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Efficient Solvers based on Hybrid High-Order (HHO) methods for flow simulations in fractured rocks.

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Work in collaboration with

Patrick Laug (Inria GAMMA3, Saclay)

Simon Legrand (Inria SED Paris)

Caroline Darcel and Romain Le Goc (ITASCA S.A.S)

Philippe Davy (Geosciences Rennes, CNRS)

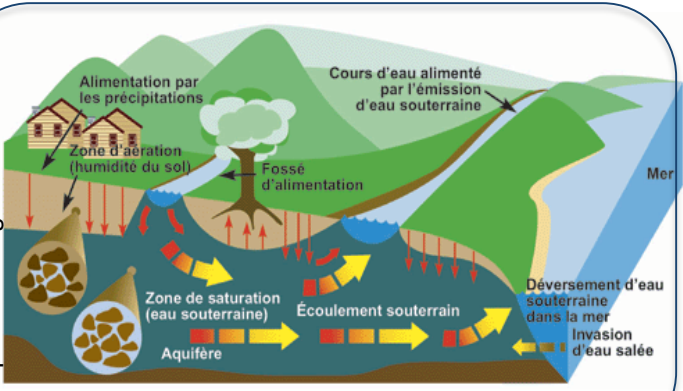
Context

Context

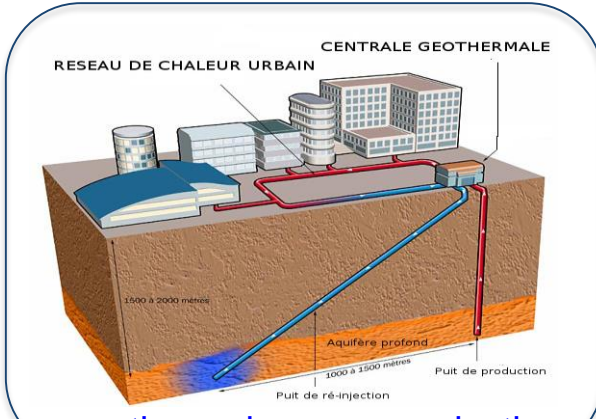
Subsurface is used in various applications covering **environment** and **energy**

- water extraction (for drinking, irrigation, industrial processes, ...)
- geothermal energy production
- oil and gas extraction, nuclear waste storage, CO₂ sequestration, ...

