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1 ANALYSIS OF THE IMPLICIT EULER TIME-DISCRETIZATION OF
2 SEMI-EXPLICIT DIFFERENTIAL-ALGEBRAIC LINEAR
3 COMPLEMENTARITY SYSTEMS*

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5 **Abstract.** This article is largely concerned with the time-discretization of differential-algebraic
6 equations (DAE) with complementarity constraints, which we name differential algebraic linear com-
7 plementarity systems (DALCS). Specifically, the Euler implicit discretization of DALCS is analysed:
8 the one-step non-smooth problem (OSNSP), that is a generalized equation, is shown to be well-
9 posed under some conditions, then the convergence of the discretized solutions is studied, and the
10 existence of solutions to the continuous-time system is shown as a consequence. Passivity of some
11 operators is pivotal to the analysis. **Examples from circuits, mechanics and switching DAE illustrate**
12 **the applicability of the developments.**

13 **Key words.** differential-algebraic system, descriptor-variable system, linear complementarity
14 system, recession function, discrete-time, well-posedness, passive system, Lur'e equations.

15 **AMS subject classifications.** 65K15, 90C33, 90C90, 93B52, 93D30, 93C30.

16 **1. Introduction.** The analysis of non-smooth dynamical systems with multi-
17 valued right-hand sides satisfying maximal monotone properties has witnessed a large
18 number of contributions [19]. Within this class one finds linear and nonlinear com-
19 plementarity dynamical systems, projected systems, differential variational inequali-
20 ties, differential inclusions with maximal monotone right-hand sides, Moreau's sweep-
21 ing process, some switching dynamics, *etc.* In this article we focus on singular, or
22 differential-algebraic linear complementarity systems (DALCS), which may be viewed
23 either as an extension of classical differential-algebraic equations (DAEs), or of "clas-
24 sical" linear complementarity systems. **As will become clear later, DALCS represent**
25 **an extension of the set-valued Lur'e systems studied, e.g., in [39, 40]. The associated**
26 **canonical form is motivated by applications in chemistry [53, 52, 62], switching DAEs**
27 **analysis [48], circuits with nonsmooth electronic components [2, section 3.5, chapter 4]**
28 **[22], optimization-constrained differential equations [19, Example 2] with applications**
29 **in atmospheric aerosol particles [38], and Lagrangian systems with both bilateral and**
30 **unilateral constraints [17] (in spite of the fact that the results presented in this article**
31 **do not apply to all of these systems).**

32 **The study of singular nonsmooth dynamical systems has not received a lot of attention**
33 **yet. Singularly perturbed differential inclusions have been tackled in [26, 56, 32, 58].**
34 **The conditions on the set-valued mappings that are imposed (compactness [56, 58],**
35 **boundedness and Hausdorff Lipschitz continuity [32], one-sided Lipschitz condition**
36 **[26]) are not satisfied by complementarity conditions (or by normal cones to convex**
37 **sets). Hence these results do not apply to DALCS. The well-posedness of DAEs**
38 **in semi-explicit form with nonsmooth constraints (which are related to the class of**
39 **systems studied in this article, see Remark 2.1 below) has been analysed in [53, 54, 52].**
40 **Passive singular set-valued systems are studied in [22], who derived well-posedness**
41 **relying on a special Weierstrass' representation. However the time-discretisation is**

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42 not tackled in the above references.

43 Our goal is to analyse the well-posedness of the one-step nonsmooth problem (OS-
 44 NSP) obtained after an implicit (backward) Euler discretization, and the convergence
 45 of the discrete-time solutions towards a continuous-time limit. The second step is use-
 46 ful not only for proving the existence of solutions to the continuous-time system, but
 47 also, perhaps most importantly, to show that the proposed time-discretization makes
 48 sense for numerical simulations. Passivity properties of suitable operators are pivotal
 49 throughout the article. It is noteworthy that implicit Euler methods are well-known
 50 in DAEs [30, 14, 33, 42, 12, 35, 61, 7], differential variational inequalities [46, 24],
 51 and complementarity systems fields [41, 21, 59, 34], as well as in the field of max-
 52 imal monotone differential inclusions [8, 47] (implicit Euler method is often called
 53 proximal algorithm, while Euler method refers to the explicit algorithm [47]), and
 54 Moreau's sweeping processes with the catching-up algorithm [19, 44, 45]. However
 55 their application to DALCS has not been studied yet.

56 The article is organised as follows: Section 2 introduces the general dynamics of
 57 the nonsmooth singular systems, section 3 presents their time-discretization, section
 58 4 studies the well-posedness of the OSNSP, section 5 is dedicated to convergence
 59 analyses: results in section 5.1 are based on the properties of the numerical scheme,
 60 while section 5.2 uses properties of the system. Section 6 is dedicated to illustrating
 61 examples and simulations, from circuits, mechanics and switching DAEs. Conclusions
 62 are drawn in section 7. The appendix is dedicated to recall various mathematical tools.

63 **Notation and definitions:** for any vector $x \in \mathbb{R}^n$ and any matrix $M \in \mathbb{R}^{m \times n}$,
 64 $\|x\|$ is the Euclidean norm and $\|M\|$ is the Frobenius norm, which are compatible
 65 norms [10, Proposition 9.3.5], *i.e.*, $\|Mx\| \leq \|M\| \|x\|$. Let $M \in \mathbb{R}^{n \times m}$, then $\text{Im}(M)$
 66 is its range, $\text{Ker}(M)$ is its null space. We use $\langle x, y \rangle = x^\top y$, so $\langle x, x \rangle = \|x\|^2$. Positive
 67 definite matrix: $M \succ 0$ if $x^\top Mx > 0$ for all $x \neq 0$, positive semidefinite matrix:
 68 $M \succcurlyeq 0$ if $x^\top Mx \geq 0$ for all x (such M is not necessarily symmetric). The maximum
 69 singular value is denoted as $\sigma_{\max}(M)$, and the minimum eigenvalue as $\lambda_{\min}(M)$. The
 70 i th row of M is denoted as $M_{i\bullet}$. The $n \times n$ identity matrix is denoted I_n . A set-valued
 71 mapping $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said monotone if for all $x_1, x_2, y_1 \in A(x_1), y_2 \in A(x_2)$, one
 72 has $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$. It is maximal monotone if its graph cannot be enlarged
 73 without destroying monotonicity. It is η -strongly monotone if there exists $\eta > 0$ such
 74 that $\langle x_1 - x_2, y_1 - y_2 \rangle \geq \eta \|x_1 - x_2\|^2$. See also Appendix A for further definitions.

75 **2. The class of nonsmooth singular dynamical systems.** Let us consider
 76 the following differential algebraic linear complementarity system (DALCS):

$$77 \quad (2.1) \quad \begin{cases} (a) P_I \dot{x}(t) = A_I x(t) + B_I z(t) + \bar{B}_I \lambda(t) + E_I(t) \\ (b) 0 = A_{II} x(t) + B_{II} z(t) + \bar{B}_{II} \lambda(t) + E_{II}(t) \\ (c) K^* \ni \lambda(t) \perp w(t) = C_I x(t) + C_{II} z(t) + D \lambda(t) + F(t) \in K, \end{cases}$$

78 with $x(t) \in \mathbb{R}^{n_1}$, $z(t) \in \mathbb{R}^{n_2}$, $\lambda(t) \in \mathbb{R}^m$, $w(t) \in \mathbb{R}^m$, $P_I \in \mathbb{R}^{n_1 \times n_1}$ is full-rank,
 79 $K \subseteq \mathbb{R}^m$ is a closed nonempty convex cone, K^* is its dual cone, $\bar{B}_{II} \in \mathbb{R}^{p \times m}$,
 80 $A_{II} \in \mathbb{R}^{p \times n_1}$, $B_{II} \in \mathbb{R}^{p \times n_2}$, $C_I \in \mathbb{R}^{m \times n_1}$, $C_{II} \in \mathbb{R}^{m \times n_2}$, $D \in \mathbb{R}^{m \times m}$, $B_I \in$
 81 $\mathbb{R}^{n_1 \times n_2}$, $A_I \in \mathbb{R}^{n_1 \times n_1}$, $\bar{B}_I \in \mathbb{R}^{n_1 \times m}$, and $E_I(\cdot)$, $E_{II}(\cdot)$, $F(\cdot)$ are bounded, Lipschitz
 82 continuous functions of time. The form in (2.1) is that of a semi-explicit DALCS. It
 83 corresponds to a kind of extension of the impulse-free Weierstrass form [20], where
 84 the transformed state matrix has non zero off-diagonal terms A_{II} and B_I .

85 *Remark 2.1.* Assume that $K = \mathbb{R}_+^m$. The complementarity constraint (2.1) (c)
 86 is equivalently rewritten as $\phi(\lambda, w) = 0$ where $\phi(\cdot)$ is a C -function [3, 29]. Therefore

87 (2.1) (b) (c) can be rewritten equivalently as $\Phi(x, z, \lambda, t) = 0$ for some function $\Phi(\cdot)$,
 88 usually nondifferentiable. The system (2.1) thus possesses the form of a semi-explicit
 89 DAE as studied in [53, 52], assuming piecewise differentiable functions $\Phi(\cdot)$, with x
 90 the differential state, z and λ the algebraic state variables. The so-called regularity
 91 conditions (which are an extension of index 1 conditions for DAEs) used in [53, 54,
 92 52, 51] may hold in particular cases for (2.1). Examples in section 6.3 illustrate the
 93 links between both approaches in [53, 54, 52, 51] and in this article.

94 Consider (2.1), then the initial data $x(0) = x_0$ and $z(0) = z_0$ are compatible with the
 95 system's constraints if and only if:

$$96 \quad (2.2) \quad \begin{aligned} 0 &\in A_{II}x_0 + B_{II}z_0 + E_{II}(0) - \bar{B}_{II}(D + \mathcal{N}_{K^*})^{-1}(C_Ix_0 + C_{II}z_0 + F(0)) \\ \lambda(0) &\in -(D + \mathcal{N}_{K^*})^{-1}(C_Ix_0 + C_{II}z_0 + F(0)). \end{aligned}$$

97 This implies that $C_Ix_0 + C_{II}z_0 + F(0) \in \text{Im}(D + \mathcal{N}_{K^*})$ and $A_{II}x_0 + B_{II}z_0 + E_{II}(0) \in$
 98 $\bar{B}_{II}\text{Dom}(D + \mathcal{N}_{K^*})$.

99 **3. Time-discretization of DALCS with implicit Euler methods.** Let us
 100 consider the time interval $[0, T]$, $T > 0$, $h = \frac{T}{N}$, $N \in \mathbb{N}_+$, $t_k = kh$, $k \in \{0, 1, \dots, N\}$,
 101 $t_0 = 0$, $t_N = T$. For a continuous function $f(\cdot)$, $f_k \triangleq f(t_k)$. The implicit Euler
 102 discretization of (2.1) reads as:

$$103 \quad (3.1) \quad \begin{cases} (a) & P_I(x_{k+1} - x_k) = hA_Ix_{k+1} + hB_Iz_{k+1} + h\bar{B}_I\lambda_{k+1} + hE_{I,k} \\ (b) & A_{II}x_{k+1} + B_{II}z_{k+1} + \bar{B}_{II}\lambda_{k+1} + E_{II,k} = 0 \\ (c) & K^* \ni \lambda_{k+1} \perp C_Ix_{k+1} + C_{II}z_{k+1} + D\lambda_{k+1} + F_k \in K. \end{cases}$$

104 The problem in (3.1) (b) (c) is a mixed LCP (MLCP). Let us now state a basic
 105 assumption.

106 ASSUMPTION 1. The matrix $P_I^h \triangleq P_I - hA_I$ is full-rank.

107 In case $P_I \succ 0$, [25, Theorem 2.11] can be used to set conditions on hA_I so that
 108 $P_I^h \succ 0$: there always exists h_{\max} such that the assumption holds for all $h \in [0, h_{\max})$
 109 (with $h_{\max} = +\infty$ if $A_I \preceq 0$). Let us define the following matrices and vectors:

- 110 1. $\tilde{A} = hA_{II}(P_I^h)^{-1}B_I + B_{II}$,
- 111 2. $\tilde{B} = hA_{II}(P_I^h)^{-1}\bar{B}_I + \bar{B}_{II}$,
- 112 3. $\tilde{E}_k = E_{II,k} + hA_{II}(P_I^h)^{-1}E_{I,k} + A_{II}(P_I^h)^{-1}P_Ix_k$,
- 113 4. $\tilde{C} = hC_I(P_I^h)^{-1}B_I + C_{II}$,
- 114 5. $\tilde{D} = hC_I(P_I^h)^{-1}\bar{B}_I + D$,
- 115 6. $\tilde{F}_k = F_k + hC_I(P_I^h)^{-1}E_{I,k} + C_I(P_I^h)^{-1}P_Ix_k$.

116 Then we deduce from (3.1) the difference equations:

$$117 \quad (3.2) \quad \begin{cases} (a) & x_{k+1} = (P_I^h)^{-1}(P_Ix_k + hB_Iz_{k+1} + h\bar{B}_I\lambda_{k+1} + hE_{I,k}) \\ (b) & \tilde{A}z_{k+1} + \tilde{B}\lambda_{k+1} + \tilde{E}_k = 0 \\ (c) & K^* \ni \lambda_{k+1} \perp \tilde{C}z_{k+1} + \tilde{D}\lambda_{k+1} + \tilde{F}_k \in K, \end{cases}$$

118 where the last two lines of (3.2) are a MLCP with unknowns z_{k+1} and λ_{k+1} . It is
 119 therefore possible to study the well-posedness of this MLCP independently of the
 120 first line in (3.2). Letting $\zeta_{k+1} \triangleq -z_{k+1}$, this MLCP can be rewritten equivalently as
 121 follows:

$$122 \quad (3.3) \quad \begin{cases} \lambda_{k+1} \in (\mathcal{N}_{K^*} + \tilde{D})^{-1}(-\tilde{C}\zeta_{k+1} - \tilde{F}_k), \\ -\tilde{A}\zeta_{k+1} + \tilde{E}_k \in -\tilde{B}(\mathcal{N}_{K^*} + \tilde{D})^{-1}(\tilde{C}\zeta_{k+1} - \tilde{F}_k), \\ x_{k+1} = (P_I^h)^{-1}(P_Ix_k + hB_Iz_{k+1} + h\bar{B}_I\lambda_{k+1} + hE_{I,k}). \end{cases}$$

123 Also the following calculations stem from the complementarity conditions in (3.2):
 124 (3.4)

$$\begin{aligned} & \lambda_{k+1} \in -\mathcal{N}_K(\tilde{C}z_{k+1} + \tilde{D}\lambda_{k+1} + \tilde{F}_k) \Leftrightarrow -\lambda_{k+1} \in \mathcal{N}_{\Gamma_k}(\tilde{C}z_{k+1} + \tilde{D}\lambda_{k+1}) \\ & \Leftrightarrow \partial\sigma_{\Gamma_k}(-\lambda_{k+1}) \ni \tilde{C}z_{k+1} + \tilde{D}\lambda_{k+1} \Leftrightarrow (\partial\sigma_{\Gamma_k} + \tilde{D})(-\lambda_{k+1}) \ni \tilde{C}z_{k+1} \\ & \Leftrightarrow \lambda_{k+1} \in -(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}z_{k+1}). \end{aligned}$$

125 where the inversion of set-valued mappings is used in the second equivalence, and
 126 where $\sigma_{\Gamma_k}(\cdot)$ is the support function of the set $\Gamma_k(t_k, x_k) \triangleq K - \tilde{F}_k(t_k, x_k)$ (which
 127 is a closed convex set, but not a cone as long as $\tilde{F}_k \neq 0$ which is the generic case).
 128 Therefore equivalently we can rewrite (3.3) as:

$$129 \quad (3.5) \quad \begin{cases} (a) \lambda_{k+1} \in -(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}z_{k+1}) \\ (b) -\tilde{A}z_{k+1} + \tilde{B}(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}z_{k+1}) \ni \tilde{E}_k \\ (c) x_{k+1} = (P_I^h)^{-1}(P_I x_k + hB_I z_{k+1} + h\tilde{B}_I \lambda_{k+1} + hE_{I,k}). \end{cases}$$

130 The second line in (3.3) is a generalized equation (GE) with unknown ζ_{k+1} . Its well-
 131 posedness depends on the properties of the operator $\mathcal{F} : \zeta \mapsto -\tilde{A}\zeta + \tilde{B}(\mathcal{N}_{K^*} +$
 132 $\tilde{D})^{-1}(\tilde{C}\zeta - \tilde{F}_k)$. Similarly for (3.5) with the operator $\mathcal{G} : \zeta \mapsto -\tilde{A}\zeta + \tilde{B}(\partial\sigma_{\Gamma_k} +$
 133 $\tilde{D})^{-1}(\tilde{C}\zeta)$, and (3.5) (b) is: $\mathcal{G}(z_{k+1}) \ni \tilde{E}_k$. Various results have been published to
 134 characterize such operators [16, 23, 6] and to guarantee that they are single-valued
 135 Lipschitz continuous [16, Propositions 1, 2, 3, Corollary 1], or maximal monotone
 136 (possibly set-valued) [23, 6]. The difficulty in the analysis of DALCS is that they
 137 involve the composition of two operators as seen in the OSNSP for advancing the
 138 scheme from step k to step $k+1$:

(3.6)

$$139 \quad P_I^h x_{k+1} \in P_I x_k + hB_I \mathcal{G}^{-1}(\tilde{E}_k(x_k)) - h\tilde{B}_I(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}\mathcal{G}^{-1}(\tilde{E}_k(x_k))) + hE_{I,k}.$$

140 Clearly the scheme can be advanced with a unique solution if both λ_{k+1} and z_{k+1}
 141 are uniquely defined (the ‘‘if and only if’’ holds with $\tilde{B}_I \lambda_{k+1}$ and $B_I z_{k+1}$ uniquely
 142 defined). Thus next we are going to characterize the GEs in (3.3) and (3.5) in order
 143 to analyse the scheme (3.6).

144 **Two main classes of results are provided in the sequel of the article: first the well-**
 145 **posedness (existence and uniqueness of solutions) for the OSNSP associated with the**
 146 **time-discretizations in (3.1) (Propositions 4.3 and 4.4, Corollary 4.6) and (3.2) (Propo-**
 147 **sitions 4.7 and 4.8); second, the convergence of the discrete-time solutions is analysed,**
 148 **and the limit functions are shown to be solutions of the DALCS (Propositions 5.2 and**
 149 **5.9).**

150 **4. OSNSP well-posedness.** The approach chosen in this article consists of
 151 considering the system as an ODE (2.1) (a) with a nonsmooth constraint (2.1) (b)
 152 (c): either (3.1) (b) (c) that is a mixed LCP with unknowns z_{k+1} , λ_{k+1} and parameter
 153 x_{k+1} (section 4.2), or (3.2) (b) (c) (equivalently (3.5) (a) (b)) that is a mixed LCP
 154 with z_{k+1} , λ_{k+1} (section 4.1). As will become clear later, the results rely heavily on
 155 the positive (semi) definiteness of B_{II} , and on specific passivity couplings between
 156 z in the complementarity constraint (matrix C_{II}) and λ in the algebraic constraint
 157 (matrix \tilde{B}_{II}). When $\tilde{D} = 0$ in (3.2) (c), let us state the following:

158 ASSUMPTION 2. *There exists a $p \times p$ matrix $\tilde{X} = \tilde{X}^\top \succ 0$ such that $\tilde{X}\tilde{B} = \tilde{C}^\top$.*

159 The alternative option when $D = 0$ in (3.1) (c):

160 ASSUMPTION 3. *There exists a $p \times p$ matrix $X = X^\top \succ 0$ such that $X\tilde{B}_{II} = C_{II}^\top$.*

161

162 Let us notice that Assumption 2 implies that $\tilde{X}\tilde{B}_{II} = C_{II}^\top + \mathcal{O}(h)$, while Assumption
 163 3 implies that $X\tilde{B} = \tilde{C}^\top + \mathcal{O}(h)$. The influence of the $\mathcal{O}(h)$ terms may not be easy
 164 to characterize precisely for finite $h > 0$. Assumption 3 is a model assumption, while
 165 Assumption 2 is a property of the numerical method. One sees that Assumption 3
 166 corresponds to a passivity constraint on the triple $(B_{II}, \tilde{B}_{II}, C_{II})$ [18]. Assumption
 167 2 imposes an input/output passivity-like constraint on the triple (A, B, \tilde{C}) , which is
 168 an $\mathcal{O}(h)$ approximation of $(B_{II}, \tilde{B}_{II}, C_{II})$. When $\tilde{D} \neq 0$ and $D \neq 0$, extensions of
 169 Assumptions 2 and 3 are as follows:

170 ASSUMPTION 4. *The quadruple $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is passive.*

171 and the counterpart of Assumption 3:

172 ASSUMPTION 5. *The quadruple $(B_{II}, \tilde{B}_{II}, C_{II}, D)$ is passive.*

173 Passivity is defined in Appendix C. Notice that the above assumptions imply that
 174 $n_2 = p$. Assumption 4 refers to the numerical method while Assumption 5 is a model
 175 property. Assumptions 2 and 4 refer to (3.2) (b) (c) or (3.5) (a) (b), while Assumptions
 176 3 and 5 refer to (3.1) (b) (c). Let us present a result about the relationships between
 177 Assumptions 4 and 5, *i.e.*, when does the plant's passivity guarantees the numerical
 178 scheme's passivity.

179 LEMMA 4.1. *The following statements are true:*

- 180 1. *Let $(B_{II}, \tilde{B}_{II}, C_{II}, D)$ be strictly passive. Then $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is strictly passive*
 181 *for small enough $h > 0$.*
- 182 2. *Let $(B_{II}, \tilde{B}_{II}, C_{II}, D)$ be strictly state passive. Then $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is strictly*
 183 *state passive if: a) $h > 0$ is small enough, b) $D + D^\top = 0$ and $C_I(P_I^h)^{-1}\tilde{B}_I +$
 184 $(C_I(P_I^h)^{-1}\tilde{B}_I)^\top \succcurlyeq 0$, c) $XA_{II}(P_I^h)^{-1}\tilde{B}_I - B_I^\top(P_I^h)^{-\top}C_I^\top = 0$, where X is a*
 185 *solution of the passivity LMI for $(B_{II}, \tilde{B}_{II}, C_{II}, D)$.*
- 186 3. *Let $h > 0$ and $(B_{II}, \tilde{B}_{II}, C_{II}, D)$ be passive. Then $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is passive if:*
 187 *b) and c) in item 2 hold, and a) $XA_{II}(P_I^h)^{-1}\tilde{B}_I + B_I^\top(P_I^h)^{-\top}A_{II}^\top X \preccurlyeq 0$,*
 188 *where X is a solution of the passivity LMI for $(B_{II}, \tilde{B}_{II}, C_{II}, D)$.*

189 **Proof:**

- 190 1. The strict passivity of $(B_{II}, \tilde{B}_{II}, C_{II}, D)$ means that there exists $X = X^\top \succ 0$
 191 such that:

$$192 \quad (4.1) \quad M \triangleq \begin{pmatrix} -XB_{II} - B_{II}^\top X & -X\tilde{B}_{II} + C_{II}^\top \\ -\tilde{B}_{II}^\top X + C_{II} & D + D^\top \end{pmatrix} \succ 0.$$

193 In particular this implies that $D + D^\top \succ 0$ [18, Theorem A.65] and the
 194 associated transfer function is strongly SPR [18, Remark 3.19, Section 3.12.2].
 195 Now we have $\tilde{A} = B_{II} + \mathcal{O}(h)$, $\tilde{B} = \tilde{B}_{II} + \mathcal{O}(h)$, $\tilde{C} = C_{II} + \mathcal{O}(h)$, $\tilde{D} =$
 196 $D + \mathcal{O}(h)$. Therefore

$$197 \quad (4.2) \quad M = \underbrace{\begin{pmatrix} -X\tilde{A} - \tilde{A}^\top X & -X\tilde{B} + \tilde{C}^\top \\ -\tilde{B}^\top X + \tilde{C}^\top & \tilde{D} + \tilde{D}^\top \end{pmatrix}}_{\triangleq \tilde{M}} + N(h)$$

198 where $N(h)$ stands for a matrix with suitable dimensions and which vanishes
 199 as $h \rightarrow 0$ (in fact $N(h)$ is $\mathcal{O}(h)$). Thus $\tilde{M} = M - N(h)$. Applying [25,
 200 Theorem 2.11], it follows that $\tilde{M} \succ 0$ provided that $\sigma_{\max}(N(h)) < \lambda_{\min}^2(M)$,
 201 an inequality which can always be verified if $h > 0$ is small enough. Under
 202 this condition, it follows that the strict passivity of $(B_{II}, \tilde{B}_{II}, C_{II}, D)$ implies
 203 the strict passivity of $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ ($\Rightarrow \tilde{D} \succ 0$).

