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1 ANALYSIS OF THE IMPLICIT EULER TIME-DISCRETIZATION OF  
2 SEMI-EXPLICIT DIFFERENTIAL-ALGEBRAIC LINEAR  
3 COMPLEMENTARITY SYSTEMS\*

4 BERNARD BROGLIATO<sup>†</sup>

5 **Abstract.** This article is largely concerned with the time-discretization of differential-algebraic  
6 equations (DAE) with complementarity constraints, which we name differential algebraic linear com-  
7 plementarity systems (DALCS). Specifically, the Euler implicit discretization of DALCS is analysed:  
8 the one-step non-smooth problem (OSNSP), that is a generalized equation, is shown to be well-  
9 posed under some conditions, then the convergence of the discretized solutions is studied, and the  
10 existence of solutions to the continuous-time system is shown as a consequence. Passivity of some  
11 operators is pivotal to the analysis. Examples from circuits, mechanics and switching DAE illustrate  
12 the applicability of the developments.

13 **Key words.** differential-algebraic system, descriptor-variable system, linear complementarity  
14 system, recession function, discrete-time, well-posedness, passive system, Lur'e equations.

15 **AMS subject classifications.** 65K15, 90C33, 90C90, 93B52, 93D30, 93C30.

16 **1. Introduction.** The analysis of non-smooth dynamical systems with multi-  
17 valued right-hand sides satisfying maximal monotone properties has witnessed a large  
18 number of contributions [19]. Within this class one finds linear and nonlinear com-  
19 plementarity dynamical systems, projected systems, differential variational inequali-  
20 ties, differential inclusions with maximal monotone right-hand sides, Moreau's sweep-  
21 ing process, some switching dynamics, *etc.* In this article we focus on singular, or  
22 differential-algebraic linear complementarity systems (DALCS), which may be viewed  
23 either as an extension of classical differential-algebraic equations (DAE), or of "clas-  
24 sical" linear complementarity systems. As will become clear later, DALCS represent  
25 an extension of the set-valued Lur'e systems studied, *e.g.*, in [39, 40]. The associated  
26 canonical form is motivated by applications in chemistry [53, 52, 62], switching DAE  
27 analysis [48], circuits with nonsmooth electronic components [2, section 3.5, chapter 4]  
28 [22], optimization-constrained differential equations [19, Example 2] with applications  
29 in atmospheric aerosol particles [38], and Lagrangian systems with both bilateral and  
30 unilateral constraints [17] (in spite of the fact that the results presented in this article  
31 do not apply to all of these systems).

32 The study of singular nonsmooth dynamical systems has not received a lot of attention  
33 yet. Singularly perturbed differential inclusions have been tackled in [26, 56, 32, 58].  
34 The conditions on the set-valued mappings that are imposed (compactness [56, 58],  
35 boundedness and Hausdorff Lipschitz continuity [32], one-sided Lipschitz condition  
36 [26]) are not satisfied by complementarity conditions (or by normal cones to convex  
37 sets). Hence these results do not apply to DALCS. The well-posedness of DAE in semi-  
38 explicit form with nonsmooth constraints (which are related to the class of systems  
39 studied in this article, see Remark 2.1 below) has been analysed in [53, 54, 52]. Passive  
40 singular set-valued systems are studied in [22], who derived well-posedness relying on  
41 a special Weierstrass' representation. However the time-discretisation is not tackled

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42 in the above references.

43 Our goal is to analyse the well-posedness of the one-step nonsmooth problem (OS-  
 44 NSP) obtained after an implicit (backward) Euler discretization, and the convergence  
 45 of the discrete-time solutions towards a continuous-time limit. The second step is use-  
 46 ful not only for proving the existence of solutions to the continuous-time system, but  
 47 also, perhaps most importantly, to show that the proposed time-discretization makes  
 48 sense for numerical simulations. Passivity properties of suitable operators are pivotal  
 49 throughout the article. It is noteworthy that implicit Euler methods are well-known  
 50 in DAE [30, 14, 33, 42, 12, 35, 61, 7], differential variational inequalities [46, 24],  
 51 and complementarity systems fields [41, 21, 59, 34], as well as in the field of max-  
 52 imal monotone differential inclusions [8, 47] (implicit Euler method is often called  
 53 proximal algorithm, while Euler method refers to the explicit algorithm [47]), and  
 54 Moreau's sweeping processes with the catching-up algorithm [19, 44, 45]. However  
 55 their application to DALCS has not been studied yet.

56 The article is organised as follows: Section 2 introduces the general dynamics of  
 57 the nonsmooth singular systems, section 3 presents their time-discretization, section  
 58 4 studies the well-posedness of the OSNSP, section 5 is dedicated to convergence  
 59 analyses: results in section 5.1 are based on the properties of the numerical scheme,  
 60 while section 5.2 uses properties of the system. Section 6 is dedicated to illustrating  
 61 examples and simulations, from circuits, mechanics and switching DAE. Conclusions  
 62 are drawn in section 7. The appendix is dedicated to recall various mathematical  
 63 tools.

64 **Notation and definitions:** for any vector  $x \in \mathbb{R}^n$  and any matrix  $M \in \mathbb{R}^{m \times n}$ ,  
 65  $\|x\|$  is the Euclidean norm and  $\|M\|$  is the Frobenius norm, which are compatible  
 66 norms [10, Proposition 9.3.5], *i.e.*,  $\|Mx\| \leq \|M\| \|x\|$ . Let  $M \in \mathbb{R}^{n \times m}$ , then  $\text{Im}(M)$   
 67 is its range,  $\text{Ker}(M)$  is its null space. We use  $\langle x, y \rangle = x^\top y$ , so  $\langle x, x \rangle = \|x\|^2$ . Positive  
 68 definite matrix:  $M \succ 0$  if  $x^\top Mx > 0$  for all  $x \neq 0$ , positive semidefinite matrix:  
 69  $M \succcurlyeq 0$  if  $x^\top Mx \geq 0$  for all  $x$  (such  $M$  is not necessarily symmetric). The maximum  
 70 singular value is denoted as  $\sigma_{\max}(M)$ , and the minimum eigenvalue as  $\lambda_{\min}(M)$ . The  
 71  $i$ th row of  $M$  is denoted as  $M_{i\bullet}$ . The  $n \times n$  identity matrix is denoted  $I_n$ . A set-valued  
 72 mapping  $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is said monotone if for all  $x_1, x_2, y_1 \in A(x_1), y_2 \in A(x_2)$ , one  
 73 has  $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$ . It is maximal monotone if its graph cannot be enlarged  
 74 without destroying monotonicity. It is  $\eta$ -strongly monotone if there exists  $\eta > 0$  such  
 75 that  $\langle x_1 - x_2, y_1 - y_2 \rangle \geq \eta \|x_1 - x_2\|^2$ . See also Appendix A for further definitions.

76 **2. The class of nonsmooth singular dynamical systems.** Let us consider  
 77 the following differential algebraic linear complementarity system (DALCS):

$$78 \quad (2.1) \quad \begin{cases} (a) & P_I \dot{x}(t) = A_I x(t) + B_I z(t) + \bar{B}_I \lambda(t) + E_I(t) \\ (b) & 0 = A_{II} x(t) + B_{II} z(t) + \bar{B}_{II} \lambda(t) + E_{II}(t) \\ (c) & K^* \ni \lambda(t) \perp w(t) = C_I x(t) + C_{II} z(t) + D \lambda(t) + F(t) \in K, \end{cases}$$

79 with  $x(t) \in \mathbb{R}^{n_1}$ ,  $z(t) \in \mathbb{R}^{n_2}$ ,  $\lambda(t) \in \mathbb{R}^m$ ,  $w(t) \in \mathbb{R}^m$ ,  $P_I \in \mathbb{R}^{n_1 \times n_1}$  is full-rank,  
 80  $K \subseteq \mathbb{R}^m$  is a closed nonempty convex cone,  $K^*$  is its dual cone,  $\bar{B}_{II} \in \mathbb{R}^{p \times m}$ ,  
 81  $A_{II} \in \mathbb{R}^{p \times n_1}$ ,  $B_{II} \in \mathbb{R}^{p \times n_2}$ ,  $C_I \in \mathbb{R}^{m \times n_1}$ ,  $C_{II} \in \mathbb{R}^{m \times n_2}$ ,  $D \in \mathbb{R}^{m \times m}$ ,  $B_I \in$   
 82  $\mathbb{R}^{n_1 \times n_2}$ ,  $A_I \in \mathbb{R}^{n_1 \times n_1}$ ,  $\bar{B}_I \in \mathbb{R}^{n_1 \times m}$ , and  $E_I(\cdot)$ ,  $E_{II}(\cdot)$ ,  $F(\cdot)$  are bounded, Lipschitz  
 83 continuous functions of time. The form in (2.1) is that of a semi-explicit DALCS. It  
 84 corresponds to a kind of extension of the impulse-free Weierstrass form [20], where  
 85 the transformed state matrix has non zero off-diagonal terms  $A_{II}$  and  $B_I$ .

86 *Remark 2.1.* Assume that  $K = \mathbb{R}_{\neq}^m$ . The complementarity constraint (2.1) (c)  
 87 is equivalently rewritten as  $\phi(\lambda, w) = 0$  where  $\phi(\cdot)$  is a  $C$ -function [3, 29]. Therefore  
 88 (2.1) (b) (c) can be rewritten equivalently as  $\Phi(x, z, \lambda, t) = 0$  for some function  $\Phi(\cdot)$ ,  
 89 usually nondifferentiable. The system (2.1) thus possesses the form of a semi-explicit  
 90 DAE as studied in [53, 52], assuming piecewise differentiable functions  $\Phi(\cdot)$ , with  $x$   
 91 the differential state,  $z$  and  $\lambda$  the algebraic state variables. The so-called regularity  
 92 conditions (which are an extension of index 1 conditions for DAE) used in [53, 54, 52,  
 93 51] may hold in particular cases for (2.1). Examples in section 6.3 illustrate the links  
 94 between both approaches in [53, 54, 52, 51] and in this article.

95 Consider (2.1), then the initial data  $x(0) = x_0$  and  $z(0) = z_0$  are compatible with the  
 96 system's constraints if and only if:

$$97 \quad (2.2) \quad \begin{aligned} 0 &\in A_{II}x_0 + B_{II}z_0 + E_{II}(0) - \bar{B}_{II}(D + \mathcal{N}_{K^*})^{-1}(C_Ix_0 + C_{II}z_0 + F(0)) \\ \lambda(0) &\in -(D + \mathcal{N}_{K^*})^{-1}(C_Ix_0 + C_{II}z_0 + F(0)). \end{aligned}$$

98 This implies that  $C_Ix_0 + C_{II}z_0 + F(0) \in \text{Im}(D + \mathcal{N}_{K^*})$  and  $A_{II}x_0 + B_{II}z_0 + E_{II}(0) \in$   
 99  $\bar{B}_{II}\text{Dom}(D + \mathcal{N}_{K^*})$ .

100 **3. Time-discretization of DALCS with implicit Euler methods.** Let us  
 101 consider the time interval  $[0, T]$ ,  $T > 0$ ,  $h = \frac{T}{N}$ ,  $N \in \mathbb{N}_+$ ,  $t_k = kh$ ,  $k \in \{0, 1, \dots, N\}$ ,  
 102  $t_0 = 0$ ,  $t_N = T$ . For a continuous function  $f(\cdot)$ ,  $f_k \triangleq f(t_k)$ . The implicit Euler  
 103 discretization of (2.1) reads as:

$$104 \quad (3.1) \quad \begin{cases} (a) & P_I(x_{k+1} - x_k) = hA_Ix_{k+1} + hB_Iz_{k+1} + h\bar{B}_I\lambda_{k+1} + hE_{I,k} \\ (b) & A_{II}x_{k+1} + B_{II}z_{k+1} + \bar{B}_{II}\lambda_{k+1} + E_{II,k} = 0 \\ (c) & K^* \ni \lambda_{k+1} \perp C_Ix_{k+1} + C_{II}z_{k+1} + D\lambda_{k+1} + F_k \in K. \end{cases}$$

105 The problem in (3.1) (b) (c) is a mixed LCP (MLCP). Let us now state a basic  
 106 assumption.

107 **ASSUMPTION 1.** *The matrix  $P_I^h \triangleq P_I - hA_I$  is full-rank.*

108 In case  $P_I \succ 0$ , [25, Theorem 2.11] can be used to set conditions on  $hA_I$  so that  
 109  $P_I^h \succ 0$ : there always exists  $h_{\max}$  such that the assumption holds for all  $h \in [0, h_{\max})$   
 110 (with  $h_{\max} = +\infty$  if  $A_I \preceq 0$ ). Let us define the following matrices and vectors:

- 111 1.  $\tilde{A} = hA_{II}(P_I^h)^{-1}B_I + B_{II}$ ,
- 112 2.  $\tilde{B} = hA_{II}(P_I^h)^{-1}\bar{B}_I + \bar{B}_{II}$ ,
- 113 3.  $\tilde{E}_k = E_{II,k} + hA_{II}(P_I^h)^{-1}E_{I,k} + A_{II}(P_I^h)^{-1}P_Ix_k$ ,
- 114 4.  $\tilde{C} = hC_I(P_I^h)^{-1}B_I + C_{II}$ ,
- 115 5.  $\tilde{D} = hC_I(P_I^h)^{-1}\bar{B}_I + D$ ,
- 116 6.  $\tilde{F}_k = F_k + hC_I(P_I^h)^{-1}E_{I,k} + C_I(P_I^h)^{-1}P_Ix_k$ .

117 Then we deduce from (3.1) the difference equations:

$$118 \quad (3.2) \quad \begin{cases} (a) & x_{k+1} = (P_I^h)^{-1}(P_Ix_k + hB_Iz_{k+1} + h\bar{B}_I\lambda_{k+1} + hE_{I,k}) \\ (b) & \tilde{A}z_{k+1} + \tilde{B}\lambda_{k+1} + \tilde{E}_k = 0 \\ (c) & K^* \ni \lambda_{k+1} \perp \tilde{C}z_{k+1} + \tilde{D}\lambda_{k+1} + \tilde{F}_k \in K, \end{cases}$$

119 where the last two lines of (3.2) are a MLCP with unknowns  $z_{k+1}$  and  $\lambda_{k+1}$ . It is  
 120 therefore possible to study the well-posedness of this MLCP independently of the  
 121 first line in (3.2). Letting  $\zeta_{k+1} \triangleq -z_{k+1}$ , this MLCP can be rewritten equivalently as

122 follows:

$$123 \quad (3.3) \quad \begin{cases} \lambda_{k+1} \in (\mathcal{N}_{K^*} + \tilde{D})^{-1}(-\tilde{C}z_{k+1} - \tilde{F}_k), \\ -\tilde{A}\zeta_{k+1} + \tilde{E}_k \in -\tilde{B}(\mathcal{N}_{K^*} + \tilde{D})^{-1}(\tilde{C}\zeta_{k+1} - \tilde{F}_k), \\ x_{k+1} = (P_I^h)^{-1}(P_I x_k + hB_I z_{k+1} + h\tilde{B}_I \lambda_{k+1} + hE_{I,k}). \end{cases}$$

124 Also the following calculations stem from the complementarity conditions in (3.2):  
 (3.4)

$$125 \quad \begin{aligned} & \lambda_{k+1} \in -\mathcal{N}_K(\tilde{C}z_{k+1} + \tilde{D}\lambda_{k+1} + \tilde{F}_k) \Leftrightarrow -\lambda_{k+1} \in \mathcal{N}_{\Gamma_k}(\tilde{C}z_{k+1} + \tilde{D}\lambda_{k+1}) \\ \Leftrightarrow & \partial\sigma_{\Gamma_k}(-\lambda_{k+1}) \ni \tilde{C}z_{k+1} + \tilde{D}\lambda_{k+1} \Leftrightarrow (\partial\sigma_{\Gamma_k} + \tilde{D})(-\lambda_{k+1}) \ni \tilde{C}z_{k+1} \\ \Leftrightarrow & \lambda_{k+1} \in -(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}z_{k+1}). \end{aligned}$$

126 where the inversion of set-valued mappings is used in the second equivalence, and  
 127 where  $\sigma_{\Gamma_k}(\cdot)$  is the support function of the set  $\Gamma_k(t_k, x_k) \triangleq K - \tilde{F}_k(t_k, x_k)$  (which  
 128 is a closed convex set, but not a cone as long as  $\tilde{F}_k \neq 0$  which is the generic case).  
 129 Therefore equivalently we can rewrite (3.3) as:

$$130 \quad (3.5) \quad \begin{cases} (a) \lambda_{k+1} \in -(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}z_{k+1}) \\ (b) -\tilde{A}z_{k+1} + \tilde{B}(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}z_{k+1}) \ni \tilde{E}_k \\ (c) x_{k+1} = (P_I^h)^{-1}(P_I x_k + hB_I z_{k+1} + h\tilde{B}_I \lambda_{k+1} + hE_{I,k}). \end{cases}$$

131 The second line in (3.3) is a generalized equation (GE) with unknown  $\zeta_{k+1}$ . Its well-  
 132 posedness depends on the properties of the operator  $\mathcal{F} : \zeta \mapsto -\tilde{A}\zeta + \tilde{B}(\mathcal{N}_{K^*} +$   
 133  $\tilde{D})^{-1}(\tilde{C}\zeta - \tilde{F}_k)$ . Similarly for (3.5) with the operator  $\mathcal{G} : \zeta \mapsto -\tilde{A}\zeta + \tilde{B}(\partial\sigma_{\Gamma_k} +$   
 134  $\tilde{D})^{-1}(\tilde{C}\zeta)$ , and (3.5) (b) is:  $\mathcal{G}(z_{k+1}) \ni \tilde{E}_k$ . Various results have been published to  
 135 characterize such operators [16, 23, 6] and to guarantee that they are single-valued  
 136 Lipschitz continuous [16, Propositions 1, 2, 3, Corollary 1], or maximal monotone  
 137 (possibly set-valued) [23, 6]. The difficulty in the analysis of DALCS is that they  
 138 involve the composition of two operators as seen in the OSNSP for advancing the  
 139 scheme from step  $k$  to step  $k+1$ :

$$140 \quad (3.6) \quad P_I^h x_{k+1} \in P_I x_k + hB_I \mathcal{G}^{-1}(\tilde{E}_k(x_k)) - h\tilde{B}_I(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}\mathcal{G}^{-1}(\tilde{E}_k(x_k))) + hE_{I,k}.$$

141 Clearly the scheme can be advanced with a unique solution if both  $\lambda_{k+1}$  and  $z_{k+1}$   
 142 are uniquely defined (the “if and only if” holds with  $\tilde{B}_I \lambda_{k+1}$  and  $B_I z_{k+1}$  uniquely  
 143 defined). Thus next we are going to characterize the GEs in (3.3) and (3.5) in order  
 144 to analyse the scheme (3.6).

145 Two main classes of results are provided in the sequel of the article: first the well-  
 146 posedness (existence and uniqueness of solutions) for the OSNSP associated with the  
 147 time-discretizations in (3.1) (Propositions 4.3 and 4.4, Corollary 4.6) and (3.2) (Propo-  
 148 sitions 4.7 and 4.8); second, the convergence of the discrete-time solutions is analysed,  
 149 and the limit functions are shown to be solutions of the DALCS (Propositions 5.2 and  
 150 5.9).

151 **4. OSNSP well-posedness.** The approach chosen in this article consists of  
 152 considering the system as an ODE (2.1) (a) with a nonsmooth constraint (2.1) (b)  
 153 (c): either (3.1) (b) (c) that is a mixed LCP with unknowns  $z_{k+1}$ ,  $\lambda_{k+1}$  and parameter  
 154  $x_{k+1}$  (section 4.2), or (3.2) (b) (c) (equivalently (3.5) (a) (b)) that is a mixed LCP  
 155 with  $z_{k+1}$ ,  $\lambda_{k+1}$  (section 4.1). As will become clear later, the results rely heavily on  
 156 the positive (semi) definiteness of  $B_{II}$ , and on specific passivity couplings between  
 157  $z$  in the complementarity constraint (matrix  $C_{II}$ ) and  $\lambda$  in the algebraic constraint  
 158 (matrix  $\tilde{B}_{II}$ ). When  $\tilde{D} = 0$  in (3.2) (c), let us state the following:

159 ASSUMPTION 2. *There exists a  $p \times p$  matrix  $\tilde{X} = \tilde{X}^\top \succ 0$  such that  $\tilde{X}\tilde{B} = \tilde{C}^\top$ .*

160 The alternative option when  $D = 0$  in (3.1) (c):

161 ASSUMPTION 3. *There exists a  $p \times p$  matrix  $X = X^\top \succ 0$  such that  $X\bar{B}_{II} = C_{II}^\top$ .*

162

163 Let us notice that Assumption 2 implies that  $\tilde{X}\bar{B}_{II} = C_{II}^\top + \mathcal{O}(h)$ , while Assumption  
 164 3 implies that  $X\tilde{B} = \tilde{C}^\top + \mathcal{O}(h)$ . The influence of the  $\mathcal{O}(h)$  terms may not be easy  
 165 to characterize precisely for finite  $h > 0$ . Assumption 3 is a model assumption, while  
 166 Assumption 2 is a property of the numerical method. One sees that Assumption 3  
 167 corresponds to a passivity constraint on the triple  $(B_{II}, \bar{B}_{II}, C_{II})$  [18]. Assumption  
 168 2 imposes an input/output passivity-like constraint on the triple  $(\tilde{A}, \tilde{B}, \tilde{C})$ , which is  
 169 an  $\mathcal{O}(h)$  approximation of  $(B_{II}, \bar{B}_{II}, C_{II})$ . When  $\tilde{D} \neq 0$  and  $D \neq 0$ , extensions of  
 170 Assumptions 2 and 3 are as follows:

171 ASSUMPTION 4. *The quadruple  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  is passive.*

172 and the counterpart of Assumption 3:

173 ASSUMPTION 5. *The quadruple  $(B_{II}, \bar{B}_{II}, C_{II}, D)$  is passive.*

174 Passivity is defined in Appendix C. Notice that the above assumptions imply that  
 175  $n_2 = p$ . Assumption 4 refers to the numerical method while Assumption 5 is a model  
 176 property. Assumptions 2 and 4 refer to (3.2) (b) (c) or (3.5) (a) (b), while Assumptions  
 177 3 and 5 refer to (3.1) (b) (c). Let us present a result about the relationships between  
 178 Assumptions 4 and 5, *i.e.*, when does the plant's passivity guarantees the numerical  
 179 scheme's passivity.

180 LEMMA 4.1. *The following statements are true:*

- 181 1. *Let  $(B_{II}, \bar{B}_{II}, C_{II}, D)$  be strictly passive. Then  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  is strictly passive*  
 182 *for small enough  $h > 0$ .*
- 183 2. *Let  $(B_{II}, \bar{B}_{II}, C_{II}, D)$  be strictly state passive. Then  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  is strictly*  
 184 *state passive if: a)  $h > 0$  is small enough, b)  $D + D^\top = 0$  and  $C_I(P_I^h)^{-1}\bar{B}_I +$   
 185  $(C_I(P_I^h)^{-1}\bar{B}_I)^\top \succcurlyeq 0$ , c)  $XA_{II}(P_I^h)^{-1}\bar{B}_I - B_I^\top(P_I^h)^{-\top}C_I^\top = 0$ , where  $X$  is a*  
 186 *solution of the passivity LMI for  $(B_{II}, \bar{B}_{II}, C_{II}, D)$ .*
- 187 3. *Let  $h > 0$  and  $(B_{II}, \bar{B}_{II}, C_{II}, D)$  be passive. Then  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  is passive if:*  
 188 *b) and c) in item 2 hold, and a)  $XA_{II}(P_I^h)^{-1}\bar{B}_I + B_I^\top(P_I^h)^{-\top}A_{II}^\top X \preccurlyeq 0$ ,*  
 189 *where  $X$  is a solution of the passivity LMI for  $(B_{II}, \bar{B}_{II}, C_{II}, D)$ .*

190 **Proof:**

- 191 1. The strict passivity of  $(B_{II}, \bar{B}_{II}, C_{II}, D)$  means that there exists  $X = X^\top \succ 0$   
 192 such that:

$$193 \quad (4.1) \quad M \triangleq \begin{pmatrix} -XB_{II} - B_{II}^\top X & -X\bar{B}_{II} + C_{II}^\top \\ -\bar{B}_{II}^\top X + C_{II} & D + D^\top \end{pmatrix} \succ 0.$$

194 In particular this implies that  $D + D^\top \succ 0$  [18, Theorem A.65] and the  
 195 associated transfer function is strongly SPR [18, Remark 3.19, Section 3.12.2].  
 196 Now we have  $\tilde{A} = B_{II} + \mathcal{O}(h)$ ,  $\tilde{B} = \bar{B}_{II} + \mathcal{O}(h)$ ,  $\tilde{C} = C_{II} + \mathcal{O}(h)$ ,  $\tilde{D} =$   
 197  $D + \mathcal{O}(h)$ . Therefore

$$198 \quad (4.2) \quad M = \underbrace{\begin{pmatrix} -X\tilde{A} - \tilde{A}^\top X & -X\tilde{B} + \tilde{C}^\top \\ -\tilde{B}^\top X + \tilde{C}^\top & \tilde{D} + \tilde{D}^\top \end{pmatrix}}_{\triangleq \tilde{M}} + N(h)$$

where  $N(h)$  stands for a matrix with suitable dimensions and which vanishes as  $h \rightarrow 0$  (in fact  $N(h)$  is  $\mathcal{O}(h)$ ). Thus  $\tilde{M} = M - N(h)$ . Applying [25, Theorem 2.11], it follows that  $\tilde{M} \succ 0$  provided that  $\sigma_{\max}(N(h)) < \lambda_{\min}^2(M)$ , an inequality which can always be verified if  $h > 0$  is small enough. Under this condition, it follows that the strict passivity of  $(B_{II}, \bar{B}_{II}, C_{II}, D)$  implies the strict passivity of  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  ( $\Rightarrow \tilde{D} \succ 0$ ).

