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# On Stability of Homogeneous Systems in Presence of Parasitic Dynamics

Jesús Mendoza-Avila *Member, IEEE,* Denis Efimov *Senior Member, IEEE,* Leonid Fridman *Member, IEEE,* and Jaime A. Moreno *Member, IEEE.*

***Abstract*—In this paper, the effect of a homogeneous parasitic dynamics on the stability of a homogeneous system, when homogeneity degrees are possibly different, is studied via ISS approach in the framework of singular perturbations. Thus, the possibilities to reduce the order of the interconnected system considering only the reduced-order dynamics and neglecting the parasitic ones are examined. Proposed analysis discovers three kinds of stability in the behavior of such an interconnection by assuming that both, reduced-order and unforced parasitic dynamics, are globally asymptotically stable. In the first case, global asymptotic stability of the interconnection can be concluded, when the homogeneity degrees of both systems coincide and the singular perturbation parameter is small enough. In the second case, only local stability of interconnection can be ensured, when the homogeneity degree of reduced-order dynamics is greater than the homogeneity degrees of the parasitic ones. Helpfully, the proposed approach allows to estimate the domain of attraction for the system's trajectories as a function of the singular perturbation parameter. In the third case, only practical stability can be guaranteed, when the homogeneity degree of reduced-order dynamics is smaller than the homogeneity degrees of the parasitic ones. Additionally, the proposed approach provides an estimation of the asymptotic bound of the system's trajectories in terms of the singular perturbation parameter.**

***Index Terms*—Homogeneity; Interconnected Systems; Singular Perturbations; Chattering Analysis.**

## I. Introduction

Homogeneous systems [1], [2], [3] constitute a subclass of nonlinear dynamics admitting special properties, e.g., scalability of solutions or different rates of convergence: rational (or nearly fixed-time), exponential and finite-time. Particularly, homogeneous controllers of negative degree are not Lipschitz continuous, possessing an infinite gain near to the origin and providing finite-time convergence [4], [5], [6], [7]. However, it is a well-known fact that the presence of linear Parasitic Dynamics (PD) deteriorates the performance of finite-time convergent systems and the so-called chattering (a high-frequency oscillatory behavior of a system in steady-state) may arise [8], [9], [10], [11], [12], [13], [14], [15]. Consequently, the problems of chattering reduction and its analysis have attracted a lot of attention, see for example [10], [11], [12], [16], [17].

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Commonly, singularly perturbed models are used in face of a decomposition of interconnected systems into the Main Dynamics (MD) and the PD (see the references of [18]). For smooth singularly perturbed systems, methods of stability analysis are based on Klimushchev-Krasvoskii Theorem [19], where asymptotic stability of the interconnection is concluded from uniform exponential stability of the linearized systems. Relaxing the last assumption, [20] addresses asymptotic and exponential stability by means of quadratic-type Lyapunov Functions (LF's) and estimates the domain of attraction in terms of the upper bound of the Singular Perturbation Parameter (SPP).

In the framework of *Input-to-State Stability* (ISS), the work in [21] has studied the properties of smooth singularly perturbed systems providing a kind of "total stability" under standard assumptions. The concept of ISS was introduced to analyze the stability of systems affected by external inputs, e.g., exogenous disturbances or measurement noises (see the list of references in [22]). The connection between Lyapunov stability and ISS has permitted the development of many stability concepts, which have been found to be very useful for the analysis and design of nonlinear control systems [23], [24]. For instance, the so-called *Small-Gain Theorem* provides a sufficient condition to guarantee the stability of interconnected systems [23], [25], [26], [27].

The results of [28] related the ISS concepts with homogeneous systems. Moreover, [29] has investigated the robustness properties of finite-time stable homogeneous systems like uniform stability for small inputs and finite-time strongly integral ISS. In addition, it is concerned with the robustness analysis of locally homogeneous systems, and finite-time stable systems in cascade connection.

Nevertheless, despite the amount of publications on singular perturbations theory, those results do not cover the general case of homogeneous systems. The reasons for this include: the homogeneous systems can be non-smooth, the velocity of convergence of homogeneous systems is parametrized by their Homogeneity Degree (HD), the homogeneous systems of negative degree usually exhibit chattering in the presence of linear PD, and around the origin, the homogeneous systems of positive degree are slower than any exponentially converging dynamics, but they are faster in the large.

In order to fill this gap, in [30] we have presented a study of the stability of the interconnection of a homogeneous MD and a linear-like homogeneous PD by means of concepts of singular perturbations, Lyapunov methods, ISS properties and Small-Gain Theorem. That analysis requires just continuity of the MD but continuous differentiability of the PD.

In the present paper, we provide a study of the stability of an interconnection where both the MD and the PD are nonlinear, continuous and homogeneous. This is an extension of our previous results to a more general class of systems by following the same methodology. The obtained results allow to conclude three kinds of stability for the interconnection depending on the relationship between the HD of MD and PD: 1) Global Asymptotic Stability (GAS) when the HD's coincide and the SPP is sufficiently small. 2) Practical GAS when the HD of the MD is smaller, where the asymptotic bound of the trajectories is a monotonically increasing function of the SPP. 3) Local asymptotic stability when the HD of the PD is smaller, where the size of the domain of attraction is a monotonically decreasing function of the SPP.

Moreover, this work provides a more complete, detailed and accurate proof of the main theorem than the previous version. In addition, some examples are presented with the aim to illustrate the different cases of the main theorem.

The rest of the paper has the following structure. In Section II some useful definitions and preparatory results are presented. Section III contains the problem statement and results about the stability of the interconnection of homogeneous systems affected by singular perturbations. In Section IV, some examples are provided in order to illustrate all above mentioned cases. Finally, the conclusions are presented in Section V.

*Notation:*

- $\mathbb{N}$ and $\mathbb{R}$ are the sets of natural and real numbers, respectively. Moreover, $\mathbb{R}_+$ represents the set of non-negative real numbers, i.e., $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$.
- $|\cdot|$ denotes the absolute value in $\mathbb{R}$, $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$.
- For any $\gamma \in \mathbb{R}$, expressions like $|\bullet|^{\gamma} sign(\bullet)$ are written as $\lceil\bullet\rfloor^{\gamma}$.
- A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. The function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is unbounded. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ if, for each fixed $t \in \mathbb{R}_+$, $\beta(\cdot, t) \in \mathcal{K}$ and, for each fixed $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ is non-increasing and it tends to zero for $t \to \infty$.
- The space $\mathcal{L}_\infty^m$ is defined as the set of measurable essentially bounded functions $u : [0, \infty) \to \mathbb{R}^m$, such that, $\|u\|_{\mathcal{L}_\infty} = ess\sup_{t \geq 0} |u(t)| < \infty$.
- $s(A)$ represents the eigenvalues of a symmetric matrix $A \in \mathbb{R}^{n \times n}$. Also, $s_{\min}(A)$ and $s_{\max}(A)$ correspond to the minimum and the maximum eigenvalue of $A$, respectively.
- For $\vartheta, \varrho \in \mathcal{K}$, the expression $\vartheta \circ \varrho(\cdot) = \vartheta(\varrho(\cdot))$ denotes the composition of $\varrho$ and $\vartheta$.

