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1 ANALYSIS OF THE IMPLICIT EULER TIME-DISCRETIZATION OF  
2 SEMI-EXPLICIT DIFFERENTIAL-ALGEBRAIC LINEAR  
3 COMPLEMENTARITY SYSTEMS\*

4 BERNARD BROGLIATO<sup>†</sup>

5 **Abstract.** This article is largely concerned with the time-discretization of differential-algebraic  
6 equations (DAE) with complementarity constraints, which we name differential algebraic linear com-  
7 plementarity systems (DALCS). Specifically, the Euler implicit discretization of DALCS is analysed:  
8 the one-step non-smooth problem (OSNSP), that is a generalized equation, is shown to be well-  
9 posed under some conditions, then the convergence of the discretized solutions is studied, and the  
10 existence of solutions to the continuous-time system is shown as a consequence. Passivity of some  
11 operators is pivotal to the analysis. **Examples from circuits, mechanics and switching DAE illustrate**  
12 **the applicability of the developments.**

13 **Key words.** differential-algebraic system, descriptor-variable system, linear complementarity  
14 system, recession function, discrete-time, well-posedness, passive system, Lur'e equations.

15 **AMS subject classifications.** 65K15, 90C33, 90C90, 93B52, 93D30, 93C30.

16 **1. Introduction.** The analysis of non-smooth dynamical systems with multi-  
17 valued right-hand sides satisfying maximal monotone properties has witnessed a large  
18 number of contributions [19]. Within this class one finds linear and nonlinear com-  
19 plementarity dynamical systems, projected systems, differential variational inequali-  
20 ties, differential inclusions with maximal monotone right-hand sides, Moreau's sweep-  
21 ing process, some switching dynamics, *etc.* In this article we focus on singular, or  
22 differential-algebraic linear complementarity systems (DALCS), which may be viewed  
23 either as an extension of classical differential-algebraic equations (DAEs), or of "clas-  
24 sical" linear complementarity systems. **As will become clear later, DALCS represent**  
25 **an extension of the set-valued Lur'e systems studied, e.g., in [39, 40]. The associated**  
26 **canonical form is motivated by applications in chemistry [53, 52, 62], switching DAEs**  
27 **analysis [48], circuits with nonsmooth electronic components [2, section 3.5, chapter 4]**  
28 **[22], optimization-constrained differential equations [19, Example 2] with applications**  
29 **in atmospheric aerosol particles [38], and Lagrangian systems with both bilateral and**  
30 **unilateral constraints [17] (in spite of the fact that the results presented in this article**  
31 **do not apply to all of these systems).**

32 **The study of singular nonsmooth dynamical systems has not received a lot of attention**  
33 **yet. Singularly perturbed differential inclusions have been tackled in [26, 56, 32, 58].**  
34 **The conditions on the set-valued mappings that are imposed (compactness [56, 58],**  
35 **boundedness and Hausdorff Lipschitz continuity [32], one-sided Lipschitz condition**  
36 **[26]) are not satisfied by complementarity conditions (or by normal cones to convex**  
37 **sets). Hence these results do not apply to DALCS. The well-posedness of DAEs**  
38 **in semi-explicit form with nonsmooth constraints (which are related to the class of**  
39 **systems studied in this article, see Remark 2.1 below) has been analysed in [53, 54, 52].**  
40 **Passive singular set-valued systems are studied in [22], who derived well-posedness**  
41 **relying on a special Weierstrass' representation. However the time-discretisation is**

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42 not tackled in the above references.

43 Our goal is to analyse the well-posedness of the one-step nonsmooth problem (OS-  
 44 NSP) obtained after an implicit (backward) Euler discretization, and the convergence  
 45 of the discrete-time solutions towards a continuous-time limit. The second step is use-  
 46 ful not only for proving the existence of solutions to the continuous-time system, but  
 47 also, perhaps most importantly, to show that the proposed time-discretization makes  
 48 sense for numerical simulations. Passivity properties of suitable operators are pivotal  
 49 throughout the article. It is noteworthy that implicit Euler methods are well-known  
 50 in DAEs [30, 14, 33, 42, 12, 35, 61, 7], differential variational inequalities [46, 24],  
 51 and complementarity systems fields [41, 21, 59, 34], as well as in the field of max-  
 52 imal monotone differential inclusions [8, 47] (implicit Euler method is often called  
 53 proximal algorithm, while Euler method refers to the explicit algorithm [47]), and  
 54 Moreau's sweeping processes with the catching-up algorithm [19, 44, 45]. However  
 55 their application to DALCS has not been studied yet.

56 The article is organised as follows: Section 2 introduces the general dynamics of  
 57 the nonsmooth singular systems, section 3 presents their time-discretization, section  
 58 4 studies the well-posedness of the OSNSP, section 5 is dedicated to convergence  
 59 analyses: results in section 5.1 are based on the properties of the numerical scheme,  
 60 while section 5.2 uses properties of the system. Section 6 is dedicated to illustrating  
 61 examples and simulations, from circuits, mechanics and switching DAEs. Conclusions  
 62 are drawn in section 7. The appendix is dedicated to recall various mathematical tools.

63 **Notation and definitions:** for any vector  $x \in \mathbb{R}^n$  and any matrix  $M \in \mathbb{R}^{m \times n}$ ,  
 64  $\|x\|$  is the Euclidean norm and  $\|M\|$  is the Frobenius norm, which are compatible  
 65 norms [10, Proposition 9.3.5], *i.e.*,  $\|Mx\| \leq \|M\| \|x\|$ . Let  $M \in \mathbb{R}^{n \times m}$ , then  $\text{Im}(M)$   
 66 is its range,  $\text{Ker}(M)$  is its null space. We use  $\langle x, y \rangle = x^\top y$ , so  $\langle x, x \rangle = \|x\|^2$ . Positive  
 67 definite matrix:  $M \succ 0$  if  $x^\top Mx > 0$  for all  $x \neq 0$ , positive semidefinite matrix:  
 68  $M \succcurlyeq 0$  if  $x^\top Mx \geq 0$  for all  $x$  (such  $M$  is not necessarily symmetric). The maximum  
 69 singular value is denoted as  $\sigma_{\max}(M)$ , and the minimum eigenvalue as  $\lambda_{\min}(M)$ . The  
 70  $i$ th row of  $M$  is denoted as  $M_{i\bullet}$ . The  $n \times n$  identity matrix is denoted  $I_n$ . A set-valued  
 71 mapping  $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is said monotone if for all  $x_1, x_2, y_1 \in A(x_1), y_2 \in A(x_2)$ , one  
 72 has  $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$ . It is maximal monotone if its graph cannot be enlarged  
 73 without destroying monotonicity. It is  $\eta$ -strongly monotone if there exists  $\eta > 0$  such  
 74 that  $\langle x_1 - x_2, y_1 - y_2 \rangle \geq \eta \|x_1 - x_2\|^2$ . See also Appendix A for further definitions.

75 **2. The class of nonsmooth singular dynamical systems.** Let us consider  
 76 the following differential algebraic linear complementarity system (DALCS):

$$77 \quad (2.1) \quad \begin{cases} (a) & P_I \dot{x}(t) = A_I x(t) + B_I z(t) + \bar{B}_I \lambda(t) + E_I(t) \\ (b) & 0 = A_{II} x(t) + B_{II} z(t) + \bar{B}_{II} \lambda(t) + E_{II}(t) \\ (c) & K^* \ni \lambda(t) \perp w(t) = C_I x(t) + C_{II} z(t) + D \lambda(t) + F(t) \in K, \end{cases}$$

78 with  $x(t) \in \mathbb{R}^{n_1}$ ,  $z(t) \in \mathbb{R}^{n_2}$ ,  $\lambda(t) \in \mathbb{R}^m$ ,  $w(t) \in \mathbb{R}^m$ ,  $P_I \in \mathbb{R}^{n_1 \times n_1}$  is full-rank,  
 79  $K \subseteq \mathbb{R}^m$  is a closed nonempty convex cone,  $K^*$  is its dual cone,  $\bar{B}_{II} \in \mathbb{R}^{p \times m}$ ,  
 80  $A_{II} \in \mathbb{R}^{p \times n_1}$ ,  $B_{II} \in \mathbb{R}^{p \times n_2}$ ,  $C_I \in \mathbb{R}^{m \times n_1}$ ,  $C_{II} \in \mathbb{R}^{m \times n_2}$ ,  $D \in \mathbb{R}^{m \times m}$ ,  $B_I \in$   
 81  $\mathbb{R}^{n_1 \times n_2}$ ,  $A_I \in \mathbb{R}^{n_1 \times n_1}$ ,  $\bar{B}_I \in \mathbb{R}^{n_1 \times m}$ , and  $E_I(\cdot)$ ,  $E_{II}(\cdot)$ ,  $F(\cdot)$  are bounded, Lipschitz  
 82 continuous functions of time. The form in (2.1) is that of a semi-explicit DALCS. It  
 83 corresponds to a kind of extension of the impulse-free Weierstrass form [20], where  
 84 the transformed state matrix has non zero off-diagonal terms  $A_{II}$  and  $B_I$ .

85 *Remark 2.1.* Assume that  $K = \mathbb{R}_+^m$ . The complementarity constraint (2.1) (c)  
 86 is equivalently rewritten as  $\phi(\lambda, w) = 0$  where  $\phi(\cdot)$  is a  $C$ -function [3, 29]. Therefore

87 (2.1) (b) (c) can be rewritten equivalently as  $\Phi(x, z, \lambda, t) = 0$  for some function  $\Phi(\cdot)$ ,  
 88 usually nondifferentiable. The system (2.1) thus possesses the form of a semi-explicit  
 89 DAE as studied in [53, 52], assuming piecewise differentiable functions  $\Phi(\cdot)$ , with  $x$   
 90 the differential state,  $z$  and  $\lambda$  the algebraic state variables. The so-called regularity  
 91 conditions (which are an extension of index 1 conditions for DAEs) used in [53, 54,  
 92 52, 51] may hold in particular cases for (2.1). Examples in section 6.3 illustrate the  
 93 links between both approaches in [53, 54, 52, 51] and in this article.

94 Consider (2.1), then the initial data  $x(0) = x_0$  and  $z(0) = z_0$  are compatible with the  
 95 system's constraints if and only if:

$$96 \quad (2.2) \quad \begin{aligned} 0 &\in A_{II}x_0 + B_{II}z_0 + E_{II}(0) - \bar{B}_{II}(D + \mathcal{N}_{K^*})^{-1}(C_Ix_0 + C_{II}z_0 + F(0)) \\ \lambda(0) &\in -(D + \mathcal{N}_{K^*})^{-1}(C_Ix_0 + C_{II}z_0 + F(0)). \end{aligned}$$

97 This implies that  $C_Ix_0 + C_{II}z_0 + F(0) \in \text{Im}(D + \mathcal{N}_{K^*})$  and  $A_{II}x_0 + B_{II}z_0 + E_{II}(0) \in$   
 98  $\bar{B}_{II}\text{Dom}(D + \mathcal{N}_{K^*})$ .

99 **3. Time-discretization of DALCS with implicit Euler methods.** Let us  
 100 consider the time interval  $[0, T]$ ,  $T > 0$ ,  $h = \frac{T}{N}$ ,  $N \in \mathbb{N}_+$ ,  $t_k = kh$ ,  $k \in \{0, 1, \dots, N\}$ ,  
 101  $t_0 = 0$ ,  $t_N = T$ . For a continuous function  $f(\cdot)$ ,  $f_k \triangleq f(t_k)$ . The implicit Euler  
 102 discretization of (2.1) reads as:

$$103 \quad (3.1) \quad \begin{cases} (a) & P_I(x_{k+1} - x_k) = hA_Ix_{k+1} + hB_Iz_{k+1} + h\bar{B}_I\lambda_{k+1} + hE_{I,k} \\ (b) & A_{II}x_{k+1} + B_{II}z_{k+1} + \bar{B}_{II}\lambda_{k+1} + E_{II,k} = 0 \\ (c) & K^* \ni \lambda_{k+1} \perp C_Ix_{k+1} + C_{II}z_{k+1} + D\lambda_{k+1} + F_k \in K. \end{cases}$$

104 The problem in (3.1) (b) (c) is a mixed LCP (MLCP). Let us now state a basic  
 105 assumption.

106 ASSUMPTION 1. The matrix  $P_I^h \triangleq P_I - hA_I$  is full-rank.

107 In case  $P_I \succ 0$ , [25, Theorem 2.11] can be used to set conditions on  $hA_I$  so that  
 108  $P_I^h \succ 0$ : there always exists  $h_{\max}$  such that the assumption holds for all  $h \in [0, h_{\max})$   
 109 (with  $h_{\max} = +\infty$  if  $A_I \preccurlyeq 0$ ). Let us define the following matrices and vectors:

- 110 1.  $\tilde{A} = hA_{II}(P_I^h)^{-1}B_I + B_{II}$ ,
- 111 2.  $\tilde{B} = hA_{II}(P_I^h)^{-1}\bar{B}_I + \bar{B}_{II}$ ,
- 112 3.  $\tilde{E}_k = E_{II,k} + hA_{II}(P_I^h)^{-1}E_{I,k} + A_{II}(P_I^h)^{-1}P_Ix_k$ ,
- 113 4.  $\tilde{C} = hC_I(P_I^h)^{-1}B_I + C_{II}$ ,
- 114 5.  $\tilde{D} = hC_I(P_I^h)^{-1}\bar{B}_I + D$ ,
- 115 6.  $\tilde{F}_k = F_k + hC_I(P_I^h)^{-1}E_{I,k} + C_I(P_I^h)^{-1}P_Ix_k$ .

116 Then we deduce from (3.1) the difference equations:

$$117 \quad (3.2) \quad \begin{cases} (a) & x_{k+1} = (P_I^h)^{-1}(P_Ix_k + hB_Iz_{k+1} + h\bar{B}_I\lambda_{k+1} + hE_{I,k}) \\ (b) & \tilde{A}z_{k+1} + \tilde{B}\lambda_{k+1} + \tilde{E}_k = 0 \\ (c) & K^* \ni \lambda_{k+1} \perp \tilde{C}z_{k+1} + \tilde{D}\lambda_{k+1} + \tilde{F}_k \in K, \end{cases}$$

118 where the last two lines of (3.2) are a MLCP with unknowns  $z_{k+1}$  and  $\lambda_{k+1}$ . It is  
 119 therefore possible to study the well-posedness of this MLCP independently of the  
 120 first line in (3.2). Letting  $\zeta_{k+1} \triangleq -z_{k+1}$ , this MLCP can be rewritten equivalently as  
 121 follows:

$$122 \quad (3.3) \quad \begin{cases} \lambda_{k+1} \in (\mathcal{N}_{K^*} + \tilde{D})^{-1}(-\tilde{C}\zeta_{k+1} - \tilde{F}_k), \\ -\tilde{A}\zeta_{k+1} + \tilde{E}_k \in -\tilde{B}(\mathcal{N}_{K^*} + \tilde{D})^{-1}(\tilde{C}\zeta_{k+1} - \tilde{F}_k), \\ x_{k+1} = (P_I^h)^{-1}(P_Ix_k + hB_Iz_{k+1} + h\bar{B}_I\lambda_{k+1} + hE_{I,k}). \end{cases}$$

123 Also the following calculations stem from the complementarity conditions in (3.2):  
 (3.4)

$$\begin{aligned}
 & \lambda_{k+1} \in -\mathcal{N}_K(\tilde{C}z_{k+1} + \tilde{D}\lambda_{k+1} + \tilde{F}_k) \Leftrightarrow -\lambda_{k+1} \in \mathcal{N}_{\Gamma_k}(\tilde{C}z_{k+1} + \tilde{D}\lambda_{k+1}) \\
 124 \quad & \Leftrightarrow \partial\sigma_{\Gamma_k}(-\lambda_{k+1}) \ni \tilde{C}z_{k+1} + \tilde{D}\lambda_{k+1} \quad \Leftrightarrow (\partial\sigma_{\Gamma_k} + \tilde{D})(-\lambda_{k+1}) \ni \tilde{C}z_{k+1} \\
 & \Leftrightarrow \lambda_{k+1} \in -(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}z_{k+1}).
 \end{aligned}$$

125 where the inversion of set-valued mappings is used in the second equivalence, and  
 126 where  $\sigma_{\Gamma_k}(\cdot)$  is the support function of the set  $\Gamma_k(t_k, x_k) \triangleq K - \tilde{F}_k(t_k, x_k)$  (which  
 127 is a closed convex set, but not a cone as long as  $\tilde{F}_k \neq 0$  which is the generic case).  
 128 Therefore equivalently we can rewrite (3.3) as:

$$129 \quad (3.5) \quad \begin{cases} (a) \lambda_{k+1} \in -(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}z_{k+1}) \\ (b) -\tilde{A}z_{k+1} + \tilde{B}(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}z_{k+1}) \ni \tilde{E}_k \\ (c) x_{k+1} = (P_I^h)^{-1}(P_I x_k + hB_I z_{k+1} + h\tilde{B}_I \lambda_{k+1} + hE_{I,k}). \end{cases}$$

130 The second line in (3.3) is a generalized equation (GE) with unknown  $\zeta_{k+1}$ . Its well-  
 131 posedness depends on the properties of the operator  $\mathcal{F} : \zeta \mapsto -\tilde{A}\zeta + \tilde{B}(\mathcal{N}_{K^*} +$   
 132  $\tilde{D})^{-1}(\tilde{C}\zeta - \tilde{F}_k)$ . Similarly for (3.5) with the operator  $\mathcal{G} : \zeta \mapsto -\tilde{A}\zeta + \tilde{B}(\partial\sigma_{\Gamma_k} +$   
 133  $\tilde{D})^{-1}(\tilde{C}\zeta)$ , and (3.5) (b) is:  $\mathcal{G}(z_{k+1}) \ni \tilde{E}_k$ . Various results have been published to  
 134 characterize such operators [16, 23, 6] and to guarantee that they are single-valued  
 135 Lipschitz continuous [16, Propositions 1, 2, 3, Corollary 1], or maximal monotone  
 136 (possibly set-valued) [23, 6]. The difficulty in the analysis of DALCS is that they  
 137 involve the composition of two operators as seen in the OSNSP for advancing the  
 138 scheme from step  $k$  to step  $k+1$ :

$$139 \quad (3.6) \quad P_I^h x_{k+1} \in P_I x_k + hB_I \mathcal{G}^{-1}(\tilde{E}_k(x_k)) - h\tilde{B}_I(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}\mathcal{G}^{-1}(\tilde{E}_k(x_k))) + hE_{I,k}.$$

140 Clearly the scheme can be advanced with a unique solution if both  $\lambda_{k+1}$  and  $z_{k+1}$   
 141 are uniquely defined (the ‘‘if and only if’’ holds with  $\tilde{B}_I \lambda_{k+1}$  and  $B_I z_{k+1}$  uniquely  
 142 defined). Thus next we are going to characterize the GEs in (3.3) and (3.5) in order  
 143 to analyse the scheme (3.6).

144 **Two main classes of results are provided in the sequel of the article: first the well-**  
 145 **posedness (existence and uniqueness of solutions) for the OSNSP associated with the**  
 146 **time-discretizations in (3.1) (Propositions 4.3 and 4.4, Corollary 4.6) and (3.2) (Propo-**  
 147 **sitions 4.7 and 4.8); second, the convergence of the discrete-time solutions is analysed,**  
 148 **and the limit functions are shown to be solutions of the DALCS (Propositions 5.2 and**  
 149 **5.9).**

150 **4. OSNSP well-posedness.** The approach chosen in this article consists of  
 151 considering the system as an ODE (2.1) (a) with a nonsmooth constraint (2.1) (b)  
 152 (c): either (3.1) (b) (c) that is a mixed LCP with unknowns  $z_{k+1}$ ,  $\lambda_{k+1}$  and parameter  
 153  $x_{k+1}$  (section 4.2), or (3.2) (b) (c) (equivalently (3.5) (a) (b)) that is a mixed LCP  
 154 with  $z_{k+1}$ ,  $\lambda_{k+1}$  (section 4.1). As will become clear later, the results rely heavily on  
 155 the positive (semi) definiteness of  $B_{II}$ , and on specific passivity couplings between  
 156  $z$  in the complementarity constraint (matrix  $C_{II}$ ) and  $\lambda$  in the algebraic constraint  
 157 (matrix  $\tilde{B}_{II}$ ). When  $\tilde{D} = 0$  in (3.2) (c), let us state the following:

158 **ASSUMPTION 2.** *There exists a  $p \times p$  matrix  $\tilde{X} = \tilde{X}^\top \succ 0$  such that  $\tilde{X}\tilde{B} = \tilde{C}^\top$ .*

159 The alternative option when  $D = 0$  in (3.1) (c):

160 **ASSUMPTION 3.** *There exists a  $p \times p$  matrix  $X = X^\top \succ 0$  such that  $X\tilde{B}_{II} = C_{II}^\top$ .*

161

162 Let us notice that Assumption 2 implies that  $\tilde{X}\tilde{B}_{II} = C_{II}^\top + \mathcal{O}(h)$ , while Assumption  
 163 3 implies that  $X\tilde{B} = \tilde{C}^\top + \mathcal{O}(h)$ . The influence of the  $\mathcal{O}(h)$  terms may not be easy  
 164 to characterize precisely for finite  $h > 0$ . Assumption 3 is a model assumption, while  
 165 Assumption 2 is a property of the numerical method. One sees that Assumption 3  
 166 corresponds to a passivity constraint on the triple  $(B_{II}, \tilde{B}_{II}, C_{II})$  [18]. Assumption  
 167 2 imposes an input/output passivity-like constraint on the triple  $(A, B, \tilde{C})$ , which is  
 168 an  $\mathcal{O}(h)$  approximation of  $(B_{II}, \tilde{B}_{II}, C_{II})$ . When  $\tilde{D} \neq 0$  and  $D \neq 0$ , extensions of  
 169 Assumptions 2 and 3 are as follows:

170 ASSUMPTION 4. *The quadruple  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  is passive.*

171 and the counterpart of Assumption 3:

172 ASSUMPTION 5. *The quadruple  $(B_{II}, \tilde{B}_{II}, C_{II}, D)$  is passive.*

173 Passivity is defined in Appendix C. Notice that the above assumptions imply that  
 174  $n_2 = p$ . Assumption 4 refers to the numerical method while Assumption 5 is a model  
 175 property. Assumptions 2 and 4 refer to (3.2) (b) (c) or (3.5) (a) (b), while Assumptions  
 176 3 and 5 refer to (3.1) (b) (c). Let us present a result about the relationships between  
 177 Assumptions 4 and 5, *i.e.*, when does the plant's passivity guarantees the numerical  
 178 scheme's passivity.