2. The result is proved by developing  $\tilde{M}$  in (4.2), then using [10, Proposition 8.2.4, Fact 8.21.22], or [18, Lemma A.69], as well as the fact that  $-XB_{II} - B_{II}^\top X \succ 0$  from the strict state passivity.
3. The proof is similar to that of item 2.

□

The converse implication in item 1 does not hold, since one may have  $\tilde{M} \succ 0$  for some  $h > 0$ , but  $M \not\succeq 0$ . It is possible to state necessary and sufficient conditions, relying on [10, Proposition 8.2.4, Fact 8.21.22] or [18, Lemma A.70]. However the obtained conditions are cumbersome and hardly usable. Lemma 4.1 can be used to determine the conditions under which Assumption 4 is verified in the results presented in the next section. The conditions b) and c) in Lemma 4.1 item 2 hold if: b)  $C_I = 0$  or  $\bar{B}_I = 0$ , c) ( $A_{II} = 0$  or  $\bar{B}_I = 0$ ) and ( $B_I = 0$  or  $C_I = 0$ ). Condition a) in Lemma 4.1 item 3 holds if  $A_{II} = 0$  or  $B_I = 0$ . Let us remind that the solution  $X = X^\top \succ 0$  of the Lyapunov equation  $XB_{II} + B_{II}^\top X = -Q$ ,  $Q = Q^\top \succ 0$ , can be written  $X = (-\frac{1}{2}Q + F)B_{II}^{-1}$ , where  $F = -F^\top$  is unique [60, Theorems 1 and 2]. Let us denote the symmetric part of a square matrix  $M$  as  $\text{sym}(M)$  and its skew-symmetric part as  $\text{skw}(M)$ . This allows us to state the following necessary conditions.

**COROLLARY 4.2.** 1) Let  $(B_{II}, \bar{B}_{II}, C_{II}, D)$  be strictly state passive with  $XB_{II} + B_{II}^\top X = -Q$ ,  $Q = Q^\top \succ 0$ , and assume that  $A_{II}(P_I^h)^{-1}\bar{B}_I$  is right-invertible ( $\Rightarrow p \leq m$ ). Then  $X = (-\frac{1}{2}Q + F)B_{II}^{-1}$ , for some  $F = -F^\top$  uniquely defined, and condition c) item 2 in Lemma 4.1 holds only if

$$(4.3) \quad \text{sym}\left(\frac{1}{2}QB_{II}^{-1}A_{II}(P_I^h)^{-1}\bar{B}_I + B_I^\top(P_I^h)^{-\top}C_I^\top\right) = F\text{skw}(B_{II}^{-1}A_{II}(P_I^h)^{-1}\bar{B}_I)F,$$

for any pair  $(Q, F)$ , and

$$(4.4) \quad \text{Ker}(C_I(P_I^h)^{-1}B_I) = \{0\}.$$

2) Let  $(B_{II}, \bar{B}_{II}, C_{II}, D)$  be passive with  $XB_{II} + B_{II}^\top X = -Q$ ,  $Q = Q^\top \succcurlyeq 0$  and  $B_{II}$  invertible. Then  $X = (-\frac{1}{2}Q + F)B_{II}^{-1}$ , for some  $F = -F^\top$ , and condition a) item 3 in Lemma 4.1 holds only if

$$(4.5) \quad \text{sym}(-QB_{II}^{-1}A_{II}(P_I^h)^{-1}B_I) + 2F\text{skw}(B_{II}^{-1}A_{II}(P_I^h)^{-1}B_I)F \preccurlyeq 0,$$

for any pair  $(Q, F)$ .

**Proof:** 1) Condition c) yields  $(-\frac{1}{2}Q + F)B_{II}^{-1}A_{II}(P_I^h)^{-1}\bar{B}_I = B_I^\top(P_I^h)^{-\top}C_I^\top$ , then (4.3) follows using  $F = -F^\top$ . Using the right-invertibility, (4.4) follows from [23, equ. (70)] and the fact that  $X$  is full rank. 2) From passivity any solution must be symmetric. The Lyapunov equation and the skew-symmetry of  $F$  yield (4.5). □

We see that the right-invertibility condition (resp. condition (4.4)) in item 1 of the lemma, characterizes the couplings between  $x$  and  $\lambda$  in the index reduction (resp. between  $x$  and  $z$  in the relative degree reduction (*i.e.*, in  $\dot{w}$ )).

241 **4.1. Analysis of the OSNSP for (3.2).** The next proposition analyses the  
 242 OSNSP well-posedness, and applies to two main cases:  $\tilde{D} = 0$ , and  $\tilde{D} \succ 0$ . Let us  
 243 remind that  $p = n_2$  as a consequence of Assumptions 2, 3, 4 and 5, so that  $\tilde{A}$  and  $B_{II}$   
 244 are  $p \times p$  matrices.

245 **PROPOSITION 4.3.** *Let  $h > 0$ ,  $k \geq 0$  be given.*

- 246 1. *Let Assumption 4 hold. Suppose that either  $\text{Im}(\tilde{C}) \cap \text{int}(\text{Im}(\partial\sigma_{\Gamma_k} + \tilde{D})) \neq \emptyset$   
 247 or  $\tilde{B}$  is square full rank. Suppose also that there is  $\zeta_0$  such that  $\tilde{C}\zeta_0 \in$   
 248  $\text{Im}(\partial\sigma_{\Gamma_k} + \tilde{D})$ . Then the operator  $\mathcal{G} : \zeta \mapsto -\tilde{A}\zeta + \tilde{B}(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}\zeta)$  is  
 249 maximal monotone with non empty domain.*
- 250 2. *Let  $\tilde{D} \succ 0$ , then both mappings  $(\mathcal{N}_{K^*} + \tilde{D})^{-1}(\cdot)$  and  $(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\cdot)$  are  
 251 well-defined, single-valued Lipschitz continuous and maximal monotone for  
 252 each  $h \geq 0$ ,  $k \geq 0$  (maximal monotonicity holds for  $\tilde{D} \succ 0$ ). The operator  
 253  $\mathcal{G}(\cdot)$  is single-valued and Lipschitz continuous also.*
- 254 3. *Let  $\tilde{D} = 0$ , Assumption 2 hold true,  $\tilde{R}^2 = \tilde{X}$ ,  $\tilde{R} = \tilde{R}^\top \succ 0$ ,  $\text{Im}(\tilde{C}) \cap \Gamma_k \neq \emptyset$   
 255 and  $K = P(G, 0)$  (see Proposition A.1 e)) for some matrix  $G \in \mathbb{R}^{m \times m}$ . Let  
 256 us assume that  $\tilde{R}\tilde{A}\tilde{R}^{-1} \preceq 0$ . Let us consider the set:*

$$257 \quad (4.6) \quad \begin{aligned} \tilde{\mathcal{S}} \triangleq & P(G\tilde{C}\tilde{R}^{-1}, 0) \\ & \cap \{ \xi \in \mathbb{R}^p \mid \tilde{R}\tilde{A}\tilde{R}^{-1}\xi = -\sum_{i=1}^m \lambda_i (G\tilde{C}\tilde{R}^{-1})_{i\bullet}^\top, \lambda_i \geq 0 \} \\ & \cap \text{Ker}(\tilde{R}\tilde{A}\tilde{R}^{-1} + \tilde{R}^{-1}\tilde{A}^\top\tilde{R}). \end{aligned}$$

258 Define  $\tilde{q}_k = -\tilde{R}\tilde{E}_k$ ,  $\tilde{f}(\cdot) = \psi_{\Gamma_k} \circ \tilde{C}\tilde{R}^{-1}(\cdot)$ ,  $\Gamma_k = K - \tilde{F}_k$ , and consider the  
 259  $VI(-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, \tilde{f})$ : Find  $\xi_{k+1} = \tilde{R}z_{k+1}$  such that

$$260 \quad (4.7) \quad \langle -\tilde{R}\tilde{A}\tilde{R}^{-1}\xi_{k+1} + \tilde{q}_k, v - \xi_{k+1} \rangle + \tilde{f}(v) - \tilde{f}(\xi_{k+1}) \geq 0, \text{ for all } v \in \text{dom}(\tilde{f}).$$

- 261 (a) *If  $\tilde{\mathcal{S}} = \{0\}$ , then the  $VI(-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, \tilde{f})$  has at least one solution.*  
 262 (b) *If  $\tilde{\mathcal{S}} \neq \{0\}$ , and if there exists  $\xi_0 \in \tilde{\Gamma}_k$  such that*

$$263 \quad (4.8) \quad \langle \tilde{q}_k - (-\tilde{R}\tilde{A}\tilde{R}^{-1})^\top \xi_0, v \rangle > 0 \text{ for all } v \in \tilde{\mathcal{S}}, v \neq 0,$$

264 where  $\tilde{\Gamma}_k = \{ \xi \in \mathbb{R}^{n_2} \mid G\tilde{C}\tilde{R}^{-1}\xi + G\tilde{F}_k \geq 0 \} = \text{dom}(\tilde{f})$ , then the  
 265  $VI(-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, \tilde{f})$  has at least one solution.

- 266 (c) *If  $\xi_{k+1}^1$  and  $\xi_{k+1}^2$  are two solutions of the  $VI(-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, \tilde{f})$ , then  $\xi_{k+1}^1 -$   
 267  $\xi_{k+1}^2 \in \text{Ker}(\tilde{R}\tilde{A}\tilde{R}^{-1} + \tilde{R}^{-1}\tilde{A}^\top\tilde{R})$ .*
- 268 (d) *Let  $\tilde{R}\tilde{A}\tilde{R}^{-1}$  be symmetric. If  $\xi_{k+1}^1$  and  $\xi_{k+1}^2$  are two solutions of the  
 269  $VI(-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, \tilde{f})$ , then  $\langle \tilde{q}_k, \xi_{k+1}^1 - \xi_{k+1}^2 \rangle = 0$ .*
- 270 (e) *Let  $\tilde{R}\tilde{A}\tilde{R}^{-1}$  be symmetric. Then any solution of the  $VI(-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, \tilde{f})$   
 271 is also a solution to the problem:  $\min_{\xi \in \tilde{\Gamma}_k} -\frac{1}{2}\xi^\top \tilde{R}\tilde{A}\tilde{R}^{-1}\xi + \langle \tilde{q}_k, \xi \rangle$ .*
- 272 (f) *Let  $-\tilde{R}\tilde{A}\tilde{R}^{-1} \succ 0$  and be symmetric. Then  $\xi_{k+1} = \text{proj}_{-\tilde{R}\tilde{A}\tilde{R}^{-1}}[\tilde{\Gamma}_k; -\tilde{q}_k]$ .*

273 **Proof:**

- 274 1. Follows from [6, Theorem 1], which is itself inspired from [23] and [16, The-  
 275 orem 2], and which states that the negative feedback interconnection of a  
 276 passive system with a maximal monotone operator, defines another maximal  
 277 monotone operator (under the stated basic conditions).
- 278 2. Follows from [16, Proposition 1] (see Proposition B.1) and from [55, Lemma  
 279 1] (or [9, Corollary 24.4 (ii)]) since  $\text{Im}(\tilde{D}) = \mathbb{R}^m$  and  $\text{int}(\text{Dom}(\mathcal{N}_{K^*})) =$   
 280  $\text{int}(K^*) \neq \emptyset$ , and  $\text{int}(\text{Dom}(\partial\sigma_{\Gamma_k})) \neq \emptyset$  because  $\Gamma_k$  is convex and  $\text{int}(\Gamma_k) \neq \emptyset$ .



281 3. The proof follows from Propositions A.1 and A.2. First notice that the MLCP  
 282  $\tilde{A}z_{k+1} + \tilde{B}\lambda_{k+1} + \tilde{E}k = 0$ ,  $K^* \ni \lambda_{k+1} \perp \tilde{C}z_{k+1} + \tilde{D}\lambda_{k+1} + \tilde{F}k \in K$ , can  
 283 be equivalently rewritten as the VI( $-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, \tilde{f}$ ) in (4.7). Indeed one  
 284 obtains  $-\tilde{R}\tilde{A}\tilde{R}^{-1}\xi_{k+1} - \tilde{R}\tilde{E}k \in -\tilde{R}\tilde{B} \partial\psi_{\Gamma_k}(\tilde{C}\tilde{R}^{-1}\xi_{k+1})$ . Using the chain rule  
 285 of Convex Analysis [49, Theorem 23.9] and the fact that  $\Gamma_k$  is polyhedral,  
 286 the MLCP can be written equivalently as in (4.7). Let us now compute the  
 287 set  $\mathcal{S}$  in (4.6). Here  $\text{dom}(\varphi) = \text{dom}(\tilde{f}) = \{z \in \mathbb{R}^p \mid \tilde{C}\tilde{R}^{-1}z \in \Gamma_k\} =$   
 288  $\{z \in \mathbb{R}^p \mid \tilde{C}\tilde{R}^{-1}z + \tilde{F}k \in K\} = \{z \in \mathbb{R}^p \mid G\tilde{C}\tilde{R}^{-1}z + G\tilde{F}k \geq 0\}$ . Hence  
 289  $(\text{dom}(\varphi))_\infty = P(G\tilde{C}\tilde{R}^{-1}, 0)$  from Proposition A.1 e). Now we have  $\tilde{f}(\xi) =$   
 290  $\psi_{\Gamma_k}(\tilde{C}\tilde{R}^{-1}\xi) = \psi_{\tilde{\Gamma}_k}(\xi)$  with  $\tilde{\Gamma}_k = \{\xi \in \mathbb{R}^p \mid \tilde{C}\tilde{R}^{-1}\xi \in K - \tilde{F}k\} = \{\xi \in$   
 291  $\mathbb{R}^p \mid G\tilde{C}\tilde{R}^{-1}\xi + G\tilde{F}k \geq 0\} = P(G\tilde{C}\tilde{R}^{-1}, -G\tilde{F}k)$ . Thus  $\tilde{f}_\infty(\cdot) = \psi_{(\tilde{\Gamma}_k)_\infty}(\cdot)$   
 292 and  $\text{dom}(\tilde{f}_\infty) = P(G\tilde{C}\tilde{R}^{-1}, 0)$ . Now we have to calculate the dual cone  
 293  $P(G\tilde{C}\tilde{R}^{-1}, 0)^*$ . Using [31, Example 15], one finds that  $P(G\tilde{C}\tilde{R}^{-1}, 0)^* = \{w \in$   
 294  $\mathbb{R}^{n_2} \mid w = \sum_{i=1}^m \lambda_i (G\tilde{C}\tilde{R}^{-1})_{i\bullet}^\top, \lambda_i \geq 0\}$ , where  $(G\tilde{C}\tilde{R}^{-1})_{i\bullet}$  denotes the  $i$ th  
 295 row of the matrix  $G\tilde{C}\tilde{R}^{-1} \in \mathbb{R}^{m \times n_2}$ . Hence it follows that  $\mathcal{K}(-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{f}) =$   
 296  $\{\xi \in \mathbb{R}^{n_2} \mid \tilde{R}\tilde{A}\tilde{R}^{-1}\xi = -\sum_{i=1}^m \lambda_i (G\tilde{C}\tilde{R}^{-1})_{i\bullet}^\top, \lambda_i \geq 0\}$ . The third set in  $\tilde{\mathcal{S}}$   
 297 is equal to  $\text{Ker}(\tilde{R}\tilde{A}\tilde{R}^{-1} + \tilde{R}^{-1}A^\top\tilde{R})$ , from  $\text{Ker}(\mathbf{M} + \mathbf{M}^\top)$  in Proposition A.2  
 298 a). Then the results in items (a) to (f) are a consequence of Proposition A.2,  
 299 where  $\varphi_\infty(v) = 0$  in item (b). Item (f) follows from item e) in Proposition  
 300 A.2.

301  $\square$

302 Item 1 encompasses cases with  $\tilde{D} \succ 0$ . If item 2 holds, then  $\lambda_{k+1}$  is uniquely defined,  
 303 provided that  $z_{k+1}$  exists (which is guaranteed if conditions of item 1 hold). From  
 304 item 3 (c), it follows that  $z_{k+1}$  is unique if  $\tilde{A}$  has full rank (recall that  $B_{II}$  and  $\tilde{A}$  are  
 305 square  $p \times p$  matrices). However from (3.2) and (3.3),  $x_{k+1}$  is uniquely defined if and  
 306 only if  $\tilde{B}_I(\mathcal{N}_{K^*})^{-1}(-\tilde{C}z_{k+1} - \tilde{F}k)$  is unique. This happens trivially when  $\tilde{B}_I = 0$  (if  
 307 one thinks of (2.1) as the limit of a singularly perturbed system, this simply means  
 308 that the multiplier enters only the  $z$ -dynamics). Let us now deal with the case  $\tilde{D} \succ 0$ .

309 ASSUMPTION 6. Let  $h > 0$ , and  $\tilde{D} = \begin{pmatrix} \tilde{D}_{11} & 0 \\ 0 & 0 \end{pmatrix}$  with  $0 \prec \tilde{D}_{11} = \tilde{D}_{11}^\top \in \mathbb{R}^{m_1 \times m_1}$ .

311 Such specific structure occurs for instance when  $D$  has the same blockdiagonal struc-  
 312 ture and  $\tilde{B}_I = 0$ , i.e., the multiplier does not intervene in the  $x$ -dynamics. Let us  
 313 split the multiplier as  $\lambda = (\lambda_1^\top, \lambda_2^\top)^\top$ ,  $\lambda_1 \in \mathbb{R}^{m_1}$ ,  $\lambda_2 \in \mathbb{R}^{m_2}$ ,  $m_1 + m_2 = m$ , and  
 314  $K = K_1 \times K_2 \subset \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$  (hence  $K^* = K_1^* \times K_2^*$ ),  $\tilde{B} = (\tilde{B}_1 \quad \tilde{B}_2)$ ,  $\tilde{B}_1 \in \mathbb{R}^{p \times m_1}$ ,  
 315  $\tilde{B}_2 \in \mathbb{R}^{p \times m_2}$ ,  $\tilde{C} = \begin{pmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{pmatrix}$ ,  $\tilde{C}_1 \in \mathbb{R}^{m_1 \times p}$ ,  $\tilde{C}_2 \in \mathbb{R}^{m_2 \times p}$ . If  $\tilde{D}_{11} = 0$  then  $\lambda = \lambda_2$ ,  
 316  $\tilde{B} = \tilde{B}_2$ ,  $\tilde{C} = \tilde{C}_2$ . If  $\tilde{D} = \tilde{D}_{11} \succ 0$  then item 2 in Proposition 4.3 applies.

317 ASSUMPTION 7. Let  $h > 0$ . There exists a  $(p \times p)$ -matrix  $\tilde{X}_2 = \tilde{X}_2^\top \succ 0$  such that  
 318  $\tilde{X}_2\tilde{B}_2 = \tilde{C}_2^\top$ .

319 This assumption implies that there are couplings between the multiplier  $\lambda_{k+1}$  in the  
 320 algebraic equation, and the algebraic variable  $z_{k+1}$  in the complementarity conditions,  
 321 see (3.2). Define  $\tilde{R}_2 = \tilde{R}_2^\top \succ 0$ ,  $\tilde{R}_2^2 = \tilde{X}_2$ , and  $\xi_{k+1} = \tilde{R}_2 z_{k+1}$ . The next result extends  
 322 Proposition 4.3 to the positive semidefinite case with blockdiagonal structure.

323 PROPOSITION 4.4. Let Assumptions 6 and 7 hold true,  $h > 0$  and  $k \geq 0$  be given.

324 Define  $\Gamma_{2,k} = K_2 - \tilde{F}_{2,k}$ . Consider the VI: find  $\xi_{k+1} \in \text{dom}(\tilde{f}_2)$  such that  
 (4.9)

$$325 \quad -\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_{k+1} - \tilde{R}_2 \tilde{E}_k + \underbrace{\tilde{R}_2 \tilde{B}_1 (\tilde{D}_{11} + \partial \psi_{K_1^*})^{-1} (\tilde{C}_1 \tilde{R}_2^{-1} \xi_{k+1} + \tilde{F}_{1,k})}_{\triangleq \tilde{G}_2(t_k, \xi_{k+1})} \in -\partial \tilde{f}_2(\xi_{k+1})$$

326 where  $\tilde{f}_2(\cdot) = \psi_{\Gamma_{2,k}} \circ \tilde{C}_2 \tilde{R}_2^{-1}(\cdot)$ .

- 327 1. Suppose that  $\text{Im}(\tilde{C}_2) \cap \Gamma_{2,k} \neq \emptyset$ . The MLCP  $\tilde{A}z_{k+1} + \tilde{B}\lambda_{k+1} + \tilde{E}_k = 0$ ,
- 328  $K^* \ni \lambda_{k+1} \perp \tilde{C}z_{k+1} + \tilde{D}\lambda_{k+1} + \tilde{F}_k \in K$ , can be equivalently rewritten as the
- 329 VI in (4.9).
- 330 2. Assume that the set  $\tilde{\Gamma}_{2,k} = \{\xi \in \mathbb{R}^p \mid \tilde{C}_2 \tilde{R}_2^{-1} \xi \in \Gamma_{2,k}\}$  is compact. Then the
- 331 set of solutions to the VI in (4.9) is compact and nonempty.
- 332 3. Assume that for each  $t_k$ , the following well-defined, single-valued and Lip-
- 333 schitz continuous mapping, is maximal monotone:

$$\tilde{G}_2 : \xi_{k+1} \mapsto \tilde{R}_2 \tilde{B}_1 (\tilde{D}_{11} + \partial \psi_{K_1^*})^{-1} (\tilde{C}_1 \tilde{R}_2^{-1} \xi_{k+1} - \tilde{F}_{1,k}).$$

332 If  $-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \succ 0$ ,  $\tilde{\Gamma}_{2,k} \cap \text{Int}(\text{Dom}(\tilde{G}_2)) \neq \emptyset$  or  $\text{Int}(\tilde{\Gamma}_{2,k}) \cap \text{Dom}(\tilde{G}_2) \neq \emptyset$ , then  
 333 the set of solutions to the VI in (4.9) is closed and convex. If  $-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \succ 0$ ,  
 334 the VI in (4.9) has a unique solution.

335 **Proof:**

- 336 1. The complementarity conditions in (3.2) (c) are rewritten as:

$$337 \quad (4.10) \quad \begin{pmatrix} K_1^* \\ K_2^* \end{pmatrix} \ni \begin{pmatrix} \lambda_{1,k+1} \\ \lambda_{2,k+1} \end{pmatrix} \perp \begin{pmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{pmatrix} z_{k+1} + \begin{pmatrix} \tilde{D}_{11} \lambda_{1,k+1} \\ 0 \end{pmatrix} + \begin{pmatrix} \tilde{F}_{1,k} \\ \tilde{F}_{2,k} \end{pmatrix} \in \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$$

338 while the equality constraint in (3.2) (b) is rewritten as:

$$339 \quad (4.11) \quad \tilde{A}z_{k+1} + \tilde{B}_1 \lambda_{1,k+1} + \tilde{B}_2 \lambda_{2,k+1} + \tilde{E}_k = 0.$$

340 From (4.10) it is deduced that:

$$341 \quad (4.12) \quad \begin{cases} \lambda_{1,k+1} = -(\mathcal{N}_{K_1^*} + \tilde{D}_{11})^{-1} (\tilde{C}_1 z_{k+1} + \tilde{F}_{1,k}) \\ \lambda_{2,k+1} \in -\mathcal{N}_{K_2 - \tilde{F}_{2,k}}(\tilde{C}_2 z_{k+1}). \end{cases}$$

342 Using (4.11) it is inferred:

$$343 \quad (4.13) \quad \begin{aligned} & \tilde{A}z_{k+1} - \tilde{B}_1 (\mathcal{N}_{K_1} + \tilde{D}_{11})^{-1} (\tilde{C}_1 z_{k+1} + \tilde{F}_{1,k}) \\ & \quad - \tilde{B}_2 \mathcal{N}_{K_2 - \tilde{F}_{2,k}}(\tilde{C}_2 z_{k+1}) + \tilde{E}_k = 0 \\ & \quad \quad \quad \updownarrow \\ & -\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_{k+1} + \tilde{R}_2 \tilde{B}_1 (\mathcal{N}_{K_1^*} + \tilde{D}_{11})^{-1} (\tilde{C}_1 z_{k+1} + \tilde{F}_{1,k}) - \tilde{R}_2 \tilde{E}_k \\ & \quad \quad \quad \in -\tilde{R}_2 \tilde{B}_2 \mathcal{N}_{K_2 - \tilde{F}_{2,k}}(\tilde{C}_2 \tilde{R}_2^{-1} \xi_{k+1}), \end{aligned}$$

344 which is (4.9) since  $\tilde{R}_2 \tilde{B}_2 = \tilde{R}_2^{-1} \tilde{X}_2 \tilde{B}_2 = \tilde{R}_2^{-1} \tilde{C}_2^\top$  and applying the chain rule  
 345 of Convex Analysis for polyhedral functions.

- 346 2. This follows from [29, Corollary 2.2.5] and since the map  $(\tilde{D}_{11} + \partial \psi_{K_1^*})^{-1}(\cdot)$
- 347 is single-valued, well defined and Lipschitz continuous (see Proposition B.1).
- 348 3. The first assertion is a consequence of [9, Corollary 24.4] (for the maximal
- 349 monotonicity of the mapping  $(-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} + \tilde{G}_2 + \partial f_2)(\cdot)$ , and of [9, Proposition
- 350 23.39]. The second assertion follows from [29, Theorem 2.3.3], noting that the

351 mapping in the left-hand side of (4.9) is maximal monotone [9, Corollary 24.4  
 352 (ii)] and strongly monotone since  $-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1} \succ 0$ . This can also be proved  
 353 using [9, Corollary 23.37]. Single-valuedness and Lipschitz continuity stem  
 354 from  $\tilde{D}_{11} \succ 0$ .  $\square$

355 Further features of the solutions to the VI in (4.9) will be studied in section 5.1.3. The  
 356 next lemma states a classical condition that guarantees the maximal monotonicity of  
 357 Proposition 4.4 item 3. The first condition in item 1 is a constraint qualification  
 358 which guarantees that the function  $\tilde{f}_2(\cdot)$  has a nonempty domain. It is usually seen  
 359 as a technical condition, however in some applications it may not be satisfied. Item 1  
 360 states conditions under which the VI in (4.9) can be constructed. Item 3 uses some  
 361 results about the zeroes of maximal monotone mappings, or of the sum of these.