## II. Preliminaries

### A. Weighted homogeneity

The presentation of this subsection follows [1], [28], [3]. For real numbers $r_i > 0 \ \ (i = 1, \ldots, n)$ called weights and $\lambda > 0$, one can define

- the vector of weights $r = (r_1, \ldots, r_n)^T$, $r_{\max} = \max_{1 \leq j \leq n} r_j$ and $r_{\min} = \min_{1 \leq j \leq n} r_j$;
- the dilation matrix function $\Lambda_r(\lambda) = \mathrm{diag}(\lambda^{r_i})_{i=1}^n$, such that, for all $x \in \mathbb{R}^n$ and for all $\lambda > 0$, $\Lambda_r(\lambda)x = (\lambda^{r_1}x_1, \ldots, \lambda^{r_i}x_i, \ldots, \lambda^{r_n}x_n)^T$ (throughout this paper the dilation matrix is represented by $\Lambda_r$ wherever $\lambda$ can be omitted);
- the $r$-homogeneous norm of $x \in \mathbb{R}^n$ is given by $||x||_r = \left(\sum_{i=1}^n |x_i|^{\frac{\rho}{r_i}}\right)^{\frac{1}{\rho}}$ for $\rho \geq r_{\max}$. It is not a norm in the usual sense, since it does not satisfy the triangle inequality;
- for $s > 0$, the sphere and the ball in the homogeneous norm are defined as $S_r(s) = \{x \in \mathbb{R}^n : ||x||_r = s\}$ and $B_r(s) = \{x \in \mathbb{R}^n : ||x||_r \leq s\}$, respectively.

**Definition 1.** *A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is $r$-homogeneous with a degree $\mu \in \mathbb{R}$ if for all $\lambda > 0$ and all $x \in \mathbb{R}^n$:*

$$\phi(\Lambda_r(\lambda)x) = \lambda^{\mu}\phi(x). \tag{1}$$

*A vector field $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is $r$-homogeneous with a degree $\nu \geq -r_{\min}$ if for all $x \in \mathbb{R}^n$ and all $\lambda > 0$,*

$$\Phi(\Lambda_r(\lambda)x) = \lambda^{\nu}\Lambda_r(\lambda)\Phi(x). \tag{2}$$

*The system $\dot{x} = \Phi(x)$ is $r$-homogeneous of degree $\nu$ if the vector field $\Phi$ satisfies the property (2).*

By its definition, the norm $||\cdot||_r$ is an $r$-homogeneous function of degree 1, besides, for $r_{max} = 1$, it is locally Lipschitz continuous [31]. Moreover, there exist $\underline{\sigma}, \bar{\sigma} \in \mathcal{K}_\infty$, such that,

$$\underline{\sigma}(||x||_r) \leq ||x|| \leq \bar{\sigma}(||x||_r) \qquad \forall x \in \mathbb{R}^n, \tag{3}$$

i.e., there is a relationship between the norms $||\cdot||$ and $||\cdot||_r$. A particular selection of $\bar{\sigma}$ and $\underline{\sigma}$ can be found in [31, Proposition 1] or [2, Lemma 7.2]. See [1], [2] for more details about homogeneous norms.

### B. Input-to-state stability

Consider a system

$$\dot{x} = f(x, u), \tag{4}$$

where $x \in \mathbb{R}^n$ is the state vector and $u \in \mathbb{R}^m$ is an input. In addition, $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is supposed to ensure forward existence and uniqueness of the system solutions at least locally in time, and $f(0, 0) = 0$. The next definitions and theorems were introduced by [23], [26], [28].

**Definition 2.** *The system (4) is said to be input-to-state practically stable (ISpS), if there exist a class $\mathcal{KL}$ function $\beta$, a class $\mathcal{K}$ function $\gamma$ and a constant $c \geq 0$, such that, for any $u \in \mathcal{L}_\infty^m$ and any $x_0 \in \mathbb{R}^n$, the solution $x(t)$ with initial condition $x(0) = x_0$ satisfies*

$$\|x(t)\| \leq \max\{\beta(\|x_0\|, t), \gamma(\|u\|_{\mathcal{L}_\infty}), c\} \tag{5}$$

*for all $t \geq 0$. The system (4) is called ISS if $c = 0$. If the estimate (5) is satisfied just for $c = 0$, $\|x_0\| \leq \rho_0$ and $\|u\|_{\mathcal{L}_\infty} \leq \rho_u$, for some $\rho_0, \rho_u > 0$, then the system (4) is called local ISS. The function $\gamma$ is called nonlinear asymptotic gain.*

If $u(t) = 0$ for all $t \geq 0$, then ISpS implies practical GAS, ISS implies GAS, and local ISS implies local asymptotic stability.

**Definition 3.** *A smooth function $V : \mathbb{R}^n \to \mathbb{R}$ is called an ISpS-Lyapunov function (ISpS-LF) for the system (4) if there exist some $d \geq 0$, $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\chi \in \mathcal{K}$, such that, for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \tag{6}$$

*and*

$$\|x\| \geq \chi(\max\{\|u\|, d\}) \implies \tfrac{\partial V}{\partial x} f(x,u) \leq -\alpha_3(\|x\|) \tag{7}$$

*hold. Moreover, if $d = 0$ then $V$ is called an ISS-LF for the system (4). If $V(x)$ satisfies* (7) *only for $d = 0$, $\|x_0\| \leq \rho_0$ and $\|u\|_\infty \leq \rho_u$, for some $\rho_0, \rho_u > 0$, then it is called a local ISS-LF for the system (4).*

**Remark 1.** *The function $\gamma(\cdot)$ in (5) can be computed from the functions $\alpha_1(\cdot), \alpha_2(\cdot)$ and $\chi(\cdot)$. It is given by*

$$\gamma(s) = \alpha_1^{-1} \circ \alpha_2 \circ \chi(s). \tag{8}$$

**Theorem 1.** *The system (4) is (local, practical) ISS if it admits an (local, practical) ISS-LF.*

**Remark 2.** *Due to the relationship of the $r$-homogeneous norm and the Euclidean one in* (3)*, we can substitute $\|\cdot\|$ by $\|\cdot\|_r$ in Definitions 2 and 3. For instance, from* (3) *we can find some functions $\beta' \in \mathcal{KL}$, $\gamma' \in \mathcal{K}_\infty$ and a constant $c' \geq 0$ such that we can rewrite the estimate* (5) *in terms of the $r$-homogeneous norm. Similarly, by using* (3) *we can replace $\|\cdot\|$ by $\|\cdot\|_r$ in inequalities* (6) *and* (7) *with different constants and functions of the corresponding classes.*

### C. Input-to-state stability of interconnected systems

Consider a system

$$\dot{x} = f(x,y), \qquad \dot{y} = g(x,y,u), \tag{9}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $u \in \mathbb{R}^p$, the vector fields $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ and $g : \mathbb{R}^{n+m} \to \mathbb{R}^m$ are continuous and ensure forward existence and uniqueness of solutions. For $u = 0$, the origin is supposed to be an equilibrium point of (9). Moreover, for some vectors of weights $r$ and $\tilde{r}$ the vector fields $f$ and $g$ (for $u = 0$) are $(r, \tilde{r})$-homogeneous of degrees $\nu$ and $\mu$, respectively.

