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Functional convergence to the local time of sticky diffusions

Alexis Anagnostakis*

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Abstract

We prove the convergence of a class of high frequency path-functionals of a sticky diffusion to its local time. First, we prove this for the sticky Brownian motion. Then, we extend the result to sticky stochastic differential equations. We combine the local time approximation with an approximation of the occupation time to set up a consistent stickiness estimator. Last, we perform numerical experiments to assess the properties of the stickiness estimator and the local time approximation.

Keywords: local time, occupation time, stickiness parameter estimation, sticky process, sticky Brownian motion, singular diffusion, rates of convergence, statistics of stochastic processes, process estimation

1 Introduction

Introduced by Feller in [13], one-dimensional sticky diffusions are continuous processes that satisfy the Markov property and spend positive amount of time at some points of their state-space. These points, called sticky points, can be located either in the interior or on the boundary of the state-space. The amount of time a process spends on average at a sticky point is governed by a parameter called stickiness parameter. Special cases of sticky behavior are sticky reflection, which occurs when the sticky point is located at an attainable boundary point of the state-space, and absorption, which corresponds to a stickiness parameter that equals infinity.

Though they have been around for some time, we are recently witnessing a resurgence of interest in sticky processes. The reason behind this is that, besides their inherent theoretical

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value, they are able to replicate accurately various diffusive behaviors found in nature. Such dynamics are typically found in finance, biology and even in quantum and classical mechanics. In particular, they can be used to describe the behavior of interest rates around 0 [19, 27], the behavior of molecules near a membrane [14], the concentration of pathogens in a healthy individual [9], the dynamics of mesoscale particles upon contact in colloids [20, 30] and the motion of quantum particles when they reach a source of emission [10]. From a theoretical standpoint, they are used to create new types of probabilistic couplings [11] and appear as the limit of storage processes [16]. Many papers have also appeared recently addressing the numerical challenges of simulating sticky diffusions [1, 2, 3, 8, 26].

A class of sticky diffusions are sticky stochastic differential equations (or sticky SDEs). These are processes that behave like a homogeneous one-dimensional SDE on the entirety of their state-space \mathbb{I} , an interval of \mathbb{R} , except on a countable set points where they exhibit sticky behavior. For simplicity we suppose that there is one unique sticky point located at $0 \in \mathbb{I}$. We will further suppose that the functions μ and $\sigma : \mathbb{I} \mapsto \mathbb{R}$ satisfy the following condition:

Condition 1.1. *Let B a standard Brownian motion defined on a probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}_x)$. We consider the following SDE:*

$$dX_t = \mu(X_t) dt + \sigma(X_t) dB_t. \quad (1)$$

- (μ, σ) are taken so that (1) has a unique non-explosive strong solution.
- Let X be the solution of (1) such that $\mathbb{P}(X_0 = x) = 1$. For $\theta = (\sigma'(X_t) - \frac{\mu(X_t)}{\sigma(X_t)})_{t \geq 0}$ and every $x \in \mathbb{I}$,

$$\mathbb{E}_x \left[\exp \left(\int_0^t \theta_s dB_s - \frac{1}{2} \int_0^t \theta_s^2 ds \right) \right] = 1, \quad (2)$$

- $\sigma \in C^1(\mathbb{I})$.

Thus, from (1) and (15), a homogeneous sticky SDE can be described by the triplet (μ, σ, ρ) , where μ and σ are the drift and diffusivity functions that govern the behavior away from the sticky point and ρ is the stickiness of 0. To abbreviate the terminology while keeping explicit the dependence of a sticky SDE on μ , σ and ρ , we call such a process a (μ, σ, ρ) -sticky SDE solution. The most elementary of these processes is the sticky Brownian motion, which is the process that behaves like a standard Brownian motion away from 0 and has a sticky point at 0.

Let X be a (μ, σ, ρ) -sticky SDE solution defined on a probability space $(\Omega, (\mathcal{F})_{t \geq 0}, \mathbb{P})$. From Proposition 4.3, X is a semi-martingale. Thus, from [28] pages 222-225, the local time of X at a level a is well defined as the continuous, strictly-increasing process $L^a(X) = (L_t^a(X))_{t \geq 0}$ such that for every $t \geq 0$,

$$|X_t - a| - |X_0 - a| = \int_0^t \text{sgn}(X_s - a) dX_s + L_t^a(X), \quad (3)$$

where $\text{sgn}(x) = \mathbb{1}_{x \geq 0} - \mathbb{1}_{x < 0}$. Let $(\Omega, (\mathcal{F})_{t \geq 0}, \mathbb{P})$ be the probability space on which X is defined. From Corollary 1.6 of [28]-Chapter VI, $L^\alpha(X)$ can alternatively be defined for every $t \geq 0$ as the limit in probability:

$$L_t^\alpha(X) = \mathbb{P}\text{-}\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^t \mathbb{1}_{0 \leq X_s - \alpha < \epsilon} d\langle X \rangle_s. \quad (4)$$

If furthermore the functions (μ, σ) satisfy Condition 1.1, for every $t \geq 0$,

$$L_t^\alpha(X) = \mathbb{P}\text{-}\lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbb{1}_{|X_s - \alpha| < \epsilon} d\langle X \rangle_s. \quad (5)$$

In this paper, we study the convergence properties of high-frequency path functionals of the type:

$$\frac{n^\alpha}{n} \sum_{i=1}^{[nt]} g(n^\alpha X_{\frac{i-1}{n}}), \quad (6)$$

when the underlying process X is a sticky SDE. We give necessary conditions under which the functional (6) converges to the local time at 0 of X . Approximating the local time with high-frequency path functionals has been studied in the case of the standard Brownian motion [5, 6], smooth SDEs [4, 17] and the oscillating/skew Brownian motion [23, 25].

In particular, we prove the following results:

Theorem 1.2. *Let X^ρ be a (μ, σ, ρ) -sticky SDE defined on a probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}_x)$ such that $\mathbb{P}_x(X_0 = x) = 1$ for any x in the state-space \mathbb{I} of X and (μ, σ) satisfy Condition 1.1. Let $g : \mathbb{R} \mapsto \mathbb{R}$ be a bounded Lebesgue-integrable function which vanishes on an open interval around 0 and $T : \mathbb{R} \mapsto \mathbb{R}$ a continuously differentiable function such that for an $\epsilon > 0$:*

$$T(0) = 0, \quad T'(0) = 1, \quad \epsilon \leq T'(x) \leq 1/\epsilon, \quad |T''(x)| \leq 1/\epsilon, \quad (7)$$

for every $x \in \mathbb{R}$. For every such function T , let $g_n[T]$ be the sequence of functions such that:

$$g_n[T](x) = g(n^\alpha T(n^{-\alpha}x)), \quad (8)$$

for every x and n . Then, for every $\alpha \in (0, 1/2)$,

$$\frac{n^\alpha}{n} \sum_{i=1}^{[nt]} g_n[T](n^\alpha X_{\frac{i-1}{n}}^\rho) \xrightarrow{n \rightarrow \infty} \frac{\lambda(g)}{\sigma(0)} L_t^0(X^\rho). \quad (9)$$

locally, uniformly in time, in \mathbb{P}_x -probability¹.

¹We say that a sequence of processes X^n converges locally uniformly in time, in \mathbb{P} -probability to a process X iff for every $t \geq 0$,

$$\sup_{s \leq t} \{|X_s^n - X_s|\} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0 \quad (10)$$

Corollary 1.3 (of Theorem 1.2 and Lemma 5.1). *Let X^ρ be a (μ, σ, ρ) -sticky SDE defined on a probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}_x)$ such that $\mathbb{P}_x(X_0 = x) = 1$ for any x in the state-space \mathbb{I} of X and (μ, σ) satisfy Condition 1.1. Then, if g is a bounded integrable function which vanishes on an open interval around 0 and $\alpha \in (0, 1/2)$,*

$$\widehat{\rho}_n[g, \alpha] := 2 \frac{\lambda(g)}{\sigma(0)} \frac{1}{n^\alpha} \frac{\sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}}=0}}{\sum_{i=1}^{[nt]} g(n^\alpha X_{\frac{i-1}{n}})} \quad (11)$$

is a consistent estimator of ρ as $n \rightarrow \infty$. In particular, this holds for $g(x) = \mathbb{1}_{1 < |x| < 2}$.

In order to address the proof of these results we need a proper characterization of sticky SDEs. A convenient way to characterize sticky diffusions is via their scale function and speed measure. This is because this characterization is local and yields an explicit representation of the sticky point as an atom in the speed measure.

In general, one can describe the law of any one-dimensional diffusion process X with state-space \mathbb{I} by a pair (s, m) comprised of a continuous increasing function $s : \mathbb{I} \mapsto \mathbb{R}$ and a strictly positive², locally finite measure m over \mathbb{I} . The function s is called scale function and expresses the propensity of the process to follow a certain direction (up or down). It can be defined as the function such that for each $a < b$ such that $a, b \in \mathbb{I}$,

$$\mathbb{P}_x(\tau_b < \tau_a) = \frac{s(x) - s(a)}{s(b) - s(a)}, \quad (12)$$

where $\tau_\zeta = \inf\{s > 0 : X_s = \zeta\}$. The measure $m(dx)$ is called speed measure and expresses the speed at which the process moves and in particular the speed at which the process exits compact intervals. It can be defined as the unique positive locally finite measure m such that for each $a < b$ with $a, b \in \mathbb{I}$,

$$\mathbb{E}_x(\tau_{ab}) = \int_{(a,b)} \frac{((s(y \wedge x)) - s(a))(s(b) - (s(y \vee x)))}{s(b) - s(a)} m(dy). \quad (13)$$

The (s, m) -formulation is particularly adapted for studying sticky one-dimensional diffusions. A point ζ of the state space is sticky if and only if $m(\{\zeta\}) > 0$. Moreover, since the (s, m) -formulation is local, one can characterize a sticky process in (s, m) -terms by adding an atom to the speed measure of the non-sticky version of this process. For example, the sticky Brownian motion of stickiness $\rho > 0$ is the diffusion process defined through s and m defined for every $x \in \mathbb{R}$ by

$$s(x) = x, \quad m(dx) = 2 dx + \rho \delta_0(dx) \quad (14)$$

²We say that a measure m on $(\mathbb{I}, \mathcal{B}(\mathbb{I}))$ is strictly positive iff $m((a, b)) > 0$ for any open interval $(a, b) \subset \mathbb{I}$.

where $s_0(x) = x$ and $m_0(dx) = 2 dx$ are the scale function and speed measure of the standard Brownian motion. Similarly, the (μ, σ, ρ) -sticky SDE solution is the diffusion process with s and m given by

$$s'(x) = e^{-\int_a^x \frac{2\mu(u)}{\sigma^2(u)} du}, \quad m(dx) = \frac{1}{s'(x)} \frac{2}{\sigma^2(x)} dx + \rho \delta_0(dx), \quad (15)$$

for every $x \in \mathbb{R}$ and with s' being the right-derivative of s .

Sticky diffusions can also be defined as time changed non-sticky process (see Theorem 2.1) or in the sticky SDE case, by their path-wise formulation (see Proposition 4.2).

Outline: In Section 2, we prove several properties of the sticky Brownian motion, its local time and its semi-group. In Section 3, we prove Theorem 3.1 which is Theorem 1.2 for the sticky Brownian motion. In Section 4, we prove the path-wise formulation of sticky SDEs and use it to prove Theorem 1.2 by extension of Theorem 3.1. In Section 5, we prove Corollary 1.3. Section 6 is dedicated to numerical experiments.