- 204 2. The result is proved by developing \tilde{M} in (4.2), then using [10, Proposition
 205 8.2.4, Fact 8.21.22], or [18, Lemma A.69], as well as the fact that $-XB_{II} -$
 206 $B_{II}^\top X \succ 0$ from the strict state passivity.
 207 3. The proof is similar to that of item 2.

208 \square

209 The converse implication in item 1 does not hold, since one may have $\tilde{M} \succ 0$ for some
 210 $h > 0$, but $M \not\succeq 0$. It is possible to state necessary and sufficient conditions, relying
 211 on [10, Proposition 8.2.4, Fact 8.21.22] or [18, Lemma A.70]. However the obtained
 212 conditions are cumbersome and hardly usable. Lemma 4.1 can be used to determine
 213 the conditions under which Assumption 4 is verified in the results presented in the next
 214 section. The conditions b) and c) in Lemma 4.1 item 2 hold if: b) $C_I = 0$ or $\bar{B}_I = 0$, c)
 215 ($A_{II} = 0$ or $\bar{B}_I = 0$) and ($B_I = 0$ or $C_I = 0$). Condition a) in Lemma 4.1 item 3 holds
 216 if $A_{II} = 0$ or $B_I = 0$. Let us remind that the solution $X = X^\top \succ 0$ of the Lyapunov
 217 equation $XB_{II} + B_{II}^\top X = -Q$, $Q = Q^\top \succ 0$, can be written $X = (-\frac{1}{2}Q + F)B_{II}^{-1}$,
 218 where $F = -F^\top$ is unique [60, Theorems 1 and 2]. Let us denote the symmetric
 219 part of a square matrix M as $\text{sym}(M)$ and its skew-symmetric part as $\text{skw}(M)$. This
 220 allows us to state the following necessary conditions.

221 **COROLLARY 4.2.** 1) Let $(B_{II}, \bar{B}_{II}, C_{II}, D)$ be strictly state passive with $XB_{II} +$
 222 $B_{II}^\top X = -Q$, $Q = Q^\top \succ 0$, and assume that $A_{II}(P_I^h)^{-1}\bar{B}_I$ is right-invertible ($\Rightarrow p \leq$
 223 m). Then $X = (-\frac{1}{2}Q + F)B_{II}^{-1}$, for some $F = -F^\top$ uniquely defined, and condition
 224 c) item 2 in Lemma 4.1 holds only if

$$225 \quad (4.3) \quad \text{sym}\left(\frac{1}{2}QB_{II}^{-1}A_{II}(P_I^h)^{-1}\bar{B}_I + B_I^\top(P_I^h)^{-\top}C_I^\top\right) = F\text{skw}(B_{II}^{-1}A_{II}(P_I^h)^{-1}\bar{B}_I)F,$$

226 for any pair (Q, F) , and

$$227 \quad (4.4) \quad \text{Ker}(C_I(P_I^h)^{-1}B_I) = \{0\}.$$

228 2) Let $(B_{II}, \bar{B}_{II}, C_{II}, D)$ be passive with $XB_{II} + B_{II}^\top X = -Q$, $Q = Q^\top \succcurlyeq 0$ and B_{II}
 229 invertible. Then $X = (-\frac{1}{2}Q + F)B_{II}^{-1}$, for some $F = -F^\top$, and condition a) item 3
 230 in Lemma 4.1 holds only if

$$231 \quad (4.5) \quad \text{sym}(-QB_{II}^{-1}A_{II}(P_I^h)^{-1}B_I) + 2F\text{skw}(B_{II}^{-1}A_{II}(P_I^h)^{-1}B_I)F \preccurlyeq 0,$$

232 for any pair (Q, F) .

233 **Proof:** 1) Condition c) yields $(-\frac{1}{2}Q + F)B_{II}^{-1}A_{II}(P_I^h)^{-1}\bar{B}_I = B_I^\top(P_I^h)^{-\top}C_I^\top$, then
 234 (4.3) follows using $F = -F^\top$. Using the right-invertibility, (4.4) follows from [23,
 235 equ. (70)] and the fact that X is full rank. 2) From passivity any solution must be
 236 symmetric. The Lyapunov equation and the skew-symmetry of F yield (4.5). \square

237 We see that the right-invertibility condition (resp. condition (4.4)) in item 1 of the
 238 lemma, characterizes the couplings between x and λ in the index reduction (resp.
 239 between x and z in the relative degree reduction (*i.e.*, in \dot{w})).

240 **4.1. Analysis of the OSNSP for (3.2).** The next proposition analyses the
 241 OSNSP well-posedness, and applies to two main cases: $\tilde{D} = 0$, and $\tilde{D} \succ 0$. Let us
 242 remind that $p = n_2$ as a consequence of Assumptions 2, 3, 4 and 5, so that \tilde{A} and B_{II}
 243 are $p \times p$ matrices.

244 **PROPOSITION 4.3.** Let $h > 0$, $k \geq 0$ be given.

- 245 1. Let Assumption 4 hold. Suppose that either $\text{Im}(\tilde{C}) \cap \text{int}(\text{Im}(\partial\sigma_{\Gamma_k} + \tilde{D})) \neq \emptyset$
 246 or \tilde{B} is square full rank. Suppose also that there is ζ_0 such that $\tilde{C}\zeta_0 \in$
 247 $\text{Im}(\partial\sigma_{\Gamma_k} + \tilde{D})$. Then the operator $\mathcal{G} : \zeta \mapsto -\tilde{A}\zeta + \tilde{B}(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}\zeta)$ is
 248 maximal monotone with non empty domain.
 249 2. Let $\tilde{D} \succ 0$, then both mappings $(\mathcal{N}_{K^*} + \tilde{D})^{-1}(\cdot)$ and $(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\cdot)$ are
 250 well-defined, single-valued Lipschitz continuous and maximal monotone for
 251 each $h \geq 0, k \geq 0$ (maximal monotonicity holds for $\tilde{D} \succ 0$). The operator
 252 $\mathcal{G}(\cdot)$ is single-valued and Lipschitz continuous also.
 253 3. Let $\tilde{D} = 0$, Assumption 2 hold true, $\tilde{R}^2 = \tilde{X}$, $\tilde{R} = \tilde{R}^\top \succ 0$, $\text{Im}(\tilde{C}) \cap \Gamma_k \neq \emptyset$
 254 and $K = P(G, 0)$ (see Proposition A.1 e)) for some matrix $G \in \mathbb{R}^{m \times m}$. Let
 255 us assume that $\tilde{R}\tilde{A}\tilde{R}^{-1} \preceq 0$. Let us consider the set:

$$(4.6) \quad \begin{aligned} \tilde{\mathcal{S}} &\triangleq P(G\tilde{C}\tilde{R}^{-1}, 0) \\ &\cap \{\xi \in \mathbb{R}^p \mid \tilde{R}\tilde{A}\tilde{R}^{-1}\xi = -\sum_{i=1}^m \lambda_i (G\tilde{C}\tilde{R}^{-1})_{i\bullet}^\top, \lambda_i \geq 0\} \\ &\cap \text{Ker}(\tilde{R}\tilde{A}\tilde{R}^{-1} + \tilde{R}^{-1}\tilde{A}^\top\tilde{R}). \end{aligned}$$

257 Define $\tilde{q}_k = -\tilde{R}\tilde{E}_k$, $\tilde{f}(\cdot) = \psi_{\Gamma_k} \circ \tilde{C}\tilde{R}^{-1}(\cdot)$, $\Gamma_k = K - \tilde{F}_k$, and consider the
 258 VI($-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, \tilde{f}$): Find $\xi_{k+1} = \tilde{R}z_{k+1}$ such that

$$(4.7) \quad \langle -\tilde{R}\tilde{A}\tilde{R}^{-1}\xi_{k+1} + \tilde{q}_k, v - \xi_{k+1} \rangle + \tilde{f}(v) - \tilde{f}(\xi_{k+1}) \geq 0, \text{ for all } v \in \text{dom}(\tilde{f}).$$

- 260 (a) If $\tilde{\mathcal{S}} = \{0\}$, then the VI($-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, \tilde{f}$) has at least one solution.
 261 (b) If $\tilde{\mathcal{S}} \neq \{0\}$, and if there exists $\xi_0 \in \tilde{\Gamma}_k$ such that

$$(4.8) \quad \langle \tilde{q}_k - (-\tilde{R}\tilde{A}\tilde{R}^{-1})^\top \xi_0, v \rangle > 0 \text{ for all } v \in \tilde{\mathcal{S}}, v \neq 0,$$

263 where $\tilde{\Gamma}_k = \{\xi \in \mathbb{R}^{m_2} \mid G\tilde{C}\tilde{R}^{-1}\xi + G\tilde{F}_k \geq 0\} = \text{dom}(\tilde{f})$, then the
 264 VI($-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, \tilde{f}$) has at least one solution.

- 265 (c) If ξ_{k+1}^1 and ξ_{k+1}^2 are two solutions of the VI($-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, \tilde{f}$), then $\xi_{k+1}^1 -$
 266 $\xi_{k+1}^2 \in \text{Ker}(\tilde{R}\tilde{A}\tilde{R}^{-1} + \tilde{R}^{-1}\tilde{A}^\top\tilde{R})$.
 267 (d) Let $\tilde{R}\tilde{A}\tilde{R}^{-1}$ be symmetric. If ξ_{k+1}^1 and ξ_{k+1}^2 are two solutions of the
 268 VI($-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, \tilde{f}$), then $\langle \tilde{q}_k, \xi_{k+1}^1 - \xi_{k+1}^2 \rangle = 0$.
 269 (e) Let $\tilde{R}\tilde{A}\tilde{R}^{-1}$ be symmetric. Then any solution of the VI($-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, \tilde{f}$)
 270 is also a solution to the problem: $\min_{\xi \in \tilde{\Gamma}_k} -\frac{1}{2}\xi^\top \tilde{R}\tilde{A}\tilde{R}^{-1}\xi + \langle \tilde{q}_k, \xi \rangle$.
 271 (f) Let $-\tilde{R}\tilde{A}\tilde{R}^{-1} \succ 0$ and be symmetric. Then $\xi_{k+1} = \text{proj}_{-\tilde{R}\tilde{A}\tilde{R}^{-1}[\tilde{\Gamma}_k; -\tilde{q}_k]}$.

272 **Proof:**

- 273 1. Follows from [6, Theorem 1], which is itself inspired from [23] and [16, The-
 274 orem 2], and which states that the negative feedback interconnection of a
 275 passive system with a maximal monotone operator, defines another maximal
 276 monotone operator (under the stated basic conditions).
 277 2. Follows from [16, Proposition 1] (see Proposition B.1) and from [55, Lemma
 278 1] (or [9, Corollary 24.4 (ii)]) since $\text{Im}(\tilde{D}) = \mathbb{R}^m$ and $\text{int}(\text{Dom}(\mathcal{N}_{K^*})) =$
 279 $\text{int}(K^*) \neq \emptyset$, and $\text{int}(\text{Dom}(\partial\sigma_{\Gamma_k})) \neq \emptyset$ because Γ_k is convex and $\text{int}(\Gamma_k) \neq \emptyset$.
 280 3. The proof follows from Propositions A.1 and A.2. First notice that the MLCP
 281 $\tilde{A}z_{k+1} + \tilde{B}\lambda_{k+1} + \tilde{E}_k = 0$, $K^* \ni \lambda_{k+1} \perp \tilde{C}z_{k+1} + \tilde{D}\lambda_{k+1} + \tilde{F}_k \in K$, can
 282 be equivalently rewritten as the VI($-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, \tilde{f}$) in (4.7). Indeed one
 283 obtains $-\tilde{R}\tilde{A}\tilde{R}^{-1}\xi_{k+1} - \tilde{R}\tilde{E}_k \in -\tilde{R}\tilde{B} \partial\psi_{\Gamma_k}(\tilde{C}\tilde{R}^{-1}\xi_{k+1})$. Using the chain rule
 284 of Convex Analysis [49, Theorem 23.9] and the fact that Γ_k is polyhedral,
 285 the MLCP can written equivalently as in (4.7). Let us now compute the
 286 set \mathcal{S} in (4.6). Here $\text{dom}(\varphi) = \text{dom}(\tilde{f}) = \{z \in \mathbb{R}^p \mid \tilde{C}\tilde{R}^{-1}z \in \Gamma_k\} =$

287 $\{z \in \mathbb{R}^p \mid \tilde{C}\tilde{R}^{-1}z + \tilde{F}_k \in K\} = \{z \in \mathbb{R}^p \mid G\tilde{C}\tilde{R}^{-1}z + G\tilde{F}_k \geq 0\}$. Hence
 288 $(\text{dom}(\varphi))_\infty = P(G\tilde{C}\tilde{R}^{-1}, 0)$ from Proposition A.1 e). Now we have $\tilde{f}(\xi) =$
 289 $\psi_{\tilde{\Gamma}_k}(\tilde{C}\tilde{R}^{-1}\xi) = \psi_{\tilde{\Gamma}_k}(\xi)$ with $\tilde{\Gamma}_k = \{\xi \in \mathbb{R}^p \mid \tilde{C}\tilde{R}^{-1}\xi \in K - \tilde{F}_k\} = \{\xi \in$
 290 $\mathbb{R}^p \mid G\tilde{C}\tilde{R}^{-1}\xi + G\tilde{F}_k \geq 0\} = P(G\tilde{C}\tilde{R}^{-1}, -G\tilde{F}_k)$. Thus $\tilde{f}_\infty(\cdot) = \psi_{(\tilde{\Gamma}_k)_\infty}(\cdot)$
 291 and $\text{dom}(\tilde{f}_\infty) = P(G\tilde{C}\tilde{R}^{-1}, 0)$. Now we have to calculate the dual cone
 292 $P(G\tilde{C}\tilde{R}^{-1}, 0)^*$. Using [31, Example 15], one finds that $P(G\tilde{C}\tilde{R}^{-1}, 0)^* = \{w \in$
 293 $\mathbb{R}^{n_2} \mid w = \sum_{i=1}^m \lambda_i (G\tilde{C}\tilde{R}^{-1})_{i\bullet}^\top, \lambda_i \geq 0\}$, where $(G\tilde{C}\tilde{R}^{-1})_{i\bullet}$ denotes the i th
 294 row of the matrix $G\tilde{C}\tilde{R}^{-1} \in \mathbb{R}^{m \times n_2}$. Hence it follows that $\mathcal{K}(-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{f}) =$
 295 $\{\xi \in \mathbb{R}^{n_2} \mid \tilde{R}\tilde{A}\tilde{R}^{-1}\xi = -\sum_{i=1}^m \lambda_i (G\tilde{C}\tilde{R}^{-1})_{i\bullet}^\top, \lambda_i \geq 0\}$. The third set in $\tilde{\mathcal{S}}$
 296 is equal to $\text{Ker}(\tilde{R}\tilde{A}\tilde{R}^{-1} + \tilde{R}^{-1}\tilde{A}^\top\tilde{R})$, from $\text{Ker}(\mathbf{M} + \mathbf{M}^\top)$ in Proposition A.2
 297 a). Then the results in items (a) to (f) are a consequence of Proposition A.2,
 298 where $\varphi_\infty(v) = 0$ in item (b). Item (f) follows from item e) in Proposition
 299 A.2.

300 \square

301 Item 1 encompasses cases with $\tilde{D} \succ 0$. If item 2 holds, then λ_{k+1} is uniquely defined,
 302 provided that z_{k+1} exists (which is guaranteed if conditions of item 1 hold). From
 303 item 3 (c), it follows that z_{k+1} is unique if \tilde{A} has full rank (recall that B_{II} and \tilde{A} are
 304 square $p \times p$ matrices). However from (3.2) and (3.3), x_{k+1} is uniquely defined if and
 305 only if $\tilde{B}_I(\mathcal{N}_{K^*})^{-1}(-\tilde{C}z_{k+1} - \tilde{F}_k)$ is unique. This happens trivially when $\tilde{B}_I = 0$ (if
 306 one thinks of (2.1) as the limit of a singularly perturbed system, this simply means
 307 that the multiplier enters only the z -dynamics). Let us now deal with the case $\tilde{D} \succ 0$.

308 ASSUMPTION 6. Let $h > 0$, and $\tilde{D} = \begin{pmatrix} \tilde{D}_{11} & 0 \\ 0 & 0 \end{pmatrix}$ with $0 \prec \tilde{D}_{11} = \tilde{D}_{11}^\top \in \mathbb{R}^{m_1 \times m_1}$.

310 Such specific structure occurs for instance when D has the same blockdiagonal struc-
 311 ture and $\tilde{B}_I = 0$, i.e., the multiplier does not intervene in the x -dynamics. Let us
 312 split the multiplier as $\lambda = (\lambda_1^\top, \lambda_2^\top)^\top$, $\lambda_1 \in \mathbb{R}^{m_1}$, $\lambda_2 \in \mathbb{R}^{m_2}$, $m_1 + m_2 = m$, and
 313 $K = K_1 \times K_2 \subset \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ (hence $K^* = K_1^* \times K_2^*$), $\tilde{B} = \begin{pmatrix} \tilde{B}_1 & \tilde{B}_2 \end{pmatrix}$, $\tilde{B}_1 \in \mathbb{R}^{p \times m_1}$,
 314 $\tilde{B}_2 \in \mathbb{R}^{p \times m_2}$, $\tilde{C} = \begin{pmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{pmatrix}$, $\tilde{C}_1 \in \mathbb{R}^{m_1 \times p}$, $\tilde{C}_2 \in \mathbb{R}^{m_2 \times p}$. If $\tilde{D}_{11} = 0$ then $\lambda = \lambda_2$,
 315 $\tilde{B} = \tilde{B}_2$, $\tilde{C} = \tilde{C}_2$. If $\tilde{D} = \tilde{D}_{11} \succ 0$ then item 2 in Proposition 4.3 applies.

316 ASSUMPTION 7. Let $h > 0$. There exists a $(p \times p)$ -matrix $\tilde{X}_2 = \tilde{X}_2^\top \succ 0$ such that
 317 $\tilde{X}_2\tilde{B}_2 = \tilde{C}_2^\top$.

318 This assumption implies that there are couplings between the multiplier λ_{k+1} in the
 319 algebraic equation, and the algebraic variable z_{k+1} in the complementarity conditions,
 320 see (3.2). Define $\tilde{R}_2 = \tilde{R}_2^\top \succ 0$, $\tilde{R}_2^2 = \tilde{X}_2$, and $\xi_{k+1} = \tilde{R}_2 z_{k+1}$. The next result extends
 321 Proposition 4.3 to the positive semidefinite case with blockdiagonal structure.

322 PROPOSITION 4.4. Let Assumptions 6 and 7 hold true, $h > 0$ and $k \geq 0$ be given.
 323 Define $\Gamma_{2,k} = K_2 - \tilde{F}_{2,k}$. Consider the VI: find $\xi_{k+1} \in \text{dom}(\tilde{f}_2)$ such that
 324 (4.9)

$$-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1}\xi_{k+1} - \tilde{R}_2\tilde{E}_k + \underbrace{\tilde{R}_2\tilde{B}_1(\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(\tilde{C}_1\tilde{R}_2^{-1}\xi_{k+1} + \tilde{F}_{1,k})}_{\triangleq \tilde{G}_2(t_k, \xi_{k+1})} \in -\partial\tilde{f}_2(\xi_{k+1})$$

325 where $\tilde{f}_2(\cdot) = \psi_{\Gamma_{2,k}} \circ \tilde{C}_2\tilde{R}_2^{-1}(\cdot)$.

326 1. Suppose that $\text{Im}(\tilde{C}_2) \cap \Gamma_{2,k} \neq \emptyset$. The MLCP $\tilde{A}z_{k+1} + \tilde{B}\lambda_{k+1} + \tilde{E}_k = 0$,
 327 $K^* \ni \lambda_{k+1} \perp \tilde{C}z_{k+1} + \tilde{D}\lambda_{k+1} + \tilde{F}_k \in K$, can be equivalently rewritten as the

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VI in (4.9).

329

2. Assume that the set $\tilde{\Gamma}_{2,k} = \{\xi \in \mathbb{R}^p \mid \tilde{C}_2 \tilde{R}_2^{-1} \xi \in \Gamma_{2,k}\}$ is compact. Then the set of solutions to the VI in (4.9) is compact and nonempty.

330

3. Assume that for each t_k , the following well-defined, single-valued and Lipschitz continuous mapping, is maximal monotone:

$$\tilde{G}_2 : \xi_{k+1} \mapsto \tilde{R}_2 \tilde{B}_1 (\tilde{D}_{11} + \partial \psi_{K_1^*})^{-1} (\tilde{C}_1 \tilde{R}_2^{-1} \xi_{k+1} - \tilde{F}_{1,k}).$$

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If $-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \succ 0$, $\tilde{\Gamma}_{2,k} \cap \text{Int}(\text{Dom}(\tilde{G}_2)) \neq \emptyset$ or $\text{Int}(\tilde{\Gamma}_{2,k}) \cap \text{Dom}(\tilde{G}_2) \neq \emptyset$, then the set of solutions to the VI in (4.9) is closed and convex. If $-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \succ 0$, the VI in (4.9) has a unique solution.

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Proof:

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1. The complementarity conditions in (3.2) (c) are rewritten as:

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$$(4.10) \quad \begin{pmatrix} K_1^* \\ K_2^* \end{pmatrix} \ni \begin{pmatrix} \lambda_{1,k+1} \\ \lambda_{2,k+1} \end{pmatrix} \perp \begin{pmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{pmatrix} z_{k+1} + \begin{pmatrix} \tilde{D}_{11} \lambda_{1,k+1} \\ 0 \end{pmatrix} + \begin{pmatrix} \tilde{F}_{1,k} \\ \tilde{F}_{2,k} \end{pmatrix} \in \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$$

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while the equality constraint in (3.2) (b) is rewritten as:

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$$(4.11) \quad \tilde{A} z_{k+1} + \tilde{B}_1 \lambda_{1,k+1} + \tilde{B}_2 \lambda_{2,k+1} + \tilde{E}_k = 0.$$

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From (4.10) it is deduced that:

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$$(4.12) \quad \begin{cases} \lambda_{1,k+1} = -(\mathcal{N}_{K_1^*} + \tilde{D}_{11})^{-1} (\tilde{C}_1 z_{k+1} + \tilde{F}_{1,k}) \\ \lambda_{2,k+1} \in -\mathcal{N}_{K_2 - \tilde{F}_{2,k}} (\tilde{C}_2 z_{k+1}). \end{cases}$$

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Using (4.11) it is inferred:

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$$(4.13) \quad \begin{aligned} & \tilde{A} z_{k+1} - \tilde{B}_1 (\mathcal{N}_{K_1} + \tilde{D}_{11})^{-1} (\tilde{C}_1 z_{k+1} + \tilde{F}_{1,k}) \\ & \quad - \tilde{B}_2 \mathcal{N}_{K_2 - \tilde{F}_{2,k}} (\tilde{C}_2 z_{k+1}) + \tilde{E}_k = 0 \\ & \quad \quad \quad \updownarrow \\ & -\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_{k+1} + \tilde{R}_2 \tilde{B}_1 (\mathcal{N}_{K_1^*} + \tilde{D}_{11})^{-1} (\tilde{C}_1 z_{k+1} + \tilde{F}_{1,k}) - \tilde{R}_2 \tilde{E}_k \\ & \quad \quad \quad \in -\tilde{R}_2 \tilde{B}_2 \mathcal{N}_{K_2 - \tilde{F}_{2,k}} (\tilde{C}_2 \tilde{R}_2^{-1} \xi_{k+1}), \end{aligned}$$

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which is (4.9) since $\tilde{R}_2 \tilde{B}_2 = \tilde{R}_2^{-1} \tilde{X}_2 \tilde{B}_2 = \tilde{R}_2^{-1} \tilde{C}_2^\top$ and applying the chain rule of Convex Analysis for polyhedral functions.

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345

2. This follows from [29, Corollary 2.2.5] and since the map $(\tilde{D}_{11} + \partial \psi_{K_1^*})^{-1}(\cdot)$ is single-valued, well defined and Lipschitz continuous (see Proposition B.1).

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347

3. The first assertion is a consequence of [9, Corollary 24.4] (for the maximal monotonicity of the mapping $(-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} + \tilde{G}_2 + \partial \tilde{f}_2)(\cdot)$, and of [9, Proposition 23.39]. The second assertion follows from [29, Theorem 2.3.3], noting that the mapping in the left-hand side of (4.9) is maximal monotone [9, Corollary 24.4 (ii)] and strongly monotone since $-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \succ 0$. This can also be proved using [9, Corollary 23.37]. Single-valuedness and Lipschitz continuity stem from $\tilde{D}_{11} \succ 0$. \square

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Further features of the solutions to the VI in (4.9) will be studied in section 5.1.3. The next lemma states a classical condition that guarantees the maximal monotonicity of Proposition 4.4 item 3. The first condition in item 1 is a constraint qualification which guarantees that the function $\tilde{f}_2(\cdot)$ has a nonempty domain. It is usually seen as a technical condition, however in some applications it may not be satisfied. Item 1 states conditions under which the VI in (4.9) can be constructed. Item 3 uses some results about the zeroes of maximal monotone mappings, or of the sum of these.