179 LEMMA 4.1. *The following statements are true:*

- 180 1. *Let  $(B_{II}, \tilde{B}_{II}, C_{II}, D)$  be strictly passive. Then  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  is strictly passive*  
 181 *for small enough  $h > 0$ .*
- 182 2. *Let  $(B_{II}, \tilde{B}_{II}, C_{II}, D)$  be strictly state passive. Then  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  is strictly*  
 183 *state passive if: a)  $h > 0$  is small enough, b)  $D + D^\top = 0$  and  $C_I(P_I^h)^{-1}\tilde{B}_I +$   
 184  $(C_I(P_I^h)^{-1}\tilde{B}_I)^\top \succcurlyeq 0$ , c)  $XA_{II}(P_I^h)^{-1}\tilde{B}_I - B_I^\top(P_I^h)^{-\top}C_I^\top = 0$ , where  $X$  is a*  
 185 *solution of the passivity LMI for  $(B_{II}, \tilde{B}_{II}, C_{II}, D)$ .*
- 186 3. *Let  $h > 0$  and  $(B_{II}, \tilde{B}_{II}, C_{II}, D)$  be passive. Then  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  is passive if:*  
 187 *b) and c) in item 2 hold, and a)  $XA_{II}(P_I^h)^{-1}\tilde{B}_I + B_I^\top(P_I^h)^{-\top}A_{II}^\top X \preccurlyeq 0$ ,*  
 188 *where  $X$  is a solution of the passivity LMI for  $(B_{II}, \tilde{B}_{II}, C_{II}, D)$ .*

189 **Proof:**

- 190 1. The strict passivity of  $(B_{II}, \tilde{B}_{II}, C_{II}, D)$  means that there exists  $X = X^\top \succ 0$   
 191 such that:

$$192 \quad (4.1) \quad M \triangleq \begin{pmatrix} -XB_{II} - B_{II}^\top X & -X\tilde{B}_{II} + C_{II}^\top \\ -\tilde{B}_{II}^\top X + C_{II} & D + D^\top \end{pmatrix} \succ 0.$$

193 In particular this implies that  $D + D^\top \succ 0$  [18, Theorem A.65] and the  
 194 associated transfer function is strongly SPR [18, Remark 3.19, Section 3.12.2].  
 195 Now we have  $\tilde{A} = B_{II} + \mathcal{O}(h)$ ,  $\tilde{B} = \tilde{B}_{II} + \mathcal{O}(h)$ ,  $\tilde{C} = C_{II} + \mathcal{O}(h)$ ,  $\tilde{D} =$   
 196  $D + \mathcal{O}(h)$ . Therefore

$$197 \quad (4.2) \quad M = \underbrace{\begin{pmatrix} -X\tilde{A} - \tilde{A}^\top X & -X\tilde{B} + \tilde{C}^\top \\ -\tilde{B}^\top X + \tilde{C}^\top & \tilde{D} + \tilde{D}^\top \end{pmatrix}}_{\triangleq \tilde{M}} + N(h)$$

198 where  $N(h)$  stands for a matrix with suitable dimensions and which vanishes  
 199 as  $h \rightarrow 0$  (in fact  $N(h)$  is  $\mathcal{O}(h)$ ). Thus  $\tilde{M} = M - N(h)$ . Applying [25,  
 200 Theorem 2.11], it follows that  $\tilde{M} \succ 0$  provided that  $\sigma_{\max}(N(h)) < \lambda_{\min}^2(\tilde{M})$ ,  
 201 an inequality which can always be verified if  $h > 0$  is small enough. Under  
 202 this condition, it follows that the strict passivity of  $(B_{II}, \tilde{B}_{II}, C_{II}, D)$  implies  
 203 the strict passivity of  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  ( $\Rightarrow \tilde{D} \succ 0$ ).

- 204 2. The result is proved by developing  $\tilde{M}$  in (4.2), then using [10, Proposition  
 205 8.2.4, Fact 8.21.22], or [18, Lemma A.69], as well as the fact that  $-XB_{II} -$   
 206  $B_{II}^\top X \succ 0$  from the strict state passivity.  
 207 3. The proof is similar to that of item 2.

208  $\square$

209 The converse implication in item 1 does not hold, since one may have  $\tilde{M} \succ 0$  for some  
 210  $h > 0$ , but  $M \not\succeq 0$ . It is possible to state necessary and sufficient conditions, relying  
 211 on [10, Proposition 8.2.4, Fact 8.21.22] or [18, Lemma A.70]. However the obtained  
 212 conditions are cumbersome and hardly usable. Lemma 4.1 can be used to determine  
 213 the conditions under which Assumption 4 is verified in the results presented in the next  
 214 section. The conditions b) and c) in Lemma 4.1 item 2 hold if: b)  $C_I = 0$  or  $\bar{B}_I = 0$ , c)  
 215 ( $A_{II} = 0$  or  $\bar{B}_I = 0$ ) and ( $B_I = 0$  or  $C_I = 0$ ). Condition a) in Lemma 4.1 item 3 holds  
 216 if  $A_{II} = 0$  or  $B_I = 0$ . Let us remind that the solution  $X = X^\top \succ 0$  of the Lyapunov  
 217 equation  $XB_{II} + B_{II}^\top X = -Q$ ,  $Q = Q^\top \succ 0$ , can be written  $X = (-\frac{1}{2}Q + F)B_{II}^{-1}$ ,  
 218 where  $F = -F^\top$  is unique [60, Theorems 1 and 2]. Let us denote the symmetric  
 219 part of a square matrix  $M$  as  $\text{sym}(M)$  and its skew-symmetric part as  $\text{skw}(M)$ . This  
 220 allows us to state the following necessary conditions.

221 **COROLLARY 4.2.** 1) Let  $(B_{II}, \bar{B}_{II}, C_{II}, D)$  be strictly state passive with  $XB_{II} +$   
 222  $B_{II}^\top X = -Q$ ,  $Q = Q^\top \succ 0$ , and assume that  $A_{II}(P_I^h)^{-1}\bar{B}_I$  is right-invertible ( $\Rightarrow p \leq$   
 223  $m$ ). Then  $X = (-\frac{1}{2}Q + F)B_{II}^{-1}$ , for some  $F = -F^\top$  uniquely defined, and condition  
 224 c) item 2 in Lemma 4.1 holds only if

$$225 \quad (4.3) \quad \text{sym}\left(\frac{1}{2}QB_{II}^{-1}A_{II}(P_I^h)^{-1}\bar{B}_I + B_I^\top(P_I^h)^{-\top}C_I^\top\right) = F\text{skw}(B_{II}^{-1}A_{II}(P_I^h)^{-1}\bar{B}_I)F,$$

226 for any pair  $(Q, F)$ , and

$$227 \quad (4.4) \quad \text{Ker}(C_I(P_I^h)^{-1}B_I) = \{0\}.$$

228 2) Let  $(B_{II}, \bar{B}_{II}, C_{II}, D)$  be passive with  $XB_{II} + B_{II}^\top X = -Q$ ,  $Q = Q^\top \succcurlyeq 0$  and  $B_{II}$   
 229 invertible. Then  $X = (-\frac{1}{2}Q + F)B_{II}^{-1}$ , for some  $F = -F^\top$ , and condition a) item 3  
 230 in Lemma 4.1 holds only if

$$231 \quad (4.5) \quad \text{sym}(-QB_{II}^{-1}A_{II}(P_I^h)^{-1}B_I) + 2F\text{skw}(B_{II}^{-1}A_{II}(P_I^h)^{-1}B_I)F \preccurlyeq 0,$$

232 for any pair  $(Q, F)$ .

233 **Proof:** 1) Condition c) yields  $(-\frac{1}{2}Q + F)B_{II}^{-1}A_{II}(P_I^h)^{-1}\bar{B}_I = B_I^\top(P_I^h)^{-\top}C_I^\top$ , then  
 234 (4.3) follows using  $F = -F^\top$ . Using the right-invertibility, (4.4) follows from [23,  
 235 equ. (70)] and the fact that  $X$  is full rank. 2) From passivity any solution must be  
 236 symmetric. The Lyapunov equation and the skew-symmetry of  $F$  yield (4.5).  $\square$

237 We see that the right-invertibility condition (resp. condition (4.4)) in item 1 of the  
 238 lemma, characterizes the couplings between  $x$  and  $\lambda$  in the index reduction (resp.  
 239 between  $x$  and  $z$  in the relative degree reduction (*i.e.*, in  $\dot{w}$ )).

240 **4.1. Analysis of the OSNSP for (3.2).** The next proposition analyses the  
 241 OSNSP well-posedness, and applies to two main cases:  $\tilde{D} = 0$ , and  $\tilde{D} \succ 0$ . Let us  
 242 remind that  $p = n_2$  as a consequence of Assumptions 2, 3, 4 and 5, so that  $\tilde{A}$  and  $B_{II}$   
 243 are  $p \times p$  matrices.

244 **PROPOSITION 4.3.** Let  $h > 0$ ,  $k \geq 0$  be given.

- 245 1. Let Assumption 4 hold. Suppose that either  $\text{Im}(\tilde{C}) \cap \text{int}(\text{Im}(\partial\sigma_{\Gamma_k} + \tilde{D})) \neq \emptyset$   
 246 or  $\tilde{B}$  is square full rank. Suppose also that there is  $\zeta_0$  such that  $\tilde{C}\zeta_0 \in$   
 247  $\text{Im}(\partial\sigma_{\Gamma_k} + \tilde{D})$ . Then the operator  $\mathcal{G} : \zeta \mapsto -\tilde{A}\zeta + \tilde{B}(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}\zeta)$  is  
 248 maximal monotone with non empty domain.  
 249 2. Let  $\tilde{D} \succ 0$ , then both mappings  $(\mathcal{N}_{K^*} + \tilde{D})^{-1}(\cdot)$  and  $(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\cdot)$  are  
 250 well-defined, single-valued Lipschitz continuous and maximal monotone for  
 251 each  $h \geq 0, k \geq 0$  (maximal monotonicity holds for  $\tilde{D} \succ 0$ ). The operator  
 252  $\mathcal{G}(\cdot)$  is single-valued and Lipschitz continuous also.  
 253 3. Let  $\tilde{D} = 0$ , Assumption 2 hold true,  $\tilde{R}^2 = \tilde{X}$ ,  $\tilde{R} = \tilde{R}^\top \succ 0$ ,  $\text{Im}(\tilde{C}) \cap \Gamma_k \neq \emptyset$   
 254 and  $K = P(G, 0)$  (see Proposition A.1 e) for some matrix  $G \in \mathbb{R}^{m \times m}$ . Let  
 255 us assume that  $\tilde{R}\tilde{A}\tilde{R}^{-1} \preceq 0$ . Let us consider the set:

$$(4.6) \quad \begin{aligned} \tilde{\mathcal{S}} &\triangleq P(G\tilde{C}\tilde{R}^{-1}, 0) \\ &\cap \{\xi \in \mathbb{R}^p \mid \tilde{R}\tilde{A}\tilde{R}^{-1}\xi = -\sum_{i=1}^m \lambda_i (G\tilde{C}\tilde{R}^{-1})_{i\bullet}^\top, \lambda_i \geq 0\} \\ &\cap \text{Ker}(\tilde{R}\tilde{A}\tilde{R}^{-1} + \tilde{R}^{-1}\tilde{A}^\top\tilde{R}). \end{aligned}$$

257 Define  $\tilde{q}_k = -\tilde{R}\tilde{E}_k$ ,  $\tilde{f}(\cdot) = \psi_{\Gamma_k} \circ \tilde{C}\tilde{R}^{-1}(\cdot)$ ,  $\Gamma_k = K - \tilde{F}_k$ , and consider the  
 258 VI $(-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, \tilde{f})$ : Find  $\xi_{k+1} = \tilde{R}z_{k+1}$  such that

$$(4.7) \quad \langle -\tilde{R}\tilde{A}\tilde{R}^{-1}\xi_{k+1} + \tilde{q}_k, v - \xi_{k+1} \rangle + \tilde{f}(v) - \tilde{f}(\xi_{k+1}) \geq 0, \text{ for all } v \in \text{dom}(\tilde{f}).$$

- 260 (a) If  $\tilde{\mathcal{S}} = \{0\}$ , then the VI $(-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, \tilde{f})$  has at least one solution.  
 261 (b) If  $\tilde{\mathcal{S}} \neq \{0\}$ , and if there exists  $\xi_0 \in \tilde{\Gamma}_k$  such that

$$(4.8) \quad \langle \tilde{q}_k - (-\tilde{R}\tilde{A}\tilde{R}^{-1})^\top \xi_0, v \rangle > 0 \text{ for all } v \in \tilde{\mathcal{S}}, v \neq 0,$$

262 where  $\tilde{\Gamma}_k = \{\xi \in \mathbb{R}^{m_2} \mid G\tilde{C}\tilde{R}^{-1}\xi + G\tilde{F}_k \geq 0\} = \text{dom}(\tilde{f})$ , then the  
 263 VI $(-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, \tilde{f})$  has at least one solution.

- 264 (c) If  $\xi_{k+1}^1$  and  $\xi_{k+1}^2$  are two solutions of the VI $(-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, \tilde{f})$ , then  $\xi_{k+1}^1 -$   
 265  $\xi_{k+1}^2 \in \text{Ker}(\tilde{R}\tilde{A}\tilde{R}^{-1} + \tilde{R}^{-1}\tilde{A}^\top\tilde{R})$ .  
 266 (d) Let  $\tilde{R}\tilde{A}\tilde{R}^{-1}$  be symmetric. If  $\xi_{k+1}^1$  and  $\xi_{k+1}^2$  are two solutions of the  
 267 VI $(-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, \tilde{f})$ , then  $\langle \tilde{q}_k, \xi_{k+1}^1 - \xi_{k+1}^2 \rangle = 0$ .  
 268 (e) Let  $\tilde{R}\tilde{A}\tilde{R}^{-1}$  be symmetric. Then any solution of the VI $(-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, \tilde{f})$   
 269 is also a solution to the problem:  $\min_{\xi \in \tilde{\Gamma}_k} -\frac{1}{2}\xi^\top \tilde{R}\tilde{A}\tilde{R}^{-1}\xi + \langle \tilde{q}_k, \xi \rangle$ .  
 270 (f) Let  $-\tilde{R}\tilde{A}\tilde{R}^{-1} \succ 0$  and be symmetric. Then  $\xi_{k+1} = \text{proj}_{-\tilde{R}\tilde{A}\tilde{R}^{-1}[\tilde{\Gamma}_k; -\tilde{q}_k]}$ .

272 **Proof:**

- 273 1. Follows from [6, Theorem 1], which is itself inspired from [23] and [16, The-  
 274 orem 2], and which states that the negative feedback interconnection of a  
 275 passive system with a maximal monotone operator, defines another maximal  
 276 monotone operator (under the stated basic conditions).  
 277 2. Follows from [16, Proposition 1] (see Proposition B.1) and from [55, Lemma  
 278 1] (or [9, Corollary 24.4 (ii)]) since  $\text{Im}(\tilde{D}) = \mathbb{R}^m$  and  $\text{int}(\text{Dom}(\mathcal{N}_{K^*})) =$   
 279  $\text{int}(K^*) \neq \emptyset$ , and  $\text{int}(\text{Dom}(\partial\sigma_{\Gamma_k})) \neq \emptyset$  because  $\Gamma_k$  is convex and  $\text{int}(\Gamma_k) \neq \emptyset$ .  
 280 3. The proof follows from Propositions A.1 and A.2. First notice that the MLCP  
 281  $\tilde{A}z_{k+1} + \tilde{B}\lambda_{k+1} + \tilde{E}_k = 0, K^* \ni \lambda_{k+1} \perp \tilde{C}z_{k+1} + \tilde{D}\lambda_{k+1} + \tilde{F}_k \in K$ , can  
 282 be equivalently rewritten as the VI $(-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, \tilde{f})$  in (4.7). Indeed one  
 283 obtains  $-\tilde{R}\tilde{A}\tilde{R}^{-1}\xi_{k+1} - \tilde{R}\tilde{E}_k \in -\tilde{R}\tilde{B} \partial\psi_{\Gamma_k}(\tilde{C}\tilde{R}^{-1}\xi_{k+1})$ . Using the chain rule  
 284 of Convex Analysis [49, Theorem 23.9] and the fact that  $\Gamma_k$  is polyhedral,  
 285 the MLCP can written equivalently as in (4.7). Let us now compute the  
 286 set  $\mathcal{S}$  in (4.6). Here  $\text{dom}(\varphi) = \text{dom}(\tilde{f}) = \{z \in \mathbb{R}^p \mid \tilde{C}\tilde{R}^{-1}z \in \Gamma_k\} =$



287  $\{z \in \mathbb{R}^p \mid \tilde{C}\tilde{R}^{-1}z + \tilde{F}_k \in K\} = \{z \in \mathbb{R}^p \mid G\tilde{C}\tilde{R}^{-1}z + G\tilde{F}_k \geq 0\}$ . Hence  
 288  $(\text{dom}(\varphi))_\infty = P(G\tilde{C}\tilde{R}^{-1}, 0)$  from Proposition A.1 e). Now we have  $\tilde{f}(\xi) =$   
 289  $\psi_{\tilde{\Gamma}_k}(\tilde{C}\tilde{R}^{-1}\xi) = \psi_{\tilde{\Gamma}_k}(\xi)$  with  $\tilde{\Gamma}_k = \{\xi \in \mathbb{R}^p \mid \tilde{C}\tilde{R}^{-1}\xi \in K - \tilde{F}_k\} = \{\xi \in$   
 290  $\mathbb{R}^p \mid G\tilde{C}\tilde{R}^{-1}\xi + G\tilde{F}_k \geq 0\} = P(G\tilde{C}\tilde{R}^{-1}, -G\tilde{F}_k)$ . Thus  $\tilde{f}_\infty(\cdot) = \psi_{(\tilde{\Gamma}_k)_\infty}(\cdot)$   
 291 and  $\text{dom}(\tilde{f}_\infty) = P(G\tilde{C}\tilde{R}^{-1}, 0)$ . Now we have to calculate the dual cone  
 292  $P(G\tilde{C}\tilde{R}^{-1}, 0)^*$ . Using [31, Example 15], one finds that  $P(G\tilde{C}\tilde{R}^{-1}, 0)^* = \{w \in$   
 293  $\mathbb{R}^{n_2} \mid w = \sum_{i=1}^m \lambda_i (G\tilde{C}\tilde{R}^{-1})_{i\bullet}^\top, \lambda_i \geq 0\}$ , where  $(G\tilde{C}\tilde{R}^{-1})_{i\bullet}$  denotes the  $i$ th  
 294 row of the matrix  $G\tilde{C}\tilde{R}^{-1} \in \mathbb{R}^{m \times n_2}$ . Hence it follows that  $\mathcal{K}(-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{f}) =$   
 295  $\{\xi \in \mathbb{R}^{n_2} \mid \tilde{R}\tilde{A}\tilde{R}^{-1}\xi = -\sum_{i=1}^m \lambda_i (G\tilde{C}\tilde{R}^{-1})_{i\bullet}^\top, \lambda_i \geq 0\}$ . The third set in  $\tilde{\mathcal{S}}$   
 296 is equal to  $\text{Ker}(\tilde{R}\tilde{A}\tilde{R}^{-1} + \tilde{R}^{-1}\tilde{A}^\top\tilde{R})$ , from  $\text{Ker}(\mathbf{M} + \mathbf{M}^\top)$  in Proposition A.2  
 297 a). Then the results in items (a) to (f) are a consequence of Proposition A.2,  
 298 where  $\varphi_\infty(v) = 0$  in item (b). Item (f) follows from item e) in Proposition  
 299 A.2.

300  $\square$

301 Item 1 encompasses cases with  $\tilde{D} \succ 0$ . If item 2 holds, then  $\lambda_{k+1}$  is uniquely defined,  
 302 provided that  $z_{k+1}$  exists (which is guaranteed if conditions of item 1 hold). From  
 303 item 3 (c), it follows that  $z_{k+1}$  is unique if  $\tilde{A}$  has full rank (recall that  $B_{II}$  and  $\tilde{A}$  are  
 304 square  $p \times p$  matrices). However from (3.2) and (3.3),  $x_{k+1}$  is uniquely defined if and  
 305 only if  $\tilde{B}_I(\mathcal{N}_{K^*})^{-1}(-\tilde{C}z_{k+1} - \tilde{F}_k)$  is unique. This happens trivially when  $\tilde{B}_I = 0$  (if  
 306 one thinks of (2.1) as the limit of a singularly perturbed system, this simply means  
 307 that the multiplier enters only the  $z$ -dynamics). Let us now deal with the case  $\tilde{D} \succ 0$ .

308 ASSUMPTION 6. Let  $h > 0$ , and  $\tilde{D} = \begin{pmatrix} \tilde{D}_{11} & 0 \\ 0 & 0 \end{pmatrix}$  with  $0 \prec \tilde{D}_{11} = \tilde{D}_{11}^\top \in \mathbb{R}^{m_1 \times m_1}$ .

310 Such specific structure occurs for instance when  $D$  has the same blockdiagonal struc-  
 311 ture and  $\tilde{B}_I = 0$ , i.e., the multiplier does not intervene in the  $x$ -dynamics. Let us  
 312 split the multiplier as  $\lambda = (\lambda_1^\top, \lambda_2^\top)^\top$ ,  $\lambda_1 \in \mathbb{R}^{m_1}$ ,  $\lambda_2 \in \mathbb{R}^{m_2}$ ,  $m_1 + m_2 = m$ , and  
 313  $K = K_1 \times K_2 \subset \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$  (hence  $K^* = K_1^* \times K_2^*$ ),  $\tilde{B} = \begin{pmatrix} \tilde{B}_1 & \tilde{B}_2 \end{pmatrix}$ ,  $\tilde{B}_1 \in \mathbb{R}^{p \times m_1}$ ,  
 314  $\tilde{B}_2 \in \mathbb{R}^{p \times m_2}$ ,  $\tilde{C} = \begin{pmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{pmatrix}$ ,  $\tilde{C}_1 \in \mathbb{R}^{m_1 \times p}$ ,  $\tilde{C}_2 \in \mathbb{R}^{m_2 \times p}$ . If  $\tilde{D}_{11} = 0$  then  $\lambda = \lambda_2$ ,  
 315  $\tilde{B} = \tilde{B}_2$ ,  $\tilde{C} = \tilde{C}_2$ . If  $\tilde{D} = \tilde{D}_{11} \succ 0$  then item 2 in Proposition 4.3 applies.

316 ASSUMPTION 7. Let  $h > 0$ . There exists a  $(p \times p)$ -matrix  $\tilde{X}_2 = \tilde{X}_2^\top \succ 0$  such that  
 317  $\tilde{X}_2\tilde{B}_2 = \tilde{C}_2^\top$ .

318 This assumption implies that there are couplings between the multiplier  $\lambda_{k+1}$  in the  
 319 algebraic equation, and the algebraic variable  $z_{k+1}$  in the complementarity conditions,  
 320 see (3.2). Define  $\tilde{R}_2 = \tilde{R}_2^\top \succ 0$ ,  $\tilde{R}_2^2 = \tilde{X}_2$ , and  $\xi_{k+1} = \tilde{R}_2 z_{k+1}$ . The next result extends  
 321 Proposition 4.3 to the positive semidefinite case with blockdiagonal structure.

322 PROPOSITION 4.4. Let Assumptions 6 and 7 hold true,  $h > 0$  and  $k \geq 0$  be given.  
 323 Define  $\Gamma_{2,k} = K_2 - \tilde{F}_{2,k}$ . Consider the VI: find  $\xi_{k+1} \in \text{dom}(\tilde{f}_2)$  such that  
 324 (4.9)

$$-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1}\xi_{k+1} - \tilde{R}_2\tilde{E}_k + \underbrace{\tilde{R}_2\tilde{B}_1(\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(\tilde{C}_1\tilde{R}_2^{-1}\xi_{k+1} + \tilde{F}_{1,k})}_{\triangleq \tilde{G}_2(t_k, \xi_{k+1})} \in -\partial\tilde{f}_2(\xi_{k+1})$$

325 where  $\tilde{f}_2(\cdot) = \psi_{\Gamma_{2,k}} \circ \tilde{C}_2\tilde{R}_2^{-1}(\cdot)$ .

326 1. Suppose that  $\text{Im}(\tilde{C}_2) \cap \Gamma_{2,k} \neq \emptyset$ . The MLCP  $\tilde{A}z_{k+1} + \tilde{B}\lambda_{k+1} + \tilde{E}_k = 0$ ,  
 327  $K^* \ni \lambda_{k+1} \perp \tilde{C}z_{k+1} + \tilde{D}\lambda_{k+1} + \tilde{F}_k \in K$ , can be equivalently rewritten as the

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VI in (4.9).

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2. Assume that the set  $\tilde{\Gamma}_{2,k} = \{\xi \in \mathbb{R}^p \mid \tilde{C}_2 \tilde{R}_2^{-1} \xi \in \Gamma_{2,k}\}$  is compact. Then the set of solutions to the VI in (4.9) is compact and nonempty.

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3. Assume that for each  $t_k$ , the following well-defined, single-valued and Lipschitz continuous mapping, is maximal monotone:

$$\tilde{G}_2 : \xi_{k+1} \mapsto \tilde{R}_2 \tilde{B}_1 (\tilde{D}_{11} + \partial \psi_{K_1^*})^{-1} (\tilde{C}_1 \tilde{R}_2^{-1} \xi_{k+1} - \tilde{F}_{1,k}).$$

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If  $-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \succ 0$ ,  $\tilde{\Gamma}_{2,k} \cap \text{Int}(\text{Dom}(\tilde{G}_2)) \neq \emptyset$  or  $\text{Int}(\tilde{\Gamma}_{2,k}) \cap \text{Dom}(\tilde{G}_2) \neq \emptyset$ , then the set of solutions to the VI in (4.9) is closed and convex. If  $-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \succ 0$ , the VI in (4.9) has a unique solution.