Assume that both systems in (9) are ISpS w.r.t. their corresponding inputs. Hence, from Definition 2 and Remark 2 there exists some functions $\beta_1, \beta_2 \in \mathcal{KL}$, $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}$ and constants $c_1, c_2 > 0$, such that for all $t \geq 0$:

$$\|x(t)\|_r \leq \max\{\beta_1(\|x_0\|_r, t), \gamma_1(\sup_{\tau\in[0,t)} \|y(\tau)\|_{\tilde{r}}), c_1\}, \tag{10}$$

$$\|y(t)\|_{\tilde{r}} \leq \max\{\beta_2(\|y_0\|_{\tilde{r}}, t), \gamma_2(\sup_{\tau\in[0,t)} \|x(\tau)\|_r), \gamma_3(\|u\|_{\mathcal{L}_\infty}), c_2\}, \tag{11}$$

where $x_0 \in \mathbb{R}^n$ and $y_0 \in \mathbb{R}^m$ are the initial conditions for each variable, and $u \in \mathcal{L}_\infty^p$ is an input. Likewise, to assume that each subsystem in (9) is locally ISS implies that there exist some $\rho_{01}, \rho_{02}, \rho_u > 0$ such that the estimates (10) and (11) are satisfied just for all $\|x_0\|_r \leq \rho_{01}$, $\|y_0\|_{\tilde{r}} \leq \rho_{02}$, and $\|u\|_{\mathcal{L}_\infty} \leq \rho_u$, for all $t \geq 0$, with $c_1 = c_2 = 0$.

**Corollary 1.** *Let each subsystem in (9) be ISpS (local ISS). If there exists some $c_0 \geq 0$ ($c_3 \geq 0$), such that,*

$$\gamma_2 \circ \gamma_1(s) < s, \quad \forall s > c_0 \quad (\forall\ s \in (0, c_3]), \tag{12}$$

*then the interconnected system (9) is ISpS. Furthermore, if $c_0 = c_1 = c_2 = 0$ then the system is (local) ISS.*

Corollary 1 fits the small-gain condition for $r$-homogeneous system when the (local, practical) ISS estimates are presented in terms of the $r$-homogeneous norm. We omit its proof due to the space limitation but the essential arguments are based on [25, Lemma 3.2], where positive definiteness is the only required property from the norm. So, this result can be seen as a corollary of the small-gain theorems presented by [25], [26], [27], [32].

Roughly speaking, small-gain theorems establish that the interconnected system (9) is ISS, if the composite function $\gamma_1 \circ \gamma_2(\cdot)$ is a simple contraction. The inequality (12) is commonly referred to as the *small-gain condition*.

## III. Stability analysis of interconnected homogeneous systems affected by singular perturbations

Consider an interconnected system

$$\dot{x} = f(x,y), \tag{13}$$

$$\epsilon\dot{y} = g(x,y), \tag{14}$$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ are the state variables, $\epsilon > 0$ is a SPP, $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ and $g : \mathbb{R}^{n+m} \to \mathbb{R}^m$ are continuous vector fields ensuring forward existence and uniqueness of system trajectories. Moreover, for some vectors of weights $r$ and $\tilde{r}$ the vector fields $f$ and $g$ are $(r, \tilde{r})$-homogeneous of degrees $\nu$ and $\mu$, respectively, i.e.,

$$\lambda^\nu \Lambda_r(\lambda) f(x,y) = f(\Lambda_r(\lambda)x, \Lambda_{\tilde{r}}(\lambda)y) \tag{15}$$

$$\lambda^\mu \Lambda_{\tilde{r}} g(x,y) = (\lambda) g(\Lambda_r(\lambda)x, \Lambda_{\tilde{r}}(\lambda)y), \tag{16}$$

for all $\lambda > 0$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. Hence, note that $f(0,0) = 0$ and $g(0,0) = 0$. Through this paper, the system (13) is called the MD and the system (14) is called the PD.

**Remark 3.** *The simplest example of a parasitic dynamics represented by a homogeneous mathematical model may be found in the family of critically-damped fast-actuators. These systems are commonly represented by a second-order linear system, which is homogeneous of degree 0. Thus, the implementation of a homogeneous controller in a plant with a critically-damped actuator matches perfectly with the characteristics that have been imposed to the system (12)-(13).*

In the following, three different problems will be addressed: analysis of the stability of the interconnected systems (13)-(14) based on their HD's, estimation of the ultimate bound for the system trajectories, and evaluation of the domain of attraction for the system trajectories.

Most of the existing results require sufficient smoothness of the vector fields involved in the analysis (see [18] and references therein). However, in the general case of homogeneous systems such a condition is quite conservative. Thus, in this study just continuity of the vector fields $f$, $g$ is assumed. Moreover, let us introduce the hypotheses for this research.

**Assumption 1.** *Consider the system (13)-(14):*

1) *The equation $g(\bar{x}, \bar{y}) = 0$ has an isolated and continuously differentiable solution $\bar{y} = h(\bar{x})$.*
2) *The Reduced-Order Dynamics (ROD)*
$$\dot{\bar{x}} = f(\bar{x}, h(\bar{x})), \tag{17}$$
*is GAS at the origin.*
3) *Consider $z = y - h(x)$. For the Boundary-Layer (BL) dynamics*
$$\tfrac{dz}{d\tau} = g(x, z + h(x)); \quad \tau = \tfrac{t}{\epsilon}, \tag{18}$$
*there exists a continuously differentiable and $(\tilde{r}, r)$-homogeneous LF $W : \mathbb{R}^{n+m} \to \mathbb{R}_+$ satisfying*
$$W(\Lambda_{\tilde{r}}(\lambda)z, \Lambda_r(\lambda)x) = \lambda^{\iota} W(z, x), \quad \forall \lambda > 0, \tag{19}$$
$$\underline{a}_z \|z\|_{\tilde{r}}^{\iota} \leq W(z, x) \leq \bar{a}_z \|z\|_{\tilde{r}}^{\iota}, \tag{20}$$
$$\tfrac{\partial W(z,x)}{\partial z} g(x, z + h(x)) \leq -b_z \|z\|_{\tilde{r}}^{\mu+\iota}, \tag{21}$$
*for all $z \in \mathbb{R}^m$, $x \in \mathbb{R}^n$ and for some $\underline{a}_z, \bar{a}_z, b_z > 0$, where $\iota > \max\{0, -\mu\}$ is the HD of $W$.*

As a consequence of (16), the function $h$ is homogeneous, i.e., $h(\Lambda_r(\lambda)x) = \Lambda_{\tilde{r}}(\lambda)h(x)$, and consequently, $h(0) = 0$. This implies that the ROD (17) is $r$-homogeneous of a degree $\nu$, i.e., the ROD inherits the homogeneity properties of the MD.