362 LEMMA 4.5. *Let  $\tilde{X}_2\tilde{B}_1 = \tilde{C}_1^\top$  for  $\tilde{X}_2$  in Assumption 7, then for each fixed  $t_k$ , the  
 363 mapping  $\tilde{G}_2 : \xi_{k+1} \mapsto \tilde{R}_2\tilde{B}_1(\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(\tilde{C}_1\tilde{R}_2^{-1}\xi_{k+1} + \tilde{F}_{1,k})$  is maximal monotone.*

364 **Proof:** Indeed  $\tilde{X}_2\tilde{B}_1 = \tilde{C}_1^\top \Rightarrow \tilde{R}_2\tilde{B}_1 = \tilde{R}_2^{-1}\tilde{C}_1^\top = (\tilde{C}_1\tilde{R}_2^{-1})^\top$ . The mapping  $(\tilde{D}_{11} +$   
 365  $\partial\psi_{K_1^*})^{-1}(\cdot)$  is maximal monotone, and the following operation:  $(\tilde{C}_1\tilde{R}_2^{-1})^\top(\tilde{D}_{11} +$   
 366  $\partial\psi_{K_1^*})^{-1}(\tilde{C}_1\tilde{R}_2^{-1} \cdot + \tilde{F}_{1,k})$  preserves the maximal monotonicity [50, Exercise 12.4].  $\square$

367 Propositions 4.3 and 4.4 provide some informations about the existence and unique-  
 368 ness of the unknown  $z_{k+1}$ , and characterize some mappings. It is noteworthy that  
 369 other well-posedness results for VIs are available in the literature. For instance if  
 370 the set  $\tilde{\Gamma}_k$  is compact, [29, Corollary 2.2.5] guarantees existence of a solution to  
 371 VI( $-\tilde{R}\tilde{A}\tilde{R}^{-1}, \tilde{q}_k, f$ ) with a compact set of solutions. Also when  $\tilde{\Gamma}_k = \mathbb{R}_+^m$  one may  
 372 rely on P-matrices and LCPs properties, when it is a closed convex cone one may use  
 373 copositive matrices [29, Theorem 2.5.10].

374 The next corollary completes the existence and uniqueness results for the OSNSP.

375 COROLLARY 4.6. *The next results hold true:*

- 376 1. (set of solutions  $z_{k+1}$ ) *Let Assumption 4 and the conditions of item 1 in*  
 377 *Proposition 4.3 hold. Then the set of solutions  $z_{k+1}$  in (3.5) (or the set of*  
 378 *solutions  $\zeta_{k+1}$  in (3.3)), is closed and convex. If in addition the conditions*  
 379 *of item 3 (a) or (b) in Proposition 4.3 hold, then the set of solutions is non*  
 380 *empty. If in addition  $\tilde{A} \prec 0$ , then  $z_{k+1}$  is unique.*
- 381 2. (set of solutions  $\lambda_{k+1}$ ) *Let  $\tilde{C} \text{ Dom}(\mathcal{G}) \subseteq \text{Im}(\partial\sigma_{\Gamma_k} + \tilde{D})$ , and the conditions of*  
 382 *item 1 in Proposition 4.3 hold true. Then the set of solutions  $\lambda_{k+1}$  in (3.5)*  
 383 *(a) is non empty for each  $z_{k+1}$ .*
- 384 3. *Let Assumption 4 hold with strict passivity, and the conditions of item 1 in*  
 385 *Proposition 4.3 be true. Then the GE in (3.5) (or the GE in (3.3)) has*  
 386 *a unique solution  $(\lambda_{k+1}, z_{k+1})$  for any  $\tilde{E}_k$ . Consequently  $x_{k+1}$  in (3.2) is*  
 387 *unique.*
- 388 4. *Let  $\tilde{D} = 0$  and  $\mathcal{G}(\cdot)$  be maximal monotone. Then if the set  $\tilde{\Gamma}_k = \{\zeta \in$   
 389  $\mathbb{R}^{n_2} \mid G\tilde{C}\tilde{R}^{-1}\zeta + G\tilde{F}_k \geq 0\}$  is bounded, the GE with unknown  $z_{k+1}$  in (3.5)*  
 390 *(or in (3.3)) has at least one solution and the set of solutions is compact.*

391 **Proof:**

- 392 1. Follows from [9, Proposition 23.39] by the maximal monotonicity of  $\mathcal{G}(\cdot)$ . If  
 393  $-\tilde{A} \succ 0$  then  $\mathcal{G}(\cdot)$  is strongly monotone (however it may not be Lipschitz  
 394 continuous and this property is used later in section 5.2), and so is  $\mathcal{G}(\cdot) + \tilde{E}_k$ .  
 395 Hence the generalized equation  $\mathcal{G}(z_{k+1}) + \tilde{E}_k \ni 0$  has a unique solution [9,  
 396 Corollary 23.37].
- 397 2. Let us consider  $\lambda_{k+1} \in -(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}z_{k+1})$ . We know that any solution

398  $z_{k+1} \in \mathcal{G}^{-1}(\tilde{E}_k) \subseteq \text{Im}(\mathcal{G}^{-1}) = \text{Dom}(\mathcal{G})$ . Since  $\text{Dom}(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1} = \text{Im}(\partial\sigma_{\Gamma_k} +$   
 399  $\tilde{D})$  the result follows.  
 400 3. Strict passivity implies that  $\tilde{D} \succ 0$  and  $\tilde{A}$  is a Hurwitz matrix, therefore  
 401 from a slight modification of [6, Theorem 1] it follows that the operator  $\mathcal{G}(\cdot)$   
 402 is strongly, hence strictly monotone (indeed there exists a real  $\alpha > 0$  small  
 403 enough such that  $(\tilde{A} + \alpha I_{n_2}, \tilde{B}, \tilde{C}, \tilde{D})$  is passive). Using [9, Proposition 23.35]  
 404 shows that  $z_{k+1}$  is unique, and item 2 in Proposition 4.3 shows that  $\lambda_{k+1}$  is  
 405 unique. The conclusion follows.  
 406 4. Under these conditions  $\text{Dom}(\mathcal{G})$  is bounded (compact). Hence from [9, Propo-  
 407 sition 23.36 (iii)] existence of a solution holds true. From [29, Corollary 2.2.5]  
 408 the compactness of the set of solutions follows. This could also be inferred  
 409 from item (e) in Proposition 4.3 in case  $\tilde{R}\tilde{A}\tilde{R}^{-1}$  is symmetric.  
 410  $\square$

411 Item 3 in Corollary 4.6 guarantees the uniqueness of solutions to the OSNSP in (3.6),  
 412 in which case the  $\in$  can be replaced by equality. Concerning item 2 in Corollary 4.6,  
 413 an important step is to characterize both  $\text{Dom}(\mathcal{G})$  and  $\text{Im}(\partial\sigma_{\Gamma_k} + \tilde{D})$ . If  $\tilde{D} = \tilde{D}^\top \succ 0$ ,  
 414 one can apply [15, Theorem 3], and infer that  $\text{Im}(\partial\sigma_{\Gamma_k} + \tilde{D}) \simeq \text{Im}(\partial\sigma_{\Gamma_k}) + \text{Im}(\tilde{D}) =$   
 415  $\text{Dom}(\partial\psi_{\Gamma_k}) + \text{Im}(\tilde{D}) \subset \text{dom}(\psi_{\Gamma_k}) + \text{Im}(\tilde{D}) = \Gamma_k + \text{Im}(\tilde{D})$  (see [15] for the exact  
 416 meaning of  $\simeq$  in this context). Similar calculations may be done for in item 1 of  
 417 Proposition 4.3. It is inferred that the basic condition in Proposition 4.3 item 1,  
 418 *i.e.*,  $\text{Im}(\tilde{C}) \cap \text{int}(\text{Im}(\partial\sigma_{\Gamma_k} + \tilde{D})) \neq \emptyset$ , or in Corollary 4.6 item 2, *i.e.*,  $\tilde{C} \text{Dom}(\mathcal{G}) \subseteq$   
 419  $\text{Im}(\partial\sigma_{\Gamma_k} + \tilde{D})$ , always are satisfied when  $\tilde{D} \succ 0$ . For the sake of brevity the conditions  
 420 for the existence and uniqueness of  $\lambda_{k+1}$  in (4.12) are not stated, similar tools as for  
 421 Corollary 4.6 can be used.

422 **4.2. Analysis of the OSNSP for (3.1).** It is also interesting to investigate  
 423 the case when Assumption 3 (resp. Assumption 5) is satisfied instead of Assumption  
 424 2 (resp. Assumption 4). The operator  $\mathcal{G}(\cdot)$  is now defined as  $\mathcal{G} : \zeta \mapsto -B_{II}\zeta +$   
 425  $(\partial\sigma_{\Gamma(t_k, x_{k+1})} + D)^{-1}(C_{II}\zeta)$ , where  $\Gamma(t_k, x_{k+1}) = \{\xi \in \mathbb{R}^m \mid \xi + C_I x_{k+1} + F_k \in$   
 426  $K\}$ . Consequently the results of Proposition 4.3 item 1, Corollary 4.6 item 3, with  
 427 Assumption 5 plus strict passivity, and Proposition 4.3 item 2 when  $D \succ 0$ , hold. For  
 428 the sake of brevity we do not develop all the arguments once again.  
 429 Let us focus now on the case  $D \succcurlyeq 0$ . From Assumption 3,  $D = 0$  and (2.1) one infers:

$$430 \quad (4.14) \quad -RB_{II}R^{-1}\xi(t) - RA_{II}x(t) - RE_{II}(t) \in -\partial\psi_{\Phi(t, x(t))}(\xi(t)),$$

431 where  $\xi = -Rz$ ,  $R^2 = X$ ,  $R = R^\top \succ 0$ ,  $\Phi(t, x) = \{\xi \in \mathbb{R}^p \mid C_{II}R^{-1}\xi + C_I x + F(t) \in$   
 432  $K\}$ , and the set-valued term is obtained using the chain rule, under the Convex  
 433 Analysis chain rule qualification condition [49, Theorem 23.9]. The first step is to  
 434 analyse the existence and uniqueness of solutions to the VI:

$$435 \quad (4.15) \quad -RB_{II}R^{-1}\xi_{k+1} - RA_{II}x_{k+1} - RE_{II,k} \in -\partial\psi_{\Phi(t_k, x_{k+1})}(\xi_{k+1}),$$

436 which is the counterpart of (4.7), relying on Proposition A.2. The unknown is  $\xi_{k+1}$   
 437 and the other terms and variables are considered as data for this VI. Similar results  
 438 to those in item 3 of Proposition 4.3 can be obtained.

439 **PROPOSITION 4.7.** *Let  $D = 0$ , Assumption 3 be true,  $h > 0$  be given, assume that*  
 440  *$\text{Im}(C_{II}) \cap (K - C_I x_{k+1} + F_k) \neq \emptyset$  and that  $K$  is polyhedral. Consider the discrete-time*  
 441 *system in (3.1), and the associated VI  $(-RB_{II}R^{-1}, -RA_{II}x_{k+1} - RE_{II,k}, \psi_{\Phi(t_k, x_{k+1})})$*

442 in (4.15). Assume that  $-RB_{II}R^{-1} \succ 0$ . Consider the set:  
 (4.16)

$$443 \quad \mathcal{S} \triangleq P(GC_{II}R^{-1}, 0) \cap \{\xi \in \mathbb{R}^{n_2} \mid RB_{II}R^{-1}\xi = \sum_{i=1}^m \lambda_i (GC_{II}R^{-1})_{i\bullet}^\top, \lambda_i \geq 0\} \\ \cap \text{Ker}(RB_{II}R^{-1} + R^{-1}B_{II}^\top R).$$

444 Then:

- 445 1. If  $\mathcal{S} = \{0\}$  the VI  $(-RB_{II}R^{-1}, -RA_{II}x_{k+1} - RE_{II,k}, \psi_{\Phi(t_k, x_{k+1})})$  has a unique  
 446 solution.
- 447 2. If  $\mathcal{S} \neq \{0\}$ , and if there exists  $\xi_0 \in \Phi(t_k, x_{k+1})$  such that  $\langle -RA_{II}x_{k+1} -$   
 448  $RE_{II,k} + RB_{II}R^{-1}\xi_0, v \rangle > 0$  for all  $v \neq 0$  and  $v \in \mathcal{S}$ , then it follows that the  
 449 VI  $(-RB_{II}R^{-1}, -RA_{II}x_{k+1} - \tilde{R}E_{II,k}, \psi_{\Phi(t_k, x_{k+1})})$  has a unique solution.

450 **Proof:** The set  $\mathcal{S}$  in (4.16) is constructed as follows. The chain rule holds from  
 451 [49, Theorem 23.9] and the fact that  $\psi_{K-C_I x_{k+1} + F_k}(\cdot)$  is polyhedral. Hence the  
 452 VI in (4.15) is equivalent to (3.1) (b) (c). Here  $\varphi = \psi_{\Phi(t_k, x_{k+1})}$ , hence  $\text{dom}(\varphi) =$   
 453  $\Phi(t_k, x_{k+1})$  and  $(\text{dom}(\varphi))_\infty$ . Assume that  $K = P(G, 0)$  (see Proposition A.1 (e)),  
 454 then  $(\text{dom}(\varphi))_\infty = \{\xi \in \mathbb{R}^p \mid GC_{II}R^{-1}\xi \geq 0\} = P(GC_{II}R^{-1}, 0)$ . Further,  $\varphi_\infty =$   
 455  $\psi_{(\Phi(t_k, x_{k+1}))_\infty} = \psi_{P(GC_{II}R^{-1}, 0)}$ , hence  $\text{dom}(\varphi_\infty) = P(GC_{II}R^{-1}, 0)$  and  $\text{dom}(\varphi_\infty)^* =$   
 456  $(P(GC_{II}R^{-1}, 0))^* = \{w \in \mathbb{R}^{n_2} \mid w = -\sum_{i=1}^m \lambda_i (GC_{II}R^{-1})_{i\bullet}^\top, \lambda_i \geq 0\}$ , where [31,  
 457 Example 15] was used as in the proof of Proposition 4.3. Thus  $\mathcal{K}(-RB_{II}R^{-1}, \varphi) =$   
 458  $\{\xi \in \mathbb{R}^{n_2} \mid -RB_{II}R^{-1}\xi = -\sum_{i=1}^m \lambda_i (GC_{II}R^{-1})_{i\bullet}^\top, \lambda_i \geq 0\}$ . Finally in Proposition  
 459 A.2:  $\text{Ker}(\mathbf{M} + \mathbf{M}^\top) = \text{Ker}(RB_{II}R^{-1} + (RB_{II}R^{-1})^\top)$  and  $R = R^\top$ . Items 1 and 2  
 460 are a direct consequence of Proposition A.2 (a) and (b).  $\square$

461 For the sake of briefness we do not state similar results as in Proposition 4.3 (c) to  
 462 (f). To enlarge the analysis let us now state the counterpart of Assumption 6:

463 ASSUMPTION 8. Let  $D = \begin{pmatrix} D_{11} & 0 \\ 0 & 0 \end{pmatrix}$  with  $0 \prec D_{11} = D_{11}^\top \in \mathbb{R}^{m_1 \times m_1}$ .

464 Similarly to Assumption 7, with the same splitting of the multiplier  $\lambda$ , the matrices  
 465  $B_{II}$  and  $C_{II}$ , and of the cone  $K$  (see the comments after Assumption 6, in particular  
 466 if  $D_{11} = 0$  the case  $D = 0$  is recovered with  $G_2(\cdot, \cdot)$  in (4.17) which vanishes, and the  
 467 set  $\Phi_2 = \Phi$ ), let us state the following:

468 ASSUMPTION 9. There exists a  $(p \times p)$ -matrix  $X_2 = X_2^\top \succ 0$  such that  $X_2 \bar{B}_{II}^2 =$   
 469  $(C_{II}^2)^\top$ .

470 Let us recall that  $p$  is the number of equality constraints in (2.1). Let us define  
 471  $R_2 = R_2^\top \succ 0$ ,  $R_2^2 = X_2$ ,  $\xi_{k+1} = R_2 z_{k+1}$ ,  $\Phi_2(t_k, x_{k+1}) = \{\xi \in \mathbb{R}^p \mid C_{II}^2 R_2^{-1} \xi +$   
 472  $C_{II}^2 x_{k+1} + F_{2,k} \in K_2\}$ . The next proposition is the counterpart of Proposition 4.4. It  
 473 states conditions that guarantee that  $z_{k+1}$  can be calculated from (3.1) (b) (c).

474 PROPOSITION 4.8. Let  $h > 0$ , and Assumptions 8 and 9 hold true. Consider the  
 475 VI: find  $\xi_{k+1} \in \text{dom}(f_2)$  such that

$$476 \quad (4.17) \quad \underbrace{-R_2 B_{II} R_2^{-1} \xi_{k+1} - R_2 A_{II} x_{k+1} - R_2 E_{II,k} + R_2 \bar{B}_{II}^1 (D_{11} + \partial \psi_{K_1^*})^{-1} (C_{II}^1 R_2^{-1} \xi_{k+1} - F_{1,k})}_{\triangleq G_2(t_k, \xi_{k+1})} \in -\partial f_2(\xi_{k+1})$$

477 with  $f_2(\cdot) = (\psi_{K_2 - C_{II}^2 x_{k+1} - F_{2,k}} \circ C_{II}^2 R_2^{-1})(\cdot) = \psi_{\Phi_2(t_k, x_{k+1})}(\cdot)$ .

- 478 1. Assume that  $\text{Im}(C_{II}^2) \cap (K_2 - C_{II}^2 x_{k+1} - F_{2,k}) \neq \emptyset$ , then the VI in (4.17) is  
 479 equivalent to the MLCP in (3.1) (b) (c).

- 480 2. Suppose that the set  $\Phi_2(t_k, x_{k+1})$  is compact and nonempty, then the set of  
 481 solutions to the VI in (4.17) is compact and nonempty.  
 482 3. Assume that for each fixed  $t_k$ , the mapping  $G_2(\cdot)$  is maximal monotone, and  
 483 that  $-R_2 B_{II} R_2^{-1} \succ 0$ , then the VI in (4.17) has a unique solution.  
 484 4. Assume that  $-R_2 B_{II} R_2^{-1} \succcurlyeq 0$ ,  $G_2(\cdot)$  is maximal monotone for each fixed  $t_k$ ,  
 485 and  $\Phi_2(t_k, x_{k+1}) \cap \text{Int}(\text{Dom}(G_2)) \neq \emptyset$  or  $\text{Int}(\Phi_2(t_k, x_{k+1})) \cap \text{Dom}(G_2) \neq \emptyset$ .  
 486 Then the set of solutions to the VI in (4.17) is closed and convex.

487 **Proof:** The proof of the first three items follows closely that of Proposition 4.4 and it  
 488 is therefore omitted. Item 4: the VI is rewritten as  $\underbrace{(-R_2 B_{II} R_2^{-1} + G_2 + \partial f_2)}_{\triangleq H_2}(\xi_{k+1}) -$

489  $R_2 A_{II} x_{k+1} - R_2 E_{II,k} \ni 0$ , and the assumption guarantees that  $H_2(\cdot)$  is maximal  
 490 monotone [9, Corollary 24.4]. The result follows from [9, Proposition 23.39].  $\square$

491 The same as Lemma 4.5 can be stated to guarantee the maximal monotonicity in  
 492 items 3 and 4. It is noteworthy that the main discrepancy between Proposition  
 493 4.4 and Proposition 4.8, is that the solution  $\xi_{k+1}$  depends on  $x_{k+1}$  in the latter,  
 494 while it depends on  $x_k$  in the former. This is a difficulty with the application of  
 495 Proposition 4.8 in some cases, for instance when  $C_{II} = 0$  (examples in sections 6.2  
 496 and 6.4 will demonstrate that this can occur). Actually, the condition  $\text{Im}(C_{II}^2) \cap (K_2 -$   
 497  $C_{II}^2 x_{k+1} - F_{2,k}) \neq \emptyset$  (that is a form of constraint qualification), or its counterparts in  
 498 Propositions 4.3, 4.4, 4.7, happens to be crucial in the setting of this work. It states  
 499 that the algebraic variable  $z$  has to enter the complementarity constraints through  
 500 a  $C_{II}$  matrix which has a big enough range. This may occur in some systems, see  
 501 section 6.5.

502 **5. Convergence results.** In this section it is assumed that the conditions for  
 503 existence of solutions in Propositions 4.3, 4.4, 4.7, or 4.8, are fulfilled, so that sequences  
 504 can be calculated. The section is divided into two main parts: in sections 5.1.1, 5.1.2  
 505 and 5.1.3, convergence is analysed from the properties of the quadruple  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$   
 506 which characterize the MLCP in (3.2) (b) (c). In section 5.2, the analysis is led with  
 507 the properties of  $(B_{II}, \bar{B}_{II}, C_{II}, D)$  which characterize the MLCP in (3.1) (b) (c).  
 508 The relationships between both approaches are explained.

509 **5.1. Convergence results from (3.2).** In this section the numerical scheme  
 510 in (3.2) and the results in section 4.1 are used. First let us notice that  $z_{k+1}$  and  
 511  $\lambda_{k+1}$  are solutions of the generalized equation in (3.5) (a) (b). Thus if existence of  
 512 solutions hold, they both are functions of  $\tilde{E}_k$  (in addition to being functions of  $\tilde{F}_k$   
 513 through the set  $\Gamma_k$ ). Under some conditions which are recalled below, like item 2 of  
 514 Proposition 4.3, they can be single-valued Lipschitz continuous functions of  $\tilde{E}_k$ . Then  
 515 the right-hand side of the first line in (3.2) may be written compactly as

$$516 \quad (5.1) \quad P_I^h x_{k+1} = P_I x_k + h \bar{E}_k(t_k, x_k)$$

517 for some  $\bar{E}_k(t_k, x_k)$  which is a linear function of  $E_{I,k}$ ,  $E_{II,k}$ ,  $\lambda_{k+1}$  and  $z_{k+1}$ . The  
 518 time-dependence of  $\bar{E}_k$  comes from  $E_{I,k}$ ,  $E_{II,k}$  and  $F_k$ , while the state dependence  
 519 comes from the third terms in  $\tilde{E}_k$  and in  $\tilde{F}_k$  (which enters  $\Gamma_k(t_k, x_k) = K - \tilde{F}_k$ ). From  
 520 (5.1) it can be calculated and checked by induction that:

$$521 \quad (5.2) \quad x_k = ((P_I^h)^{-1} P_I)^k x_0 + h \sum_{j=0}^{k-1} ((P_I^h)^{-1} P_I)^{k-1-j} (P_I^h)^{-1} \bar{E}_j(t_j, x_j), \quad k \geq 1,$$

522 Since  $P_I^h = P_I - hA_I$ , it follows that for  $h > 0$  small enough,  $(P_I^h)^{-1}P_I = I_p + \mathcal{O}(h)$   
 523 [10, Proposition 9.4.13] hence  $(P_I^h)^{-1} = P_I^{-1} + \mathcal{O}(h)$ . Thus

$$\begin{aligned}
 x_k &= x_0 + \mathcal{O}(h) + h \sum_{j=0}^{k-1} (I_p + \mathcal{O}(h))^{k-1-j} (P_I^{-1} + \mathcal{O}(h)) \bar{E}_j(t_j, x_j) \\
 &= x_0 + \mathcal{O}(h) + h \sum_{j=0}^{k-1} (I_p + \mathcal{O}(h)) (P_I^{-1} + \mathcal{O}(h)) \bar{E}_j(t_j, x_j) \\
 524 \quad (5.3) \quad &= x_0 + \mathcal{O}(h) + h \sum_{j=0}^{k-1} (P_I^{-1} + \mathcal{O}(h)) \bar{E}_j(t_j, x_j) \\
 &= x_0 + \mathcal{O}(h) + h(P_I^{-1} + \mathcal{O}(h)) \sum_{j=0}^{k-1} \bar{E}_j(t_j, x_j),
 \end{aligned}$$

525 for all  $k \geq 1$ . It is inferred that:

$$\begin{aligned}
 526 \quad (5.4) \quad \dot{x}_k &\stackrel{\Delta}{=} \frac{x_k - x_{k-1}}{h} = \frac{\mathcal{O}(h)}{h} + (P_I^{-1} + \mathcal{O}(h)) \bar{E}(t_{k-1}, x_{k-1}) \\
 &= \mathcal{O}(1) + (P_I^{-1} + \mathcal{O}(h)) \bar{E}(t_{k-1}, x_{k-1}).
 \end{aligned}$$

527 Our aim is now to prove that both  $\{x_k\}$  and  $\{\dot{x}_k\}$  are uniformly bounded sequences,  
 528 which will allow us to conclude about the convergence of the sequence  $\{x^h(\cdot)\}$  (see the  
 529 definition in (5.7)). For this it is necessary and sufficient that  $\bar{E}_k(t_k, x_k)$  be uniformly  
 530 bounded for all  $k \geq 0$ , since  $P_I$  is full rank. It follows from (3.5) (a) and (b), that  
 531 there are two operators which are crucial to characterize the right-hand side in (3.6).