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**Proof:**

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1. The complementarity conditions in (3.2) (c) are rewritten as:

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$$(4.10) \quad \begin{pmatrix} K_1^* \\ K_2^* \end{pmatrix} \ni \begin{pmatrix} \lambda_{1,k+1} \\ \lambda_{2,k+1} \end{pmatrix} \perp \begin{pmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{pmatrix} z_{k+1} + \begin{pmatrix} \tilde{D}_{11} \lambda_{1,k+1} \\ 0 \end{pmatrix} + \begin{pmatrix} \tilde{F}_{1,k} \\ \tilde{F}_{2,k} \end{pmatrix} \in \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$$

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while the equality constraint in (3.2) (b) is rewritten as:

338

$$(4.11) \quad \tilde{A} z_{k+1} + \tilde{B}_1 \lambda_{1,k+1} + \tilde{B}_2 \lambda_{2,k+1} + \tilde{E}_k = 0.$$

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From (4.10) it is deduced that:

340

$$(4.12) \quad \begin{cases} \lambda_{1,k+1} = -(\mathcal{N}_{K_1^*} + \tilde{D}_{11})^{-1} (\tilde{C}_1 z_{k+1} + \tilde{F}_{1,k}) \\ \lambda_{2,k+1} \in -\mathcal{N}_{K_2 - \tilde{F}_{2,k}} (\tilde{C}_2 z_{k+1}). \end{cases}$$

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Using (4.11) it is inferred:

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$$(4.13) \quad \begin{aligned} & \tilde{A} z_{k+1} - \tilde{B}_1 (\mathcal{N}_{K_1^*} + \tilde{D}_{11})^{-1} (\tilde{C}_1 z_{k+1} + \tilde{F}_{1,k}) \\ & \quad - \tilde{B}_2 \mathcal{N}_{K_2 - \tilde{F}_{2,k}} (\tilde{C}_2 z_{k+1}) + \tilde{E}_k = 0 \\ & \quad \quad \quad \updownarrow \\ & -\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_{k+1} + \tilde{R}_2 \tilde{B}_1 (\mathcal{N}_{K_1^*} + \tilde{D}_{11})^{-1} (\tilde{C}_1 z_{k+1} + \tilde{F}_{1,k}) - \tilde{R}_2 \tilde{E}_k \\ & \quad \quad \quad \in -\tilde{R}_2 \tilde{B}_2 \mathcal{N}_{K_2 - \tilde{F}_{2,k}} (\tilde{C}_2 \tilde{R}_2^{-1} \xi_{k+1}), \end{aligned}$$

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which is (4.9) since  $\tilde{R}_2 \tilde{B}_2 = \tilde{R}_2^{-1} \tilde{X}_2 \tilde{B}_2 = \tilde{R}_2^{-1} \tilde{C}_2^\top$  and applying the chain rule of Convex Analysis for polyhedral functions.

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345

2. This follows from [29, Corollary 2.2.5] and since the map  $(\tilde{D}_{11} + \partial \psi_{K_1^*})^{-1}(\cdot)$  is single-valued, well defined and Lipschitz continuous (see Proposition B.1).

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347

3. The first assertion is a consequence of [9, Corollary 24.4] (for the maximal monotonicity of the mapping  $(-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} + \tilde{G}_2 + \partial \tilde{f}_2)(\cdot)$ , and of [9, Proposition 23.39]. The second assertion follows from [29, Theorem 2.3.3], noting that the mapping in the left-hand side of (4.9) is maximal monotone [9, Corollary 24.4 (ii)] and strongly monotone since  $-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \succ 0$ . This can also be proved using [9, Corollary 23.37]. Single-valuedness and Lipschitz continuity stem from  $\tilde{D}_{11} \succ 0$ .  $\square$

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Further features of the solutions to the VI in (4.9) will be studied in section 5.1.3. The next lemma states a classical condition that guarantees the maximal monotonicity of Proposition 4.4 item 3. The first condition in item 1 is a constraint qualification which guarantees that the function  $\tilde{f}_2(\cdot)$  has a nonempty domain. It is usually seen as a technical condition, however in some applications it may not be satisfied. Item 1 states conditions under which the VI in (4.9) can be constructed. Item 3 uses some results about the zeroes of maximal monotone mappings, or of the sum of these.

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361 LEMMA 4.5. Let  $\tilde{X}_2 \tilde{B}_1 = \tilde{C}_1^\top$  for  $\tilde{X}_2$  in Assumption 7, then for each fixed  $t_k$ , the  
 362 mapping  $\tilde{G}_2 : \xi_{k+1} \mapsto \tilde{R}_2 \tilde{B}_1 (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1} (\tilde{C}_1 \tilde{R}_2^{-1} \xi_{k+1} + \tilde{F}_{1,k})$  is maximal monotone.

363 **Proof:** Indeed  $\tilde{X}_2 \tilde{B}_1 = \tilde{C}_1^\top \Rightarrow \tilde{R}_2 \tilde{B}_1 = \tilde{R}_2^{-1} \tilde{C}_1^\top = (\tilde{C}_1 \tilde{R}_2^{-1})^\top$ . The mapping  $(\tilde{D}_{11} +$   
 364  $\partial\psi_{K_1^*})^{-1}(\cdot)$  is maximal monotone, and the following operation:  $(\tilde{C}_1 \tilde{R}_2^{-1})^\top (\tilde{D}_{11} +$   
 365  $\partial\psi_{K_1^*})^{-1}(\tilde{C}_1 \tilde{R}_2^{-1} \cdot + \tilde{F}_{1,k})$  preserves the maximal monotonicity [50, Exercise 12.4].  $\square$

366 Propositions 4.3 and 4.4 provide some informations about the existence and unique-  
 367 ness of the unknown  $z_{k+1}$ , and characterize some mappings. It is noteworthy that  
 368 other well-posedness results for VIs are available in the literature. For instance if  
 369 the set  $\tilde{\Gamma}_k$  is compact, [29, Corollary 2.2.5] guarantees existence of a solution to  
 370 VI( $-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, \tilde{f}$ ) with a compact set of solutions. Also when  $\tilde{\Gamma}_k = \mathbb{R}_+^m$  one may  
 371 rely on P-matrices and LCPs properties, when it is a closed convex cone one may use  
 372 copositive matrices [29, Theorem 2.5.10].

373 The next corollary completes the existence and uniqueness results for the OSNSP.

374 COROLLARY 4.6. *The next results hold true:*

- 375 1. (set of solutions  $z_{k+1}$ ) Let Assumption 4 and the conditions of item 1 in  
 376 Proposition 4.3 hold. Then the set of solutions  $z_{k+1}$  in (3.5) (or the set of  
 377 solutions  $\zeta_{k+1}$  in (3.3)), is closed and convex. If in addition the conditions  
 378 of item 3 (a) or (b) in Proposition 4.3 hold, then the set of solutions is non  
 379 empty. If in addition  $\tilde{A} \prec 0$ , then  $z_{k+1}$  is unique.
- 380 2. (set of solutions  $\lambda_{k+1}$ ) Let  $\tilde{C} \text{Dom}(\mathcal{G}) \subseteq \text{Im}(\partial\sigma_{\Gamma_k} + \tilde{D})$ , and the conditions of  
 381 item 1 in Proposition 4.3 hold true. Then the set of solutions  $\lambda_{k+1}$  in (3.5)  
 382 (a) is non empty for each  $z_{k+1}$ .
- 383 3. Let Assumption 4 hold with strict passivity, and the conditions of item 1 in  
 384 Proposition 4.3 be true. Then the GE in (3.5) (or the GE in (3.3)) has  
 385 a unique solution  $(\lambda_{k+1}, z_{k+1})$  for any  $\tilde{E}_k$ . Consequently  $x_{k+1}$  in (3.2) is  
 386 unique.
- 387 4. Let  $\tilde{D} = 0$  and  $\mathcal{G}(\cdot)$  be maximal monotone. Then if the set  $\tilde{\Gamma}_k = \{\zeta \in$   
 388  $\mathbb{R}^{n_2} \mid G\tilde{C}\tilde{R}^{-1}\zeta + G\tilde{F}_k \geq 0\}$  is bounded, the GE with unknown  $z_{k+1}$  in (3.5)  
 389 (or in (3.3)) has at least one solution and the set of solutions is compact.

390 **Proof:**

- 391 1. Follows from [9, Proposition 23.39] by the maximal monotonicity of  $\mathcal{G}(\cdot)$ . If  
 392  $-\tilde{A} \succ 0$  then  $\mathcal{G}(\cdot)$  is strongly monotone (however it may not be Lipschitz  
 393 continuous and this property is used later in section 5.2), and so is  $\mathcal{G}(\cdot) + \tilde{E}_k$ .  
 394 Hence the generalized equation  $\mathcal{G}(z_{k+1}) + \tilde{E}_k \ni 0$  has a unique solution [9,  
 395 Corollary 23.37].
- 396 2. Let us consider  $\lambda_{k+1} \in -(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}z_{k+1})$ . We know that any solution  
 397  $z_{k+1} \in \mathcal{G}^{-1}(\tilde{E}_k) \subseteq \text{Im}(\mathcal{G}^{-1}) = \text{Dom}(\mathcal{G})$ . Since  $\text{Dom}(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1} = \text{Im}(\partial\sigma_{\Gamma_k} +$   
 398  $\tilde{D})$  the result follows.
- 399 3. Strict passivity implies that  $\tilde{D} \succ 0$  and  $\tilde{A}$  is a Hurwitz matrix, therefore  
 400 from a slight modification of [6, Theorem 1] it follows that the operator  $\mathcal{G}(\cdot)$   
 401 is strongly, hence strictly monotone (indeed there exists a real  $\alpha > 0$  small  
 402 enough such that  $(\tilde{A} + \alpha I_{n_2}, \tilde{B}, \tilde{C}, \tilde{D})$  is passive). Using [9, Proposition 23.35]  
 403 shows that  $z_{k+1}$  is unique, and item 2 in Proposition 4.3 shows that  $\lambda_{k+1}$  is  
 404 unique. The conclusion follows.
- 405 4. Under these conditions  $\text{Dom}(\mathcal{G})$  is bounded (compact). Hence from [9, Propo-  
 406 sition 23.36 (iii)] existence of a solution holds true. From [29, Corollary 2.2.5]  
 407 the compactness of the set of solutions follows. This could also be inferred  
 408 from item (e) in Proposition 4.3 in case  $\tilde{R}\tilde{A}\tilde{R}^{-1}$  is symmetric.

409  $\boxtimes$

410 Item 3 in Corollary 4.6 guarantees the uniqueness of solutions to the OSNSP in (3.6),  
 411 in which case the  $\in$  can be replaced by equality. Concerning item 2 in Corollary 4.6,  
 412 an important step is to characterize both  $\text{Dom}(\mathcal{G})$  and  $\text{Im}(\partial\sigma_{\Gamma_k} + \tilde{D})$ . If  $\tilde{D} = \tilde{D}^\top \succcurlyeq 0$ ,  
 413 one can apply [15, Theorem 3], and infer that  $\text{Im}(\partial\sigma_{\Gamma_k} + \tilde{D}) \simeq \text{Im}(\partial\sigma_{\Gamma_k}) + \text{Im}(\tilde{D}) =$   
 414  $\text{Dom}(\partial\psi_{\Gamma_k}) + \text{Im}(\tilde{D}) \subset \text{dom}(\psi_{\Gamma_k}) + \text{Im}(\tilde{D}) = \Gamma_k + \text{Im}(\tilde{D})$  (see [15] for the exact  
 415 meaning of  $\simeq$  in this context). Similar calculations may be done for in item 1 of  
 416 Proposition 4.3. It is inferred that the basic condition in Proposition 4.3 item 1,  
 417 i.e.,  $\text{Im}(\tilde{C}) \cap \text{int}(\text{Im}(\partial\sigma_{\Gamma_k} + \tilde{D})) \neq \emptyset$ , or in Corollary 4.6 item 2, i.e.,  $\tilde{C} \text{Dom}(\mathcal{G}) \subseteq$   
 418  $\text{Im}(\partial\sigma_{\Gamma_k} + \tilde{D})$ , always are verified when  $\tilde{D} \succ 0$ . For the sake of brevity the conditions  
 419 for the existence and uniqueness of  $\lambda_{k+1}$  in (4.12) are not stated, similar tools as for  
 420 Corollary 4.6 can be used.

421 **4.2. Analysis of the OSNSP for (3.1).** It is also interesting to investigate  
 422 the case when Assumption 3 (resp. Assumption 5) is satisfied instead of Assumption  
 423 2 (resp. Assumption 4). The operator  $\mathcal{G}(\cdot)$  is now defined as  $\mathcal{G} : \zeta \mapsto -B_{II}\zeta +$   
 424  $(\partial\sigma_{\Gamma(t_k, x_{k+1})} + D)^{-1}(C_{II}\zeta)$ , where  $\Gamma(t_k, x_{k+1}) = \{\xi \in \mathbb{R}^m \mid \xi + C_I x_{k+1} + F_k \in$   
 425  $K\}$ . Consequently the results of Proposition 4.3 item 1, Corollary 4.6 item 3, with  
 426 Assumption 5 plus strict passivity, and Proposition 4.3 item 2 when  $D \succ 0$ , hold. For  
 427 the sake of brevity we do not develop all the arguments once again.  
 428 Let us focus now on the case  $D \succcurlyeq 0$ . From Assumption 3,  $D = 0$  and (2.1) one infers:

$$429 \quad (4.14) \quad -RB_{II}R^{-1}\xi(t) - RA_{II}x(t) - RE_{II}(t) \in -\partial\psi_{\Phi(t, x(t))}(\xi(t)),$$

430 where  $\xi = -Rz$ ,  $R^2 = X$ ,  $R = R^\top \succ 0$ ,  $\Phi(t, x) = \{\xi \in \mathbb{R}^p \mid C_{II}R^{-1}\xi + C_I x + F(t) \in$   
 431  $K\}$ , and the set-valued term is obtained using the chain rule, under the Convex  
 432 Analysis chain rule qualification condition [49, Theorem 23.9]. The first step is to  
 433 analyse the existence and uniqueness of solutions to the VI:

$$434 \quad (4.15) \quad -RB_{II}R^{-1}\xi_{k+1} - RA_{II}x_{k+1} - RE_{II,k} \in -\partial\psi_{\Phi(t_k, x_{k+1})}(\xi_{k+1}),$$

435 which is the counterpart of (4.7), relying on Proposition A.2. The unknown is  $\xi_{k+1}$   
 436 and the other terms and variables are considered as data for this VI. Similar results  
 437 to those in item 3 of Proposition 4.3 can be obtained.

438 **PROPOSITION 4.7.** *Let  $D = 0$ , Assumption 3 be true,  $h > 0$  be given, assume that*  
 439  *$\text{Im}(C_{II}) \cap (K - C_I x_{k+1} + F_k) \neq \emptyset$  and that  $K$  is polyhedral. Consider the discrete-time*  
 440 *system in (3.1), and the associated VI( $-RB_{II}R^{-1}$ ,  $-RA_{II}x_{k+1} - RE_{II,k}$ ,  $\psi_{\Phi(t_k, x_{k+1})}$ )*  
 441 *in (4.15). Assume that  $-RB_{II}R^{-1} \succcurlyeq 0$ . Consider the set:*

$$442 \quad (4.16) \quad \mathcal{S} \triangleq P(GC_{II}R^{-1}, 0) \cap \{\xi \in \mathbb{R}^{n_2} \mid RB_{II}R^{-1}\xi = \sum_{i=1}^m \lambda_i (GC_{II}R^{-1})_{i\bullet}^\top, \lambda_i \geq 0\}$$

$$\cap \text{Ker}(RB_{II}R^{-1} + R^{-1}B_{II}^\top R).$$

443 *Then:*

- 444 1. *If  $\mathcal{S} = \{0\}$  the VI( $-RB_{II}R^{-1}$ ,  $-RA_{II}x_{k+1} - RE_{II,k}$ ,  $\psi_{\Phi(t_k, x_{k+1})}$ ) has a unique*  
 445 *solution.*
- 446 2. *If  $\mathcal{S} \neq \{0\}$ , and if there exists  $\xi_0 \in \Phi(t_k, x_{k+1})$  such that  $\langle -RA_{II}x_{k+1} -$*   
 447  *$RE_{II,k} + RB_{II}R^{-1}\xi_0, v \rangle > 0$  for all  $v \neq 0$  and  $v \in \mathcal{S}$ , then it follows that the*  
 448 *VI( $-RB_{II}R^{-1}$ ,  $-RA_{II}x_{k+1} - RE_{II,k}$ ,  $\psi_{\Phi(t_k, x_{k+1})}$ ) has a unique solution.*

449 **Proof:** The set  $\mathcal{S}$  in (4.16) is constructed as follows. The chain rule holds from  
 450 [49, Theorem 23.9] and the fact that  $\psi_{K-C_I x_{k+1}+F_k}(\cdot)$  is polyhedral. Hence the  
 451 VI in (4.15) is equivalent to (3.1) (b) (c). Here  $\varphi = \psi_{\Phi(t_k, x_{k+1})}$ , hence  $\text{dom}(\varphi) =$   
 452  $\Phi(t_k, x_{k+1})$  and  $(\text{dom}(\varphi))_\infty$ . Assume that  $K = P(G, 0)$  (see Proposition A.1 (e)),  
 453 then  $(\text{dom}(\varphi))_\infty = \{\xi \in \mathbb{R}^p \mid GC_{II}R^{-1}\xi \geq 0\} = P(GC_{II}R^{-1}, 0)$ . Further,  $\varphi_\infty =$   
 454  $\psi_{(\Phi(t_k, x_{k+1}))_\infty} = \psi_{P(GC_{II}R^{-1}, 0)}$ , hence  $\text{dom}(\varphi_\infty) = P(GC_{II}R^{-1}, 0)$  and  $\text{dom}(\varphi_\infty)^* =$   
 455  $(P(GC_{II}R^{-1}, 0))^* = \{w \in \mathbb{R}^{n_2} \mid w = -\sum_{i=1}^m \lambda_i (GC_{II}R^{-1})_{i\bullet}^\top, \lambda_i \geq 0\}$ , where [31,  
 456 Example 15] was used as in the proof of Proposition 4.3. Thus  $\mathcal{K}(-RB_{II}R^{-1}, \varphi) =$   
 457  $\{\xi \in \mathbb{R}^{n_2} \mid -RB_{II}R^{-1}\xi = -\sum_{i=1}^m \lambda_i (GC_{II}R^{-1})_{i\bullet}^\top, \lambda_i \geq 0\}$ . Finally in Proposition  
 458 A.2:  $\text{Ker}(\mathbf{M} + \mathbf{M}^\top) = \text{Ker}(RB_{II}R^{-1} + (RB_{II}R^{-1})^\top)$  and  $R = R^\top$ . Items 1 and 2  
 459 are a direct consequence of Proposition A.2 (a) and (b).  $\square$

460 For the sake of brevity we do not state similar results as in Proposition 4.3 (c) to  
 461 (f). To enlarge the analysis let us now state the counterpart of Assumption 6:

462 ASSUMPTION 8. *Let  $D = \begin{pmatrix} D_{11} & 0 \\ 0 & 0 \end{pmatrix}$  with  $0 \prec D_{11} = D_{11}^\top \in \mathbb{R}^{m_1 \times m_1}$ .*

463 Similarly to Assumption 7, with the same splitting of the multiplier  $\lambda$ , the matrices  
 464  $B_{II}$  and  $C_{II}$ , and of the cone  $K$  (see the comments after Assumption 6, in particular  
 465 if  $D_{11} = 0$  the case  $D = 0$  is recovered with  $G_2(\cdot, \cdot)$  in (4.17) which vanishes, and the  
 466 set  $\Phi_2 = \Phi$ ), let us state the following:

467 ASSUMPTION 9. *There exists a  $(p \times p)$ -matrix  $X_2 = X_2^\top \succ 0$  such that  $X_2 \bar{B}_{II}^2 =$   
 468  $(C_{II}^2)^\top$ .*

469 Let us recall that  $p$  is the number of equality constraints in (2.1). Let us define  
 470  $R_2 = R_2^\top \succ 0$ ,  $R_2^2 = X_2$ ,  $\xi_{k+1} = R_2 z_{k+1}$ ,  $\Phi_2(t_k, x_{k+1}) = \{\xi \in \mathbb{R}^p \mid C_{II}^2 R_2^{-1} \xi +$   
 471  $C_I^2 x_{k+1} + F_{2,k} \in K_2\}$ . The next proposition is the counterpart of Proposition 4.4. It  
 472 states conditions that guarantee that  $z_{k+1}$  can be calculated from (3.1) (b) (c).

473 PROPOSITION 4.8. *Let  $h > 0$ , and Assumptions 8 and 9 hold true. Consider the  
 474 VI: find  $\xi_{k+1} \in \text{dom}(f_2)$  such that*

$$475 \quad (4.17) \quad \underbrace{-R_2 B_{II} R_2^{-1} \xi_{k+1} - R_2 A_{II} x_{k+1} - R_2 E_{II, k} + R_2 \bar{B}_{II}^1 (D_{11} + \partial \psi_{K_1^*})^{-1} (C_{II}^1 R_2^{-1} \xi_{k+1} - F_{1, k})}_{\triangleq G_2(t_k, \xi_{k+1})} \in -\partial f_2(\xi_{k+1})$$

476 with  $f_2(\cdot) = (\psi_{K_2 - C_I^2 x_{k+1} - F_{2, k}} \circ C_{II}^2 R_2^{-1})(\cdot) = \psi_{\Phi_2(t_k, x_{k+1})}(\cdot)$ .

- 477 1. Assume that  $\text{Im}(C_{II}^2) \cap (K_2 - C_I^2 x_{k+1} - F_{2, k}) \neq \emptyset$ , then the VI in (4.17) is  
 478 equivalent to the MLCP in (3.1) (b) (c).
- 479 2. Suppose that the set  $\Phi_2(t_k, x_{k+1})$  is compact and nonempty, then the set of  
 480 solutions to the VI in (4.17) is compact and nonempty.
- 481 3. Assume that for each fixed  $t_k$ , the mapping  $G_2(\cdot)$  is maximal monotone, and  
 482 that  $-R_2 B_{II} R_2^{-1} \succ 0$ , then the VI in (4.17) has a unique solution.
- 483 4. Assume that  $-R_2 B_{II} R_2^{-1} \succcurlyeq 0$ ,  $G_2(\cdot)$  is maximal monotone for each fixed  $t_k$ ,  
 484 and  $\Phi_2(t_k, x_{k+1}) \cap \text{Int}(\text{Dom}(G_2)) \neq \emptyset$  or  $\text{Int}(\Phi_2(t_k, x_{k+1})) \cap \text{Dom}(G_2) \neq \emptyset$ .  
 485 Then the set of solutions to the VI in (4.17) is closed and convex.

486 **Proof:** The proof of the first three items follows closely that of Proposition 4.4 and it  
 487 is therefore omitted. Item 4: the VI is rewritten as  $\underbrace{(-R_2 B_{II} R_2^{-1} + G_2 + \partial f_2)}_{\triangleq H_2}(\xi_{k+1}) -$

488  $R_2 A_{II} x_{k+1} - R_2 E_{II, k} \ni 0$ , and the assumption guarantees that  $H_2(\cdot)$  is maximal  
 489 monotone [9, Corollary 24.4]. The result follows from [9, Proposition 23.39].  $\square$

490 The same as Lemma 4.5 can be stated to guarantee the maximal monotonicity in  
 491 items 3 and 4. It is noteworthy that the main discrepancy between Proposition  
 492 4.4 and Proposition 4.8, is that the solution  $\xi_{k+1}$  depends on  $x_{k+1}$  in the latter,  
 493 while it depends on  $x_k$  in the former. This is a difficulty with the application of  
 494 Proposition 4.8 in some cases, for instance when  $C_{II} = 0$  (examples in sections 6.2  
 495 and 6.4 will demonstrate that this can occur). Actually, the condition  $\text{Im}(C_{II}^2) \cap (K_2 -$   
 496  $C_I^2 x_{k+1} - F_{2,k}) \neq \emptyset$  (that is a form of constraint qualification), or its counterparts in  
 497 Propositions 4.3, 4.4, 4.7, happens to be crucial in the setting of this work. It states  
 498 that the algebraic variable  $z$  has to enter the complementarity constraints through  
 499 a  $C_{II}$  matrix which has a big enough range. This may occur in some systems, see  
 500 section 6.5.