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361 LEMMA 4.5. Let $\tilde{X}_2 \tilde{B}_1 = \tilde{C}_1^\top$ for \tilde{X}_2 in Assumption 7, then for each fixed t_k , the
 362 mapping $\tilde{G}_2 : \xi_{k+1} \mapsto \tilde{R}_2 \tilde{B}_1 (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1} (\tilde{C}_1 \tilde{R}_2^{-1} \xi_{k+1} + \tilde{F}_{1,k})$ is maximal monotone.

363 **Proof:** Indeed $\tilde{X}_2 \tilde{B}_1 = \tilde{C}_1^\top \Rightarrow \tilde{R}_2 \tilde{B}_1 = \tilde{R}_2^{-1} \tilde{C}_1^\top = (\tilde{C}_1 \tilde{R}_2^{-1})^\top$. The mapping $(\tilde{D}_{11} +$
 364 $\partial\psi_{K_1^*})^{-1}(\cdot)$ is maximal monotone, and the following operation: $(\tilde{C}_1 \tilde{R}_2^{-1})^\top (\tilde{D}_{11} +$
 365 $\partial\psi_{K_1^*})^{-1}(\tilde{C}_1 \tilde{R}_2^{-1} \cdot + \tilde{F}_{1,k})$ preserves the maximal monotonicity [50, Exercise 12.4]. \square

366 Propositions 4.3 and 4.4 provide some informations about the existence and unique-
 367 ness of the unknown z_{k+1} , and characterize some mappings. It is noteworthy that
 368 other well-posedness results for VIs are available in the literature. For instance if
 369 the set $\tilde{\Gamma}_k$ is compact, [29, Corollary 2.2.5] guarantees existence of a solution to
 370 VI $(-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, \tilde{f})$ with a compact set of solutions. Also when $\tilde{\Gamma}_k = \mathbb{R}_+^m$ one may
 371 rely on P-matrices and LCPs properties, when it is a closed convex cone one may use
 372 copositive matrices [29, Theorem 2.5.10].

373 The next corollary completes the existence and uniqueness results for the OSNSP.

374 COROLLARY 4.6. *The next results hold true:*

- 375 1. (set of solutions z_{k+1}) Let Assumption 4 and the conditions of item 1 in
 376 Proposition 4.3 hold. Then the set of solutions z_{k+1} in (3.5) (or the set of
 377 solutions ζ_{k+1} in (3.3)), is closed and convex. If in addition the conditions
 378 of item 3 (a) or (b) in Proposition 4.3 hold, then the set of solutions is non
 379 empty. If in addition $\tilde{A} \prec 0$, then z_{k+1} is unique.
- 380 2. (set of solutions λ_{k+1}) Let $\tilde{C} \text{Dom}(\mathcal{G}) \subseteq \text{Im}(\partial\sigma_{\Gamma_k} + \tilde{D})$, and the conditions of
 381 item 1 in Proposition 4.3 hold true. Then the set of solutions λ_{k+1} in (3.5)
 382 (a) is non empty for each z_{k+1} .
- 383 3. Let Assumption 4 hold with strict passivity, and the conditions of item 1 in
 384 Proposition 4.3 be true. Then the GE in (3.5) (or the GE in (3.3)) has
 385 a unique solution (λ_{k+1}, z_{k+1}) for any \tilde{E}_k . Consequently x_{k+1} in (3.2) is
 386 unique.
- 387 4. Let $\tilde{D} = 0$ and $\mathcal{G}(\cdot)$ be maximal monotone. Then if the set $\tilde{\Gamma}_k = \{\zeta \in$
 388 $\mathbb{R}^{n_2} \mid G\tilde{C}\tilde{R}^{-1}\zeta + G\tilde{F}_k \geq 0\}$ is bounded, the GE with unknown z_{k+1} in (3.5)
 389 (or in (3.3)) has at least one solution and the set of solutions is compact.

390 **Proof:**

- 391 1. Follows from [9, Proposition 23.39] by the maximal monotonicity of $\mathcal{G}(\cdot)$. If
 392 $-\tilde{A} \succ 0$ then $\mathcal{G}(\cdot)$ is strongly monotone (however it may not be Lipschitz
 393 continuous and this property is used later in section 5.2), and so is $\mathcal{G}(\cdot) + \tilde{E}_k$.
 394 Hence the generalized equation $\mathcal{G}(z_{k+1}) + \tilde{E}_k \ni 0$ has a unique solution [9,
 395 Corollary 23.37].
- 396 2. Let us consider $\lambda_{k+1} \in -(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}z_{k+1})$. We know that any solution
 397 $z_{k+1} \in \mathcal{G}^{-1}(\tilde{E}_k) \subseteq \text{Im}(\mathcal{G}^{-1}) = \text{Dom}(\mathcal{G})$. Since $\text{Dom}(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1} = \text{Im}(\partial\sigma_{\Gamma_k} +$
 398 $\tilde{D})$ the result follows.
- 399 3. Strict passivity implies that $\tilde{D} \succ 0$ and \tilde{A} is a Hurwitz matrix, therefore
 400 from a slight modification of [6, Theorem 1] it follows that the operator $\mathcal{G}(\cdot)$
 401 is strongly, hence strictly monotone (indeed there exists a real $\alpha > 0$ small
 402 enough such that $(\tilde{A} + \alpha I_{n_2}, \tilde{B}, \tilde{C}, \tilde{D})$ is passive). Using [9, Proposition 23.35]
 403 shows that z_{k+1} is unique, and item 2 in Proposition 4.3 shows that λ_{k+1} is
 404 unique. The conclusion follows.
- 405 4. Under these conditions $\text{Dom}(\mathcal{G})$ is bounded (compact). Hence from [9, Propo-
 406 sition 23.36 (iii)] existence of a solution holds true. From [29, Corollary 2.2.5]
 407 the compactness of the set of solutions follows. This could also be inferred
 408 from item (e) in Proposition 4.3 in case $\tilde{R}\tilde{A}\tilde{R}^{-1}$ is symmetric.

409 \boxtimes

410 Item 3 in Corollary 4.6 guarantees the uniqueness of solutions to the OSNSP in (3.6),
 411 in which case the \in can be replaced by equality. Concerning item 2 in Corollary 4.6,
 412 an important step is to characterize both $\text{Dom}(\mathcal{G})$ and $\text{Im}(\partial\sigma_{\Gamma_k} + \tilde{D})$. If $\tilde{D} = \tilde{D}^\top \succcurlyeq 0$,
 413 one can apply [15, Theorem 3], and infer that $\text{Im}(\partial\sigma_{\Gamma_k} + \tilde{D}) \simeq \text{Im}(\partial\sigma_{\Gamma_k}) + \text{Im}(\tilde{D}) =$
 414 $\text{Dom}(\partial\psi_{\Gamma_k}) + \text{Im}(\tilde{D}) \subset \text{dom}(\psi_{\Gamma_k}) + \text{Im}(\tilde{D}) = \Gamma_k + \text{Im}(\tilde{D})$ (see [15] for the exact
 415 meaning of \simeq in this context). Similar calculations may be done for in item 1 of
 416 Proposition 4.3. It is inferred that the basic condition in Proposition 4.3 item 1,
 417 i.e., $\text{Im}(\tilde{C}) \cap \text{int}(\text{Im}(\partial\sigma_{\Gamma_k} + \tilde{D})) \neq \emptyset$, or in Corollary 4.6 item 2, i.e., $\tilde{C} \text{Dom}(\mathcal{G}) \subseteq$
 418 $\text{Im}(\partial\sigma_{\Gamma_k} + \tilde{D})$, always are verified when $\tilde{D} \succ 0$. For the sake of brevity the conditions
 419 for the existence and uniqueness of λ_{k+1} in (4.12) are not stated, similar tools as for
 420 Corollary 4.6 can be used.

421 **4.2. Analysis of the OSNSP for (3.1).** It is also interesting to investigate
 422 the case when Assumption 3 (resp. Assumption 5) is satisfied instead of Assumption
 423 2 (resp. Assumption 4). The operator $\mathcal{G}(\cdot)$ is now defined as $\mathcal{G} : \zeta \mapsto -B_{II}\zeta +$
 424 $(\partial\sigma_{\Gamma(t_k, x_{k+1})} + D)^{-1}(C_{II}\zeta)$, where $\Gamma(t_k, x_{k+1}) = \{\xi \in \mathbb{R}^m \mid \xi + C_I x_{k+1} + F_k \in$
 425 $K\}$. Consequently the results of Proposition 4.3 item 1, Corollary 4.6 item 3, with
 426 Assumption 5 plus strict passivity, and Proposition 4.3 item 2 when $D \succ 0$, hold. For
 427 the sake of brevity we do not develop all the arguments once again.
 428 Let us focus now on the case $D \succcurlyeq 0$. From Assumption 3, $D = 0$ and (2.1) one infers:

$$429 \quad (4.14) \quad -RB_{II}R^{-1}\xi(t) - RA_{II}x(t) - RE_{II}(t) \in -\partial\psi_{\Phi(t, x(t))}(\xi(t)),$$

430 where $\xi = -Rz$, $R^2 = X$, $R = R^\top \succ 0$, $\Phi(t, x) = \{\xi \in \mathbb{R}^p \mid C_{II}R^{-1}\xi + C_I x + F(t) \in$
 431 $K\}$, and the set-valued term is obtained using the chain rule, under the Convex
 432 Analysis chain rule qualification condition [49, Theorem 23.9]. The first step is to
 433 analyse the existence and uniqueness of solutions to the VI:

$$434 \quad (4.15) \quad -RB_{II}R^{-1}\xi_{k+1} - RA_{II}x_{k+1} - RE_{II,k} \in -\partial\psi_{\Phi(t_k, x_{k+1})}(\xi_{k+1}),$$

435 which is the counterpart of (4.7), relying on Proposition A.2. The unknown is ξ_{k+1}
 436 and the other terms and variables are considered as data for this VI. Similar results
 437 to those in item 3 of Proposition 4.3 can be obtained.

438 **PROPOSITION 4.7.** *Let $D = 0$, Assumption 3 be true, $h > 0$ be given, assume that*
 439 *$\text{Im}(C_{II}) \cap (K - C_I x_{k+1} + F_k) \neq \emptyset$ and that K is polyhedral. Consider the discrete-time*
 440 *system in (3.1), and the associated VI($-RB_{II}R^{-1}$, $-RA_{II}x_{k+1} - RE_{II,k}$, $\psi_{\Phi(t_k, x_{k+1})}$)*
 441 *in (4.15). Assume that $-RB_{II}R^{-1} \succcurlyeq 0$. Consider the set:*

$$442 \quad (4.16) \quad \mathcal{S} \triangleq P(GC_{II}R^{-1}, 0) \cap \{\xi \in \mathbb{R}^{n_2} \mid RB_{II}R^{-1}\xi = \sum_{i=1}^m \lambda_i (GC_{II}R^{-1})_{i\bullet}^\top, \lambda_i \geq 0\}$$

$$\cap \text{Ker}(RB_{II}R^{-1} + R^{-1}B_{II}^\top R).$$

443 *Then:*

- 444 1. *If $\mathcal{S} = \{0\}$ the VI($-RB_{II}R^{-1}$, $-RA_{II}x_{k+1} - RE_{II,k}$, $\psi_{\Phi(t_k, x_{k+1})}$) has a unique*
 445 *solution.*
- 446 2. *If $\mathcal{S} \neq \{0\}$, and if there exists $\xi_0 \in \Phi(t_k, x_{k+1})$ such that $\langle -RA_{II}x_{k+1} -$*
 447 *$RE_{II,k} + RB_{II}R^{-1}\xi_0, v \rangle > 0$ for all $v \neq 0$ and $v \in \mathcal{S}$, then it follows that the*
 448 *VI($-RB_{II}R^{-1}$, $-RA_{II}x_{k+1} - RE_{II,k}$, $\psi_{\Phi(t_k, x_{k+1})}$) has a unique solution.*

449 **Proof:** The set \mathcal{S} in (4.16) is constructed as follows. The chain rule holds from
 450 [49, Theorem 23.9] and the fact that $\psi_{K-C_I x_{k+1}+F_k}(\cdot)$ is polyhedral. Hence the
 451 VI in (4.15) is equivalent to (3.1) (b) (c). Here $\varphi = \psi_{\Phi(t_k, x_{k+1})}$, hence $\text{dom}(\varphi) =$
 452 $\Phi(t_k, x_{k+1})$ and $(\text{dom}(\varphi))_\infty$. Assume that $K = P(G, 0)$ (see Proposition A.1 (e)),
 453 then $(\text{dom}(\varphi))_\infty = \{\xi \in \mathbb{R}^p \mid GC_{II}R^{-1}\xi \geq 0\} = P(GC_{II}R^{-1}, 0)$. Further, $\varphi_\infty =$
 454 $\psi_{(\Phi(t_k, x_{k+1}))_\infty} = \psi_{P(GC_{II}R^{-1}, 0)}$, hence $\text{dom}(\varphi_\infty) = P(GC_{II}R^{-1}, 0)$ and $\text{dom}(\varphi_\infty)^* =$
 455 $(P(GC_{II}R^{-1}, 0))^* = \{w \in \mathbb{R}^{n_2} \mid w = -\sum_{i=1}^m \lambda_i (GC_{II}R^{-1})_{i\bullet}^\top, \lambda_i \geq 0\}$, where [31,
 456 Example 15] was used as in the proof of Proposition 4.3. Thus $\mathcal{K}(-RB_{II}R^{-1}, \varphi) =$
 457 $\{\xi \in \mathbb{R}^{n_2} \mid -RB_{II}R^{-1}\xi = -\sum_{i=1}^m \lambda_i (GC_{II}R^{-1})_{i\bullet}^\top, \lambda_i \geq 0\}$. Finally in Proposition
 458 A.2: $\text{Ker}(\mathbf{M} + \mathbf{M}^\top) = \text{Ker}(RB_{II}R^{-1} + (RB_{II}R^{-1})^\top)$ and $R = R^\top$. Items 1 and 2
 459 are a direct consequence of Proposition A.2 (a) and (b). \square

460 For the sake of brevity we do not state similar results as in Proposition 4.3 (c) to
 461 (f). To enlarge the analysis let us now state the counterpart of Assumption 6:

462 ASSUMPTION 8. *Let $D = \begin{pmatrix} D_{11} & 0 \\ 0 & 0 \end{pmatrix}$ with $0 \prec D_{11} = D_{11}^\top \in \mathbb{R}^{m_1 \times m_1}$.*

463 Similarly to Assumption 7, with the same splitting of the multiplier λ , the matrices
 464 B_{II} and C_{II} , and of the cone K (see the comments after Assumption 6, in particular
 465 if $D_{11} = 0$ the case $D = 0$ is recovered with $G_2(\cdot, \cdot)$ in (4.17) which vanishes, and the
 466 set $\Phi_2 = \Phi$), let us state the following:

467 ASSUMPTION 9. *There exists a $(p \times p)$ -matrix $X_2 = X_2^\top \succ 0$ such that $X_2 \bar{B}_{II}^2 =$
 468 $(C_{II}^2)^\top$.*

469 Let us recall that p is the number of equality constraints in (2.1). Let us define
 470 $R_2 = R_2^\top \succ 0$, $R_2^2 = X_2$, $\xi_{k+1} = R_2 z_{k+1}$, $\Phi_2(t_k, x_{k+1}) = \{\xi \in \mathbb{R}^p \mid C_{II}^2 R_2^{-1} \xi +$
 471 $C_I^2 x_{k+1} + F_{2,k} \in K_2\}$. The next proposition is the counterpart of Proposition 4.4. It
 472 states conditions that guarantee that z_{k+1} can be calculated from (3.1) (b) (c).

473 PROPOSITION 4.8. *Let $h > 0$, and Assumptions 8 and 9 hold true. Consider the
 474 VI: find $\xi_{k+1} \in \text{dom}(f_2)$ such that*

$$475 \quad (4.17) \quad \underbrace{-R_2 B_{II} R_2^{-1} \xi_{k+1} - R_2 A_{II} x_{k+1} - R_2 E_{II,k} + R_2 \bar{B}_{II}^1 (D_{11} + \partial \psi_{K_1^*})^{-1} (C_{II}^1 R_2^{-1} \xi_{k+1} - F_{1,k})}_{\triangleq G_2(t_k, \xi_{k+1})} \in -\partial f_2(\xi_{k+1})$$

476 with $f_2(\cdot) = (\psi_{K_2 - C_I^2 x_{k+1} - F_{2,k}} \circ C_{II}^2 R_2^{-1})(\cdot) = \psi_{\Phi_2(t_k, x_{k+1})}(\cdot)$.

- 477 1. Assume that $\text{Im}(C_{II}^2) \cap (K_2 - C_I^2 x_{k+1} - F_{2,k}) \neq \emptyset$, then the VI in (4.17) is
 478 equivalent to the MLCP in (3.1) (b) (c).
- 479 2. Suppose that the set $\Phi_2(t_k, x_{k+1})$ is compact and nonempty, then the set of
 480 solutions to the VI in (4.17) is compact and nonempty.
- 481 3. Assume that for each fixed t_k , the mapping $G_2(\cdot)$ is maximal monotone, and
 482 that $-R_2 B_{II} R_2^{-1} \succ 0$, then the VI in (4.17) has a unique solution.
- 483 4. Assume that $-R_2 B_{II} R_2^{-1} \succcurlyeq 0$, $G_2(\cdot)$ is maximal monotone for each fixed t_k ,
 484 and $\Phi_2(t_k, x_{k+1}) \cap \text{Int}(\text{Dom}(G_2)) \neq \emptyset$ or $\text{Int}(\Phi_2(t_k, x_{k+1})) \cap \text{Dom}(G_2) \neq \emptyset$.
 485 Then the set of solutions to the VI in (4.17) is closed and convex.

486 **Proof:** The proof of the first three items follows closely that of Proposition 4.4 and it
 487 is therefore omitted. Item 4: the VI is rewritten as $\underbrace{(-R_2 B_{II} R_2^{-1} + G_2 + \partial f_2)}_{\triangleq H_2}(\xi_{k+1}) -$

488 $R_2 A_{II} x_{k+1} - R_2 E_{II,k} \ni 0$, and the assumption guarantees that $H_2(\cdot)$ is maximal
 489 monotone [9, Corollary 24.4]. The result follows from [9, Proposition 23.39]. \square

490 The same as Lemma 4.5 can be stated to guarantee the maximal monotonicity in
 491 items 3 and 4. It is noteworthy that the main discrepancy between Proposition
 492 4.4 and Proposition 4.8, is that the solution ξ_{k+1} depends on x_{k+1} in the latter,
 493 while it depends on x_k in the former. This is a difficulty with the application of
 494 Proposition 4.8 in some cases, for instance when $C_{II} = 0$ (examples in sections 6.2
 495 and 6.4 will demonstrate that this can occur). Actually, the condition $\text{Im}(C_{II}^2) \cap (K_2 -$
 496 $C_I^2 x_{k+1} - F_{2,k}) \neq \emptyset$ (that is a form of constraint qualification), or its counterparts in
 497 Propositions 4.3, 4.4, 4.7, happens to be crucial in the setting of this work. It states
 498 that the algebraic variable z has to enter the complementarity constraints through
 499 a C_{II} matrix which has a big enough range. This may occur in some systems, see
 500 section 6.5.

501 **5. Convergence results.** In this section it is assumed that the conditions for
 502 existence of solutions in Propositions 4.3, 4.4, 4.7, or 4.8, are fulfilled, so that sequences
 503 can be calculated. The section is divided into two main parts: in sections 5.1.1, 5.1.2
 504 and 5.1.3, convergence is analysed from the properties of the quadruple $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$
 505 which characterize the MLCP in (3.2) (b) (c). In section 5.2, the analysis is led with
 506 the properties of $(B_{II}, \tilde{B}_{II}, C_{II}, D)$ which characterize the MLCP in (3.1) (b) (c).
 507 The relationships between both approaches are explained.

508 **5.1. Convergence results from (3.2).** In this section the numerical scheme
 509 in (3.2) and the results in section 4.1 are used. First let us notice that z_{k+1} and
 510 λ_{k+1} are solutions of the generalized equation in (3.5) (a) (b). Thus if existence of
 511 solutions hold, they both are functions of \tilde{E}_k (in addition to being functions of \tilde{F}_k
 512 through the set Γ_k). Under some conditions which are recalled below, like item 2 of
 513 Proposition 4.3, they can be single-valued Lipschitz continuous functions of \tilde{E}_k . Then
 514 the right-hand side of the first line in (3.2) may be written compactly as

$$515 \quad (5.1) \quad P_I^h x_{k+1} = P_I x_k + h \bar{E}_k(t_k, x_k)$$

516 for some $\bar{E}_k(t_k, x_k)$ which is a linear function of $E_{I,k}$, $E_{II,k}$, λ_{k+1} and z_{k+1} . The
 517 time-dependence of \bar{E}_k comes from $E_{I,k}$, $E_{II,k}$ and F_k , while the state dependence
 518 comes from the third terms in \tilde{E}_k and in \tilde{F}_k (which enters $\Gamma_k(t_k, x_k) = K - \tilde{F}_k$). From
 519 (5.1) it can be calculated and checked by induction that:

$$520 \quad (5.2) \quad x_k = ((P_I^h)^{-1} P_I)^k x_0 + h \sum_{j=0}^{k-1} ((P_I^h)^{-1} P_I)^{k-1-j} (P_I^h)^{-1} \bar{E}_j(t_j, x_j), \quad k \geq 1,$$

521 Since $P_I^h = P_I - hA_I$, it follows that for $h > 0$ small enough, $(P_I^h)^{-1} P_I = I_p + \mathcal{O}(h)$
 522 [10, Proposition 9.4.13] hence $(P_I^h)^{-1} = P_I^{-1} + \mathcal{O}(h)$. Thus

$$523 \quad (5.3) \quad \begin{aligned} x_k &= x_0 + \mathcal{O}(h) + h \sum_{j=0}^{k-1} (I_p + \mathcal{O}(h))^{k-1-j} (P_I^{-1} + \mathcal{O}(h)) \bar{E}_j(t_j, x_j) \\ &= x_0 + \mathcal{O}(h) + h \sum_{j=0}^{k-1} (I_p + \mathcal{O}(h)) (P_I^{-1} + \mathcal{O}(h)) \bar{E}_j(t_j, x_j) \\ &= x_0 + \mathcal{O}(h) + h \sum_{j=0}^{k-1} (P_I^{-1} + \mathcal{O}(h)) \bar{E}_j(t_j, x_j) \\ &= x_0 + \mathcal{O}(h) + h (P_I^{-1} + \mathcal{O}(h)) \sum_{j=0}^{k-1} \bar{E}_j(t_j, x_j), \end{aligned}$$

524 for all $k \geq 1$. It is inferred that:

$$525 \quad (5.4) \quad \begin{aligned} \dot{x}_k &\triangleq \frac{x_k - x_{k-1}}{h} = \frac{\mathcal{O}(h)}{h} + (P_I^{-1} + \mathcal{O}(h)) \bar{E}(t_{k-1}, x_{k-1}) \\ &= \mathcal{O}(1) + (P_I^{-1} + \mathcal{O}(h)) \bar{E}(t_{k-1}, x_{k-1}). \end{aligned}$$

526 Our aim is now to prove that both $\{x_k\}$ and $\{\dot{x}_k\}$ are uniformly bounded sequences,
 527 which will allow us to conclude about the convergence of the sequence $\{x^h(\cdot)\}$ (see the
 528 **definition in (5.7)**). For this it is necessary and sufficient that $\bar{E}_k(t_k, x_k)$ be uniformly
 529 bounded for all $k \geq 0$, since P_I is full rank. It follows from (3.5) (a) and (b), that
 530 there are two operators which are crucial to characterize the right-hand side in (3.6).