2 Some preliminary results

In this section we prove several properties of the sticky Brownian motion that will be used in the latter sections. The first property is the time-scaling of the sticky Brownian motion which is the analogue of (26) for the standard Brownian motion. Then, we prove a characterization of the local time of the sticky Brownian motion. Finally, we prove that the transition kernel of the sticky Brownian motion is bounded by the transition kernel of the non-sticky one. The transition kernel of a diffusion process X with speed measure m is the positive measurable function $p : \mathbb{R}_+ \times \mathbb{I}^2 \mapsto \mathbb{R}_+$ (see [28] page 80 or [7] page 13) such that for every positive measurable function h ,

$$\mathbb{E}_x(h(X_t)) = \int_{\mathbb{R}} h(y) p(t, x, y) m(dy). \quad (16)$$

The left hand side of (16) defines a family of functionals $(P_t)_{t \geq 0}$ as

$$P_t h(x) = \mathbb{E}_x(h(X_t)), \quad (17)$$

for every $t \geq 0$ and positive measurable h and $x \in \mathbb{I}$. It turns out that $(P_t)_{t \geq 0}$ is a family of linear operators from C_b to C_b that characterize the law of X (see Chapter 3.1 of [29]) and satisfy the semi-group property with respect to t , i.e. $P_{t+s} = P_t P_s$. We thus call $(P_t)_{t \geq 0}$ the semi-group of the diffusion X .

In our proofs we will use several fundamental results on time-changes of semi-martingales. The first one is the characterization of diffusions on natural scale in terms of a time-changed Brownian motion.

Theorem 2.1 (Theorem V.47.1 of [29]). *Let $X = (X_t)_{t \geq 0}$ be a diffusion on natural scale with speed measure m defined on a probability space $\mathcal{P} = (\Omega, \mathcal{F}, P_x)$ such that $P_x(X_0 = x) = 1$. Then, there exists a Brownian motion $B = (B_t)_{t \geq 0}$ defined on an extension of \mathcal{P} such that for every $t \geq 0$,*

$$X_t = B_{\gamma(t)}, \quad (18)$$

where $\gamma(t)$ is the right inverse of $A(t) = \frac{1}{2} \int_{\mathbb{R}} L_t^y(B) m(dy)$.

Corollary 2.2 (of Theorem 2.1). *Let m_X be a locally finite strictly positive measure on \mathbb{R} and B a standard Brownian motion defined on a probability space $\mathcal{P}_x = (\Omega, \mathcal{F}, P_x)$ such that $P_x(B_0 = x) = 1$. Let $L^0(B)$ the time-continuous version of the local time of B at 0, $A_X(t) = \frac{1}{2} \int_{\mathbb{R}} L_t^0(B) m_X(dy)$ and $\gamma_X(t)$ the right-inverse of $A_X(t)$. Then, the process X such that $X_t = B_{\gamma_X(t)}$ for every $t \geq 0$ is a diffusion process on natural scale with speed measure $m_X(dx)$ and such that $P_x(X_0 = x) = 1$.*

The other results regard changes of variables in stochastic integrals with respect to a time change $(T_t)_{t \geq 0}$. An $\{\mathcal{F}_t\}_{t \geq 0}$ -time change is any almost surely increasing, right-continuous family of $\{\mathcal{F}_t\}_{t \geq 0}$ -stopping times $(T_t)_{t \geq 0}$ such that $T_0 = 0$.

Theorem 2.3 (Theorem 3.1 of [22]). *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a filtered probability space, M an $\{\mathcal{F}_t\}_{t \geq 0}$ -semi-martingale and T an $\{\mathcal{F}_t\}_{t \geq 0}$ -time change that admits a strictly increasing version and whose right-inverse also admits a strictly increasing version. Then, if H is a predictable adapted process with respect to $\{\mathcal{F}_t\}_{t \geq 0}$, P -almost surely,*

$$\int_0^t H_s dM_{T_s} = \int_0^{T_t^{-1}} H_{T_s} dM_s, \quad (19)$$

for every $t \geq 0$.

The last results are given as exercises in [28] and allow us to characterize the local time of re-scaled and time-changed processes respectively.

Lemma 2.4 (Exercise 1.23 of [28]-Chapter VI). *Let X be a continuous semi-martingale and f a strictly increasing difference of two convex functions. If $f(X) = (f(X_t))_{t \geq 0}$, then for every $a \in \mathbb{I}$ and $t \geq 0$,*

$$L_t^{f(a)}(f(X)) = f'(a) L_t^a(X) \quad (20)$$

almost surely, where f' is the right-derivative of f .

Lemma 2.5 (Exercise 1.27 of [28]-Chapter VI). *Let X be a semi-martingale with state-space \mathbb{I} , T a time change and Y is the time changed process such that $Y_t = X_{T(t)}$ for every $t \geq 0$. Then for every $a \in \mathbb{I}$ and $t \geq 0$,*

$$L_t^a(Y) = L_{T(t)}^a(X) \quad (21)$$

almost surely.

Lemma 2.6 ((2.2.16) of [18]). *Let $\{X^n\}_{n \in \mathbb{N}}$ and X be a sequence of increasing processes and an almost surely continuous bounded process defined on a probability space $(\Omega, (\mathcal{F})_{t \geq 0}, \mathbb{P})$ such that:*

$$X_t^n \xrightarrow[n \rightarrow \infty]{} X_t \quad (22)$$

in probability, for every $t \geq 0$. Then the convergence is uniform in time, in probability.

2.1 Scaling properties of the sticky Brownian motion

Let us recall that the sticky Brownian motion of stickiness $\rho > 0$ is defined through s and m given by (14).

Lemma 2.7. *Let $\{(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}_x^\rho); x \in \mathbb{R}, \rho \geq 0\}$ be a family of filtered probability spaces and $X^\rho = (X_t^\rho)_{t \geq 0}$ a process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0})$ such that under \mathbb{P}_x^ρ it is the sticky Brownian motion of stickiness parameter ρ and $\mathbb{P}_x^\rho(X_0^\rho = x)$. Then,*

$$\text{Law}_{\mathbb{P}_x^\rho}(X_{ct}^\rho, L_{ct}^0(X^\rho); t \geq 0) = \text{Law}_{\mathbb{P}_x^{\rho/\sqrt{c}}}(\sqrt{c}X_t^{\rho/\sqrt{c}}, \sqrt{c}L_t^0(X^{\rho/\sqrt{c}}); t \geq 0), \quad (23)$$

where $L_t^0(X^\rho)$ and $L_t^0(X^{\rho/\sqrt{c}})$ are the local times of X^ρ and $X^{\rho/\sqrt{c}}$ respectively.

Proof. Let X^ρ be a process defined on $\mathcal{P}_x^\rho = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}_x^\rho)$ such that under \mathbb{P}_x^ρ it is a sticky Brownian motion of stickiness ρ . From Theorem 2.1, there exists a Brownian motion B defined on an extension of \mathcal{P}_x^ρ such that for every $t \geq 0$,

$$X_t^\rho = B_{\gamma(t)} \quad (24)$$

where $\gamma(t)$ is the right inverse of $A(t) = \frac{1}{2} \int_{\mathbb{R}} L_t^y(B) m(dy)$. Moreover from (21), \mathbb{P}_x^ρ -almost surely, for every $t \geq 0$,

$$L_t^0(X^\rho) = L_{\gamma(t)}^0(B). \quad (25)$$

From Proposition 1.10 of [28]-Chapter I, for every $c > 0$, the process B' defined as,

$$B'_t = B_{ct}/\sqrt{c}, \quad (26)$$

is a standard Brownian motion. Moreover, \mathbb{P}_x^ρ -almost surely,

$$\begin{aligned} L_{ct}^0(B) &= \mathbb{P}_x\text{-}\lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^{ct} \mathbb{1}_{|B_s| < \epsilon} ds = \mathbb{P}_x\text{-}\lim_{\epsilon \rightarrow 0} \frac{c}{2\epsilon} \int_0^t \mathbb{1}_{|B_{cs}| < \epsilon} ds \\ &= \mathbb{P}_x\text{-}\lim_{\epsilon \rightarrow 0} \frac{c}{2\epsilon} \int_0^t \mathbb{1}_{|B'_s| < \epsilon/\sqrt{c}} ds = \sqrt{c}L_t^0(B'). \end{aligned} \quad (27)$$

Let $A_\rho(t) = t + \frac{\rho}{2}L_t^0(B)$ and $A'_\rho(t) = t + \frac{\rho}{2}L_t^0(B')$ with $\gamma_\rho(t)$ and $\gamma'_\rho(t)$ be their respective right inverses. From (27), \mathbb{P}_x^ρ -almost surely,

$$A_\rho(ct) = ct + \frac{\rho}{2}L_{ct}^0(B) = ct + \sqrt{c}\frac{\rho}{2}L_t^0(B') = c\left(t + \frac{\rho}{2\sqrt{c}}L_t^0(B')\right) = cA'_{\rho/\sqrt{c}}(t), \quad (28)$$

$$\gamma_\rho(ct) = \inf \{s > 0 : A_\rho(s) > ct\} = \inf \{s > 0 : A'_{\rho/\sqrt{c}}(s/c) > t\} = c\gamma'_{\rho/\sqrt{c}}(t). \quad (29)$$

From (26) and (29), P_x^ρ -almost surely,

$$B_{\gamma_\rho(ct)} = \sqrt{c}B'_{\gamma'_{\rho/\sqrt{c}}(t)}. \quad (30)$$

Let $X^{\rho/\sqrt{c}}$ be the process such that

$$X_t^{\rho/\sqrt{c}} = B'_{\gamma'_{\rho/\sqrt{c}}(t)}, \quad (31)$$

which is a martingale from the martingale stopping theorem. As such its local time $L^0(X^{\rho/\sqrt{c}})$ is a well defined object. From (21), P_x -almost surely, for every $t \geq 0$,

$$L_t^0(X^{\rho/\sqrt{c}}) = L_{\gamma'_{\rho/\sqrt{c}}(t)}^0(B'). \quad (32)$$

We observe that for a measure $m_\rho(dy) = 2dy + \rho\delta_0(dy)$,

$$A_\rho(t) = \int_{\mathbb{R}} L_t^y(B)m_\rho(dy) \quad \text{and} \quad A'_\rho(t) = \int_{\mathbb{R}} L_t^y(B')m_\rho(dy). \quad (33)$$

Thus, from (31) and Corollary 2.2, $\sqrt{c}X_t^{\rho/\sqrt{c}} = X_{ct}^\rho$ is a sticky Brownian motion of stickiness parameter ρ/\sqrt{c} and whose local time process $L^0(X^{\rho/\sqrt{c}})$, from (25), (27) and (32), satisfies P_x^ρ -almost-surely

$$L_t^0(X^{\rho/\sqrt{c}}) = L_t^0(X^\rho)/\sqrt{c}, \quad (34)$$

for every $t \geq 0$. Thus, setting $P_x^{\rho/\sqrt{c}} = P_{x\sqrt{c}}^\rho$, (23) is proven. \square

Corollary 2.8. *Let $X^\rho = (X_t^\rho)_{t \geq 0}$ be the sticky Brownian motion of stickiness parameter $\rho > 0$ and $\{P_t^\rho\}_{t \geq 0}$ be its semi-group. Then, for every measurable function $h : \mathbb{R} \mapsto \mathbb{R}$,*

$$P_t^{\rho\sqrt{n}}h(x\sqrt{n}) = E_x(h(\sqrt{n}X_{\frac{t}{n}}^\rho)). \quad (35)$$

2.2 Estimates on the sticky semi-group

Lemma 2.9. *Let $\{P_t^\rho\}_{t \geq 0}$ be the semi-group of the sticky Brownian motion of parameter $\rho > 0$. There exists a constant $K > 0$ that does not depend on ρ such that for every real-valued function $h(x)$ such that $h(0) = 0$,*

$$|P_t^\rho h(x)| \leq K \frac{\lambda(|h|)}{\sqrt{t}}, \quad (36)$$

for every $t > 0$, where $\lambda(g) = \int_{\mathbb{R}} g(x) dx$.