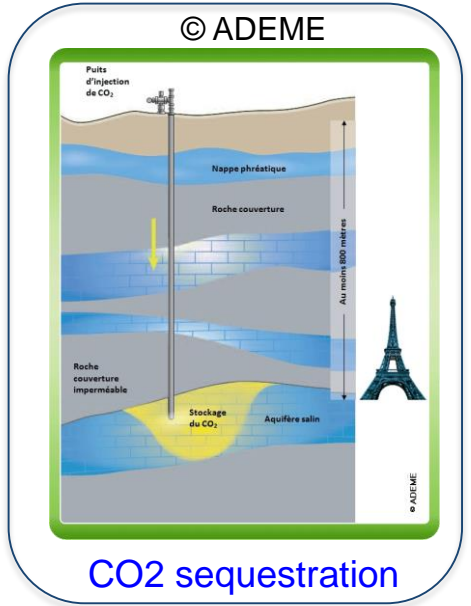
© <http://www.ec.gc.ca/eau-water/>



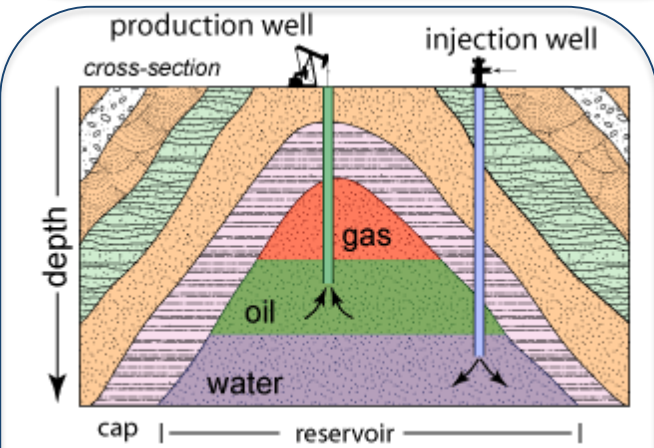
water extraction



geothermal energy production



CO2 sequestration

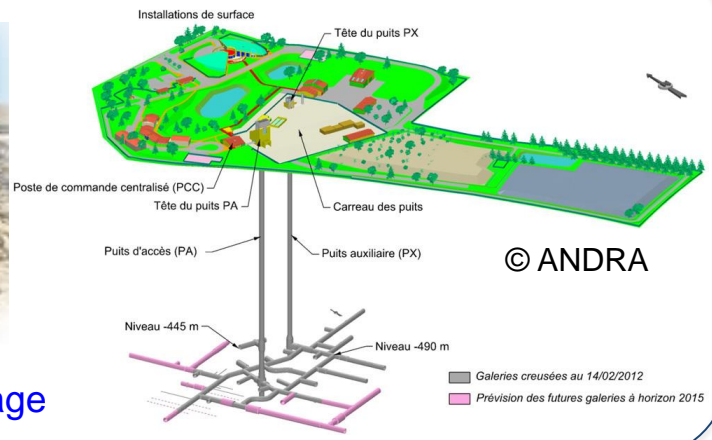


oil and gas extraction
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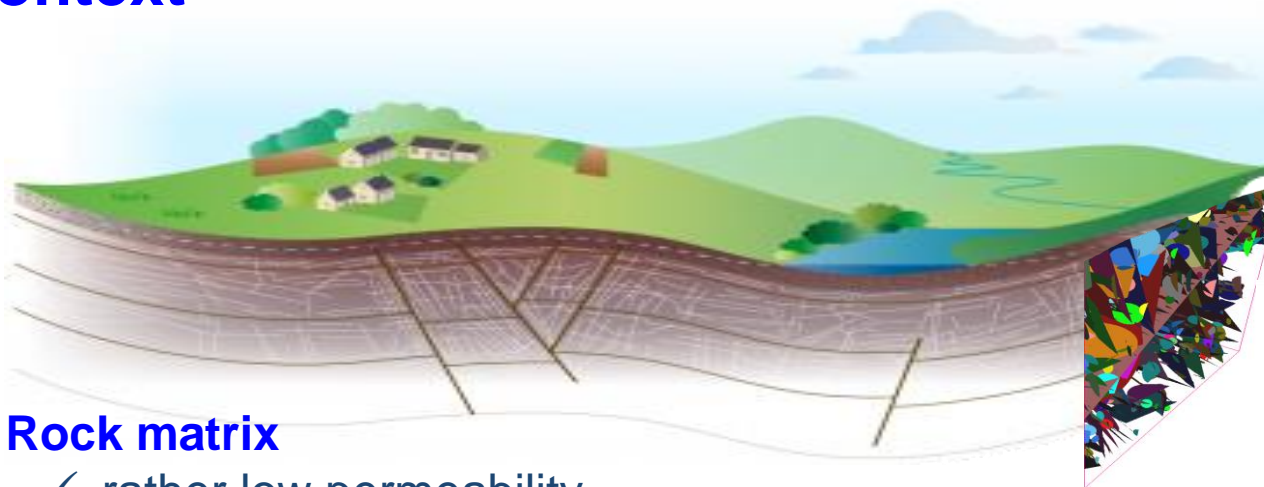
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waste storage



© ANDRA

Context



- **Rock matrix**
 - ✓ rather low permeability
- **Fractures**
 - ✓ usually high permeability
 - ✓ diversity in size and organization
 - ✓ play a key role in many physical and chemical phenomena (they cannot be neglected in the modelling)

coupled physical and chemical phenomena



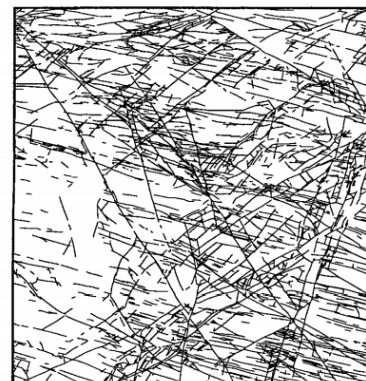
Coal cleats (Australia)
5 cm



Granite (Sweden)
1 m

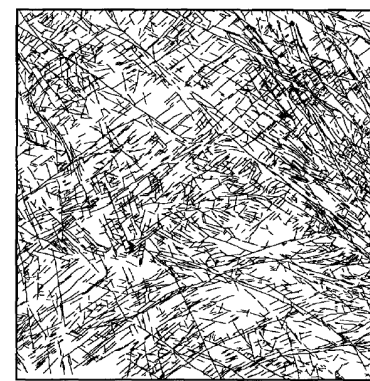
Sandstones (Norway)

MAP 4: $H = 35m$, Area = $90m \times 90m$



100 m

MAP 7: $H = 370m$, Area = $720m \times 720m$



1 km

Groundwater velocity:

- Gravel, sand: a few m / h
- fractured zones: up to several m / hour
- Clay: <1 m / 100 years and for some, a few cm / 10,000 years!

Challenges

- **Challenging objective:** realistic models and flow simulations of networks with thousands to millions of fractures ($\sim 10^3$ to $\sim 10^6$)
- **Main goal:** to develop strategies to reduce computational costs and increase robustness for accurate large scale numerical simulations
- **A combination of expertise:**
 1. **Designing realistic models** of fractured media
 - ✓ Géosciences – Rennes
 - ✓ Itasca Consultants – Lyon / Rennes
 2. **Advanced mesh generation:** intersecting patches, mesh gradation, matching and non-matching meshes at the intersections
 - ✓ Inria team GAMMA – Saclay
 3. **Solving efficiently the flow problem:** mixed hybrid methods, mortar techniques, domain decomposition methods
 - ✓ Inria team SERENA – Paris
 - Seek **collaborations** regarding the linear solvers (direct or iterative)

Modelling

UFM (Unified Fracture Model)

Davy, P., R. Le Goc, and C. Darcel (2013), A model of fracture nucleation, growth and arrest, and consequences for fracture density and scaling, *Journal of Geophysical Research: Solid Earth*, 118(4), 1393-1407

In 2012,
6,845 fractures

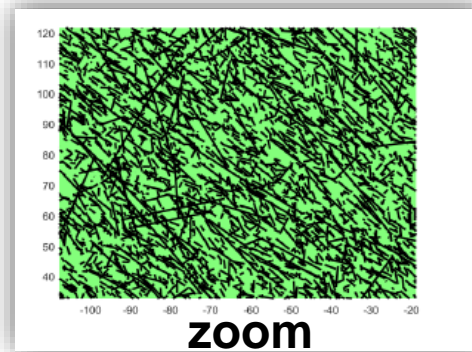
de Dreuzy, J.-R., et al., Synthetic benchmark for modeling flow in 3D fractured media. *Computers & Geosciences* (2012),