531 **5.1.1. Analysis of the operator $\mathcal{G}(\cdot)$.** Notice that the mapping $\mathcal{Z} : \zeta \mapsto \eta =$
 532 $(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\zeta)$ is under the conditions of Proposition 4.3 item 2, a well-defined
 533 $\mathbb{R}^m \rightarrow \mathbb{R}^m$ single-valued Lipschitz continuous mapping which is parameterized by
 534 t_k and x_k through the set $\Gamma_k(t_k, x_k)$. Assume that x_k is bounded, then redoing
 535 the proof of **Proposition B.1**, it follows that the Lipschitz constant of $\mathcal{Z}(\cdot)$ does not
 536 depend on $\partial\sigma_{\Gamma_k}(\cdot)$ but depends only on \tilde{D} , being equal to $\frac{2}{\lambda_{\min}(\tilde{D} + \tilde{D}^\top)}$. In turn
 537 $\lambda_{\min}(\tilde{D} + \tilde{D}^\top) > 0$ uniformly in $h > 0$ only if $D \succ 0$. Thus the operator $\mathcal{G} :$
 538 $\zeta \mapsto -\tilde{A}\zeta + \tilde{B}(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}\zeta)$ is also single-valued Lipschitz continuous (being the
 539 composition of Lipschitz continuous mappings) with a Lipschitz constant that does
 540 not depend on x_k , and which is bounded for bounded h . Let us now assume the
 541 following which reinforces Assumption 4 and guarantees that $\tilde{D} \succ 0$:

542 **ASSUMPTION 10.** *The system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is strictly passive, uniformly in $h \geq 0$.*
 543

544 Then if the basic conditions stated in item 1 of Proposition 4.3 are verified in addition
 545 to Assumption 10, the operator $\mathcal{G}(\cdot)$ is strongly monotone for some constant $\epsilon > 0$,
 546 and we know from Corollary 4.6 item 3, that x_{k+1} is unique. As stated in Proposition
 547 4.3 item 1, maximal monotonicity follows from [6, Theorem 1]. Strong monotonicity
 548 follows from the fact that under Assumption 10, the quadruple $(\tilde{A} + \epsilon I_{n_2}, \tilde{B}, \tilde{C}, \tilde{D})$
 549 is passive for a small enough $\epsilon > 0$, so that $(\mathcal{G} - \epsilon I_{n_2})(\cdot)$ is maximal monotone for
 550 such ϵ . Thus if Assumption 10 and the basic conditions in item 1 of Proposition
 551 4.3 are true, $\mathcal{G} : \mathbb{R}^p \rightarrow \mathbb{R}^p$ is well-defined and single-valued, Lipschitz continuous
 552 and strongly monotone. Consequently it is inferred from [50, Proposition 12.54] that
 553 under the above set of conditions, $\mathcal{G}^{-1}(\cdot)$ is single-valued, well defined and globally
 554 Lipschitz, and the solution z_{k+1} of the generalized equation $\bar{E}_k(t_k, x_k) \in \mathcal{G}(z_{k+1}) \Leftrightarrow$
 555 $z_{k+1} = \mathcal{G}^{-1}(\bar{E}_k(t_k, x_k))$ in (3.5) is Lipschitz continuous in x_k and t_k . Thus λ_{k+1} is
 556 also Lipschitz continuous in x_k , and we conclude that the same applies to x_{k+1} in
 557 (3.2).

558 *Remark 5.1.* Assumption 10 is stringent. Actually the strict state passivity of
 559 $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is sufficient to guarantee that $\mathcal{G}(\cdot)$ is strongly monotone. However in
 560 the body of sections 5.1.1 and 5.1.2 we need also that λ_{k+1} be a Lipschitz continuous
 561 function of its argument, a sufficient condition being that $\tilde{D} \succ 0$, which in turn implies
 562 that strict state passivity and strict passivity are identical. Thus strict passivity could
 563 be relaxed to strict state passivity if it is assured that $\bar{B}_I(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}\cdot)$ is Lipschitz
 564 continuous. This may occur in certain cases (the trivial case is $\bar{B}_I = 0$).

565 **5.1.2. Boundedness and convergence analysis.** Under the above conditions,
 566 the term $\bar{E}_k(t_k, x_k)$ and consequently $x_{k+1}(t_k, x_k)$ in (3.2) and $\dot{x}_k(t_k, x_k)$ in (5.4) are
 567 single-valued and Lipschitz continuous in x_k and t_k (provided $E_I(\cdot)$, $E_{II}(\cdot)$ and $F(\cdot)$
 568 are Lipschitz continuous functions of t , which is a mild requirement on the data),
 569 since \bar{E}_k and \bar{F}_k are linear in the argument x_k . We have $\|\bar{E}(t_k^1, x_k^1) - \bar{E}(t_k^2, x_k^2)\| \leq$
 570 $L_t|t_k^1 - t_k^2| + L_x\|x_k^1 - x_k^2\|$ for any $t_k^1, t_k^2, x_k^1, x_k^2$, and for some $L_t > 0$ and $L_x > 0$. Thus
 571 $\|\bar{E}(t_k, x_k^1) - \bar{E}(t_k, x_k^2)\| \leq L_x\|x_k^1 - x_k^2\|$ and $\|\bar{E}(t_k, x_k) - \bar{E}(t_k, 0)\| \leq L_x\|x_k\|$. Notice
 572 that due to the Lipschitz continuity both z_{k+1} and λ_{k+1} are uniformly bounded in

573 time for fixed x_k (provided $E_I(\cdot)$, $E_{II}(\cdot)$ and $F(\cdot)$ are uniformly bounded, which is
 574 a mild requirement on the data). Therefore $\bar{E}(t_k, 0)$ is uniformly bounded also, and
 575 $\|\bar{E}(t_k, x_k)\| \leq \|\bar{E}(t_k, 0)\| + L_x \|x_k\| \leq m + L_x \|x_k\|$. Now using (5.2) we get:
 (5.5)

$$576 \quad \|x_k\| \leq \sigma_{\max}^k((P_I^h)^{-1}P_I)\|x_0\| + h\sigma_{\min}(P_I^h) \sum_{j=1}^{k-1} \sigma_{\max}^{k-j-1}(P_I(P_I^h)^{-1})(m + L_x \|x_{j-1}\|),$$

577 where we used the Hölder induced matrix norm $\|\cdot\|_{2,2} = \sigma_{\max}(\cdot)$ and [10, Proposition
 578 9.4.9, Corollary 9.4.12]. Let us make the following:

579 **ASSUMPTION 11.** *One has $\sigma_{\max}((P_I^h)^{-1}P_I) \leq 1$ for all $h \geq 0$.*

580 Assumption 11 is a technical assumption which is satisfied if for instance the $n_1 \times n_1$
 581 matrix $A_I \prec 0$ and $P_I \succ 0$. Indeed this implies that $P_I^h \succ P_I$, thus $(P_I^h)^{-1}P_I \prec I_{n_1}$.
 582 Notice that under Assumption 11 one no longer needs $h > 0$ small enough to obtain
 583 the developments in (5.4), because the spectral radius of $(P_I^h)^{-1}P_I$ is smaller than
 584 one. It is also noteworthy that changing $A_I x_{k+1}$ to $A_I x_k$ in the right-hand side of
 585 (3.1) (a), allows one to dispense with Assumption 11, with a slight modification of the
 586 matrices \tilde{A} , \tilde{B} , \tilde{C} , \tilde{D} , \tilde{E}_k and \tilde{F}_k (where $(P_I^h)^{-1}$ is replaced by P_I^{-1} in all of them).
 587 Consequently from (5.5) one infers that:

$$588 \quad (5.6) \quad \|x_k\| \leq \|x_0\| + h\sigma_{\min}(P_I^h) \sum_{j=1}^k (m + L_x \|x_{j-1}\|),$$

589 Assume now that x_0, x_1, \dots, x_{k-1} are uniformly bounded by some x_{\max} , then from (5.6)
 590 one has $\|x_k\| \leq x_{\max} + h\sigma_{\min}(P_I^h)k(m + L_x x_{\max}) \leq x_{\max} + h\sigma_{\min}(P_I^h)n(m + L_x x_{\max}) =$
 591 $x_{\max} + h\sigma_{\min}(P_I^h)\frac{T}{h}(m + L_x x_{\max}) = x_{\max} + \sigma_{\min}(P_I^h)T(m + L_x x_{\max})$. Therefore x_k
 592 is also uniformly bounded (its upperbound depends only on T which is fixed), and
 593 we conclude that all x_k , $1 \leq k \leq n$, are bounded provided x_0 is bounded. Then it is
 594 inferred from (5.4) that \hat{x}_k is also uniformly bounded. Let us define the sequences of
 595 piecewise-linear approximations $\{x^h(\cdot)\}$ and $\{\hat{x}^h(\cdot)\}$ as follows:

$$596 \quad (5.7) \quad \begin{cases} x^h(t) = x_{k+1} + \frac{t_{k+1}-t}{h}(x_k - x_{k+1}) & t \in [t_k, t_{k+1}). \\ \hat{x}^h(t) = \frac{x_{k+1}-x_k}{h} \end{cases}$$

597 We have shown that $x^h(\cdot)$ and $\hat{x}^h(\cdot)$ are uniformly bounded on $[0, T]$ and for any
 598 $h > 0$, hence $x^h(\cdot)$ has a Lipschitz constant that does not depend on h , consequently
 599 the family $\{x^h\}_{h>0}$ is equi-Lipschitz continuous. Thus from Arzela-Ascoli Theorem
 600 $\{x^h(\cdot)\}$ stays in a compact subset of $C^0([0, T]; \mathbb{R}^{n_1})$ and there exist subsequences
 601 which converge uniformly in $C^0([0, T]; \mathbb{R}^{n_1})$ towards a Lipschitz continuous limit $x(\cdot)$.
 602 Let $\tilde{E}^h(\cdot)$ and $\tilde{F}^h(\cdot)$ be defined in a similar way as $x^h(\cdot)$ in (5.7). Due to the Lip-
 603 schitz continuity of $E_I(\cdot)$, $E_{II}(\cdot)$, and $F(\cdot)$, the sequences of functions $\{\tilde{E}^h(\cdot)\}_{h>0}$
 604 and $\{\tilde{F}^h(\cdot)\}_{h>0}$ also converge uniformly in $C^0([0, T]; \mathbb{R}^p)$ and $C^0([0, T]; \mathbb{R}^m)$ to limits
 605 $\tilde{E}(\cdot) = E_{II}(\cdot) + A_I x(\cdot)$ and $\tilde{F}(\cdot) = F(\cdot) + C_I x(\cdot)$ as $h \rightarrow 0$, respectively. From (5.4) it
 606 is deduced that $\hat{x}^h(\cdot)$ is uniformly bounded. Therefore from the Banach-Alaoglu The-
 607 orem, it follows that $\hat{x}^h(\cdot)$ converges weakly* to a limit $v(\cdot)$. Since the domain $[0, T]$ is
 608 bounded, one has $\mathcal{L}^2([0, T]; \mathbb{R}^{n_1}) \subseteq \mathcal{L}^1([0, T]; \mathbb{R}^{n_1})$ [36, Corollary 1, p.80], and conse-
 609 quently $\hat{x}^h(\cdot)$ converges weakly in $\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})$. Since $x^h(t) = x^h(0) + \int_0^t \hat{x}^h(\tau)d\tau$,
 610 we infer that $v(\cdot) = \dot{x}(\cdot)$ almost everywhere.

611 Now notice that using (3.2) the following holds:

$$612 \quad (5.8) \quad \begin{cases} (a) P_I \dot{x}^h(t) = A_I x_h^*(t) + B_I z_h^*(t) + \bar{B}_I \lambda_h^*(t) + E_I^h(t) \\ (b) \bar{A} z_h^*(t) + \bar{B} \lambda_h^*(t) + \bar{E}_{II}^h(t) + A_{II} x^h(t) = 0 \\ (c) 0 \leq \lambda_h^*(t) \perp \bar{C} z_h^*(t) + \bar{D} \lambda_h^*(t) + F^h(t) + C_I x^h(t) \geq 0, \end{cases}$$

613 for all $t \in [t_k, t_{k+1})$, $k \geq 0$, $x_h^*(t) = x_{k+1}$ for all $t \in [t_k, t_{k+1})$ and similarly for the
 614 other terms are step-functions. From the boundedness of $\|\dot{x}_k\| \triangleq \|\frac{x_{k+1} - x_k}{h}\| \leq C$ for
 615 some $C > 0$, we infer that $x_h^*(\cdot)$ converges strongly in $\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})$ towards $x(\cdot)$.
 616 Indeed:

$$617 \quad (5.9) \quad \begin{aligned} \|x^h - x_h^*\|_{\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})}^2 &\leq \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_{k+1} - t_k)^2 \|\dot{x}_k\|^2 dt \\ &\leq C^2 \sum_{k=0}^{n-1} (t_{k+1} - t_k)^3 = \frac{C^2 n h^3}{3} = \frac{C^2 T h^2}{3} \longrightarrow 0 \text{ as } h \rightarrow 0. \end{aligned}$$

618 To conclude one notices that $\|x^h - x_h^*\|_{\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})}^2 = \|x^h \pm x - x_h^*\|_{\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})}^2$. Now
 619 we have $z_h^*(t) \in \mathcal{G}^{-1}(E_{II, \star}^h(t) + A_{II} x_h^*(t))$ for all $t \in [t_k, t_{k+1})$, $k \geq 0$. Since $E_I(\cdot)$,
 620 $E_{II}(\cdot)$ are Lipschitz continuous, it follows that $\|\dot{E}_k\| \triangleq \|\frac{E_{k+1} - E_k}{h}\| \leq k_E$ where k_E
 621 is its Lipschitz constant. We infer that $E_{II, \star}^h(\cdot)$ converges strongly in $\mathcal{L}^2([0, T]; \mathbb{R}^p)$
 622 towards $E_{II}(\cdot)$, and using the Lipschitz continuity of $\mathcal{G}^{-1}(\cdot)$ it follows that $z_h^*(\cdot)$
 623 converges strongly in $\mathcal{L}^2([0, T]; \mathbb{R}^p)$ towards a limit $z(\cdot) = \mathcal{G}^{-1}(E_{II}(\cdot) + A_{II} x(\cdot))$.
 624 Then from (3.3) $\lambda_h^*(t) \in (\mathcal{N}_{\mathbb{R}_+^m} + \bar{D})^{-1}(-\bar{C} z_h^*(t) - F_h^*(t) + C_I x_h^*(t))$, so that $\lambda_h^*(\cdot)$
 625 converges towards a limit $\lambda(\cdot) = (\mathcal{N}_{\mathbb{R}_+^m} + \bar{D})^{-1}(-\bar{C} z(\cdot) - F(\cdot) + C_I x(\cdot))$ strongly in
 626 $\mathcal{L}^2([0, T]; \mathbb{R}^m)$, due to the single-valuedness, Lipschitzness and maximal monotonicity
 627 of $(\mathcal{N}_{\mathbb{R}_+^m} + \bar{D})^{-1}(\cdot)$. It is concluded that $P_I \dot{x}^h(t) = A_I x_h^*(t) + B_I z_h^*(t) + \bar{B}_I \lambda_h^*(t) +$
 628 $E_I^h(t) \rightharpoonup P_I \dot{x}(t) = A_I x(t) + B_I z(t) + \bar{B}_I \lambda(t) + E_I(t)$ weakly in $\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})$ as $h \rightarrow$
 629 0 , $\bar{A} z_h^*(t) + \bar{B} \lambda_h^*(t) + \bar{E}_{II}^h(t) + A_{II} x^h(t) \rightarrow \bar{B}_I z(t) + \bar{B}_{II} \lambda(t) + E_{II}(t) + A_{II} x(t) (= 0)$,
 630 and $0 \leq \lambda_h^*(t) \perp \bar{C} z_h^*(t) + \bar{D} \lambda_h^*(t) + F^h(t) + C_I x^h(t) \geq 0 \rightarrow 0 \leq \lambda(t) \perp C_I z(t) +$
 631 $D \lambda(t) + F(t) + C_I x(t) \geq 0$ as $h \rightarrow 0$ strongly in $\mathcal{L}^2([0, T]; \mathbb{R}^l)$ where l depends on the
 632 variables dimensions. Hence we have proved the existence of a continuous solution
 633 (x, z, λ) to (2.1) on $[0, T]$. Consequently the following result is proved:

634 **PROPOSITION 5.2.** *Let the conditions of item 2 in Corollary 4.6 hold true uni-*
 635 *formly in h ($\Rightarrow D \succ 0$). Assume that Assumptions 10 and 11 hold, $E(\cdot)$ and $F(\cdot)$ are*
 636 *bounded Lipschitz continuous functions, initial data are bounded. Then the DALCS*
 637 *in (2.1) possesses for each admissible bounded initial data, a continuous solution*
 638 *$(x, z, \lambda) : [0, T] \rightarrow \mathbb{R}^{n_1 + n_2 + m}$, $T > 0$. Moreover the Euler approximations converge*
 639 *to a solution as $h \rightarrow 0$.*

640 **5.1.3. The case blockdiagonal $\bar{D} = \bar{D}^\top \succcurlyeq 0$.** This section is dedicated to
 641 **analyse** an alternative to the conditions stated in sections 5.1.1 and 5.1.2. Let us
 642 consider the case of Proposition 4.4 with Assumption 6. Then it follows from the
 643 complementarity conditions that $\lambda_{1, k+1} = (\bar{D}_{11} + \partial\psi_{K_1^*})^{-1}(-\bar{C}_1 z_{k+1} - \bar{F}_{1, k})$, where
 644 the mapping $(\bar{D}_{11} + \partial\psi_{K_1^*})^{-1}(\cdot)$ is maximal monotone (being the sum of two maximal
 645 monotone mappings satisfying [9, Corollary 24.4 (ii)]), single-valued, well-defined and
 646 Lipschitz continuous, see Proposition B.1. Thus if $\bar{B}_I = (\bar{B}_I^1, 0)$ with $\bar{B}_I^1 \in \mathbb{R}^{n_1 \times m_1}$
 647 one obtains $\bar{B}_I \lambda_{k+1} = \bar{B}_I^1 \lambda_{1, k+1} = \bar{B}_I^1 (\bar{D}_{11} + \partial\psi_{K_1^*})^{-1}(-\bar{C}_1 z_{k+1} - \bar{F}_{1, k})$ in the right-
 648 hand side of (3.5) (c).

649 It may be less restrictive to compute upperbounds using the maximal monotonicity.
 650 To start with let us consider the VI in (4.9) and let us state several features of its
 651 solutions. Here Assumption 7 has to be reinforced a little:

652 ASSUMPTION 12. *The conditions of Assumption 7 hold where $\tilde{X}_2 \succ 0$ for any*
 653 *$h \geq 0$.*

654 This secures that the variable change $\xi_{k+1} = -\tilde{R}_2^{-1}z_{k+1}$ remains well-defined as $h \rightarrow$
 655 0 . It is also important that item 1 in Proposition 4.4 holds, so that the VI in (4.9) is
 656 **equivalent to the MLCP in (3.2) (b) (c).**

657 LEMMA 5.3. *Assume that $\lambda_{\min}(-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1}) (= \lambda_{\min}(-\tilde{A}))$ is large enough, then*
 658 *any solution $\xi_{k+1} = -\tilde{R}_2^{-1}z_{k+1}$ of the VI in (4.9) is upperbounded by an affine function*
 659 *of $\|x_k\|$.*

660 **Proof:** Let us manipulate the VI in (4.9) as follows:

$$(5.10) \quad \begin{aligned} & -\tilde{R}_2\tilde{A}\tilde{R}_2^{-1}\xi_{k+1} + \tilde{R}_2\tilde{E}_k + \tilde{G}_2(t_k, \xi_{k+1}) \in -\partial\tilde{f}_2(\xi_{k+1}) \\ \Leftrightarrow & -\tilde{R}_2\tilde{A}\tilde{R}_2^{-1}(\xi_{k+1} - \xi_0) + \tilde{R}_2\tilde{E}_k + \tilde{G}_2(t_k, \xi_{k+1}) + \zeta_0 - \tilde{R}_2\tilde{A}\tilde{R}_2^{-1}\xi_0 = -\zeta_{k+1} + \zeta_0 \\ & \text{with } \zeta_{k+1} \in \partial\tilde{f}_2(\xi_{k+1}), \zeta_0 \in \partial\tilde{f}_2(\xi_0) \\ \Rightarrow & \langle -\tilde{R}_2\tilde{A}\tilde{R}_2^{-1}(\xi_{k+1} - \xi_0), \xi_{k+1} - \xi_0 \rangle + \langle \tilde{R}_2\tilde{E}_k, \xi_{k+1} - \xi_0 \rangle \\ & + \langle \tilde{G}_2(t_k, \xi_{k+1}) - \tilde{G}_2(t_k, \xi_0), \xi_{k+1} - \xi_0 \rangle + \langle \zeta_0 - \tilde{R}_2\tilde{A}\tilde{R}_2^{-1}\xi_0, \xi_{k+1} - \xi_0 \rangle \\ & + \langle \tilde{G}_2(t_k, \xi_0), \xi_{k+1} - \xi_0 \rangle = -\langle \zeta_{k+1} - \zeta_0, \xi_{k+1} - \xi_0 \rangle \leq 0, \end{aligned}$$

662 where we have denoted $\tilde{G}_2(t_k, \xi_{k+1})$ to highlight the dependency of this mapping on t_k
 663 and x_k through \tilde{F}_k . We have $\|\tilde{R}_2\tilde{B}_1(\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(z_1) - \tilde{R}_2\tilde{B}_1(\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(z_2)\| \leq$
 664 $\eta \|z_1 - z_2\|$ for any z_1 and z_2 , so that $\|\tilde{G}_2(t_k, \xi_{k+1}) - \tilde{G}_2(t_k, \xi_0)\| \leq \eta \|\tilde{C}_1\tilde{R}_2^{-1}(\xi_{k+1} -$
 665 $\xi_0)\| \leq \eta' \|\xi_{k+1} - \xi_0\|$, where η depends on \tilde{D}_{11} , \tilde{R}_2 and \tilde{B}_1 (see the proof of **Propo-**
 666 **sition B.1 in [16]**), and η' depends on \tilde{D}_{11} , \tilde{R}_2 , \tilde{B}_1 , and \tilde{C}_1 . Thus we obtain:

$$(5.11) \quad \begin{aligned} & \langle -\tilde{R}_2\tilde{A}\tilde{R}_2^{-1}(\xi_{k+1} - \xi_0), \xi_{k+1} - \xi_0 \rangle \leq \langle -\tilde{R}_2\tilde{E}_k, \xi_{k+1} - \xi_0 \rangle - \langle \tilde{G}_2(t_k, \xi_0), \xi_{k+1} - \xi_0 \rangle \\ & + \langle \zeta_0 + \tilde{R}_2\tilde{A}\tilde{R}_2^{-1}\xi_0, \xi_{k+1} - \xi_0 \rangle + \eta' \|\xi_{k+1} - \xi_0\|^2 \end{aligned}$$

668 which implies

$$(5.12) \quad \begin{aligned} & \langle (-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1} - \eta'I_{n_2})(\xi_{k+1} - \xi_0), \xi_{k+1} - \xi_0 \rangle \leq \|\tilde{R}_2\tilde{E}_k\| \|\xi_{k+1} - \xi_0\| \\ & + \|\tilde{G}_2(t_k, \xi_0) \pm \tilde{G}_2(t_0, \xi_0)\| \|\xi_{k+1} - \xi_0\| + \|\zeta_0 + \tilde{R}_2\tilde{A}\tilde{R}_2^{-1}\xi_0\| \|\xi_{k+1} - \xi_0\|. \end{aligned}$$

670 Now one has: $\|\tilde{E}_k\| \leq \|E_{II,k} + hA_{II}E_{I,k}\| + \|A_{II}(P_I^h)^{-1}P_I\| \|x_k\|$, $\tilde{G}_2(t_k, \xi_0) =$
 671 $\tilde{R}_2\tilde{B}_1(\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(\tilde{C}_1\tilde{R}_2^{-1}\xi_0 - \tilde{F}_{1,k})$, $\|\tilde{G}_2(t_k, \xi_0) - \tilde{G}_2(t_0, \xi_0)\| = \|\tilde{R}_2\tilde{B}_1(\tilde{D}_{11} +$
 672 $\partial\psi_{K_1^*})^{-1}(\tilde{C}_1\tilde{R}_2^{-1}\xi_0 - \tilde{F}_{1,k}) - \tilde{R}_2\tilde{B}_1(\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(\tilde{C}_1\tilde{R}_2^{-1}\xi_0 - \tilde{F}_{1,0})\| \leq \eta \|\tilde{F}_{1,k} -$
 673 $\tilde{F}_{1,0}\| \leq \eta \|F_k + hC_I(P_I^h)^{-1}E_{I,k}\| + \eta \|C_I(P_I^h)^{-1}P_I\| \|x_k\|$. Let us denote $\alpha_k =$
 674 $\|\tilde{R}_2\| \|E_{II,k} + hA_{II}E_{I,k}\| + \eta \|F_k + hC_I(P_I^h)^{-1}E_{I,k}\|$, $\beta_0 = \|\tilde{G}_2(t_0, \xi_0)\| + \|\zeta_0 +$
 675 $\tilde{R}_2\tilde{A}\tilde{R}_2^{-1}\xi_0\|$, and $\gamma = \eta \|C_I(P_I^h)^{-1}P_I\| + \|\tilde{R}_2\| \|A_{II}(P_I^h)^{-1}P_I\|$. Then:

$$(5.13) \quad \begin{aligned} & \langle (-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1} - \eta'I_{n_2})(\xi_{k+1} - \xi_0), \xi_{k+1} - \xi_0 \rangle \leq (\alpha_k + \beta_0 + \gamma \|x_k\|) \|\xi_{k+1} - \xi_0\| \\ & \leq \frac{1}{2}\alpha_k^2 + \frac{1}{2}\beta_0^2 + \frac{1}{2}\gamma^2 \|x_k\|^2 + \frac{1}{2}\|\xi_{k+1} - \xi_0\|^2 \end{aligned}$$

677 which implies:

$$(5.14) \quad \langle (-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1} - \eta'I_{n_2} - \frac{1}{2}I_{n_2})(\xi_{k+1} - \xi_0), \xi_{k+1} - \xi_0 \rangle \leq \frac{1}{2}\alpha_k^2 + \frac{1}{2}\beta_0^2 + \frac{1}{2}\gamma^2 \|x_k\|^2.$$

679 Therefore if $-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1} - \eta'I_{n_2} - \frac{1}{2}I_{n_2} \succ 0$, one obtains

$$(5.15) \quad \begin{aligned} \|\xi_{k+1} - \xi_0\| & \leq \frac{1}{\sqrt{\lambda_{\min}(-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1} - \eta'I_{n_2} - \frac{1}{2}I_{n_2})}} \sqrt{\frac{1}{2}\alpha_k^2 + \frac{1}{2}\beta_0^2 + \frac{1}{2}\gamma^2 \|x_k\|^2} \\ & \leq \frac{1}{\sqrt{2}\sqrt{\lambda_{\min}(-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1} - \eta'I_{n_2} - \frac{1}{2}I_{n_2})}} \left(\sqrt{\alpha_k^2 + \beta_0^2} + \gamma \|x_k\| \right) \end{aligned}$$

681 which ends the proof. \square

682 Therefore the term $hB_I z_{k+1}$ in the right-hand side of (3.5) (c), admits also an affine
683 upperbound in $\|x_k\|$. A further characterization of the iterates is given now.