Proof. Let $p_\rho(t, x, y)$ be the probability transition kernel of the sticky Brownian motion of parameter $\rho > 0$ with respect to its speed measure $m(dy) = 2 dy + \frac{\rho}{2} \delta_0(dy)$. From page 108 of [7],

$$p_\rho(t, x, y) = u_1(t, x, y) - u_2(t, x, y) + v_\rho(t, x, y), \quad (37)$$

for every $x, y \in \mathbb{R}$ and $t > 0$, where

$$\begin{cases} u_1(t, x, y) &= \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}, \\ u_2(t, x, y) &= \frac{1}{\sqrt{2\pi t}} e^{-(|x|+|y|)^2/2t}, \\ v_\rho(t, x, y) &= \frac{2}{\rho} e^{4(|x|+|y|)/\rho + 8t/\rho^2} \operatorname{erfc}\left(\frac{|x|+|y|}{\sqrt{2t}} + \frac{2\sqrt{2t}}{\rho}\right). \end{cases}$$

We observe that for $x, y \in \mathbb{R}$ and $t \geq 0$: $\exp(-(|x| + |y|)^2/2t) < \exp(-(x - y)^2/2t)$ and

$$u_1(t, x, y) - u_2(t, x, y) \leq \frac{1}{2\sqrt{2\pi t}} e^{-(x-y)^2/2t}. \quad (38)$$

The Mills ratio of a Gaussian random variable (see [15] page 98) yields $\operatorname{erfc}(x) \sim e^{-x^2}/x$. Thus, there exists a constant $K_{Mills} > 0$ such that

$$v_\rho(t, x, y) \leq K_{Mills} \frac{2\sqrt{2t}}{\rho(|x| + |y|) + 8t} e^{-(|x|+|y|)^2/2t} \leq K_{Mills} \frac{1}{2\sqrt{2t}} e^{-(x-y)^2/2t}. \quad (39)$$

From (38) and (39), for $K = 1 + K_{Mills}\sqrt{\pi}/2$,

$$p_\rho(t, x, y) \leq K \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t} \quad (40)$$

and

$$|P_t h(x)| \leq \int_{\mathbb{R}} |h(y)| p_\rho(t, x, y) dy \leq K \int_{\mathbb{R}} |h(y)| \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t} dy. \quad (41)$$

Observing $e^{-(x-y)^2/2t} \leq 1$ yields (36). \square

3 The case of the sticky Brownian motion

In this section we prove Theorem 3.1 which is Theorem 1.2 in the case of the sticky Brownian motion. This result will then extended to the case of driftless sticky SDEs. Thus, along with an adapted version of Girsanov theorem, we will be able to prove Theorem 1.2 for any sticky SDE. The latter is done in Section 4.2.

We thus prove the following:

Theorem 3.1. *Let $X^\rho = (X_t^\rho)_{t \geq 0}$ be the sticky Brownian motion of stickiness parameter $\rho > 0$, g a bounded integrable function that vanishes on an open interval around 0 and T a*

continuously differentiable real-valued function that satisfies (7) for an $\epsilon > 0$. Moreover, let g_n be the sequence of real-valued function defined as:

$$g_n(x) = g(n^\alpha T(n^{-\alpha}x)), \quad (42)$$

for every $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then, for every $\alpha \in (0, 1/2)$,

$$\frac{n^\alpha}{n} \sum_{i=1}^{[nt]} g_n(n^\alpha X_{\frac{i-1}{n}}^\rho) \xrightarrow{n \rightarrow \infty} \lambda(g) L_t^0(X^\rho). \quad (43)$$

locally, uniformly in time, in P_x -probability.

As the proof of Theorem 3.1 is rather tedious, we have isolated parts of it in Lemmas 3.2, 3.3 and 3.4.

Lemma 3.2. *Let X^ρ be the sticky Brownian motion of stickiness $\rho > 0$ and g be an integrable function such that $g(0) = 0$. Then, there exists a constant $K > 0$ such that for every $x \in \mathbb{R}$, $t > 0$ and $n \in \mathbb{N}$:*

$$\mathbb{E}_x \left(\sup_{s \leq t} \left| \frac{u_n}{n} \sum_{i=1}^{[ns]} g(u_n X_{\frac{i-1}{n}}^\rho) \right| \right) \leq K \left(\frac{u_n}{n} |g(u_n x)| + \lambda(|g|) \sqrt{t} \right). \quad (44)$$

Proof. We observe that

$$\begin{aligned} \mathbb{E}_x \left(\sup_{s \leq t} \left| \frac{u_n}{n} \sum_{i=1}^{[ns]} g(u_n X_{\frac{i-1}{n}}^\rho) \right| \right) &\leq \mathbb{E}_x \left(\frac{u_n}{n} \sum_{i=1}^{[nt]} |g(u_n X_{\frac{i-1}{n}}^\rho)| \right) \\ &= \frac{u_n}{n} |g(u_n x)| + \frac{u_n}{n} \sum_{i=2}^{[nt]} \mathbb{E}_x (|g(u_n X_{\frac{i-1}{n}}^\rho)|). \end{aligned} \quad (45)$$

Then, if $h_n(x) = g(u_n x / \sqrt{n})$,

$$\sum_{i=1}^{[nt]} h_n(\sqrt{n} X_{\frac{i-1}{n}}^\rho) = \sum_{i=1}^{[nt]} g(u_n X_{\frac{i-1}{n}}^\rho), \quad \lambda(|h_n|) = \frac{\sqrt{n}}{u_n} \lambda(|g|), \quad h_n(0) = 0. \quad (46)$$

From (45) and (46),

$$\mathbb{E}_x \left(\sup_{s \leq t} \left| \frac{u_n}{n} \sum_{i=1}^{[ns]} g(u_n X_{\frac{i-1}{n}}^\rho) \right| \right) \leq \frac{u_n}{n} |g(u_n x)| + \frac{u_n}{n} \sum_{i=2}^{[nt]} \mathbb{E}_x (|h_n(\sqrt{n} X_{\frac{i-1}{n}}^\rho)|). \quad (47)$$

As $h_n(0) = 0$, from (35) and (36),

$$\mathbb{E}_x (|h_n(\sqrt{n} X_{\frac{i-1}{n}}^\rho)|) = P_{i-1}^{\rho\sqrt{n}} |h_n(x\sqrt{n})| \leq K \frac{\lambda(|h_n|)}{\sqrt{i-1}}. \quad (48)$$

From (46), (48) and as $\sum_{i=1}^{[nt]} \frac{1}{\sqrt{i}} \leq 2\sqrt{nt}$,

$$\sum_{i=2}^{[nt]} \mathbb{E}_x \left(\left| h_n(\sqrt{n} X_{\frac{i-1}{n}}^\rho) \right| \right) \leq 2K\lambda(|h_n|)\sqrt{nt} = 2K \frac{n}{u_n} \lambda(|g|)\sqrt{t}. \quad (49)$$

From (47) and (49) we get (44). \square

Lemma 3.3. *Let X be a sticky Brownian motion with local time $L^a(X)$. Moreover, let g and T be two real-valued functions such that g is bounded and integrable and T satisfies (7). Then, for any $t \geq 0$,*

$$\int_{\mathbb{R}} g_n(x) L_{[nt]/n}^{x/n^\alpha}(X) dx \xrightarrow{n \rightarrow \infty} \lambda(g) L_t^0(X), \quad (50)$$

where g_n is given by (42).

Proof. From Trotter's theorem [31], the local time of the standard Brownian motion $L^x(B)$ admits a version that is (t, x) -jointly continuous. As the time-change γ also admits a continuous version, from (21), the local time of X $L^x(X)$ admits a version that is (t, x) -jointly continuous. Thus,

$$|L_{[nt]/n}^{x/n^\alpha}(X) - L_t^0(X)| \xrightarrow{n \rightarrow \infty} 0, \quad (51)$$

for every $t \geq 0$ and $x \in \mathbb{R}$. Moreover, there exists a positive random variable U such that

$$|L_{[nt]/n}^{x/n^\alpha}(X) - L_t^0(X)| \leq U, \quad (52)$$

for every x and n . Thus, as g is bounded,

$$\left| \int_{\mathbb{R}} g_n(x) (L_{[nt]/n}^{x/n^\alpha}(X) - L_t^0(X)) dx \right| \leq \|g\|_\infty \int_{|x| \leq q} |L_{[nt]/n}^{x/n^\alpha}(X) - L_t^0(X)| dx + U \int_{|x| > q} |g_n(x)| dx. \quad (53)$$

From (51), (52) and Lebesgue convergence theorem,

$$\int_{|x| \leq q} |L_{[nt]/n}^{x/n^\alpha}(X) - L_t^0(X)| \xrightarrow{n \rightarrow \infty} 0. \quad (54)$$

With a change of variables,

$$\begin{aligned} \int_{|x| > q} |g_n(x)| dx &= \int_{|x| > q} |g(n^\alpha T(x/n^\alpha))| dx \\ &= \int_{u_n T(q/u_n)}^\infty g(y) \frac{1}{T'(y/u_n)} dy + \int_{-\infty}^{u_n T(-q/u_n)} g(y) \frac{1}{T'(y/u_n)} dy \end{aligned} \quad (55)$$

From (7) and (55),

$$\limsup_n \int_{|x| > q} |g_n(x)| dx \leq \frac{1}{\epsilon} \left(\int_q^\infty g(y) dy + \int_{-q}^0 g(y) dy \right) \quad (56)$$

which since g is integrable converges to 0 as $q \rightarrow \infty$. From (53), (54) and (55),

$$\left| \int_{\mathbb{R}} g_n(x) (L_{[nt]/n}^{x/n^\alpha}(X) - L_t^0(X)) dx \right| \xrightarrow{n \rightarrow \infty} 0. \quad (57)$$

Using again the same change of variables as in (55),

$$\int_{\mathbb{R}} g_n(x) dx = \int_{\mathbb{R}} g(n^\alpha T(x/n^\alpha)) dx = \int_{\mathbb{R}} g(x) \frac{1}{T'(T^{-1}(x/n^\alpha))} dx. \quad (58)$$

Thus, as g is integrable and $T'(x) \geq \epsilon$ for every $x \in \mathbb{R}$, from Lebesgue convergence theorem,

$$\int_{\mathbb{R}} g_n(x) dx \xrightarrow{n \rightarrow \infty} \int_{\mathbb{R}} g(x) dx. \quad (59)$$

Equations (57) and (59) yield (50). \square

Lemma 3.4. *Let T_n be the functional define for each real-valued function h and $x \in \mathbb{R}$ by*

$$T_n[h](x) = \int_0^1 (P_{n^{2\alpha}s/n}^{n^\alpha\rho} h(x) - h(x)) ds, \quad (60)$$

where $\{P_t^\rho\}_{t \geq 0}$ is the semi-group of the sticky Brownian motion of stickiness parameter ρ . Then, for every bounded integrable Lipschitz function k such that k vanishes on an open interval around 0,

$$\lambda(|T_n[k_n]|) \xrightarrow{n \rightarrow \infty} 0, \quad (61)$$

where $k_n(x) = k(n^\alpha T(x/n^\alpha))$.