532 **5.1.1. Analysis of the operator  $\mathcal{G}(\cdot)$ .** Notice that the mapping  $\mathcal{Z} : \zeta \mapsto \eta =$   
 533  $(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\zeta)$  is under the conditions of Proposition 4.3 item 2, a well-defined  
 534  $\mathbb{R}^m \rightarrow \mathbb{R}^m$  single-valued Lipschitz continuous mapping which is parameterized by  
 535  $t_k$  and  $x_k$  through the set  $\Gamma_k(t_k, x_k)$ . Assume that  $x_k$  is bounded, then redoing  
 536 the proof of Proposition B.1, it follows that the Lipschitz constant of  $\mathcal{Z}(\cdot)$  does not  
 537 depend on  $\partial\sigma_{\Gamma_k}(\cdot)$  but depends only on  $\tilde{D}$ , being equal to  $\frac{2}{\lambda_{\min}(\tilde{D} + \tilde{D}^\top)}$ . In turn  
 538  $\lambda_{\min}(\tilde{D} + \tilde{D}^\top) > 0$  uniformly in  $h > 0$  only if  $D \succ 0$ . Thus the operator  $\mathcal{G} :$   
 539  $\zeta \mapsto -\tilde{A}\zeta + \tilde{B}(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}\zeta)$  is also single-valued Lipschitz continuous (being the  
 540 composition of Lipschitz continuous mappings) with a Lipschitz constant that does  
 541 not depend on  $x_k$ , and which is bounded for bounded  $h$ . Let us now assume the  
 542 following which reinforces Assumption 4 and guarantees that  $\tilde{D} \succ 0$ :

543 ASSUMPTION 10. *The system  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  is strictly passive, uniformly in  $h \geq 0$ .*  
 544

545 Then if the basic conditions stated in item 1 of Proposition 4.3 are verified in addition  
 546 to Assumption 10, the operator  $\mathcal{G}(\cdot)$  is strongly monotone for some constant  $\epsilon > 0$ ,  
 547 and we know from Corollary 4.6 item 3, that  $x_{k+1}$  is unique. As stated in Proposition  
 548 4.3 item 1, maximal monotonicity follows from [6, Theorem 1]. Strong monotonicity  
 549 follows from the fact that under Assumption 10, the quadruple  $(\tilde{A} + \epsilon I_{n_2}, \tilde{B}, \tilde{C}, \tilde{D})$   
 550 is passive for a small enough  $\epsilon > 0$ , so that  $(\mathcal{G} - \epsilon I_{n_2})(\cdot)$  is maximal monotone for  
 551 such  $\epsilon$ . Thus if Assumption 10 and the basic conditions in item 1 of Proposition  
 552 4.3 are true,  $\mathcal{G} : \mathbb{R}^p \rightarrow \mathbb{R}^p$  is well-defined and single-valued, Lipschitz continuous  
 553 and strongly monotone. Consequently it is inferred from [50, Proposition 12.54] that  
 554 under the above set of conditions,  $\mathcal{G}^{-1}(\cdot)$  is single-valued, well defined and globally  
 555 Lipschitz, and the solution  $z_{k+1}$  of the generalized equation  $\bar{E}_k(t_k, x_k) \in \mathcal{G}(z_{k+1}) \Leftrightarrow$   
 556  $z_{k+1} = \mathcal{G}^{-1}(\bar{E}_k(t_k, x_k))$  in (3.5) is Lipschitz continuous in  $x_k$  and  $t_k$ . Thus  $\lambda_{k+1}$  is  
 557 also Lipschitz continuous in  $x_k$ , and we conclude that the same applies to  $x_{k+1}$  in  
 558 (3.2).

559 *Remark 5.1.* Assumption 10 is stringent. Actually the strict state passivity of  
 560  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  is sufficient to guarantee that  $\mathcal{G}(\cdot)$  is strongly monotone. However in  
 561 the body of sections 5.1.1 and 5.1.2 we need also that  $\lambda_{k+1}$  be a Lipschitz continuous

562 function of its argument, a sufficient condition being that  $\tilde{D} \succ 0$ , which in turn implies  
 563 that strict state passivity and strict passivity are identical. Thus strict passivity could  
 564 be relaxed to strict state passivity if it is assured that  $\tilde{B}_I(\partial\sigma_{\Gamma_k} + \tilde{D})^{-1}(\tilde{C}\cdot)$  is Lipschitz  
 565 continuous. This may occur in certain cases (the trivial case is  $\tilde{B}_I = 0$ ).

566 **5.1.2. Boundedness and convergence analysis.** Under the above conditions,  
 567 the term  $\tilde{E}_k(t_k, x_k)$  and consequently  $x_{k+1}(t_k, x_k)$  in (3.2) and  $\dot{x}_k(t_k, x_k)$  in (5.4) are  
 568 single-valued and Lipschitz continuous in  $x_k$  and  $t_k$  (provided  $E_I(\cdot)$ ,  $E_{II}(\cdot)$  and  $F(\cdot)$   
 569 are Lipschitz continuous functions of  $t$ , which is a mild requirement on the data),  
 570 since  $\tilde{E}_k$  and  $\tilde{F}_k$  are linear in the argument  $x_k$ . We have  $\|\tilde{E}(t_k^1, x_k^1) - \tilde{E}(t_k^2, x_k^2)\| \leq$   
 571  $L_t|t_k^1 - t_k^2| + L_x\|x_k^1 - x_k^2\|$  for any  $t_k^1, t_k^2, x_k^1, x_k^2$ , and for some  $L_t > 0$  and  $L_x > 0$ . Thus  
 572  $\|\tilde{E}(t_k, x_k^1) - \tilde{E}(t_k, x_k^2)\| \leq L_x\|x_k^1 - x_k^2\|$  and  $\|\tilde{E}(t_k, x_k) - \tilde{E}(t_k, 0)\| \leq L_x\|x_k\|$ . Notice  
 573 that due to the Lipschitz continuity both  $z_{k+1}$  and  $\lambda_{k+1}$  are uniformly bounded in  
 574 time for fixed  $x_k$  (provided  $E_I(\cdot)$ ,  $E_{II}(\cdot)$  and  $F(\cdot)$  are uniformly bounded, which is  
 575 a mild requirement on the data). Therefore  $\tilde{E}(t_k, 0)$  is uniformly bounded also, and  
 576  $\|\tilde{E}(t_k, x_k)\| \leq \|\tilde{E}(t_k, 0)\| + L_x\|x_k\| \leq m + L_x\|x_k\|$ . Now using (5.2) we get:  
 (5.5)

$$577 \quad \|x_k\| \leq \sigma_{\max}^k((P_I^h)^{-1}P_I)\|x_0\| + h\sigma_{\min}(P_I^h) \sum_{j=1}^{k-1} \sigma_{\max}^{k-j-1}(P_I(P_I^h)^{-1})(m + L_x\|x_{j-1}\|),$$

578 where we used the Hölder induced matrix norm  $\|\cdot\|_{2,2} = \sigma_{\max}(\cdot)$  and [10, Proposition  
 579 9.4.9, Corollary 9.4.12]. Let us make the following:

580 **ASSUMPTION 11.** *One has  $\sigma_{\max}((P_I^h)^{-1}P_I) \leq 1$  for all  $h \geq 0$ .*

581 Assumption 11 is a technical assumption which is satisfied if for instance the  $n_1 \times n_1$   
 582 matrix  $A_I \prec 0$  and  $P_I \succ 0$ . Indeed this implies that  $P_I^h \succ P_I$ , thus  $(P_I^h)^{-1}P_I \prec I_{n_1}$ .  
 583 Notice that under Assumption 11 one no longer needs  $h > 0$  small enough to obtain  
 584 the developments in (5.4), because the spectral radius of  $(P_I^h)^{-1}P_I$  is smaller than  
 585 one. It is also noteworthy that changing  $A_I x_{k+1}$  to  $A_I x_k$  in the right-hand side of  
 586 (3.1) (a), allows one to dispense with Assumption 11, with a slight modification of the  
 587 matrices  $\tilde{A}$ ,  $\tilde{B}$ ,  $\tilde{C}$ ,  $\tilde{D}$ ,  $\tilde{E}_k$  and  $\tilde{F}_k$  (where  $(P_I^h)^{-1}$  is replaced by  $P_I^{-1}$  in all of them).  
 588 Consequently from (5.5) one infers that:

$$589 \quad (5.6) \quad \|x_k\| \leq \|x_0\| + h\sigma_{\min}(P_I^h) \sum_{j=1}^k (m + L_x\|x_{j-1}\|),$$

590 Assume now that  $x_0, x_1, \dots, x_{k-1}$  are uniformly bounded by some  $x_{\max}$ , then from (5.6)  
 591 one has  $\|x_k\| \leq x_{\max} + h\sigma_{\min}(P_I^h)k(m + L_x x_{\max}) \leq x_{\max} + h\sigma_{\min}(P_I^h)n(m + L_x x_{\max}) =$   
 592  $x_{\max} + h\sigma_{\min}(P_I^h)\frac{T}{h}(m + L_x x_{\max}) = x_{\max} + \sigma_{\min}(P_I^h)T(m + L_x x_{\max})$ . Therefore  $x_k$   
 593 is also uniformly bounded (its upperbound depends only on  $T$  which is fixed), and  
 594 we conclude that all  $x_k$ ,  $1 \leq k \leq n$ , are bounded provided  $x_0$  is bounded. Then it is  
 595 inferred from (5.4) that  $\dot{x}_k$  is also uniformly bounded. Let us define the sequences of  
 596 piecewise-linear approximations  $\{x^h(\cdot)\}$  and  $\{\dot{x}^h(\cdot)\}$  as follows:

$$597 \quad (5.7) \quad \begin{cases} x^h(t) = x_{k+1} + \frac{t_{k+1}-t}{h}(x_k - x_{k+1}) \\ \dot{x}^h(t) = \frac{x_{k+1}-x_k}{h} \end{cases} \quad t \in [t_k, t_{k+1}).$$

598 We have shown that  $x^h(\cdot)$  and  $\dot{x}^h(\cdot)$  are uniformly bounded on  $[0, T]$  and for any  
 599  $h > 0$ , hence  $x^h(\cdot)$  has a Lipschitz constant that does not depend on  $h$ , consequently  
 600 the family  $\{x^h\}_{h \geq 0}$  is equi-Lipschitz continuous. Thus from Arzela-Ascoli Theorem



601  $\{x^h(\cdot)\}$  stays in a compact subset of  $\mathcal{C}^0([0, T]; \mathbb{R}^{n_1})$  and there exist subsequences  
 602 which converge uniformly in  $\mathcal{C}^0([0, T]; \mathbb{R}^{n_1})$  towards a Lipschitz continuous limit  $x(\cdot)$ .  
 603 Let  $\tilde{E}^h(\cdot)$  and  $\tilde{F}^h(\cdot)$  be defined in a similar way as  $x^h(\cdot)$  in (5.7). Due to the Lip-  
 604 schitz continuity of  $E_I(\cdot)$ ,  $E_{II}(\cdot)$ , and  $F(\cdot)$ , the sequences of functions  $\{\tilde{E}^h(\cdot)\}_{h \geq 0}$   
 605 and  $\{\tilde{F}^h(\cdot)\}_{h \geq 0}$  also converge uniformly in  $\mathcal{C}^0([0, T]; \mathbb{R}^p)$  and  $\mathcal{C}^0([0, T]; \mathbb{R}^m)$  to limits  
 606  $\tilde{E}(\cdot) = E_{II}(\cdot) + A_{II}x(\cdot)$  and  $\tilde{F}(\cdot) = F(\cdot) + C_Ix(\cdot)$  as  $h \rightarrow 0$ , respectively. From (5.4) it  
 607 is deduced that  $\dot{x}^h(\cdot)$  is uniformly bounded. Therefore from the Banach-Alaoglu The-  
 608 orem, it follows that  $\dot{x}^h(\cdot)$  converges weakly\* to a limit  $v(\cdot)$ . Since the domain  $[0, T]$  is  
 609 bounded, one has  $\mathcal{L}^2([0, T]; \mathbb{R}^{n_1}) \subseteq \mathcal{L}^1([0, T]; \mathbb{R}^{n_1})$  [36, Corollary 1, p.80], and conse-  
 610 quently  $\dot{x}^h(\cdot)$  converges weakly in  $\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})$ . Since  $x^h(t) = x^h(0) + \int_0^t \dot{x}^h(\tau) d\tau$ ,  
 611 we infer that  $v(\cdot) = \dot{x}(\cdot)$  almost everywhere.  
 612 Now notice that using (3.2) the following holds:

$$613 \quad (5.8) \quad \begin{cases} (a) P_I \dot{x}^h(t) = A_I x_h^*(t) + B_I z_h^*(t) + \bar{B}_I \lambda_h^*(t) + E_I^h(t) \\ (b) \tilde{A} z_h^*(t) + \tilde{B} \lambda_h^*(t) + \tilde{E}_{II}^h(t) + A_{II} x^h(t) = 0 \\ (c) 0 \leq \lambda_h^*(t) \perp C z_h^*(t) + \tilde{D} \lambda_h^*(t) + F^h(t) + C_I x^h(t) \geq 0, \end{cases}$$

614 for all  $t \in [t_k, t_{k+1})$ ,  $k \geq 0$ ,  $x_h^*(t) = x_{k+1}$  for all  $t \in [t_k, t_{k+1})$  and similarly for the  
 615 other terms are step-functions. From the boundedness of  $\|\dot{x}_k\| \triangleq \|\frac{x_{k+1} - x_k}{h}\| \leq C$  for  
 616 some  $C > 0$ , we infer that  $x_h^*(\cdot)$  converges strongly in  $\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})$  towards  $x(\cdot)$ .  
 617 Indeed:

$$618 \quad (5.9) \quad \begin{aligned} \|x^h - x_h^*\|_{\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})}^2 &\leq \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_{k+1} - t_k)^2 \|\dot{x}_k\|^2 dt \\ &\leq C^2 \sum_{k=0}^{n-1} (t_{k+1} - t_k)^3 = \frac{C^2 n h^3}{3} = \frac{C^2 T h^2}{3} \rightarrow 0 \text{ as } h \rightarrow 0. \end{aligned}$$

619 To conclude one notices that  $\|x^h - x_h^*\|_{\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})}^2 = \|x^h \pm x - x_h^*\|_{\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})}^2$ . Now  
 620 we have  $z_h^*(t) \in \mathcal{G}^{-1}(E_{II, \star}^h(t) + A_{II} x_h^*(t))$  for all  $t \in [t_k, t_{k+1})$ ,  $k \geq 0$ . Since  $E_I(\cdot)$ ,  
 621  $E_{II}(\cdot)$  are Lipschitz continuous, it follows that  $\|\dot{E}_k\| \triangleq \|\frac{E_{k+1} - E_k}{h}\| \leq k_E$  where  $k_E$   
 622 is its Lipschitz constant. We infer that  $E_{II, \star}^h(\cdot)$  converges strongly in  $\mathcal{L}^2([0, T]; \mathbb{R}^p)$   
 623 towards  $E_{II}(\cdot)$ , and using the Lipschitz continuity of  $\mathcal{G}^{-1}(\cdot)$  it follows that  $z_h^*(\cdot)$   
 624 converges strongly in  $\mathcal{L}^2([0, T]; \mathbb{R}^p)$  towards a limit  $z(\cdot) = \mathcal{G}^{-1}(E_{II}(\cdot) + A_{II}x(\cdot))$ .  
 625 Then from (3.3)  $\lambda_h^*(t) \in (\mathcal{N}_{\mathbb{R}_+^m} + \tilde{D})^{-1}(-\tilde{C} z_h^*(t) - F_h^*(t) + C_I x_h^*(t))$ , so that  $\lambda_h^*(\cdot)$   
 626 converges towards a limit  $\lambda(\cdot) = (\mathcal{N}_{\mathbb{R}_+^m} + \tilde{D})^{-1}(-\tilde{C} z(\cdot) - F(\cdot) + C_I x(\cdot))$  strongly in  
 627  $\mathcal{L}^2([0, T]; \mathbb{R}^m)$ , due to the single-valuedness, Lipschitzness and maximal monotonicity  
 628 of  $(\mathcal{N}_{\mathbb{R}_+^m} + \tilde{D})^{-1}(\cdot)$ . It is concluded that  $P_I \dot{x}^h(t) = A_I x_h^*(t) + B_I z_h^*(t) + \bar{B}_I \lambda_h^*(t) +$   
 629  $E_I^h(t) \rightarrow P_I \dot{x}(t) = A_I x(t) + B_I z(t) + \bar{B}_I \lambda(t) + E_I(t)$  weakly in  $\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})$  as  $h \rightarrow$   
 630  $0$ ,  $\tilde{A} z_h^*(t) + \tilde{B} \lambda_h^*(t) + \tilde{E}_{II}^h(t) + A_{II} x^h(t) \rightarrow B_{II} z(t) + \bar{B}_{II} \lambda(t) + E_{II}(t) + A_{II} x(t) (= 0)$ ,  
 631 and  $0 \leq \lambda_h^*(t) \perp \tilde{C} z_h^*(t) + \tilde{D} \lambda_h^*(t) + F^h(t) + C_I x^h(t) \geq 0 \rightarrow 0 \leq \lambda(t) \perp C_{II} z(t) +$   
 632  $D \lambda(t) + F(t) + C_I x(t) \geq 0$  as  $h \rightarrow 0$  strongly in  $\mathcal{L}^2([0, T]; \mathbb{R}^l)$  where  $l$  depends on the  
 633 variables dimensions. Hence we have proved the existence of a continuous solution  
 634  $(x, z, \lambda)$  to (2.1) on  $[0, T]$ . Consequently the following result is proved:

635 **PROPOSITION 5.2.** *Let the conditions of item 2 in Corollary 4.6 hold true uni-*  
 636 *formly in  $h$  ( $\Rightarrow D \succ 0$ ). Assume that Assumptions 10 and 11 hold,  $E(\cdot)$  and  $F(\cdot)$  are*  
 637 *bounded Lipschitz continuous functions, initial data are bounded. Then the DALCS*  
 638 *in (2.1) possesses for each admissible bounded initial data, a continuous solution*  
 639  *$(x, z, \lambda) : [0, T] \rightarrow \mathbb{R}^{n_1 + n_2 + m}$ ,  $T > 0$ . Moreover the Euler approximations converge*  
 640 *to a solution as  $h \rightarrow 0$ .*

641 **5.1.3. The case blockdiagonal**  $\tilde{D} = \tilde{D}^\top \succcurlyeq 0$ . This section is dedicated to  
 642 analyse an alternative to the conditions stated in sections 5.1.1 and 5.1.2. Let us  
 643 consider the case of Proposition 4.4 with Assumption 6. Then it follows from the  
 644 complementarity conditions that  $\lambda_{1,k+1} = (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(-\tilde{C}_1 z_{k+1} - \tilde{F}_{1,k})$ , where  
 645 the mapping  $(\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(\cdot)$  is maximal monotone (being the sum of two maximal  
 646 monotone mappings satisfying [9, Corollary 24.4 (ii)]), single-valued, well-defined and  
 647 Lipschitz continuous, see Proposition B.1. Thus if  $\tilde{B}_I = (\tilde{B}_I^1, 0)$  with  $\tilde{B}_I^1 \in \mathbb{R}^{n_1 \times m_1}$   
 648 one obtains  $\tilde{B}_I \lambda_{k+1} = \tilde{B}_I^1 \lambda_{1,k+1} = \tilde{B}_I^1 (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(-\tilde{C}_1 z_{k+1} - \tilde{F}_{1,k})$  in the right-  
 649 hand side of (3.5) (c).

650 It may be less restrictive to compute upperbounds using the maximal monotonicity.  
 651 To start with let us consider the VI in (4.9) and let us state several features of its  
 652 solutions. Here Assumption 7 has to be reinforced a little:

653 ASSUMPTION 12. *The conditions of Assumption 7 hold where  $\tilde{X}_2 \succ 0$  for any*  
 654  *$h \geq 0$ .*

655 This secures that the variable change  $\xi_{k+1} = -\tilde{R}_2^{-1} z_{k+1}$  remains well-defined as  $h \rightarrow$   
 656  $0$ . It is also important that item 1 in Proposition 4.4 holds, so that the VI in (4.9) is  
 657 equivalent to the MLCP in (3.2) (b) (c).

658 LEMMA 5.3. *Assume that  $\lambda_{\min}(-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1}) (= \lambda_{\min}(-\tilde{A}))$  is large enough, then*  
 659 *any solution  $\xi_{k+1} = -\tilde{R}_2^{-1} z_{k+1}$  of the VI in (4.9) is upperbounded by an affine function*  
 660 *of  $\|x_k\|$ .*

661 **Proof:** Let us manipulate the VI in (4.9) as follows:

$$(5.10) \quad \begin{aligned} & -\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_{k+1} + \tilde{R}_2 \tilde{E}_k + \tilde{G}_2(t_k, \xi_{k+1}) \in -\partial \tilde{f}_2(\xi_{k+1}) \\ \Leftrightarrow & -\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} (\xi_{k+1} - \xi_0) + \tilde{R}_2 \tilde{E}_k + \tilde{G}_2(t_k, \xi_{k+1}) + \zeta_0 - \tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_0 = -\zeta_{k+1} + \zeta_0 \\ & \text{with } \zeta_{k+1} \in \partial \tilde{f}_2(\xi_{k+1}), \zeta_0 \in \partial \tilde{f}_2(\xi_0) \\ \Rightarrow & \langle -\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} (\xi_{k+1} - \xi_0), \xi_{k+1} - \xi_0 \rangle + \langle \tilde{R}_2 \tilde{E}_k, \xi_{k+1} - \xi_0 \rangle \\ & + \langle \tilde{G}_2(t_k, \xi_{k+1}) - \tilde{G}_2(t_k, \xi_0), \xi_{k+1} - \xi_0 \rangle + \langle \zeta_0 - \tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_0, \xi_{k+1} - \xi_0 \rangle \\ & + \langle \tilde{G}_2(t_k, \xi_0), \xi_{k+1} - \xi_0 \rangle = -\langle \zeta_{k+1} - \zeta_0, \xi_{k+1} - \xi_0 \rangle \leq 0, \end{aligned}$$

663 where we have denoted  $\tilde{G}_2(t_k, \xi_{k+1})$  to highlight the dependency of this mapping on  $t_k$   
 664 and  $x_k$  through  $\tilde{F}_k$ . We have  $\|\tilde{R}_2 \tilde{B}_1 (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(z_1) - \tilde{R}_2 \tilde{B}_1 (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(z_2)\| \leq$   
 665  $\eta \|z_1 - z_2\|$  for any  $z_1$  and  $z_2$ , so that  $\|\tilde{G}_2(t_k, \xi_{k+1}) - \tilde{G}_2(t_k, \xi_0)\| \leq \eta \|\tilde{C}_1 \tilde{R}_2^{-1} (\xi_{k+1} -$   
 666  $\xi_0)\| \leq \eta' \|\xi_{k+1} - \xi_0\|$ , where  $\eta$  depends on  $\tilde{D}_{11}$ ,  $\tilde{R}_2$  and  $\tilde{B}_1$  (see the proof of Propo-  
 667 sition B.1 in [16]), and  $\eta'$  depends on  $\tilde{D}_{11}$ ,  $\tilde{R}_2$ ,  $\tilde{B}_1$ , and  $\tilde{C}_1$ . Thus we obtain:

$$(5.11) \quad \begin{aligned} & \langle -\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} (\xi_{k+1} - \xi_0), \xi_{k+1} - \xi_0 \rangle \leq \langle -\tilde{R}_2 \tilde{E}_k, \xi_{k+1} - \xi_0 \rangle - \langle \tilde{G}_2(t_k, \xi_0), \xi_{k+1} - \xi_0 \rangle \\ & + \langle \zeta_0 + \tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_0, \xi_{k+1} - \xi_0 \rangle + \eta' \|\xi_{k+1} - \xi_0\|^2 \end{aligned}$$

669 which implies

$$(5.12) \quad \begin{aligned} & \langle (-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} - \eta' I_{n_2})(\xi_{k+1} - \xi_0), \xi_{k+1} - \xi_0 \rangle \leq \|\tilde{R}_2 \tilde{E}_k\| \|\xi_{k+1} - \xi_0\| \\ & + \|\tilde{G}_2(t_k, \xi_0) \pm \tilde{G}_2(t_0, \xi_0)\| \|\xi_{k+1} - \xi_0\| + \|\zeta_0 + \tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_0\| \|\xi_{k+1} - \xi_0\|. \end{aligned}$$

671 Now one has:  $\|\tilde{E}_k\| \leq \|E_{II,k} + h A_{II} E_{I,k}\| + \|A_{II} (P_I^h)^{-1} P_I\| \|x_k\|$ ,  $\tilde{G}_2(t_k, \xi_0) =$   
 672  $\tilde{R}_2 \tilde{B}_1 (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1} (\tilde{C}_1 \tilde{R}_2^{-1} \xi_0 - \tilde{F}_{1,k})$ ,  $\|\tilde{G}_2(t_k, \xi_0) - \tilde{G}_2(t_0, \xi_0)\| = \|\tilde{R}_2 \tilde{B}_1 (\tilde{D}_{11} +$   
 673  $\partial\psi_{K_1^*})^{-1} (\tilde{C}_1 \tilde{R}_2^{-1} \xi_0 - \tilde{F}_{1,k}) - \tilde{R}_2 \tilde{B}_1 (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1} (\tilde{C}_1 \tilde{R}_2^{-1} \xi_0 - \tilde{F}_{1,0})\| \leq \eta \|\tilde{F}_{1,k} +$   
 674  $\tilde{F}_{1,0}\| \leq \eta \|F_k + h C_I (P_I^h)^{-1} E_{I,k}\| + \eta \|C_I (P_I^h)^{-1} P_I\| \|x_k\|$ . Let us denote  $\alpha_k =$

675  $\|\tilde{R}_2\| \|E_{II,k} + hA_{II}E_{I,k}\| + \eta \|F_k + hC_I(P_I^h)^{-1}E_{I,k}\|$ ,  $\beta_0 = \|\tilde{G}_2(t_0, \xi_0)\| + \|\zeta_0 +$   
 676  $\tilde{R}_2\tilde{A}\tilde{R}_2^{-1}\xi_0\|$ , and  $\gamma = \eta \|C_I(P_I^h)^{-1}P_I\| + \|\tilde{R}_2\| \|A_{II}(P_I^h)^{-1}P_I\|$ . Then:  
 (5.13)

$$677 \quad \langle (-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1} - \eta'I_{n_2})(\xi_{k+1} - \xi_0), \xi_{k+1} - \xi_0 \rangle \leq (\alpha_k + \beta_0 + \gamma \|x_k\|) \|\xi_{k+1} - \xi_0\|$$

$$\leq \frac{1}{2}\alpha_k^2 + \frac{1}{2}\beta_0^2 + \frac{1}{2}\gamma^2 \|x_k\|^2 + \frac{1}{2}\|\xi_{k+1} - \xi_0\|^2$$

678 which implies:

$$(5.14)$$

$$679 \quad \langle (-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1} - \eta'I_{n_2} - \frac{1}{2}I_{n_2})(\xi_{k+1} - \xi_0), \xi_{k+1} - \xi_0 \rangle \leq \frac{1}{2}\alpha_k^2 + \frac{1}{2}\beta_0^2 + \frac{1}{2}\gamma^2 \|x_k\|^2.$$

680 Therefore if  $-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1} - \eta'I_{n_2} - \frac{1}{2}I_{n_2} \succ 0$ , one obtains

$$681 \quad \|\xi_{k+1} - \xi_0\| \leq \frac{1}{\sqrt{\lambda_{\min}(-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1} - \eta'I_{n_2} - \frac{1}{2}I_{n_2})}} \sqrt{\frac{1}{2}\alpha_k^2 + \frac{1}{2}\beta_0^2 + \frac{1}{2}\gamma^2 \|x_k\|^2}$$

$$\leq \frac{1}{\sqrt{2}\sqrt{\lambda_{\min}(-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1} - \eta'I_{n_2} - \frac{1}{2}I_{n_2})}} \left( \sqrt{\alpha_k^2 + \beta_0^2} + \gamma \|x_k\| \right)$$

682 which ends the proof.  $\square$

683 Therefore the term  $hB_I z_{k+1}$  in the right-hand side of (3.5) (c), admits also an affine  
 684 upperbound in  $\|x_k\|$ . A further characterization of the iterates is given now.