684 LEMMA 5.4. Assume that $-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \succ \frac{3}{2} I_{n_2}$. Assume also that the mapping
685 $\tilde{G}_2(\cdot)$ is monotone (see Lemma 4.5 for conditions), and that $E_I(\cdot)$, $E_{II}(\cdot)$ and $F(\cdot)$
686 are uniformly bounded and Lipschitz continuous. Then for any k and $p \geq 0$ and any
687 solution of the VI in (4.9), one has:

$$688 \quad (5.16) \quad \|\xi_k - \xi_p\| \leq \alpha + \beta \|x_{k-1} - x_{p-1}\|$$

689 for some bounded constants $\alpha \geq 0$ and $\beta \geq 0$, and α is proportional to $(t_{k-1} - t_{p-1})$.

690 **Proof:** From

$$691 \quad (5.17) \quad \begin{cases} -\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_k + \tilde{R}_2 \tilde{E}_{k-1} + \tilde{G}_2(t_{k-1}, \xi_k) = -\zeta_k \\ -\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_p + \tilde{R}_2 \tilde{E}_{p-1} + \tilde{G}_2(t_{p-1}, \xi_p) = -\zeta_p \end{cases}$$

692 with $\zeta_k \in \partial f_2(\xi_k)$, $\zeta_p \in \partial f_2(\xi_p)$, it follows that

$$693 \quad (5.18) \quad \begin{aligned} & \langle -\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} (\xi_k - \xi_p), \xi_k - \xi_p \rangle + \langle \tilde{R}_2 (\tilde{E}_{k-1} - \tilde{E}_{p-1}), \xi_k - \xi_p \rangle \\ & + \langle \tilde{G}_2(t_{k-1}, \xi_k) - \tilde{G}_2(t_{k-1}, \xi_p) + \tilde{G}_2(t_{k-1}, \xi_p) - \tilde{G}_2(t_{p-1}, \xi_p), \xi_k - \xi_p \rangle \\ & = -\langle \zeta_k - \zeta_p, \xi_k - \xi_p \rangle \end{aligned}$$

694 which implies

$$695 \quad (5.19) \quad \begin{aligned} & \langle -\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} (\xi_k - \xi_p), \xi_k - \xi_p \rangle + \langle \tilde{R}_2 (\tilde{E}_{k-1} - \tilde{E}_{p-1}), \xi_k - \xi_p \rangle \\ & + \langle \tilde{G}_2(t_{k-1}, \xi_p) - \tilde{G}_2(t_{p-1}, \xi_p), \xi_k - \xi_p \rangle \leq 0. \end{aligned}$$

696 One has:

$$697 \quad (5.20) \quad \begin{aligned} & \langle -\tilde{R}_2 (\tilde{E}_{k-1} - \tilde{E}_{p-1}), \xi_k - \xi_p \rangle \\ & \leq \langle -\tilde{R}_2 (E_{II,k-1} - E_{II,p-1}), \xi_k - \xi_p \rangle - \langle \tilde{R}_2 (hA_{II}(E_{I,k-1} - E_{I,p-1})), \xi_k - \xi_p \rangle \\ & \quad - \langle A_{II}(P_I^h)^{-1} P_I (x_{k-1} - x_{p-1}), \xi_k - \xi_p \rangle \\ & \leq \frac{1}{2} \|\tilde{R}_2 ((E_{II,k-1} - E_{II,p-1}) + hA_{II}(E_{I,k-1} - E_{I,p-1}))\|^2 + \frac{1}{2} \|\xi_k - \xi_p\|^2 \\ & \quad + \frac{1}{2} \|A_{II}(P_I^h)^{-1} P_I\|^2 \|x_{k-1} - x_{p-1}\|^2 + \frac{1}{2} \|\xi_k - \xi_p\|^2 \\ & \leq \|\tilde{R}_2\|^2 \|E_{II,k-1} - E_{II,p-1}\|^2 + \|\xi_k - \xi_p\|^2 + h \|A_{II}\|^2 \|E_{I,k-1} - E_{I,p-1}\|^2 \\ & \quad + \frac{1}{2} \|A_{II}(P_I^h)^{-1} P_I\|^2 \|x_{k-1} - x_{p-1}\|^2 \leq \alpha' + \beta' \|x_{k-1} - x_{p-1}\|^2 + \|\xi_k - \xi_p\|^2, \end{aligned}$$

698 for some $\alpha' \geq 0$ proportional to $(t_{k-1} - t_{p-1})^2$ since $E(\cdot)$ is Lipschitz continuous,
699 $\beta' \geq 0$, and it was used that $x^\top y \leq \|x\| \|y\| \leq \frac{1}{2} (\|x\|^2 + \|y\|^2)$. Also:

$$700 \quad (5.21) \quad \begin{aligned} & \langle -\tilde{G}_2(t_{k-1}, \xi_p) + \tilde{G}_2(t_{p-1}, \xi_p), \xi_k - \xi_p \rangle \\ & \leq \|\tilde{G}_2(t_{k-1}, \xi_p) - \tilde{G}_2(t_{p-1}, \xi_p)\| \|\xi_k - \xi_p\| \\ & \leq \frac{1}{2} k_{\tilde{G}_2} \|\tilde{F}_{1,k-1} - \tilde{F}_{p-1}\|^2 + \frac{1}{2} \|\xi_k - \xi_p\|^2 \\ & \leq k_{\tilde{G}_2} \|F_{k-1} - F_{p-1}\|^2 + h k_{\tilde{G}_2} \|C_I(P_I^h)^{-1} (E_{I,k-1} - E_{I,p-1})\|^2 \\ & \quad + k_{\tilde{G}_2} \|C_I(P_I^h)^{-1} P_I (x_{k-1} - x_{p-1})\|^2 + \frac{1}{2} \|\xi_k - \xi_p\|^2 \\ & \leq \alpha'' + \beta'' \|x_{k-1} - x_{p-1}\|^2 + \frac{1}{2} \|\xi_k - \xi_p\|^2, \end{aligned}$$

701 for some $\alpha'' \geq 0$ proportional to $(t_{k-1} - t_{p-1})^2$ (since $E(\cdot)$ and $F(\cdot)$ are Lipschitz
702 continuous), and $\beta'' \geq 0$. Hence using (5.19):

$$703 \quad (5.22) \quad \begin{aligned} \|\xi_k - \xi_p\| & \leq \frac{1}{\sqrt{\lambda_{\min}(-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} - \frac{3}{2} I_p)}} \sqrt{\alpha' + \alpha'' + (\beta' + \beta'') \|x_{k-1} - x_{p-1}\|^2} \\ & \leq \frac{1}{\sqrt{\lambda_{\min}(-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} - \frac{3}{2} I_p)}} (\sqrt{\alpha' + \alpha''} + \sqrt{\beta' + \beta''} \|x_{k-1} - x_{p-1}\|) \end{aligned}$$

704 and the lemma is proved. \square

705

706 **COROLLARY 5.5.** *Let $h > 0$. Assume that the conditions and assumptions of*
 707 *Lemma 5.4 are fulfilled. For all $k \geq 1$ and any solution of the VI in (4.9), one has:*

$$708 \quad (5.23) \quad \left\| \frac{\xi_{k+1} - \xi_k}{h} \right\| \leq \alpha + \beta \left\| \frac{x_k - x_{k-1}}{h} \right\|$$

709 *for some bounded constants $\alpha \geq 0$ and $\beta \geq 0$.*

710 **Proof:** Let us consider the proof of Lemma 5.4, replacing k by $k + 1$ and p by $k - 1$.
 711 Then α in (5.16) is proportional to h . Redoing the calculations in (5.20) and (5.22),
 712 one obtains (5.23). \square

713 Clearly for all $h > 0$ (5.23) can be written equivalently as: $\|\xi_{k+1} - \xi_k\| \leq h\alpha +$
 714 $\beta\|x_k - x_{k-1}\|$. The bounds calculated for the variable ξ_k also hold for the variable
 715 $z_k = -\tilde{R}_2 \xi_k$. Let us now analyse λ_{k+1} in (3.5) (a). As seen above, $\lambda_{1,k+1} = (\tilde{D}_{11} +$
 716 $\partial\psi_{K_1^*})^{-1}(-\tilde{C}_1 z_{k+1} - \tilde{F}_{1,k})$ and $\lambda_{2,k+1} \in -\mathcal{N}_{K_2}(\tilde{C}_2 z_{k+1} + \tilde{F}_{2,k})$. If the matrix \tilde{B}_I has
 717 the above structure (i.e., $\tilde{B}_I = (\tilde{B}_I^1, 0)$), then the term $\tilde{B}_I \lambda_{k+1} = \tilde{B}_I^1$ is Lipschitz
 718 continuous with respect to the variables z_{k+1} , x_k and F_k . In a more general setting
 719 one obtains:

$$720 \quad (5.24) \quad P_I^h x_{k+1} \in P_I x_k + h B_I z_{k+1} + h \tilde{B}_I^1 (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1} (-\tilde{C}_1 z_{k+1} - \tilde{F}_{1,k}) \\ - h \tilde{B}_I^2 \mathcal{N}_{K_2}(\tilde{C}_2 z_{k+1} + \tilde{F}_{2,k}).$$

721 The difficulty is that the last term in the right-hand side of (5.24) is set-valued in
 722 general, and this is intrinsically due to the structure of the implicit scheme in (2.1).
 723 A solution is to consider the VI in (4.9). Its right-hand side matches with the last
 724 term in the right-hand side of (5.24) and it is set-valued. However provided the VI
 725 is well-posed, we know that one element of this set is equal to the left-hand side of
 726 (4.9). We may assume that the scheme in (5.24) is implemented with this element.
 727 This element appears to define a Lipschitz continuous function of ξ_{k+1} (let us recall
 728 that the Lipschitz constant of the third term is equal to $\frac{1}{\lambda_{\min}(\tilde{D}_{11})}$, see Proposition
 729 B.1). If this element is chosen then the numerical scheme (5.24) becomes:

$$730 \quad (5.25) \quad P_I^h x_{k+1} = P_I x_k + h B_I z_{k+1} + h \tilde{B}_I^1 (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1} (\tilde{C}_1 z_{k+1} - \tilde{F}_{1,k}) \\ + h \tilde{B}_I^2 (-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_{k+1} - \tilde{R}_2 \tilde{E}_k + \tilde{G}_2(t_k, \xi_{k+1})).$$

731 Let us now study (5.25), using Lemmata 5.3, 5.4, Corollary 5.5. It is noteworthy that
 732 the constant α in both Lemmata and in Corollary can be made arbitrarily small by
 733 increasing $-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1}$ arbitrarily (in the set of positive definite matrices).

734 **PROPOSITION 5.6.** *Assume that the conditions of Lemmata 5.3, 5.4 and of Corol-*
 735 *lary 5.5 are verified, $E_I(\cdot)$, $E_{II}(\cdot)$ and $F(\cdot)$ are bounded Lipschitz continuous func-*
 736 *tions of time, and Assumption 13 below is satisfied. Then the sequences $\{x^h\}_{h \geq 0}$ and*
 737 *$\{\dot{x}^h\}_{h \geq 0}$ in (5.7) are uniformly bounded in h on $[0, T]$. Thus $\{x^h\}_{h \geq 0}$ is equi-Lipschitz*
 738 *continuous and there is a subsequence of it that converges uniformly in $\mathcal{C}^0([0, T]; \mathbb{R}^{n_1})$*
 739 *towards a Lipschitz continuous limit denoted as $x(\cdot)$.*

740 **Proof:** In the proof, use is made of positive bounded constants denoted generically as
 741 k_i . Consider (5.25). First recall that the operator $\zeta \mapsto (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(\zeta)$ is Lipschitz

742 continuous with Lipschitz constant $\frac{1}{\lambda_{\min}(\tilde{D}_{11})}$. Hence we obtain:

$$\begin{aligned}
& \|(P_I^h)^{-1} \tilde{B}_I^1 (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1} (-\tilde{C}_1 z_{k+1} - \tilde{F}_{1,k}) \\
& \quad - (P_I^h)^{-1} \tilde{B}_I^1 (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1} (-\tilde{C}_1 z_{p+1} - \tilde{F}_{1,p})\| \\
743 \quad (5.26) \quad & \leq k_1 \|\tilde{C}_1(z_{k+1} - z_{p+1} + \tilde{F}_{1,k} - \tilde{F}_{1,p})\| \leq k_2 \|z_{k+1} - z_{p+1}\| + k_3 \|\tilde{F}_{1,k} - \tilde{F}_{1,p}\| \\
& \leq k_4 \|\xi_{k+1} - \xi_{p+1}\| + k_4 \|F_{1,k} - F_{1,p}\| + k_5 \|E_{I,k} - E_{I,p}\| + k_6 \|x_k - x_p\| \\
& \leq k_4(\alpha + \beta \|x_k - x_p\|) + k_7 |t_k - t_p| + k_6 \|x_k - x_p\| \leq k_8 + k_9 \|x_k - x_p\| \\
& \quad + k_7 |t_k - t_p|,
\end{aligned}$$

744 for any k and $p \geq 0$, where we used Lemma 5.4 and the Lipschitzness property. In a
745 quite similar way one has $\|(P_I^h)^{-1} \tilde{B}_I^2 (\tilde{G}_2(\xi_{k+1}) - \tilde{G}_2(\xi_{p+1}))\| \leq k_8 + k_9 \|x_k - x_p\| +$
746 $k_7 |t_k - t_p|$, where k_7, k_8 and k_9 depend on the Lipschitz constants of $\tilde{G}_2(\cdot, \cdot)$, $F(\cdot)$, and
747 on $\|(P_I^h)^{-1} \tilde{B}_I^2\|$. One also has $\|(P_I^h)^{-1} (B_I + \tilde{B}_I^2 (\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1})(z_{k+1} - z_{p+1}))\| \leq k_{10}(\alpha +$
748 $\beta \|x_k - x_p\|) \leq k_{11} + k_{12} \|x_k - x_p\|$, and $\|\tilde{B}_I^2 \tilde{R}_2 (\tilde{E}_k - \tilde{E}_p)\| \leq k_{13} |t_k - t_p| + k_{14} \|x_k - x_p\|$.
749 It is deduced that $\|x_{k+1} - x_{p+1}\| \leq (\|(P_I^h)^{-1} P_I\| + hk_{15}) \|x_k - x_p\| + hk_{16} |t_k - t_p| + hk_{17}$,
750 where the term k_{17} stems from $k_8 + k_{11} = \alpha(k_4 + k_{10})$. Here we are led to make the
751 following which is more stringent than Assumption 11:

752 ASSUMPTION 13. *There exists $h_{\max} > 0$ such that $\|(P_I^h)^{-1} P_I\| + hk_{15} < 1$ for all*
753 $0 < h \leq h_{\max}$.

754 Obviously for $h = 0$ and the Hölder-induced norm one has $\|(P_I^h)^{-1} P_I\|_{2,2} + hk_{15} = 1$.
755 Let us choose $p = 0$, it follows that $\|x_{k+1} - x_1\| \leq (\|(P_I^h)^{-1} P_I\| + hk_{15}) \|x_k -$
756 $x_0\| + hk_{16} T + hk_{17}$, hence $\|x_{k+1}\| \leq (\|(P_I^h)^{-1} P_I\| + hk_{15}) \|x_k\| + hk_{16} T + hk_{17} +$
757 $\|x_1\| + (\|(P_I^h)^{-1} P_I\| + hk_{15}) \|x_0\|$. Now it has to be proved that $\|x_1\|$ is bounded.
758 From Lemma 5.3 and (5.15), $\|\xi_1\|$ and $\|z_1\|$ are upperbounded by an affine function of
759 initial data $\|x_0\|$ and $\|z_0\|$. The operator $(\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(\cdot)$ is well-defined and single-
760 valued. Taking $k = 0$ in (5.25) one infers that indeed $\|x_1\|$ is bounded for bounded
761 initial data. Consequently it follows that $\|x_{k+1}\| \leq (\|(P_I^h)^{-1} P_I\| + hk_{15}) \|x_k\| +$
762 $hk_{16} T + hk_{17} + hf(x_0, z_0) + \|(P_I^h)^{-1} P_I\| \|x_0\|$ for some function $f(\cdot)$ that is bounded
763 for bounded initial data. To simplify the notations let us set $h_1 \triangleq \|(P_I^h)^{-1} P_I\| + hk_{15}$
764 and $h_2 \triangleq k_{16} T + k_{17} + f(x_0, z_0)$. We obtain for all $k \geq 1$:

$$\begin{aligned}
765 \quad (5.27) \quad \|x_k\| & \leq h_1^k \|x_0\| + \sum_{j=0}^{k-1} h_1^j (hh_2 + \|(P_I^h)^{-1} P_I\| \|x_0\|) \\
& \leq h_1^k \|x_0\| + (hh_2 + \|(P_I^h)^{-1} P_I\| \|x_0\|) \frac{1-h_1^{k+1}}{1-h_1}.
\end{aligned}$$

766 Thus in view of Assumption 13 the sequence $\{x_k\}_{k \geq 0}$ generated by (5.25) is uniformly
767 bounded for bounded initial data.

768 Let us now reconsider the inequality $\|x_{k+1} - x_{p+1}\| \leq (\|(P_I^h)^{-1} P_I\| + hk_{15}) \|x_k -$
769 $x_p\| + hk_{16} |t_k - t_p| + hk_{17}$ obtained above. Setting $p = k - 1$ yields $\|x_{k+1} - x_k\| \leq$
770 $(\|(P_I^h)^{-1} P_I\| + hk_{15}) \|x_k - x_{k-1}\| + h^2 k_{16} + hk_{17} \Rightarrow \frac{\|x_{k+1} - x_k\|}{h} \leq (\|(P_I^h)^{-1} P_I\| +$
771 $hk_{15}) \frac{\|x_k - x_{k-1}\|}{h} + hk_{16} + k_{17}$. Now using Lemma 5.4 and the Lipschitzness it follows
772 that α , hence k_{17} , are proportional to $k_E^2 h^2$. We may therefore write the inequality
773 as $\frac{\|x_{k+1} - x_k\|}{h} \leq (\|(P_I^h)^{-1} P_I\| + hk_{15}) \frac{\|x_k - x_{k-1}\|}{h} + hk_{16} + k_{18} h^2$ for some $k_{18} \geq 0$.
774 Proceeding as for (5.27) one infers that if $\frac{\|x_1 - x_0\|}{h}$ is bounded then $\frac{\|x_{k+1} - x_k\|}{h}$ is
775 bounded for all $k \geq 1$. Evaluating (5.25) at $k = 0$ and using the boundedness of x_1
776 as shown above, proves that this is indeed the case. The proposition is proved using
777 the Arzela-Ascoli Theorem. \square

778 Clearly in practice one needs to compute the constants k_i , for instance k_{15} is needed
779 to determine h_{\max} in Assumption 13.

780 COROLLARY 5.7. *Under the conditions of Proposition 5.6, the sequences $\{z^h\}_{h \geq 0}$
 781 and $\{z^h\}_{h \geq 0}$ (respectively $\{\xi^h\}_{h \geq 0}$ and $\{\xi^h\}_{h \geq 0}$) are uniformly bounded in h on $[0, T]$,
 782 $0 < T < +\infty$. Thus $\{z^h\}_{h \geq 0}$ (respectively $\{\xi^h\}_{h \geq 0}$) is equi-Lipschitz continuous and
 783 there is a subsequence of it that converges uniformly in $C^0([0, T]; \mathbb{R}^{n_2})$ towards a
 784 Lipschitz continuous limit denoted as $z(\cdot)$ (respectively $\xi(\cdot)$).*

785 **Proof:** This follows from Proposition 5.6 and using [Lemmata 5.3, 5.4](#), [Corollary 5.5](#).
 786 \square

787 Let us now pass to the analysis of the multiplier convergence. Recall that $\lambda_{1,k+1} =$
 788 $(\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(-\tilde{C}_1 z_{k+1} - \tilde{F}_{1,k})$ and $\lambda_{2,k+1} = -\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_{k+1} + \tilde{R}_2 \tilde{E}_k + \tilde{G}_2(\xi_{k+1}) \in$
 789 $-\mathcal{N}_{K_2}(\tilde{C}_2 z_{k+1} + \tilde{F}_{2,k})$, therefore for all $t \geq 0$, $\lambda_{1,\star}^h(t) = (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(-\tilde{C}_1 z_\star^h(t) -$
 790 $\tilde{F}_{1,\star}^h(t))$ and $\lambda_{2,\star}^h(t) = -\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_\star^h(t) + \tilde{R}_2 \tilde{E}_\star^h(t) + \tilde{G}_2(\xi_\star^h(t)) \in -\mathcal{N}_{K_2}(\tilde{C}_2 z_\star^h(t) +$
 791 $\tilde{F}_{2,\star}^h(t))$, where we recall that the step-functions are defined as $f_\star^h(t) = f_{k+1}$ for
 792 $t \in [t_k, t_{k+1})$. We know that $\dot{\xi}_k \triangleq \|\frac{\xi_{k+1} - \xi_k}{h}\| \leq C$ for some $C > 0$, hence:

$$793 \quad (5.28) \quad \|\xi^h - \xi_\star^h\|_{\mathcal{L}_2([0, T]; \mathbb{R}^{n_2})}^2 \leq \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_{k+1} - t_k)^2 \|\dot{\xi}_k\|^2 dt$$

$$\leq C^2 \sum_{k=0}^{n-1} (t_{k+1} - t_k)^3 = \frac{C^2 n h^3}{3} = \frac{C^2 T h^2}{3}.$$

794 Consequently $\|\xi^h - \xi_\star^h\|_{\mathcal{L}_2([0, T]; \mathbb{R}^{n_2})} \rightarrow 0$ as $h \rightarrow 0$. Since $\|\xi^h - \xi_\star^h\|_{\mathcal{L}_2([0, T]; \mathbb{R}^{n_2})} =$
 795 $\|\xi^h \pm \xi - \xi_\star^h\|_{\mathcal{L}_2([0, T]; \mathbb{R}^{n_2})} \rightarrow \|\xi - \xi_\star^h\|_{\mathcal{L}_2([0, T]; \mathbb{R}^{n_2})} \rightarrow 0$, we conclude that the sequence
 796 $\{\xi_\star^h\}_{h \geq 0}$ converges strongly in $\mathcal{L}_2([0, T]; \mathbb{R}^{n_2})$ towards the limit $\xi(\cdot)$. The same result
 797 holds for the sequence $\{x_\star^h\}_{h \geq 0}$ with the limit $x(\cdot)$. Therefore the sequence $\{-\tilde{C}_1 z_\star^h -$
 798 $\tilde{F}_{1,\star}^h\}_{h \geq 0}$ also converges strongly in \mathcal{L}_2 towards a limit, and we infer that the same
 799 conclusion holds for both $\{\lambda_{1,\star}^h(\cdot)\}_{h \geq 0}$ and $\{\lambda_{2,\star}^h(\cdot)\}_{h \geq 0}$. Since $\mathcal{N}_{K_2}(\cdot)$ is maximal
 800 monotone, it follows from [\[9, Proposition 20.33\]](#) that the limit $\lambda_2(\cdot)$ satisfies $\lambda_2(t) \in$
 801 $-\mathcal{N}_{K_2}(C_{II,2} z(t) + F_2(t) + C_{I,2} x(t))$. Thus we proved the following result.