**Remark 4.** *The existence of such a LF $W(x, z)$, as it is described in Assumption 1, implies that the BL* (18) *is GAS at the origin, uniformly with respect to $x$. The latter is a usual assumption related to the stability of singularly perturbed systems.*

Under Assumption 1, the next theorem presents the main result of this paper:

**Theorem 2.** *Let the interconnected system (13)-(14) satisfy Assumption 1. There exist a constant $\epsilon^* > 0$ and some functions $\upsilon_1, \upsilon_2 \in \mathcal{K}_\infty$, such that the origin of the system (13)-(14) is*

- *Globally asymptotically stable if $\mu = \nu$ and $\epsilon < \epsilon^*$.*
- *Locally asymptotically stable if $\mu < \nu$, with a domain of attraction:*
$$\|x_0\|_r < 1/\upsilon_1(\epsilon), \quad \text{and} \quad \|y_0 - h(x_0)\|_{\tilde{r}} \leq 1/\upsilon_2(\epsilon).$$
- *Globally asymptotically practically stable if $\mu > \nu$, with asymptotic bounds:*
$$\|x(t)\|_r < \upsilon_1(\epsilon), \quad \text{and} \quad \|y(t) - h(x(t))\|_{\tilde{r}} \leq \upsilon_2(\epsilon),$$
*for all $t > T$, with some $T > 0$, all $x_0 \in \mathbb{R}^n$ and all $y_0 \in \mathbb{R}^m$.*

Hence, the condition $\nu = \mu$ implies the existence of a critical value $\epsilon^*$, such that, the stability of the system (13)-(14) is ensured for all $\epsilon < \epsilon^*$. The value of $\epsilon$ determines the amount of deviations for the trajectories of the system (13)-(14) from the trajectories of the ROD (17). Note that this case illustrates the concept of motion separation predicted by classical results on smooth (at least Lipschitz continuous) singularly perturbed systems [18]. So, it allows to ensure GAS.

For $\mu \neq \nu$, the system (13)-(14) always possesses some kind of stability (local or practical, independently in SPP), and by decreasing the value of $\epsilon$ it is possible to enlarge the domain of attraction for $\mu < \nu$, or to decrease the size of the asymptotic bounds for $\mu > \nu$. Systems with a smaller HD are faster around the origin and slower in the large, and vice versa. Therefore, the motion separation follows the relations of HD's (contrarily the standard theory of singular perturbations or the case $\nu = \mu$, where it is regulated by SPP) and it only appears near to the origin for $\nu > \mu$ and outside of a neighborhood of the origin for $\nu < \mu$.

## A. *Proof of Theorem 2*

Under a hypothesis like Assumption 1, the conventional singular perturbation theory aims to show that the behavior of the system (13)-(14) is pretty similar to the behavior of the ROD (17), such that, the presence of the PD can be neglected (see [18]). Thus, the stability of the system (13)-(14) is guaranteed by proving that the trajectories $y$ of the PD converge to the equilibrium manifold $h(x)$. However, since the initial condition $y_0$ differs from the initial value $h(x_0)$, there exists a transient response of the system (14) before it can reach the desired behavior. Let us represent such a transient by the variable $z$ such that $y = z + h(x)$ and the system (13) is rewritten as
$$\dot{x} = f(x, z + h(x)). \tag{22}$$
It can be readily seen that if $z = 0$, the system (22) collapses to the ROD. Therefore, (global, local or practical) asymptotic stability of the origin of the system (22) can be concluded from ISS stability of the system (22) w.r.t. $z$, plus (global, local or practical) vanishing of $z$ to zero.

Following [33], since the ROD (17) is assumed to be GAS at the origin and $r$-homogeneous of degree $\nu$, there exists a continuously differentiable LF $V : \mathbb{R}^n \to \mathbb{R}_+$, which satisfies
$$V(x) = \lambda^{-\kappa} V(\Lambda_r(\lambda)x), \quad \forall \lambda > 0, \tag{23}$$
$$\underline{a}_x \|x\|_r^{\kappa} \leq V(x) \leq \bar{a}_x \|x\|_r^{\kappa}, \tag{24}$$
$$\tfrac{\partial V(x)}{\partial x} f(x, h(x)) \leq -b_x \|x\|_r^{\nu+\kappa}, \tag{25}$$
$$\sup_{x \in B_r(1)} \|\tfrac{\partial V(x)}{\partial x}\| \leq c_x, \tag{26}$$
for all $x \in \mathbb{R}^n$ and for some $\underline{a}_x, \bar{a}_x, b_x, c_x > 0$, where $\kappa > \max\{0, -\nu\}$ is the HD of $V$. So, let us take $V$ as an ISS-LF candidate for the system (22). The derivative of $V$ along the trajectories of the system (22) is given by
$$\begin{aligned}\dot{V} =& \tfrac{\partial V(x)}{\partial x} f(x, z + h(x)) \\ =& \tfrac{\partial V(x)}{\partial x} f(x, h(x)) + \tfrac{\partial V(x)}{\partial x} \left(f(x, z + h(x)) - f(x, h(x))\right).\end{aligned}$$

Then, considering (15) and (23), and by applying the dilations $\Lambda_r(\lambda)$ and $\Lambda_{\tilde{r}}(\lambda)$, where $\lambda = \|x\|_r^{-1}$, it is obtained

$$\begin{aligned}\dot{V} =& \tfrac{\partial V(x)}{\partial x} f(x,h(x)) + \|x\|_r^{\nu+\kappa} \tfrac{\partial V(\xi)}{\partial \xi} \\ & \times \left( f(\xi, \Lambda_{\tilde{r}}(\|x\|_r^{-1})z + h(\xi)) - f(\xi, h(\xi)\right) \\ \leq & \tfrac{\partial V(x)}{\partial x} f(x,h(x)) + \|x\|_r^{\nu+\kappa} \left\| \tfrac{\partial V(\xi)}{\partial \xi} \right\| \\ & \times \left\| f(\xi, \Lambda_{\tilde{r}}(\|x\|_r^{-1})z + h(\xi)) - f(\xi, h(\xi) \right\|, \end{aligned} \tag{27}$$

where $\xi = \Lambda_r(\|x\|_r^{-1})x \in S_r(1)$.