501 **5. Convergence results.** In this section it is assumed that the conditions for  
 502 existence of solutions in Propositions 4.3, 4.4, 4.7, or 4.8, are fulfilled, so that sequences  
 503 can be calculated. The section is divided into two main parts: in sections 5.1.1, 5.1.2  
 504 and 5.1.3, convergence is analysed from the properties of the quadruple  $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$   
 505 which characterize the MLCP in (3.2) (b) (c). In section 5.2, the analysis is led with  
 506 the properties of  $(B_{II}, \bar{B}_{II}, C_{II}, D)$  which characterize the MLCP in (3.1) (b) (c).  
 507 The relationships between both approaches are explained.

508 **5.1. Convergence results from (3.2).** In this section the numerical scheme  
 509 in (3.2) and the results in section 4.1 are used. First let us notice that  $z_{k+1}$  and  
 510  $\lambda_{k+1}$  are solutions of the generalized equation in (3.5) (a) (b). Thus if existence of  
 511 solutions hold, they both are functions of  $\bar{E}_k$  (in addition to being functions of  $\bar{F}_k$   
 512 through the set  $\Gamma_k$ ). Under some conditions which are recalled below, like item 2 of  
 513 Proposition 4.3, they can be single-valued Lipschitz continuous functions of  $\bar{E}_k$ . Then  
 514 the right-hand side of the first line in (3.2) may be written compactly as

$$515 \quad (5.1) \quad P_I^h x_{k+1} = P_I x_k + h \bar{E}_k(t_k, x_k)$$

516 for some  $\bar{E}_k(t_k, x_k)$  which is a linear function of  $E_{I,k}$ ,  $E_{II,k}$ ,  $\lambda_{k+1}$  and  $z_{k+1}$ . The  
 517 time-dependence of  $\bar{E}_k$  comes from  $E_{I,k}$ ,  $E_{II,k}$  and  $F_k$ , while the state dependence  
 518 comes from the third terms in  $\bar{E}_k$  and in  $\bar{F}_k$  (which enters  $\Gamma_k(t_k, x_k) = K - \bar{F}_k$ ). From  
 519 (5.1) it can be calculated and checked by induction that:

$$520 \quad (5.2) \quad x_k = ((P_I^h)^{-1} P_I)^k x_0 + h \sum_{j=0}^{k-1} ((P_I^h)^{-1} P_I)^{k-1-j} (P_I^h)^{-1} \bar{E}_j(t_j, x_j), \quad k \geq 1,$$

521 Since  $P_I^h = P_I - hA_I$ , it follows that for  $h > 0$  small enough,  $(P_I^h)^{-1} P_I = I_p + \mathcal{O}(h)$   
 522 [10, Proposition 9.4.13] hence  $(P_I^h)^{-1} = P_I^{-1} + \mathcal{O}(h)$ . Thus

$$523 \quad (5.3) \quad \begin{aligned} x_k &= x_0 + \mathcal{O}(h) + h \sum_{j=0}^{k-1} (I_p + \mathcal{O}(h))^{k-1-j} (P_I^{-1} + \mathcal{O}(h)) \bar{E}_j(t_j, x_j) \\ &= x_0 + \mathcal{O}(h) + h \sum_{j=0}^{k-1} (I_p + \mathcal{O}(h)) (P_I^{-1} + \mathcal{O}(h)) \bar{E}_j(t_j, x_j) \\ &= x_0 + \mathcal{O}(h) + h \sum_{j=0}^{k-1} (P_I^{-1} + \mathcal{O}(h)) \bar{E}_j(t_j, x_j) \\ &= x_0 + \mathcal{O}(h) + h (P_I^{-1} + \mathcal{O}(h)) \sum_{j=0}^{k-1} \bar{E}_j(t_j, x_j), \end{aligned}$$

524 for all  $k \geq 1$ . It is inferred that:

$$525 \quad (5.4) \quad \begin{aligned} \dot{x}_k &\triangleq \frac{x_k - x_{k-1}}{h} = \frac{\mathcal{O}(h)}{h} + (P_I^{-1} + \mathcal{O}(h)) \bar{E}(t_{k-1}, x_{k-1}) \\ &= \mathcal{O}(1) + (P_I^{-1} + \mathcal{O}(h)) \bar{E}(t_{k-1}, x_{k-1}). \end{aligned}$$

526 Our aim is now to prove that both  $\{x_k\}$  and  $\{\dot{x}_k\}$  are uniformly bounded sequences,  
 527 which will allow us to conclude about the convergence of the sequence  $\{x^h(\cdot)\}$  (see the  
 528 **definition in (5.7)**). For this it is necessary and sufficient that  $\bar{E}_k(t_k, x_k)$  be uniformly  
 529 bounded for all  $k \geq 0$ , since  $P_I$  is full rank. It follows from (3.5) (a) and (b), that  
 530 there are two operators which are crucial to characterize the right-hand side in (3.6).

531 **5.1.1. Analysis of the operator  $\mathcal{G}(\cdot)$ .** Notice that the mapping  $\mathcal{Z} : \zeta \mapsto \eta =$   
 532  $(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\zeta)$  is under the conditions of Proposition 4.3 item 2, a well-defined  
 533  $\mathbb{R}^m \rightarrow \mathbb{R}^m$  single-valued Lipschitz continuous mapping which is parameterized by  
 534  $t_k$  and  $x_k$  through the set  $\Gamma_k(t_k, x_k)$ . Assume that  $x_k$  is bounded, then redoing  
 535 the proof of **Proposition B.1**, it follows that the Lipschitz constant of  $\mathcal{Z}(\cdot)$  does not  
 536 depend on  $\partial\sigma_{\Gamma_k}(\cdot)$  but depends only on  $\tilde{D}$ , being equal to  $\frac{2}{\lambda_{\min}(\tilde{D} + \tilde{D}^\top)}$ . In turn  
 537  $\lambda_{\min}(\tilde{D} + \tilde{D}^\top) > 0$  uniformly in  $h > 0$  only if  $D \succ 0$ . Thus the operator  $\mathcal{G} :$   
 538  $\zeta \mapsto -\tilde{A}\zeta + \tilde{B}(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}\zeta)$  is also single-valued Lipschitz continuous (being the  
 539 composition of Lipschitz continuous mappings) with a Lipschitz constant that does  
 540 not depend on  $x_k$ , and which is bounded for bounded  $h$ . Let us now assume the  
 541 following which reinforces Assumption 4 and guarantees that  $\tilde{D} \succ 0$ :

542 **ASSUMPTION 10.** *The system  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  is strictly passive, uniformly in  $h \geq 0$ .*  
 543

544 Then if the basic conditions stated in item 1 of Proposition 4.3 are verified in addition  
 545 to Assumption 10, the operator  $\mathcal{G}(\cdot)$  is strongly monotone for some constant  $\epsilon > 0$ ,  
 546 and we know from Corollary 4.6 item 3, that  $x_{k+1}$  is unique. As stated in Proposition  
 547 4.3 item 1, maximal monotonicity follows from [6, Theorem 1]. Strong monotonicity  
 548 follows from the fact that under Assumption 10, the quadruple  $(\tilde{A} + \epsilon I_{n_2}, \tilde{B}, \tilde{C}, \tilde{D})$   
 549 is passive for a small enough  $\epsilon > 0$ , so that  $(\mathcal{G} - \epsilon I_{n_2})(\cdot)$  is maximal monotone for  
 550 such  $\epsilon$ . Thus if Assumption 10 and the basic conditions in item 1 of Proposition  
 551 4.3 are true,  $\mathcal{G} : \mathbb{R}^p \rightarrow \mathbb{R}^p$  is well-defined and single-valued, Lipschitz continuous  
 552 and strongly monotone. Consequently it is inferred from [50, Proposition 12.54] that  
 553 under the above set of conditions,  $\mathcal{G}^{-1}(\cdot)$  is single-valued, well defined and globally  
 554 Lipschitz, and the solution  $z_{k+1}$  of the generalized equation  $\bar{E}_k(t_k, x_k) \in \mathcal{G}(z_{k+1}) \Leftrightarrow$   
 555  $z_{k+1} = \mathcal{G}^{-1}(\bar{E}_k(t_k, x_k))$  in (3.5) is Lipschitz continuous in  $x_k$  and  $t_k$ . Thus  $\lambda_{k+1}$  is  
 556 also Lipschitz continuous in  $x_k$ , and we conclude that the same applies to  $x_{k+1}$  in  
 557 (3.2).

558 *Remark 5.1.* Assumption 10 is stringent. Actually the strict state passivity of  
 559  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  is sufficient to guarantee that  $\mathcal{G}(\cdot)$  is strongly monotone. However in  
 560 the body of sections 5.1.1 and 5.1.2 we need also that  $\lambda_{k+1}$  be a Lipschitz continuous  
 561 function of its argument, a sufficient condition being that  $\tilde{D} \succ 0$ , which in turn implies  
 562 that strict state passivity and strict passivity are identical. Thus strict passivity could  
 563 be relaxed to strict state passivity if it is assured that  $\bar{B}_I(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}\cdot)$  is Lipschitz  
 564 continuous. This may occur in certain cases (the trivial case is  $\bar{B}_I = 0$ ).

565 **5.1.2. Boundedness and convergence analysis.** Under the above conditions,  
 566 the term  $\bar{E}_k(t_k, x_k)$  and consequently  $x_{k+1}(t_k, x_k)$  in (3.2) and  $\dot{x}_k(t_k, x_k)$  in (5.4) are  
 567 single-valued and Lipschitz continuous in  $x_k$  and  $t_k$  (provided  $E_I(\cdot)$ ,  $E_{II}(\cdot)$  and  $F(\cdot)$   
 568 are Lipschitz continuous functions of  $t$ , which is a mild requirement on the data),  
 569 since  $\bar{E}_k$  and  $\bar{F}_k$  are linear in the argument  $x_k$ . We have  $\|\bar{E}(t_k^1, x_k^1) - \bar{E}(t_k^2, x_k^2)\| \leq$   
 570  $L_t|t_k^1 - t_k^2| + L_x\|x_k^1 - x_k^2\|$  for any  $t_k^1, t_k^2, x_k^1, x_k^2$ , and for some  $L_t > 0$  and  $L_x > 0$ . Thus  
 571  $\|\bar{E}(t_k, x_k^1) - \bar{E}(t_k, x_k^2)\| \leq L_x\|x_k^1 - x_k^2\|$  and  $\|\bar{E}(t_k, x_k) - \bar{E}(t_k, 0)\| \leq L_x\|x_k\|$ . Notice  
 572 that due to the Lipschitz continuity both  $z_{k+1}$  and  $\lambda_{k+1}$  are uniformly bounded in

573 time for fixed  $x_k$  (provided  $E_I(\cdot)$ ,  $E_{II}(\cdot)$  and  $F(\cdot)$  are uniformly bounded, which is  
 574 a mild requirement on the data). Therefore  $\bar{E}(t_k, 0)$  is uniformly bounded also, and  
 575  $\|\bar{E}(t_k, x_k)\| \leq \|\bar{E}(t_k, 0)\| + L_x \|x_k\| \leq m + L_x \|x_k\|$ . Now using (5.2) we get:  
 (5.5)

$$576 \quad \|x_k\| \leq \sigma_{\max}^k((P_I^h)^{-1}P_I)\|x_0\| + h\sigma_{\min}(P_I^h) \sum_{j=1}^{k-1} \sigma_{\max}^{k-j-1}(P_I(P_I^h)^{-1})(m + L_x \|x_{j-1}\|),$$

577 where we used the Hölder induced matrix norm  $\|\cdot\|_{2,2} = \sigma_{\max}(\cdot)$  and [10, Proposition  
 578 9.4.9, Corollary 9.4.12]. Let us make the following:

579 **ASSUMPTION 11.** *One has  $\sigma_{\max}((P_I^h)^{-1}P_I) \leq 1$  for all  $h \geq 0$ .*

580 Assumption 11 is a technical assumption which is satisfied if for instance the  $n_1 \times n_1$   
 581 matrix  $A_I \prec 0$  and  $P_I \succ 0$ . Indeed this implies that  $P_I^h \succ P_I$ , thus  $(P_I^h)^{-1}P_I \prec I_{n_1}$ .  
 582 Notice that under Assumption 11 one no longer needs  $h > 0$  small enough to obtain  
 583 the developments in (5.4), because the spectral radius of  $(P_I^h)^{-1}P_I$  is smaller than  
 584 one. It is also noteworthy that changing  $A_I x_{k+1}$  to  $A_I x_k$  in the right-hand side of  
 585 (3.1) (a), allows one to dispense with Assumption 11, with a slight modification of the  
 586 matrices  $\tilde{A}$ ,  $\tilde{B}$ ,  $\tilde{C}$ ,  $\tilde{D}$ ,  $\tilde{E}_k$  and  $\tilde{F}_k$  (where  $(P_I^h)^{-1}$  is replaced by  $P_I^{-1}$  in all of them).  
 587 Consequently from (5.5) one infers that:

$$588 \quad (5.6) \quad \|x_k\| \leq \|x_0\| + h\sigma_{\min}(P_I^h) \sum_{j=1}^k (m + L_x \|x_{j-1}\|),$$

589 Assume now that  $x_0, x_1, \dots, x_{k-1}$  are uniformly bounded by some  $x_{\max}$ , then from (5.6)  
 590 one has  $\|x_k\| \leq x_{\max} + h\sigma_{\min}(P_I^h)k(m + L_x x_{\max}) \leq x_{\max} + h\sigma_{\min}(P_I^h)n(m + L_x x_{\max}) =$   
 591  $x_{\max} + h\sigma_{\min}(P_I^h)\frac{T}{h}(m + L_x x_{\max}) = x_{\max} + \sigma_{\min}(P_I^h)T(m + L_x x_{\max})$ . Therefore  $x_k$   
 592 is also uniformly bounded (its upperbound depends only on  $T$  which is fixed), and  
 593 we conclude that all  $x_k$ ,  $1 \leq k \leq n$ , are bounded provided  $x_0$  is bounded. Then it is  
 594 inferred from (5.4) that  $\hat{x}_k$  is also uniformly bounded. Let us define the sequences of  
 595 piecewise-linear approximations  $\{x^h(\cdot)\}$  and  $\{\hat{x}^h(\cdot)\}$  as follows:

$$596 \quad (5.7) \quad \begin{cases} x^h(t) = x_{k+1} + \frac{t_{k+1}-t}{h}(x_k - x_{k+1}) & t \in [t_k, t_{k+1}). \\ \hat{x}^h(t) = \frac{x_{k+1}-x_k}{h} \end{cases}$$

597 We have shown that  $x^h(\cdot)$  and  $\hat{x}^h(\cdot)$  are uniformly bounded on  $[0, T]$  and for any  
 598  $h > 0$ , hence  $x^h(\cdot)$  has a Lipschitz constant that does not depend on  $h$ , consequently  
 599 the family  $\{x^h\}_{h>0}$  is equi-Lipschitz continuous. Thus from Arzela-Ascoli Theorem  
 600  $\{x^h(\cdot)\}$  stays in a compact subset of  $C^0([0, T]; \mathbb{R}^{n_1})$  and there exist subsequences  
 601 which converge uniformly in  $C^0([0, T]; \mathbb{R}^{n_1})$  towards a Lipschitz continuous limit  $x(\cdot)$ .  
 602 Let  $\tilde{E}^h(\cdot)$  and  $\tilde{F}^h(\cdot)$  be defined in a similar way as  $x^h(\cdot)$  in (5.7). Due to the Lip-  
 603 schitz continuity of  $E_I(\cdot)$ ,  $E_{II}(\cdot)$ , and  $F(\cdot)$ , the sequences of functions  $\{\tilde{E}^h(\cdot)\}_{h>0}$   
 604 and  $\{\tilde{F}^h(\cdot)\}_{h>0}$  also converge uniformly in  $C^0([0, T]; \mathbb{R}^p)$  and  $C^0([0, T]; \mathbb{R}^m)$  to limits  
 605  $\tilde{E}(\cdot) = E_{II}(\cdot) + A_I x(\cdot)$  and  $\tilde{F}(\cdot) = F(\cdot) + C_I x(\cdot)$  as  $h \rightarrow 0$ , respectively. From (5.4) it  
 606 is deduced that  $\hat{x}^h(\cdot)$  is uniformly bounded. Therefore from the Banach-Alaoglu The-  
 607 orem, it follows that  $\hat{x}^h(\cdot)$  converges weakly\* to a limit  $v(\cdot)$ . Since the domain  $[0, T]$  is  
 608 bounded, one has  $\mathcal{L}^2([0, T]; \mathbb{R}^{n_1}) \subseteq \mathcal{L}^1([0, T]; \mathbb{R}^{n_1})$  [36, Corollary 1, p.80], and conse-  
 609 quently  $\hat{x}^h(\cdot)$  converges weakly in  $\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})$ . Since  $x^h(t) = x^h(0) + \int_0^t \hat{x}^h(\tau) d\tau$ ,  
 610 we infer that  $v(\cdot) = \dot{x}(\cdot)$  almost everywhere.



611 Now notice that using (3.2) the following holds:

$$612 \quad (5.8) \quad \begin{cases} (a) P_I \dot{x}^h(t) = A_I x_h^*(t) + B_I z_h^*(t) + \bar{B}_I \lambda_h^*(t) + E_I^h(t) \\ (b) \bar{A} z_h^*(t) + \bar{B} \lambda_h^*(t) + \bar{E}_{II}^h(t) + A_{II} x^h(t) = 0 \\ (c) 0 \leq \lambda_h^*(t) \perp \bar{C} z_h^*(t) + \bar{D} \lambda_h^*(t) + F^h(t) + C_I x^h(t) \geq 0, \end{cases}$$

613 for all  $t \in [t_k, t_{k+1})$ ,  $k \geq 0$ ,  $x_h^*(t) = x_{k+1}$  for all  $t \in [t_k, t_{k+1})$  and similarly for the  
614 other terms are step-functions. From the boundedness of  $\|\dot{x}_k\| \triangleq \|\frac{x_{k+1} - x_k}{h}\| \leq C$  for  
615 some  $C > 0$ , we infer that  $x_h^*(\cdot)$  converges strongly in  $\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})$  towards  $x(\cdot)$ .  
616 Indeed:

$$617 \quad (5.9) \quad \begin{aligned} \|x^h - x_h^*\|_{\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})}^2 &\leq \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_{k+1} - t_k)^2 \|\dot{x}_k\|^2 dt \\ &\leq C^2 \sum_{k=0}^{n-1} (t_{k+1} - t_k)^3 = \frac{C^2 n h^3}{3} = \frac{C^2 T h^2}{3} \longrightarrow 0 \text{ as } h \rightarrow 0. \end{aligned}$$

618 To conclude one notices that  $\|x^h - x_h^*\|_{\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})}^2 = \|x^h \pm x - x_h^*\|_{\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})}^2$ . Now  
619 we have  $z_h^*(t) \in \mathcal{G}^{-1}(E_{II, \star}^h(t) + A_{II} x_h^*(t))$  for all  $t \in [t_k, t_{k+1})$ ,  $k \geq 0$ . Since  $E_I(\cdot)$ ,  
620  $E_{II}(\cdot)$  are Lipschitz continuous, it follows that  $\|\dot{E}_k\| \triangleq \|\frac{E_{k+1} - E_k}{h}\| \leq k_E$  where  $k_E$   
621 is its Lipschitz constant. We infer that  $E_{II, \star}^h(\cdot)$  converges strongly in  $\mathcal{L}^2([0, T]; \mathbb{R}^p)$   
622 towards  $E_{II}(\cdot)$ , and using the Lipschitz continuity of  $\mathcal{G}^{-1}(\cdot)$  it follows that  $z_h^*(\cdot)$   
623 converges strongly in  $\mathcal{L}^2([0, T]; \mathbb{R}^p)$  towards a limit  $z(\cdot) = \mathcal{G}^{-1}(E_{II}(\cdot) + A_{II} x(\cdot))$ .  
624 Then from (3.3)  $\lambda_h^*(t) \in (\mathcal{N}_{\mathbb{R}_+^m} + \bar{D})^{-1}(-\bar{C} z_h^*(t) - F_h^*(t) + C_I x_h^*(t))$ , so that  $\lambda_h^*(\cdot)$   
625 converges towards a limit  $\lambda(\cdot) = (\mathcal{N}_{\mathbb{R}_+^m} + \bar{D})^{-1}(-\bar{C} z(\cdot) - F(\cdot) + C_I x(\cdot))$  strongly in  
626  $\mathcal{L}^2([0, T]; \mathbb{R}^m)$ , due to the single-valuedness, Lipschitzness and maximal monotonicity  
627 of  $(\mathcal{N}_{\mathbb{R}_+^m} + \bar{D})^{-1}(\cdot)$ . It is concluded that  $P_I \dot{x}^h(t) = A_I x_h^*(t) + B_I z_h^*(t) + \bar{B}_I \lambda_h^*(t) +$   
628  $E_I^h(t) \rightharpoonup P_I \dot{x}(t) = A_I x(t) + B_I z(t) + \bar{B}_I \lambda(t) + E_I(t)$  weakly in  $\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})$  as  $h \rightarrow$   
629  $0$ ,  $\bar{A} z_h^*(t) + \bar{B} \lambda_h^*(t) + \bar{E}_{II}^h(t) + A_{II} x^h(t) \rightarrow \bar{B}_I z(t) + \bar{B}_{II} \lambda(t) + E_{II}(t) + A_{II} x(t) (= 0)$ ,  
630 and  $0 \leq \lambda_h^*(t) \perp \bar{C} z_h^*(t) + \bar{D} \lambda_h^*(t) + F^h(t) + C_I x^h(t) \geq 0 \rightarrow 0 \leq \lambda(t) \perp C_I z(t) +$   
631  $D \lambda(t) + F(t) + C_I x(t) \geq 0$  as  $h \rightarrow 0$  strongly in  $\mathcal{L}^2([0, T]; \mathbb{R}^l)$  where  $l$  depends on the  
632 variables dimensions. Hence we have proved the existence of a continuous solution  
633  $(x, z, \lambda)$  to (2.1) on  $[0, T]$ . Consequently the following result is proved:

634 **PROPOSITION 5.2.** *Let the conditions of item 2 in Corollary 4.6 hold true uni-*  
635 *formly in  $h$  ( $\Rightarrow D \succ 0$ ). Assume that Assumptions 10 and 11 hold,  $E(\cdot)$  and  $F(\cdot)$  are*  
636 *bounded Lipschitz continuous functions, initial data are bounded. Then the DALCS*  
637 *in (2.1) possesses for each admissible bounded initial data, a continuous solution*  
638  *$(x, z, \lambda) : [0, T] \rightarrow \mathbb{R}^{n_1 + n_2 + m}$ ,  $T > 0$ . Moreover the Euler approximations converge*  
639 *to a solution as  $h \rightarrow 0$ .*

640 **5.1.3. The case blockdiagonal  $\bar{D} = \bar{D}^\top \succcurlyeq 0$ .** This section is dedicated to  
641 **analyse** an alternative to the conditions stated in sections 5.1.1 and 5.1.2. Let us  
642 consider the case of Proposition 4.4 with Assumption 6. Then it follows from the  
643 complementarity conditions that  $\lambda_{1, k+1} = (\bar{D}_{11} + \partial\psi_{K_1^*})^{-1}(-\bar{C}_1 z_{k+1} - \bar{F}_{1, k})$ , where  
644 the mapping  $(\bar{D}_{11} + \partial\psi_{K_1^*})^{-1}(\cdot)$  is maximal monotone (being the sum of two maximal  
645 monotone mappings satisfying [9, Corollary 24.4 (ii)]), single-valued, well-defined and  
646 Lipschitz continuous, see Proposition B.1. Thus if  $\bar{B}_I = (\bar{B}_I^1, 0)$  with  $\bar{B}_I^1 \in \mathbb{R}^{n_1 \times m_1}$   
647 one obtains  $\bar{B}_I \lambda_{k+1} = \bar{B}_I^1 \lambda_{1, k+1} = \bar{B}_I^1 (\bar{D}_{11} + \partial\psi_{K_1^*})^{-1}(-\bar{C}_1 z_{k+1} - \bar{F}_{1, k})$  in the right-  
648 hand side of (3.5) (c).

