results in sections 5.1.3 and 5.2 apply.

1067 **7. Conclusion.** This article is devoted to analysing the implicit Euler discreti-
 1068 sation of a class of nonsmooth nonlinear systems with equality constraints, named
 1069 differential-algebraic linear complementarity systems. The well-posedness of the one-
 1070 step nonsmooth problem is studied first, then the convergence of the discrete-time
 1071 solutions is analysed. This article only scratches the surface of singular nonsmooth
 1072 systems analysis. The above results could be extended in various directions. Most
 1073 importantly, it would be worth extending the results to more general indices and rela-
 1074 tive degrees, to deeper analyse how the algebraic and the complementarity constraints
 1075 are coupled, and to consider state discontinuities and solutions of bounded variation.

1076 **Appendix A. Well-posedness of Variational Inequalities.** Some basic
 1077 notions of Convex Analysis and Complementarity Theory are briefly introduce now
 1078 [11, 29, 49, 31]. Let $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper convex and lower semicontin-
 1079 uous (lsc) function, we denote by $\text{dom}(f) \triangleq \{x \in \mathbb{R}^n \mid f(x) < +\infty\}$ the domain of
 1080 the function $f(\cdot)$. The Fenchel transform $f^*(\cdot)$ of $f(\cdot)$ is the proper, convex and lsc
 1081 function defined by: for all $z \in \mathbb{R}^n$, $f^*(z) = \sup_{x \in \text{dom}(f)} \{\langle x, z \rangle - f(x)\}$. The sub-
 1082 differential $\partial f(x)$ of $f(\cdot)$ at $x \in \mathbb{R}^n$ is defined by: $\partial f(x) = \{\omega \in \mathbb{R}^n \mid f(v) - f(x) \geq$
 1083 $\langle \omega, v - x \rangle, \forall v \in \mathbb{R}^n\}$. We denote by $\text{Dom}(\partial f) \triangleq \{x \in \mathbb{R}^n \mid \partial f(x) \neq \emptyset\}$ the domain of
 1084 the subdifferential operator $\partial f : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Let x_0 be any element in the domain $\text{dom}(f)$, the recession function $f_\infty(\cdot)$ of $f(\cdot)$
 is defined by: for all $x \in \mathbb{R}^n$: $f_\infty(x) = \lim_{\lambda \rightarrow +\infty} \frac{1}{\lambda} f(x_0 + \lambda x)$. The function $f_\infty :$
 $\mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper convex and lsc function which describes the asymptotic
 behavior of $f(\cdot)$. Let $K \subset \mathbb{R}^n$ be a nonempty closed convex set. Let x_0 be any
 element in K . The recession cone of K is defined by [49]:

$$K_\infty = \bigcap_{\lambda > 0} \frac{1}{\lambda} (K - x_0) = \{u \in \mathbb{R}^n \mid x + \lambda u \in K, \forall \lambda \geq 0, \forall x \in K\}.$$

1085 The set K_∞ is a nonempty closed convex cone that is described in terms of the
 1086 directions which recede from K . The indicator function of a set $K \subseteq \mathbb{R}^n$ is $\Psi_K(x) = 0$
 1087 if $x \in K$, $\Psi_K(x) = +\infty$ if $x \notin K$. If K is closed non empty convex, we have
 1088 $\partial \Psi_K(x) = \mathcal{N}_K(x)$, the so-called normal cone to K at x , defined as $\mathcal{N}_K(x) = \{v \in$
 1089 $\mathbb{R}^n \mid v^\top (s - x) \leq 0 \text{ for all } s \in K\}$.

1090 Important properties of the recession function and recession cone are recalled now:

1091 PROPOSITION A.1. [11, Proposition 1.4.8] *The following statements hold:*

1092 a) Let $f_1 : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ and $f_2 : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be two proper, convex and
 1093 lsc functions. Suppose that $f_1 + f_2$ is proper. Then for all $x \in \mathbb{R}^n$: $(f_1 + f_2)_\infty(x) =$
 1094 $(f_1)_\infty(x) + (f_2)_\infty(x)$.

1095 b) Let $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lsc function, and let K be a
 1096 nonempty closed convex set, such that $f + \Psi_K$ is proper (equivalently $\text{dom}(f) \cap K$ is
 1097 non empty). Then for all $x \in \mathbb{R}^n$: $(f + \Psi_K)_\infty(x) = f_\infty(x) + (\Psi_K)_\infty(x)$.