685 LEMMA 5.4. Assume that  $-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1} \succ \frac{3}{2}I_{n_2}$ . Assume also that the mapping  
 686  $\tilde{G}_2(\cdot)$  is monotone (see Lemma 4.5 for conditions), and that  $E_I(\cdot)$ ,  $E_{II}(\cdot)$  and  $F(\cdot)$   
 687 are uniformly bounded and Lipschitz continuous. Then for any  $k$  and  $p \geq 0$  and any  
 688 solution of the VI in (4.9), one has:

$$689 \quad (5.16) \quad \|\xi_k - \xi_p\| \leq \alpha + \beta \|x_{k-1} - x_{p-1}\|$$

690 for some bounded constants  $\alpha \geq 0$  and  $\beta \geq 0$ , and  $\alpha$  is proportional to  $(t_{k-1} - t_{p-1})$ .

691 **Proof:** From

$$692 \quad (5.17) \quad \begin{cases} -\tilde{R}_2\tilde{A}\tilde{R}_2^{-1}\xi_k + \tilde{R}_2\tilde{E}_{k-1} + \tilde{G}_2(t_{k-1}, \xi_k) = -\zeta_k \\ -\tilde{R}_2\tilde{A}\tilde{R}_2^{-1}\xi_p + \tilde{R}_2\tilde{E}_{p-1} + \tilde{G}_2(t_{p-1}, \xi_p) = -\zeta_p \end{cases}$$

693 with  $\zeta_k \in \partial f_2(\xi_k)$ ,  $\zeta_p \in \partial f_2(\xi_p)$ , it follows that

$$694 \quad (5.18) \quad \begin{aligned} & \langle -\tilde{R}_2\tilde{A}\tilde{R}_2^{-1}(\xi_k - \xi_p), \xi_k - \xi_p \rangle + \langle \tilde{R}_2(\tilde{E}_{k-1} - \tilde{E}_{p-1}), \xi_k - \xi_p \rangle \\ & + \langle \tilde{G}_2(t_{k-1}, \xi_k) - \tilde{G}_2(t_{k-1}, \xi_p) + \tilde{G}_2(t_{k-1}, \xi_p) - \tilde{G}_2(t_{p-1}, \xi_p), \xi_k - \xi_p \rangle \\ & = -\langle \zeta_k - \zeta_p, \xi_k - \xi_p \rangle \end{aligned}$$

695 which implies

$$696 \quad (5.19) \quad \begin{aligned} & \langle -\tilde{R}_2\tilde{A}\tilde{R}_2^{-1}(\xi_k - \xi_p), \xi_k - \xi_p \rangle + \langle \tilde{R}_2(\tilde{E}_{k-1} - \tilde{E}_{p-1}), \xi_k - \xi_p \rangle \\ & + \langle \tilde{G}_2(t_{k-1}, \xi_p) - \tilde{G}_2(t_{p-1}, \xi_p), \xi_k - \xi_p \rangle \leq 0. \end{aligned}$$

697 One has:

$$(5.20)$$

$$\begin{aligned} & \langle -\tilde{R}_2(\tilde{E}_{k-1} - \tilde{E}_{p-1}), \xi_k - \xi_p \rangle \\ & \leq \langle -\tilde{R}_2(E_{II,k-1} - E_{II,p-1}), \xi_k - \xi_p \rangle - \langle \tilde{R}_2(hA_{II}(E_{I,k-1} - E_{I,p-1})), \xi_k - \xi_p \rangle \\ & \quad - \langle A_{II}(P_I^h)^{-1}P_I(x_{k-1} - x_{p-1}), \xi_k - \xi_p \rangle \\ 698 \quad & \leq \frac{1}{2}\|\tilde{R}_2((E_{II,k-1} - E_{II,p-1}) + hA_{II}(E_{I,k-1} - E_{I,p-1}))\|^2 + \frac{1}{2}\|\xi_k - \xi_p\|^2 \\ & \quad + \frac{1}{2}\|A_{II}(P_I^h)^{-1}P_I\|^2\|x_{k-1} - x_{p-1}\|^2 + \frac{1}{2}\|\xi_k - \xi_p\|^2 \\ & \leq \|\tilde{R}_2\|^2\|E_{II,k-1} - E_{II,p-1}\|^2 + \|\xi_k - \xi_p\|^2 + h\|A_{II}\|^2\|E_{I,k-1} - E_{I,p-1}\|^2 \\ & \quad + \frac{1}{2}\|A_{II}(P_I^h)^{-1}P_I\|^2\|x_{k-1} - x_{p-1}\|^2 \leq \alpha' + \beta'\|x_{k-1} - x_{p-1}\|^2 + \|\xi_k - \xi_p\|^2, \end{aligned}$$

699 for some  $\alpha' \geq 0$  proportional to  $(t_{k-1} - t_{p-1})^2$  since  $E(\cdot)$  is Lipschitz continuous,  
 700  $\beta' \geq 0$ , and it was used that  $x^\top y \leq \|x\| \|y\| \leq \frac{1}{2}(\|x\|^2 + \|y\|^2)$ . Also:

$$\begin{aligned}
 & \langle -\tilde{G}_2(t_{k-1}, \xi_p) + \tilde{G}_2(t_{p-1}, \xi_p), \xi_k - \xi_p \rangle \\
 & \leq \|\tilde{G}_2(t_{k-1}, \xi_p) - \tilde{G}_2(t_{p-1}, \xi_p)\| \|\xi_k - \xi_p\| \\
 701 \quad (5.21) \quad & \leq \frac{1}{2} k \tilde{G}_2 \|\tilde{F}_{1,k-1} - \tilde{F}_{p-1}\|^2 + \frac{1}{2} \|\xi_k - \xi_p\|^2 \\
 & \leq k \tilde{G}_2 \|F_{k-1} - F_{p-1}\|^2 + h k \tilde{G}_2 \|C_I(P_I^h)^{-1}(E_{I,k-1} - E_{I,p-1})\|^2 \\
 & \quad + k \tilde{G}_2 \|C_I(P_I^h)^{-1}P_I(x_{k-1} - x_{p-1})\|^2 + \frac{1}{2} \|\xi_k - \xi_p\|^2 \\
 & \leq \alpha'' + \beta'' \|x_{k-1} - x_{p-1}\|^2 + \frac{1}{2} \|\xi_k - \xi_p\|^2,
 \end{aligned}$$

702 for some  $\alpha'' \geq 0$  proportional to  $(t_{k-1} - t_{p-1})^2$  (since  $E(\cdot)$  and  $F(\cdot)$  are Lipschitz  
 703 continuous), and  $\beta'' \geq 0$ . Hence using (5.19):

$$\begin{aligned}
 704 \quad (5.22) \quad \|\xi_k - \xi_p\| & \leq \frac{1}{\sqrt{\lambda_{\min}(-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} - \frac{3}{2} I_p)}} \sqrt{\alpha' + \alpha'' + (\beta' + \beta'') \|x_{k-1} - x_{p-1}\|^2} \\
 & \leq \frac{1}{\sqrt{\lambda_{\min}(-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} - \frac{3}{2} I_p)}} (\sqrt{\alpha' + \alpha''} + \sqrt{\beta' + \beta''} \|x_{k-1} - x_{p-1}\|)
 \end{aligned}$$

705 and the lemma is proved.  $\square$

706

707 **COROLLARY 5.5.** *Let  $h > 0$ . Assume that the conditions and assumptions of*  
 708 *Lemma 5.4 are fulfilled. For all  $k \geq 1$  and any solution of the VI in (4.9), one has:*

$$709 \quad (5.23) \quad \left\| \frac{\xi_{k+1} - \xi_k}{h} \right\| \leq \alpha + \beta \left\| \frac{x_k - x_{k-1}}{h} \right\|$$

710 for some bounded constants  $\alpha \geq 0$  and  $\beta \geq 0$ .

711 **Proof:** Let us consider the proof of Lemma 5.4, replacing  $k$  by  $k+1$  and  $p$  by  $k-1$ .  
 712 Then  $\alpha$  in (5.16) is proportional to  $h$ . Redoing the calculations in (5.20) and (5.22),  
 713 one obtains (5.23).  $\square$

714 Clearly for all  $h > 0$  (5.23) can be written equivalently as:  $\|\xi_{k+1} - \xi_k\| \leq h\alpha +$   
 715  $\beta \|x_k - x_{k-1}\|$ . The bounds calculated for the variable  $\xi_k$  also hold for the variable  
 716  $z_k = -\tilde{R}_2 \xi_k$ . Let us now analyse  $\lambda_{k+1}$  in (3.5) (a). As seen above,  $\lambda_{1,k+1} = (\tilde{D}_{11} +$   
 717  $\partial\psi_{K_1^*})^{-1}(-\tilde{C}_1 z_{k+1} - \tilde{F}_{1,k})$  and  $\lambda_{2,k+1} \in -\mathcal{N}_{K_2}(\tilde{C}_2 z_{k+1} + \tilde{F}_{2,k})$ . If the matrix  $\tilde{B}_I$  has  
 718 the above structure (i.e.,  $\tilde{B}_I = (\tilde{B}_I^1, 0)$ ), then the term  $\tilde{B}_I \lambda_{k+1} = \tilde{B}_I^1$  is Lipschitz  
 719 continuous with respect to the variables  $z_{k+1}$ ,  $x_k$  and  $F_k$ . In a more general setting  
 720 one obtains:

$$721 \quad (5.24) \quad P_I^h x_{k+1} \in P_I x_k + h B_I z_{k+1} + h \tilde{B}_I^1 (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1} (-\tilde{C}_1 z_{k+1} - \tilde{F}_{1,k}) \\
 \quad \quad \quad - h \tilde{B}_I^2 \mathcal{N}_{K_2}(\tilde{C}_2 z_{k+1} + \tilde{F}_{2,k}).$$

722 The difficulty is that the last term in the right-hand side of (5.24) is set-valued in  
 723 general, and this is intrinsically due to the structure of the implicit scheme in (2.1).  
 724 A solution is to consider the VI in (4.9). Its right-hand side matches with the last  
 725 term in the right-hand side of (5.24) and it is set-valued. However provided the VI  
 726 is well-posed, we know that one element of this set is equal to the left-hand side of  
 727 (4.9). We may assume that the scheme in (5.24) is implemented with this element.  
 728 This element appears to define a Lipschitz continuous function of  $\xi_{k+1}$  (let us recall  
 729 that the Lipschitz constant of the third term is equal to  $\frac{1}{\lambda_{\min}(\tilde{D}_{11})}$ , see Proposition  
 730 B.1). If this element is chosen then the numerical scheme (5.24) becomes:

$$731 \quad (5.25) \quad P_I^h x_{k+1} = P_I x_k + h B_I z_{k+1} + h \tilde{B}_I^1 (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1} (\tilde{C}_1 z_{k+1} - \tilde{F}_{1,k}) \\
 \quad \quad \quad + h \tilde{B}_I^2 (-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_{k+1} - \tilde{R}_2 \tilde{E}_k + \tilde{G}_2(t_k, \xi_{k+1})).$$

732 Let us now study (5.25), using Lemmata 5.3, 5.4, Corollary 5.5. It is noteworthy that  
 733 the constant  $\alpha$  in both Lemmata and in Corollary can be made arbitrarily small by  
 734 increasing  $-\tilde{R}_2\tilde{A}\tilde{R}_2^{-1}$  arbitrarily (in the set of positive definite matrices).

735 **PROPOSITION 5.6.** *Assume that the conditions of Lemmata 5.3, 5.4 and of Corol-*  
 736 *lary 5.5 are verified,  $E_I(\cdot)$ ,  $E_{II}(\cdot)$  and  $F(\cdot)$  are bounded Lipschitz continuous func-*  
 737 *tions of time, and Assumption 13 below is satisfied. Then the sequences  $\{x^h\}_{h \geq 0}$  and*  
 738  *$\{\dot{x}^h\}_{h \geq 0}$  in (5.7) are uniformly bounded in  $h$  on  $[0, T]$ . Thus  $\{x^h\}_{h \geq 0}$  is equi-Lipschitz*  
 739 *continuous and there is a subsequence of it that converges uniformly in  $\mathcal{C}^0([0, T]; \mathbb{R}^{n_1})$*   
 740 *towards a Lipschitz continuous limit denoted as  $x(\cdot)$ .*

741 **Proof:** In the proof, use is made of positive bounded constants denoted generically as  
 742  $k_i$ . Consider (5.25). First recall that the operator  $\zeta \mapsto (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(\zeta)$  is Lipschitz  
 743 continuous with Lipschitz constant  $\frac{1}{\lambda_{\min}(\tilde{D}_{11})}$ . Hence we obtain:

$$\begin{aligned}
 & \| (P_I^h)^{-1} \tilde{B}_I^1 (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1} (-\tilde{C}_1 z_{k+1} - \tilde{F}_{1,k}) \\
 & \quad - (P_I^h)^{-1} \tilde{B}_I^1 (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1} (-\tilde{C}_1 z_{p+1} - \tilde{F}_{1,p}) \| \\
 744 \quad (5.26) \quad & \leq k_1 |\tilde{C}_1(z_{k+1} - z_{p+1} + \tilde{F}_{1,k} - \tilde{F}_{1,p})| \leq k_2 \|z_{k+1} - z_{p+1}\| + k_3 \|\tilde{F}_{1,k} - \tilde{F}_{1,p}\| \\
 & \leq k_4 \|\xi_{k+1} - \xi_{p+1}\| + k_4 \|F_{1,k} - F_{1,p}\| + k_5 \|E_{I,k} - E_{I,p}\| + k_6 \|x_k - x_p\| \\
 & \leq k_4(\alpha + \beta \|x_k - x_p\|) + k_7 |t_k - t_p| + k_6 \|x_k - x_p\| \leq k_8 + k_9 \|x_k - x_p\| \\
 & \quad + k_7 |t_k - t_p|,
 \end{aligned}$$

745 for any  $k$  and  $p \geq 0$ , where we used Lemma 5.4 and the Lipschitzness property. In a  
 746 quite similar way one has  $\| (P_I^h)^{-1} \tilde{B}_I^2 (\tilde{G}_2(\xi_{k+1}) - \tilde{G}_2(\xi_{p+1})) \| \leq k_8 + k_9 \|x_k - x_p\| +$   
 747  $k_7 |t_k - t_p|$ , where  $k_7, k_8$  and  $k_9$  depend on the Lipschitz constants of  $\tilde{G}_2(\cdot, \cdot)$ ,  $F(\cdot)$ , and  
 748 on  $\| (P_I^h)^{-1} \tilde{B}_I^2 \|$ . One also has  $\| (P_I^h)^{-1} (B_I + \tilde{B}_I^2 (\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1})(z_{k+1} - z_{p+1})) \| \leq k_{10}(\alpha +$   
 749  $\beta \|x_k - x_p\|) \leq k_{11} + k_{12} \|x_k - x_p\|$ , and  $\| \tilde{B}_I^2 \tilde{R}_2 (\tilde{E}_k - \tilde{E}_p) \| \leq k_{13} |t_k - t_p| + k_{14} \|x_k - x_p\|$ .  
 750 It is deduced that  $\|x_{k+1} - x_{p+1}\| \leq (\| (P_I^h)^{-1} P_I \| + hk_{15}) \|x_k - x_p\| + hk_{16} |t_k - t_p| + hk_{17}$ ,  
 751 where the term  $k_{17}$  stems from  $k_8 + k_{11} = \alpha(k_4 + k_{10})$ . Here we are led to make the  
 752 following which is more stringent than Assumption 11:

753 **ASSUMPTION 13.** *There exists  $h_{\max} > 0$  such that  $\| (P_I^h)^{-1} P_I \| + hk_{15} < 1$  for all*  
 754  $0 < h \leq h_{\max}$ .

755 Obviously for  $h = 0$  and the Hölder-induced norm one has  $\| (P_I^h)^{-1} P_I \|_{2,2} + hk_{15} = 1$ .  
 756 Let us choose  $p = 0$ , it follows that  $\|x_{k+1} - x_1\| \leq (\| (P_I^h)^{-1} P_I \| + hk_{15}) \|x_k -$   
 757  $x_0\| + hk_{16}T + hk_{17}$ , hence  $\|x_{k+1}\| \leq (\| (P_I^h)^{-1} P_I \| + hk_{15}) \|x_k\| + hk_{16}T + hk_{17} +$   
 758  $\|x_1\| + (\| (P_I^h)^{-1} P_I \| + hk_{15}) \|x_0\|$ . Now it has to be proved that  $\|x_1\|$  is bounded.  
 759 From Lemma 5.3 and (5.15),  $\|\xi_1\|$  and  $\|z_1\|$  are upperbounded by an affine function of  
 760 initial data  $\|x_0\|$  and  $\|z_0\|$ . The operator  $(\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(\cdot)$  is well-defined and single-  
 761 valued. Taking  $k = 0$  in (5.25) one infers that indeed  $\|x_1\|$  is bounded for bounded  
 762 initial data. Consequently it follows that  $\|x_{k+1}\| \leq (\| (P_I^h)^{-1} P_I \| + hk_{15}) \|x_k\| +$   
 763  $hk_{16}T + hk_{17} + hf(x_0, z_0) + \| (P_I^h)^{-1} P_I \| \|x_0\|$  for some function  $f(\cdot)$  that is bounded  
 764 for bounded initial data. To simplify the notations let us set  $h_1 \triangleq \| (P_I^h)^{-1} P_I \| + hk_{15}$   
 765 and  $h_2 \triangleq k_{16}T + k_{17} + f(x_0, z_0)$ . We obtain for all  $k \geq 1$ :

$$\begin{aligned}
 766 \quad (5.27) \quad \|x_k\| & \leq h_1^k \|x_0\| + \sum_{j=0}^{k-1} h_1^j (hh_2 + \| (P_I^h)^{-1} P_I \| \|x_0\|) \\
 & \leq h_1^k \|x_0\| + (hh_2 + \| (P_I^h)^{-1} P_I \| \|x_0\|) \frac{1 - h_1^{k+1}}{1 - h_1}.
 \end{aligned}$$

767 Thus in view of Assumption 13 the sequence  $\{x_k\}_{k \geq 0}$  generated by (5.25) is uniformly  
 768 bounded for bounded initial data.

769 Let us now reconsider the inequality  $\|x_{k+1} - x_{p+1}\| \leq (\|(P_I^h)^{-1}P_I\| + hk_{15}) \|x_k -$   
 770  $x_p\| + hk_{16}|t_k - t_p| + hk_{17}$  obtained above. Setting  $p = k - 1$  yields  $\|x_{k+1} - x_k\| \leq$   
 771  $(\|(P_I^h)^{-1}P_I\| + hk_{15}) \|x_k - x_{k-1}\| + h^2k_{16} + hk_{17} \Rightarrow \frac{\|x_{k+1} - x_k\|}{h} \leq (\|(P_I^h)^{-1}P_I\| +$   
 772  $hk_{15}) \frac{\|x_k - x_{k-1}\|}{h} + hk_{16} + k_{17}$ . Now using Lemma 5.4 and the Lipschitzness it follows  
 773 that  $\alpha$ , hence  $k_{17}$ , are proportional to  $k_E^2 h^2$ . We may therefore write the inequality  
 774 as  $\frac{\|x_{k+1} - x_k\|}{h} \leq (\|(P_I^h)^{-1}P_I\| + hk_{15}) \frac{\|x_k - x_{k-1}\|}{h} + hk_{16} + k_{18}h^2$  for some  $k_{18} \geq 0$ .  
 775 Proceeding as for (5.27) one infers that if  $\frac{\|x_1 - x_0\|}{h}$  is bounded then  $\frac{\|x_{k+1} - x_k\|}{h}$  is  
 776 bounded for all  $k \geq 1$ . Evaluating (5.25) at  $k = 0$  and using the boundedness of  $x_1$   
 777 as shown above, proves that this is indeed the case. The proposition is proved using  
 778 the Arzela-Ascoli Theorem.  $\square$

779 Clearly in practice one needs to compute the constants  $k_i$ , for instance  $k_{15}$  is needed  
 780 to determine  $h_{\max}$  in Assumption 13.

781 **COROLLARY 5.7.** *Under the conditions of Proposition 5.6, the sequences  $\{z^h\}_{h \geq 0}$*   
 782 *and  $\{z^h\}_{h \geq 0}$  (respectively  $\{\xi^h\}_{h \geq 0}$  and  $\{\xi^h\}_{h \geq 0}$ ) are uniformly bounded in  $h$  on  $[0, T]$ ,*  
 783  *$0 < T < +\infty$ . Thus  $\{z^h\}_{h \geq 0}$  (respectively  $\{\xi^h\}_{h \geq 0}$ ) is equi-Lipschitz continuous and*  
 784 *there is a subsequence of it that converges uniformly in  $C^0([0, T]; \mathbb{R}^{n_2})$  towards a*  
 785 *Lipschitz continuous limit denoted as  $z(\cdot)$  (respectively  $\xi(\cdot)$ ).*

786 **Proof:** This follows from Proposition 5.6 and using Lemmata 5.3, 5.4, Corollary 5.5.  
 787  $\square$

788 Let us now pass to the analysis of the multiplier convergence. Recall that  $\lambda_{1,k+1} =$   
 789  $(\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(-\tilde{C}_1 z_{k+1} - \tilde{F}_{1,k})$  and  $\lambda_{2,k+1} = -\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_{k+1} + \tilde{R}_2 \tilde{E}_k + \tilde{G}_2(\xi_{k+1}) \in$   
 790  $-\mathcal{N}_{K_2}(\tilde{C}_2 z_{k+1} + \tilde{F}_{2,k})$ , therefore for all  $t \geq 0$ ,  $\lambda_{1,\star}^h(t) = (\tilde{D}_{11} + \partial\psi_{K_1^*})^{-1}(-\tilde{C}_1 z_\star^h(t) -$   
 791  $\tilde{F}_{1,\star}^h(t))$  and  $\lambda_{2,\star}^h(t) = -\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1} \xi_\star^h(t) + \tilde{R}_2 \tilde{E}_\star^h(t) + \tilde{G}_2(\xi_\star^h(t)) \in -\mathcal{N}_{K_2}(\tilde{C}_2 z_\star^h(t) +$   
 792  $\tilde{F}_{2,\star}^h(t))$ , where we recall that the step-functions are defined as  $f_\star^h(t) = f_{k+1}$  for  
 793  $t \in [t_k, t_{k+1})$ . We know that  $\dot{\xi}_k \triangleq \frac{\xi_{k+1} - \xi_k}{h} \leq C$  for some  $C > 0$ , hence:

$$\begin{aligned}
 794 \quad (5.28) \quad \|\xi^h - \xi_\star^h\|_{\mathcal{L}_2([0, T]; \mathbb{R}^{n_2})}^2 &\leq \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_{k+1} - t_k)^2 \|\dot{\xi}_k\|^2 dt \\
 &\leq C^2 \sum_{k=0}^{n-1} (t_{k+1} - t_k)^3 = \frac{C^2 n h^3}{3} = \frac{C^2 T h^2}{3}.
 \end{aligned}$$

795 Consequently  $\|\xi^h - \xi_\star^h\|_{\mathcal{L}_2([0, T]; \mathbb{R}^{n_2})} \rightarrow 0$  as  $h \rightarrow 0$ . Since  $\|\xi^h - \xi_\star^h\|_{\mathcal{L}_2([0, T]; \mathbb{R}^{n_2})} =$   
 796  $\|\xi^h \pm \xi - \xi_\star^h\|_{\mathcal{L}_2([0, T]; \mathbb{R}^{n_2})} \rightarrow \|\xi - \xi_\star^h\|_{\mathcal{L}_2([0, T]; \mathbb{R}^{n_2})} \rightarrow 0$ , we conclude that the sequence  
 797  $\{\xi_\star^h\}_{h \geq 0}$  converges strongly in  $\mathcal{L}_2([0, T]; \mathbb{R}^{n_2})$  towards the limit  $\xi(\cdot)$ . The same result  
 798 holds for the sequence  $\{x_\star^h\}_{h \geq 0}$  with the limit  $x(\cdot)$ . Therefore the sequence  $\{-\tilde{C}_1 z_\star^h -$   
 799  $\tilde{F}_{1,\star}^h\}_{h \geq 0}$  also converges strongly in  $\mathcal{L}_2$  towards a limit, and we infer that the same  
 800 conclusion holds for both  $\{\lambda_{1,\star}^h(\cdot)\}_{h \geq 0}$  and  $\{\lambda_{2,\star}^h(\cdot)\}_{h \geq 0}$ . Since  $\mathcal{N}_{K_2}(\cdot)$  is maximal  
 801 monotone, it follows from [9, Proposition 20.33] that the limit  $\lambda_2(\cdot)$  satisfies  $\lambda_2(t) \in$   
 802  $-\mathcal{N}_{K_2}(C_{II,2} z(t) + F_2(t) + C_{I,2} x(t))$ . Thus we proved the following result.