802 LEMMA 5.8. *Let the conditions of [Lemmata 5.3, 5.4](#), [Corollary 5.5](#) be satisfied for
 803 all $h > 0$. The sequences $\{\xi_\star^h\}_{h \geq 0}$, $\{x_\star^h\}_{h \geq 0}$, $\{\lambda_{1,\star}^h(\cdot)\}_{h \geq 0}$ and $\{\lambda_{2,\star}^h(\cdot)\}_{h \geq 0}$ converge
 804 strongly in $\mathcal{L}_2([0, T]; \mathbb{R}^{n_2})$ towards the limits $\xi(\cdot)$, $x(\cdot)$, $\lambda_1(\cdot)$ and $\lambda_2(\cdot)$, respectively.*

805 To finish this section, one can consider again [\(5.8\)](#) and the analysis which follows it,
 806 to obtain a proposition similar to [Proposition 5.2](#), relying on [Lemma 5.8](#), [Proposition](#)
 807 [5.6](#), and on [Corollary 5.5](#). It is noteworthy that the three fundamental assumptions
 808 in this section, are that $-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1}$ be sufficiently large as a positive definite matrix,
 809 that a suitable selection is chosen for the multiplier $\lambda_{2,k}$, and that a coupling as stated
 810 in [Assumptions 2 or 7](#) exists.

811 PROPOSITION 5.9. *Assume that the conditions of [Proposition 5.6](#), [Corollary 5.7](#)
 812 and [Lemma 5.8](#) are verified. Then the limit functions $x(\cdot)$, $z(\cdot)$, $\lambda(\cdot)$, are solutions of
 813 the DALCS [\(2.1\)](#).*

814 *Proof.* Let us consider [\(5.8\)](#). From [Lemma 5.8](#), $\{x_\star^h(\cdot)\}$, $\{z_\star^h(\cdot)\}$, $\{\lambda_\star^h(\cdot)\}$ converge
 815 strongly in $\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})$ towards limits $x(\cdot)$, $z(\cdot)$ and $\lambda(\cdot)$. Using the same argu-
 816 ments as those developed before [Proposition 5.2](#), it is inferred from [\(5.8\)](#) (b) (c) that
 817 the limits verify [\(2.1\)](#) (b) (c). Using the same arguments as those developed just above
 818 [\(5.8\)](#), it is deduced that $\{\dot{x}^h(\cdot)\}$ converges weakly* to a limit $v(\cdot)$ where $v(\cdot) = \dot{x}(\cdot)$
 819 almost everywhere. Consequently $P_I \dot{x}^h(t) = A_I x_\star^h(t) + B_I z_\star^h(t) + \bar{B}_I \lambda_\star^h(t) + E_I^h(t) \rightarrow$
 820 $P_I \dot{x}(t) = A_I x(t) + B_I z(t) + \bar{B}_I \lambda(t) + E_I(t)$ weakly in $\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})$ as $h \rightarrow 0$. \square

821 **5.2. Convergence results from [\(3.1\)](#).** In this section the properties of the
 822 plant's model [\(2.1\)](#) are used, thus the numerical scheme in [\(3.1\)](#) and the results in

823 section 4.2 are analysed. This section is devoted to the case $D \succcurlyeq 0$ as in Proposition
 824 4.8. Indeed, since the strict passivity of $(B_{II}, \bar{B}_{II}, C_{II}, D)$ implies Assumption 10 for
 825 small enough $h > 0$, it is unnecessary to repeat the developments in sections 5.1.1 and
 826 5.1.2. However, it is not possible in general to determine the implications between
 827 Assumption 7 and Assumption 9, nor the relations between $\lambda_{\min}(\tilde{A})$ and $\lambda_{\min}(B_{II})$
 828 for $h > 0$. Therefore let us deal with the VI in (4.17) where x_{k+1} is replaced by
 829 $x_{k+\theta} = \theta x_k + (1-\theta)x_{k+1}$, $\theta \in [0, 1]$. The following holds, which is the counterpart of
 830 Lemma 5.3.

831 **LEMMA 5.10.** *Assume that $\lambda_{\min}(-R_2 B_{II} R_2^{-1})$ ($= \lambda_{\min}(-B_{II})$) is large enough,
 832 $E_I(\cdot)$, $E_{II}(\cdot)$ and $F(\cdot)$ are uniformly bounded and Lipschitz continuous functions of
 833 time. Then any solution $\xi_{k+1} = R_2^{-1} z_{k+1}$ of the VI in (4.17) is upperbounded by an
 834 affine function of $\|x_{k+\theta}\|$.*

835 The proof follows the same steps as the proof of Lemma 5.3 and it is therefore omitted.
 836 Lemma 5.10 implies that the term $B_{II} z_{k+1}$ in the right-hand side of (3.1) (a), is
 837 upperbounded by an affine function of $\|x_{k+\theta}\|$. The coefficients of the affine function
 838 depend only on the system's data (constant matrices, Lipschitz constants) and on the
 839 initial conditions. Let us now propose the counterpart of Lemma 5.4.

840 **LEMMA 5.11.** *Assume that $-R_2 B_{II} R_2^{-1} \succcurlyeq I_{n_2}$. Assume also that the well-defined,
 841 single-valued and Lipschitz continuous mapping $G_2(\cdot)$ is monotone, and that $E_I(\cdot)$,
 842 $E_{II}(\cdot)$ and $F(\cdot)$ are Lipschitz continuous functions of time. Then for any k and $p \geq 0$
 843 and any solution of the VI in (4.17), one has:*

$$844 \quad (5.29) \quad \|\xi_k - \xi_p\| \leq \alpha + \beta \|x_{k-1+\theta} - x_{p-1+\theta}\|$$

845 for all $k \geq 1$ and some bounded constants $\alpha \geq 0$ (proportional to $|t_{k-1} - t_{p-1}|$) and
 846 $\beta \geq 0$.

847 **Proof:** The proof is similar to the proof of Lemma 5.4, since both VIs in (4.17) and
 848 (4.9) possess similar structures. Using (4.17) one finds:

$$849 \quad (5.30) \quad \begin{cases} -R_2 B_{II} R_2^{-1} \xi_{k+1} - R_2 A_{II} x_{k+\theta} - R_2 E_{II,k} + G_2(t_k, \xi_{k+1}) = -\zeta_{k+1} \\ -R_2 B_{II} R_2^{-1} \xi_{p+1} - R_2 A_{II} x_{p+\theta} - R_2 E_{II,p} + G_2(t_p, \xi_{p+1}) = -\zeta_{p+1} \end{cases}$$

850 with $\zeta_k \in \partial f_2(\xi_k)$, $k \geq 0$. Using the monotonicity of $G_2(t_k, \cdot)$ and of $\partial f_2(\cdot)$ it follows
 851 that:

$$852 \quad (5.31) \quad \begin{aligned} & \langle -R_2 B_{II} R_2^{-1} (\xi_{k+1} - \xi_{p+1}), \xi_{k+1} - \xi_{p+1} \rangle \\ & \leq \langle R_2 A_{II} (x_{k+\theta} - x_{p+\theta}) + R_2 (E_{II,k} - E_{II,p}), \xi_{k+1} - \xi_{p+1} \rangle \\ & \quad - \langle G_2(t_k, \xi_{k+1}) - G_2(t_k, \xi_{p+1}), \xi_{k+1} - \xi_{p+1} \rangle \\ & \quad + \langle G_2(t_k, \xi_{p+1}) - G_2(t_p, \xi_{p+1}), \xi_{k+1} - \xi_{p+1} \rangle \\ & \leq \|R_2 A_{II}\|^2 \|x_{k+\theta} - x_{p+\theta}\|^2 + \|R_2\|^2 k_E^2 (t_k - t_p)^2 \\ & \quad + \frac{1}{2} k_{G_2}^2 \|F_k - F_p\|^2 + \|\xi_{k+1} - \xi_{p+1}\|^2 \\ & \leq \|R_2 A_{II}\|^2 \|x_{k+\theta} - x_{p+\theta}\|^2 + \|R_2\|^2 k_E^2 (t_k - t_p)^2 + \frac{1}{2} k_{G_2}^2 k_F^2 (t_k - t_p)^2 \\ & \quad + \|\xi_{k+1} - \xi_{p+1}\|^2. \end{aligned}$$

853 The lemma is proved. \square

854 Thus Corollary 5.5 extends as well, and in particular one has $\|\xi_{k+1} - \xi_k\| \leq h\alpha +$
 855 $\beta \|x_{k+\theta} - x_{k-1+\theta}\|$ for all $k \geq 1$, for bounded constants $\alpha \geq 0$ and $\beta \geq 0$. The scheme
 856 in (5.25) is now replaced by:

$$857 \quad (5.32) \quad \begin{aligned} P_I^h x_{k+1} = & P_I x_k + h B_I z_{k+1} + h \bar{B}_I^1 (D_{11} + \partial \psi_{K_1^*})^{-1} (C_{II}^1 R_2^{-1} \xi_{k+1} + F_{1,k}) \\ & + h \bar{B}_I^2 (-R_2 B_{II} R_2^{-1} \xi_{k+1} - R_2 A_{II} x_{k+\theta} - R_2 E_{II,k} + G_2(t_k, \xi_{k+1})), \end{aligned}$$

858 equivalently:

(5.33)

$$\begin{aligned}
 859 \quad & (P_I^h + h(1 - \theta)\bar{B}_I^2 R_2 A_{II})x_{k+1} = P_I x_k + hB_I z_{k+1} \\
 & + h\bar{B}_I^1 (D_{11} + \partial\psi_{K_1^*})^{-1} (C_{II}^1 R_2^{-1} \xi_{k+1} + F_{1,k}) + h\bar{B}_I^2 (-R_2 B_{II} R_2^{-1} \xi_{k+1} \\
 & - \theta R_2 A_{II} x_k - R_2 \bar{E}_{II,k} + G_2(t_k, \xi_{k+1})),
 \end{aligned}$$

860 and it appears that both schemes in (5.25) and in (5.33) possess the same structure.
 861 Consequently the results of convergence obtained for the foregoing implicit scheme
 862 (Proposition 5.6, Corollary 5.7, Lemma 5.8) extend to the scheme in this section, for
 863 suitably chosen numerical and system parameters. For the sake of brevity we do not
 864 repeat the developments.

865 *Remark 5.12.* Using Lemma 4.1 item 1, Assumption 10 could be replaced by
 866 assuming the strict passivity of the quadruple $(B_{II}, \bar{B}_{II}, C_{II}, D)$ (a system property)
 867 and $h > 0$ small enough.

868 *Remark 5.13.* Obviously when B_{II} is invertible it is possible to recast (2.1) into
 869 an LCS by solving the equality constraint for z . This yields the LCS quadruple
 870 $(P_I^{-1}(A_I - B_I B_{II}^{-1} A_{II}), P_I^{-1}(\bar{B}_I - B_I B_{II}^{-1} \bar{B}_{II}), C_I - C_{II} B_{II}^{-1} A_{II}, D - C_{II} B_{II}^{-1} \bar{B}_{II})$.
 871 It is unclear how to relate in general the passivity of this quadruple, to the above
 872 assumptions which focus on the MLCP part (2.1) (b) (c) of the system. Especially
 873 no specific assumptions are made on the matrices A_I, B_I, \bar{B}_I, C_I in the above analy-
 874 ses. For instance Assumption 3 implies that the feedthrough matrix of the LCS is
 875 $D - C_{II} B_{II}^{-1} X^{-1} C_{II}^T$, which may be indefinite in general. It is also noteworthy that
 876 transforming the MLCP into an LCP may be undesirable [59].

877 **5.3. Recapitulation.** Let us recapitulate the various assumptions upon which
 878 the results in this article rely. Assumptions 2, 4, 7, 10, 12 (respectively Assumptions
 879 3, 5, 9), characterize the passivity properties of an auxiliary operator defined by the
 880 DALCS constraints in (3.2) (b) (c) (respectively in (2.1) (b) (c)). These passivity
 881 properties constrain the relative degree between the “input” λ_{k+1} and the “output”
 882 $w_{k+1} = \tilde{C}z_{k+1} + \tilde{D}\lambda_{k+1}$ (respectively λ and $w = C_{II}z + D\lambda$). Assumptions 6 and 8
 883 have practical importance (because some systems possess such D matrix), they are
 884 also related to passivity in the sense that passive systems possess positive semidefinite
 885 throughout matrices. Thus all these assumptions are fundamental ones for the devel-
 886 opments in this article, which basically deals with the analysis of two types of gener-
 887 alised equations (GE): $0 \in A(u) + B(u)$ in (4.7) and (4.15), and $0 \in A(u) + B(u) + C(u)$
 888 in (4.9) and (4.17), where the mappings $A(\cdot), B(\cdot)$ and $C(\cdot)$ are maximal monotone
 889 (notice in passing that due to time-dependency, these GEs do not fit with those in [28,
 890 Chapter 2], so that in particular [28, Theorem 2F.6] cannot be applied in general).
 891 Most importantly, the convergence results (Proposition 5.2, Lemmata 5.3 and 5.4,
 892 Corollary 5.5, Proposition 5.6, Corollary 5.7, Lemmata 5.8 and 5.10) rely upon the
 893 positive definiteness of $-R_2 B_{II} R_2^{-1}$ and $-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1}$, which is an index one condition
 894 and guarantees strong monotonicity properties, allowing us to characterize the prop-
 895 erties of the solutions to the GEs. It would be worth relaxing it in future studies.
 896 One path could be to use least norm elements [34]. Finally, it is worth noting that
 897 many practical systems like circuits with nonlinear resistors and inductances contain
 898 nonlinearities [19, section 4.1]. An extension of the above developments, consists of
 899 replacing the linear term $A_I x$ in (2.1), by a nonlinear term $A_I(x)$, where $A_I(\cdot)$ is
 900 globally Lipschitz continuous, and setting $A_I(x_k)$ in (3.1). The matrices $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$
 901 keep the same structure with P_I^{-1} instead of $(P_I^h)^{-1}$, while both \tilde{E}_k and \tilde{F}_k contain

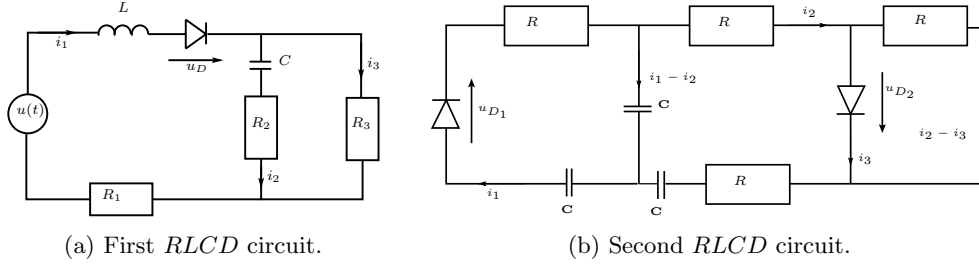


Fig. 1: Two circuits with ideal diodes.

902 $A_I(x_k)$. Nonlinearities in the equality and in the complementarity constraints should
 903 also be considered, where the results in [27] could be useful.

904 **6. Examples and Numerical Simulations.** The next examples are taken
 905 from circuits, mechanics and state-dependent-switch DAEs. They aim at illustrat-
 906 ing the above developments, and the limitations imposed by some assumptions.

907 **6.1. Example 1 (circuit).** Let us consider the circuit in Figure 1b. Using
 908 Kirchoff laws its dynamics are given by:

$$909 \quad (6.1) \quad \begin{cases} \dot{x}_1(t) = \frac{-2}{RC}x_1(t) + \frac{1}{RC}x_2(t) + \frac{1}{R}u_{D_1}(t) \\ \dot{x}_2(t) = \frac{3}{RC}x_2(t) + \frac{1}{2RC}x_1(t) + \frac{1}{2R}u_{D_2}(t) \\ 0 = \frac{3}{2}u_{D_2}(t) - Rx_3(t) - \frac{1}{C}x_2(t) + \frac{1}{2C}x_1(t) \\ 0 \leq u_{D_1}(t) \perp \frac{-2}{RC}x_1(t) + \frac{1}{RC}x_2(t) + \frac{1}{R}u_{D_1}(t) \geq 0 \\ 0 \leq u_{D_2}(t) \perp x_3(t) \geq 0, \end{cases}$$

910 with $x_1(t) = \int_0^t i_1(s)ds$, $x_2(t) = \int_0^t i_2(s)ds$, $x_3 = i_3 = z$, $\lambda = (u_{D_1} \ u_{D_2})^\top$ (the
 911 voltages across the diodes), $K = K^* = \mathbb{R}_+^2$, $n_1 = 2$, $p = n_2 = 1$, $m = 2$. One
 912 has $D = \begin{pmatrix} \frac{1}{R} & 0 \\ 0 & 0 \end{pmatrix}$, $P_I = I_2$, $A_I = \begin{pmatrix} \frac{-2}{RC} & \frac{1}{RC} \\ \frac{3}{2RC} & \frac{1}{RC} \end{pmatrix}$, $B_I = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\bar{B}_I = \begin{pmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{2R} \end{pmatrix}$,

913 $A_{II} = \begin{pmatrix} \frac{1}{2C} & -\frac{1}{C} \end{pmatrix}$, $B_{II} = -R$, $\bar{B}_{II} = \begin{pmatrix} 0 & \frac{3}{2} \end{pmatrix}$, $C_I = \begin{pmatrix} \frac{-2}{RC} & \frac{1}{RC} \\ 0 & 0 \end{pmatrix}$, $C_{II} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

914 Assumption 8 holds with $D_{11} = \frac{1}{R}$. Thus $\bar{B}_{II}^2 = \frac{3}{2}$, $C_{II}^2 = 1$, $X_2 = \frac{2}{3}$ and Assumption
 915 9 is verified. Item 1 in Proposition 4.8 applies since $\text{Im}(C_{II}^2) = \mathbb{R}$. Since $\bar{B}_{II}^1 = 0$
 916 the mapping $G_2(\cdot)$ is zero, and item 3 in Proposition 4.8 applies. The quadruple
 917 $(B_{II}, \bar{B}_{II}, C_{II}, D)$ is passive because the LMI (C.1) is verified with $X = \frac{2}{3}$, hence
 918 Assumption 5 is satisfied, but it is not strictly passive since $D \succcurlyeq 0$. It can be checked

919 that the transfer matrix $H(s) = D + C_{II}(s - B_{II})^{-1}\bar{B}_{II} = \begin{pmatrix} \frac{1}{R} & 0 \\ 0 & \frac{3}{2(s+R)} \end{pmatrix}$ is positive

920 real [18, Theorem 2.48]. The conditions of Lemmata 5.10 and 5.11 are verified by
 921 taking R large enough (notice that all other parameters depend on $\frac{1}{R}$). Here $\tilde{A} =$
 922 $B_{II} = -R$, but Assumptions 2 and 6 do not hold, hence the material in Proposition
 923 4.4 cannot be used.

924 **6.2. Example 2 (circuit).** Let us consider now the circuit in Figure 1a. Using
 925 Kirchhoff laws its dynamics may be written as:

$$926 \quad (6.2) \quad \begin{cases} \dot{x}_1(t) = -\frac{R_1}{L}x_1(t) - \frac{R_3}{L}z(t) + \frac{1}{L}u(t) + \frac{1}{L}u_D(t) \\ \dot{x}_2(t) = \frac{R_3}{R_2}z(t) - \frac{1}{R_2\mathbf{C}}x_2(t) \\ 0 = x_1(t) + \frac{1}{R_2\mathbf{C}}x_2(t) - (1 + \frac{R_3}{R_2})z(t) \\ 0 \leq u_D(t) \perp x_1(t) \geq 0, \end{cases}$$

927 where $x_1 = i_1$, $x_2(t) = \int_0^t i_2(s)ds$, $z = i_3$, $\lambda = u_D$ (the voltage across the diode),
 928 $K = K^* = \mathbb{R}_+$, $n_1 = 2$, $p = n_2 = 1$, $m = 1$. Let apply the feedback control $u = 2R_3z$,

929 which modifies B_I . We get: $A_I = \begin{pmatrix} -\frac{R_1}{L} & 0 \\ 0 & -\frac{1}{R_2\mathbf{C}} \end{pmatrix}$, $B_I = \begin{pmatrix} \frac{R_3}{R_2} \\ \frac{1}{R_2} \end{pmatrix}$, $\bar{B}_I = \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix}$,

930 $A_{II} = (1 \ \frac{1}{R_2\mathbf{C}})$, $B_{II} = -(1 + \frac{R_3}{R_2})$, $\bar{B}_{II} = 0$, $C_I = (1 \ 0)$, $C_{II} = 0$, $D = 0$, $E_I = \begin{pmatrix} \frac{1}{L}u \\ 0 \end{pmatrix}$,

931 $E_{II} = 0$, $F(t) = 0$, $P_I = I_2$, $P_I^h = \begin{pmatrix} 1 + h\frac{R_1}{L} & 0 \\ 0 & 1 + h\frac{1}{R_2\mathbf{C}} \end{pmatrix} \succ 0$ for all $h \geq 0$. Thus

932 $\tilde{B} = \frac{h}{hR_1 + L}$, $\tilde{C} = \frac{hR_3}{L + hR_1}$, $\tilde{D} = \frac{h}{L + hR_1}$, $\tilde{A} = -\frac{R_2 + R_3}{R_2} + hR_3 \left(\frac{1}{L + hR_1} + \frac{1}{R_2(h + R_2\mathbf{C})} \right)$.

933 Assumption 2 holds with $\tilde{X} = R_3$, $\tilde{D} > 0$ for all $h > 0$, and Assumption 4 holds
 934 for $h > 0$ small enough such that $\tilde{A} < 0$. For $h > 0$ small enough the set $\tilde{S} = \{0\}$
 935 in (4.6) since $\tilde{R}\tilde{A}\tilde{R}^{-1} = \tilde{A} < 0$, hence the VI in (4.7) always has a solution z_{k+1} by
 936 Proposition 4.3 item 3 (a). By item (c) (or (f)) of the same proposition, the solution is
 937 unique. When $\tilde{A} < 0$, the quadruple $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is strictly passive for all $h > 0$, hence
 938 items 1 and 3 of Corollary 4.6 apply. However Assumption 10 does not hold since
 939 $\tilde{D} \rightarrow 0$ as $h \rightarrow 0$. Consider now the quadruple $(B_{II}, \bar{B}_{II}, C_{II}, D) = (B_{II}, 0, 0, 0)$.
 940 It is passive since (C.1) is verified, it is even strictly state passive but not strictly
 941 passive since $D = 0$. Assumptions 3 and 9 are trivially satisfied since both sides of
 942 the passivity equality are null. Provided that $\{0\} \in \mathbb{R}_+ - x_{1,k+1} \Leftrightarrow x_{1,k+1} \geq 0$, item 1
 943 in Proposition 4.7 applies and the OSNSP for (3.1) has a unique solution (but clearly
 944 there is no guarantee that this inequality is satisfied along the integration process,
 945 rendering Proposition 4.7 unapplicable). Proposition 4.8 item 3 also applies here for
 946 the VI in (4.17), however one has to check the equivalence stated in item 1 of the
 947 same proposition.

948 As noted above, the matrix A_I intervenes only in Assumption 1, hence its modification
 949 does not alter the results. Setting $u = 2R_3z + R_1x_1 + aLx_1 + v(t)$ yields the modified

950 $A_I = \begin{pmatrix} a & 0 \\ 0 & -\frac{1}{R_2\mathbf{C}} \end{pmatrix}$, and B_I as above. This yields (see Remark 5.13): $P_I^{-1}(A_I -$

951 $B_I B_{II}^{-1} A_{II}) = \begin{pmatrix} a + \frac{R_2 R_3}{L(R_2 + R_3)} & \frac{R_3}{L\mathbf{C}(R_2 + R_3)} \\ \frac{R_3}{R_2 + R_3} & -\frac{1}{(R_2 + R_3)\mathbf{C}} \end{pmatrix}$, which is unstable if $a + \frac{R_2 R_3}{(R_2 + R_3)L} >$

952 $\frac{1}{(R_2 + R_3)\mathbf{C}}$. The other elements of the reduced LCS quadruple are given by $P_I^{-1}(\bar{B}_I -$

953 $B_I B_{II}^{-1} \bar{B}_{II}) = \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix}$, $C_I - C_{II} B_{II}^{-1} A_{II} = (1 \ 0)$, $D - C_{II} B_{II}^{-1} \bar{B}_{II} = 0$. Consequently

954 the reduced LCS is not passive with the above feedback controller. Also $P_I^h = \text{diag}(1 -$
 955 $ha, 1 + \frac{h}{R_2\mathbf{C}}) \succ 0$ for all $h < \frac{1}{a}$.

956 To simplify the calculations let us set $R_2 = R_3 = 1\Omega$, $L = 1\text{H}$, $\mathbf{C} = 1\text{F}$, $a = 1$. Then

957 the discretization (3.1) of (6.2) writes as:

$$958 \quad (6.3) \quad \begin{cases} x_{1,k+1} - x_{1,k} = hx_{1,k+1} + hz_{k+1} + h\lambda_{k+1} + hv_k \\ x_{2,k+1} - x_{2,k} = -hx_{2,k+1} + hz_{k+1} \\ 0 = x_{1,k+1} + x_{2,k+1} - 2z_{k+1} \\ 0 \leq \lambda_{k+1} \perp x_{1,k+1} \geq 0. \end{cases}$$

959 Equivalently the discretization in (3.2) can be used with the above matrices $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$.