1098 c) Let $K \subset \mathbb{R}^n$ be a nonempty, closed and convex set. Then for all $x \in \mathbb{R}^n$:
 1099 $(\Psi_K)_\infty(x) = \Psi_{K_\infty}(x)$. Moreover for all $x \in K$ and $e \in K_\infty$: $x + e \in K$.

1100 d) If $K \subseteq \mathbb{R}^n$ is a nonempty closed and convex cone, then $K_\infty = K$.

1101 e) Let $K = P(A, b) \triangleq \{x \in \mathbb{R}^n \mid Ax \geq b\}$ for $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. If $K \neq \emptyset$ then
 1102 $K_\infty = P(A, 0) = \{x \in \mathbb{R}^n \mid Ax \geq 0\}$.

1103 Let us now concatenate [4, Theorem 3, Corollaries 3 and 4]. They concern variational
 1104 inequalities (VIs) of the form: Find $u \in \mathbb{R}^n$ such that

$$1105 \quad (\text{A.1}) \quad \langle \mathbf{M}u + \mathbf{q}, v - u \rangle + \varphi(v) - \varphi(u) \geq 0, \text{ for all } v \in \mathbb{R}^n$$

1106 where $\mathbf{M} \in \mathbb{R}^{n \times n}$ is a real matrix, $\mathbf{q} \in \mathbb{R}^n$ a vector and $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$
 1107 a proper convex and lsc function. The VI in (A.1) is equivalent to the inclusion
 1108 $\mathbf{M}u + \mathbf{q} \in -\partial\varphi(u)$, which is a generalized equation (GE) of the form: $0 \in A(u) + B(u)$
 1109 where $A(\cdot)$ and $B(\cdot)$ are two maximal monotone operators (since it will be assumed
 1110 $\mathbf{M} \succcurlyeq 0$). In fact the problems encountered in this paper (see (4.7), (4.9), (4.14), (4.15),
 1111 (4.17)) all belong to this class of GEs. Numerical schemes that compute solutions of
 1112 such GEs have been studied for a long time in the mathematical literature, see [57]
 1113 and references therein. The next proposition states conditions about existence and
 1114 uniqueness of solutions.

1115 The problem in (A.1) is denoted as $VI(\mathbf{M}, \mathbf{q}, \varphi)$, and we set:

$$1116 \quad (\text{A.2}) \quad \mathcal{K}(\mathbf{M}, \varphi) = \{x \in \mathbb{R}^n \mid \mathbf{M}x \in (\text{dom}(\varphi_\infty))^*\}.$$

1117 Note that $(\text{dom}(\varphi_\infty))^*$ is the dual cone of the domain of the recession function φ_∞
 1118 while $(\text{dom}(\varphi))_\infty$ (that we may denote also as $\text{dom}(\varphi)_\infty$) is the recession cone of
 1119 $\text{dom}(\varphi)$.

1120 **PROPOSITION A.2.** [4] Let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lsc
 1121 function with closed domain, $\mathbf{M} \in \mathbb{R}^{n \times n}$, and suppose that $\mathbf{M} \succcurlyeq 0$.

1122 a) If $(\text{dom}(\varphi))_\infty \cap \text{Ker}\{\mathbf{M} + \mathbf{M}^\top\} \cap \mathcal{K}(\mathbf{M}, \varphi) = \{0\}$ then for each $\mathbf{q} \in \mathbb{R}^n$, problem
 1123 $VI(\mathbf{M}, \mathbf{q}, \varphi)$ has at least one solution.

1124 b) Suppose that $(\text{dom}(\varphi))_\infty \cap \text{Ker}\{\mathbf{M} + \mathbf{M}^\top\} \cap \mathcal{K}(\mathbf{M}, \varphi) \neq \{0\}$. If there exists $x_0 \in$
 1125 $\text{dom}(\varphi)$ such that

$$1126 \quad (\text{A.3}) \quad \langle \mathbf{q} - \mathbf{M}^\top x_0, v \rangle + \varphi_\infty(v) > 0, \forall v \in (\text{dom}(\varphi))_\infty \cap \text{Ker}\{\mathbf{M} + \mathbf{M}^\top\} \cap \mathcal{K}(\mathbf{M}, \varphi), v \neq 0,$$

1127 then problem $VI(\mathbf{M}, \mathbf{q}, \varphi)$ has at least one solution.