Proof. From Jensen's inequality,

$$\lambda(|T_n[k_n]|) \leq \int_0^1 \int_{\mathbb{R}} |P_{n^{2\alpha}s/n}^{n^\alpha\rho} k_n(x) - k_n(x)| dx ds. \quad (62)$$

From (40),

$$\begin{aligned} |P_{n^{2\alpha}s/n}^{n^\alpha\rho} k_n(x) - k_n(x)| &\leq \int_{\mathbb{R}} |k_n(y) - k_n(x)| p_{n^\alpha\rho}(n^{2\alpha}s/n, x, y) m(dy) \\ &= \int_{\mathbb{R}} |k_n(y) - k_n(x)| p_{n^\alpha\rho}(n^{2\alpha}s/n, x, y) dy + \frac{n^\alpha\rho}{2} |k_n(x)| p_{n^\alpha\rho}(n^{2\alpha}s/n, x, 0) \\ &\leq K \left[\int_{\mathbb{R}} |k_n(y) - k_n(x)| \frac{1}{n^\alpha \sqrt{2\pi s/n}} e^{-(x-y)^2 n/2sn^{2\alpha}} dy + |k_n(x)| \frac{\sqrt{n}\rho}{2\sqrt{2\pi s}} e^{-x^2 n/2sn^{2\alpha}} \right], \end{aligned} \quad (63)$$

where $\frac{1}{n^\alpha \sqrt{2\pi s/n}} e^{-(x-y)^2 n/2sn^{2\alpha}}$ is the probability density function of a Gaussian $\mathcal{N}(x, n^{2\alpha}s/n)$.

Thus from positive Fubini,

$$\begin{aligned} \int_{\mathbb{R}} |P_{n^{2\alpha}s/n}^{n^\alpha\rho} k_n(x) - k_n(x)| dx &\leq K \left[\mathbb{E} \left[\int_{\mathbb{R}} |k_n(x + n^\alpha \sqrt{s/n} Z) - k_n(x)| dx \right] \right. \\ &\quad \left. + \int_{\mathbb{R}} |k_n(x)| \frac{\sqrt{n}\rho}{2\sqrt{2\pi s}} e^{-x^2 n/2sn^{2\alpha}} dx \right] \end{aligned} \quad (64)$$

where $Z \sim \mathcal{N}(0, 1)$ under P_x . For the first additive term of right-hand side of (64), with the same argument as (58),

$$\int_{\mathbb{R}} k_n(x) dx \leq \frac{1}{\epsilon} \int_{\mathbb{R}} k(x) dx \quad (65)$$

Thus, for every $\omega \in \Omega$,

$$\int_{\mathbb{R}} |k_n(x + n^\alpha \sqrt{s/n} Z) - k_n(x)| dx \leq \int_{\mathbb{R}} |k_n(x + n^\alpha \sqrt{s/n} Z)| dx + \int_{\mathbb{R}} |k_n(x)| dx \leq \frac{2}{\epsilon} \lambda(|k|). \quad (66)$$

Moreover since $T \in C^1$ and k is Lipschitz, P_x -almost surely,

$$\begin{aligned} |k_n(x + n^\alpha \sqrt{s/n} Z) - k_n(x)| &= |k(n^\alpha T(x/n^\alpha + \sqrt{s/n} Z)) - k(n^\alpha T(x/n^\alpha))| \\ &\leq |k(n^\alpha T(x/n^\alpha) + \sqrt{s/n} Z \|T'\|_\infty) - k(n^\alpha T(x/n^\alpha))| \leq |k|_{\text{Lip}} \sqrt{s/n} Z \|T'\|_\infty, \end{aligned} \quad (67)$$

which converges to 0 as $n \rightarrow \infty$. From (66), (67) and Lebesgue convergence theorem,

$$\mathbb{E} \left[\int_{\mathbb{R}} |k_n(x + n^\alpha \sqrt{s/n} Z) - k_n(x)| dx \right] \xrightarrow{n \rightarrow \infty} 0. \quad (68)$$

For the second additive term of right-hand side of (64): Let $\delta > 0$ be a positive real number such that $k(x) = 0$ for every $x \geq 0$ such that $x \notin (\delta, 1/\delta)$. From (7), T is strictly increasing, thus,

$$\delta \leq u_n T(x/u_n) \leq 1/\delta, \quad (69)$$

is equivalent to

$$u_n T^{-1}(\delta/u_n) \leq x \leq u_n T^{-1}(1/u_n \delta). \quad (70)$$

From (7),

$$\begin{aligned} \liminf_n u_n T^{-1}(\delta/u_n) &\geq \delta \epsilon, \\ \limsup_n u_n T^{-1}(1/u_n \delta) &\leq 1/\delta \epsilon. \end{aligned} \quad (71)$$

Thus, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $\text{supp } k_n \subset (\epsilon \delta/2, 2/\epsilon \delta)$. Thus, since k is bounded,

$$\int_{\mathbb{R}} |k_n(x)| \frac{\sqrt{n} \rho}{2\sqrt{2\pi s}} e^{-x^2 n / 2s n^{2\alpha}} dx \leq 2\|k\|_\infty \int_{\epsilon \delta/2}^{2/\epsilon \delta} \frac{\sqrt{n} \rho}{2\sqrt{2\pi s}} e^{-x^2 n / 2s n^{2\alpha}} dx \leq \frac{\rho \|k\|_\infty}{\sqrt{2\pi s}} \frac{2}{\epsilon \delta} \sqrt{n} e^{-n^{1-2\alpha} (\epsilon \delta)^2 / 8s}, \quad (72)$$

which converges to 0 as $n \rightarrow \infty$. From (62), (64), (68) and (72), the convergence (61) is proven. \square

Proof (of Theorem 3.1). Let X be the sticky Brownian motion of parameter $\rho > 0$, $(P_t^\rho)_{t \geq 0}$ its semi-group and $L^x(X)$ its local time at x . From the occupation times formula and the characterization of $\langle X^\rho \rangle_t$ in [12],

$$\int_0^t f(X_s) \mathbb{1}_{X_s \neq 0} ds = \int_{\mathbb{R}} f(y) L_t^y(X) dy.$$

Thus, if $f(0) = 0$,

$$\int_0^t f(X_s) ds = \int_{\mathbb{R}} f(y) L_t^y(X) dy. \quad (73)$$

By applying consecutive change of variables and from (73),

$$\begin{aligned} \int_{\mathbb{R}} g_n(x) L_{[nt]/n}^{x/n^\alpha}(X) dx &= n^\alpha \int_{\mathbb{R}} g_n(n^\alpha x) L_{[nt]/n}^x(X) dx \\ &= n^\alpha \int_0^{[nt]/n} g_n(n^\alpha X_s) ds = \frac{n^\alpha}{n} \int_0^{[nt]} g_n(n^\alpha X_{s/n}) ds. \end{aligned} \quad (74)$$

Thus,

$$\begin{aligned} \frac{n^\alpha}{n} \sum_{i=1}^{[nt]} g_n(n^\alpha X_{\frac{i-1}{n}}) &= \frac{n^\alpha}{n} \sum_{i=1}^{[nt]} g_n(n^\alpha X_{\frac{i-1}{n}}) - \frac{n^\alpha}{n} \int_0^{[nt]} g_n(n^\alpha X_{s/n}) ds + \int_{\mathbb{R}} g_n(x) L_{[nt]/n}^{x/n^\alpha}(X) dx \\ &= \sum_{i=1}^{[nt]} \frac{n^\alpha}{n} \int_0^1 (g_n(n^\alpha X_{\frac{i-1}{n}}) - g_n(n^\alpha X_{\frac{i-1}{n} + \frac{s}{n}})) ds \\ &\quad + \frac{n^\alpha}{n} \int_0^{nt-[nt]} (g_n(n^\alpha X_{\frac{[nt]}{n}}) - g_n(n^\alpha X_{\frac{[nt]}{n} + \frac{s}{n}})) ds \\ &\quad + \int_{\mathbb{R}} g_n(x) L_{[nt]/n}^{x/n^\alpha}(X) dx. \end{aligned} \quad (75)$$

For the second additive term at the right hand side of (75),

$$\begin{aligned} \left| \frac{n^\alpha}{n} \int_0^{nt-[nt]} (g_n(n^\alpha X_{\frac{[nt]}{n}}) - g_n(n^\alpha X_{\frac{[nt]}{n} + \frac{s}{n}})) ds \right| \\ \leq \frac{n^\alpha}{n} \int_0^{nt-[nt]} |g_n(n^\alpha X_{\frac{[nt]}{n}}) - g_n(n^\alpha X_{\frac{[nt]}{n} + \frac{s}{n}})| ds \leq 2\|g\|_\infty \frac{n^\alpha}{n}, \end{aligned} \quad (76)$$

which converges to 0 as $n \rightarrow \infty$.

For the first additive term at the right hand side of (75), let

$$A_t^n := \sum_{i=1}^{[nt]} \frac{n^\alpha}{n} \int_0^1 (g_n(n^\alpha X_{\frac{i-1}{n} + \frac{s}{n}}) - g_n(n^\alpha X_{\frac{i-1}{n}})) ds, \quad (77)$$

$$B_t^n := \sum_{i=1}^{[nt]} \frac{n^\alpha}{n} \int_0^1 \mathbb{E}_x \left(g_n(n^\alpha X_{\frac{i-1}{n} + \frac{s}{n}}) - g_n(n^\alpha X_{\frac{i-1}{n}}) \middle| \mathcal{F}_{\frac{i-1}{n}} \right) ds. \quad (78)$$

As the cross terms have expectancy 0, from Minkowski's inequality,

$$\mathbb{E}_x (|A_t^n - B_t^n|^2) \leq 2 \frac{n^{2\alpha}}{n^2} \sum_{i=1}^{[nt]} \mathbb{E}_x \left(\int_0^1 (g_n(n^\alpha X_{\frac{i-1}{n} + \frac{s}{n}}) - g_n(n^\alpha X_{\frac{i-1}{n}})) ds \right)^2 \leq 4\|g\|_\infty^2 \frac{n^{2\alpha}}{n} \frac{[nt]}{n}, \quad (79)$$

which converges to 0 as $n \rightarrow \infty$. Thus, $A_t^n - B_t^n$ converges to 0 in $L^2(\mathbb{P}_x)$ and consequently in $L^1(\mathbb{P}_x)$. As such, proving that $A_t^n \xrightarrow[n \rightarrow \infty]{} 0$ in $L^1(\mathbb{P}_x)$ is equivalent to proving that $B_t^n \xrightarrow[n \rightarrow \infty]{} 0$ in $L^1(\mathbb{P}_x)$. To prove the latter, we define for each real-valued function h the functional,

$$T_n[h](x) = \int_0^1 (P_{n^{2\alpha}s/n}^{n^\alpha \rho} h(x) - h(x)) \, ds. \quad (80)$$

From (23),

$$B_t^n = \frac{n^\alpha}{n} \sum_{i=1}^{[nt]} T_n[g_n](n^\alpha X_{\frac{i-1}{n}}). \quad (81)$$

From (44), there exists a constant $K' > 0$ that does not depend on n or ρ such that

$$\mathbb{E}_x(|B_t^n|) \leq K' \left(\frac{u_n}{n} |T_n[g_n](u_n x)| + \sqrt{t} \lambda(|T_n[g_n]|) \right). \quad (82)$$

From (17) and (42), for every $t > 0$ and $\rho > 0$,

$$|P_t^\rho g_n(x)| \leq \|g\|_\infty, \quad (83)$$

By taking $K'' = K'(2\|g\|_\infty \vee 1)$,

$$\mathbb{E}_x(|B_t^n|) \leq K'' \left(\frac{n^\alpha}{n} + \sqrt{t} \lambda(|T_n[g_n]|) \right). \quad (84)$$