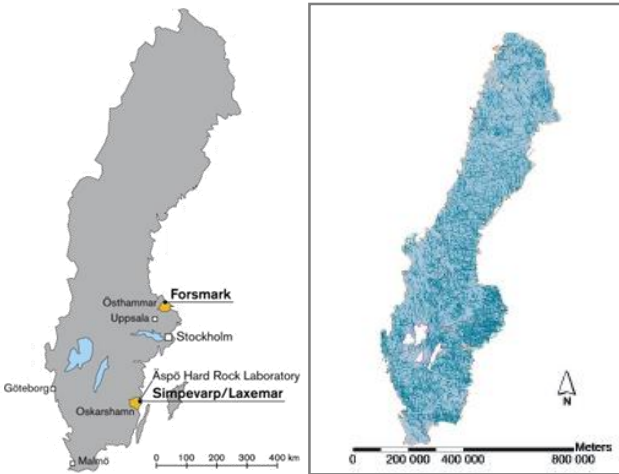
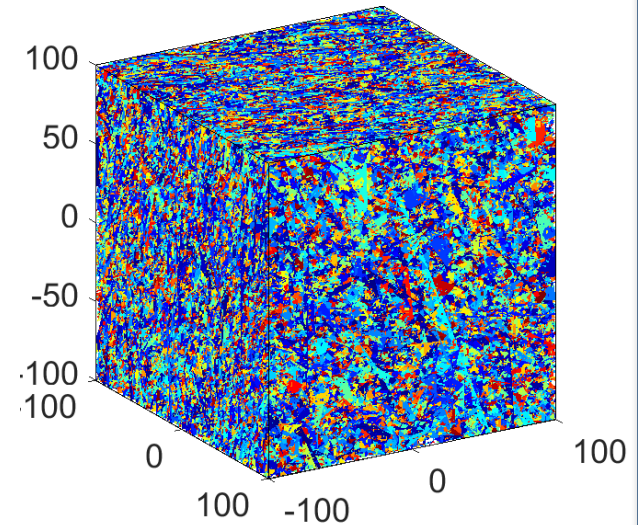
DFN test cases
provided by the
LabCom factory

L=200 m
1,176,566 fractures
2,410,539 intersections

Transmissivity range (one value per fracture):
[3.35-06; 25.8]



The largest fracture:
11,710 intersections



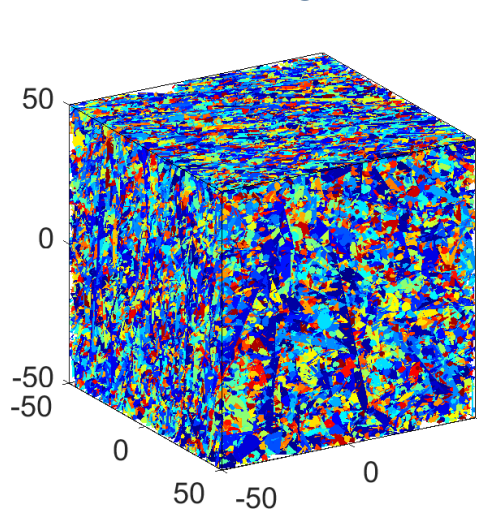
- **Itasca Consultants (France):** expertise and software in geomechanics
- **SKB (Sweden):** nuclear fuel and waste management

MAP 7: $H = 370m$, Area = $720m \times 720m$

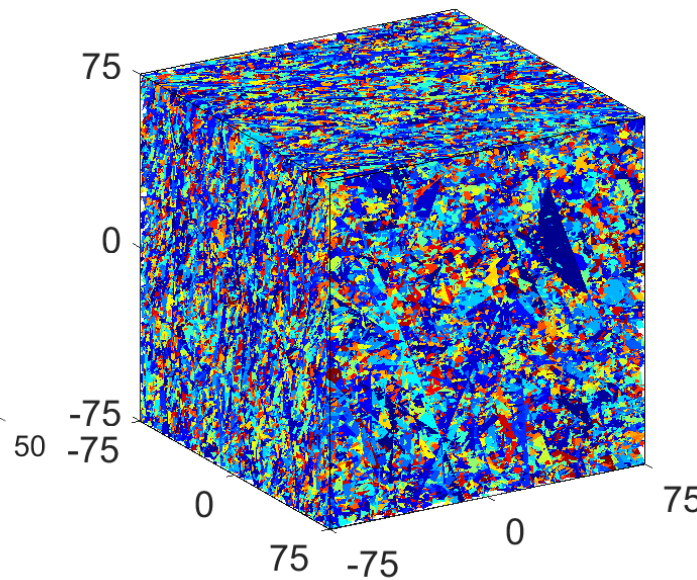


DFN test cases provided by the LabCom factory

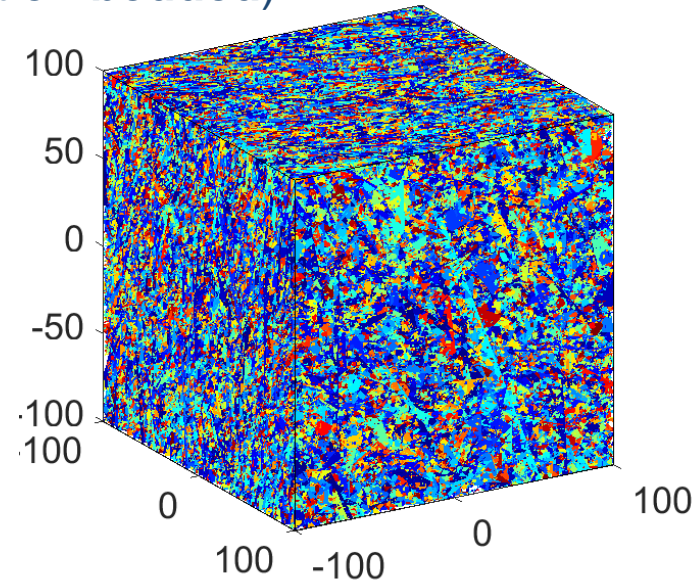
□ We change the cube size L to work on different (but **embedded**) DFN