649 It may be less restrictive to compute upperbounds using the maximal monotonicity.  
650 To start with let us consider the VI in (4.9) and let us state several features of its  
651 solutions. Here Assumption 7 has to be reinforced a little:

652 ASSUMPTION 12. *The conditions of Assumption 7 hold where  $\tilde{X}_2 \succ 0$  for any*  
 653  *$h \geq 0$ .*

654 This secures that the variable change  $\xi_{k+1} = -\tilde{R}_2^{-1}z_{k+1}$  remains well-defined as  $h \rightarrow$   
 655  $0$ . It is also important that item 1 in Proposition 4.4 holds, so that the VI in (4.9) is  
 656 **equivalent to the MLCP in (3.2) (b) (c).**

657 LEMMA 5.3. *Assume that  $\lambda_{\min}(-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1}) (= \lambda_{\min}(-\tilde{A}))$  is large enough, then*  
 658 *any solution  $\xi_{k+1} = -\tilde{R}_2^{-1}z_{k+1}$  of the VI in (4.9) is upperbounded by an affine function*  
 659 *of  $\|x_k\|$ .*

660 **Proof:** Let us manipulate the VI in (4.9) as follows:

$$(5.10) \quad \begin{aligned} & -\tilde{R}_2\tilde{A}\tilde{R}_2^{-1}\xi_{k+1} + \tilde{R}_2\tilde{E}_k + \tilde{G}_2(t_k, \xi_{k+1}) \in -\partial\tilde{f}_2(\xi_{k+1}) \\ \Leftrightarrow & -\tilde{R}_2\tilde{A}\tilde{R}_2^{-1}(\xi_{k+1} - \xi_0) + \tilde{R}_2\tilde{E}_k + \tilde{G}_2(t_k, \xi_{k+1}) + \zeta_0 - \tilde{R}_2\tilde{A}\tilde{R}_2^{-1}\xi_0 = -\zeta_{k+1} + \zeta_0 \\ & \text{with } \zeta_{k+1} \in \partial\tilde{f}_2(\xi_{k+1}), \zeta_0 \in \partial\tilde{f}_2(\xi_0) \\ \Rightarrow & \langle -\tilde{R}_2\tilde{A}\tilde{R}_2^{-1}(\xi_{k+1} - \xi_0), \xi_{k+1} - \xi_0 \rangle + \langle \tilde{R}_2\tilde{E}_k, \xi_{k+1} - \xi_0 \rangle \\ & + \langle \tilde{G}_2(t_k, \xi_{k+1}) - \tilde{G}_2(t_k, \xi_0), \xi_{k+1} - \xi_0 \rangle + \langle \zeta_0 - \tilde{R}_2\tilde{A}\tilde{R}_2^{-1}\xi_0, \xi_{k+1} - \xi_0 \rangle \\ & + \langle \tilde{G}_2(t_k, \xi_0), \xi_{k+1} - \xi_0 \rangle = -\langle \zeta_{k+1} - \zeta_0, \xi_{k+1} - \xi_0 \rangle \leq 0, \end{aligned}$$

662 where we have denoted  $\tilde{G}_2(t_k, \xi_{k+1})$  to highlight the dependency of this mapping on  $t_k$   
 663 and  $x_k$  through  $\tilde{F}_k$ . We have  $\|\tilde{R}_2\tilde{B}_1(\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(z_1) - \tilde{R}_2\tilde{B}_1(\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(z_2)\| \leq$   
 664  $\eta \|z_1 - z_2\|$  for any  $z_1$  and  $z_2$ , so that  $\|\tilde{G}_2(t_k, \xi_{k+1}) - \tilde{G}_2(t_k, \xi_0)\| \leq \eta \|\tilde{C}_1\tilde{R}_2^{-1}(\xi_{k+1} -$   
 665  $\xi_0)\| \leq \eta' \|\xi_{k+1} - \xi_0\|$ , where  $\eta$  depends on  $\tilde{D}_{11}$ ,  $\tilde{R}_2$  and  $\tilde{B}_1$  (see the proof of **Propo-**  
 666 **sition B.1 in [16]**), and  $\eta'$  depends on  $\tilde{D}_{11}$ ,  $\tilde{R}_2$ ,  $\tilde{B}_1$ , and  $\tilde{C}_1$ . Thus we obtain:

$$(5.11) \quad \begin{aligned} & \langle -\tilde{R}_2\tilde{A}\tilde{R}_2^{-1}(\xi_{k+1} - \xi_0), \xi_{k+1} - \xi_0 \rangle \leq \langle -\tilde{R}_2\tilde{E}_k, \xi_{k+1} - \xi_0 \rangle - \langle \tilde{G}_2(t_k, \xi_0), \xi_{k+1} - \xi_0 \rangle \\ & + \langle \zeta_0 + \tilde{R}_2\tilde{A}\tilde{R}_2^{-1}\xi_0, \xi_{k+1} - \xi_0 \rangle + \eta' \|\xi_{k+1} - \xi_0\|^2 \end{aligned}$$

668 which implies

$$(5.12) \quad \begin{aligned} & \langle (-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1} - \eta'I_{n_2})(\xi_{k+1} - \xi_0), \xi_{k+1} - \xi_0 \rangle \leq \|\tilde{R}_2\tilde{E}_k\| \|\xi_{k+1} - \xi_0\| \\ & + \|\tilde{G}_2(t_k, \xi_0) \pm \tilde{G}_2(t_0, \xi_0)\| \|\xi_{k+1} - \xi_0\| + \|\zeta_0 + \tilde{R}_2\tilde{A}\tilde{R}_2^{-1}\xi_0\| \|\xi_{k+1} - \xi_0\|. \end{aligned}$$

670 Now one has:  $\|\tilde{E}_k\| \leq \|E_{II,k} + hA_{II}E_{I,k}\| + \|A_{II}(P_I^h)^{-1}P_I\| \|x_k\|$ ,  $\tilde{G}_2(t_k, \xi_0) =$   
 671  $\tilde{R}_2\tilde{B}_1(\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(\tilde{C}_1\tilde{R}_2^{-1}\xi_0 - \tilde{F}_{1,k})$ ,  $\|\tilde{G}_2(t_k, \xi_0) - \tilde{G}_2(t_0, \xi_0)\| = \|\tilde{R}_2\tilde{B}_1(\tilde{D}_{11} +$   
 672  $\partial\psi_{K_1^*})^{-1}(\tilde{C}_1\tilde{R}_2^{-1}\xi_0 - \tilde{F}_{1,k}) - \tilde{R}_2\tilde{B}_1(\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(\tilde{C}_1\tilde{R}_2^{-1}\xi_0 - \tilde{F}_{1,0})\| \leq \eta \|\tilde{F}_{1,k} +$   
 673  $\tilde{F}_{1,0}\| \leq \eta \|F_k + hC_I(P_I^h)^{-1}E_{I,k}\| + \eta \|C_I(P_I^h)^{-1}P_I\| \|x_k\|$ . Let us denote  $\alpha_k =$   
 674  $\|\tilde{R}_2\| \|E_{II,k} + hA_{II}E_{I,k}\| + \eta \|F_k + hC_I(P_I^h)^{-1}E_{I,k}\|$ ,  $\beta_0 = \|\tilde{G}_2(t_0, \xi_0)\| + \|\zeta_0 +$   
 675  $\tilde{R}_2\tilde{A}\tilde{R}_2^{-1}\xi_0\|$ , and  $\gamma = \eta \|C_I(P_I^h)^{-1}P_I\| + \|\tilde{R}_2\| \|A_{II}(P_I^h)^{-1}P_I\|$ . Then:

$$(5.13) \quad \begin{aligned} & \langle (-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1} - \eta'I_{n_2})(\xi_{k+1} - \xi_0), \xi_{k+1} - \xi_0 \rangle \leq (\alpha_k + \beta_0 + \gamma \|x_k\|) \|\xi_{k+1} - \xi_0\| \\ & \leq \frac{1}{2}\alpha_k^2 + \frac{1}{2}\beta_0^2 + \frac{1}{2}\gamma^2 \|x_k\|^2 + \frac{1}{2}\|\xi_{k+1} - \xi_0\|^2 \end{aligned}$$

677 which implies:

$$(5.14) \quad \langle (-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1} - \eta'I_{n_2} - \frac{1}{2}I_{n_2})(\xi_{k+1} - \xi_0), \xi_{k+1} - \xi_0 \rangle \leq \frac{1}{2}\alpha_k^2 + \frac{1}{2}\beta_0^2 + \frac{1}{2}\gamma^2 \|x_k\|^2.$$

679 Therefore if  $-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1} - \eta'I_{n_2} - \frac{1}{2}I_{n_2} \succ 0$ , one obtains

$$(5.15) \quad \begin{aligned} \|\xi_{k+1} - \xi_0\| & \leq \frac{1}{\sqrt{\lambda_{\min}(-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1} - \eta'I_{n_2} - \frac{1}{2}I_{n_2})}} \sqrt{\frac{1}{2}\alpha_k^2 + \frac{1}{2}\beta_0^2 + \frac{1}{2}\gamma^2 \|x_k\|^2} \\ & \leq \frac{1}{\sqrt{2}\sqrt{\lambda_{\min}(-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1} - \eta'I_{n_2} - \frac{1}{2}I_{n_2})}} \left( \sqrt{\alpha_k^2 + \beta_0^2} + \gamma \|x_k\| \right) \end{aligned}$$

681 which ends the proof.  $\square$

682 Therefore the term  $hB_I z_{k+1}$  in the right-hand side of (3.5) (c), admits also an affine  
683 upperbound in  $\|x_k\|$ . A further characterization of the iterates is given now.

684 LEMMA 5.4. *Assume that  $-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \succ \frac{3}{2} I_{n_2}$ . Assume also that the mapping  
685  $\tilde{G}_2(\cdot)$  is monotone (see Lemma 4.5 for conditions), and that  $E_I(\cdot)$ ,  $E_{II}(\cdot)$  and  $F(\cdot)$   
686 are uniformly bounded and Lipschitz continuous. Then for any  $k$  and  $p \geq 0$  and any  
687 solution of the VI in (4.9), one has:*

$$688 \quad (5.16) \quad \|\xi_k - \xi_p\| \leq \alpha + \beta \|x_{k-1} - x_{p-1}\|$$

689 for some bounded constants  $\alpha \geq 0$  and  $\beta \geq 0$ , and  $\alpha$  is proportional to  $(t_{k-1} - t_{p-1})$ .

690 **Proof:** From

$$691 \quad (5.17) \quad \begin{cases} -\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_k + \tilde{R}_2 \tilde{E}_{k-1} + \tilde{G}_2(t_{k-1}, \xi_k) = -\zeta_k \\ -\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_p + \tilde{R}_2 \tilde{E}_{p-1} + \tilde{G}_2(t_{p-1}, \xi_p) = -\zeta_p \end{cases}$$

692 with  $\zeta_k \in \partial f_2(\xi_k)$ ,  $\zeta_p \in \partial f_2(\xi_p)$ , it follows that

$$693 \quad (5.18) \quad \begin{aligned} & \langle -\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} (\xi_k - \xi_p), \xi_k - \xi_p \rangle + \langle \tilde{R}_2 (\tilde{E}_{k-1} - \tilde{E}_{p-1}), \xi_k - \xi_p \rangle \\ & + \langle \tilde{G}_2(t_{k-1}, \xi_k) - \tilde{G}_2(t_{k-1}, \xi_p) + \tilde{G}_2(t_{k-1}, \xi_p) - \tilde{G}_2(t_{p-1}, \xi_p), \xi_k - \xi_p \rangle \\ & = -\langle \zeta_k - \zeta_p, \xi_k - \xi_p \rangle \end{aligned}$$

694 which implies

$$695 \quad (5.19) \quad \begin{aligned} & \langle -\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} (\xi_k - \xi_p), \xi_k - \xi_p \rangle + \langle \tilde{R}_2 (\tilde{E}_{k-1} - \tilde{E}_{p-1}), \xi_k - \xi_p \rangle \\ & + \langle \tilde{G}_2(t_{k-1}, \xi_p) - \tilde{G}_2(t_{p-1}, \xi_p), \xi_k - \xi_p \rangle \leq 0. \end{aligned}$$

696 One has:

$$697 \quad (5.20) \quad \begin{aligned} & \langle -\tilde{R}_2 (\tilde{E}_{k-1} - \tilde{E}_{p-1}), \xi_k - \xi_p \rangle \\ & \leq \langle -\tilde{R}_2 (E_{II,k-1} - E_{II,p-1}), \xi_k - \xi_p \rangle - \langle \tilde{R}_2 (hA_{II}(E_{I,k-1} - E_{I,p-1})), \xi_k - \xi_p \rangle \\ & \quad - \langle A_{II}(P_I^h)^{-1} P_I (x_{k-1} - x_{p-1}), \xi_k - \xi_p \rangle \\ & \leq \frac{1}{2} \|\tilde{R}_2 ((E_{II,k-1} - E_{II,p-1}) + hA_{II}(E_{I,k-1} - E_{I,p-1}))\|^2 + \frac{1}{2} \|\xi_k - \xi_p\|^2 \\ & \quad + \frac{1}{2} \|A_{II}(P_I^h)^{-1} P_I\|^2 \|x_{k-1} - x_{p-1}\|^2 + \frac{1}{2} \|\xi_k - \xi_p\|^2 \\ & \leq \|\tilde{R}_2\|^2 \|E_{II,k-1} - E_{II,p-1}\|^2 + \|\xi_k - \xi_p\|^2 + h \|A_{II}\|^2 \|E_{I,k-1} - E_{I,p-1}\|^2 \\ & \quad + \frac{1}{2} \|A_{II}(P_I^h)^{-1} P_I\|^2 \|x_{k-1} - x_{p-1}\|^2 \leq \alpha' + \beta' \|x_{k-1} - x_{p-1}\|^2 + \|\xi_k - \xi_p\|^2, \end{aligned}$$

698 for some  $\alpha' \geq 0$  proportional to  $(t_{k-1} - t_{p-1})^2$  since  $E(\cdot)$  is Lipschitz continuous,  
699  $\beta' \geq 0$ , and it was used that  $x^\top y \leq \|x\| \|y\| \leq \frac{1}{2} (\|x\|^2 + \|y\|^2)$ . Also:

$$700 \quad (5.21) \quad \begin{aligned} & \langle -\tilde{G}_2(t_{k-1}, \xi_p) + \tilde{G}_2(t_{p-1}, \xi_p), \xi_k - \xi_p \rangle \\ & \leq \|\tilde{G}_2(t_{k-1}, \xi_p) - \tilde{G}_2(t_{p-1}, \xi_p)\| \|\xi_k - \xi_p\| \\ & \leq \frac{1}{2} k_{\tilde{G}_2} \|\tilde{F}_{1,k-1} - \tilde{F}_{p-1}\|^2 + \frac{1}{2} \|\xi_k - \xi_p\|^2 \\ & \leq k_{\tilde{G}_2} \|F_{k-1} - F_{p-1}\|^2 + h k_{\tilde{G}_2} \|C_I (P_I^h)^{-1} (E_{I,k-1} - E_{I,p-1})\|^2 \\ & \quad + k_{\tilde{G}_2} \|C_I (P_I^h)^{-1} P_I (x_{k-1} - x_{p-1})\|^2 + \frac{1}{2} \|\xi_k - \xi_p\|^2 \\ & \leq \alpha'' + \beta'' \|x_{k-1} - x_{p-1}\|^2 + \frac{1}{2} \|\xi_k - \xi_p\|^2, \end{aligned}$$

701 for some  $\alpha'' \geq 0$  proportional to  $(t_{k-1} - t_{p-1})^2$  (since  $E(\cdot)$  and  $F(\cdot)$  are Lipschitz  
702 continuous), and  $\beta'' \geq 0$ . Hence using (5.19):

$$703 \quad (5.22) \quad \begin{aligned} \|\xi_k - \xi_p\| & \leq \frac{1}{\sqrt{\lambda_{\min}(-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} - \frac{3}{2} I_p)}} \sqrt{\alpha' + \alpha'' + (\beta' + \beta'') \|x_{k-1} - x_{p-1}\|^2} \\ & \leq \frac{1}{\sqrt{\lambda_{\min}(-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} - \frac{3}{2} I_p)}} (\sqrt{\alpha' + \alpha''} + \sqrt{\beta' + \beta''} \|x_{k-1} - x_{p-1}\|) \end{aligned}$$

704 and the lemma is proved.  $\square$

705

706 **COROLLARY 5.5.** *Let  $h > 0$ . Assume that the conditions and assumptions of*  
 707 *Lemma 5.4 are fulfilled. For all  $k \geq 1$  and any solution of the VI in (4.9), one has:*

$$708 \quad (5.23) \quad \left\| \frac{\xi_{k+1} - \xi_k}{h} \right\| \leq \alpha + \beta \left\| \frac{x_k - x_{k-1}}{h} \right\|$$

709 *for some bounded constants  $\alpha \geq 0$  and  $\beta \geq 0$ .*

710 **Proof:** Let us consider the proof of Lemma 5.4, replacing  $k$  by  $k + 1$  and  $p$  by  $k - 1$ .  
 711 Then  $\alpha$  in (5.16) is proportional to  $h$ . Redoing the calculations in (5.20) and (5.22),  
 712 one obtains (5.23).  $\square$

713 Clearly for all  $h > 0$  (5.23) can be written equivalently as:  $\|\xi_{k+1} - \xi_k\| \leq h\alpha +$   
 714  $\beta\|x_k - x_{k-1}\|$ . The bounds calculated for the variable  $\xi_k$  also hold for the variable  
 715  $z_k = -\tilde{R}_2 \xi_k$ . Let us now analyse  $\lambda_{k+1}$  in (3.5) (a). As seen above,  $\lambda_{1,k+1} = (\tilde{D}_{11} +$   
 716  $\partial\psi_{K_1^*})^{-1}(-\tilde{C}_1 z_{k+1} - \tilde{F}_{1,k})$  and  $\lambda_{2,k+1} \in -\mathcal{N}_{K_2}(\tilde{C}_2 z_{k+1} + \tilde{F}_{2,k})$ . If the matrix  $\tilde{B}_I$  has  
 717 the above structure (i.e.,  $\tilde{B}_I = (\tilde{B}_I^1, 0)$ ), then the term  $\tilde{B}_I \lambda_{k+1} = \tilde{B}_I^1$  is Lipschitz  
 718 continuous with respect to the variables  $z_{k+1}$ ,  $x_k$  and  $F_k$ . In a more general setting  
 719 one obtains:

$$720 \quad (5.24) \quad P_I^h x_{k+1} \in P_I x_k + h B_I z_{k+1} + h \tilde{B}_I^1 (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1} (-\tilde{C}_1 z_{k+1} - \tilde{F}_{1,k}) \\ - h \tilde{B}_I^2 \mathcal{N}_{K_2}(\tilde{C}_2 z_{k+1} + \tilde{F}_{2,k}).$$

721 The difficulty is that the last term in the right-hand side of (5.24) is set-valued in  
 722 general, and this is intrinsically due to the structure of the implicit scheme in (2.1).  
 723 A solution is to consider the VI in (4.9). Its right-hand side matches with the last  
 724 term in the right-hand side of (5.24) and it is set-valued. However provided the VI  
 725 is well-posed, we know that one element of this set is equal to the left-hand side of  
 726 (4.9). We may assume that the scheme in (5.24) is implemented with this element.  
 727 This element appears to define a Lipschitz continuous function of  $\xi_{k+1}$  (let us recall  
 728 that the Lipschitz constant of the third term is equal to  $\frac{1}{\lambda_{\min}(\tilde{D}_{11})}$ , see Proposition  
 729 B.1). If this element is chosen then the numerical scheme (5.24) becomes:

$$730 \quad (5.25) \quad P_I^h x_{k+1} = P_I x_k + h B_I z_{k+1} + h \tilde{B}_I^1 (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1} (\tilde{C}_1 z_{k+1} - \tilde{F}_{1,k}) \\ + h \tilde{B}_I^2 (-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_{k+1} - \tilde{R}_2 \tilde{E}_k + \tilde{G}_2(t_k, \xi_{k+1})).$$

731 Let us now study (5.25), using Lemmata 5.3, 5.4, Corollary 5.5. It is noteworthy that  
 732 the constant  $\alpha$  in both Lemmata and in Corollary can be made arbitrarily small by  
 733 increasing  $-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1}$  arbitrarily (in the set of positive definite matrices).

734 **PROPOSITION 5.6.** *Assume that the conditions of Lemmata 5.3, 5.4 and of Corol-*  
 735 *lary 5.5 are verified,  $E_I(\cdot)$ ,  $E_{II}(\cdot)$  and  $F(\cdot)$  are bounded Lipschitz continuous func-*  
 736 *tions of time, and Assumption 13 below is satisfied. Then the sequences  $\{x^h\}_{h \geq 0}$  and*  
 737  *$\{\dot{x}^h\}_{h \geq 0}$  in (5.7) are uniformly bounded in  $h$  on  $[0, T]$ . Thus  $\{x^h\}_{h \geq 0}$  is equi-Lipschitz*  
 738 *continuous and there is a subsequence of it that converges uniformly in  $\mathcal{C}^0([0, T]; \mathbb{R}^{n_1})$*   
 739 *towards a Lipschitz continuous limit denoted as  $x(\cdot)$ .*

740 **Proof:** In the proof, use is made of positive bounded constants denoted generically as  
 741  $k_i$ . Consider (5.25). First recall that the operator  $\zeta \mapsto (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(\zeta)$  is Lipschitz

742 continuous with Lipschitz constant  $\frac{1}{\lambda_{\min}(\tilde{D}_{11})}$ . Hence we obtain:

$$\begin{aligned}
& \|(P_I^h)^{-1} \tilde{B}_I^1 (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1} (-\tilde{C}_1 z_{k+1} - \tilde{F}_{1,k}) \\
& \quad - (P_I^h)^{-1} \tilde{B}_I^1 (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1} (-\tilde{C}_1 z_{p+1} - \tilde{F}_{1,p})\| \\
743 \quad (5.26) \quad & \leq k_1 \|\tilde{C}_1(z_{k+1} - z_{p+1} + \tilde{F}_{1,k} - \tilde{F}_{1,p})\| \leq k_2 \|z_{k+1} - z_{p+1}\| + k_3 \|\tilde{F}_{1,k} - \tilde{F}_{1,p}\| \\
& \leq k_4 \|\xi_{k+1} - \xi_{p+1}\| + k_4 \|F_{1,k} - F_{1,p}\| + k_5 \|E_{I,k} - E_{I,p}\| + k_6 \|x_k - x_p\| \\
& \leq k_4(\alpha + \beta \|x_k - x_p\|) + k_7 |t_k - t_p| + k_6 \|x_k - x_p\| \leq k_8 + k_9 \|x_k - x_p\| \\
& \quad + k_7 |t_k - t_p|,
\end{aligned}$$

744 for any  $k$  and  $p \geq 0$ , where we used Lemma 5.4 and the Lipschitzness property. In a  
745 quite similar way one has  $\|(P_I^h)^{-1} \tilde{B}_I^2 (\tilde{G}_2(\xi_{k+1}) - \tilde{G}_2(\xi_{p+1}))\| \leq k_8 + k_9 \|x_k - x_p\| +$   
746  $k_7 |t_k - t_p|$ , where  $k_7, k_8$  and  $k_9$  depend on the Lipschitz constants of  $\tilde{G}_2(\cdot, \cdot)$ ,  $F(\cdot)$ , and  
747 on  $\|(P_I^h)^{-1} \tilde{B}_I^2\|$ . One also has  $\|(P_I^h)^{-1} (B_I + \tilde{B}_I^2 (\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1})(z_{k+1} - z_{p+1}))\| \leq k_{10}(\alpha +$   
748  $\beta \|x_k - x_p\|) \leq k_{11} + k_{12} \|x_k - x_p\|$ , and  $\|\tilde{B}_I^2 \tilde{R}_2 (\tilde{E}_k - \tilde{E}_p)\| \leq k_{13} |t_k - t_p| + k_{14} \|x_k - x_p\|$ .  
749 It is deduced that  $\|x_{k+1} - x_{p+1}\| \leq (\|(P_I^h)^{-1} P_I\| + hk_{15}) \|x_k - x_p\| + hk_{16} |t_k - t_p| + hk_{17}$ ,  
750 where the term  $k_{17}$  stems from  $k_8 + k_{11} = \alpha(k_4 + k_{10})$ . Here we are led to make the  
751 following which is more stringent than Assumption 11:

752 ASSUMPTION 13. *There exists  $h_{\max} > 0$  such that  $\|(P_I^h)^{-1} P_I\| + hk_{15} < 1$  for all*  
753  $0 < h \leq h_{\max}$ .