Thus, as $0 < \alpha < 1/2$, it remains to prove that $\lambda(|T_n[g_n]|) \rightarrow 0$. For this we use a Lipschitz approximation of g . In particular, as g is bounded and in $L^1(dx)$, for each p it is possible to find a Lipschitz function k_p such that, $k_p(0) = 0$ and $\lambda(|g - k_p|) < 1/p$. Let $k_p^n(x) = k_p(n^\alpha T(x/n^\alpha))$, from (7),

$$\lambda(|g_n - k_p^n|) < 1/p\epsilon. \quad (85)$$

Let $p_\rho(t, x, y)$ be the sticky Brownian motion transition kernel given in (37). As $p_\rho(t, x, y) = p_\rho(t, y, x)$ for every $x, y \in \mathbb{R}$,

$$\begin{aligned} \lambda(|P_t^\rho g_n - P_t^\rho k_p^n|) &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |g_n(y) - k_p^n(y)| p_\rho(t, x, y) \, dy \, dx = \\ &\int_{\mathbb{R}} |g_n(y) - k_p^n(y)| \int_{\mathbb{R}} p_\rho(t, x, y) \, dx \, dy = \int_{\mathbb{R}} |g_n(y) - k_p^n(y)| \int_{\mathbb{R}} p_\rho(t, y, x) \, dx \, dy \\ &\leq \int_{\mathbb{R}} |g_n(y) - k_p^n(y)| \, dy = \lambda(|g_n - k_p^n|). \end{aligned} \quad (86)$$

From (85) and (86),

$$\lambda(|T_n[g_n]|) \leq \frac{2}{p\epsilon} + \lambda(|T_n[k_p^n]|). \quad (87)$$

Thus, from (61),

$$\lambda(|T_n[g_n]|) \xrightarrow[n \rightarrow \infty]{} 0. \quad (88)$$

From (79), (84) and (88),

$$A_t^n = \sum_{i=1}^{[nt]} \frac{n^\alpha}{n} \int_0^1 (g_n(n^\alpha X_{\frac{i-1}{n} + \frac{s}{n}}) - g_n(n^\alpha X_{\frac{i-1}{n}})) ds \xrightarrow[n \rightarrow \infty]{P_x} 0. \quad (89)$$

From (50), (75), (76) and (89), for every $t \geq 0$,

$$\frac{n^\alpha}{n} \sum_{i=1}^{[nt]} g_n(n^\alpha X_{\frac{i-1}{n}}) \xrightarrow[n \rightarrow \infty]{P_x} \lambda(g) L_t^0(X). \quad (90)$$

If g is a positive function the processes $\frac{n^\alpha}{n} \sum_{i=1}^{[nt]} g_n(n^\alpha X_{\frac{i-1}{n}})$ are non-decreasing with P_x -almost surely, a continuous limit. Thus, from Lemma 2.6, the convergence is locally uniform in time, in probability. For an arbitrary g satisfying the conditions of Theorem 3.1, let $g = g^+ - g^-$, where $g^+(x) = \max\{g(x), 0\}$ and $g^-(x) = \max\{-g(x), 0\}$. Then as g^+ and g^- are both positive function and thus,

$$\frac{n^\alpha}{n} \sum_{i=1}^{[nt]} g_n^+(n^\alpha X_{\frac{i-1}{n}}) \xrightarrow[n \rightarrow \infty]{} \lambda(g^+) L_t^0(X), \quad \frac{n^\alpha}{n} \sum_{i=1}^{[nt]} g_n^-(n^\alpha X_{\frac{i-1}{n}}) \xrightarrow[n \rightarrow \infty]{} \lambda(g^-) L_t^0(X), \quad (91)$$

locally uniformly in time, in P_x -probability. Using the triangle inequality for the absolute value and the $L^\infty(0, t)$ -norm, the locally uniform convergence of (43) in P_x -probability is proven. \square

4 The case of sticky SDE solutions

The proof of Theorem 1.2 works by combining Theorem 3.1 with Girsanov theorem. In order to use the latter, one needs a proper path-wise characterization of sticky SDEs. In Section 4.1, we prove that a sticky SDEs can be expressed as the solution of a system of the form (92)-(93). Section 4.2 is dedicated to the proof of Theorem 1.2.

4.1 Path-wise description of a sticky SDE

In this section, we first prove the existence and uniqueness of the solution of (92)-(93). The system (92)-(93) has no strong solution (see [12]) as the amount of time they spend at 0 is random and not entirely determined by the driving Brownian motion. We then prove the characterization the law of the solution of (92)-(93). Finally, we prove that any sticky SDE can be expressed as the unique solution an SDE system of the form (92)-(93). The latter will allow us to use path-wise tools on sticky SDEs. Special cases of this are proven in [12] and [27] for the sticky Brownian motion and Ornstein-Uhlenbeck process with sticky reflection respectively.

Theorem 4.1. *We consider the following system*

$$dX_t = \mu(X_t) \mathbb{1}_{X_t \neq 0} dt + \sigma(X_t) \mathbb{1}_{X_t \neq 0} dB_t, \quad (92)$$

$$\mathbb{1}_{X_t=0} dt = \frac{\rho}{2} dL_t^0(X), \quad (93)$$

where B is a standard Brownian motion, $L^0(X)$ is the local time process of X at 0 and (μ, σ) are a pair of real-valued functions over \mathbb{I} that satisfy Condition (1.1). Then, the system (92)-(93) has a jointly unique weak solution.

Proof. Let $(\Omega, (\mathcal{F})_{t \geq 0}, P_x)$ be a probability space, B^1 a Brownian motion defined on $(\Omega, (\mathcal{F})_{t \geq 0}, P_x)$ and Y be the strong solution of

$$dY_t = \mu(Y_t) dt + \sigma(Y_t) dB_t^1, \quad (94)$$

such that

$$P_x(Y_0 = 0) = 1. \quad (95)$$

Let A and its right-inverse γ be the continuous and strictly increasing time-transforms defined for every $t \geq 0$ by

$$A(t) = t + \frac{\rho}{2} L_t^0(Y), \quad \gamma(t) = \inf \{s > 0 : A(s) > t\}. \quad (96)$$

Let X and B be the processes defined for every $t \geq 0$ by

$$X_t = Y_{\gamma(t)}, \quad (97)$$

$$B_t = B_{\gamma(t)}^1 + \int_0^t \mathbb{1}_{X_s=0} dB_s^0, \quad (98)$$

where B^0 is a Brownian motion independent of B^1 . From (19), for every $t \geq 0$,

$$B_{\gamma(t)}^1 = \int_0^{\gamma(t)} \mathbb{1}_{Y_s \neq 0} dB_s^1 = \int_0^t \mathbb{1}_{X_s \neq 0} dB_{\gamma(s)}^1 \quad (99)$$

and

$$\begin{aligned} \gamma(t) &= \langle B_{\gamma(\cdot)}^1 \rangle_t = \int_0^{\gamma(t)} \mathbb{1}_{Y_s \neq 0} ds \\ &= \int_0^{\gamma(t)} \mathbb{1}_{Y_s \neq 0} \left(ds + \frac{\rho}{2} dL_s^0(Y) \right) = \int_0^{\gamma(t)} \mathbb{1}_{Y_s \neq 0} dA(s) = \int_0^t \mathbb{1}_{X_s \neq 0} ds. \end{aligned} \quad (100)$$

Thus,

$$\langle B \rangle_t = \langle B_{\gamma(\cdot)}^1 + \int_0^{\cdot} \mathbb{1}_{X_s=0} dB_s^0 \rangle_t = \int_0^t \mathbb{1}_{X_s \neq 0} ds + \int_0^t \mathbb{1}_{X_s=0} ds = t. \quad (101)$$

From Lévy's characterization (see Theorem 3.16 of [21]-Chapter 3), B is a standard Brownian motion. From (19),

$$\begin{aligned}
X_t = Y_{\gamma(t)} &= \int_0^{\gamma(t)} dY_s = \int_0^{\gamma(t)} \mathbb{1}_{Y_s \neq 0} dY_s \\
&= \int_0^{\gamma(t)} \mathbb{1}_{Y_s \neq 0} \mu(Y_s) ds + \int_0^{\gamma(t)} \mathbb{1}_{Y_s \neq 0} \sigma(Y_s) dB_s^1 \\
&= \int_0^{\gamma(t)} \mathbb{1}_{Y_s \neq 0} \mu(Y_s) \left(ds + \frac{\rho}{2} dL_s^0(Y) \right) + \int_0^{\gamma(t)} \mathbb{1}_{Y_s \neq 0} \sigma(Y_s) dB_s^1 \\
&= \int_0^{\gamma(t)} \mathbb{1}_{Y_s \neq 0} \mu(Y_s) dA(s) + \int_0^{\gamma(t)} \mathbb{1}_{Y_s \neq 0} \sigma(Y_s) dB_s^1 \\
&= \int_0^t \mathbb{1}_{X_s \neq 0} \mu(X_s) ds + \int_0^t \mathbb{1}_{X_s \neq 0} \sigma(X_s) dB_{\gamma(s)}^1 \\
&= \int_0^t \mathbb{1}_{X_s \neq 0} \mu(X_s) ds + \int_0^t \mathbb{1}_{X_s \neq 0} \sigma(X_s) \{ dB_{\gamma(s)}^1 + \mathbb{1}_{X_s=0} dB_s^0 \} \\
&= \int_0^t \mathbb{1}_{X_s \neq 0} \mu(X_s) ds + \int_0^t \mathbb{1}_{X_s \neq 0} \sigma(X_s) dB_s. \quad (102)
\end{aligned}$$

Moreover,

$$\begin{aligned}
\int_0^t \mathbb{1}_{X_s=0} ds &= \int_0^t \mathbb{1}_{Y_{\gamma(s)}=0} dA(\gamma(s)) = \int_0^{\gamma(t)} \mathbb{1}_{Y_s=0} dA(s) \\
&= \int_0^{\gamma(t)} \mathbb{1}_{Y_s=0} \left(ds + \frac{\rho}{2} dL_s^0(Y) \right) = \frac{\rho}{2} \int_0^{\gamma(t)} \mathbb{1}_{Y_s=0} dL_s^0(Y) = \frac{\rho}{2} L_{\gamma(t)}^0(Y) = \frac{\rho}{2} L_t^0(X). \quad (103)
\end{aligned}$$

From (95) and (97), $P_x(X_0 = x) = 1$. From (102) and (103), the pair $(X_t, B_t)_{t \geq 0}$ solves (92)-(93), proving the existence of a solution.