L=100 m



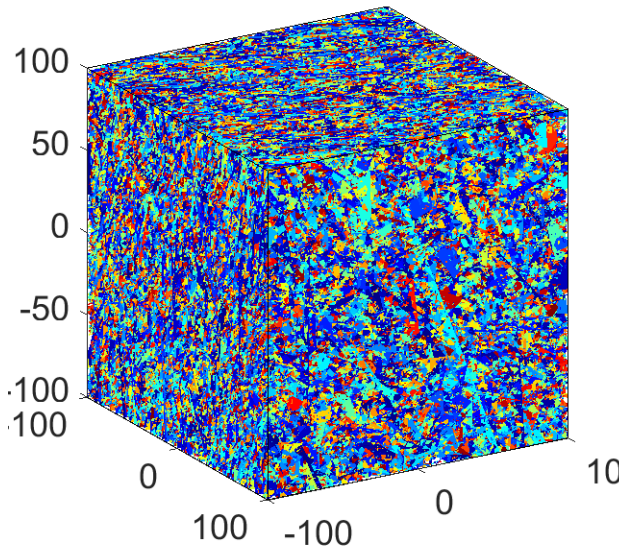
L=150 m



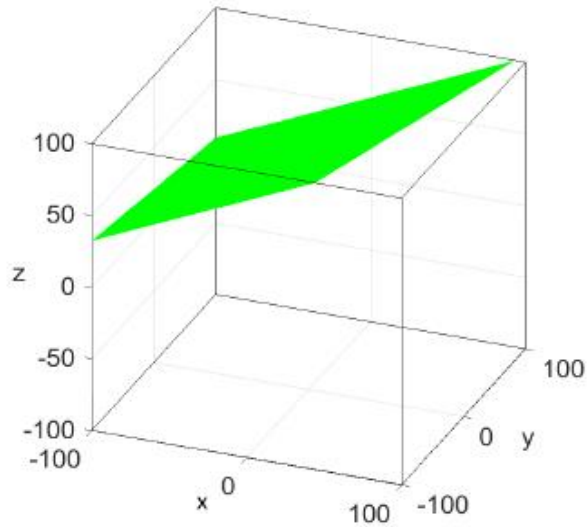
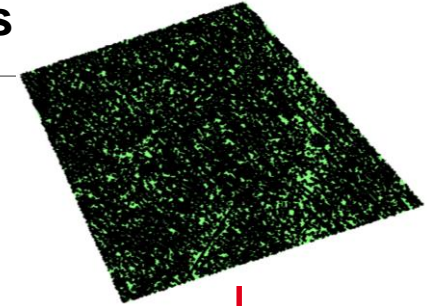
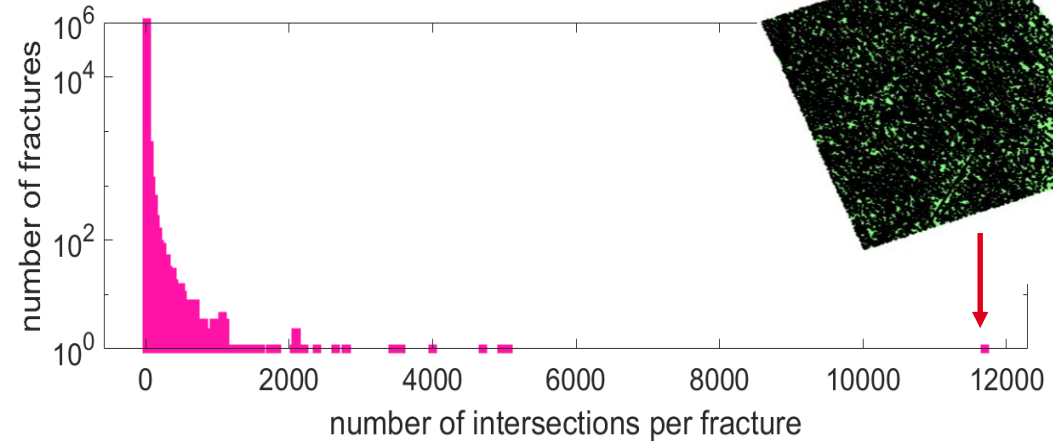
L=200 m

Cube size L	# fractures	# intersections	Max(#inter per fracture)	Transmissivity range [$\text{m}^2 \cdot \text{s}^{-1}$]
100	152,405	302,907	1,022	[3.5e-06; 20.33]
150	508,339	1,031,231	4,930	[3.38-06; 25.8]
200	1,176,566	2,410,539	11,710	[3.35-06; 25.8]

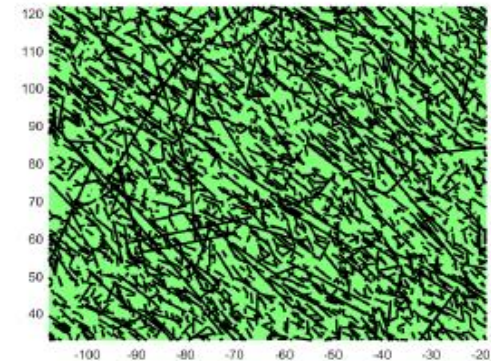
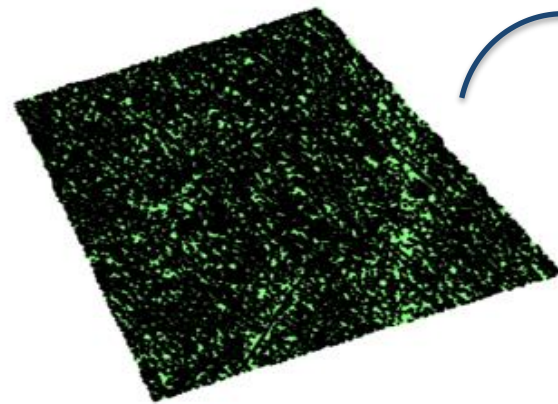
DFN test cases provided by the LabCom factory



L=200 m
1,176,566 fractures



zoom



11,710 intersections

The flow problem

J. Erhel, J.-R. de Dreuzy, and B. Poirriez. SIAM J. Sci. Comput. (2009)

J. Maryska, O. Severyn, and M. Vohralik, Computational Geosciences (2004)

Assumptions:

- The rock matrix is impervious: flow is **only simulated in the fractures**
- Study of steady state flow
- There is no longitudinal flux in the intersections of fractures