By the continuity of $f(x,y)$, for any $b_x, c_x$ and $0 < \theta < 1$ there exists $\delta > 0$, such that, if $\|\Lambda_{\tilde{r}}(\|x\|_r^{-1})z\|_{\tilde{r}} \leq \delta$, then

$$\|f(\xi, h(\xi) + \Lambda_{\tilde{r}}(\|x\|_r^{-1})z) - f(\xi, h(\xi))\| \leq \frac{\theta b_x}{c_x}, \tag{28}$$

for all $\xi \in S_r(1)$. Recall that $\|\cdot\|_{\tilde{r}}$ is $\tilde{r}$-homogeneous of degree 1, then the expression $\|\Lambda_{\tilde{r}}(\|x\|_r^{-1})z\|_{\tilde{r}} \leq \delta$, can be rewritten as $\|z\|_{\tilde{r}} \leq \delta \|x\|_r$.

Therefore, by substituting (25), (26) and (28) in (27), it can be concluded that

$$\begin{aligned}\dot{V} \leq & - b_x \|x\|_r^{\nu+\kappa} + \theta b_x \|x\|_r^{\nu+\kappa} \\ \leq & - (1-\theta) b_x \|x\|_r^{\nu+\kappa}, \quad \text{if} \quad \|x\|_r \geq \delta^{-1}\|z\|_{\tilde{r}}. \end{aligned} \tag{29}$$

Through the relationship (3), it is a straightforward exercise to check that $V$ satisfies the conditions of Definition 3, so it is an ISS-LF for the system (22). Hence, from Definition 2 and considering the relationship (3) the solution $x$ of the system (22) satisfies

$$\|x(t)\|_r \leq \max\{\beta_1(\|x_0\|_r, t), \gamma_1(\sup_{\tau\in[0,t)} \|z(\tau)\|_{\tilde{r}})\}, \tag{30}$$

for all $t \geq 0$, where $\beta_1$ is a $\mathcal{KL}$ function and, from Remark 1 and inequalities (24) and (29), $\gamma_1$ is a $\mathcal{K}$ function given by

$$\gamma_1(s) = \delta^{-1} \frac{\bar{a}_x}{\underline{a}_x} s. \tag{31}$$

Now, let us investigate the scenarios where vanishing of the transient $z$ to zero can be guaranteed such that the convergence of the trajectories $y$ to the desired behavior $h(x)$ can be concluded. The dynamics of the variable $z$ is given by

$$\dot{z} = \tfrac{1}{\epsilon} g(x, z+h(x)) - \tfrac{\partial h(x)}{\partial x} f(x, z+h(x)), \tag{32}$$

where $x$ can be seen as an input.

The stability of the system (32) can be investigated by using $W$, described in (19)-(21) as an ISS-LF candidate. So, the derivative of $W$ along the trajectories of the system (32) is given by

$$\begin{aligned}\dot{W} =& \tfrac{1}{\epsilon} \tfrac{\partial W(z,x)}{\partial z} g(x, z+h(x)) \\ & + \left( \tfrac{\partial W(z,x)}{\partial x} - \tfrac{\partial W(z,x)}{\partial z} \tfrac{\partial h(x)}{\partial x} \right) f(x, z+h(x)). \end{aligned} \tag{33}$$

By homogeneity of each component in (33), applying the dilations $\Lambda_r(\lambda^{-1})$ and $\Lambda_{\tilde{r}}(\lambda^{-1})$, where $\lambda = \max\{\|z\|_{\tilde{r}}, \|x\|_r\}$, we obtain:

$$\begin{aligned}\dot{W} =& \tfrac{1}{\epsilon} \tfrac{\partial W(z,x)}{\partial z} g(x, z+h(x)) + \lambda^{\nu+\iota} \Big( \tfrac{\partial W(\zeta,\xi)}{\partial \xi} \\ & - \tfrac{\partial W(\zeta,\xi)}{\partial \zeta} \tfrac{\partial h(\xi)}{\partial \xi} \Big) f(\xi, \zeta + h(\xi)) \\ \leq & \tfrac{1}{\epsilon} \tfrac{\partial W(z,x)}{\partial z} g(x, z+h(x)) + \lambda^{\nu+\iota} \Big( \left\| \tfrac{\partial W(\zeta,\xi)}{\partial \xi} \right\| \\ & + \left\| \tfrac{\partial W(\zeta,\xi)}{\partial \zeta} \right\| \left\| \tfrac{\partial h(\xi)}{\partial \xi} \right\| \Big) \|f(\xi, \zeta+h(\xi))\|, \end{aligned} \tag{34}$$

where $\xi = \Lambda_r^{-1}(\lambda)x$ and $\zeta = \Lambda_{\tilde{r}}^{-1}(\lambda)z$ (i.e., $\xi \in B_r(1)$ and $\zeta \in B_{\tilde{r}}(1)$).

So, by substituting (21) in (34), we obtain

$$\dot{W} \leq -\tfrac{b_z}{\epsilon} \|z\|_{\tilde{r}}^{\mu+\iota} + b_z \eta \lambda^{\nu+\iota},$$

where $\eta$ is given by

$$\begin{aligned}\eta = \frac{1}{b_z} \sup_{\substack{\xi\in B_r(1) \\ \zeta \in B_{\tilde{r}}(1)}} & \Big\{ \Big( \left\| \tfrac{\partial W(\zeta,\xi)}{\partial \xi} \right\| \\ & + \left\| \tfrac{\partial W(\zeta,\xi)}{\partial \zeta} \right\| \left\| \tfrac{\partial h(\xi)}{\partial \xi} \right\| \Big) \|f(\xi, \zeta + h(\xi))\| \Big\}. \end{aligned} \tag{35}$$

Hence, for any $0 < \tilde{\theta} < 1$, it is obtained that

$$\begin{aligned}\dot{W} \leq & -(1-\tilde{\theta}) \tfrac{b_z}{\epsilon} \|z\|_{\tilde{r}}^{\mu+\iota} - \tilde{\theta} \tfrac{b_z}{\epsilon} \|z\|_{\tilde{r}}^{\mu+\iota} \\ & + b_z \eta \max\{\|z\|_{\tilde{r}}^{\nu+\iota}, \|x\|_r^{\nu+\iota}\}. \end{aligned}$$

Since $\max\{a,b\} \leq a+b$ for any $a, b \in \mathbb{R}_+$, the last expression can be rewritten as

$$\begin{aligned}\dot{W} \leq & -(1-\tilde{\theta}) \tfrac{b_z}{\epsilon} \|z\|_{\tilde{r}}^{\mu+\iota} - \tfrac{(\tilde{\theta}_1 + \tilde{\theta}_2) b_z}{\epsilon} \|z\|_{\tilde{r}}^{\mu+\iota} \\ & + b_z \eta \|z\|_{\tilde{r}}^{\nu+\iota} + b_z \eta \|x\|_r^{\nu+\iota}, \end{aligned}$$

where $\tilde{\theta}_1, \tilde{\theta}_2 > 0$ are such that $\tilde{\theta}_1 + \tilde{\theta}_2 = \tilde{\theta}$. Hence, if

$$\|z\|_{\tilde{r}}^{\mu+\iota} \geq \tfrac{\eta\epsilon}{\tilde{\theta}_2} \|x\|_r^{\nu+\iota}, \tag{36}$$

then

$$\dot{W} \leq -(1-\tilde{\theta}) \tfrac{b_z}{\epsilon} \|z\|_{\tilde{r}}^{\mu+\iota} - \tfrac{\tilde{\theta}_1 b_z}{\epsilon} \|z\|_{\tilde{r}}^{\mu+\iota} + b_z \eta \|z\|_{\tilde{r}}^{\nu+\iota}.$$