754 Obviously for  $h = 0$  and the Hölder-induced norm one has  $\|(P_I^h)^{-1} P_I\|_{2,2} + hk_{15} = 1$ .  
755 Let us choose  $p = 0$ , it follows that  $\|x_{k+1} - x_1\| \leq (\|(P_I^h)^{-1} P_I\| + hk_{15}) \|x_k -$   
756  $x_0\| + hk_{16} T + hk_{17}$ , hence  $\|x_{k+1}\| \leq (\|(P_I^h)^{-1} P_I\| + hk_{15}) \|x_k\| + hk_{16} T + hk_{17} +$   
757  $\|x_1\| + (\|(P_I^h)^{-1} P_I\| + hk_{15}) \|x_0\|$ . Now it has to be proved that  $\|x_1\|$  is bounded.  
758 From Lemma 5.3 and (5.15),  $\|\xi_1\|$  and  $\|z_1\|$  are upperbounded by an affine function of  
759 initial data  $\|x_0\|$  and  $\|z_0\|$ . The operator  $(\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(\cdot)$  is well-defined and single-  
760 valued. Taking  $k = 0$  in (5.25) one infers that indeed  $\|x_1\|$  is bounded for bounded  
761 initial data. Consequently it follows that  $\|x_{k+1}\| \leq (\|(P_I^h)^{-1} P_I\| + hk_{15}) \|x_k\| +$   
762  $hk_{16} T + hk_{17} + hf(x_0, z_0) + \|(P_I^h)^{-1} P_I\| \|x_0\|$  for some function  $f(\cdot)$  that is bounded  
763 for bounded initial data. To simplify the notations let us set  $h_1 \triangleq \|(P_I^h)^{-1} P_I\| + hk_{15}$   
764 and  $h_2 \triangleq k_{16} T + k_{17} + f(x_0, z_0)$ . We obtain for all  $k \geq 1$ :

$$\begin{aligned}
765 \quad (5.27) \quad \|x_k\| & \leq h_1^k \|x_0\| + \sum_{j=0}^{k-1} h_1^j (hh_2 + \|(P_I^h)^{-1} P_I\| \|x_0\|) \\
& \leq h_1^k \|x_0\| + (hh_2 + \|(P_I^h)^{-1} P_I\| \|x_0\|) \frac{1-h_1^{k+1}}{1-h_1}.
\end{aligned}$$

766 Thus in view of Assumption 13 the sequence  $\{x_k\}_{k \geq 0}$  generated by (5.25) is uniformly  
767 bounded for bounded initial data.

768 Let us now reconsider the inequality  $\|x_{k+1} - x_{p+1}\| \leq (\|(P_I^h)^{-1} P_I\| + hk_{15}) \|x_k -$   
769  $x_p\| + hk_{16} |t_k - t_p| + hk_{17}$  obtained above. Setting  $p = k - 1$  yields  $\|x_{k+1} - x_k\| \leq$   
770  $(\|(P_I^h)^{-1} P_I\| + hk_{15}) \|x_k - x_{k-1}\| + h^2 k_{16} + hk_{17} \Rightarrow \frac{\|x_{k+1} - x_k\|}{h} \leq (\|(P_I^h)^{-1} P_I\| +$   
771  $hk_{15}) \frac{\|x_k - x_{k-1}\|}{h} + hk_{16} + k_{17}$ . Now using Lemma 5.4 and the Lipschitzness it follows  
772 that  $\alpha$ , hence  $k_{17}$ , are proportional to  $k_E^2 h^2$ . We may therefore write the inequality  
773 as  $\frac{\|x_{k+1} - x_k\|}{h} \leq (\|(P_I^h)^{-1} P_I\| + hk_{15}) \frac{\|x_k - x_{k-1}\|}{h} + hk_{16} + k_{18} h^2$  for some  $k_{18} \geq 0$ .  
774 Proceeding as for (5.27) one infers that if  $\frac{\|x_1 - x_0\|}{h}$  is bounded then  $\frac{\|x_{k+1} - x_k\|}{h}$  is  
775 bounded for all  $k \geq 1$ . Evaluating (5.25) at  $k = 0$  and using the boundedness of  $x_1$   
776 as shown above, proves that this is indeed the case. The proposition is proved using  
777 the Arzela-Ascoli Theorem.  $\square$

778 Clearly in practice one needs to compute the constants  $k_i$ , for instance  $k_{15}$  is needed  
779 to determine  $h_{\max}$  in Assumption 13.

780 **COROLLARY 5.7.** *Under the conditions of Proposition 5.6, the sequences  $\{z^h\}_{h \geq 0}$*   
 781 *and  $\{z^h\}_{h \geq 0}$  (respectively  $\{\xi^h\}_{h \geq 0}$  and  $\{\xi^h\}_{h \geq 0}$ ) are uniformly bounded in  $h$  on  $[0, T]$ ,*  
 782  *$0 < T < +\infty$ . Thus  $\{z^h\}_{h \geq 0}$  (respectively  $\{\xi^h\}_{h \geq 0}$ ) is equi-Lipschitz continuous and*  
 783 *there is a subsequence of it that converges uniformly in  $C^0([0, T]; \mathbb{R}^{n_2})$  towards a*  
 784 *Lipschitz continuous limit denoted as  $z(\cdot)$  (respectively  $\xi(\cdot)$ ).*

785 **Proof:** This follows from Proposition 5.6 and using **Lemmata 5.3, 5.4**, Corollary 5.5.  
 786  $\square$

787 Let us now pass to the analysis of the multiplier convergence. Recall that  $\lambda_{1,k+1} =$   
 788  $(\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(-\tilde{C}_1 z_{k+1} - \tilde{F}_{1,k})$  and  $\lambda_{2,k+1} = -\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_{k+1} + \tilde{R}_2 \tilde{E}_k + \tilde{G}_2(\xi_{k+1}) \in$   
 789  $-\mathcal{N}_{K_2}(\tilde{C}_2 z_{k+1} + \tilde{F}_{2,k})$ , therefore for all  $t \geq 0$ ,  $\lambda_{1,\star}^h(t) = (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(-\tilde{C}_1 z_\star^h(t) -$   
 790  $\tilde{F}_{1,\star}^h(t))$  and  $\lambda_{2,\star}^h(t) = -\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_\star^h(t) + \tilde{R}_2 \tilde{E}_\star^h(t) + \tilde{G}_2(\xi_\star^h(t)) \in -\mathcal{N}_{K_2}(\tilde{C}_2 z_\star^h(t) +$   
 791  $\tilde{F}_{2,\star}^h(t))$ , where we recall that the step-functions are defined as  $f_\star^h(t) = f_{k+1}$  for  
 792  $t \in [t_k, t_{k+1})$ . We know that  $\dot{\xi}_k \triangleq \|\frac{\xi_{k+1} - \xi_k}{h}\| \leq C$  for some  $C > 0$ , hence:

$$793 \quad (5.28) \quad \|\xi^h - \xi_\star^h\|_{\mathcal{L}_2([0, T]; \mathbb{R}^{n_2})}^2 \leq \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_{k+1} - t_k)^2 \|\dot{\xi}_k\|^2 dt$$

$$\leq C^2 \sum_{k=0}^{n-1} (t_{k+1} - t_k)^3 = \frac{C^2 n h^3}{3} = \frac{C^2 T h^2}{3}.$$

794 Consequently  $\|\xi^h - \xi_\star^h\|_{\mathcal{L}_2([0, T]; \mathbb{R}^{n_2})} \rightarrow 0$  as  $h \rightarrow 0$ . Since  $\|\xi^h - \xi_\star^h\|_{\mathcal{L}_2([0, T]; \mathbb{R}^{n_2})} =$   
 795  $\|\xi^h \pm \xi - \xi_\star^h\|_{\mathcal{L}_2([0, T]; \mathbb{R}^{n_2})} \rightarrow \|\xi - \xi_\star^h\|_{\mathcal{L}_2([0, T]; \mathbb{R}^{n_2})} \rightarrow 0$ , we conclude that the sequence  
 796  $\{\xi_\star^h\}_{h \geq 0}$  converges strongly in  $\mathcal{L}_2([0, T]; \mathbb{R}^{n_2})$  towards the limit  $\xi(\cdot)$ . The same result  
 797 holds for the sequence  $\{x_\star^h\}_{h \geq 0}$  with the limit  $x(\cdot)$ . Therefore the sequence  $\{-\tilde{C}_1 z_\star^h -$   
 798  $\tilde{F}_{1,\star}^h\}_{h \geq 0}$  also converges strongly in  $\mathcal{L}_2$  towards a limit, and we infer that the same  
 799 conclusion holds for both  $\{\lambda_{1,\star}^h(\cdot)\}_{h \geq 0}$  and  $\{\lambda_{2,\star}^h(\cdot)\}_{h \geq 0}$ . Since  $\mathcal{N}_{K_2}(\cdot)$  is maximal  
 800 monotone, it follows from [9, Proposition 20.33] that the limit  $\lambda_2(\cdot)$  satisfies  $\lambda_2(t) \in$   
 801  $-\mathcal{N}_{K_2}(C_{II,2} z(t) + F_2(t) + C_{I,2} x(t))$ . Thus we proved the following result.

802 **LEMMA 5.8.** *Let the conditions of **Lemmata 5.3, 5.4**, Corollary 5.5 be satisfied for*  
 803 *all  $h > 0$ . The sequences  $\{\xi_\star^h\}_{h \geq 0}$ ,  $\{x_\star^h\}_{h \geq 0}$ ,  $\{\lambda_{1,\star}^h(\cdot)\}_{h \geq 0}$  and  $\{\lambda_{2,\star}^h(\cdot)\}_{h \geq 0}$  converge*  
 804 *strongly in  $\mathcal{L}_2([0, T]; \mathbb{R}^{n_2})$  towards the limits  $\xi(\cdot)$ ,  $x(\cdot)$ ,  $\lambda_1(\cdot)$  and  $\lambda_2(\cdot)$ , respectively.*

805 To finish this section, one can consider again (5.8) and the analysis which follows it,  
 806 to obtain a proposition similar to Proposition 5.2, relying on Lemma 5.8, Proposition  
 807 5.6, and on Corollary 5.5. It is noteworthy that the three fundamental assumptions  
 808 in this section, are that  $-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1}$  be sufficiently large as a positive definite matrix,  
 809 that a suitable selection is chosen for the multiplier  $\lambda_{2,k}$ , and that a coupling as stated  
 810 in Assumptions 2 or 7 exists.

811 **PROPOSITION 5.9.** *Assume that the conditions of Proposition 5.6, Corollary 5.7*  
 812 *and Lemma 5.8 are verified. Then the limit functions  $x(\cdot)$ ,  $z(\cdot)$ ,  $\lambda(\cdot)$ , are solutions of*  
 813 *the DALCS (2.1).*

814 **Proof.** Let us consider (5.8). From Lemma 5.8,  $\{x_\star^h(\cdot)\}$ ,  $\{z_\star^h(\cdot)\}$ ,  $\{\lambda_\star^h(\cdot)\}$  converge  
 815 strongly in  $\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})$  towards limits  $x(\cdot)$ ,  $z(\cdot)$  and  $\lambda(\cdot)$ . Using the same argu-  
 816 ments as those developed before Proposition 5.2, it is inferred from (5.8) (b) (c) that  
 817 the limits verify (2.1) (b) (c). Using the same arguments as those developed just above  
 818 (5.8), it is deduced that  $\{\dot{x}^h(\cdot)\}$  converges weakly\* to a limit  $v(\cdot)$  where  $v(\cdot) = \dot{x}(\cdot)$   
 819 almost everywhere. Consequently  $P_I \dot{x}^h(t) = A_I x_\star^h(t) + B_I z_\star^h(t) + \bar{B}_I \lambda_\star^h(t) + E_I^h(t) \rightarrow$   
 820  $P_I \dot{x}(t) = A_I x(t) + B_I z(t) + \bar{B}_I \lambda(t) + E_I(t)$  weakly in  $\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})$  as  $h \rightarrow 0$ .  $\square$

821 **5.2. Convergence results from (3.1).** In this section the properties of the  
 822 plant's model (2.1) are used, thus the numerical scheme in (3.1) and the results in

823 section 4.2 are analysed. This section is devoted to the case  $D \succcurlyeq 0$  as in Proposition  
 824 4.8. Indeed, since the strict passivity of  $(B_{II}, \bar{B}_{II}, C_{II}, D)$  implies Assumption 10 for  
 825 small enough  $h > 0$ , it is unnecessary to repeat the developments in sections 5.1.1 and  
 826 5.1.2. However, it is not possible in general to determine the implications between  
 827 Assumption 7 and Assumption 9, nor the relations between  $\lambda_{\min}(\tilde{A})$  and  $\lambda_{\min}(B_{II})$   
 828 for  $h > 0$ . Therefore let us deal with the VI in (4.17) where  $x_{k+1}$  is replaced by  
 829  $x_{k+\theta} = \theta x_k + (1 - \theta)x_{k+1}$ ,  $\theta \in [0, 1]$ . The following holds, which is the counterpart of  
 830 Lemma 5.3.

831 **LEMMA 5.10.** *Assume that  $\lambda_{\min}(-R_2 B_{II} R_2^{-1})$  ( $= \lambda_{\min}(-B_{II})$ ) is large enough,  
 832  $E_I(\cdot)$ ,  $E_{II}(\cdot)$  and  $F(\cdot)$  are uniformly bounded and Lipschitz continuous functions of  
 833 time. Then any solution  $\xi_{k+1} = R_2^{-1} z_{k+1}$  of the VI in (4.17) is upperbounded by an  
 834 affine function of  $\|x_{k+\theta}\|$ .*

835 The proof follows the same steps as the proof of Lemma 5.3 and it is therefore omitted.  
 836 Lemma 5.10 implies that the term  $B_{II} z_{k+1}$  in the right-hand side of (3.1) (a), is  
 837 upperbounded by an affine function of  $\|x_{k+\theta}\|$ . The coefficients of the affine function  
 838 depend only on the system's data (constant matrices, Lipschitz constants) and on the  
 839 initial conditions. Let us now propose the counterpart of Lemma 5.4.

840 **LEMMA 5.11.** *Assume that  $-R_2 B_{II} R_2^{-1} \succcurlyeq I_{n_2}$ . Assume also that the well-defined,  
 841 single-valued and Lipschitz continuous mapping  $G_2(\cdot)$  is monotone, and that  $E_I(\cdot)$ ,  
 842  $E_{II}(\cdot)$  and  $F(\cdot)$  are Lipschitz continuous functions of time. Then for any  $k$  and  $p \geq 0$   
 843 and any solution of the VI in (4.17), one has:*

$$844 \quad (5.29) \quad \|\xi_k - \xi_p\| \leq \alpha + \beta \|x_{k-1+\theta} - x_{p-1+\theta}\|$$

845 for all  $k \geq 1$  and some bounded constants  $\alpha \geq 0$  (proportional to  $|t_{k-1} - t_{p-1}|$ ) and  
 846  $\beta \geq 0$ .

847 **Proof:** The proof is similar to the proof of Lemma 5.4, since both VIs in (4.17) and  
 848 (4.9) possess similar structures. Using (4.17) one finds:

$$849 \quad (5.30) \quad \begin{cases} -R_2 B_{II} R_2^{-1} \xi_{k+1} - R_2 A_{II} x_{k+\theta} - R_2 E_{II,k} + G_2(t_k, \xi_{k+1}) = -\zeta_{k+1} \\ -R_2 B_{II} R_2^{-1} \xi_{p+1} - R_2 A_{II} x_{p+\theta} - R_2 E_{II,p} + G_2(t_p, \xi_{p+1}) = -\zeta_{p+1} \end{cases}$$

850 with  $\zeta_k \in \partial f_2(\xi_k)$ ,  $k \geq 0$ . Using the monotonicity of  $G_2(t_k, \cdot)$  and of  $\partial f_2(\cdot)$  it follows  
 851 that:

$$852 \quad (5.31) \quad \begin{aligned} & \langle -R_2 B_{II} R_2^{-1} (\xi_{k+1} - \xi_{p+1}), \xi_{k+1} - \xi_{p+1} \rangle \\ & \leq \langle R_2 A_{II} (x_{k+\theta} - x_{p+\theta}) + R_2 (E_{II,k} - E_{II,p}), \xi_{k+1} - \xi_{p+1} \rangle \\ & \quad - \langle G_2(t_k, \xi_{k+1}) - G_2(t_k, \xi_{p+1}), \xi_{k+1} - \xi_{p+1} \rangle \\ & \quad + \langle G_2(t_k, \xi_{p+1}) - G_2(t_p, \xi_{p+1}), \xi_{k+1} - \xi_{p+1} \rangle \\ & \leq \|R_2 A_{II}\|^2 \|x_{k+\theta} - x_{p+\theta}\|^2 + \|R_2\|^2 k_E^2 (t_k - t_p)^2 \\ & \quad + \frac{1}{2} k_{G_2}^2 \|F_k - F_p\|^2 + \|\xi_{k+1} - \xi_{p+1}\|^2 \\ & \leq \|R_2 A_{II}\|^2 \|x_{k+\theta} - x_{p+\theta}\|^2 + \|R_2\|^2 k_E^2 (t_k - t_p)^2 + \frac{1}{2} k_{G_2}^2 k_F^2 (t_k - t_p)^2 \\ & \quad + \|\xi_{k+1} - \xi_{p+1}\|^2. \end{aligned}$$

853 The lemma is proved.  $\square$

854 Thus Corollary 5.5 extends as well, and in particular one has  $\|\xi_{k+1} - \xi_k\| \leq h\alpha +$   
 855  $\beta \|x_{k+\theta} - x_{k-1+\theta}\|$  for all  $k \geq 1$ , for bounded constants  $\alpha \geq 0$  and  $\beta \geq 0$ . The scheme  
 856 in (5.25) is now replaced by:

$$857 \quad (5.32) \quad \begin{aligned} P_I^h x_{k+1} = & P_I x_k + h B_I z_{k+1} + h \bar{B}_I^1 (D_{11} + \partial \psi_{K_1^*})^{-1} (C_{II}^1 R_2^{-1} \xi_{k+1} + F_{1,k}) \\ & + h \bar{B}_I^2 (-R_2 B_{II} R_2^{-1} \xi_{k+1} - R_2 A_{II} x_{k+\theta} - R_2 E_{II,k} + G_2(t_k, \xi_{k+1})), \end{aligned}$$

858 equivalently:

(5.33)

$$\begin{aligned}
 859 \quad & (P_I^h + h(1 - \theta)\bar{B}_I^2 R_2 A_{II})x_{k+1} = P_I x_k + hB_I z_{k+1} \\
 & + h\bar{B}_I^1 (D_{11} + \partial\psi_{K_1^*})^{-1} (C_{II}^1 R_2^{-1} \xi_{k+1} + F_{1,k}) + h\bar{B}_I^2 (-R_2 B_{II} R_2^{-1} \xi_{k+1} \\
 & - \theta R_2 A_{II} x_k - R_2 \bar{E}_{II,k} + G_2(t_k, \xi_{k+1})),
 \end{aligned}$$

860 and it appears that both schemes in (5.25) and in (5.33) possess the same structure.  
 861 Consequently the results of convergence obtained for the foregoing implicit scheme  
 862 (Proposition 5.6, Corollary 5.7, Lemma 5.8) extend to the scheme in this section, for  
 863 suitably chosen numerical and system parameters. For the sake of brevity we do not  
 864 repeat the developments.

865 *Remark 5.12.* Using Lemma 4.1 item 1, Assumption 10 could be replaced by  
 866 assuming the strict passivity of the quadruple  $(B_{II}, \bar{B}_{II}, C_{II}, D)$  (a system property)  
 867 and  $h > 0$  small enough.

868 *Remark 5.13.* Obviously when  $B_{II}$  is invertible it is possible to recast (2.1) into  
 869 an LCS by solving the equality constraint for  $z$ . This yields the LCS quadruple  
 870  $(P_I^{-1}(A_I - B_I B_{II}^{-1} A_{II}), P_I^{-1}(\bar{B}_I - B_I B_{II}^{-1} \bar{B}_{II}), C_I - C_{II} B_{II}^{-1} A_{II}, D - C_{II} B_{II}^{-1} \bar{B}_{II})$ .  
 871 It is unclear how to relate in general the passivity of this quadruple, to the above  
 872 assumptions which focus on the MLCP part (2.1) (b) (c) of the system. Especially  
 873 no specific assumptions are made on the matrices  $A_I, B_I, \bar{B}_I, C_I$  in the above analy-  
 874 ses. For instance Assumption 3 implies that the feedthrough matrix of the LCS is  
 875  $D - C_{II} B_{II}^{-1} X^{-1} C_{II}^T$ , which may be indefinite in general. It is also noteworthy that  
 876 transforming the MLCP into an LCP may be undesirable [59].

877 **5.3. Recapitulation.** Let us recapitulate the various assumptions upon which  
 878 the results in this article rely. Assumptions 2, 4, 7, 10, 12 (respectively Assumptions  
 879 3, 5, 9), characterize the passivity properties of an auxiliary operator defined by the  
 880 DALCS constraints in (3.2) (b) (c) (respectively in (2.1) (b) (c)). These passivity  
 881 properties constrain the relative degree between the “input”  $\lambda_{k+1}$  and the “output”  
 882  $w_{k+1} = \tilde{C}z_{k+1} + \tilde{D}\lambda_{k+1}$  (respectively  $\lambda$  and  $w = C_{II}z + D\lambda$ ). Assumptions 6 and 8  
 883 have practical importance (because some systems possess such  $D$  matrix), they are  
 884 also related to passivity in the sense that passive systems possess positive semidefinite  
 885 throughout matrices. Thus all these assumptions are fundamental ones for the devel-  
 886 opments in this article, which basically deals with the analysis of two types of gener-  
 887 alised equations (GE):  $0 \in A(u) + B(u)$  in (4.7) and (4.15), and  $0 \in A(u) + B(u) + C(u)$   
 888 in (4.9) and (4.17), where the mappings  $A(\cdot), B(\cdot)$  and  $C(\cdot)$  are maximal monotone  
 889 (notice in passing that due to time-dependency, these GEs do not fit with those in [28,  
 890 Chapter 2], so that in particular [28, Theorem 2F.6] cannot be applied in general).  
 891 Most importantly, the convergence results (Proposition 5.2, Lemmata 5.3 and 5.4,  
 892 Corollary 5.5, Proposition 5.6, Corollary 5.7, Lemmata 5.8 and 5.10) rely upon the  
 893 positive definiteness of  $-R_2 B_{II} R_2^{-1}$  and  $-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1}$ , which is an index one condition  
 894 and guarantees strong monotonicity properties, allowing us to characterize the prop-  
 895 erties of the solutions to the GEs. It would be worth relaxing it in future studies.  
 896 One path could be to use least norm elements [34]. Finally, it is worth noting that  
 897 many practical systems like circuits with nonlinear resistors and inductances contain  
 898 nonlinearities [19, section 4.1]. An extension of the above developments, consists of  
 899 replacing the linear term  $A_I x$  in (2.1), by a nonlinear term  $A_I(x)$ , where  $A_I(\cdot)$  is  
 900 globally Lipschitz continuous, and setting  $A_I(x_k)$  in (3.1). The matrices  $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$   
 901 keep the same structure with  $P_I^{-1}$  instead of  $(P_I^h)^{-1}$ , while both  $\tilde{E}_k$  and  $\tilde{F}_k$  contain



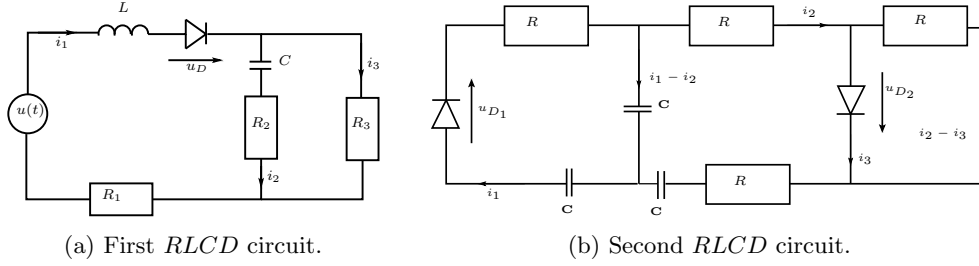


Fig. 1: Two circuits with ideal diodes.