Flow equations within each fracture f_i with κ_i a positive definite tensor

$$\nabla \cdot \mathbf{u}_i(\mathbf{x}) = g_i(\mathbf{x}), \quad \text{for } \mathbf{x} \in f_i, \quad (\text{Continuity equation})$$

$$\mathbf{u}_i(\mathbf{x}) = -\kappa_i \nabla p_i(\mathbf{x}), \quad \text{for } \mathbf{x} \in f_i. \quad (\text{Poiseuille's law})$$

$$p_i(\mathbf{x}) = p_i^D(\mathbf{x}), \quad \text{for } \mathbf{x} \in \Gamma^D \cap \Gamma_i, \quad (\text{Dirichlet boundary conditions})$$

$$\mathbf{u}_i(\mathbf{x}) \cdot \mathbf{n} = q_i^N(\mathbf{x}), \quad \text{for } \mathbf{x} \in \Gamma^N \cap \Gamma_i, \quad (\text{Neumann boundary conditions})$$

$$\mathbf{u}_i(\mathbf{x}) \cdot \mathbf{n} = 0, \quad \text{for } \mathbf{x} \in \Gamma_i \setminus (\Gamma_i \cap (\Gamma^D \cup \Gamma^N)). \quad (\text{Impervious rock matrix})$$

Continuity conditions at each intersection I_m with I_m the m^{th} intersection, $m = 1, \dots, N_I$ N_I total number of intersections

$$p_k^{m,+} = p_l^{m,+} = p_k^{m,-} = p_l^{m,-}, \quad \text{on } I_m, \quad (\text{Continuity of the hydraulic heads})$$

$$\sum_{j \in \mathcal{S}_m} \mathbf{u}_j \cdot \mathbf{n}_{j,m}^+ + \mathbf{u}_j \cdot \mathbf{n}_{j,m}^- = 0, \quad \text{on } I_m. \quad (\text{Conservation of fluxes})$$

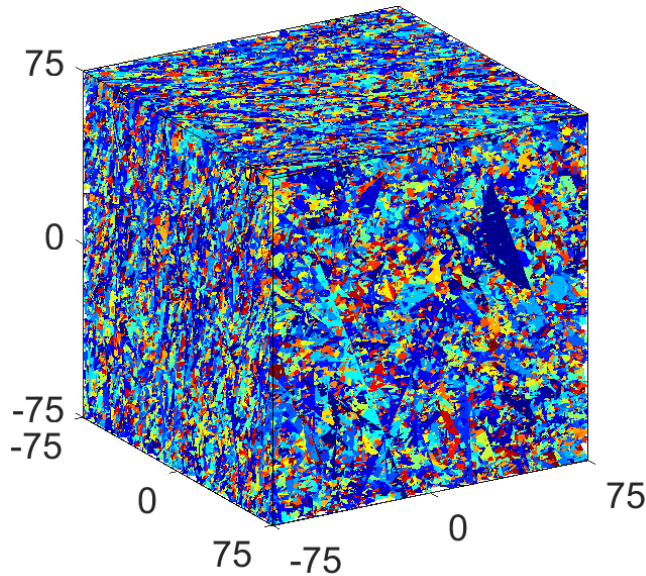
Mesh generation

Mesh generation

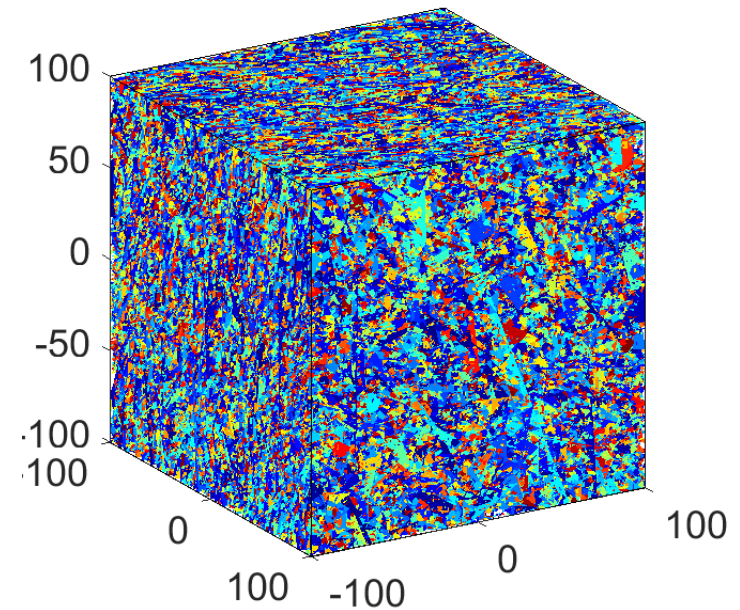
□ MODFRAC software
(Inria Gamma & UTT)

H. Borouchaki, P. Laug, P.L. George, Parametric surface meshing using a combined advancing-front & generalized-Delaunay approach, Int. J. for Numerical Methods in Engineering, 2000