So, through the relationship (3), it can be readily check that $W$ satisfies Definition 3, hence it is an ISS-LF for the system (32). Therefore the system (32) is

- ISS w.r.t. $x$, if $\mu = \nu$ and
$$\epsilon \leq \tfrac{\tilde{\theta}_1}{\eta}. \tag{37}$$
- locally ISS w.r.t. $x$, if $\mu < \nu$ and
$$\|z\|_{\tilde{r}} \leq \left( \tfrac{\tilde{\theta}_1}{\epsilon\eta} \right)^{\frac{1}{\nu-\mu}}. \tag{38}$$
- ISpS w.r.t. $x$, if $\mu > \nu$ and
$$\|z\|_{\tilde{r}} \geq \left( \tfrac{\epsilon\eta}{\tilde{\theta}_1} \right)^{\frac{1}{\mu-\nu}}. \tag{39}$$

Accordingly, from Definition 2 and the relationship (3), the trajectories of the system (32) are bounded by

$$\|z(t)\|_{\tilde{r}} \leq \max\{\beta_2(\|z_0\|_{\tilde{r}}, t), \gamma_2(\sup_{\tau\in[0,t)} \|x(\tau)\|_r), \rho\},$$

for all $t \geq 0$, where $\beta_2$ is a $\mathcal{KL}$ function, $\rho$ is a constant given by $\rho = 0$ for $\mu \leq \nu$, and $\rho = \frac{\bar{a}_z}{\underline{a}_z} \left( \frac{\epsilon\eta}{\tilde{\theta}_1} \right)^{\frac{1}{\mu-\nu}}$ for $\mu > \nu$, also considering Definition 3, Remark 1 and inequalities (20) and (36), $\gamma_2$ is a class $\mathcal{K}$ function given by

$$\gamma_2(s) = \tfrac{\bar{a}_z}{\underline{a}_z} \left( \tfrac{\eta\epsilon}{\tilde{\theta}_2} s^{\nu+\iota} \right)^{\frac{1}{\mu+\iota}}. \tag{40}$$

Now, let's analyze the internal stability of the interconnected system (22),(32) by using Corollary 1. The functions (31) and (40) are the nonlinear asymptotic gains for the systems (22) and (32), respectively. Note that (31) and (40) were obtained in terms of the $r$-homogeneous norm as it is required in the hypothesis of Corollary 1. So, according to the small-gain condition (12), the stability of the interconnection is ensured if the composition

$$\gamma_1(\gamma_2(s)) = \delta^{-1}\frac{\bar{a}_x\bar{a}_z}{\underline{a}_x\underline{a}_z}\left(\frac{\eta\epsilon}{\tilde{\theta}_2}s^{\nu+\iota}\right)^{\frac{1}{\mu+\iota}},$$

is a contraction, i.e., $\gamma_1(\gamma_2(s)) < s$, that is,

$$\frac{\epsilon\eta}{\tilde{\theta}_2}\left(\delta^{-1}\frac{\bar{a}_x\bar{a}_z}{\underline{a}_x\underline{a}_z}\right)^{\mu+\iota} < s^{\mu-\nu}. \qquad (41)$$

Finally, according to the HD's of the systems (13) and (14), there are three different cases of stability. For the cases where $\nu \geq \mu$, vanishing of the transient $z$ can be concluded (at east locally), which guarantees GAS (or local asymptotic stability) of the interconnected system (13)-(14) at the origin. However, for the case $\nu < \mu$ only practical stability can be proven, but since the system (22) is ISS w.r.t. $z$ then the same property can be concluded for the MD (13) in presence of the PD (14).

For the case $\nu = \mu$, we have from inequalities (41) and (37) that

$$\epsilon < \epsilon^* = \min\left\{\frac{\tilde{\theta}_1}{\eta}, \frac{\tilde{\theta}_2}{\eta}\left(\delta\frac{\underline{a}_x\underline{a}_z}{\bar{a}_x\bar{a}_z}\right)^{\mu+\iota}\right\}. \qquad (42)$$

Whereas for the cases where $\nu \neq \mu$ we obtain

$$\upsilon_1(\epsilon) = \left(\frac{\epsilon\eta}{\tilde{\theta}_2}\left(\delta^{-1}\frac{\bar{a}_x\bar{a}_z}{\underline{a}_x\underline{a}_z}\right)^{\mu+\iota}\right)^{\frac{1}{|\mu-\nu|}}. \qquad (43)$$

Similarly, since $z = y - h(x)$ and $h(0) = 0$, a $\mathcal{K}_\infty$-function $\upsilon_2$ can be readily obtained from the composition $\gamma_2(\gamma_1(s)) < s$, and considering inequalities (38) and (39), hence

$$\upsilon_2(\epsilon) = \max\left\{\left(\frac{\epsilon\eta}{\tilde{\theta}_1}\right)^{\frac{1}{|\mu-\nu|}}, \left(\frac{\epsilon\eta}{\tilde{\theta}_2}\left(\delta^{-1}\frac{\bar{a}_x}{\underline{a}_x}\right)^{\nu+\iota}\left(\frac{\bar{a}_z}{\underline{a}_z}\right)^{\mu+\iota}\right)^{\frac{1}{|\mu-\nu|}}\right\}.$$

Thus, Theorem 2 is proven. ■

## IV. Illustrative Examples

The following examples have the purpose to illustrate the different kinds of stability predicted by Theorem 2. To this end, some simplifications are introduced in order to exhibit that results nicely.

### A. Case $\nu < \mu$

Consider the system

$$\dot{x} = -\lceil y_1 \rfloor^{\frac{2}{3}} \qquad (44)$$

$$\epsilon\dot{y}_1 = y_2, \quad \epsilon\dot{y}_2 = -y_1 - 2y_2 + \alpha^{\frac{3}{2}}x, \qquad (45)$$

where $x$ is the state of the MD (44), $y_1, y_2$ are the states of the PD (45), and $\epsilon$ is a small parameter. For $\epsilon = 0$, the solution $h(x)$ is given by $y_1 = \alpha^{\frac{3}{2}}x$ and $y_2 = 0$, such that, the ROD

$$\dot{x} = -\alpha\lceil x \rfloor^{\frac{2}{3}} \qquad (46)$$

is continuous and $r$-homogeneous of degree $\nu = -\frac{1}{2}$ for the weight $r = \frac{3}{2}$. Also, for any $\alpha > 0$, it is finite-time stable at the origin. On the other hand, for $\epsilon > 0$ define $z_1 = y_1 - \alpha^{\frac{3}{2}}x$, $z_2 = y_2$ and $\tau = \epsilon^{-1}t$, such that, the BL

$$\frac{dz_1}{d\tau} = z_2, \quad \frac{dz_2}{d\tau} = -z_1 - 2z_2, \qquad (47)$$

is continuous, $\tilde{r}$-homogeneous of degree $\mu = 0$ for the weights $\tilde{r} = [1, 1]$, and exponentially stable. Then, $\nu < \mu$ and according to Theorem 2 the system (44)-(45) is globally asymptotically practically stable as it is depicted in Figure 1.