803 **LEMMA 5.8.** *Let the conditions of Lemmata 5.3, 5.4, Corollary 5.5 be satisfied for*  
 804 *all  $h > 0$ . The sequences  $\{\xi_\star^h\}_{h \geq 0}$ ,  $\{x_\star^h\}_{h \geq 0}$ ,  $\{\lambda_{1,\star}^h(\cdot)\}_{h \geq 0}$  and  $\{\lambda_{2,\star}^h(\cdot)\}_{h \geq 0}$  converge*  
 805 *strongly in  $\mathcal{L}_2([0, T]; \mathbb{R}^{n_2})$  towards the limits  $\xi(\cdot)$ ,  $x(\cdot)$ ,  $\lambda_1(\cdot)$  and  $\lambda_2(\cdot)$ , respectively.*

806 To finish this section, one can consider again (5.8) and the analysis which follows it,  
 807 to obtain a proposition similar to Proposition 5.2, relying on Lemma 5.8, Proposition  
 808 5.6, and on Corollary 5.5. It is noteworthy that the three fundamental assumptions  
 809 in this section, are that  $-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1}$  be sufficiently large as a positive definite matrix,  
 810 that a suitable selection is chosen for the multiplier  $\lambda_{2,k}$ , and that a coupling as stated  
 811 in Assumptions 2 or 7 exists.

812 PROPOSITION 5.9. *Assume that the conditions of Proposition 5.6, Corollary 5.7*  
 813 *and Lemma 5.8 are verified. Then the limit functions  $x(\cdot)$ ,  $z(\cdot)$ ,  $\lambda(\cdot)$ , are solutions of*  
 814 *the DALCS (2.1).*

815 *Proof.* Let us consider (5.8). From Lemma 5.8,  $\{x_\star^h(\cdot)\}$ ,  $\{z_\star^h(\cdot)\}$ ,  $\{\lambda_\star^h(\cdot)\}$  converge  
 816 strongly in  $\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})$  towards limits  $x(\cdot)$ ,  $z(\cdot)$  and  $\lambda(\cdot)$ . Using the same argu-  
 817 ments as those developed before Proposition 5.2, it is inferred from (5.8) (b) (c) that  
 818 the limits verify (2.1) (b) (c). Using the same arguments as those developed just above  
 819 (5.8), it is deduced that  $\{\dot{x}^h(\cdot)\}$  converges weakly\* to a limit  $v(\cdot)$  where  $v(\cdot) = \dot{x}(\cdot)$   
 820 almost everywhere. Consequently  $P_I \dot{x}^h(t) = A_I x_\star^h(t) + B_I z_\star^h(t) + \bar{B}_I \lambda_\star^h(t) + E_I^h(t) \rightarrow$   
 821  $P_I \dot{x}(t) = A_I x(t) + B_I z(t) + \bar{B}_I \lambda(t) + E_I(t)$  weakly in  $\mathcal{L}^2([0, T]; \mathbb{R}^{n_1})$  as  $h \rightarrow 0$ .  $\square$

822 **5.2. Convergence results from (3.1).** In this section the properties of the  
 823 plant's model (2.1) are used, thus the numerical scheme in (3.1) and the results in  
 824 section 4.2 are analysed. This section is devoted to the case  $D \succcurlyeq 0$  as in Proposition  
 825 4.8. Indeed, since the strict passivity of  $(B_{II}, \bar{B}_{II}, C_{II}, D)$  implies Assumption 10 for  
 826 small enough  $h > 0$ , it is unnecessary to repeat the developments in sections 5.1.1 and  
 827 5.1.2. However, it is not possible in general to determine the implications between  
 828 Assumption 7 and Assumption 9, nor the relations between  $\lambda_{\min}(\hat{A})$  and  $\lambda_{\min}(B_{II})$   
 829 for  $h > 0$ . Therefore let us deal with the VI in (4.17) where  $x_{k+1}$  is replaced by  
 830  $x_{k+\theta} = \theta x_k + (1 - \theta)x_{k+1}$ ,  $\theta \in [0, 1]$ . The following holds, which is the counterpart of  
 831 Lemma 5.3.

832 LEMMA 5.10. *Assume that  $\lambda_{\min}(-R_2 B_{II} R_2^{-1}) (= \lambda_{\min}(-B_{II}))$  is large enough,*  
 833  *$E_I(\cdot)$ ,  $E_{II}(\cdot)$  and  $F(\cdot)$  are uniformly bounded and Lipschitz continuous functions of*  
 834 *time. Then any solution  $\xi_{k+1} = R_2^{-1} z_{k+1}$  of the VI in (4.17) is upperbounded by an*  
 835 *affine function of  $\|x_{k+\theta}\|$ .*

836 The proof follows the same steps as the proof of Lemma 5.3 and it is therefore omitted.  
 837 Lemma 5.10 implies that the term  $B_I z_{k+1}$  in the right-hand side of (3.1) (a), is  
 838 upperbounded by an affine function of  $\|x_{k+\theta}\|$ . The coefficients of the affine function  
 839 depend only on the system's data (constant matrices, Lipschitz constants) and on the  
 840 initial conditions. Let us now propose the counterpart of Lemma 5.4.

841 LEMMA 5.11. *Assume that  $-R_2 B_{II} R_2^{-1} \succcurlyeq I_{n_2}$ . Assume also that the well-defined,*  
 842 *single-valued and Lipschitz continuous mapping  $G_2(\cdot)$  is monotone, and that  $E_I(\cdot)$ ,*  
 843  *$E_{II}(\cdot)$  and  $F(\cdot)$  are Lipschitz continuous functions of time. Then for any  $k$  and  $p \geq 0$*   
 844 *and any solution of the VI in (4.17), one has:*

$$845 \quad (5.29) \quad \|\xi_k - \xi_p\| \leq \alpha + \beta \|x_{k-1+\theta} - x_{p-1+\theta}\|$$

846 *for all  $k \geq 1$  and some bounded constants  $\alpha \geq 0$  (proportional to  $|t_{k-1} - t_{p-1}|$ ) and*  
 847  *$\beta \geq 0$ .*

848 **Proof:** The proof is similar to the proof of Lemma 5.4, since both VIs in (4.17) and  
 849 (4.9) possess similar structures. Using (4.17) one finds:

$$850 \quad (5.30) \quad \begin{cases} -R_2 B_{II} R_2^{-1} \xi_{k+1} - R_2 A_{II} x_{k+\theta} - R_2 E_{II,k} + G_2(t_k, \xi_{k+1}) = -\zeta_{k+1} \\ -R_2 B_{II} R_2^{-1} \xi_{p+1} - R_2 A_{II} x_{p+\theta} - R_2 E_{II,p} + G_2(t_p, \xi_{p+1}) = -\zeta_{p+1} \end{cases}$$

851 with  $\zeta_k \in \partial f_2(\xi_k)$ ,  $k \geq 0$ . Using the monotonicity of  $G_2(t_k, \cdot)$  and of  $\partial f_2(\cdot)$  it follows

852 that:

$$\begin{aligned}
 & \langle -R_2 B_{II} R_2^{-1} (\xi_{k+1} - \xi_{p+1}), \xi_{k+1} - \xi_{p+1} \rangle \\
 & \leq \langle R_2 A_{II} (x_{k+\theta} - x_{p+\theta}) + R_2 (E_{II,k} - E_{II,p}), \xi_{k+1} - \xi_{p+1} \rangle \\
 & \quad - \langle G_2(t_k, \xi_{k+1}) - G_2(t_k, \xi_{p+1}), \xi_{k+1} - \xi_{p+1} \rangle \\
 853 \quad (5.31) \quad & \quad + \langle G_2(t_k, \xi_{p+1}) - G_2(t_p, \xi_{p+1}), \xi_{k+1} - \xi_{p+1} \rangle \\
 & \leq \|R_2 A_{II}\|^2 \|x_{k+\theta} - x_{p+\theta}\|^2 + \|R_2\|^2 k_E^2 (t_k - t_p)^2 \\
 & \quad + \frac{1}{2} k_{G_2}^2 \|F_k - F_p\|^2 + \|\xi_{k+1} - \xi_{p+1}\|^2 \\
 & \leq \|R_2 A_{II}\|^2 \|x_{k+\theta} - x_{p+\theta}\|^2 + \|R_2\|^2 k_E^2 (t_k - t_p)^2 + \frac{1}{2} k_{G_2}^2 k_F^2 (t_k - t_p)^2 \\
 & \quad + \|\xi_{k+1} - \xi_{p+1}\|^2.
 \end{aligned}$$

854 The lemma is proved.  $\square$

855 Thus Corollary 5.5 extends as well, and in particular one has  $\|\xi_{k+1} - \xi_k\| \leq h\alpha +$   
 856  $\beta\|x_{k+\theta} - x_{k-1+\theta}\|$  for all  $k \geq 1$ , for bounded constants  $\alpha \geq 0$  and  $\beta \geq 0$ . The scheme  
 857 in (5.25) is now replaced by:

$$\begin{aligned}
 858 \quad (5.32) \quad P_I^h x_{k+1} = & P_I x_k + h B_I z_{k+1} + h \bar{B}_I^1 (D_{11} + \partial\psi_{K_1^*})^{-1} (C_{II}^1 R_2^{-1} \xi_{k+1} + F_{1,k}) \\
 & + h \bar{B}_I^2 (-R_2 B_{II} R_2^{-1} \xi_{k+1} - R_2 A_{II} x_{k+\theta} - R_2 E_{II,k} + G_2(t_k, \xi_{k+1})),
 \end{aligned}$$

859 equivalently:

$$\begin{aligned}
 860 \quad (5.33) \quad (P_I^h + h(1 - \theta)\bar{B}_I^2 R_2 A_{II}) x_{k+1} = & P_I x_k + h B_I z_{k+1} \\
 & + h \bar{B}_I^1 (D_{11} + \partial\psi_{K_1^*})^{-1} (C_{II}^1 R_2^{-1} \xi_{k+1} + F_{1,k}) + h \bar{B}_I^2 (-R_2 B_{II} R_2^{-1} \xi_{k+1} \\
 & - \theta R_2 A_{II} x_k - R_2 E_{II,k} + G_2(t_k, \xi_{k+1})),
 \end{aligned}$$

861 and it appears that both schemes in (5.25) and in (5.33) possess the same structure.  
 862 Consequently the results of convergence obtained for the foregoing implicit scheme  
 863 (Proposition 5.6, Corollary 5.7, Lemma 5.8) extend to the scheme in this section, for  
 864 suitably chosen numerical and system parameters. For the sake of brevity we do not  
 865 repeat the developments.

866 *Remark 5.12.* Using Lemma 4.1 item 1, Assumption 10 could be replaced by  
 867 assuming the strict passivity of the quadruple  $(B_{II}, \bar{B}_{II}, C_{II}, D)$  (a system property)  
 868 and  $h > 0$  small enough.

869 *Remark 5.13.* Obviously when  $B_{II}$  is invertible it is possible to recast (2.1) into  
 870 an LCS by solving the equality constraint for  $z$ . This yields the LCS quadruple  
 871  $(P_I^{-1}(A_I - B_I B_{II}^{-1} A_{II}), P_I^{-1}(\bar{B}_I - B_I B_{II}^{-1} \bar{B}_{II}), C_I - C_{II} B_{II}^{-1} A_{II}, D - C_{II} B_{II}^{-1} \bar{B}_{II})$ .  
 872 It is unclear how to relate in general the passivity of this quadruple, to the above  
 873 assumptions which focus on the MLCP part (2.1) (b) (c) of the system. Especially  
 874 no specific assumptions are made on the matrices  $A_I, B_I, \bar{B}_I, C_I$  in the above analy-  
 875 ses. For instance Assumption 3 implies that the feedthrough matrix of the LCS is  
 876  $D - C_{II} B_{II}^{-1} X^{-1} C_{II}^\top$ , which may be indefinite in general. It is also noteworthy that  
 877 transforming the MLCP into an LCP may be undesirable [59].

878 **5.3. Recapitulation.** Let us recapitulate the various assumptions upon which  
 879 the results in this article rely. Assumptions 2, 4, 7, 10, 12 (respectively Assumptions  
 880 3, 5, 9), characterize the passivity properties of an auxiliary operator defined by the  
 881 DALCS constraints in (3.2) (b) (c) (respectively in (2.1) (b) (c)). These passivity  
 882 properties constrain the relative degree between the ‘‘input’’  $\lambda_{k+1}$  and the ‘‘output’’  
 883  $w_{k+1} = \tilde{C} z_{k+1} + \tilde{D} \lambda_{k+1}$  (respectively  $\lambda$  and  $w = C_{II} z + D \lambda$ ). Assumptions 6 and 8  
 884 have practical importance (because some systems possess such  $D$  matrix), they are



885 also related to passivity in the sense that passive systems possess positive semidefinite  
 886 throughout matrices. Thus all these assumptions are fundamental ones for the devel-  
 887 opments in this article, which basically deals with the analysis of two types of gener-  
 888 alised equations (GE):  $0 \in A(u)+B(u)$  in (4.7) and (4.15), and  $0 \in A(u)+B(u)+C(u)$   
 889 in (4.9) and (4.17), where the mappings  $A(\cdot)$ ,  $B(\cdot)$  and  $C(\cdot)$  are maximal monotone  
 890 (notice in passing that due to time-dependency, these GEs do not fit with those in [28,  
 891 Chapter 2], so that in particular [28, Theorem 2F.6] cannot be applied in general).  
 892 Most importantly, the convergence results (Proposition 5.2, Lemmata 5.3 and 5.4,  
 893 Corollary 5.5, Proposition 5.6, Corollary 5.7, Lemmata 5.8 and 5.10) rely upon the  
 894 positive definiteness of  $-R_2 B_{II} R_2^{-1}$  and  $-\tilde{R}_2 \tilde{A} \tilde{R}_2^{-1}$ , which is an index one condition  
 895 and guarantees strong monotonicity properties, allowing us to characterize the prop-  
 896 erties of the solutions to the GEs. It would be worth relaxing it in future studies.  
 897 One path could be to use least norm elements [34]. Finally, it is worth noting that  
 898 many practical systems like circuits with nonlinear resistors and inductances contain  
 899 nonlinearities [19, section 4.1]. An extension of the above developments, consists of  
 900 replacing the linear term  $A_I x$  in (2.1), by a nonlinear term  $A_I(x)$ , where  $A_I(\cdot)$  is  
 901 globally Lipschitz continuous, and setting  $A_I(x_k)$  in (3.1). The matrices  $\tilde{A}_2, \tilde{B}, \tilde{C}, \tilde{D}$   
 902 keep the same structure with  $P_I^{-1}$  instead of  $(P_I^h)^{-1}$ , while both  $\tilde{E}_k$  and  $\tilde{F}_k$  contain  
 903  $A_I(x_k)$ . Nonlinearities in the equality and in the complementarity constraints should  
 904 also be considered, where the results in [27] could be useful.

905 **6. Examples and Numerical Simulations.** The next examples are taken  
 906 from circuits, mechanics and state-dependent-switch DAE. They aim at illustrating  
 907 the above developments, and the limitations imposed by some assumptions.

908 **6.1. Example 1 (circuit).** Let us consider the circuit in Figure 1b. Using  
 909 Kirchoff laws its dynamics are given by:

$$910 \quad (6.1) \quad \begin{cases} \dot{x}_1(t) = \frac{-2}{RC}x_1(t) + \frac{1}{RC}x_2(t) + \frac{1}{R}u_{D_1}(t) \\ \dot{x}_2(t) = \frac{-1}{RC}x_2(t) + \frac{1}{2RC}x_1(t) + \frac{1}{2R}u_{D_2}(t) \\ 0 = \frac{3}{2}u_{D_2}(t) - Rx_3(t) - \frac{1}{C}x_2(t) + \frac{1}{2C}x_1(t) \\ 0 \leq u_{D_1}(t) \perp \frac{-2}{RC}x_1(t) + \frac{1}{RC}x_2(t) + \frac{1}{R}u_{D_1}(t) \geq 0 \\ 0 \leq u_{D_2}(t) \perp x_3(t) \geq 0, \end{cases}$$

911 with  $x_1(t) = \int_0^t i_1(s)ds$ ,  $x_2(t) = \int_0^t i_2(s)ds$ ,  $x_3 = i_3 = z$ ,  $\lambda = (u_{D_1} \ u_{D_2})^\top$  (the  
 912 voltages across the diodes),  $K = K^* = \mathbb{R}^2_+$ ,  $n_1 = 2$ ,  $p = n_2 = 1$ ,  $m = 2$ . One  
 913 has  $D = \begin{pmatrix} \frac{1}{R} & 0 \\ 0 & 0 \end{pmatrix}$ ,  $P_I = I_2$ ,  $A_I = \begin{pmatrix} \frac{-2}{RC} & \frac{1}{RC} \\ \frac{1}{2RC} & \frac{1}{RC} \end{pmatrix}$ ,  $B_I = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ,  $\bar{B}_I = \begin{pmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{2R} \end{pmatrix}$ ,  
 914  $A_{II} = \begin{pmatrix} \frac{1}{2C} & -\frac{1}{C} \end{pmatrix}$ ,  $B_{II} = -R$ ,  $\bar{B}_{II} = \begin{pmatrix} 0 & \frac{3}{2} \end{pmatrix}$ ,  $C_I = \begin{pmatrix} \frac{-2}{RC} & \frac{1}{RC} \\ 0 & 0 \end{pmatrix}$ ,  $C_{II} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ .  
 915 Assumption 8 holds with  $D_{11} = \frac{1}{R}$ . Thus  $\bar{B}_{II}^2 = \frac{3}{2}$ ,  $C_{II}^2 = 1$ ,  $X_2 = \frac{2}{3}$  and Assumption  
 916 9 is verified. Item 1 in Proposition 4.8 applies since  $\text{Im}(C_{II}^2) = \mathbb{R}$ . Since  $\bar{B}_{II}^1 = 0$   
 917 the mapping  $G_2(\cdot)$  is zero, and item 3 in Proposition 4.8 applies. The quadruple  
 918  $(B_{II}, \bar{B}_{II}, C_{II}, D)$  is passive because the LMI (C.1) is verified with  $X = \frac{2}{3}$ , hence  
 919 Assumption 5 is satisfied, but it is not strictly passive since  $D \not\succ 0$ . It can be checked  
 920 that the transfer matrix  $H(s) = D + C_{II}(s - B_{II})^{-1}\bar{B}_{II} = \begin{pmatrix} \frac{1}{R} & 0 \\ 0 & \frac{3}{2(s+R)} \end{pmatrix}$  is positive  
 921 real [18, Theorem 2.48]. The conditions of Lemmata 5.10 and 5.11 are verified by  
 922 taking  $R$  large enough (notice that all other parameters depend on  $\frac{1}{R}$ ). Here  $\tilde{A} =$   
 923  $B_{II} = -R$ , but Assumptions 2 and 6 do not hold, hence the material in Proposition  
 924 4.4 cannot be used.

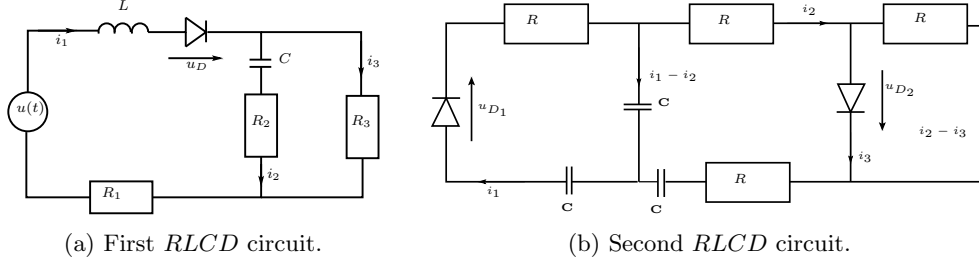


Fig. 1: Two circuits with ideal diodes.

925 **6.2. Example 2 (circuit).** Let us consider now the circuit in Figure 1a. Using  
 926 Kirchhoff laws its dynamics may be written as:

$$927 \quad (6.2) \quad \begin{cases} \dot{x}_1(t) = -\frac{R_1}{L}x_1(t) - \frac{R_3}{L}z(t) + \frac{1}{L}u(t) + \frac{1}{L}u_D(t) \\ \dot{x}_2(t) = \frac{R_3}{R_2}z(t) - \frac{1}{R_2C}x_2(t) \\ 0 = x_1(t) + \frac{1}{R_2C}x_2(t) - (1 + \frac{R_3}{R_2})z(t) \\ 0 \leq u_D(t) \perp x_1(t) \geq 0, \end{cases}$$

928 where  $x_1 = i_1$ ,  $x_2(t) = \int_0^t i_2(s)ds$ ,  $z = i_3$ ,  $\lambda = u_D$  (the voltage across the diode),  
 929  $K = K^* = \mathbb{R}_+$ ,  $n_1 = 2$ ,  $p = n_2 = 1$ ,  $m = 1$ . Let apply the feedback control  $u = 2R_3z$ ,  
 930 which modifies  $B_I$ . We get:  $A_I = \begin{pmatrix} -\frac{R_1}{L} & 0 \\ 0 & -\frac{1}{R_2C} \end{pmatrix}$ ,  $B_I = \begin{pmatrix} \frac{R_3}{L} \\ \frac{R_3}{R_2} \end{pmatrix}$ ,  $\bar{B}_I = \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix}$ ,  
 931  $A_{II} = (1 - \frac{1}{R_2C})$ ,  $B_{II} = -(1 + \frac{R_3}{R_2})$ ,  $\bar{B}_{II} = 0$ ,  $C_I = (1 \ 0)$ ,  $C_{II} = 0$ ,  $D = 0$ ,  $E_I = \begin{pmatrix} \frac{1}{L}u \\ 0 \end{pmatrix}$ ,

932  $E_{II} = 0$ ,  $F(t) = 0$ ,  $P_I = I_2$ ,  $P_I^h = \begin{pmatrix} 1 + h\frac{R_1}{L} & 0 \\ 0 & 1 + h\frac{1}{R_2C} \end{pmatrix} \succ 0$  for all  $h \geq 0$ . Thus

$$933 \quad \tilde{B} = \frac{h}{hR_1 + L}, \quad \tilde{C} = \frac{hR_3}{L + hR_1}, \quad \tilde{D} = \frac{h}{L + hR_1}, \quad \tilde{A} = -\frac{R_2 + R_3}{R_2} + hR_3 \left( \frac{1}{L + hR_1} + \frac{1}{R_2(h + R_2C)} \right).$$

934 Assumption 2 holds with  $\tilde{X} = R_3$ ,  $\tilde{D} > 0$  for all  $h > 0$ , and Assumption 4 holds  
 935 for  $h > 0$  small enough such that  $\tilde{A} < 0$ . For  $h > 0$  small enough the set  $\tilde{S} = \{0\}$   
 936 in (4.6) since  $\tilde{R}\tilde{A}\tilde{R}^{-1} = \tilde{A} < 0$ , hence the VI in (4.7) always has a solution  $z_{k+1}$  by  
 937 Proposition 4.3 item 3 (a). By item (c) (or (f)) of the same proposition, the solution is  
 938 unique. When  $\tilde{A} < 0$ , the quadruple  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  is strictly passive for all  $h > 0$ , hence  
 939 items 1 and 3 of Corollary 4.6 apply. However Assumption 10 does not hold since  
 940  $\tilde{D} \rightarrow 0$  as  $h \rightarrow 0$ . Consider now the quadruple  $(B_{II}, \bar{B}_{II}, C_{II}, D) = (B_{II}, 0, 0, 0)$ .  
 941 It is passive since (C.1) is verified, it is even strictly state passive but not strictly  
 942 passive since  $D = 0$ . Assumptions 3 and 9 are trivially satisfied since both sides of  
 943 the passivity equality are null. Provided that  $\{0\} \in \mathbb{R}_+ - x_{1,k+1} \Leftrightarrow x_{1,k+1} \geq 0$ , item 1  
 944 in Proposition 4.7 applies and the OSNSP for (3.1) has a unique solution (but clearly  
 945 there is no guarantee that this inequality is satisfied along the integration process,  
 946 rendering Proposition 4.7 unapplicable). Proposition 4.8 item 3 also applies here for  
 947 the VI in (4.17), however one has to check the equivalence stated in item 1 of the  
 948 same proposition.

949 As noted above, the matrix  $A_I$  intervenes only in Assumption 1, hence its modification  
 950 does not alter the results. Setting  $u = 2R_3z + R_1x_1 + aLx_1 + v(t)$  yields the modified

$$951 \quad A_I = \begin{pmatrix} a & 0 \\ 0 & -\frac{1}{R_2C} \end{pmatrix}, \text{ and } B_I \text{ as above. This yields (see Remark 5.13): } P_I^{-1}(A_I -$$

952  $B_I B_{II}^{-1} A_{II}) = \begin{pmatrix} a + \frac{R_2 R_3}{L(R_2 + R_3)} & \frac{R_3}{L\mathbf{C}(R_2 + R_3)} \\ \frac{R_3}{R_2 + R_3} & -\frac{1}{(R_2 + R_3)\mathbf{C}} \end{pmatrix}$ , which is unstable if  $a + \frac{R_2 R_3}{(R_2 + R_3)L} >$   
953  $\frac{1}{(R_2 + R_3)\mathbf{C}}$ . The other elements of the reduced LCS quadruple are given by  $P_I^{-1}(\bar{B}_I -$   
954  $B_I B_{II}^{-1} \bar{B}_{II}) = \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix}$ ,  $C_I - C_{II} B_{II}^{-1} A_{II} = (1 \ 0)$ ,  $D - C_{II} B_{II}^{-1} \bar{B}_{II} = 0$ . Consequently  
955 the reduced LCS is not passive with the above feedback controller. Also  $P_I^h = \text{diag}(1 -$   
956  $ha, 1 + \frac{h}{R_2 \mathbf{C}}) \succ 0$  for all  $h < \frac{1}{a}$ .  
957 To simplify the calculations let us set  $R_2 = R_3 = 1\Omega$ ,  $L = 1\text{H}$ ,  $\mathbf{C} = 1\text{F}$ ,  $a = 1$ . Then  
958 the discretization (3.1) of (6.2) writes as:

$$959 \quad (6.3) \quad \begin{cases} x_{1,k+1} - x_{1,k} = hx_{1,k+1} + hz_{k+1} + h\lambda_{k+1} + hv_k \\ x_{2,k+1} - x_{2,k} = -hx_{2,k+1} + hz_{k+1} \\ 0 = x_{1,k+1} + x_{2,k+1} - 2z_{k+1} \\ 0 \leq \lambda_{k+1} \perp x_{1,k+1} \geq 0. \end{cases}$$

960 Equivalently the discretization in (3.2) can be used with the above matrices  $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ .

961 **6.3. Example 3 (switching DAE).** Let us consider the following system,  
962 which is a nonsmooth DAE (similar to those studied in [53, 54, 52, 51], and which  
963 can be interpreted as a DAE with state-dependent switches):

$$964 \quad (6.4) \quad \begin{cases} (a) \dot{x}(t) = A_I x(t) + B_I z(t) + E_I(t) \\ (b) 0 = a|x(t)| + b|z(t)| - cz(t) + E_{II}(t), \end{cases}$$

965 with  $x(t) \in \mathbb{R}$ ,  $z(t) \in \mathbb{R}$ ,  $a, b, c \in \mathbb{R}$ ,  $c > 0$ . The constraint in (6.4) (b) is equivalently  
966 rewritten as:

$$967 \quad (6.5) \quad \begin{aligned} 0 &= a\lambda_1(t) + a\lambda_2(t) + b\lambda_3(t) + b\lambda_4(t) - cz(t) + E_{II}(t) \\ 0 &\leq \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \lambda_3(t) \\ \lambda_4(t) \end{pmatrix} \perp \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \lambda_3(t) \\ \lambda_4(t) \end{pmatrix} + \begin{pmatrix} -x(t) \\ x(t) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -z(t) \\ z(t) \end{pmatrix} \geq 0. \end{aligned}$$

968 Therefore (6.4) (a) (6.5) fits with (2.1) with  $A_{II} = 0$ ,  $B_{II} = -c$ ,  $\bar{B}_{II} = (a \ a \ b \ b)$ ,  
969  $D = I_4$ ,  $C_I = (-1 \ 1 \ 0 \ 0)^\top$ ,  $C_{II} = (0 \ 0 \ -1 \ 1)^\top$ ,  $F(t) = 0$ ,  $\bar{B}_I = 0$ . Let us check the  
970 strict passivity of the quadruple  $(B_{II}, \bar{B}_{II}, C_{II}, D)$ . This boils down to checking that  
971 (see (C.1)):

$$972 \quad (6.6) \quad M \triangleq \begin{pmatrix} 2cX & -aX & -aX & -bX - 1 & -bX + 1 \\ -aX & 2 & 0 & 0 & 0 \\ -aX & 0 & 2 & 0 & 0 \\ -bX - 1 & 0 & 0 & 2 & 0 \\ -bX + 1 & 0 & 0 & 0 & 2 \end{pmatrix} \succ 0,$$

973 for some  $X > 0$ . Let us denote  $M = P + N$  with  $P = \text{diag}(2cX, 2I_4) \succ 0$ , and  
974  $N_{00}$  the matrix obtained for  $a = b = 0$ . Applying [25, Theorem 2.11] or [10, Fact  
975 8.19.2], it follows that  $P + N_{00} \succ 0$  if  $\sigma_{\max}^2(N_{00}) = 2 < \lambda_{\min}(2cX)\lambda_{\min}(2I_4) = 4cX$ .  
976 Using continuity of the eigenvalues [10, Fact 10.11.8], it is deduced that given  $c > 0$ ,  
977  $X > 0$ , there exists  $a$  and  $b$  small enough such that (6.6) holds (or, given  $a, b$  and  
978  $X > 0$ , there exists  $c > 0$  large enough). Thus, item 1 in Lemma 4.1 holds, item 2  
979 in Proposition 4.3 holds for  $h > 0$  small enough (guaranteeing  $\tilde{D} \succ 0$ ), Assumption

980 10 holds true for small enough  $h > 0$  (see the proof of item 1 in Lemma 4.1), and  
 981 the arguments stated at the beginning of section 5.2 about the convergence analysis  
 982 led in sections 5.1.1 and 5.1.2, hold. We have:  $\tilde{A} = -c$ ,  $\tilde{B} = (a \ a \ b \ b)$ ,  $\tilde{C} =$   
 983  $(0 \ 0 \ -1 \ 1)^\top + h(-1 \ 1 \ 0 \ 0)^\top(1 - hA_I)^{-1}B_I = (\frac{-hB_I}{1-hA_I} \ \frac{hB_I}{1-hA_I} \ 0 \ 0)^\top$ ,  $\tilde{D} = D$ .  
 984 Integration can proceed using (3.2).  
 985 Let us now pass to a second example of state-dependent-switch DAE:

$$986 \quad (6.7) \quad \begin{cases} (a) \ \dot{x}_1(t) = a_1x_1(t) + b_1x_2(t) + c_1z(t) + e_1(t) \\ (b) \ \dot{x}_2(t) = a_2x_1(t) + b_2x_2(t) + c_2z(t) + e_2(t) \\ (c) \ 0 = a_3x_1(t) + b_3x_2(t) + c_3z(t) + e_3(t) \text{ if } x_1(t) < 0 \\ (d) \ 0 = a_4x_1(t) + b_3x_2(t) + c_3z(t) + e_3(t) \text{ if } x_1(t) > 0, \end{cases}$$

987 where the switching conditions in (c) and (d) are equivalently rewritten as:

$$988 \quad (6.8) \quad \begin{cases} 0 = -a_3\lambda_1(t) + a_4\lambda_2(t) + b_3x_2(t) + c_3z(t) + e_3(t) \\ 0 \leq \lambda_1(t) \perp \lambda_1(t) + x_1(t) \geq 0 \\ 0 \leq \lambda_2(t) \perp \lambda_2(t) - x_1(t) \geq 0. \end{cases}$$

989 Thus we have:  $A_I = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ ,  $B_I = (c_1 \ c_2)^\top$ ,  $\bar{B}_I = (0 \ 0)^\top$ ,  $E_I = (e_1 \ e_2)^\top$ ,  $\bar{B}_{II} =$   
 990  $(-a_3 \ a_4)^\top$ ,  $A_{II} = (0 \ b_3)$ ,  $B_{II} = c_3$ ,  $E_{II} = e_3$ ,  $C_I = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$ ,  $C_{II} = (0 \ 0)^\top$ ,  $D = I_2$ ,  
 991  $F(t) = (0 \ 0)^\top$ . Also  $\tilde{D} = D = I_2$ ,  $\tilde{A} = c_3 + h(0 \ b_3)(I_2 - hA_I)^{-1} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_3 + \mathcal{O}(h)$ ,

992  $\tilde{B} = \bar{B}_{II} = (-a_3 \ a_4)^\top$ ,  $\tilde{C} = h \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} (I_2 - hA_I)^{-1} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathcal{O}(h)$ . It can be checked

993 that Proposition 4.3 items 1 and 2, Corollary 4.6 items 1, 2 and 3 (if the passivity LMI  
 994 holds strictly) apply. The passivity LMI  $\begin{pmatrix} -2X\tilde{A} & -X(a_3 \ a_4) + \tilde{C} \\ (-X(a_3 \ a_4) + \tilde{C})^\top & 2I_2 \end{pmatrix} \succ 0$

995 is verified for  $a_3$ ,  $a_4$  and  $h > 0$  small enough, given  $c_3 < 0$ ,  $X > 0$ , implying the  
 996 strict passivity of the quadruple  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ , uniformly as  $h \rightarrow 0$  (Assumption 10  
 997 holds true). It is of interest to investigate what happens if  $c_3 = 0$ , which makes  
 998 the “index” increase. Let us assume for simplicity that  $A_I = 0$ , the matrix of the

999 passivity LMI becomes  $\begin{pmatrix} -2hXb_3c_2 & -X(-a_3 \ a_4) + h(c_1 \ -c_1) \\ (-X(-a_3 \ a_4) + h(c_1 \ -c_1))^\top & 2I_2 \end{pmatrix}$ .

1000 For given  $X > 0$  and  $h > 0$ , there always exists  $b_3c_2 < 0$  such that this matrix is  $\succ 0$ ,  
 1001 showing that there exists conditions under which the OSNP is still well-posed when  
 1002  $c_3 = 0$ . It is interesting to note that the term  $b_3c_2z(t)$  appears when an index reduction  
 1003 is applied to the equality in (6.8) by differentiating it once. Simulations are shown  
 1004 in Figures 2 and 3. They are performed with the INRIA software package SICONOS<sup>1</sup>  
 1005 [1]. The MLCP is solved with an enumeration method and the differential variables  
 1006  $x_1(\cdot)$  and  $x_2(\cdot)$  are calculated in a second step (hence we follow the philosophy of the  
 1007 foregoing sections). The parameters are chosen as follows:  $a_1 = a_2 = b_1 = b_2 = a_3 =$   
 1008  $a_4 = b_3 = c_3 = 1$ ,  $c_1 = 10$ ,  $c_2 = 20$ ,  $e_1(t) = 40 \sin(25t)$ ,  $e_2(t) = \sin(t)$ ,  $e_3(t) = \sin(10t)$ ,  
 1009  $h = 2 \cdot 10^{-3}$ s,  $x_1(0) = 0$ ,  $x_2(0) = 0.1$ . Simulations are done on  $[0, 2]$ s. The errors are  
 1010 calculated as  $\max_{t \in [0, 2]} \|x_i(t) - x_i^{ref}(t)\|$ ,  $i = 1, 2$ , with 14 different values of step  
 1011 size  $h$  between  $10^{-1}$ s and  $10^{-4}$ s, where  $x_i^{ref}(\cdot)$  is calculated with  $h = 10^{-4}$ s. It is  
 1012 observed in Figure 2 that the order is approximately equal to one for small enough

<sup>1</sup><https://nonsmooth.gricad-pages.univ-grenoble-alpes.fr/siconos/index.html>

1013 step size. Figure 3 depicts the time evolution of various variables, demonstrating that  
 1014 the complementarity constraints are respected. Clearly the multipliers are continuous  
 1015 but non differentiable when the complementarity switches from one mode to another.

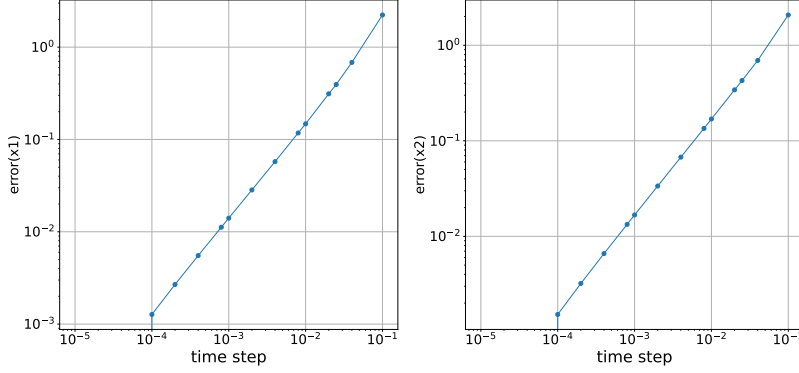


Fig. 2: Log-log error diagram:  $x_1$  (left) and  $x_2$  (right).

1016

1017 **6.4. Example 4 (mechanical system).** As an extension of the systems studied  
 1018 in [37], let us consider a linear mechanical system with both unilateral springs and  
 1019 bilateral constraints, whose dynamics are:

$$1020 \quad (6.9) \quad \begin{cases} (a) M\ddot{q}(t) + K_v\dot{q}(t) + K_pq(t) = \tau(t) + H^\top \lambda_u(t) + G^\top \lambda_b(t) \\ (b) 0 = Gq(t) - f(t) \\ (c) 0 \leq \lambda_u(t) \perp D\lambda_u(t) + C_Ix(t) + F(t) \geq 0, \end{cases}$$

1021 where  $M = M^\top \succ 0$ ,  $D \succ 0$ ,  $x = (q^\top \dot{q}^\top)^\top$ ,  $q(t) \in \mathbb{R}^{\frac{n_1}{2}}$ ,  $z = \lambda_b \in \mathbb{R}^{n_2}$ ,  $\lambda = \lambda_u \in$   
 1022  $\mathbb{R}^m$ ,  $G \in \mathbb{R}^{n_2 \times \frac{n_1}{2}}$ ,  $f: \mathbb{R}^+ \rightarrow \mathbb{R}^{n_2}$  is twice differentiable,  $\tau(\cdot)$  is an exogenous force.  
 1023 Here an index reduction is possible, in order to replace (6.9) (b) by:

$$1024 \quad (6.10) \quad 0 = G\ddot{q}(t) - \ddot{f}(t) = GM^{-1}H^\top \lambda(t) + GM^{-1}G^\top z(t) - GM^{-1}(K_p K_v)x(t) - \ddot{f}(t),$$

1025 and  $Gq(0) - f(0) = 0$ ,  $G\dot{q}(0) - \dot{f}(0) = 0$ . Thus  $B_{II} = -GM^{-1}G^\top$  ( $< 0$  if bilateral con-  
 1026 straints are independent, *i.e.*,  $\text{rank}(G) = p = n_2 \Rightarrow n_1 \geq 2n_2$ ),  $A_{II} = GM^{-1}(K_p K_v)$ ,

1027  $\bar{B}_{II} = -GM^{-1}H^\top$ ,  $B_I = \begin{pmatrix} 0 \\ -M^{-1}G^\top \end{pmatrix}$ , and  $C_{II} = 0$  (similarly to the example in

1028 section 6.2, one major difference being that here  $D \succ 0$ ). Let us examine the OSNSP  
 1029 for (6.9) on a particular example as depicted in Figure 4, whose dynamics are given  
 1030 by:

$$1031 \quad (6.11) \quad \begin{cases} m\ddot{q}_1(t) = \tau(t) + \lambda_1(t) - \lambda_2(t) \\ m\ddot{q}_2(t) = z(t) \\ 0 = q_2(t) - f(t) \Leftrightarrow \frac{-z(t)}{m} + \ddot{f}(t) = 0, q_2(0) - f(0) = 0, \dot{q}_2(0) - \dot{f}(0) = 0 \\ 0 \leq \lambda_1(t) \perp \lambda_1(t) + k(q_1(t) - L - \alpha(t) - l) \geq 0 \\ 0 \leq \lambda_2(t) \perp \lambda_2(t) - k(q_1(t) + L - \beta(t) + l) \geq 0. \end{cases}$$

1032 where  $q = (q_1 \ q_2)^\top$  are the mass coordinates,  $k > 0$  is a stiffness constant,  $\alpha(\cdot)$   
 1033 and  $\beta(\cdot)$  are the massless springs' varying positions,  $l > 0$  is the spring's free length,

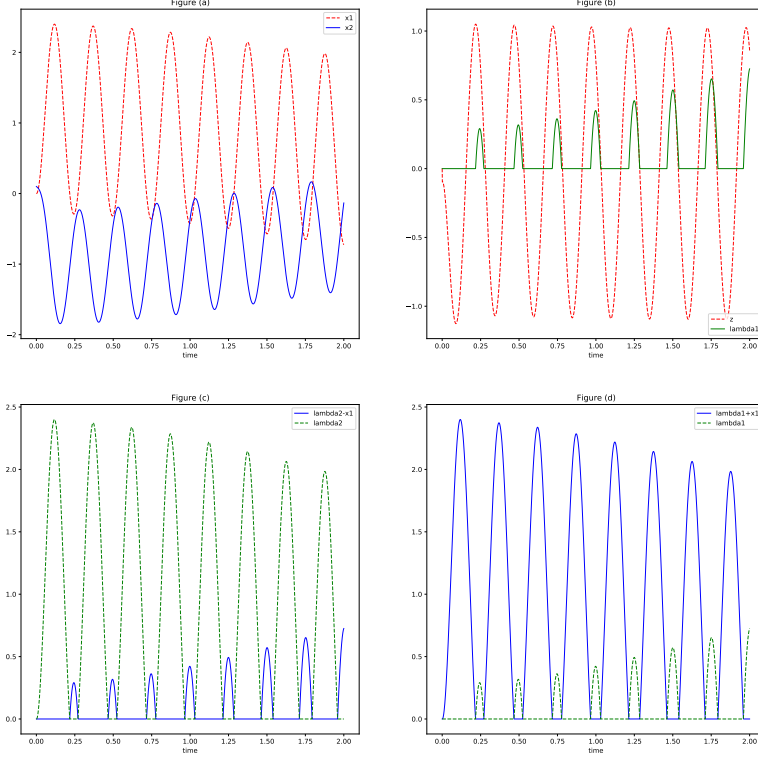


Fig. 3: (a)  $x_1(t)$  (dashed) and  $x_2(t)$ , (b)  $z(t)$  (dashed) and  $\lambda_1(t)$ , (c)  $\lambda_2(t) - x_1(t)$  and  $\lambda_2(t)$  (dashed), (d)  $\lambda_1(t) + x_1(t)$  and  $\lambda_1(t)$  (dashed).

1034  $\beta(t) > \alpha(t) + 2L + 2l$  for all  $t \geq 0$ . We have:  $x = (q_1 \dot{q}_1 \ q_2 \ \dot{q}_2)^\top$ ,  $G = (0 \ 1)$ ,  $H =$   
 1035  $\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$ ,  $M = \text{diag}(m, m)$ ,  $D = I_2$ ,  $A_{II} = 0$ ,  $B_{II} = \frac{-1}{m}$ ,  $\bar{B}_{II} = 0$ ,  $E_{II}(t) = \ddot{f}(t)$ ,  
 1036  $C_I = \begin{pmatrix} k & 0 & 0 & 0 \\ -k & 0 & 0 & 0 \end{pmatrix}$ ,  $F(t) = \begin{pmatrix} k(-L - \alpha(t) - l) \\ -k(L - \beta(t) + l) \end{pmatrix}$ ,  $C_{II} = 0$ ,  $B_I = (0 \ 0 \ 0 \ \frac{1}{m})^\top$ ,  
 1037  $P_I = I_4$  (and Assumption 1 holds for  $h > 0$  small enough),  $\bar{B}_I = \begin{pmatrix} 0 & \frac{1}{m} & 0 & 0 \\ 0 & \frac{m}{1} & 0 & 0 \end{pmatrix}^\top$ ,  
 1038  $\tilde{A} = \frac{-1}{m}$ ,  $\tilde{B} = 0$ ,  $\tilde{C} = hC_I(P_I^h)^{-1}B_I$ ,  $\tilde{D} = I_2 + hC_I(P_I^h)^{-1}\bar{B}_I \succ 0$  if  $h > 0$  is small  
 1039 enough. Let us check Assumption 4. The passivity LMI is  $\begin{pmatrix} \frac{2X}{C^\top} & \tilde{C} \\ C^\top & \tilde{D} + \tilde{D}^\top \end{pmatrix} \succ 0$ ,  
 1040  $X > 0$ , which is verified for  $h > 0$  small enough and  $X > 0$  large enough. Hence under  
 1041 these conditions the quadruple  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  is strictly passive. It can be checked that  
 1042 items 1 and 2 in Proposition 4.3 hold, as well as items 1, 2 and 3 in Corollary 4.6.

1043 Therefore the dynamics can be integrated using (3.2) with  $q_{2,0} + f_0 = 0$ ,  $\dot{q}_{2,0} + \dot{f}_0 = 0$ .  
 1044 We note also that the quadruple  $(B_{II}, \bar{B}_{II}, C_{II}, D) = (\frac{-1}{m}, 0, 0, I_2)$  is strictly passive  
 1045 (Assumption 5 is verified), hence item 1 in Lemma 4.1 holds. Assumption 10 is  
 1046 also verified (for  $h > 0$  small enough but uniformly in  $h \geq 0$ ). Let us analyse the  
 1047 convergence as in section 5.1. Assumption 11 may not be verified in this case, however  
 1048 as noted after this assumption it is easy to modify slightly the numerical scheme so  
 1049 that it is satisfied with  $P_I = I_4$ . Hence it is inferred that Proposition 5.2 applies.

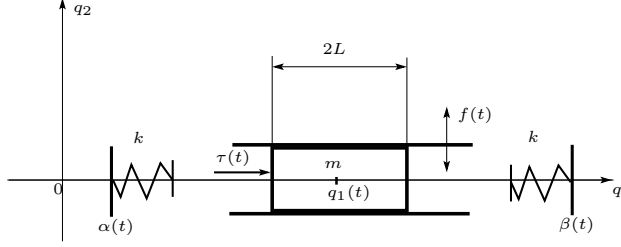


Fig. 4: A mechanical DALCS.

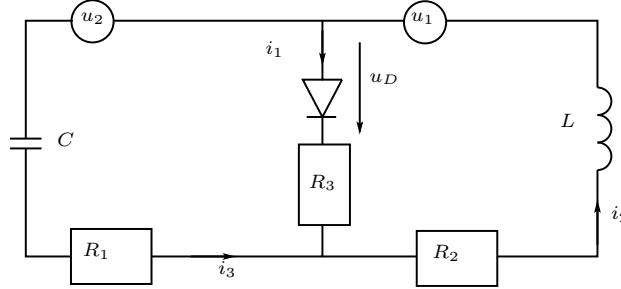


Fig. 5: An  $RLCD$  passive circuit.

1050 **6.5. Example 5 (circuit).** Let us consider the circuit in Figure 5, with  $R_1 =$   
 1051  $R_3 = 0 \Omega$ . Let  $x_1 = i_2$ ,  $x_2 = \int_0^t i_3(t) dt$ ,  $z = i_1$ ,  $\lambda = u_D$  is the voltage across the  
 1052 diode,  $u_1(\cdot)$  and  $u_2(\cdot)$  are two voltage sources. One way to write its dynamics is:

$$1053 \quad (6.12) \quad \begin{cases} \dot{x}_1(t) = -\frac{R_2}{L}x_1(t) + \frac{1}{L}\lambda(t) + \frac{1}{L}u_1(t) \\ \dot{x}_2(t) = x_1(t) - z(t) \\ 0 = \frac{1}{C}x_2(t) + \lambda(t) + u_2(t) \\ 0 \leq \lambda(t) \perp z(t) \geq 0. \end{cases}$$

1054 Assume that the feedback  $u_2 = -az + u_3$  is applied, so that the equality constraint  
 1055 becomes  $0 = \frac{1}{C}x_2(t) + u_D(t) - az(t) + u_3(t)$ . We have:  $A_{II} = (0 \quad \frac{1}{C})$ ,  $C_I = 0$ ,  
 1056  $C_{II} = 1$ ,  $B_{II} = -a$ ,  $\bar{B}_{II} = 1$ ,  $D = 0$ ,  $\bar{B}_I = (\frac{1}{L} \quad 0)^\top$ ,  $B_I = (0 \quad -1)^\top$ , and  $\bar{D} = D = 0$ ,  
 1057  $\tilde{C} = C_{II} = 1$ ,  $\tilde{A} = -a + hA_{II}(P_I^h)^{-1}B_I$ ,  $\tilde{B} = 1 + hA_{II}(P_I^h)^{-1}\bar{B}_I$ . The condition  
 1058  $\text{Im}(C_{II}) \cap (K - C_I x_{k+1} + F_k) = \mathbb{R} \cap \mathbb{R}_+ - \frac{1}{C}x_{2,k+1} \neq \emptyset$  is verified always. The set  $\mathcal{S}$  in  
 1059 (4.16) is equal to  $\{0\}$  hence Proposition 4.7 item 1 applies and the VI in (4.15) always  
 1060 has a solution. Proposition 4.8 items 1, 2, 3 apply, with  $D_{11} = 0$  in Assumption 8.

1061 The system  $(B_{II}, \bar{B}_{II}, C_{II}) = (-a, 1, 1)$  is passive and Assumptions 3 and 5 hold true.  
 1062 Also the system  $(\bar{A}, \bar{B}, \bar{C})$  is passive for  $h > 0$  small enough and Assumptions 2 and  
 1063 4 hold true. The set  $\mathcal{S}$  in (4.6) is equal to  $\{0\}$  so Proposition 4.3 item 3 (a) applies.  
 1064 For  $h > 0$  small enough,  $\bar{A} < 0$  so Corollary 4.6 items 1 and 2 apply. Assumption 12  
 1065 is verified for  $h \geq 0$  small enough with  $\bar{X} = \bar{B}^{-1}$ . It is inferred that if  $a > 0$  is large  
 1066 enough, the convergence results in sections 5.1.3 and 5.2 apply.

1067 **7. Conclusion.** This article is devoted to analysing the implicit Euler discreti-  
 1068 sation of a class of nonsmooth nonlinear systems with equality constraints, named  
 1069 differential-algebraic linear complementarity systems. The well-posedness of the one-  
 1070 step nonsmooth problem is studied first, then the convergence of the discrete-time  
 1071 solutions is analysed. This article only scratches the surface of singular nonsmooth  
 1072 systems analysis. The above results could be extended in various directions. Most  
 1073 importantly, it would be worth extending the results to more general indices and rela-  
 1074 tive degrees, to deeper analyse how the algebraic and the complementarity constraints  
 1075 are coupled, and to consider state discontinuities and solutions of bounded variation.

1076 **Appendix A. Well-posedness of Variational Inequalities.** Some basic  
 1077 notions of Convex Analysis and Complementarity Theory are briefly introduce now  
 1078 [11, 29, 49, 31]. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a proper convex and lower semicontin-  
 1079 uous (lsc) function, we denote by  $\text{dom}(f) \triangleq \{x \in \mathbb{R}^n \mid f(x) < +\infty\}$  the domain of  
 1080 the function  $f(\cdot)$ . The Fenchel transform  $f^*(\cdot)$  of  $f(\cdot)$  is the proper, convex and lsc  
 1081 function defined by: for all  $z \in \mathbb{R}^n$ ,  $f^*(z) = \sup_{x \in \text{dom}(f)} \{\langle x, z \rangle - f(x)\}$ . The sub-  
 1082 differential  $\partial f(x)$  of  $f(\cdot)$  at  $x \in \mathbb{R}^n$  is defined by:  $\partial f(x) = \{\omega \in \mathbb{R}^n \mid f(v) - f(x) \geq$   
 1083  $\langle \omega, v - x \rangle, \forall v \in \mathbb{R}^n\}$ . We denote by  $\text{Dom}(\partial f) \triangleq \{x \in \mathbb{R}^n \mid \partial f(x) \neq \emptyset\}$  the domain of  
 1084 the subdifferential operator  $\partial f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ .

Let  $x_0$  be any element in the domain  $\text{dom}(f)$ , the recession function  $f_\infty(\cdot)$  of  $f(\cdot)$   
 is defined by: for all  $x \in \mathbb{R}^n$  :  $f_\infty(x) = \lim_{\lambda \rightarrow +\infty} \frac{1}{\lambda} f(x_0 + \lambda x)$ . The function  $f_\infty : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  is a proper convex and lsc function which describes the asymptotic behavior of  $f(\cdot)$ . Let  $K \subset \mathbb{R}^n$  be a nonempty closed convex set. Let  $x_0$  be any element in  $K$ . The recession cone of  $K$  is defined by [49]:

$$K_\infty = \bigcap_{\lambda > 0} \frac{1}{\lambda} (K - x_0) = \{u \in \mathbb{R}^n \mid x + \lambda u \in K, \forall \lambda \geq 0, \forall x \in K\}.$$

1085 The set  $K_\infty$  is a nonempty closed convex cone that is described in terms of the  
 1086 directions which recede from  $K$ . The indicator function of a set  $K \subseteq \mathbb{R}^n$  is  $\Psi_K(x) = 0$   
 1087 if  $x \in K$ ,  $\Psi_K(x) = +\infty$  if  $x \notin K$ . If  $K$  is closed non empty convex, we have  
 1088  $\partial \Psi_K(x) = \mathcal{N}_K(x)$ , the so-called normal cone to  $K$  at  $x$ , defined as  $\mathcal{N}_K(x) = \{v \in$   
 1089  $\mathbb{R}^n \mid v^\top (s - x) \leq 0 \text{ for all } s \in K\}$ .

1090 Important properties of the recession function and recession cone are recalled now:

1091 PROPOSITION A.1. [11, Proposition 1.4.8] *The following statements hold:*

1092 a) Let  $f_1 : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  and  $f_2 : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be two proper, convex and  
 1093 lsc functions. Suppose that  $f_1 + f_2$  is proper. Then for all  $x \in \mathbb{R}^n$  :  $(f_1 + f_2)_\infty(x) =$   
 1094  $(f_1)_\infty(x) + (f_2)_\infty(x)$ .

1095 b) Let  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a proper, convex and lsc function, and let  $K$  be a  
 1096 nonempty closed convex set, such that  $f + \Psi_K$  is proper (equivalently  $\text{dom}(f) \cap K$  is  
 1097 non empty). Then for all  $x \in \mathbb{R}^n$  :  $(f + \Psi_K)_\infty(x) = f_\infty(x) + (\Psi_K)_\infty(x)$ .



1098 c) Let  $K \subset \mathbb{R}^n$  be a nonempty, closed and convex set. Then for all  $x \in \mathbb{R}^n$ :  
 1099  $(\Psi_K)_\infty(x) = \Psi_{K_\infty}(x)$ . Moreover for all  $x \in K$  and  $e \in K_\infty$ :  $x + e \in K$ .

1100 d) If  $K \subseteq \mathbb{R}^n$  is a nonempty closed and convex cone, then  $K_\infty = K$ .

1101 e) Let  $K = P(A, b) \triangleq \{x \in \mathbb{R}^n | Ax \geq b\}$  for  $A \in \mathbb{R}^{m \times n}$  and  $b \in \mathbb{R}^m$ . If  $K \neq \emptyset$  then  
 1102  $K_\infty = P(A, 0) = \{x \in \mathbb{R}^n | Ax \geq 0\}$ .

1103 Let us now concatenate [4, Theorem 3, Corollaries 3 and 4]. They concern variational  
 1104 inequalities (VIs) of the form: Find  $u \in \mathbb{R}^n$  such that

$$1105 \quad (\text{A.1}) \quad \langle \mathbf{M}u + \mathbf{q}, v - u \rangle + \varphi(v) - \varphi(u) \geq 0, \text{ for all } v \in \mathbb{R}^n$$

1106 where  $\mathbf{M} \in \mathbb{R}^{n \times n}$  is a real matrix,  $\mathbf{q} \in \mathbb{R}^n$  a vector and  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$   
 1107 a proper convex and lsc function. The VI in (A.1) is equivalent to the inclusion  
 1108  $\mathbf{M}u + \mathbf{q} \in -\partial\varphi(u)$ , which is a generalized equation (GE) of the form:  $0 \in A(u) + B(u)$   
 1109 where  $A(\cdot)$  and  $B(\cdot)$  are two maximal monotone operators (since it will be assumed  
 1110  $\mathbf{M} \succcurlyeq 0$ ). In fact the problems encountered in this paper (see (4.7), (4.9), (4.14), (4.15),  
 1111 (4.17)) all belong to this class of GEs. Numerical schemes that compute solutions of  
 1112 such GEs have been studied for a long time in the mathematical literature, see [57]  
 1113 and references therein. The next proposition states conditions about existence and  
 1114 uniqueness of solutions.

1115 The problem in (A.1) is denoted as  $VI(\mathbf{M}, \mathbf{q}, \varphi)$ , and we set:

$$1116 \quad (\text{A.2}) \quad \mathcal{K}(\mathbf{M}, \varphi) = \{x \in \mathbb{R}^n | \mathbf{M}x \in (\text{dom}(\varphi_\infty))^*\}.$$

1117 Note that  $(\text{dom}(\varphi_\infty))^*$  is the dual cone of the domain of the recession function  $\varphi_\infty$   
 1118 while  $(\text{dom}(\varphi))_\infty$  (that we may denote also as  $\text{dom}(\varphi)_\infty$ ) is the recession cone of  
 1119  $\text{dom}(\varphi)$ .

1120 **PROPOSITION A.2.** [4] Let  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a proper, convex and lsc  
 1121 function with closed domain,  $\mathbf{M} \in \mathbb{R}^{n \times n}$ , and suppose that  $\mathbf{M} \succcurlyeq 0$ .

1122 a) If  $(\text{dom}(\varphi))_\infty \cap \text{Ker}\{\mathbf{M} + \mathbf{M}^\top\} \cap \mathcal{K}(\mathbf{M}, \varphi) = \{0\}$  then for each  $\mathbf{q} \in \mathbb{R}^n$ , problem  
 1123  $VI(\mathbf{M}, \mathbf{q}, \varphi)$  has at least one solution.

1124 b) Suppose that  $(\text{dom}(\varphi))_\infty \cap \text{Ker}\{\mathbf{M} + \mathbf{M}^\top\} \cap \mathcal{K}(\mathbf{M}, \varphi) \neq \{0\}$ . If there exists  $x_0 \in$   
 1125  $\text{dom}(\varphi)$  such that

$$1126 \quad (\text{A.3}) \quad \langle \mathbf{q} - \mathbf{M}^\top x_0, v \rangle + \varphi_\infty(v) > 0, \forall v \in (\text{dom}(\varphi))_\infty \cap \text{Ker}\{\mathbf{M} + \mathbf{M}^\top\} \cap \mathcal{K}(\mathbf{M}, \varphi), v \neq 0,$$

1127 then problem  $VI(\mathbf{M}, \mathbf{q}, \varphi)$  has at least one solution.

1128 c) If  $u_1$  and  $u_2$  are two solutions of problem  $VI(\mathbf{M}, \mathbf{q}, \varphi)$  then  $u_1 - u_2 \in \text{Ker}\{\mathbf{M} +$   
 1129  $\mathbf{M}^\top\}$ .

1130 d) If  $\mathbf{M} = \mathbf{M}^\top$  and  $u_1$  and  $u_2$  denote two solutions of problem  $VI(\mathbf{M}, \mathbf{q}, \varphi)$ , then  
 1131  $\langle \mathbf{q}, u_1 - u_2 \rangle = \varphi(u_2) - \varphi(u_1)$ .

1132 e) If  $\mathbf{M} = \mathbf{M}^\top$ , then  $u$  is a solution of  $VI(\mathbf{M}, \mathbf{q}, \varphi)$  if and only if it is a solution of  
 1133 the optimization problem  $\min_{x \in \mathbb{R}^n} \frac{1}{2}x^\top \mathbf{M}x + \langle \mathbf{q}, x \rangle + \varphi(x)$ .

1134 **Appendix B. Some Convex Analysis and Complementarity Theory**  
 1135 **tools.** If  $K \subseteq \mathbb{R}^n$  is a set, then  $K^* = \{z \in \mathbb{R}^n | \langle z, x \rangle \geq 0 \text{ for all } x \in K\}$  is  
 1136 its dual cone. Let  $K$  be a nonempty closed convex cone, then:

$$1137 \quad (\text{B.1}) \quad K^* \ni x \perp y \in K \Leftrightarrow x \in -\mathcal{N}_K(y) \iff y \in -\mathcal{N}_{K^*}(x).$$

1138 Let  $M = M^\top \succ 0$ ,  $x$  and  $y$  two vectors, then [29]

1139 (B.2)  $M(x - y) \in -\mathcal{N}_K(x) \Leftrightarrow x = \text{proj}_M[K; y] \Leftrightarrow x = \min_{z \in K} \frac{1}{2}(z - y)^\top M(z - y).$

1140 We note that this is a particular case of (A.1), so that Proposition A.2 can be con-  
 1141 sidered as the characterization of a generalized projection operator  $VI(\mathbf{M}, \mathbf{q}, \varphi)$ . The  
 1142 next proposition is used several times in the article:

1143 PROPOSITION B.1. *Suppose that the  $m \times m$  matrix  $D \succ 0$ , and  $\mathcal{M} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$   
 1144 is a maximal monotone operator. Then the operator  $(D + \mathcal{M})^{-1}(\cdot)$  is well-defined  
 1145 (i.e., for each  $y \in \mathbb{R}^m$  there is an  $x \in \mathbb{R}^m$  such that  $y \in Dx + \mathcal{M}(x)$ ), single-valued  
 1146 and Lipschitz continuous with modulus  $L = \frac{2}{\lambda_{\min}(D + D^\top)}$ .*

1147 It was proposed in [16, Proposition 1] when  $\mathcal{M} = \partial\phi$  for some convex proper lsc  $\phi(\cdot)$ ,  
 1148 extended to maximal monotone  $\mathcal{M}(\cdot)$  in [5].

1149 **Appendix C. Passive Systems.** A quadruple  $(A, B, C, D)$  is said passive if:

1150 (C.1) 
$$\begin{pmatrix} -XA - A^\top X & -XB + C^\top \\ -B^\top X + C & D + D^\top \end{pmatrix} \succcurlyeq 0, \quad X = X^\top \succ 0.$$

1151 for some  $X$ . It is said *strictly* passive if the first matrix inequality is satisfied with  $\succ 0$   
 1152 (this implies that  $D \succ 0$ ). This is not to be confused with the *strict state* passivity [18,  
 1153 Definition 4.54] (strict passivity implies strict state passivity). In fact strict passivity  
 1154 is directly related to strongly SPR transfer matrixess [18, p.62 and section 3.12.2],  
 1155 while SPR transfer matrices with minimal realization are strictly state passive [18,  
 1156 Theorem 4.73] [43]. Strict state passivity holds when  $-XA - A^\top X \succ 0$  in (C.1), with  
 1157  $D + D^\top \succcurlyeq 0$ . The two inequalities in (C.1) make a linear matrix inequality (LMI) with  
 1158 unknown  $X$ , equivalently a Riccati inequality using the Schur complement Theorem  
 1159 [18, Theorem A.65]. Numerical algorithms exist that efficiently solve LMIs, see [13],  
 1160 and can be used to test whether or not a quadruple is passive.

1161 REFERENCES

1162 [1] V. Acary, O. Bonnefon, M. Brémond, O. Huber, F. Péron, and S. Sinclair. An introduction  
 1163 to siconos. Technical Report RT-0340, INRIA, Univ. Grenoble Alpes, Grenoble, France,  
 1164 2019. <https://hal.inria.fr/inria-00162911v3/document>.  
 1165 [2] V. Acary, O. Bonnefon, and B. Brogliato. *Nonsmooth Modeling and Simulation for Switched*  
 1166 *Circuits*, volume 69 of *LNEE*. Springer, 2011. Erratum at <https://hal.inria.fr/hal-01311078>.  
 1167 [3] V. Acary and B. Brogliato. *Numerical Methods for Nonsmooth Dynamical Systems. Applica-*  
 1168 *tions in Mechanics and Electronics*, volume 35 of *Lecture Notes in Applied and Computa-*  
 1169 *tional Mechanics*. Springer Verlag, Berlin Heidelberg, 2008.  
 1170 [4] K. Addi, B. Brogliato, and D. Goeleven. A qualitative mathematical analysis of a class of linear  
 1171 variational inequalities via semi-complementarity problems: applications in electronics.  
 1172 *Mathematical Programming A*, 126:31–67, 2011.  
 1173 [5] S. Adly, D. Goeleven, and R. Oujja. Well-posedness of nonsmooth Lurie dynamical systems  
 1174 involving maximal monotone operators. In I.N. Parasidis, E. Providis, and T.M. Rassias,  
 1175 editors, *Mathematical Analysis in Interdisciplinary Research*, volume 179 of *Optimization*  
 1176 *and its Applications*, pages 47–68. Springer Cham, 2021.  
 1177 [6] S. Adly, A. Hantoute, and B.K. Le. Maximal monotonicity and cyclic monotonicity arising in  
 1178 nonsmooth Lur’e dynamical systems. *Journal of Mathematical Analysis and Applications*,  
 1179 448:691–706, 2017.  
 1180 [7] C. Arevalo and P. Lötstedt. Improving the accuracy of BDF methods for index 3 differential-  
 1181 algebraic equations. *BIT*, 35:297–308, 1995.  
 1182 [8] J. Bastien and M. Schatzman. Numerical precision for differential inclusions with uniqueness.  
 1183 *ESAIM Mathematical Modelling and Numerical Analysis*, 36(3):427–460, 2002.

- 1184 [9] H.H. Bauschke and P.L. Combettes. *Convex Analysis and Monotone Operator Theory in Hilbert*  
1185 *Spaces*. CMS Books in Mathematics. Canadian Mathematical Society. Springer, 2011.
- 1186 [10] D.S. Bernstein. *Matrix Mathematics. Theory, Facts, and Formulas*. Princeton University Press,  
1187 2nd edition, 2009.
- 1188 [11] D.P. Bertsekas. *Convex Optimization Theory*. Athena Scientific, Belmont, 2009.
- 1189 [12] W. Blajer and K. Kolodziejczyk. A geometric approach to solving problems of control con-  
1190 straints: Theory and a DAE framework. *Multibody System Dynamics*, 11:343–364, 2004.
- 1191 [13] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakishnan. *Linear Matrix Inequalities in System*  
1192 *and Control Theory*. Society for Industrial and Applied Mathematics, Philadelphia, USA,  
1193 1994.
- 1194 [14] K.E. Brenan. Numerical simulation of trajectory prescribed path control problems by the  
1195 backward differentiation formulas. *IEEE Trans. on Aut. Control*, 31(3):266–269, 1986.
- 1196 [15] H. Brézis and A. Haraux. Image d’une somme d’opérateurs monotones et applications. *Israel*  
1197 *Journal of Mathematics*, 23(2):165–186, 1976.
- 1198 [16] B. Brogliato and D. Goeleven. Well-posedness, stability and invariance results for a class of mul-  
1199 tivalued Lur’e dynamical systems. *Nonlinear Analysis. Theory, Methods and Applications*,  
1200 74:195–212, 2011.
- 1201 [17] B. Brogliato and D. Goeleven. Singular mass matrix and redundant constraints in unilaterally  
1202 constrained Lagrangian and Hamiltonian systems. *Multibody System Dynamics*, 35:39–61,  
1203 2015.
- 1204 [18] B. Brogliato, R. Lozano, B. Maschke, and O. Egeland. *Dissipative Systems Analysis and*  
1205 *Control. Theory and Applications*. Springer Nature Switzerland AG, Cham, 3rd edition,  
1206 2020. Erratum at <https://hal.inria.fr/hal-03364924/document>.
- 1207 [19] B. Brogliato and A. Tanwani. Dynamical systems coupled with monotone set-valued operators:  
1208 Formalisms, applications, well-posedness, and stability. *SIAM Review*, 62(1):3–129, 2020.
- 1209 [20] M.K. Camlibel and R. Frasca. Extension of Kalman-Yakubovich-Popov Lemma to descriptor  
1210 systems. *Systems and Control Letters*, 58:795–803, 2009.
- 1211 [21] M.K. Camlibel, W.P.M.H. Heemels, and J.M. Schumacher. Consistency of a time-stepping  
1212 method for a class of piecewise linear networks. *IEEE Transactions on Circuits and Sys-*  
1213 *tems I*, 49(3):349–357, 2002.
- 1214 [22] M.K. Camlibel, L. Iannelli, A. Tanwani, and S. Trenn. Differential-algebraic inclusions with  
1215 maximal monotone operators. In *Proceedings of the 55th IEEE Conference on Decision*  
1216 *and Control*, pages 610–615, Las Vegas, USA, 2016.
- 1217 [23] M.K. Camlibel and J.M. Schumacher. Linear passive and maximal monotone mappings. *Math-*  
1218 *ematical Programming Ser. B*, 157:397–420, 2016.
- 1219 [24] X. Chen and Z. Wang. Convergence of regularized time-stepping methods for differential vari-  
1220 ational inequalities. *SIAM Journal on Optimization*, 23(3):1647–1671, 2013.
- 1221 [25] X. Chen and S. Xiang. Perturbation bounds of P-matrix linear complementarity problems.  
1222 *SIAM Journal on Optimization*, 18(4):1250–1265, 2007.
- 1223 [26] A. Dontchev, T.Z. Dontchev, and I. Slavov. A Tikhonov theorem for singularly perturbed  
1224 differential inclusions. *Nonlinear Analysis, Theory, Methods and Applications*, 26(9):1547–  
1225 1554, 1996.
- 1226 [27] A.L. Dontchev. On existence of solutions of parametrized generalized equations. *Set-Valued*  
1227 *and Variational Analysis*, 29:735–744, 2021.
- 1228 [28] A.L. Dontchev and R.T. Rockafellar. *Implicit Functions and Solution Mappings. A View*  
1229 *from Variational Analysis*. Operations Research and Financial Engineering. Springer Sci-  
1230 ence+Business Media, New York, 2nd edition, 2014.
- 1231 [29] F. Facchinei and J.S. Pang. *Finite-Dimensional Variational Inequalities and Complementarity*  
1232 *Problems, Volume I*. Series in Operations Research. Springer-Verlag, 2003.
- 1233 [30] C.W. Gear and L.R. Petzold. ODE methods for the solution of differential/algebraic systems.  
1234 *SIAM J. Numer. Analysis*, 21(4):716–728, 1984.
- 1235 [31] D. Goeleven. *Complementarity and Variational Inequalities in Electronics*. Mathematical  
1236 Analysis and its Applications. Academic Press, 2017.
- 1237 [32] G. Grammel. Singularly perturbed differential inclusions: an averaging approach. *Set-Valued*  
1238 *Analysis*, 4:361–374, 1996.
- 1239 [33] C. Hachtel, A. Bartel, M. Günter, and A. Sandu. Multirate implicit Euler schemes for a  
1240 class of differential-algebraic equations of index-1. *Journal of Computational and Applied*  
1241 *Mathematics*, 387:112499, 2021.
- 1242 [34] L. Han, A. Tiwari, M.K. Camlibel, and J.S. Pang. Convergence of time-stepping schemes  
1243 for passive and extended linear complementarity systems. *SIAM Journal on Numerical*  
1244 *Analysis*, 47(5):3768–3796, 2009.
- 1245 [35] E. Hockerdal, E. Frisk, and L. Ericsson. Real-time performance of DAE and ODE

1246 based estimators evaluated on a diesel engine. *Science China. Information Sciences*,  
 1247 61:70202:1–070202:13, 2018.

1248 [36] A.N. Kolmogorov and S.V. Fomin. *Elements of the Theory of Functions and Functional Analy-*  
 1249 *sis*, volume 1 and 2. Graylock Press, 1957, 1961. also Dover Inc., 1999.

1250 [37] H. Krishnan and N.H. McClamroch. Computation of state realizations for control systems  
 1251 described by a class of linear differential-algebraic equations. *International Journal of*  
 1252 *Control*, 55(6):1425–1441, 1992.

1253 [38] C. Landry, A. Caboussat, and E. Hairer. Solving optimization-constrained differential equations  
 1254 with discontinuity points, with application to atmospheric chemistry. *SIAM Journal on*  
 1255 *Scientific Computing*, 31(5):3806–3826, 2009.

1256 [39] B.K. Le. On a class of Lur’e dynamical systems with state-dependent set-valued feedback.  
 1257 *Set-Valued and Variational Analysis*, 28(3):537–557, 2020.

1258 [40] B.K. Le. Well-posedness and nonsmooth Lyapunov pairs for state-dependent maximal mono-  
 1259 tone differential inclusions. *Optimization. A Journal of Mathematical Programming and*  
 1260 *Operations Research*, 69(6):1187–1217, 2020.

1261 [41] D.M.W. Leenaerts. On linear dynamic complementary system. *IEEE Transactions on Circuits*  
 1262 *and Systems I: Fundamental Theory and Applications*, 46(8):1022–1026, 1999.

1263 [42] P. Lötstedt and L. Petzold. Numerical solution of nonlinear differential equations with algebraic  
 1264 constraints I: Convergence results for backward differentiation formulas. *Mathematics of*  
 1265 *Computation*, 46(174):491–516, 1986.

1266 [43] D.S. Madeira and J. Adamy. On the equivalence between strict positive realness and strict  
 1267 passivity of linear systems. *IEEE Transactions on Automatic Control*, 61(10):3091–3095,  
 1268 2016.

1269 [44] M.D.P. Monteiro-Marques. *Differential Inclusions in Nonsmooth Mechanical Problems. Shocks*  
 1270 *and Dry Friction*, volume 9 of *Prog. Nonlinear Differential Equations Appl.* Birkhauser,  
 1271 Basel, CH, 1993.

1272 [45] J.J. Moreau. Evolution problem associated with a moving convex set in a Hilbert space. *Journal*  
 1273 *of Differential Equations*, 26(3):347–374, 1977.

1274 [46] J.S. Pang and D.E. Stewart. Differential variational inequalities. *Mathematical Programming*  
 1275 *ser. A*, 113:345–424, 2008.

1276 [47] J. Peypouquet and S. Sorin. Evolution equations for maximal monotone operators: Asymptotic  
 1277 analysis in continuous and discrete time. *J. of Convex Analysis*, 17(3-4):1113–1163, 2010.

1278 [48] A. Rocca, V. Acary, and B. Brogliato. Index-2 hybrid DAE: a case study with well-posedness  
 1279 and numerical analysis. In *Proc. 21st IFAC World Congress*, Berlin, Germany, 2020. IFAC.

1280 [49] R.T. Rockafellar. *Convex Analysis*. Princeton University Press, 1970.

1281 [50] R.T. Rockafellar and R.J.B. Wets. *Variational Analysis*, volume 317 of *Grundlehren der math-*  
 1282 *ematischen Wissenschaften*. Springer Verlag, Berlin Heidelberg, 1998.

1283 [51] P. Stechliniski, M. Patrascu, and B.I. Barton. Nonsmooth DAEs with applications in mod-  
 1284 eling phase changes. In Campbell S., Ilchmann A., Mehrmann V., and Reis T., editors,  
 1285 *Applications of Differential-Algebraic Equations: Examples and Benchmarks*, Differential-  
 1286 Algebraic Equations Forum, pages 243–275. Springer, Cham, CH 2018.

1287 [52] P. Stechliniski, M. Patrascu, and P.I. Barton. Nonsmooth differential-algebraic equations in  
 1288 chemical engineering. *Computers and Chemical Engineering*, 114:52–68, 2018.

1289 [53] P.G. Stechliniski and P.I. Barton. Generalized derivatives of differential-algebraic equations.  
 1290 *Journal of Optimization Theory and Applications*, 171:1–26, 2016.

1291 [54] P.G. Stechliniski and P.I. Barton. Dependence of solutions of nonsmooth differential-algebraic  
 1292 equations on parameters. *Journal of Differential Equations*, 262:2254–2285, 2017.

1293 [55] A. Tanwani, B. Brogliato, and C. Prieur. Well-posedness and output regulation for implicit  
 1294 time-varying evolution variational inequalities. *SIAM Journal on Control and Optimiza-*  
 1295 *tion*, 56(2):751–781, 2018.

1296 [56] V. Veliov. A generalization of the Tikhonov theorem for singularly perturbed differential in-  
 1297 clusions. *Journal of Dynamical and Control Systems*, 3(3):291–319, 1997.

1298 [57] P.T. Vuong. A second order dynamical system and its discretization for strongly pseudo-  
 1299 monotone variational inequalities. *SIAM J. on Control and Optimization*, 59(4):2875–2897,  
 1300 2021.

1301 [58] F. Watbled. On singular perturbations for differential inclusions on the infinite interval. *Journal*  
 1302 *of Mathematical Analysis and Applications*, 310:362–37, 2005.

1303 [59] S.L. Wu, T. Zhou, and X. Chen. A Gauss-Seidel type method for dynamic nonlinear comple-  
 1304 mentarity problems. *SIAM Journal on Control and Optimization*, 58(6):3389–3412,  
 1305 2020.

1306 [60] W. Xing, Q. Zhang, J. Zhang, and Q. Wang. Some geometric properties of lyapunov equation  
 1307 and lti system. *Automatica*, 37:313–316, 2001.

- 1308 [61] A. Zelei, L.L. Kovacs, and G. Stepan. Computed torque control of an under-actuated service  
1309 robot platform modeled by natural coordinates. *Commun. Nonlinear Sci. Numer. Simulat.*,  
1310 16:2205–2217, 2011.
- 1311 [62] X. Zhao, T. Ploch, S. Noack, W. Wiechert, A. Mitsos, and E. von Lieres. Analysis of the  
1312 local well-posedness of optimization-constrained differential equations by local optimality  
1313 conditions. *AIChE Journal*, 66:e16548, 2020.