Input: DFN (up to millions of disks), delimiting cube, mesh size specifications



L=150 m
508,339 fractures

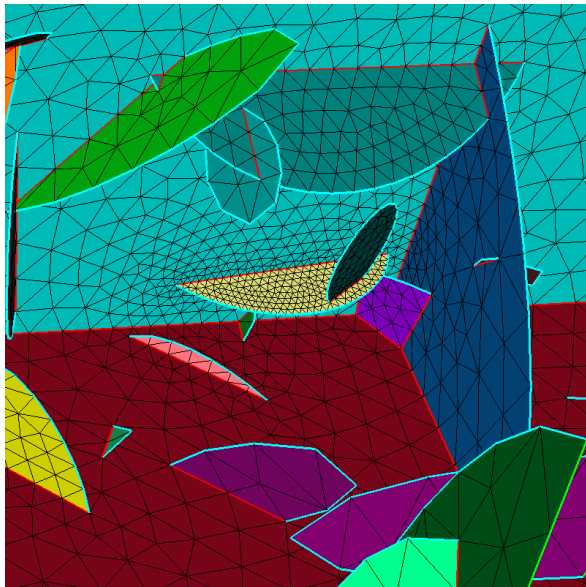


L=200 m
1,176,566 fractures

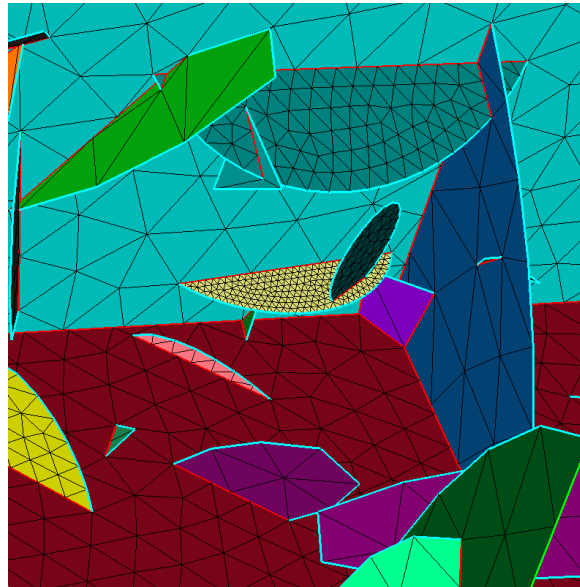
Mesh generation of the test cases

□ MODFRAC software
(Inria Gamma & UTT)

Cube size L	# fractures	#triangles	Min area	Mean(Q)	Min(Q)	Time hh:mm:ss
100	152,405	2,770,604	1.4e-09	0.66	3.1e-05	00:01:34
150	508,339	12,890,943	1.16e-11	0.77	1e-4	00:06:29
200	1,176,566	20,522,575	1.16e-11	0.63	9.25e-05	00:23:25



Matching surface mesh



Non-matching surface mesh

$$Q(T) := 4\sqrt{3} \frac{|T|}{l_1^2 + l_2^2 + l_3^2}$$

On a Laptop Intel Core i7
4 cores CPU
32GiB RAM

Parallelism using
4 POSIX-threads

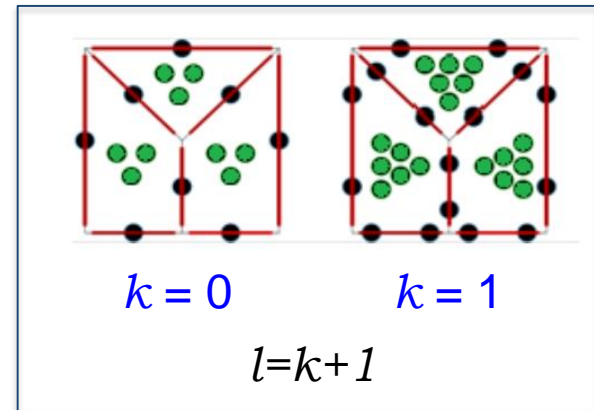
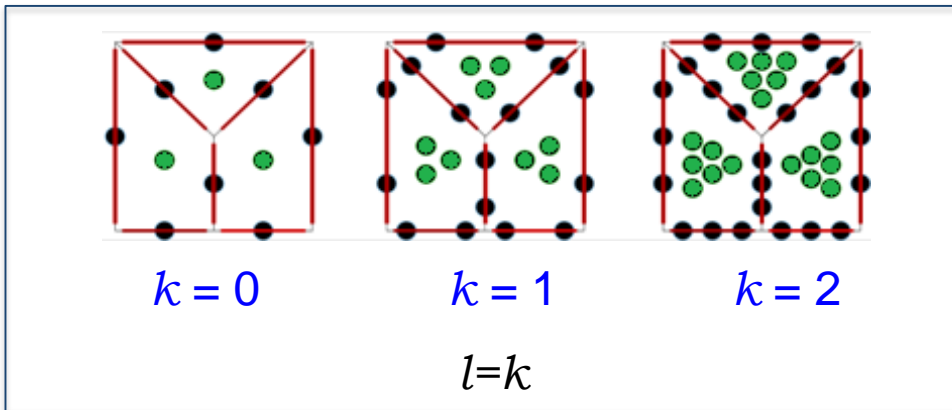
H. Borouchaki, P. Laug, P.L. George,
Parametric surface meshing
using a combined advancing-front &
generalized-Delaunay approach,
Int. J. for Numerical Methods in
Engineering, 2000

Numerical method: the HHO method

The Hybrid High Order (HHO) method

Di Pietro, D.A and Ern, A. and Lemaire, S.
Computational Methods in Applied
Mathematics (2014)

- Discrete unknowns on each finite element are:
 - On mesh faces
 - Inside the elements
- HHO unknowns are polynomial coefficients (no physical interpretation)
 - ✓ On mesh faces of order k
 - ✓ Inside the elements of order l



In our DFN applications, we choose $l = k + 1$ to get a simpler stabilization operator than the equal-order choice $l = k$.

- Algebraically, we obtain a Schur complement system for only the faces unknowns (static condensation) and known to be symmetric positive definite.

The Hybrid High Order (HHO) method

Di Pietro, D.A and Ern, A. and Lemaire, S.
Computational Methods in Applied
Mathematics (2014)

- HHO required two main ingredients:
 - ✓ Local gradient reconstruction from cell and face unknowns
 - ✓ A stabilization term ensures that the traces of the cell unknowns and the face unknowns match in a least-squares sense.
- Conservative fluxes are built from the sum of two terms
 - ✓ the normal component of the reconstructed gradient
 - ✓ A correction term coming from the stabilization
- Main advantages of HHO:
 - ✓ **Feature 1:** Globally and locally **mass conservative method** (whatever $k \geq 0$)
 - ✓ **Feature 2:** Face polynomials of **order $k \geq 0$**
 - ✓ **Feature 3:** Support **general meshes** (polygonal/polyhedral cells)
- Implementation:

Cockburn, B. and Di Pietro, D.A. and Ern, A.
ESAIM: M2AN (2016)