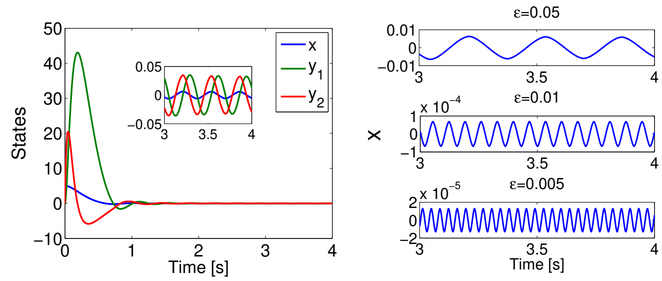


Fig. 1: On the left: behavior of the interconnected system (44)-(45) with $\alpha = 5$ and $\epsilon = 0.05$. On the right: level of chattering in the state of the plant for different values of $\epsilon$.

Note that the states of the system (44)-(45) exhibit oscillations in the steady state. To the intuition of the authors, this behavior is due to the fact that the PD is not fast enough to reach the quasi-stationary state $h(x) = [\alpha^{\frac{3}{2}}x, 0]^\top$. According to Theorem 2, the amplitude of the oscillations depends on the parameter $\epsilon$, and it is illustrated by Figure 1 (on the right).

Moreover, from Theorem 2 the chattering level for the trajectories $x(t)$ is computed as follows. The stability of the ROD (46) can be proven by the LF $V(x) = \frac{1}{2}x^2$, which fulfills the inequalities (23)-(26) with the constants $\kappa = 2$, $\underline{a}_x = \bar{a}_x = 0.5$, $b_x = \alpha$ and $c_x = 1$. On the other hand, a LF for the BL (47) is given by $W(z) = \frac{1}{2}z^\top Pz$, where $P = P^\top > 0$ is a solution of the equation $\bar{A}^\top P + P\bar{A} = -Q$ with $Q > 0$. Selecting

$$P = \begin{bmatrix} 3 & 0.5 \\ 0.5 & 3 \end{bmatrix}, \quad \text{and} \quad Q = \begin{bmatrix} 1 & 1 \\ 1 & 11 \end{bmatrix},$$

the function $W(z)$ satisfies (19)-(21) with $\iota = 2$, $\underline{a}_z = s_{\min}(P) = 2.5$, $\bar{a}_z = s_{\max}(P) = 3.5$, $b_z = s_{\min}(Q) \approx 0.9$. Furthermore, for the system (44)-(45), $\delta = 0.875\alpha^{\frac{3}{2}}$, $\eta = 5.13\alpha^{\frac{3}{2}}(\alpha^{\frac{3}{2}} + 1)^{\frac{2}{3}}$, $\theta = 0.75$ and $\tilde{\theta}_2 = 0.75$ satisfy (28) and (35). Finally, by substituting all the parameters in equation (43), we obtain that $\lim_{t\to\infty}\|x(t)\|_r \leq 39.53\epsilon^2$. Table I provides the estimation of chattering level for different values of $\epsilon$. Note that the results presented by Figure 1 (on the right) fit well with the estimations provided by Table I.

| $\epsilon$ | 0.05 | 0.01 | 0.005 |
|---|---|---|---|
| $\|x\|$ | 0.0988 | 0.00395 | 0.000988 |

TABLE I: Estimation of ultimate bounds for different values of $\epsilon$.

### B. Case $\nu = \mu$

Now, let us consider the following interconnection

$$\dot{x} = -\alpha\lceil y_1 \rfloor^{\frac{2}{3}} \qquad (48)$$

$$\epsilon\dot{y}_1 = y_2, \quad \epsilon\dot{y}_2 = -\lceil y_1 - x \rfloor^{\frac{1}{3}} - 3\lceil y_2 \rfloor^{\frac{1}{2}}, \qquad (49)$$

where $x$ is the state of the plant (48), $y_1, y_2$ are the states of the PD (49), and $\epsilon$ is a parameter. For $\epsilon = 0$, $y_1 = x$ and $y_2 = 0$

give the expression of $h(x)$, hence, the reduced order system (46) is recovered, which is continuous and $r$-homogeneous of degree $\nu = -\frac{1}{2}$ for the weight $r = \frac{3}{2}$, and for any $\alpha > 0$, finite-time stable at the origin. On the other hand, for $\epsilon > 0$ define $z_1 = y_1 - x$, $z_2 = y_2$ and $\tau = \epsilon^{-1}t$, such that, the BL

$$\tfrac{dz_1}{d\tau} = z_2, \quad \tfrac{dz_2}{d\tau} = -\lceil z_1 \rfloor^{\frac{1}{3}} - 2\lceil z_2 \rfloor^{\frac{1}{2}}, \tag{50}$$

is continuous, $\tilde{r}$-homogeneous of degree $\mu = -\frac{1}{2}$ for the weights $\tilde{r} = [\frac{3}{2}, 1]$, and finite-time stable at the origin. In this case, $\nu = \mu$, hence by Theorem 2, the system (48)-(49) is expected to be globally finite-time stable as it is illustrated in Figure 2. A critical value $\epsilon^*$ is computed as

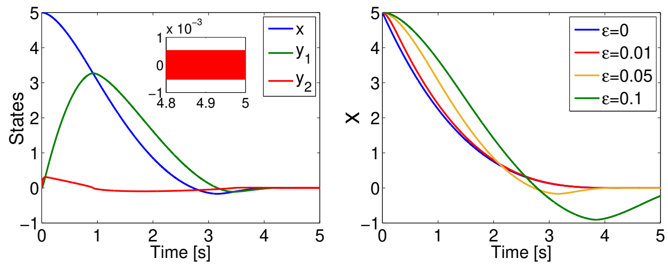


Fig. 2: On the left: behavior of the closed-loop system (48)-(49) where $\alpha = 1.2$ and $\epsilon = 0.05$. On the right: comparison of the deviation of the state of the plant for different values of the parameter $\epsilon$.