902  $A_I(x_k)$ . Nonlinearities in the equality and in the complementarity constraints should  
 903 also be considered, where the results in [27] could be useful.

904 **6. Examples and Numerical Simulations.** The next examples are taken  
 905 from circuits, mechanics and state-dependent-switch DAEs. They aim at illustrat-  
 906 ing the above developments, and the limitations imposed by some assumptions.

907 **6.1. Example 1 (circuit).** Let us consider the circuit in Figure 1b. Using  
 908 Kirchoff laws its dynamics are given by:

$$909 \quad (6.1) \quad \begin{cases} \dot{x}_1(t) = \frac{-2}{RC}x_1(t) + \frac{1}{RC}x_2(t) + \frac{1}{R}u_{D_1}(t) \\ \dot{x}_2(t) = \frac{3}{RC}x_2(t) + \frac{1}{2RC}x_1(t) + \frac{1}{2R}u_{D_2}(t) \\ 0 = \frac{3}{2}u_{D_2}(t) - Rx_3(t) - \frac{1}{C}x_2(t) + \frac{1}{2C}x_1(t) \\ 0 \leq u_{D_1}(t) \perp \frac{-2}{RC}x_1(t) + \frac{1}{RC}x_2(t) + \frac{1}{R}u_{D_1}(t) \geq 0 \\ 0 \leq u_{D_2}(t) \perp x_3(t) \geq 0, \end{cases}$$

910 with  $x_1(t) = \int_0^t i_1(s)ds$ ,  $x_2(t) = \int_0^t i_2(s)ds$ ,  $x_3 = i_3 = z$ ,  $\lambda = (u_{D_1} \ u_{D_2})^\top$  (the  
 911 voltages across the diodes),  $K = K^* = \mathbb{R}_+^2$ ,  $n_1 = 2$ ,  $p = n_2 = 1$ ,  $m = 2$ . One  
 912 has  $D = \begin{pmatrix} \frac{1}{R} & 0 \\ 0 & 0 \end{pmatrix}$ ,  $P_I = I_2$ ,  $A_I = \begin{pmatrix} \frac{-2}{RC} & \frac{1}{RC} \\ \frac{3}{2RC} & \frac{1}{RC} \end{pmatrix}$ ,  $B_I = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ,  $\bar{B}_I = \begin{pmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{2R} \end{pmatrix}$ ,

913  $A_{II} = \begin{pmatrix} \frac{1}{2C} & -\frac{1}{C} \end{pmatrix}$ ,  $B_{II} = -R$ ,  $\bar{B}_{II} = \begin{pmatrix} 0 & \frac{3}{2} \end{pmatrix}$ ,  $C_I = \begin{pmatrix} \frac{-2}{RC} & \frac{1}{RC} \\ 0 & 0 \end{pmatrix}$ ,  $C_{II} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ .

914 Assumption 8 holds with  $D_{11} = \frac{1}{R}$ . Thus  $\bar{B}_{II}^2 = \frac{3}{2}$ ,  $C_{II}^2 = 1$ ,  $X_2 = \frac{2}{3}$  and Assumption  
 915 9 is verified. Item 1 in Proposition 4.8 applies since  $\text{Im}(C_{II}^2) = \mathbb{R}$ . Since  $\bar{B}_{II}^1 = 0$   
 916 the mapping  $G_2(\cdot)$  is zero, and item 3 in Proposition 4.8 applies. The quadruple  
 917  $(B_{II}, \bar{B}_{II}, C_{II}, D)$  is passive because the LMI (C.1) is verified with  $X = \frac{2}{3}$ , hence  
 918 Assumption 5 is satisfied, but it is not strictly passive since  $D \succcurlyeq 0$ . It can be checked

919 that the transfer matrix  $H(s) = D + C_{II}(s - B_{II})^{-1}\bar{B}_{II} = \begin{pmatrix} \frac{1}{R} & 0 \\ 0 & \frac{3}{2(s+R)} \end{pmatrix}$  is positive

920 real [18, Theorem 2.48]. The conditions of Lemmata 5.10 and 5.11 are verified by  
 921 taking  $R$  large enough (notice that all other parameters depend on  $\frac{1}{R}$ ). Here  $\tilde{A} =$   
 922  $B_{II} = -R$ , but Assumptions 2 and 6 do not hold, hence the material in Proposition  
 923 4.4 cannot be used.

924 **6.2. Example 2 (circuit).** Let us consider now the circuit in Figure 1a. Using  
 925 Kirchoff laws its dynamics may be written as:

$$926 \quad (6.2) \quad \begin{cases} \dot{x}_1(t) = -\frac{R_1}{L}x_1(t) - \frac{R_3}{L}z(t) + \frac{1}{L}u(t) + \frac{1}{L}u_D(t) \\ \dot{x}_2(t) = \frac{R_3}{R_2}z(t) - \frac{1}{R_2\mathbf{C}}x_2(t) \\ 0 = x_1(t) + \frac{1}{R_2\mathbf{C}}x_2(t) - (1 + \frac{R_3}{R_2})z(t) \\ 0 \leq u_D(t) \perp x_1(t) \geq 0, \end{cases}$$

927 where  $x_1 = i_1$ ,  $x_2(t) = \int_0^t i_2(s)ds$ ,  $z = i_3$ ,  $\lambda = u_D$  (the voltage across the diode),  
 928  $K = K^* = \mathbb{R}_+$ ,  $n_1 = 2$ ,  $p = n_2 = 1$ ,  $m = 1$ . Let apply the feedback control  $u = 2R_3z$ ,

929 which modifies  $B_I$ . We get:  $A_I = \begin{pmatrix} -\frac{R_1}{L} & 0 \\ 0 & -\frac{1}{R_2\mathbf{C}} \end{pmatrix}$ ,  $B_I = \begin{pmatrix} \frac{R_3}{R_2} \\ \frac{1}{R_2} \end{pmatrix}$ ,  $\bar{B}_I = \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix}$ ,

930  $A_{II} = (1 \ \frac{1}{R_2\mathbf{C}})$ ,  $B_{II} = -(1 + \frac{R_3}{R_2})$ ,  $\bar{B}_{II} = 0$ ,  $C_I = (1 \ 0)$ ,  $C_{II} = 0$ ,  $D = 0$ ,  $E_I = \begin{pmatrix} \frac{1}{L}u \\ 0 \end{pmatrix}$ ,

931  $E_{II} = 0$ ,  $F(t) = 0$ ,  $P_I = I_2$ ,  $P_I^h = \begin{pmatrix} 1 + h\frac{R_1}{L} & 0 \\ 0 & 1 + h\frac{1}{R_2\mathbf{C}} \end{pmatrix} \succ 0$  for all  $h \geq 0$ . Thus

932  $\tilde{B} = \frac{h}{hR_1 + L}$ ,  $\tilde{C} = \frac{hR_3}{L + hR_1}$ ,  $\tilde{D} = \frac{h}{L + hR_1}$ ,  $\tilde{A} = -\frac{R_2 + R_3}{R_2} + hR_3 \left( \frac{1}{L + hR_1} + \frac{1}{R_2(h + R_2\mathbf{C})} \right)$ .

933 Assumption 2 holds with  $\tilde{X} = R_3$ ,  $\tilde{D} > 0$  for all  $h > 0$ , and Assumption 4 holds  
 934 for  $h > 0$  small enough such that  $\tilde{A} < 0$ . For  $h > 0$  small enough the set  $\tilde{S} = \{0\}$   
 935 in (4.6) since  $\tilde{R}\tilde{A}\tilde{R}^{-1} = \tilde{A} < 0$ , hence the VI in (4.7) always has a solution  $z_{k+1}$  by  
 936 Proposition 4.3 item 3 (a). By item (c) (or (f)) of the same proposition, the solution is  
 937 unique. When  $\tilde{A} < 0$ , the quadruple  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  is strictly passive for all  $h > 0$ , hence  
 938 items 1 and 3 of Corollary 4.6 apply. However Assumption 10 does not hold since  
 939  $\tilde{D} \rightarrow 0$  as  $h \rightarrow 0$ . Consider now the quadruple  $(B_{II}, \bar{B}_{II}, C_{II}, D) = (B_{II}, 0, 0, 0)$ .  
 940 It is passive since (C.1) is verified, it is even strictly state passive but not strictly  
 941 passive since  $D = 0$ . Assumptions 3 and 9 are trivially satisfied since both sides of  
 942 the passivity equality are null. Provided that  $\{0\} \in \mathbb{R}_+ - x_{1,k+1} \Leftrightarrow x_{1,k+1} \geq 0$ , item 1  
 943 in Proposition 4.7 applies and the OSNSP for (3.1) has a unique solution (but clearly  
 944 there is no guarantee that this inequality is satisfied along the integration process,  
 945 rendering Proposition 4.7 unapplicable). Proposition 4.8 item 3 also applies here for  
 946 the VI in (4.17), however one has to check the equivalence stated in item 1 of the  
 947 same proposition.

948 As noted above, the matrix  $A_I$  intervenes only in Assumption 1, hence its modification  
 949 does not alter the results. Setting  $u = 2R_3z + R_1x_1 + aLx_1 + v(t)$  yields the modified

950  $A_I = \begin{pmatrix} a & 0 \\ 0 & -\frac{1}{R_2\mathbf{C}} \end{pmatrix}$ , and  $B_I$  as above. This yields (see Remark 5.13):  $P_I^{-1}(A_I -$

951  $B_I B_{II}^{-1} A_{II}) = \begin{pmatrix} a + \frac{R_2 R_3}{L(R_2 + R_3)} & \frac{R_3}{L\mathbf{C}(R_2 + R_3)} \\ \frac{R_3}{R_2 + R_3} & -\frac{1}{(R_2 + R_3)\mathbf{C}} \end{pmatrix}$ , which is unstable if  $a + \frac{R_2 R_3}{(R_2 + R_3)L} >$

952  $\frac{1}{(R_2 + R_3)\mathbf{C}}$ . The other elements of the reduced LCS quadruple are given by  $P_I^{-1}(\bar{B}_I -$

953  $B_I B_{II}^{-1} \bar{B}_{II}) = \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix}$ ,  $C_I - C_{II} B_{II}^{-1} A_{II} = (1 \ 0)$ ,  $D - C_{II} B_{II}^{-1} \bar{B}_{II} = 0$ . Consequently

954 the reduced LCS is not passive with the above feedback controller. Also  $P_I^h = \text{diag}(1 -$   
 955  $ha, 1 + \frac{h}{R_2\mathbf{C}}) \succ 0$  for all  $h < \frac{1}{a}$ .

956 To simplify the calculations let us set  $R_2 = R_3 = 1\Omega$ ,  $L = 1\text{H}$ ,  $\mathbf{C} = 1\text{F}$ ,  $a = 1$ . Then

957 the discretization (3.1) of (6.2) writes as:

$$958 \quad (6.3) \quad \begin{cases} x_{1,k+1} - x_{1,k} = hx_{1,k+1} + hz_{k+1} + h\lambda_{k+1} + hv_k \\ x_{2,k+1} - x_{2,k} = -hx_{2,k+1} + hz_{k+1} \\ 0 = x_{1,k+1} + x_{2,k+1} - 2z_{k+1} \\ 0 \leq \lambda_{k+1} \perp x_{1,k+1} \geq 0. \end{cases}$$

959 Equivalently the discretization in (3.2) can be used with the above matrices  $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ .

960 **6.3. Example 3 (switching DAEs).** Let us consider the following system,  
961 which is a nonsmooth DAE (similar to those studied in [53, 54, 52, 51], and which  
962 can be interpreted as a DAE with state-dependent switches):

$$963 \quad (6.4) \quad \begin{cases} (a) \dot{x}(t) = A_I x(t) + B_I z(t) + E_I(t) \\ (b) 0 = a|x(t)| + b|z(t)| - cz(t) + E_{II}(t), \end{cases}$$

964 with  $x(t) \in \mathbb{R}, z(t) \in \mathbb{R}, a, b, c \in \mathbb{R}, c > 0$ . The constraint in (6.4) (b) is equivalently  
965 rewritten as:

$$966 \quad (6.5) \quad \begin{aligned} 0 &= a\lambda_1(t) + a\lambda_2(t) + b\lambda_3(t) + b\lambda_4(t) - cz(t) + E_{II}(t) \\ 0 &\leq \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \lambda_3(t) \\ \lambda_4(t) \end{pmatrix} \perp \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \lambda_3(t) \\ \lambda_4(t) \end{pmatrix} + \begin{pmatrix} -x(t) \\ x(t) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -z(t) \\ z(t) \end{pmatrix} \geq 0. \end{aligned}$$

967 Therefore (6.4) (a) (6.5) fits with (2.1) with  $A_{II} = 0, B_{II} = -c, \bar{B}_{II} = (a \ a \ b \ b),$   
968  $D = I_4, C_I = (-1 \ 1 \ 0 \ 0)^\top, C_{II} = (0 \ 0 \ -1 \ 1)^\top, F(t) = 0, \bar{B}_I = 0$ . Let us check the  
969 strict passivity of the quadruple  $(B_{II}, \bar{B}_{II}, C_{II}, D)$ . This boils down to checking that  
970 (see (C.1)):

$$971 \quad (6.6) \quad M \triangleq \begin{pmatrix} 2cX & -aX & -aX & -bX - 1 & -bX + 1 \\ -aX & 2 & 0 & 0 & 0 \\ -aX & 0 & 2 & 0 & 0 \\ -bX - 1 & 0 & 0 & 2 & 0 \\ -bX + 1 & 0 & 0 & 0 & 2 \end{pmatrix} \succ 0,$$

972 for some  $X > 0$ . Let us denote  $M = P + N$  with  $P = \text{diag}(2cX, 2I_4) \succ 0$ , and  
973  $N_{00}$  the matrix obtained for  $a = b = 0$ . Applying [25, Theorem 2.11] or [10, Fact  
974 8.19.2], it follows that  $P + N_{00} \succ 0$  if  $\sigma_{\max}^2(N_{00}) = 2 < \lambda_{\min}(2cX)\lambda_{\min}(2I_4) = 4cX$ .  
975 Using continuity of the eigenvalues [10, Fact 10.11.8], it is deduced that given  $c > 0,$   
976  $X > 0$ , there exists  $a$  and  $b$  small enough such that (6.6) holds (or, given  $a, b$  and  
977  $X > 0$ , there exists  $c > 0$  large enough). Thus, item 1 in Lemma 4.1 holds, item 2  
978 in Proposition 4.3 holds for  $h > 0$  small enough (guaranteeing  $\tilde{D} \succ 0$ ), Assumption  
979 10 holds true for small enough  $h > 0$  (see the proof of item 1 in Lemma 4.1), and  
980 the arguments stated at the beginning of section 5.2 about the convergence analysis  
981 led in sections 5.1.1 and 5.1.2, hold. We have:  $\tilde{A} = -c, \tilde{B} = (a \ a \ b \ b), \tilde{C} =$   
982  $(0 \ 0 \ -1 \ 1)^\top + h(-1 \ 1 \ 0 \ 0)^\top(1 - hA_I)^{-1}B_I = (\frac{-hB_I}{1-hA_I} \ \frac{hB_I}{1-hA_I} \ 0 \ 0)^\top, \tilde{D} = D$ .  
983 Integration can proceed using (3.2).

984 Let us now pass to a second example of state-dependent-switch DAE:

$$985 \quad (6.7) \quad \begin{cases} (a) \dot{x}_1(t) = a_1x_1(t) + b_1x_2(t) + c_1z(t) + e_1(t) \\ (b) \dot{x}_2(t) = a_2x_1(t) + b_2x_2(t) + c_2z(t) + e_2(t) \\ (c) 0 = a_3x_1(t) + b_3x_2(t) + c_3z(t) + e_3(t) \text{ if } x_1(t) < 0 \\ (d) 0 = a_4x_1(t) + b_3x_2(t) + c_3z(t) + e_3(t) \text{ if } x_1(t) > 0, \end{cases}$$

986 where the switching conditions in (c) and (d) are equivalently rewritten as:

$$987 \quad (6.8) \quad \begin{cases} 0 = -a_3\lambda_1(t) + a_4\lambda_2(t) + b_3x_2(t) + c_3z(t) + e_3(t) \\ 0 \leq \lambda_1(t) \perp \lambda_1(t) + x_1(t) \geq 0 \\ 0 \leq \lambda_2(t) \perp \lambda_2(t) - x_1(t) \geq 0. \end{cases}$$

988 Thus we have:  $A_I = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ ,  $B_I = (c_1 \ c_2)^\top$ ,  $\bar{B}_I = (0 \ 0)^\top$ ,  $E_I = (e_1 \ e_2)^\top$ ,  $\bar{B}_{II} =$

989  $(-a_3 \ a_4)^\top$ ,  $A_{II} = (0 \ b_3)$ ,  $B_{II} = c_3$ ,  $E_{II} = e_3$ ,  $C_I = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$ ,  $C_{II} = (0 \ 0)^\top$ ,  $D = I_2$ ,

990  $F(t) = (0 \ 0)^\top$ . Also  $\tilde{D} = D = I_2$ ,  $\tilde{A} = c_3 + h(0 \ b_3)(I_2 - hA_I)^{-1} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_3 + \mathcal{O}(h)$ ,

991  $\tilde{B} = \bar{B}_{II} = (-a_3 \ a_4)^\top$ ,  $\tilde{C} = h \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} (I_2 - hA_I)^{-1} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathcal{O}(h)$ . It can be checked

992 that Proposition 4.3 items 1 and 2, Corollary 4.6 items 1, 2 and 3 (if the passivity LMI  
993 holds strictly) apply. The passivity LMI  $\begin{pmatrix} -2X\tilde{A} & -X(a_3 \ a_4) + \tilde{C} \\ (-X(a_3 \ a_4) + \tilde{C})^\top & 2I_2 \end{pmatrix} \succ 0$

994 is verified for  $a_3$ ,  $a_4$  and  $h > 0$  small enough, given  $c_3 < 0$ ,  $X > 0$ , implying the  
995 strict passivity of the quadruple  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ , uniformly as  $h \rightarrow 0$  (Assumption 10  
996 holds true). It is of interest to investigate what happens if  $c_3 = 0$ , which makes  
997 the “index” increase. Let us assume for simplicity that  $A_I = 0$ , the matrix of the

998 passivity LMI becomes  $\begin{pmatrix} -2hXb_3c_2 & -X(-a_3 \ a_4) + h(c_1 \ -c_1) \\ (-X(-a_3 \ a_4) + h(c_1 \ -c_1))^\top & 2I_2 \end{pmatrix}$ .

999 For given  $X > 0$  and  $h > 0$ , there always exists  $b_3c_2 < 0$  such that this matrix is  $\succ 0$ ,

1000 showing that there exists conditions under which the OSNP is still well-posed when  
1001  $c_3 = 0$ . It is interesting to note that the term  $b_3c_2z(t)$  appears when an index reduction

1002 is applied to the equality in (6.8) by differentiating it once. Simulations are shown  
1003 in Figures 2 and 3. They are performed with the INRIA software package SICONOS<sup>1</sup>

1004 [1]. The MLCP is solved with an enumeration method and the differential variables  
1005  $x_1(\cdot)$  and  $x_2(\cdot)$  are calculated in a second step (hence we follow the philosophy of the  
1006 foregoing sections). The parameters are chosen as follows:  $a_1 = a_2 = b_1 = b_2 = a_3 =$

1007  $a_4 = b_3 = c_3 = 1$ ,  $c_1 = 10$ ,  $c_2 = 20$ ,  $e_1(t) = 40 \sin(25t)$ ,  $e_2(t) = \sin(t)$ ,  $e_3(t) = \sin(10t)$ ,

1008  $h = 2 \cdot 10^{-3}$ s,  $x_1(0) = 0$ ,  $x_2(0) = 0.1$ . Simulations are done on  $[0, 2]$ s. The errors are  
1009 calculated as  $\max_{t \in [0, 2]} \|x_i(t) - x_i^{ref}(t)\|$ ,  $i = 1, 2$ , with 14 different values of step

1010 size  $h$  between  $10^{-1}$ s and  $10^{-4}$ s, where  $x_i^{ref}(\cdot)$  is calculated with  $h = 10^{-4}$ s. It  
1011 is observed in Figure 2 that the order is approximately equal to one for small enough  
1012 step size. Figure 3 depicts the time evolution of various variables, demonstrating that  
1013 the complementarity constraints are respected. Clearly the multipliers are continuous  
1014 but non differentiable when the complementarity switches from one mode to another.

1015

1016 **6.4. Example 4 (mechanical system).** As an extension of the systems studied  
1017 in [37], let us consider a linear mechanical system with both unilateral springs and  
1018 bilateral constraints, whose dynamics are:

$$1019 \quad (6.9) \quad \begin{cases} (a) \ M\ddot{q}(t) + K_v\dot{q}(t) + K_pq(t) = \tau(t) + H^\top \lambda_u(t) + G^\top \lambda_b(t) \\ (b) \ 0 = Gq(t) - f(t) \\ (c) \ 0 \leq \lambda_u(t) \perp D\lambda_u(t) + C_Ix(t) + F(t) \geq 0, \end{cases}$$

<sup>1</sup><https://nonsmooth.gricad-pages.univ-grenoble-alpes.fr/siconos/index.html>

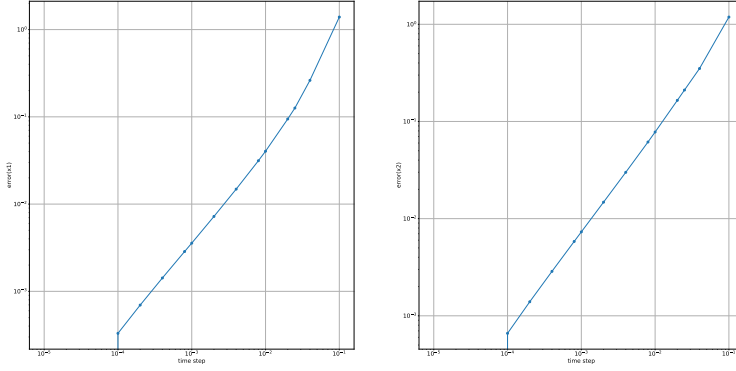


Fig. 2: Log-log error diagram:  $x_1$  (left) and  $x_2$  (right).