960 **6.3. Example 3 (switching DAEs).** Let us consider the following system,
961 which is a nonsmooth DAE (similar to those studied in [53, 54, 52, 51], and which
962 can be interpreted as a DAE with state-dependent switches):

$$963 \quad (6.4) \quad \begin{cases} (a) \dot{x}(t) = A_I x(t) + B_I z(t) + E_I(t) \\ (b) 0 = a|x(t)| + b|z(t)| - cz(t) + E_{II}(t), \end{cases}$$

964 with $x(t) \in \mathbb{R}, z(t) \in \mathbb{R}, a, b, c \in \mathbb{R}, c > 0$. The constraint in (6.4) (b) is equivalently
965 rewritten as:

$$966 \quad (6.5) \quad \begin{aligned} 0 &= a\lambda_1(t) + a\lambda_2(t) + b\lambda_3(t) + b\lambda_4(t) - cz(t) + E_{II}(t) \\ 0 &\leq \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \lambda_3(t) \\ \lambda_4(t) \end{pmatrix} \perp \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \lambda_3(t) \\ \lambda_4(t) \end{pmatrix} + \begin{pmatrix} -x(t) \\ x(t) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -z(t) \\ z(t) \end{pmatrix} \geq 0. \end{aligned}$$

967 Therefore (6.4) (a) (6.5) fits with (2.1) with $A_{II} = 0, B_{II} = -c, \bar{B}_{II} = (a \ a \ b \ b),$
968 $D = I_4, C_I = (-1 \ 1 \ 0 \ 0)^\top, C_{II} = (0 \ 0 \ -1 \ 1)^\top, F(t) = 0, \bar{B}_I = 0$. Let us check the
969 strict passivity of the quadruple $(B_{II}, \bar{B}_{II}, C_{II}, D)$. This boils down to checking that
970 (see (C.1)):

$$971 \quad (6.6) \quad M \triangleq \begin{pmatrix} 2cX & -aX & -aX & -bX - 1 & -bX + 1 \\ -aX & 2 & 0 & 0 & 0 \\ -aX & 0 & 2 & 0 & 0 \\ -bX - 1 & 0 & 0 & 2 & 0 \\ -bX + 1 & 0 & 0 & 0 & 2 \end{pmatrix} \succ 0,$$

972 for some $X > 0$. Let us denote $M = P + N$ with $P = \text{diag}(2cX, 2I_4) \succ 0$, and
973 N_{00} the matrix obtained for $a = b = 0$. Applying [25, Theorem 2.11] or [10, Fact
974 8.19.2], it follows that $P + N_{00} \succ 0$ if $\sigma_{\max}^2(N_{00}) = 2 < \lambda_{\min}(2cX)\lambda_{\min}(2I_4) = 4cX$.
975 Using continuity of the eigenvalues [10, Fact 10.11.8], it is deduced that given $c > 0,$
976 $X > 0$, there exists a and b small enough such that (6.6) holds (or, given a, b and
977 $X > 0$, there exists $c > 0$ large enough). Thus, item 1 in Lemma 4.1 holds, item 2
978 in Proposition 4.3 holds for $h > 0$ small enough (guaranteeing $\tilde{D} \succ 0$), Assumption
979 10 holds true for small enough $h > 0$ (see the proof of item 1 in Lemma 4.1), and
980 the arguments stated at the beginning of section 5.2 about the convergence analysis
981 led in sections 5.1.1 and 5.1.2, hold. We have: $\tilde{A} = -c, \tilde{B} = (a \ a \ b \ b), \tilde{C} =$
982 $(0 \ 0 \ -1 \ 1)^\top + h(-1 \ 1 \ 0 \ 0)^\top(1 - hA_I)^{-1}B_I = \begin{pmatrix} -hB_I & hB_I \\ 1-hA_I & 1-hA_I \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}^\top, \tilde{D} = D$.
983 Integration can proceed using (3.2).

984 Let us now pass to a second example of state-dependent-switch DAE:

$$985 \quad (6.7) \quad \begin{cases} (a) \dot{x}_1(t) = a_1x_1(t) + b_1x_2(t) + c_1z(t) + e_1(t) \\ (b) \dot{x}_2(t) = a_2x_1(t) + b_2x_2(t) + c_2z(t) + e_2(t) \\ (c) 0 = a_3x_1(t) + b_3x_2(t) + c_3z(t) + e_3(t) \text{ if } x_1(t) < 0 \\ (d) 0 = a_4x_1(t) + b_3x_2(t) + c_3z(t) + e_3(t) \text{ if } x_1(t) > 0, \end{cases}$$

986 where the switching conditions in (c) and (d) are equivalently rewritten as:

$$987 \quad (6.8) \quad \begin{cases} 0 = -a_3\lambda_1(t) + a_4\lambda_2(t) + b_3x_2(t) + c_3z(t) + e_3(t) \\ 0 \leq \lambda_1(t) \perp \lambda_1(t) + x_1(t) \geq 0 \\ 0 \leq \lambda_2(t) \perp \lambda_2(t) - x_1(t) \geq 0. \end{cases}$$

988 Thus we have: $A_I = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$, $B_I = (c_1 \ c_2)^\top$, $\bar{B}_I = (0 \ 0)^\top$, $E_I = (e_1 \ e_2)^\top$, $\bar{B}_{II} =$

989 $(-a_3 \ a_4)^\top$, $A_{II} = (0 \ b_3)$, $B_{II} = c_3$, $E_{II} = e_3$, $C_I = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$, $C_{II} = (0 \ 0)^\top$, $D = I_2$,

990 $F(t) = (0 \ 0)^\top$. Also $\tilde{D} = D = I_2$, $\tilde{A} = c_3 + h(0 \ b_3)(I_2 - hA_I)^{-1} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_3 + \mathcal{O}(h)$,

991 $\tilde{B} = \bar{B}_{II} = (-a_3 \ a_4)^\top$, $\tilde{C} = h \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} (I_2 - hA_I)^{-1} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathcal{O}(h)$. It can be checked

992 that Proposition 4.3 items 1 and 2, Corollary 4.6 items 1, 2 and 3 (if the passivity LMI
993 holds strictly) apply. The passivity LMI $\begin{pmatrix} -2X\tilde{A} & -X(a_3 \ a_4) + \tilde{C} \\ (-X(a_3 \ a_4) + \tilde{C})^\top & 2I_2 \end{pmatrix} \succ 0$

994 is verified for a_3 , a_4 and $h > 0$ small enough, given $c_3 < 0$, $X > 0$, implying the
995 strict passivity of the quadruple $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$, uniformly as $h \rightarrow 0$ (Assumption 10
996 holds true). It is of interest to investigate what happens if $c_3 = 0$, which makes
997 the “index” increase. Let us assume for simplicity that $A_I = 0$, the matrix of the

998 passivity LMI becomes $\begin{pmatrix} -2hXb_3c_2 & -X(-a_3 \ a_4) + h(c_1 \ -c_1) \\ (-X(-a_3 \ a_4) + h(c_1 \ -c_1))^\top & 2I_2 \end{pmatrix}$.

999 For given $X > 0$ and $h > 0$, there always exists $b_3c_2 < 0$ such that this matrix is $\succ 0$,

1000 showing that there exists conditions under which the OSNP is still well-posed when
1001 $c_3 = 0$. It is interesting to note that the term $b_3c_2z(t)$ appears when an index reduction

1002 is applied to the equality in (6.8) by differentiating it once. Simulations are shown
1003 in Figures 2 and 3. They are performed with the INRIA software package SICONOS¹

1004 [1]. The MLCP is solved with an enumeration method and the differential variables
1005 $x_1(\cdot)$ and $x_2(\cdot)$ are calculated in a second step (hence we follow the philosophy of the
1006 foregoing sections). The parameters are chosen as follows: $a_1 = a_2 = b_1 = b_2 = a_3 =$

1007 $a_4 = b_3 = c_3 = 1$, $c_1 = 10$, $c_2 = 20$, $e_1(t) = 40 \sin(25t)$, $e_2(t) = \sin(t)$, $e_3(t) = \sin(10t)$,

1008 $h = 2 \cdot 10^{-3}$ s, $x_1(0) = 0$, $x_2(0) = 0.1$. Simulations are done on $[0, 2]$ s. The errors are
1009 calculated as $\max_{t \in [0, 2]} \|x_i(t) - x_i^{ref}(t)\|$, $i = 1, 2$, with 14 different values of step

1010 size h between 10^{-1} s and 10^{-4} s, where $x_i^{ref}(\cdot)$ is calculated with $h = 10^{-4}$ s. It
1011 is observed in Figure 2 that the order is approximately equal to one for small enough
1012 step size. Figure 3 depicts the time evolution of various variables, demonstrating that
1013 the complementarity constraints are respected. Clearly the multipliers are continuous
1014 but non differentiable when the complementarity switches from one mode to another.

1015

1016 **6.4. Example 4 (mechanical system).** As an extension of the systems studied
1017 in [37], let us consider a linear mechanical system with both unilateral springs and
1018 bilateral constraints, whose dynamics are:

$$1019 \quad (6.9) \quad \begin{cases} (a) \ M\ddot{q}(t) + K_v\dot{q}(t) + K_pq(t) = \tau(t) + H^\top \lambda_u(t) + G^\top \lambda_b(t) \\ (b) \ 0 = Gq(t) - f(t) \\ (c) \ 0 \leq \lambda_u(t) \perp D\lambda_u(t) + C_Ix(t) + F(t) \geq 0, \end{cases}$$

¹<https://nonsmooth.gricad-pages.univ-grenoble-alpes.fr/siconos/index.html>

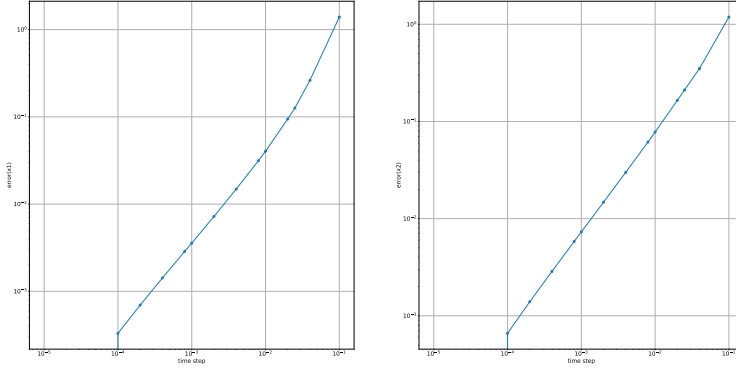


Fig. 2: Log-log error diagram: x_1 (left) and x_2 (right).

1020 where $M = M^\top \succ 0$, $D \succ 0$, $x = (q^\top \dot{q}^\top)^\top$, $q(t) \in \mathbb{R}^{\frac{n_1}{2}}$, $z = \lambda_b \in \mathbb{R}^{n_2}$, $\lambda = \lambda_u \in$
 1021 \mathbb{R}^m , $G \in \mathbb{R}^{n_2 \times \frac{n_1}{2}}$, $f: \mathbb{R}^+ \rightarrow \mathbb{R}^{n_2}$ is twice differentiable, $\tau(\cdot)$ is an exogenous force.
 1022 Here an index reduction is possible, in order to replace (6.9) (b) by:

$$1023 \quad (6.10) \quad 0 = G\dot{q}(t) - \ddot{f}(t) = GM^{-1}H^\top \lambda(t) + GM^{-1}G^\top z(t) - GM^{-1}(K_p K_v)x(t) - \ddot{f}(t),$$

1024 and $Gq(0) - f(0) = 0$, $G\dot{q}(0) - \dot{f}(0) = 0$. Thus $B_{II} = -GM^{-1}G^\top$ (< 0 if bilateral con-
 1025 straints are independent, *i.e.*, $\text{rank}(G) = p = n_2 \Rightarrow n_1 \geq 2n_2$), $A_{II} = GM^{-1}(K_p K_v)$,
 1026 $\bar{B}_{II} = -GM^{-1}H^\top$, $B_I = \begin{pmatrix} 0 \\ -M^{-1}G^\top \end{pmatrix}$, and $C_{II} = 0$ (similarly to the example in
 1027 section 6.2, one major difference being that here $D \succ 0$). Let us examine the OSNSP
 1028 for (6.9) on a particular example as depicted in Figure 4, whose dynamics are given
 1029 by:

$$1030 \quad (6.11) \quad \begin{cases} m\ddot{q}_1(t) = \tau(t) + \lambda_1(t) - \lambda_2(t) \\ m\ddot{q}_2(t) = z(t) \\ 0 = q_2(t) - f(t) \Leftrightarrow \frac{-z(t)}{m} + \ddot{f}(t) = 0, q_2(0) - f(0) = 0, \dot{q}_2(0) - \dot{f}(0) = 0 \\ 0 \leq \lambda_1(t) \perp \lambda_1(t) + k(q_1(t) - L - \alpha(t) - l) \geq 0 \\ 0 \leq \lambda_2(t) \perp \lambda_2(t) - k(q_1(t) + L - \beta(t) + l) \geq 0. \end{cases}$$

1031 where $q = (q_1 \ q_2)^\top$ are the mass coordinates, $k > 0$ is a stiffness constant, $\alpha(\cdot)$
 1032 and $\beta(\cdot)$ are the massless springs' varying positions, $l > 0$ is the spring's free length,
 1033 $\beta(t) > \alpha(t) + 2L + 2l$ for all $t \geq 0$. We have: $x = (q_1 \ \dot{q}_1 \ q_2 \ \dot{q}_2)^\top$, $G = (0 \ 1)$, $H =$
 1034 $\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$, $M = \text{diag}(m, m)$, $D = I_2$, $A_{II} = 0$, $B_{II} = \frac{-1}{m}$, $\bar{B}_{II} = 0$, $E_{II}(t) = \ddot{f}(t)$,
 1035 $C_I = \begin{pmatrix} k & 0 & 0 & 0 \\ -k & 0 & 0 & 0 \end{pmatrix}$, $F(t) = \begin{pmatrix} k(-L - \alpha(t) - l) \\ -k(L - \beta(t) + l) \end{pmatrix}$, $C_{II} = 0$, $B_I = (0 \ 0 \ 0 \ \frac{1}{m})^\top$,
 1036 $P_I = I_4$ (and Assumption 1 holds for $h > 0$ small enough), $\bar{B}_I = \begin{pmatrix} 0 & \frac{1}{m} & 0 & 0 \\ 0 & \frac{m_1}{m} & 0 & 0 \end{pmatrix}^\top$,
 1037 $\tilde{A} = \frac{-1}{m}$, $\tilde{B} = 0$, $\tilde{C} = hC_I(P_I^h)^{-1}B_I$, $\tilde{D} = I_2 + hC_I(P_I^h)^{-1}\bar{B}_I \succ 0$ if $h > 0$ is small
 1038 enough. Let us check Assumption 4. The passivity LMI is $\begin{pmatrix} \frac{2X}{C^\top} & \tilde{C} \\ \tilde{C}^\top & \tilde{D} + \tilde{D}^\top \end{pmatrix} \succ 0$,

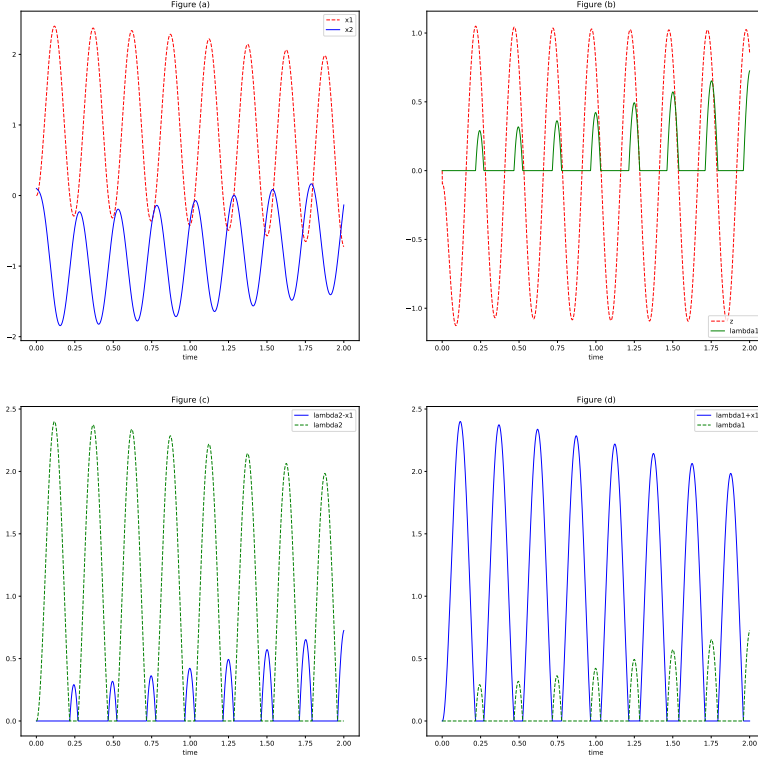


Fig. 3: (a) $x_1(t)$ (dashed) and $x_2(t)$, (b) $z(t)$ (dashed) and $\lambda_1(t)$, (c) $\lambda_2(t) - x_1(t)$ and $\lambda_2(t)$ (dashed), (d) $\lambda_1(t) + x_1(t)$ and $\lambda_1(t)$ (dashed).

1039 $X > 0$, which is verified for $h > 0$ small enough and $X > 0$ large enough. Hence under
 1040 these conditions the quadruple $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is strictly passive. It can be checked that
 1041 items 1 and 2 in Proposition 4.3 hold, as well as items 1, 2 and 3 in Corollary 4.6.
 1042 Therefore the dynamics can be integrated using (3.2) with $q_{2,0} + f_0 = 0$, $\dot{q}_{2,0} + \dot{f}_0 = 0$.
 1043 We note also that the quadruple $(B_{II}, \bar{B}_{II}, C_{II}, D) = (\frac{-1}{m}, 0, 0, I_2)$ is strictly passive
 1044 (Assumption 5 is verified), hence item 1 in Lemma 4.1 holds. Assumption 10 is
 1045 also verified (for $h > 0$ small enough but uniformly in $h \geq 0$). Let us analyse the
 1046 convergence as in section 5.1. Assumption 11 may not be verified in this case, however
 1047 as noted after this assumption it is easy to modify slightly the numerical scheme so
 1048 that it is satisfied with $P_I = I_4$. Hence it is inferred that Proposition 5.2 applies.

1049 **6.5. Example 5 (circuit).** Let us consider the circuit in Figure 5, with $R_1 =$
 1050 $R_3 = 0 \Omega$. Let $x_1 = i_2$, $x_2 = \int_0^t i_3(t)dt$, $z = i_1$, $\lambda = u_D$ is the voltage across the

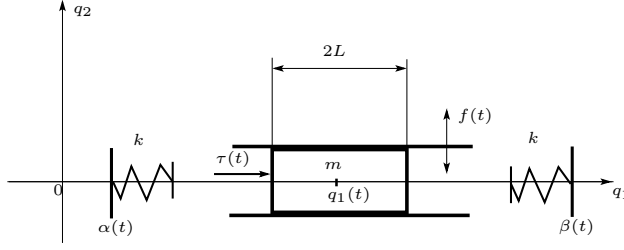
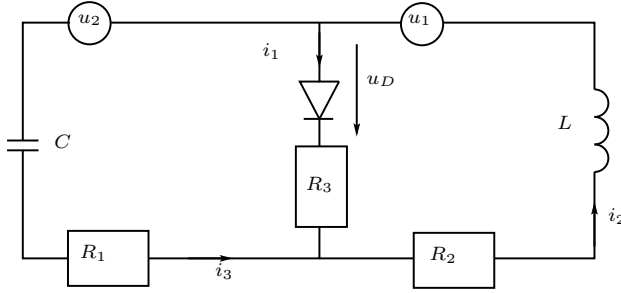


Fig. 4: A mechanical DALCS.

Fig. 5: An *RLCD* passive circuit.

1051 diode, $u_1(\cdot)$ and $u_2(\cdot)$ are two voltage sources. One way to write its dynamics is:

$$1052 \quad (6.12) \quad \begin{cases} \dot{x}_1(t) = -\frac{R_2}{L}x_1(t) + \frac{1}{L}\lambda(t) + \frac{1}{L}u_1(t) \\ \dot{x}_2(t) = x_1(t) - z(t) \\ 0 = \frac{1}{C}x_2(t) + \lambda(t) + u_2(t) \\ 0 \leq \lambda(t) \perp z(t) \geq 0. \end{cases}$$

1053 Assume that the feedback $u_2 = -az + u_3$ is applied, so that the equality constraint
 1054 becomes $0 = \frac{1}{C}x_2(t) + u_D(t) - az(t) + u_3(t)$. We have: $A_{II} = (0 \quad \frac{1}{C})$, $C_I = 0$,
 1055 $C_{II} = 1$, $B_{II} = -a$, $\bar{B}_{II} = 1$, $D = 0$, $\bar{B}_I = (\frac{1}{L} \quad 0)^\top$, $B_I = (0 \quad -1)^\top$, and $\bar{D} = D = 0$,
 1056 $\tilde{C} = C_{II} = 1$, $\tilde{A} = -a + hA_{II}(P_I^h)^{-1}B_I$, $\tilde{B} = 1 + hA_{II}(P_I^h)^{-1}\bar{B}_I$. The condition
 1057 $\text{Im}(C_{II}) \cap (K - C_I x_{k+1} + F_k) = \mathbb{R} \cap \mathbb{R}_+ - \frac{1}{C}x_{2,k+1} \neq \emptyset$ is verified always. The set \mathcal{S} in
 1058 (4.16) is equal to $\{0\}$ hence Proposition 4.7 item 1 applies and the VI in (4.15) always
 1059 has a solution. Proposition 4.8 items 1, 2, 3 apply, with $D_{11} = 0$ in Assumption 8.
 1060 The system $(B_{II}, \bar{B}_{II}, C_{II}) = (-a, 1, 1)$ is passive and Assumptions 3 and 5 hold true.
 1061 Also the system $(\tilde{A}, \tilde{B}, \tilde{C})$ is passive for $h > 0$ small enough and Assumptions 2 and
 1062 4 hold true. The set \mathcal{S} in (4.6) is equal to $\{0\}$ so Proposition 4.3 item 3 (a) applies.
 1063 For $h > 0$ small enough, $\tilde{A} < 0$ so Corollary 4.6 items 1 and 2 apply. Assumption 12
 1064 is verified for $h \geq 0$ small enough with $\tilde{X} = \tilde{B}^{-1}$. It is inferred that if $a > 0$ is large
 1065 enough, the convergence results in sections 5.1.3 and 5.2 apply.

1066 **7. Conclusion.** This article is devoted to analysing the implicit Euler discreti-
 1067 sation of a class of nonsmooth nonlinear systems with equality constraints, named
 1068 differential-algebraic linear complementarity systems. The well-posedness of the one-
 1069 step nonsmooth problem is studied first, then the convergence of the discrete-time
 1070 solutions is analysed. This article only scratches the surface of singular nonsmooth

1071 systems analysis. The above results could be extended in various directions. Most
 1072 importantly, it would be worth extending the results to more general indices and rela-
 1073 tive degrees, to deeper analyse how the algebraic and the complementarity constraints
 1074 are coupled, and to consider state discontinuities and solutions of bounded variation.

1075 **Appendix A. Well-posedness of Variational Inequalities.** Some basic
 1076 notions of Convex Analysis and Complementarity Theory are briefly introduce now
 1077 [11, 29, 49, 31]. Let $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper convex and lower semicontin-
 1078 uous (lsc) function, we denote by $\text{dom}(f) \triangleq \{x \in \mathbb{R}^n \mid f(x) < +\infty\}$ the domain of
 1079 the function $f(\cdot)$. The Fenchel transform $f^*(\cdot)$ of $f(\cdot)$ is the proper, convex and lsc
 1080 function defined by: for all $z \in \mathbb{R}^n$, $f^*(z) = \sup_{x \in \text{dom}(f)} \{\langle x, z \rangle - f(x)\}$. The sub-
 1081 differential $\partial f(x)$ of $f(\cdot)$ at $x \in \mathbb{R}^n$ is defined by: $\partial f(x) = \{\omega \in \mathbb{R}^n \mid f(v) - f(x) \geq$
 1082 $\langle \omega, v - x \rangle, \forall v \in \mathbb{R}^n\}$. We denote by $\text{Dom}(\partial f) \triangleq \{x \in \mathbb{R}^n \mid \partial f(x) \neq \emptyset\}$ the domain of
 1083 the subdifferential operator $\partial f : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Let x_0 be any element in the domain $\text{dom}(f)$, the recession function $f_\infty(\cdot)$ of $f(\cdot)$
 is defined by: for all $x \in \mathbb{R}^n$: $f_\infty(x) = \lim_{\lambda \rightarrow +\infty} \frac{1}{\lambda} f(x_0 + \lambda x)$. The function $f_\infty : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper convex and lsc function which describes the asymptotic
 behavior of $f(\cdot)$. Let $K \subset \mathbb{R}^n$ be a nonempty closed convex set. Let x_0 be any
 element in K . The recession cone of K is defined by [49]:

$$K_\infty = \bigcap_{\lambda > 0} \frac{1}{\lambda} (K - x_0) = \{u \in \mathbb{R}^n \mid x + \lambda u \in K, \forall \lambda \geq 0, \forall x \in K\}.$$

1084 The set K_∞ is a nonempty closed convex cone that is described in terms of the
 1085 directions which recede from K . The indicator function of a set $K \subseteq \mathbb{R}^n$ is $\psi_K(x) = 0$
 1086 if $x \in K$, $\psi_K(x) = +\infty$ if $x \notin K$. If K is closed non empty convex, we have
 1087 $\partial \psi_K(x) = \mathcal{N}_K(x)$, the so-called normal cone to K at x , defined as $\mathcal{N}_K(x) = \{v \in$
 1088 $\mathbb{R}^n \mid v^\top (s - x) \leq 0 \text{ for all } s \in K\}$.