For the uniqueness, we reset all notations. Let $(\Omega, (\mathcal{F}_t)_{t \geq 0}, P_x)$ be a probability space, $(\tilde{X}_t, \tilde{B}_t)_{t \geq 0}$ a solution of (92)-(93) such that $P_x(X_0 = x) = 1$ and \tilde{A} being the right-inverse of $\tilde{\gamma}$ which is the time-transform defined for every $t \geq 0$ by

$$\tilde{\gamma}(t) = \int_0^t \mathbb{1}_{\tilde{X}_s \neq 0} ds. \quad (104)$$

Let \tilde{Y} be the process defined for every $t \geq 0$ by

$$\tilde{Y}_t = \tilde{X}_{\tilde{A}(t)}. \quad (105)$$

From the definition of \tilde{Y} , $P_x(Y_0 = x) = 1$. From (93), if $L^0(\tilde{X})$ and $L^0(\tilde{Y})$ are the local times at 0 of \tilde{X} and \tilde{Y} respectively,

$$\begin{aligned}
t = \tilde{\gamma}(\tilde{A}(t)) &= \int_0^{\tilde{A}(t)} \mathbb{1}_{\tilde{X}_s \neq 0} ds = \tilde{A}(t) - \int_0^{\tilde{A}(t)} \mathbb{1}_{\tilde{X}_s=0} ds \\
&= \tilde{A}(t) - \frac{\rho}{2} L_{\tilde{A}(t)}^0(\tilde{X}) = \tilde{A}(t) - \frac{\rho}{2} L_t^0(\tilde{Y}). \quad (106)
\end{aligned}$$

The time-transform \tilde{A} is continuous and strictly increasing, thus $\tilde{\gamma}$ is its proper inverse. From (92),

$$\begin{aligned}\tilde{Y}_t &= \tilde{X}_{\tilde{A}(t)} = \tilde{X}_0 + \int_0^{\tilde{A}(t)} \mu(\tilde{X}_s) \mathbb{1}_{\tilde{X}_s \neq 0} ds + \int_0^{\tilde{A}(t)} \sigma(\tilde{X}_s) \mathbb{1}_{\tilde{X}_s \neq 0} d\tilde{B}_s \\ &= \tilde{X}_0 + \int_0^{\tilde{A}(t)} \mu(\tilde{X}_s) d\tilde{\gamma}(s) + \int_0^{\tilde{A}(t)} \sigma(\tilde{X}_s) \mathbb{1}_{\tilde{X}_s \neq 0} dB_s \\ &= \tilde{X}_0 + \int_0^t \mu(\tilde{Y}_s) ds + \int_0^{\tilde{A}(t)} \sigma(\tilde{X}_s) \mathbb{1}_{\tilde{X}_s \neq 0} d\tilde{B}_s. \quad (107)\end{aligned}$$

Let $\tilde{B}_t^1 = \int_0^{\tilde{A}(t)} \mathbb{1}_{\tilde{X}_s \neq 0} d\tilde{B}_s$ where,

$$\langle \tilde{B}^1 \rangle_t = \int_0^{\tilde{A}(t)} \mathbb{1}_{\tilde{X}_s \neq 0} ds = \tilde{\gamma}(\tilde{A}(t)) = t. \quad (108)$$

Thus, from Levy's characterization \tilde{B}^1 is a standard Brownian motion and as $d\tilde{B}_t^1 = \mathbb{1}_{\tilde{Y}_s \neq 0} d\tilde{B}_{\tilde{A}(t)}$,

$$\tilde{Y}_t = \tilde{X}_0 + \int_0^t \mu(\tilde{Y}_s) ds + \int_0^t \sigma(\tilde{Y}_s) \mathbb{1}_{\tilde{Y}_s \neq 0} d\tilde{B}_{\tilde{A}(s)} = \tilde{X}_0 + \int_0^t \mu(\tilde{Y}_s) ds + \int_0^t \sigma(\tilde{Y}_s) d\tilde{B}_s^1. \quad (109)$$

Let (Y, \tilde{B}^1) be the solution of (94) such that $P_x(Y_0 = x) = 1$, γ the right-inverse of $A(t) = t + \frac{\rho}{2} L_t^0(Y)$ and X the process such that $X_t = Y_{\gamma(t)}$ for every $t \geq 0$. From (100) and (104)

$$\gamma(t) = \tilde{\gamma}(t) \quad \text{and} \quad A(t) = \tilde{A}(t), \quad (110)$$

for every $t \geq 0$. From (105) and (110) and as $P(X_0 = Y_0 = \tilde{Y}_0 = x) = 1$, almost surely

$$Y_{\gamma(t)} = X_t = \tilde{Y}_{\tilde{\gamma}(t)}, \quad (111)$$

for every $t \geq 0$. Hence X is uniquely determined by $(B_t^1, L_t^0(Y))_{t \geq 0}$ and (92)-(93) has a unique solution. \square

Proposition 4.2. *If (X, B) is the joint solution of (92)-(93), then X is the (μ, σ, ρ) -sticky SDE solution.*

Proof. Let X be the solution of (92)-(93) and γ the time transform

$$\gamma(t) = \int_0^t \mathbb{1}_{X_s \neq 0} ds. \quad (112)$$

Moreover, let A be the right-inverse of γ and Y the process such that for every $t \geq 0$,

$$Y_t = X_{A(t)}. \quad (113)$$

From (94), (104), (110) and (111), the process Y solves (94). From Proposition 2.6 of [28]-Chapter VII and page 17 of [7], Y is the diffusion process defined through s and m where

$$s'(x) = e^{-\int_a^x \frac{2\mu(u)}{\sigma^2(u)} du}, \quad m(dx) = \frac{1}{s'(x)} \frac{2}{\sigma^2(x)} dx. \quad (114)$$

Then, $s(Y) = (s(Y_t))_{t \geq 0}$ is a diffusion process on natural scale with speed measure $m_{s(Y)}(dx) = m_Y(s^{-1}(dx))$. From Theorem 2.1, there exists a Brownian motion W such that,

$$s(Y_t) = W_{\gamma_{s(Y)}(t)}, \quad \text{for every } t \geq 0, \quad (115)$$

where $\gamma_{s(Y)}(t)$ is the right-inverse of $A_{s(Y)}(t) = \int_{\mathbb{R}} L_t^y(W) m_{s(Y)}(dy)$. From (113) and (115),

$$s(X_t) = W_{\gamma_{s(Y)}(\gamma(t))}, \quad \text{for every } t \geq 0. \quad (116)$$

From (106), the right-inverse of $\gamma_{s(Y)}(\gamma(t))$ is

$$A(A_{s(Y)}(t)) = A_{s(Y)}(t) + \frac{\rho}{2} L_{A_{s(Y)}(t)}^0(Y) = A_{s(Y)}(t) + \frac{\rho}{2} L_t^0(W) = \frac{1}{2} \int_{\mathbb{R}} L_t^y(W) \nu(dy), \quad (117)$$

where $\nu(dx) = m_{s(Y)}(dx) + \rho \delta_0(dx)$. Thus, from Corollary 2.2, $(s(X_t))_{t \geq 0}$ is a diffusion process on natural scale of speed measure $m(s^{-1}(dx)) + \rho \delta_0(dx)$. As s is continuous and invertible, X is a diffusion process with scale function s_X and speed measure m_X where

$$s_X(x) = s(x), \quad m_X(dx) = m(dx) + \rho \delta_0(dx) \quad (118)$$

or a (ρ, μ, σ) -sticky SDE solution. \square

Proposition 4.3. *Let X be a (μ, σ, ρ) -sticky SDE defined on a probability space $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}_x)$. Then, there exists a Brownian motion W such that under \mathbb{P}_x ,*

$$X_t = x + \int_0^t \mu(X_s) \mathbb{1}_{X_s \neq 0} ds + \int_0^t \sigma(X_s) \mathbb{1}_{X_s \neq 0} dW_s, \quad (119)$$

$$\int_0^t \mathbb{1}_{X_s=0} ds = \frac{\rho}{2} L_t^0(X). \quad (120)$$

Proof. Let A be the right-inverse of $\gamma = [t \mapsto \int_0^t \mathbb{1}_{X_s \neq 0} ds]$ and Y the process such that,

$$Y_t = X_{A(t)}, \quad (121)$$

for every $t \geq 0$. Let (s_X, m_X) and (s_Y, m_Y) be the scale function and speed measure pairs of X and Y respectively. From (114) and (118)

$$s_Y = s_X, \quad m_Y = m_X - \rho \delta_0. \quad (122)$$

From (15) and (122),

$$s'_Y(x) = e^{-\int_a^x \frac{2\mu(u)}{\sigma^2(u)} du}, \quad m_Y(dx) = \frac{1}{s'(x)} \frac{2}{\sigma^2(x)} dx. \quad (123)$$

Thus, from Theorem 3.12 of [28]-Chapter VII, the infinitesimal generator L_Y of Y is

$$L_Y f(x) = \mu(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x), \quad (124)$$

for every $x \in \mathbb{I}$ and $f \in \text{dom}(L_Y)$ where $\text{dom}(L_Y) = C^2(\mathbb{I})$. From Theorem 2.7 of [28]-Chapter VII, there exists a Brownian motion W^1 on an extension of $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, P_x)$ such that Y almost surely solves,

$$dY_t = \mu(Y_t) dt + \sigma(Y_t) dW_t^1, \quad (125)$$

for every $t \geq 0$. Let W be the process such that

$$W_t = W_{\gamma(t)}^1 + \int_0^t \mathbb{1}_{X_s=0} dW_s^0. \quad (126)$$

From (98) and (101) and Lévy's characterization, W is a standard Brownian motion. From (94), (104), (110) and (111), (X, W) jointly solve (119)-(120). \square

4.2 Proof of Theorem 1.2

Lemma 4.4 (Sticky Girsanov). *Let $(\Omega, (\mathcal{F})_{t \geq 0}, P)$ be a probability space and X the process that solves*

$$dX_t = \mu(X_t) \mathbb{1}_{X_t \neq 0} dt + \sigma(X_t) \mathbb{1}_{X_t \neq 0} dB_t, \quad (127)$$

$$\mathbb{1}_{X_t=0} dt = \frac{\rho}{2} dL_t^0(X), \quad (128)$$

where B is a P -Brownian motion. Let θ be a processes such that $P(\int_0^T \theta_s ds < \infty) = 1$, $\mathcal{E}(\theta)$ the process such that

$$\mathcal{E}_t(\theta) = \exp \left(\int_0^t \theta_s dB_s - \frac{1}{2} \int_0^t \theta_s^2 ds \right), \quad (129)$$

for every $t \geq 0$ and Q the probability measure such that $dQ = \mathcal{E}_t(\theta) dP$. Then, if E_P is the expectancy under P and $E_P(\mathcal{E}_t(\theta)) = 1$, the process X solves

$$d\tilde{X}_t = (\mu(\tilde{X}_t) + \theta_t \sigma(\tilde{X}_t)) \mathbb{1}_{\tilde{X}_t \neq 0} dt + \sigma(\tilde{X}_t) \mathbb{1}_{\tilde{X}_t \neq 0} d\tilde{B}_t, \quad (130)$$

$$\mathbb{1}_{\tilde{X}_t=0} dt = \frac{\rho}{2} dL_t^0(\tilde{X}), \quad (131)$$

where $\tilde{B}_t = B_t - \int_0^t \theta_s ds$ is a standard Brownian motion under Q .

Proof. Let X be the the solution of (127)-(128), γ the time-change $\gamma(t) = \int_0^t \mathbb{1}_{X_s \neq 0} ds$ for every $t \geq 0$, A its right-inverse and $Y = (X_{A(t)})_{t \geq 0}$. Let \tilde{B} be the process defined by

$\tilde{B}_t = B_t - \int_0^t \theta_s ds$ for every $t \geq 0$. Then, from Theorem 6.3 of [24], \tilde{B} is a standard Brownian motion under \mathbb{Q} and the probability measures \mathbb{P} and \mathbb{Q} are equivalent. By substitution,

$$dX_t = (\mu(X_t) + \theta_t \sigma(X_t)) \mathbb{1}_{X_t \neq 0} dt + \sigma(X_t) \mathbb{1}_{X_t \neq 0} d\tilde{B}_t. \quad (132)$$

Moreover, since: the local time $L_t^0(X)$ and the quadratic variation $\langle X \rangle_t$ are defined as limits in probability, $\mathbb{P} \sim \mathbb{Q}$ and

$$\langle X \rangle_t = \int_0^t \mathbb{1}_{X_s \neq 0} ds = t - \int_0^t \mathbb{1}_{X_s = 0} ds \quad (133)$$

Thus, (128) holds also under \mathbb{Q} and X solves (130)-(131). □

Lemma 4.5 (Sticky Itô formula). *Let X be a process defined on a probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}_x)$ such that $\mathbb{P}_x(X_0 = x) = 1$ and*

$$dX_t = \mu(X_t) \mathbb{1}_{X_t \neq 0} dt + \sigma(X_t) \mathbb{1}_{X_t \neq 0} dB_t, \quad (134)$$

$$\mathbb{1}_{X_t = 0} dt = \frac{\rho}{2} dL_t^0(X), \quad (135)$$

where B is a $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}_x)$ -standard Brownian motion. Then, for every real valued C^2 function f such that $f(0) = 0$ and $f'(0) \neq 0$, the process $f(X) = (f(X_t))_{t \geq 0}$ is solution of

$$df(X_t) = \left(f'(X_t) \mu(X_t) + \frac{1}{2} f''(X_t) \sigma^2(X_t) \right) \mathbb{1}_{X_t \neq 0} dt + f'(X_t) \sigma(X_t) \mathbb{1}_{X_t \neq 0} dB_t, \quad (136)$$

$$\mathbb{1}_{f(X_t) = 0} dt = f'(0) \frac{\rho}{2} dL_t^0(f(X)) \quad (137)$$

and $\mathbb{P}_x(f(X_0) = f(x)) = 1$.