1128 c) If u_1 and u_2 are two solutions of problem $VI(\mathbf{M}, \mathbf{q}, \varphi)$ then $u_1 - u_2 \in \text{Ker}\{\mathbf{M} +$
 1129 $\mathbf{M}^\top\}$.

1130 d) If $\mathbf{M} = \mathbf{M}^\top$ and u_1 and u_2 denote two solutions of problem $VI(\mathbf{M}, \mathbf{q}, \varphi)$, then
 1131 $\langle \mathbf{q}, u_1 - u_2 \rangle = \varphi(u_2) - \varphi(u_1)$.

1132 e) If $\mathbf{M} = \mathbf{M}^\top$, then u is a solution of $VI(\mathbf{M}, \mathbf{q}, \varphi)$ if and only if it is a solution of
 1133 the optimization problem $\min_{x \in \mathbb{R}^n} \frac{1}{2}x^\top \mathbf{M}x + \langle \mathbf{q}, x \rangle + \varphi(x)$.

1134 **Appendix B. Some Convex Analysis and Complementarity Theory**
 1135 **tools.** If $K \subseteq \mathbb{R}^n$ is a set, then $K^* = \{z \in \mathbb{R}^n \mid \langle z, x \rangle \geq 0 \text{ for all } x \in K\}$ is
 1136 its dual cone. Let K be a nonempty closed convex cone, then:

$$1137 \quad (\text{B.1}) \quad K^* \ni x \perp y \in K \Leftrightarrow x \in -\mathcal{N}_K(y) \iff y \in -\mathcal{N}_{K^*}(x).$$

1138 Let $M = M^\top \succ 0$, x and y two vectors, then [29]

1139 (B.2) $M(x - y) \in -\mathcal{N}_K(x) \Leftrightarrow x = \text{proj}_M[K; y] \Leftrightarrow x = \min_{z \in K} \frac{1}{2}(z - y)^\top M(z - y).$

1140 We note that this is a particular case of (A.1), so that Proposition A.2 can be con-
 1141 sidered as the characterization of a generalized projection operator $VI(\mathbf{M}, \mathbf{q}, \varphi)$. The
 1142 next proposition is used several times in the article:

1143 PROPOSITION B.1. *Suppose that the $m \times m$ matrix $D \succ 0$, and $\mathcal{M} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$
 1144 is a maximal monotone operator. Then the operator $(D + \mathcal{M})^{-1}(\cdot)$ is well-defined
 1145 (i.e., for each $y \in \mathbb{R}^m$ there is an $x \in \mathbb{R}^m$ such that $y \in Dx + \mathcal{M}(x)$), single-valued
 1146 and Lipschitz continuous with modulus $L = \frac{2}{\lambda_{\min}(D + D^\top)}$.*

1147 It was proposed in [16, Proposition 1] when $\mathcal{M} = \partial\phi$ for some convex proper lsc $\phi(\cdot)$,
 1148 extended to maximal monotone $\mathcal{M}(\cdot)$ in [5].

1149 **Appendix C. Passive Systems.** A quadruple (A, B, C, D) is said passive if:

1150 (C.1)
$$\begin{pmatrix} -XA - A^\top X & -XB + C^\top \\ -B^\top X + C & D + D^\top \end{pmatrix} \succcurlyeq 0, \quad X = X^\top \succ 0.$$

1151 for some X . It is said *strictly* passive if the first matrix inequality is satisfied with $\succ 0$
 1152 (this implies that $D \succ 0$). This is not to be confused with the *strict state* passivity [18,
 1153 Definition 4.54] (strict passivity implies strict state passivity). In fact strict passivity
 1154 is directly related to strongly SPR transfer matrixess [18, p.62 and section 3.12.2],
 1155 while SPR transfer matrices with minimal realization are strictly state passive [18,
 1156 Theorem 4.73] [43]. Strict state passivity holds when $-XA - A^\top X \succ 0$ in (C.1), with
 1157 $D + D^\top \succcurlyeq 0$. The two inequalities in (C.1) make a linear matrix inequality (LMI) with
 1158 unknown X , equivalently a Riccati inequality using the Schur complement Theorem
 1159 [18, Theorem A.65]. Numerical algorithms exist that efficiently solve LMIs, see [13],
 1160 and can be used to test whether or not a quadruple is passive.

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