follows. The stability of the ROD (46) can be proven by the LF $V(x) = \frac{1}{2}x^2$, which fulfills the inequalities (23)-(26) with the constants $\kappa = 2$, $\underline{a}_x = \bar{a}_x = 0.5$, $b_x = \alpha$ and $c_x = 1$. On the other hand, a LF for the BL (47) is given by $W(z,x) = 10.5|z_1|^{\frac{5}{3}} + 8.5|z_2|^{\frac{5}{2}} + 3.5z_1z_2$, which satisfies (19)-(21) with $\iota = \frac{5}{2}$, $\underline{a}_z = 6.462$, $\bar{a}_z = 10.809$ and $b_z = 2.654$. Furthermore, solving (28) and (35) for the system (48)-(49), $\delta = 0.875$ and $\eta = 13.22\alpha$, where $\theta = 0.75$, $\tilde{\theta}_1 = 0.25$ and $\tilde{\theta}_2 = 0.75$ were used. Then, substituting all the parameter in (42) a critical value $\epsilon^* = 0.0129$ is obtained, hence, the stability of the interconnection (48)-(49) is guaranteed for any $\epsilon < 0.0129$. In this case the value of $\epsilon$ determines how the trajectories of (48)-(49) deviate from the trajectories of the reduced order dynamics (46) as it is shown in Figure 2 (on the right). In the same figure, note that our estimation of $\epsilon^*$ is quite conservative and the stability of the interconnected system (48)-(49) is kept for a wider range of $\epsilon$.

### C. Case $\nu > \mu$

Finally, consider an interconnection given by

$$\dot{x} = -y_1, \tag{51}$$

$$\epsilon\dot{y}_1 = y_2, \quad \epsilon\dot{y}_2 = -\lceil y_1 - \alpha x \rfloor^{\frac{1}{3}} - 2\lceil y_2 \rfloor^{\frac{1}{2}}, \tag{52}$$

where $x$ is the state of the system (51), $y_1, y_2$ are the states of the PD (52), and $\epsilon$ is a parameter. For $\epsilon = 0$, $y_1 = \alpha x$ and $y_2 = 0$, and the reduced order dynamics is given by

$$\dot{x} = -\alpha x, \tag{53}$$

which is $r$-homogeneous of degree $\nu = 0$ for the weight $r = 1$, and also, asymptotically stable at the origin for any $\alpha > 0$. On the other hand, for $\epsilon > 0$ define $z_1 = y_1 - \alpha x$, $z_2 = y_2$ and $\tau = \epsilon^{-1}t$, such that, the BL (50) is obtained and it is continuous, $\tilde{r}$-homogeneous of degree $\mu = -\frac{1}{2}$ for the weights $\tilde{r} = [\frac{3}{2}, 1]$, and finite-time stable at the origin. In this case, $\nu > \mu$, hence, the interconnected system (51)-(52) is locally asymptotically stable at the origin as it is predicted by Theorem 2 and confirmed in Figure 3, where for an initial condition $x_0 = 3$ and $y_0 = [0, 0]$ the states converge to zero but for an initial condition $x_0 = 5$ and $y_0 = [0, 0]$ the stability cannot be ensured. The domain of attraction for the trajectories

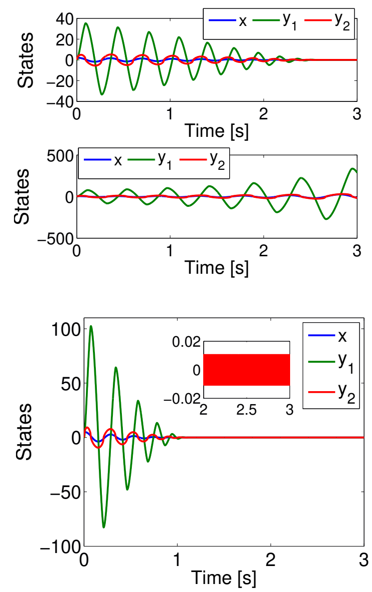


Fig. 3: Behavior of the closed-loop systems (51)-(52): On the top with $\alpha = 60$, $\epsilon = 0.01$, $x_0 = 2$ and $y_0 = [0, 0]$ (top), or $x_0 = 5$ and $y_0 = [0, 0]$ (bottom); on the bottom with $\alpha = 60$, $\epsilon = 0.005$, $x_0 = 5$ and $y_0 = [0, 0]$.

of the system (51)-(52) can be evaluated from Theorem 2 as follows. The stability of the ROD (53) can be proven by the LF $V(x) = \frac{1}{2}x^2$, which fulfills the inequalities (23)-(26) with the constants $\kappa = 2$, $\underline{a}_x = \bar{a}_x = 0.5$, $b_x = \alpha$ and $c_x = 1$. On the other hand, a LF for the BL (50) is proposed as $W(z,x) = 10.5|z_1|^{\frac{5}{3}} + 8.5|z_2|^{\frac{5}{2}} + 3.5z_1z_2$, which satisfies (19)-(21) with $\iota = \frac{5}{2}$, $\underline{a}_z = 6.462$, $\bar{a}_z = 10.809$ and $b_z = 2.654$. Furthermore, by solving (28) and (35) for the system (51)-(52), we obtain $\delta = 0.75\alpha$ and $\eta = 8.33\alpha(\alpha + 1)$, where $\theta = 0.75$ and $\tilde{\theta}_2 = 0.75$ were used.

So, substituting all the parameters in equation (43), $\|x_0\|_r \leq 0.000317\epsilon^{-2}$. Therefore, for $\epsilon = 0.01$, it is obtained $\|x_0\|_r \leq 3.17$ supporting the simulation results shown in Figure 3. Now, if the value of the parameter $\epsilon$ decreases then the domain of attraction for trajectories $x(t)$ increases, such that, for $\epsilon = 0.005$ it turns out that $\|x_0\|_r \leq 12.68$. Accordingly, the stability of the interconnected system (51)-(52) is ensured for $x_0 = 5$ and $y_0 = [0, 0]$ as Figure 3 (on the right) shows.

## V. Conclusions

This paper presented a study of the effect of a stable homogeneous PD on the stability of a homogeneous MD, which is based on classical concepts of ISS and Small-Gain

Theorem by assuming only continuity of the considered vector fields. Three types of stability for such an interconnection were discovered depending on the relation between HDs of PD and MD:

- Global asymptotic stability when both dynamics have the same HD and the SPP is sufficiently small.
- Global asymptotic practical stability, when the PD has a greater HD than the MD. The estimation of the asymptotic bound of the trajectories is provided and its size grows with the SPP. In this case, the chattering may appear if a finite-time convergent MD is considered.
- Local asymptotic stability with an estimation of the domain of attraction, when the PD has a smaller HD than the MD. The size of the domain of attraction decreases if the SPP is increased.

The first case can be interpreted as a validation of the concept of motion separation, predicted by classical results on smooth (at least Lipschitz continuous) singularly perturbed systems, for a wider class of homogeneous systems. On the other hand, such a concept of motion separation is only valid outside of a neighborhood of the origin for the second case, and near to the origin for the third one, hence, just local and practical stability can be concluded, respectively.

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