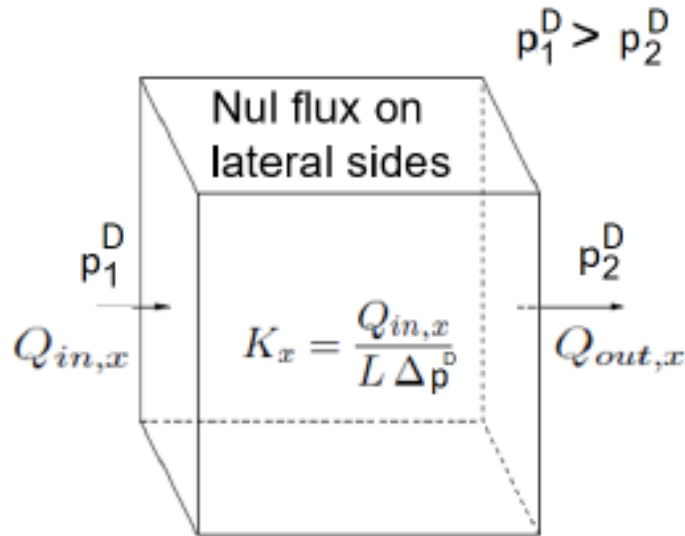
Library Disk++ for HHO
(open-source)

Cicuttin, M. and Di Pietro, D.A. and Ern, A.
Journal of Computational and Applied Mathematics (2018)

Computational performance

Computational performance of HHO method in the context of extremely large DFNs.

- Evaluation of the equivalent permeability of a DFN by running a permeameter test case



- ✓ Software:
 - Library Disk++ for HHO (open-source)
 - NEF++ (Inria) for an application to DFNs
 - Prune (Inria) for automatic parameter studies

Inria Cluster CLEPS: 4x Intel Xeon E7-4860 v2; 12 cores, 2.6-3.2GHz; **3T RAM**

- Objectives:
 - ✓ Choice of the **basis functions**
 - ✓ **Trade-off** between increasing the **polynomial order** and **refining the mesh**
 - ✓ Can resources be saved by the means of **polygonal cells** ?

Choice of the basis functions

- HHO unknowns are **polynomial coefficients**
- **On mesh faces:** scaled monomials x_F^α with $\alpha = 0, \dots, k$ and

$$x_F = 2 \frac{(x - \bar{x}_F)}{|F|}$$

- **On every mesh cell T:** we compare two choices of monomials

$$\alpha, \beta = 0, \dots, l = k + 1, \alpha + \beta \leq l$$

[Cartesian] $x_T^\alpha y_T^\beta$

$$x_T = 2 \frac{(x - \bar{x}_T)}{h_x^T}, \quad y_T = 2 \frac{(y - \bar{y}_T)}{h_y^T}$$

[Rotated] $\xi_T^\alpha \zeta_T^\beta$

$$\xi_T = 2 \frac{(x - \bar{x}_T, y - \bar{y}_T) \cdot a_1^T}{h_1^T}$$

$$\zeta_T = 2 \frac{(x - \bar{x}_T, y - \bar{y}_T) \cdot a_2^T}{h_2^T}$$

$$|Q_{in,x} + Q_{out,x}|$$

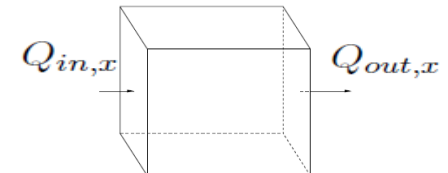
	k	[Cartesian]	[Rotated]
L100	0	1.44e-07	9.82e-11
	1	1.66e-4	5.26e-10
	2	2.31e-2	4.63e-09
L150	0	3.12E-07	7.32e-10
	1	1.58e-3	1.31e-08
	2	38.22	1.85e-08
	3	45.73	4.85e-08

□ Mass conservation

✘ **[Cartesian]**

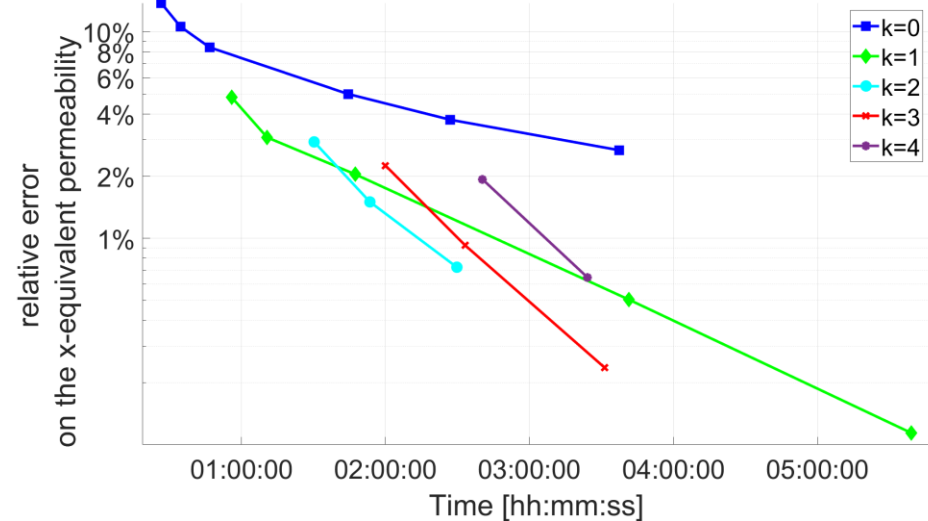
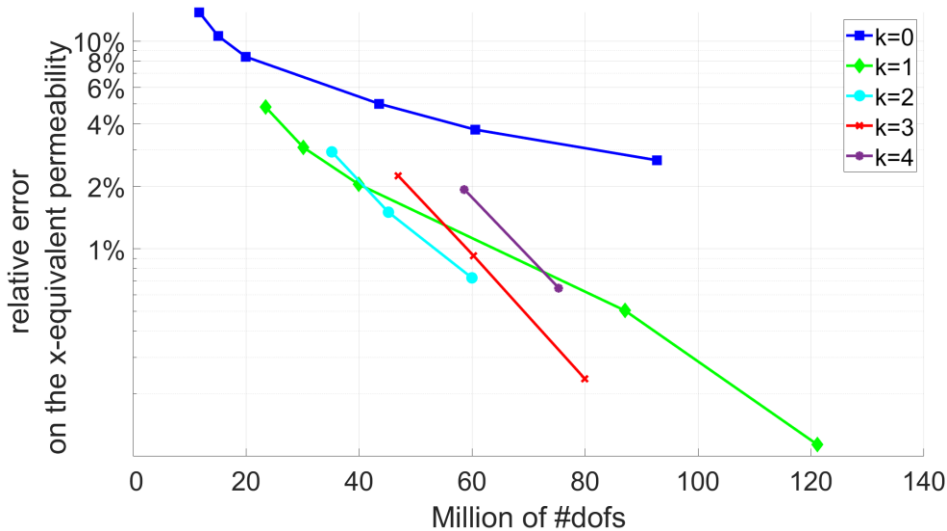
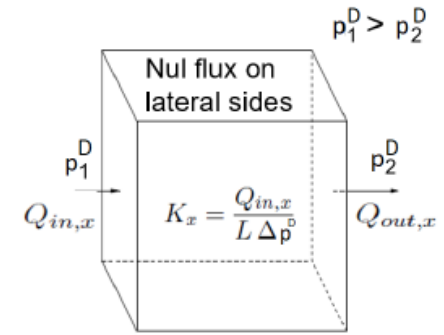
✔ **[Rotated]**