1020 where  $M = M^\top \succ 0$ ,  $D \succ 0$ ,  $x = (q^\top \dot{q}^\top)^\top$ ,  $q(t) \in \mathbb{R}^{\frac{n_1}{2}}$ ,  $z = \lambda_b \in \mathbb{R}^{n_2}$ ,  $\lambda = \lambda_u \in$   
 1021  $\mathbb{R}^m$ ,  $G \in \mathbb{R}^{n_2 \times \frac{n_1}{2}}$ ,  $f: \mathbb{R}^+ \rightarrow \mathbb{R}^{n_2}$  is twice differentiable,  $\tau(\cdot)$  is an exogenous force.  
 1022 Here an index reduction is possible, in order to replace (6.9) (b) by:

$$1023 \quad (6.10) \quad 0 = G\dot{q}(t) - \dot{f}(t) = GM^{-1}H^\top \lambda(t) + GM^{-1}G^\top z(t) - GM^{-1}(K_p K_v)x(t) - \dot{f}(t),$$

1024 and  $Gq(0) - f(0) = 0$ ,  $G\dot{q}(0) - \dot{f}(0) = 0$ . Thus  $B_{II} = -GM^{-1}G^\top$  ( $< 0$  if bilateral con-  
 1025 straints are independent, *i.e.*,  $\text{rank}(G) = p = n_2 \Rightarrow n_1 \geq 2n_2$ ),  $A_{II} = GM^{-1}(K_p K_v)$ ,  
 1026  $\bar{B}_{II} = -GM^{-1}H^\top$ ,  $B_I = \begin{pmatrix} 0 \\ -M^{-1}G^\top \end{pmatrix}$ , and  $C_{II} = 0$  (similarly to the example in  
 1027 section 6.2, one major difference being that here  $D \succ 0$ ). Let us examine the OSNSP  
 1028 for (6.9) on a particular example as depicted in Figure 4, whose dynamics are given  
 1029 by:

$$1030 \quad (6.11) \quad \begin{cases} m\ddot{q}_1(t) = \tau(t) + \lambda_1(t) - \lambda_2(t) \\ m\ddot{q}_2(t) = z(t) \\ 0 = q_2(t) - f(t) \Leftrightarrow \frac{-z(t)}{m} + \dot{f}(t) = 0, q_2(0) - f(0) = 0, \dot{q}_2(0) - \dot{f}(0) = 0 \\ 0 \leq \lambda_1(t) \perp \lambda_1(t) + k(q_1(t) - L - \alpha(t) - l) \geq 0 \\ 0 \leq \lambda_2(t) \perp \lambda_2(t) - k(q_1(t) + L - \beta(t) + l) \geq 0. \end{cases}$$

1031 where  $q = (q_1 \ q_2)^\top$  are the mass coordinates,  $k > 0$  is a stiffness constant,  $\alpha(\cdot)$   
 1032 and  $\beta(\cdot)$  are the massless springs' varying positions,  $l > 0$  is the spring's free length,  
 1033  $\beta(t) > \alpha(t) + 2L + 2l$  for all  $t \geq 0$ . We have:  $x = (q_1 \ \dot{q}_1 \ q_2 \ \dot{q}_2)^\top$ ,  $G = (0 \ 1)$ ,  $H =$   
 1034  $\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$ ,  $M = \text{diag}(m, m)$ ,  $D = I_2$ ,  $A_{II} = 0$ ,  $B_{II} = \frac{-1}{m}$ ,  $\bar{B}_{II} = 0$ ,  $E_{II}(t) = \dot{f}(t)$ ,  
 1035  $C_I = \begin{pmatrix} k & 0 & 0 & 0 \\ -k & 0 & 0 & 0 \end{pmatrix}$ ,  $F(t) = \begin{pmatrix} k(-L - \alpha(t) - l) \\ -k(L - \beta(t) + l) \end{pmatrix}$ ,  $C_{II} = 0$ ,  $B_I = (0 \ 0 \ 0 \ \frac{1}{m})^\top$ ,  
 1036  $P_I = I_4$  (and Assumption 1 holds for  $h > 0$  small enough),  $\bar{B}_I = \begin{pmatrix} 0 & \frac{1}{m} & 0 & 0 \\ 0 & \frac{m_1}{m} & 0 & 0 \end{pmatrix}^\top$ ,  
 1037  $\tilde{A} = \frac{-1}{m}$ ,  $\tilde{B} = 0$ ,  $\tilde{C} = hC_I(P_I^h)^{-1}B_I$ ,  $\tilde{D} = I_2 + hC_I(P_I^h)^{-1}\bar{B}_I \succ 0$  if  $h > 0$  is small  
 1038 enough. Let us check Assumption 4. The passivity LMI is  $\begin{pmatrix} \frac{2X}{C^\top} & \tilde{C} \\ \tilde{C}^\top & \tilde{D} + \tilde{D}^\top \end{pmatrix} \succ 0$ ,

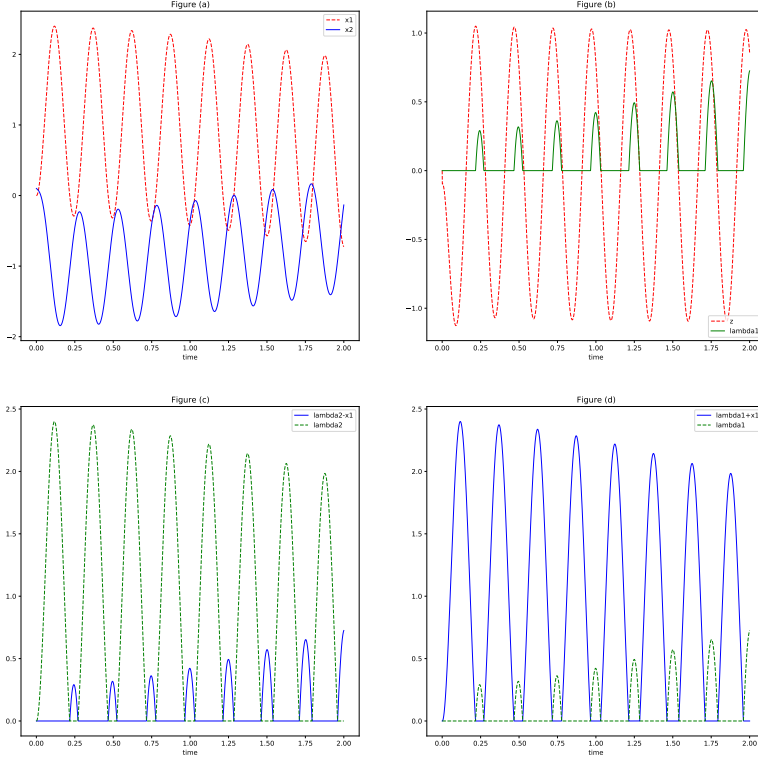


Fig. 3: (a)  $x_1(t)$  (dashed) and  $x_2(t)$ , (b)  $z(t)$  (dashed) and  $\lambda_1(t)$ , (c)  $\lambda_2(t) - x_1(t)$  and  $\lambda_2(t)$  (dashed), (d)  $\lambda_1(t) + x_1(t)$  and  $\lambda_1(t)$  (dashed).

1039  $X > 0$ , which is verified for  $h > 0$  small enough and  $X > 0$  large enough. Hence under  
 1040 these conditions the quadruple  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  is strictly passive. It can be checked that  
 1041 items 1 and 2 in Proposition 4.3 hold, as well as items 1, 2 and 3 in Corollary 4.6.  
 1042 Therefore the dynamics can be integrated using (3.2) with  $q_{2,0} + f_0 = 0$ ,  $\dot{q}_{2,0} + \dot{f}_0 = 0$ .  
 1043 We note also that the quadruple  $(B_{II}, \bar{B}_{II}, C_{II}, D) = (\frac{-1}{m}, 0, 0, I_2)$  is strictly passive  
 1044 (Assumption 5 is verified), hence item 1 in Lemma 4.1 holds. Assumption 10 is  
 1045 also verified (for  $h > 0$  small enough but uniformly in  $h \geq 0$ ). Let us analyse the  
 1046 convergence as in section 5.1. Assumption 11 may not be verified in this case, however  
 1047 as noted after this assumption it is easy to modify slightly the numerical scheme so  
 1048 that it is satisfied with  $P_I = I_4$ . Hence it is inferred that Proposition 5.2 applies.

1049 **6.5. Example 5 (circuit).** Let us consider the circuit in Figure 5, with  $R_1 =$   
 1050  $R_3 = 0 \Omega$ . Let  $x_1 = i_2$ ,  $x_2 = \int_0^t i_3(t)dt$ ,  $z = i_1$ ,  $\lambda = u_D$  is the voltage across the

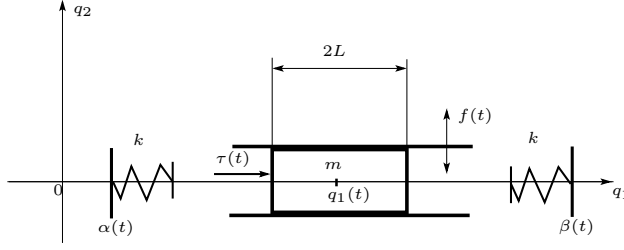
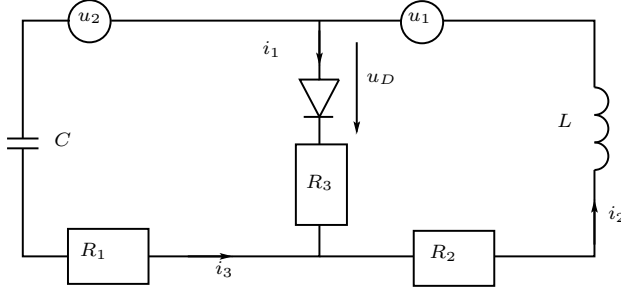


Fig. 4: A mechanical DALCS.

Fig. 5: An  $RLCD$  passive circuit.

1051 diode,  $u_1(\cdot)$  and  $u_2(\cdot)$  are two voltage sources. One way to write its dynamics is:

$$1052 \quad (6.12) \quad \begin{cases} \dot{x}_1(t) = -\frac{R_2}{L}x_1(t) + \frac{1}{L}\lambda(t) + \frac{1}{L}u_1(t) \\ \dot{x}_2(t) = x_1(t) - z(t) \\ 0 = \frac{1}{C}x_2(t) + \lambda(t) + u_2(t) \\ 0 \leq \lambda(t) \perp z(t) \geq 0. \end{cases}$$

1053 Assume that the feedback  $u_2 = -az + u_3$  is applied, so that the equality constraint  
 1054 becomes  $0 = \frac{1}{C}x_2(t) + u_D(t) - az(t) + u_3(t)$ . We have:  $A_{II} = (0 \quad \frac{1}{C})$ ,  $C_I = 0$ ,  
 1055  $C_{II} = 1$ ,  $B_{II} = -a$ ,  $\bar{B}_{II} = 1$ ,  $D = 0$ ,  $\bar{B}_I = (\frac{1}{L} \quad 0)^\top$ ,  $B_I = (0 \quad -1)^\top$ , and  $\bar{D} = D = 0$ ,  
 1056  $\tilde{C} = C_{II} = 1$ ,  $\tilde{A} = -a + hA_{II}(P_I^h)^{-1}B_I$ ,  $\tilde{B} = 1 + hA_{II}(P_I^h)^{-1}\bar{B}_I$ . The condition  
 1057  $\text{Im}(C_{II}) \cap (K - C_I x_{k+1} + F_k) = \mathbb{R} \cap \mathbb{R}_+ - \frac{1}{C}x_{2,k+1} \neq \emptyset$  is verified always. The set  $\mathcal{S}$  in  
 1058 (4.16) is equal to  $\{0\}$  hence Proposition 4.7 item 1 applies and the VI in (4.15) always  
 1059 has a solution. Proposition 4.8 items 1, 2, 3 apply, with  $D_{11} = 0$  in Assumption 8.  
 1060 The system  $(B_{II}, \bar{B}_{II}, C_{II}) = (-a, 1, 1)$  is passive and Assumptions 3 and 5 hold true.  
 1061 Also the system  $(\tilde{A}, \tilde{B}, \tilde{C})$  is passive for  $h > 0$  small enough and Assumptions 2 and  
 1062 4 hold true. The set  $\mathcal{S}$  in (4.6) is equal to  $\{0\}$  so Proposition 4.3 item 3 (a) applies.  
 1063 For  $h > 0$  small enough,  $\tilde{A} < 0$  so Corollary 4.6 items 1 and 2 apply. Assumption 12  
 1064 is verified for  $h \geq 0$  small enough with  $\tilde{X} = \tilde{B}^{-1}$ . It is inferred that if  $a > 0$  is large  
 1065 enough, the convergence results in sections 5.1.3 and 5.2 apply.

1066 **7. Conclusion.** This article is devoted to analysing the implicit Euler discreti-  
 1067 sation of a class of nonsmooth nonlinear systems with equality constraints, named  
 1068 differential-algebraic linear complementarity systems. The well-posedness of the one-  
 1069 step nonsmooth problem is studied first, then the convergence of the discrete-time  
 1070 solutions is analysed. This article only scratches the surface of singular nonsmooth

1071 systems analysis. The above results could be extended in various directions. Most  
 1072 importantly, it would be worth extending the results to more general indices and rela-  
 1073 tive degrees, to deeper analyse how the algebraic and the complementarity constraints  
 1074 are coupled, and to consider state discontinuities and solutions of bounded variation.

1075 **Appendix A. Well-posedness of Variational Inequalities.** Some basic  
 1076 notions of Convex Analysis and Complementarity Theory are briefly introduce now  
 1077 [11, 29, 49, 31]. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a proper convex and lower semicontin-  
 1078 uous (lsc) function, we denote by  $\text{dom}(f) \triangleq \{x \in \mathbb{R}^n \mid f(x) < +\infty\}$  the domain of  
 1079 the function  $f(\cdot)$ . The Fenchel transform  $f^*(\cdot)$  of  $f(\cdot)$  is the proper, convex and lsc  
 1080 function defined by: for all  $z \in \mathbb{R}^n$ ,  $f^*(z) = \sup_{x \in \text{dom}(f)} \{\langle x, z \rangle - f(x)\}$ . The sub-  
 1081 differential  $\partial f(x)$  of  $f(\cdot)$  at  $x \in \mathbb{R}^n$  is defined by:  $\partial f(x) = \{\omega \in \mathbb{R}^n \mid f(v) - f(x) \geq$   
 1082  $\langle \omega, v - x \rangle, \forall v \in \mathbb{R}^n\}$ . We denote by  $\text{Dom}(\partial f) \triangleq \{x \in \mathbb{R}^n \mid \partial f(x) \neq \emptyset\}$  the domain of  
 1083 the subdifferential operator  $\partial f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ .

Let  $x_0$  be any element in the domain  $\text{dom}(f)$ , the recession function  $f_\infty(\cdot)$  of  $f(\cdot)$   
 is defined by: for all  $x \in \mathbb{R}^n$  :  $f_\infty(x) = \lim_{\lambda \rightarrow +\infty} \frac{1}{\lambda} f(x_0 + \lambda x)$ . The function  $f_\infty : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  is a proper convex and lsc function which describes the asymptotic  
 behavior of  $f(\cdot)$ . Let  $K \subset \mathbb{R}^n$  be a nonempty closed convex set. Let  $x_0$  be any  
 element in  $K$ . The recession cone of  $K$  is defined by [49]:

$$K_\infty = \bigcap_{\lambda > 0} \frac{1}{\lambda} (K - x_0) = \{u \in \mathbb{R}^n \mid x + \lambda u \in K, \forall \lambda \geq 0, \forall x \in K\}.$$

1084 The set  $K_\infty$  is a nonempty closed convex cone that is described in terms of the  
 1085 directions which recede from  $K$ . The indicator function of a set  $K \subseteq \mathbb{R}^n$  is  $\psi_K(x) = 0$   
 1086 if  $x \in K$ ,  $\psi_K(x) = +\infty$  if  $x \notin K$ . If  $K$  is closed non empty convex, we have  
 1087  $\partial \psi_K(x) = \mathcal{N}_K(x)$ , the so-called normal cone to  $K$  at  $x$ , defined as  $\mathcal{N}_K(x) = \{v \in$   
 1088  $\mathbb{R}^n \mid v^\top (s - x) \leq 0 \text{ for all } s \in K\}$ .

1089 Important properties of the recession function and recession cone are recalled now:

1090 PROPOSITION A.1. [11, Proposition 1.4.8] *The following statements hold:*

1091 a) Let  $f_1 : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  and  $f_2 : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be two proper, convex and  
 1092 lsc functions. Suppose that  $f_1 + f_2$  is proper. Then for all  $x \in \mathbb{R}^n$  :  $(f_1 + f_2)_\infty(x) =$   
 1093  $(f_1)_\infty(x) + (f_2)_\infty(x)$ .

1094 b) Let  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a proper, convex and lsc function, and let  $K$  be a  
 1095 nonempty closed convex set, such that  $f + \Psi_K$  is proper (equivalently  $\text{dom}(f) \cap K$  is  
 1096 non empty). Then for all  $x \in \mathbb{R}^n$  :  $(f + \Psi_K)_\infty(x) = f_\infty(x) + (\Psi_K)_\infty(x)$ .

1097 c) Let  $K \subset \mathbb{R}^n$  be a nonempty, closed and convex set. Then for all  $x \in \mathbb{R}^n$  :  
 1098  $(\Psi_K)_\infty(x) = \Psi_{K_\infty}(x)$ . Moreover for all  $x \in K$  and  $e \in K_\infty$  :  $x + e \in K$ .

1099 d) If  $K \subseteq \mathbb{R}^n$  is a nonempty closed and convex cone, then  $K_\infty = K$ .

1100 e) Let  $K = P(A, b) \triangleq \{x \in \mathbb{R}^n \mid Ax \geq b\}$  for  $A \in \mathbb{R}^{m \times n}$  and  $b \in \mathbb{R}^m$ . If  $K \neq \emptyset$  then  
 1101  $K_\infty = P(A, 0) = \{x \in \mathbb{R}^n \mid Ax \geq 0\}$ .

1102 Let us now concatenate [4, Theorem 3, Corollaries 3 and 4]. They concern variational  
 1103 inequalities (VIs) of the form: Find  $u \in \mathbb{R}^n$  such that

$$1104 \quad (\text{A.1}) \quad \langle \mathbf{M}u + \mathbf{q}, v - u \rangle + \varphi(v) - \varphi(u) \geq 0, \text{ for all } v \in \mathbb{R}^n$$

1105 where  $\mathbf{M} \in \mathbb{R}^{n \times n}$  is a real matrix,  $\mathbf{q} \in \mathbb{R}^n$  a vector and  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$   
 1106 a proper convex and lsc function. The VI in (A.1) is equivalent to the inclusion



1107  $\mathbf{M}u + \mathbf{q} \in -\partial\varphi(u)$ , which is a generalized equation (GE) of the form:  $0 \in A(u) + B(u)$   
 1108 where  $A(\cdot)$  and  $B(\cdot)$  are two maximal monotone operators (since it will be assumed  
 1109  $\mathbf{M} \succcurlyeq 0$ ). In fact the problems encountered in this paper (see (4.7), (4.9), (4.14), (4.15),  
 1110 (4.17)) all belong to this class of GEs. Numerical schemes that compute solutions of  
 1111 such GEs have been studied for a long time in the mathematical literature, see [57]  
 1112 and references therein. The next proposition states conditions about existence and  
 1113 uniqueness of solutions.

1114 The problem in (A.1) is denoted as  $VI(\mathbf{M}, \mathbf{q}, \varphi)$ , and we set:

$$1115 \quad (\text{A.2}) \quad \mathcal{K}(\mathbf{M}, \varphi) = \{x \in \mathbb{R}^n \mid \mathbf{M}x \in (\text{dom}(\varphi_\infty))^*\}.$$

1116 Note that  $(\text{dom}(\varphi_\infty))^*$  is the dual cone of the domain of the recession function  $\varphi_\infty$   
 1117 while  $(\text{dom}(\varphi))_\infty$  (that we may denote also as  $\text{dom}(\varphi)_\infty$ ) is the recession cone of  
 1118  $\text{dom}(\varphi)$ .

1119 **PROPOSITION A.2.** [4] *Let  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a proper, convex and lsc*  
 1120 *function with closed domain,  $\mathbf{M} \in \mathbb{R}^{n \times n}$ , and suppose that  $\mathbf{M} \succcurlyeq 0$ .*

1121 *a) If  $(\text{dom}(\varphi))_\infty \cap \text{Ker}\{\mathbf{M} + \mathbf{M}^\top\} \cap \mathcal{K}(\mathbf{M}, \varphi) = \{0\}$  then for each  $\mathbf{q} \in \mathbb{R}^n$ , problem*  
 1122  *$VI(\mathbf{M}, \mathbf{q}, \varphi)$  has at least one solution.*

1123 *b) Suppose that  $(\text{dom}(\varphi))_\infty \cap \text{Ker}\{\mathbf{M} + \mathbf{M}^\top\} \cap \mathcal{K}(\mathbf{M}, \varphi) \neq \{0\}$ . If there exists  $x_0 \in$   
 1124  *$\text{dom}(\varphi)$  such that**

$$1125 \quad (\text{A.3}) \quad \langle \mathbf{q} - \mathbf{M}^\top x_0, v \rangle + \varphi_\infty(v) > 0, \quad \forall v \in (\text{dom}(\varphi))_\infty \cap \text{Ker}\{\mathbf{M} + \mathbf{M}^\top\} \cap \mathcal{K}(\mathbf{M}, \varphi), \quad v \neq 0,$$

1126 *then problem  $VI(\mathbf{M}, \mathbf{q}, \varphi)$  has at least one solution.*

1127 *c) If  $u_1$  and  $u_2$  are two solutions of problem  $VI(\mathbf{M}, \mathbf{q}, \varphi)$  then  $u_1 - u_2 \in \text{Ker}\{\mathbf{M} +$   
 1128  $\mathbf{M}^\top\}$ .*

1129 *d) If  $\mathbf{M} = \mathbf{M}^\top$  and  $u_1$  and  $u_2$  denote two solutions of problem  $VI(\mathbf{M}, \mathbf{q}, \varphi)$ , then*  
 1130  *$\langle \mathbf{q}, u_1 - u_2 \rangle = \varphi(u_2) - \varphi(u_1)$ .*

1131 *e) If  $\mathbf{M} = \mathbf{M}^\top$ , then  $u$  is a solution of  $VI(\mathbf{M}, \mathbf{q}, \varphi)$  if and only if it is a solution of*  
 1132 *the optimization problem  $\min_{x \in \mathbb{R}^n} \frac{1}{2}x^\top \mathbf{M}x + \langle \mathbf{q}, x \rangle + \varphi(x)$ .*

1133 **Appendix B. Some Convex Analysis and Complementarity Theory**  
 1134 **tools.** If  $K \subseteq \mathbb{R}^n$  is a set, then  $K^* = \{z \in \mathbb{R}^n \mid \langle z, x \rangle \geq 0 \text{ for all } x \in K\}$  is  
 1135 its dual cone. Let  $K$  be a nonempty closed convex cone, then:

$$1136 \quad (\text{B.1}) \quad K^* \ni x \perp y \in K \Leftrightarrow x \in -\mathcal{N}_K(y) \iff y \in -\mathcal{N}_{K^*}(x).$$

1137 Let  $M = M^\top \succcurlyeq 0$ ,  $x$  and  $y$  two vectors, then [29]

$$1138 \quad (\text{B.2}) \quad M(x - y) \in -\mathcal{N}_K(x) \Leftrightarrow x = \text{proj}_M[K; y] \Leftrightarrow x = \min_{z \in K} \frac{1}{2}(z - y)^\top M(z - y).$$

1139 We note that this is a particular case of (A.1), so that Proposition A.2 can be con-  
 1140 sidered as the characterization of a generalized projection operator  $VI(\mathbf{M}, \mathbf{q}, \varphi)$ . The  
 1141 next proposition is used several times in the article:

1142 **PROPOSITION B.1.** *Suppose that the  $m \times m$  matrix  $D \succcurlyeq 0$ , and  $\mathcal{M} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$*   
 1143 *is a maximal monotone operator. Then the operator  $(D + \mathcal{M})^{-1}(\cdot)$  is well-defined*  
 1144 *(i.e., for each  $y \in \mathbb{R}^m$  there is an  $x \in \mathbb{R}^m$  such that  $y \in Dx + \mathcal{M}(x)$ ), single-valued*  
 1145 *and Lipschitz continuous with modulus  $L = \frac{2}{\lambda_{\min}(D + D^\top)}$ .*

1146 It was proposed in [16, Proposition 1] when  $\mathcal{M} = \partial\phi$  for some convex proper lsc  $\phi(\cdot)$ ,  
 1147 extended to maximal monotone  $\mathcal{M}(\cdot)$  in [5].

1148 **Appendix C. Passive Systems.** A quadruple  $(A, B, C, D)$  is said passive if:

1149 (C.1) 
$$\begin{pmatrix} -XA - A^\top X & -XB + C^\top \\ -B^\top X + C & D + D^\top \end{pmatrix} \succcurlyeq 0, \quad X = X^\top \succ 0.$$

1150 for some  $X$ . It is said *strictly* passive if the first matrix inequality is satisfied with  $\succ 0$   
 1151 (this implies that  $D \succ 0$ ). This is not to be confused with the *strict state* passivity [18,  
 1152 Definition 4.54] (strict passivity implies strict state passivity). In fact strict passivity  
 1153 is directly related to strongly SPR transfer matrices [18, p.62 and section 3.12.2],  
 1154 while SPR transfer matrices with minimal realization are strictly state passive [18,  
 1155 Theorem 4.73] [43]. Strict state passivity holds when  $-XA - A^\top X \succ 0$  in (C.1), with  
 1156  $D + D^\top \succcurlyeq 0$ . The two inequalities in (C.1) make a linear matrix inequality (LMI) with  
 1157 unknown  $X$ , equivalently a Riccati inequality using the Schur complement Theorem  
 1158 [18, Theorem A.65]. Numerical algorithms exist that efficiently solve LMIs, see [13],  
 1159 and can be used to test whether or not a quadruple is passive.

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