Proof. The process X is a semi-martingale as,

$$X_t = X_0 + \int_0^t \mu(X_s) \mathbb{1}_{X_s \neq 0} ds + \int_0^t \sigma(X_s) \mathbb{1}_{X_s \neq 0} dB_s, \quad (138)$$

where $\int_0^t \mu(X_s) \mathbb{1}_{X_s \neq 0} ds$ is a process of bounded variation and $\int_0^t \sigma(X_s) \mathbb{1}_{X_s \neq 0} dB_s$ is a local martingale. Thus, we may apply the standard Itô formula for $f \in C^2(\mathbb{R})$,

$$\begin{aligned} df(X_t) &= f'(X_t) dX_t + \frac{1}{2} f''(X_t) d\langle X \rangle_t = \\ &= \left(f'(X_t) \mu(X_t) + \frac{1}{2} f''(X_t) \sigma^2(X_t) \right) \mathbb{1}_{X_t \neq 0} dt + f'(X_t) \sigma(X_t) \mathbb{1}_{X_t \neq 0} dB_t, \end{aligned} \quad (139)$$

thus proving (136). Lemma 2.4 and (135) yield (137), thus the result is proven. □

Proof (of Theorem 1.2). We suppose there exists a $\delta > 0$ such that

$$\delta \leq \sigma(x) \leq 1/\delta, \quad |\sigma'(x)| \leq 1/\delta, \quad (140)$$

for every $x \in \mathbb{I}$. From Proposition 4.3, there exists a Brownian motion B such that (X, B) jointly solves

$$dX_t = \mu(X_t) \mathbb{1}_{X_t \neq 0} dt + \sigma(X_t) \mathbb{1}_{X_t \neq 0} dB_t, \quad (141)$$

$$\mathbb{1}_{X_t=0} dt = \frac{\rho}{2} dL_t^0(X). \quad (142)$$

Let \mathbb{Q}_x be the probability measure such that $d\mathbb{Q}_x = \mathcal{E}(X)_t d\mathbb{P}_x$ where

$$\mathcal{E}(X)_t = \exp \left(\int_0^t \theta_s dB_s - \frac{1}{2} \int_0^t \theta_s^2 ds \right), \quad (143)$$

$$\theta_t = \sigma'(X_t) - \frac{\mu(X_t)}{\sigma(X_t)}. \quad (144)$$

From Lemma 4.4, (X, \tilde{B}) jointly solve

$$dX_t = \frac{1}{2} \sigma(X_t) \sigma'(X_t) \mathbb{1}_{X_t \neq 0} dt + \sigma(X_t) \mathbb{1}_{X_t \neq 0} d\tilde{B}_t, \quad (145)$$

$$\mathbb{1}_{X_t=0} dt = \frac{\rho}{2} dL_t^0(X), \quad (146)$$

where $\tilde{B} = B_t - \int_0^t \theta_s ds$ is a standard Brownian motion under \mathbb{Q}_x . Let S be the function defined for every $x \in \mathbb{R}$ by

$$S(x) = \int_0^x \frac{1}{\sigma(y)} dy. \quad (147)$$

We observe that S is strictly increasing and $S(0) = 0$. Thus, from (136)-(137), the process $X' = (S(X_t))_{t \geq 0}$ solves

$$dX'_t = \frac{1}{\sigma(X_t)} dX_t - \frac{1}{2} \frac{\sigma'(X_t)}{\sigma^2(X_t)} d\langle X \rangle_t = \mathbb{1}_{X'_t \neq 0} d\tilde{B}_t. \quad (148)$$

$$\mathbb{1}_{X'_t=0} dt = \frac{\rho}{2\sigma(0)} dL_t^0(X'). \quad (149)$$

From Proposition 4.2, X' is a sticky Brownian motion of parameter $\rho > 0$. Let U and T_0 be the functions defined for every $x \in \mathbb{R}$ by

$$U(x) = S^{-1}(x)/\sigma(0), \quad T_0(x) = T(U(x)). \quad (150)$$

Then,

$$T'_0(x) = T' \left(\frac{S^{-1}(x)}{\sigma(0)} \right) \frac{\sigma(x)}{\sigma(0)}, \quad T''_0(x) = T'' \left(\frac{S^{-1}(x)}{\sigma(0)} \right) \frac{\sigma^2(x)}{\sigma^2(0)} + T' \left(\frac{S^{-1}(x)}{\sigma(0)} \right) \frac{\sigma'(x)}{\sigma(0)}. \quad (151)$$

From (7), (140) and (151), for every $x \in \mathbb{R}$,

$$T_0(0) = T(0) = 0, \quad (152)$$

$$T'_0(0) = T'(0) = 1, \quad (153)$$

$$\epsilon\delta/\sigma(0) \leq T'_0(x) \leq 1/\epsilon\delta\sigma(0), \quad (154)$$

$$|T''_0(x)| \leq \|T''\|_\infty \|\sigma\|_\infty^2 / \sigma^2(0) + \|T'\|_\infty \|\sigma'\|_\infty / \sigma(0) \leq \frac{1}{\epsilon\delta\sigma(0)} \left(\frac{1}{\delta\sigma(0)} + 1 \right). \quad (155)$$

Thus, T_0 satisfies (7) for $\epsilon_{T_0} = \epsilon\delta(\sigma(0) \wedge 1/\sigma(0))$ and from Theorem 3.1 and (148)-(149),

$$\frac{n^\alpha}{n} \sum_{i=1}^{[nt]} g_n[T_0](n^\alpha X'_{\frac{i-1}{n}}) \xrightarrow{n \rightarrow \infty} \lambda(g)L_t^0(X'), \quad (156)$$

locally, uniformly in time, in \mathbb{Q}_x -probability, where $g_n[T_0](x) = g(n^\alpha T_0(x/n^\alpha))$. From (150) and since

$$g_n[T_0](n^\alpha x) = g(n^\alpha T_0(x)) = g(n^\alpha T(U(x))) = g_n[T](U(x)) \quad (157)$$

we get

$$\frac{n^\alpha}{n} \sum_{i=1}^{[nt]} g_n[T](n^\alpha X_{\frac{i-1}{n}}) \xrightarrow{n \rightarrow \infty} \lambda(g)L_t^0(X'), \quad (158)$$

locally, uniformly in time, in \mathbb{Q}_x -probability. From (4), the local time is defined as a limit in probability. Thus, from (158) and as $P_x \sim \mathbb{Q}_x$,

$$\frac{n^\alpha}{n} \sum_{i=1}^{[nt]} g_n[T](n^\alpha X_{\frac{i-1}{n}}) \xrightarrow{n \rightarrow \infty} \lambda(g)L_t^0(X'), \quad (159)$$

locally, uniformly in time, in P_x -probability. From Lemma 2.4, $L_t^0(X)/\sigma(0)$ is a version of $L_t^0(X')$. Thus, since we supposed (140), (9) is proven in the case of bounded σ , $1/\sigma$ and σ' and from section 2-5 of [17], the proof is extended to any $\sigma \in C^1$. \square

5 Proof of Corollary 1.3

From (93), the occupation/local time ratio is the stickiness parameter. Thus having an estimation of the two aforementioned quantities that does not require the knowledge of ρ allow us to use the ratio as a consistent estimator. We first show that the occupation time can be consistently approximated by a Riemann sum. Then, we use it along with Theorem 1.2 to prove Corollary 1.3.

Lemma 5.1. *Let X be a semi-martingale and $A^0(X)$ be its occupation time of 0 defined for every $t \geq 0$ by*

$$A_t^0(X) = \int_0^t \mathbb{1}_{X_s=0} ds. \quad (160)$$

Then,

$$\frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}}=0} \xrightarrow{n \rightarrow \infty} A_t^0(X), \quad (161)$$

locally uniformly in time, in probability.

Proof. As both $t \mapsto \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}}=0}$ and $A_t^0(X)$ are increasing processes, with the same argument as in the proof of Theorem 3.1, it suffices to prove the convergence in probability for each $t > 0$. If $\delta > 0$ and $\epsilon > 0$ are two positive numbers,

$$\begin{aligned} & \mathbb{P}_x \left(\left| \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}}=0} - A_t^0(X) \right| > \delta \right) \\ &= \mathbb{P}_x \left(\left| \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{|X_{\frac{i-1}{n}}| < \epsilon} - \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{0 < |X_{\frac{i-1}{n}}| < \epsilon} - A_t^0(X) \right| > \delta \right) \\ &\leq \mathbb{P}_x \left(\left| \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{|X_{\frac{i-1}{n}}| < \epsilon} - A_t^0(X) \right| + \left| \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{0 < |X_{\frac{i-1}{n}}| < \epsilon} \right| > \delta \right) \\ &\leq \mathbb{P}_x \left(\left| \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{0 < |X_{\frac{i-1}{n}}| < \epsilon} \right| > \frac{\delta}{2} \right) + \mathbb{P}_x \left(\left| \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{|X_{\frac{i-1}{n}}| < \epsilon} - A_t^0(X) \right| > \frac{\delta}{2} \right). \end{aligned} \quad (162)$$

From (36) for $h(x) = \mathbb{1}_{0 < |x| < \epsilon}$ and as $\sum_{i=1}^{[nt]} \frac{1}{\sqrt{i}} \leq 2\sqrt{nt}$,

$$\mathbb{E}_x \left(\frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{0 < |X_{\frac{i-1}{n}}| < \epsilon} \right) = \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{E}_x (h(X_{\frac{i-1}{n}})) \leq \frac{1}{n} 2\epsilon K \sqrt{n} \sum_{i=1}^{[nt]} \frac{1}{\sqrt{i}} \leq 4\epsilon K \sqrt{t} \quad (163)$$

Thus, from Markov's inequality,

$$\mathbb{P}_x \left(\left| \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{0 < |X_{\frac{i-1}{n}}| < \epsilon} \right| > \frac{\delta}{2} \right) \leq \frac{8\epsilon K \sqrt{t}}{\delta}. \quad (164)$$

We consider the following functions:

$$\phi(x) = (2 - |x|) \mathbb{1}_{1 \leq |x| < 2} + \mathbb{1}_{|x| < 1}, \quad (165)$$

$$\psi(x) = 2(1 - |x|) \mathbb{1}_{0.5 \leq |x| < 1} + \mathbb{1}_{|x| < 0.5}. \quad (166)$$

The functions ϕ and ψ are both continuous and bounded with compact support and

$$\psi(x) \leq \mathbb{1}_{|x| < 1} \leq \phi(x). \quad (167)$$

The composed function $\phi(X_t)$ and $\psi(X_t)$ are both a.s. continuous functions of t , hence a.s. Riemann integrable. Thus,

$$\frac{1}{n} \sum_{i=1}^{[nt]} \phi\left(\frac{1}{\epsilon} X_{\frac{i-1}{n}}\right) \xrightarrow{n \rightarrow \infty} \int_0^t \phi\left(\frac{1}{\epsilon} X_s\right) ds \quad (168)$$

$$\frac{1}{n} \sum_{i=1}^{[nt]} \psi\left(\frac{1}{\epsilon} X_{\frac{i-1}{n}}\right) \xrightarrow{n \rightarrow \infty} \int_0^t \psi\left(\frac{1}{\epsilon} X_s\right) ds \quad (169)$$