1089 Important properties of the recession function and recession cone are recalled now:

1090 PROPOSITION A.1. [11, Proposition 1.4.8] *The following statements hold:*

1091 a) Let $f_1 : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ and $f_2 : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be two proper, convex and
 1092 lsc functions. Suppose that $f_1 + f_2$ is proper. Then for all $x \in \mathbb{R}^n$: $(f_1 + f_2)_\infty(x) =$
 1093 $(f_1)_\infty(x) + (f_2)_\infty(x)$.

1094 b) Let $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lsc function, and let K be a
 1095 nonempty closed convex set, such that $f + \Psi_K$ is proper (equivalently $\text{dom}(f) \cap K$ is
 1096 non empty). Then for all $x \in \mathbb{R}^n$: $(f + \Psi_K)_\infty(x) = f_\infty(x) + (\Psi_K)_\infty(x)$.

1097 c) Let $K \subset \mathbb{R}^n$ be a nonempty, closed and convex set. Then for all $x \in \mathbb{R}^n$:
 1098 $(\Psi_K)_\infty(x) = \Psi_{K_\infty}(x)$. Moreover for all $x \in K$ and $e \in K_\infty$: $x + e \in K$.

1099 d) If $K \subseteq \mathbb{R}^n$ is a nonempty closed and convex cone, then $K_\infty = K$.

1100 e) Let $K = P(A, b) \triangleq \{x \in \mathbb{R}^n \mid Ax \geq b\}$ for $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. If $K \neq \emptyset$ then
 1101 $K_\infty = P(A, 0) = \{x \in \mathbb{R}^n \mid Ax \geq 0\}$.

1102 Let us now concatenate [4, Theorem 3, Corollaries 3 and 4]. They concern variational
 1103 inequalities (VIs) of the form: Find $u \in \mathbb{R}^n$ such that

$$1104 \quad (\text{A.1}) \quad \langle \mathbf{M}u + \mathbf{q}, v - u \rangle + \varphi(v) - \varphi(u) \geq 0, \text{ for all } v \in \mathbb{R}^n$$

1105 where $\mathbf{M} \in \mathbb{R}^{n \times n}$ is a real matrix, $\mathbf{q} \in \mathbb{R}^n$ a vector and $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$
 1106 a proper convex and lsc function. The VI in (A.1) is equivalent to the inclusion

1107 $\mathbf{M}u + \mathbf{q} \in -\partial\varphi(u)$, which is a generalized equation (GE) of the form: $0 \in A(u) + B(u)$
 1108 where $A(\cdot)$ and $B(\cdot)$ are two maximal monotone operators (since it will be assumed
 1109 $\mathbf{M} \succcurlyeq 0$). In fact the problems encountered in this paper (see (4.7), (4.9), (4.14), (4.15),
 1110 (4.17)) all belong to this class of GEs. Numerical schemes that compute solutions of
 1111 such GEs have been studied for a long time in the mathematical literature, see [57]
 1112 and references therein. The next proposition states conditions about existence and
 1113 uniqueness of solutions.

1114 The problem in (A.1) is denoted as $VI(\mathbf{M}, \mathbf{q}, \varphi)$, and we set:

$$1115 \quad (\text{A.2}) \quad \mathcal{K}(\mathbf{M}, \varphi) = \{x \in \mathbb{R}^n \mid \mathbf{M}x \in (\text{dom}(\varphi_\infty))^*\}.$$

1116 Note that $(\text{dom}(\varphi_\infty))^*$ is the dual cone of the domain of the recession function φ_∞
 1117 while $(\text{dom}(\varphi))_\infty$ (that we may denote also as $\text{dom}(\varphi)_\infty$) is the recession cone of
 1118 $\text{dom}(\varphi)$.

1119 **PROPOSITION A.2.** [4] *Let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lsc*
 1120 *function with closed domain, $\mathbf{M} \in \mathbb{R}^{n \times n}$, and suppose that $\mathbf{M} \succcurlyeq 0$.*

1121 *a) If $(\text{dom}(\varphi))_\infty \cap \text{Ker}\{\mathbf{M} + \mathbf{M}^\top\} \cap \mathcal{K}(\mathbf{M}, \varphi) = \{0\}$ then for each $\mathbf{q} \in \mathbb{R}^n$, problem*
 1122 *$VI(\mathbf{M}, \mathbf{q}, \varphi)$ has at least one solution.*

1123 *b) Suppose that $(\text{dom}(\varphi))_\infty \cap \text{Ker}\{\mathbf{M} + \mathbf{M}^\top\} \cap \mathcal{K}(\mathbf{M}, \varphi) \neq \{0\}$. If there exists $x_0 \in$
 1124 *$\text{dom}(\varphi)$ such that**

$$1125 \quad (\text{A.3}) \quad \langle \mathbf{q} - \mathbf{M}^\top x_0, v \rangle + \varphi_\infty(v) > 0, \quad \forall v \in (\text{dom}(\varphi))_\infty \cap \text{Ker}\{\mathbf{M} + \mathbf{M}^\top\} \cap \mathcal{K}(\mathbf{M}, \varphi), \quad v \neq 0,$$

1126 *then problem $VI(\mathbf{M}, \mathbf{q}, \varphi)$ has at least one solution.*

1127 *c) If u_1 and u_2 are two solutions of problem $VI(\mathbf{M}, \mathbf{q}, \varphi)$ then $u_1 - u_2 \in \text{Ker}\{\mathbf{M} +$
 1128 $\mathbf{M}^\top\}$.*

1129 *d) If $\mathbf{M} = \mathbf{M}^\top$ and u_1 and u_2 denote two solutions of problem $VI(\mathbf{M}, \mathbf{q}, \varphi)$, then*
 1130 *$\langle \mathbf{q}, u_1 - u_2 \rangle = \varphi(u_2) - \varphi(u_1)$.*

1131 *e) If $\mathbf{M} = \mathbf{M}^\top$, then u is a solution of $VI(\mathbf{M}, \mathbf{q}, \varphi)$ if and only if it is a solution of*
 1132 *the optimization problem $\min_{x \in \mathbb{R}^n} \frac{1}{2}x^\top \mathbf{M}x + \langle \mathbf{q}, x \rangle + \varphi(x)$.*

1133 **Appendix B. Some Convex Analysis and Complementarity Theory**
 1134 **tools.** If $K \subseteq \mathbb{R}^n$ is a set, then $K^* = \{z \in \mathbb{R}^n \mid \langle z, x \rangle \geq 0 \text{ for all } x \in K\}$ is
 1135 its dual cone. Let K be a nonempty closed convex cone, then:

$$1136 \quad (\text{B.1}) \quad K^* \ni x \perp y \in K \Leftrightarrow x \in -\mathcal{N}_K(y) \iff y \in -\mathcal{N}_{K^*}(x).$$

1137 Let $M = M^\top \succ 0$, x and y two vectors, then [29]

$$1138 \quad (\text{B.2}) \quad M(x - y) \in -\mathcal{N}_K(x) \Leftrightarrow x = \text{proj}_M[K; y] \Leftrightarrow x = \min_{z \in K} \frac{1}{2}(z - y)^\top M(z - y).$$

1139 We note that this is a particular case of (A.1), so that Proposition A.2 can be con-
 1140 sidered as the characterization of a generalized projection operator $VI(\mathbf{M}, \mathbf{q}, \varphi)$. The
 1141 next proposition is used several times in the article:

1142 **PROPOSITION B.1.** *Suppose that the $m \times m$ matrix $D \succ 0$, and $\mathcal{M} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$*
 1143 *is a maximal monotone operator. Then the operator $(D + \mathcal{M})^{-1}(\cdot)$ is well-defined*
 1144 *(i.e., for each $y \in \mathbb{R}^m$ there is an $x \in \mathbb{R}^m$ such that $y \in Dx + \mathcal{M}(x)$), single-valued*
 1145 *and Lipschitz continuous with modulus $L = \frac{2}{\lambda_{\min}(D + D^\top)}$.*

1146 It was proposed in [16, Proposition 1] when $\mathcal{M} = \partial\phi$ for some convex proper lsc $\phi(\cdot)$,
 1147 extended to maximal monotone $\mathcal{M}(\cdot)$ in [5].

1148 **Appendix C. Passive Systems.** A quadruple (A, B, C, D) is said passive if:

1149 (C.1)
$$\begin{pmatrix} -XA - A^\top X & -XB + C^\top \\ -B^\top X + C & D + D^\top \end{pmatrix} \succcurlyeq 0, \quad X = X^\top \succ 0.$$

1150 for some X . It is said *strictly* passive if the first matrix inequality is satisfied with $\succ 0$
 1151 (this implies that $D \succ 0$). This is not to be confused with the *strict state* passivity [18,
 1152 Definition 4.54] (strict passivity implies strict state passivity). In fact strict passivity
 1153 is directly related to strongly SPR transfer matrices [18, p.62 and section 3.12.2],
 1154 while SPR transfer matrices with minimal realization are strictly state passive [18,
 1155 Theorem 4.73] [43]. Strict state passivity holds when $-XA - A^\top X \succ 0$ in (C.1), with
 1156 $D + D^\top \succcurlyeq 0$. The two inequalities in (C.1) make a linear matrix inequality (LMI) with
 1157 unknown X , equivalently a Riccati inequality using the Schur complement Theorem
 1158 [18, Theorem A.65]. Numerical algorithms exist that efficiently solve LMIs, see [13],
 1159 and can be used to test whether or not a quadruple is passive.

1160 REFERENCES

1161 [1] V. Acary, O. Bonnefon, M. Brémond, O. Huber, F. Péron, and S. Sinclair. **An introduction**
 1162 **to SICONOS**. Technical Report RT-0340, INRIA, Univ. Grenoble Alpes, Grenoble, France,
 1163 2019. <https://hal.inria.fr/inria-00162911v3/document>.
 1164 [2] V. Acary, O. Bonnefon, and B. Brogliato. *Nonsmooth Modeling and Simulation for Switched*
 1165 *Circuits*, volume 69 of *Lecture Notes in Electrical Engineering*. Springer Science+Business
 1166 Media, 2011. Erratum at <https://hal.inria.fr/hal-01311078>.
 1167 [3] V. Acary and B. Brogliato. *Numerical Methods for Nonsmooth Dynamical Systems. Applica-*
 1168 *tions in Mechanics and Electronics*, volume 35 of *Lecture Notes in Applied and Computa-*
 1169 *tional Mechanics*. Springer Verlag, Berlin Heidelberg, 2008.
 1170 [4] K. Addi, B. Brogliato, and D. Goeleven. A qualitative mathematical analysis of a class of linear
 1171 variational inequalities via semi-complementarity problems: applications in electronics.
 1172 *Mathematical Programming A*, 126:31–67, 2011.
 1173 [5] S. Adly, D. Goeleven, and R. Oujja. **Well-posedness of nonsmooth Lurie dynamical systems**
 1174 **involving maximal monotone operators**. In I.N. Parasidis, E. Providis, and T.M. Rassias,
 1175 editors, *Mathematical Analysis in Interdisciplinary Research*, volume 179. Springer Inter-
 1176 national Publishing, 2021.
 1177 [6] S. Adly, A. Hantoute, and B.K. Le. Maximal monotonicity and cyclic monotonicity arising in
 1178 nonsmooth Lur’e dynamical systems. *Journal of Mathematical Analysis and Applications*,
 1179 448:691–706, 2017.
 1180 [7] C. Arevalo and P. Lötstedt. **Improving the accuracy of BDF methods for index 3 differential-**
 1181 **algebraic equations**. *BIT*, 35:297–308, 1995.
 1182 [8] J. Bastien and M. Schatzman. **Numerical precision for differential inclusions with uniqueness**.
 1183 *ESAIM Mathematical Modelling and Numerical Analysis*, 36(3):427–460, 2002.
 1184 [9] H.H. Bauschke and P.L. Combettes. *Convex Analysis and Monotone Operator Theory in*
 1185 *Hilbert Spaces*. CMS Books in Mathematics. Canadian Mathematical Society. Société
 1186 Mathématique du Canada. Springer Science+Business Media, 2011.
 1187 [10] D.S. Bernstein. *Matrix Mathematics. Theory, Facts and Formulas*. Princeton University Press,
 1188 2nd edition, 2009.
 1189 [11] D.P. Bertsekas. *Convex Optimization Theory*. Athena Scientific, Belmont, 2009.
 1190 [12] W. Blajer and K. Kolodziejczyk. **A geometric approach to solving problems of control con-**
 1191 **straints: Theory and a DAE framework**. *Multibody System Dynamics*, 11:343–364, 2004.
 1192 [13] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakishnan. *Linear Matrix Inequalities in System*
 1193 *and Control Theory*. Society for Industrial and Applied Mathematics, Philadelphia, USA,
 1194 1994.
 1195 [14] K.E. Brenan. **Numerical simulation of trajectory prescribed path control problems by the**
 1196 **backward differentiation formulas**. *IEEE Transactions on Automatic Control*, 31(3):266–
 1197 269, 1986.

- 1198 [15] H. Brézis and A. Haraux. Image d'une somme d'opérateurs monotones et applications. *Israel*
1199 *Journal of Mathematics*, 23(2):165–186, 1976.
- 1200 [16] B. Brogliato and D. Goeleven. Well-posedness, stability and invariance results for a class of mul-
1201 ti-valued Lur'e dynamical systems. *Nonlinear Analysis. Theory, Methods and Applications*,
1202 74:195–212, 2011.
- 1203 [17] B. Brogliato and D. Goeleven. Singular mass matrix and redundant constraints in unilaterally
1204 constrained Lagrangian and Hamiltonian systems. *Multibody System Dynamics*, 35:39–61,
1205 2015.
- 1206 [18] B. Brogliato, R. Lozano, B. Maschke, and O. Egeland. *Dissipative Systems Analysis and*
1207 *Control. Theory and Applications*. Communications and Control Engineering. Springer
1208 Nature Switzerland AG, Cham, CH, third edition, 2020. Erratum/Addendum at <https://hal.inria.fr/hal-03364924/document>.
- 1209 [19] B. Brogliato and A. Tanwani. Dynamical systems coupled with monotone set-valued operators:
1210 Formalisms, applications, well-posedness, and stability. *SIAM Review*, 62(1):3–129, 2020.
- 1211 [20] M.K. Camlibel and R. Frasca. Extension of Kalman-Yakubovich-Popov Lemma to descriptor
1212 systems. *Systems and Control Letters*, 58:795–803, 2009.
- 1213 [21] M.K. Camlibel, W.P.M.H. Heemels, and J.M. Schumacher. Consistency of a time-stepping
1214 method for a class of piecewise linear networks. *IEEE Transactions on Circuits and Sys-*
1215 *tems I*, 49(3):349–357, 2002.
- 1216 [22] M.K. Camlibel, L. Iannelli, A. Tanwani, and S. Trenn. Differential-algebraic inclusions with
1217 maximal monotone operators. In *Proceedings of the 55th IEEE Conference on Decision*
1218 *and Control*, pages 610–615, Las Vegas, USA, 2016.
- 1219 [23] M.K. Camlibel and J.M. Schumacher. Linear passive and maximal monotone mappings. *Math-*
1220 *ematical Programming Ser. B*, 157:397–420, 2016.
- 1221 [24] X. Chen and Z. Wang. **Convergence of regularized time-stepping methods for differential vari-**
1222 **ational inequalities**. *SIAM Journal on Optimization*, 23(3):1647–1671, 2013.
- 1223 [25] X. Chen and S. Xiang. Perturbation bounds of P-matrix linear complementarity problems.
1224 *SIAM Journal on Optimization*, 18(4):1250–1265, 2007.
- 1225 [26] A. Dontchev, T.Z. Dontchev, and I. Slavov. A Tikhonov theorem for singularly perturbed
1226 differential inclusions. *Nonlinear Analysis, Theory, Methods and Applications*, 26(9):1547–
1227 1554, 1996.
- 1228 [27] A.L. Dontchev. **On existence of solutions of parametrized generalized equations**. *Set-Valued*
1229 *and Variational Analysis*, 29:735–744, 2021.
- 1230 [28] A.L. Dontchev and R.T. Rockafellar. *Implicit Functions and Solution Mappings. A View*
1231 *from Variational Analysis*. Operations Research and Financial Engineering. Springer Sci-
1232 ence+Business Media, New York, 2nd edition, 2014.
- 1233 [29] F. Facchinei and J.S. Pang. *Finite-Dimensional Variational Inequalities and Complementarity*
1234 *Problems, Volume I*. Series in Operations Research. Springer-Verlag, New York, Berlin,
1235 Heidelberg, 2003.
- 1236 [30] C.W. Gear and L.R. Petzold. **ODE methods for the solution of differential/algebraic systems**.
1237 *SIAM J. Numer. Analysis*, 21(4):716–728, 1984.
- 1238 [31] D. Goeleven. *Complementarity and Variational Inequalities in Electronics*. Mathematical
1239 Analysis and its Applications. Academic Press, 2017.
- 1240 [32] G. Grammel. Singularly perturbed differential inclusions: an averaging approach. *Set-Valued*
1241 *Analysis*, 4:361–374, 1996.
- 1242 [33] C. Hachtel, A. Bartel, M. Günter, and A. Sandu. Multirate implicit Euler schemes for a
1243 class of differential-algebraic equations of index-1. *Journal of Computational and Applied*
1244 *Mathematics*, 387:112499, 2021.
- 1245 [34] L. Han, A. Tiwari, M.K. Camlibel, and J.S. Pang. Convergence of time-stepping schemes
1246 for passive and extended linear complementarity systems. *SIAM Journal on Numerical*
1247 *Analysis*, 47(5):3768–3796, 2009.
- 1248 [35] E. Hockerdal, E. Frisk, and L. Ericsson. **Real-time performance of DAE and ODE**
1249 **based estimators evaluated on a diesel engine**. *Science China. Information Sciences*,
1250 61:70202:1070202:13, 2018.
- 1251 [36] A.N. Kolmogorov and S.V. Fomin. *Elements of the Theory of Functions and Functional Analy-*
1252 *sis*, volume 1 and 2. Graylock Press, 1957, 1961. also Dover Inc., 1999.
- 1253 [37] H. Krishnan and N.H. McClamroch. **Computation of state realizations for control systems**
1254 **described by a class of linear differential-algebraic equations**. *International Journal of*
1255 *Control*, 55(6):1425–1441, 1992.
- 1256 [38] C. Landry, A. Caboussat, and E. Hairer. Solving optimization-constrained differential equations
1257 with discontinuity points, with application to atmospheric chemistry. *SIAM Journal on*
1258 *Scientific Computing*, 31(5):3806–3826, 2009.

- 1260 [39] B.K. Le. On a class of Lur'e dynamical systems with state-dependent set-valued feedback.
 1261 *Set-Valued and Variational Analysis*, 28(3):537–557, 2020.
- 1262 [40] B.K. Le. Well-posedness and nonsmooth Lyapunov pairs for state-dependent maximal mono-
 1263 tone differential inclusions. *Optimization. A Journal of Mathematical Programming and*
 1264 *Operations Research*, 69(6):1187–1217, 2020.
- 1265 [41] D.M.W. Leenaerts. **On Linear Dynamic Complementary System**. *IEEE Transactions on Cir-*
 1266 *cuits and Systems I: Fundamental Theory and Applications*, 46(8):1022–1026, 1999.
- 1267 [42] P. Lötstedt and L. Petzold. **Numerical solution of nonlinear differential equations with algebraic**
 1268 **constraints I: Convergence results for backward differentiation formulas**. *Mathematics of*
 1269 *Computation*, 46(174):491–516, 1986.
- 1270 [43] D.S. Madeira and J. Adamy. On the equivalence between strict positive realness and strict
 1271 passivity of linear systems. *IEEE Transactions on Automatic Control*, 61(10):3091–3095,
 1272 2016.
- 1273 [44] M.D.P. Monteiro-Marques. *Differential Inclusions in Nonsmooth Mechanical Problems. Shocks*
 1274 *and Dry Friction*, volume 9 of *Prog. Nonlinear Differential Equations Appl.* Birkhauser,
 1275 Basel, CH, 1993.
- 1276 [45] J.J. Moreau. Evolution problem associated with a moving convex set in a Hilbert space. *Journal*
 1277 *of Differential Equations*, 26(3):347–374, 1977.
- 1278 [46] J.S. Pang and D.E. Stewart. **Differential variational inequalities**. *Mathematical Programming*
 1279 *ser. A*, 113:345–424, 2008.
- 1280 [47] J. Peypouquet and S. Sorin. Evolution equations for maximal monotone operators: Asymptotic
 1281 analysis in continuous and discrete time. *Journal of Convex Analysis*, 17(3-4):1113–1163,
 1282 2010.
- 1283 [48] A. Rocca, V. Acary, and B. Brogliato. Index-2 hybrid DAE: a case study with well-posedness
 1284 and numerical analysis. In *Proc. 21st IFAC World Congress*, Berlin, Germany, 2020. IFAC.
- 1285 [49] R.T. Rockafellar. *Convex Analysis*. Princeton Landmarks in Mathematics. Princeton University
 1286 Press, 1970.
- 1287 [50] R.T. Rockafellar and R.J.B. Wets. *Variational Analysis*, volume 317 of *Grundlehren der math-*
 1288 *ematischen Wissenschaften*. Springer Verlag, Berlin Heidelberg, 1998.
- 1289 [51] P. Stechliniski, M. Patrascu, and B.I. Barton. **Nonsmooth DAEs with Applications in Mod-**
 1290 **eling Phase Changes**. In Campbell S., Ilchmann A., Mehrmann V., and Reis T., editors,
 1291 *Applications of Differential-Algebraic Equations: Examples and Benchmarks*, Differential-
 1292 Algebraic Equations Forum, pages 243–275. Springer, Cham, CH 2018.
- 1293 [52] P. Stechliniski, M. Patrascu, and P.I. Barton. Nonsmooth differential-algebraic equations in
 1294 chemical engineering. *Computers and Chemical Engineering*, 114:52–68, 2018.
- 1295 [53] P.G. Stechliniski and P.I. Barton. **Generalized derivatives of differential-algebraic equations**.
 1296 *Journal of Optimization Theory and Applications*, 171:1–26, 2016.
- 1297 [54] P.G. Stechliniski and P.I. Barton. **Dependence of solutions of nonsmooth differential-algebraic**
 1298 **equations on parameters**. *Journal of Differential Equations*, 262:2254–2285, 2017.
- 1299 [55] A. Tanwani, B. Brogliato, and C. Prieur. Well-posedness and output regulation for implicit
 1300 time-varying evolution variational inequalities. *SIAM Journal on Control and Optimiza-*
 1301 *tion*, 56(2):751–781, 2018.
- 1302 [56] V. Veliov. A generalization of the Tikhonov theorem for singularly perturbed differential in-
 1303 clusions. *Journal of Dynamical and Control Systems*, 3(3):291–319, 1997.
- 1304 [57] P.T. Vuong. **A second order dynamical system and its discretization for strongly pseudo-**
 1305 **monotone variational inequalities**. *SIAM J. on Control and Optimization*, 59(4):2875–2897,
 1306 2021.
- 1307 [58] F. Watbled. On singular perturbations for differential inclusions on the infinite interval. *Journal*
 1308 *of Mathematical Analysis and Applications*, 310:362–37, 2005.
- 1309 [59] S.L. Wu, T. Zhou, and X. Chen. A Gauss-Seidel type method for dynamic nonlinear comple-
 1310 mentarity problems. *SIAM Journal on Control and Optimization*, 58(6):3389–3412,
 1311 2020.
- 1312 [60] W. Xing, Q. Zhang, J. Zhang, and Q. Wang. **Some geometric properties of Lyapunov equation**
 1313 **and LTI system**. *Automatica*, 37:313–316, 2001.
- 1314 [61] A. Zelei, L.L. Kovacs, and G. Stepan. **Computed torque control of an under-actuated service**
 1315 **robot platform modeled by natural coordinates**. *Commun. Nonlinear Sci. Numer. Simulat.*,
 1316 16:2205–2217, 2011.
- 1317 [62] X. Zhao, T. Ploch, S. Noack, W. Wiechert, A. Mitsos, and E. von Lieres. Analysis of the
 1318 local well-posedness of optimization-constrained differential equations by local optimality
 1319 conditions. *AIChE Journal*, 66:e16548, 2020.