From Lebesgue convergence theorem both $\int_0^t \phi\left(\frac{1}{\epsilon} X_s\right) ds$ and $\int_0^t \psi\left(\frac{1}{\epsilon} X_s\right) ds$ converge to $A_t^0(X)$ as $\epsilon \rightarrow 0$. Thus for each $\delta > 0$ there exists an $\epsilon > 0$ such that

$$\mathbb{P}_x\text{-}\limsup_n \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{|X_{\frac{i-1}{n}}| < \epsilon} \leq A_t^0(X) + \delta, \quad (170)$$

$$\mathbb{P}_x\text{-}\liminf_n \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{|X_{\frac{i-1}{n}}| < \epsilon} \geq A_t^0(X) - \delta. \quad (171)$$

Thus, for each $\delta > 0$ there exists an $\epsilon_0 > 0$ such that

$$\mathbb{P}_x \left(\left| \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{|X_{\frac{i-1}{n}}| < \epsilon} - A_t^0(X) \right| > \frac{\delta}{2} \right) \xrightarrow{n \rightarrow \infty} 0. \quad (172)$$

From (162), (164) and (172), for each $\epsilon' > 0$ by choosing $\epsilon' = \epsilon\delta$ in (164), there exists a $\delta > 0$ such that

$$0 \leq \limsup_n \mathbb{P}_x \left(\left| \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}}=0} - A_t^0(X) \right| > \delta \right) \leq 4\epsilon K\sqrt{t}. \quad (173)$$

Thus, $\frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}}=0}$ converges in probability to $A_t^0(X)$. From (??), $A_t^0(X)$ admits almost surely a continuous version. Thus, from Lemma 2.6, (161) is proven. \square

Proof (of Corollary 1.3). We consider that we are on the event $\mathcal{L} = \{L_t^0(X) > 0\}$. By integrating (93) we observe that

$$A_t^0(X) = \frac{\rho}{2} L_t^0(X). \quad (174)$$

Thus,

$$\mathcal{L} = \{L_t^0(X) > 0\} = \{A_t^0(X) > 0\}. \quad (175)$$

From (9) and (161),

$$\frac{(1/n) \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}}=0}}{(n^\alpha/n) \sum_{i=1}^{[nt]} g(n^\alpha X_{\frac{i-1}{n}})} \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} \frac{A_t^0(X)}{(\lambda(g)/\sigma(0))L_t^0(X)}. \quad (176)$$

On the event \mathcal{L} , both terms in the ratio on the right-hand side of (176) are strictly positive. As such, the ratio is a well-defined real-valued random variable. Thus, from (174) we get (11) on \mathcal{L} . \square

6 Numerical experiments

In this section, we present numerical simulations to assess the asymptotic behavior of the local time approximation (9) and the stickiness parameter estimator (11). We simulate trajectories of an approximation process of the sticky Brownian motion of parameter ρ , we then compare the stickiness parameter estimations (11) with the true value of ρ . For the numerical simulations we use the Space-time Markov chain approximation or STMCA Algorithm [2]. This algorithm uses grid-valued random walks to approximate the law of any general diffusion process. The STMCA Algorithm is particularly adapted to our problematic as:

- It is well suited for the simulation of sticky singular one-dimensional diffusions.
- By suitable choice of the grid, we can control the amount of path-observations of X observed through the test function g in (177). In particular, for grids that satisfy the condition in Corollary 2.5 of [2], the convergence speed of the STMCA algorithm is optimal. Thus, if one increases the precision of such a grid \underline{g} around 0, we approximate accurately X with \underline{g} -valued STMCA random walks and feed more path-observations to the statistic $T^{(1)}$ in (177).

The statistic: Let X be a sticky Brownian motion of parameter ρ , g the function defined for every $x \in \mathbb{R}$ by $g(x) = \mathbb{1}_{1 < |x| < 5}/8$ and $\alpha \in (0, 1/2)$. From (9), and (11), the statistics

$$T_n^{(1)}(X) := \frac{n^\alpha}{n} \sum_{i=1}^{[nt]} g(n^\alpha X_{\frac{i-1}{n}}^\rho) \quad \text{and} \quad T_n^{(2)}(X) := \frac{2 \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}}=0}}{T_n^{(1)}(X)} \quad (177)$$

converge to $L_t^0(X)$ and ρ respectively uniformly in time, in probability. We use the latter statistic as proxy to assess the properties of the local time approximation.

The grid: In the simulations we use the grid:

$$\underline{g}_1(h) = \{0\} \cup \{\pm x_j(h); j \in \mathbb{N}\}, \quad (178)$$

where $x_0(h) = 0$ and $\{x_j(h)\}_{j \in \mathbb{N}}$ is recursively defined for every $j \in \mathbb{N}$ by:

$$x_j(h) = x_{j-1}(h) + \left(h^2 \frac{1}{|x_{j-1}| + 1} + h \left(1 - \frac{1}{|x_{j-1}| + 1} \right) \right) \mathbb{1}_{|x_{j-1}| < 1} + h \mathbb{1}_{|x_{j-1}| \geq 1}. \quad (179)$$

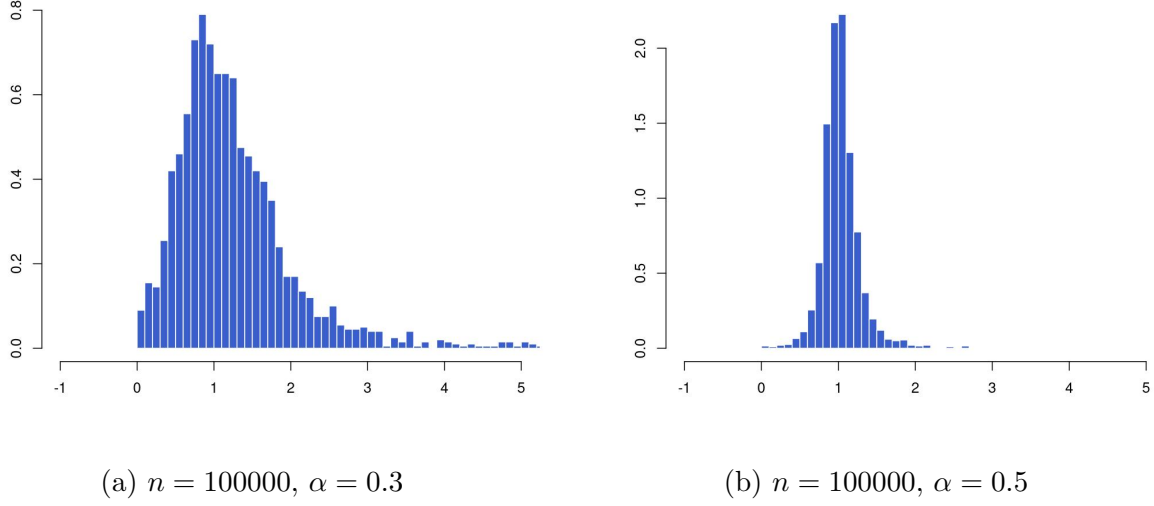


Figure 1: Stickiness parameter estimations histograms. (true value $\rho = 1$)

Simulation results

In this section we present Monte Carlo simulation results. The integer N_{MC} will be the Monte Carlo simulation size. For every $j \leq N_{\text{MC}}$ we simulate the path of an approximation process X_t^j . To assess the quality of each Monte Carlo estimation we use the following metrics:

- $(\hat{\rho}_{\text{MC}}, \hat{S}_{\text{MC}}^2, \hat{\sigma}_{\text{MC}})$: Monte Carlo estimation, variance and standard deviation of the estimated stickiness parameters,
- $\widehat{\text{acc}}$: average number of path-values observed by g , i.e

$$\widehat{\text{acc}} = \frac{1}{N_{\text{MC}}} \sum_{j=1}^{N_{\text{MC}}} \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{g(X_{(i-1)/n}^j) \neq 0}, \quad (180)$$

- rej : percentage of trajectories where the local time estimation equals 0, i.e

$$\text{rej} = \# \left\{ j \leq N_{\text{MC}} : \frac{n^\alpha}{n} \sum_{i=1}^{[nt]} g(n^\alpha X_{\frac{i-1}{n}}^j) = 0 \right\}. \quad (181)$$

Within each of the following tables we use the same simulated STMCA trajectories of the sticky Brownian motion of parameter $\rho = 1$:

α	n	$\hat{\rho}_{\text{MC}}$	\hat{S}_{MC}^2	$\hat{\sigma}_{\text{MC}}$	$\widehat{\text{acc}}$	rej/ N_{MC}
0.6	20000	1.04379	0.14514	0.380973	0.0121671	0/1000
0.6	40000	1.02967	0.03178	0.178288	0.0079693	0/1000
0.6	100000	1.00135	0.01592	0.126199	0.0046799	0/1000
0.6	300000	1.00638	0.00850	0.092235	0.0023960	0/1000

Table 1: Simulation results for fixed α and different values of n .

α	n	$\hat{\rho}_{\text{MC}}$	\hat{S}_{MC}^2	$\hat{\sigma}_{\text{MC}}$	$\widehat{\text{acc}}$	rej/ N_{MC}
0.3	100000	1.28303	0.66712	0.81677	0.126344	0/2000
0.4	100000	1.12398	0.38812	0.62299	0.043843	0/2000
0.5	100000	1.04014	0.09153	0.30253	0.014309	0/2000
0.6	100000	1.01083*	0.04976*	0.22307*	0.004598*	1/2000
0.7	100000	1.06961*	0.14296*	0.37810*	0.001367*	13/2000
0.8	100000	1.01483*	0.12386*	0.35194*	0.000467*	23/2000

Table 2: Simulation results for fixed n and different values of α . Estimation with an asterisk were performed removing the trajectories with $T_n^{(1)}(X) = 0$.

α	n	$\hat{\rho}_{\text{MC}}$	\hat{S}_{MC}^2	$\hat{\sigma}_{\text{MC}}$	$\widehat{\text{acc}}$	rej/ N_{MC}
0.55536	20000	1.04813	0.094319	0.307115	0.0190297	0/1000
0.519033	40000	1.04893	0.095243	0.308616	0.0190053	0/1000
0.477724	100000	1.04946	0.100884	0.317623	0.0190204	0/1000
0.436109	300000	1.04931	0.096736	0.311024	0.0190139	0/1000

Table 3: Simulation results for (n, α) satisfying $\log n = 5.5/\alpha$.

Observations

We observe the following:

- The higher the value of $\alpha \in (0, 1)$, the lower the estimated variance and the better the approximation of the local time,
- Also, the higher the α , the more the trajectory of X^ρ is inflated and the less things are observed through g . Thus, having a finite set of path-wise observations of X^ρ , one must find an $\alpha \in (0, 1)$ large enough to trigger the asymptotic regime of (6) and low enough so we do not dump too much path-wise observations.
- In Table 3 we see that for a fixed $c > 0$, every (n, α) such that $\log n = c/\alpha$ yield the same Monte Carlo variance. This relation can be guessed from (5) and (6),

- The convergence (9) seems to hold for $\alpha \in [1/2, 1)$, valued not covered by Theorem 1.2. We thus conjecture the following:

Conjecture 6.1. *Theorem 1.2 holds for any $\alpha \in (0, 1)$ and the rate of convergence of both (11) and (9) are increasing in terms of α .*

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