- **Global and local mass conservation ensured whatever k**



Trade-off between polynomial order and mesh refinement

- Evaluation of the equivalent permeability

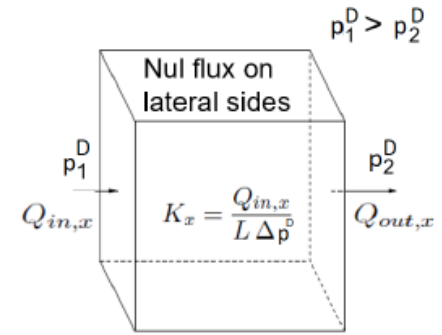
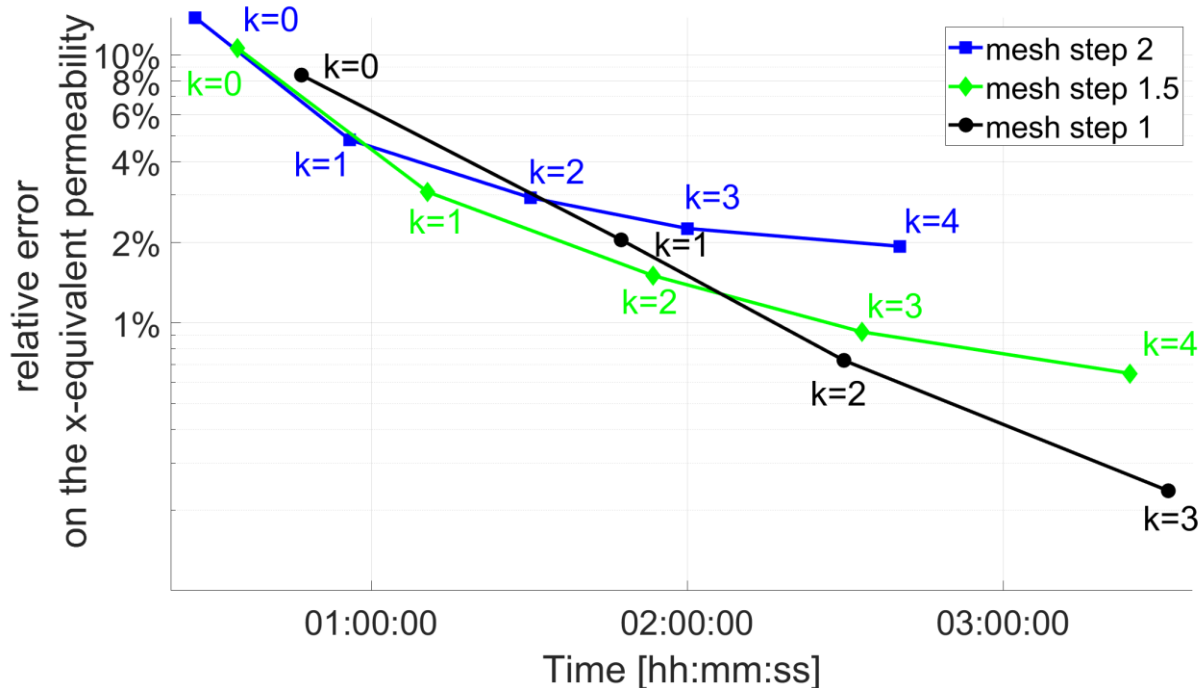


Recommendations

- ✓ use a face polynomial order k at least equal to 1, instead of refining a mesh while keeping $k = 0$

Trade-off between polynomial order and mesh refinement

➤ Evaluation of the equivalent permeability



❑ Recommendations

- ✓ use a face polynomial order k at least equal to 1, instead of refining a mesh while keeping $k = 0$
- ✓ choose $k = 1$ on a mesh fine rather than $k > 1$ on a coarser mesh

Trade-off between polynomial order and mesh refinement

- Computational resources for approximately the same number of dofs

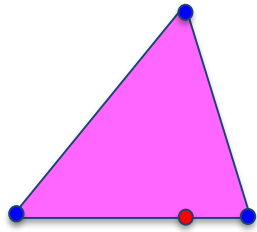
L150: 508,339 fractures

k	#dofs	Total Time (hh:mm)	RAM (GiB)	Rel. Error K_x	<i>DFN</i> Mesh step
0	43,531,537	01:51	133	4.8%	0.6
1	39,998,460	01:47	176	2.0%	1
2	45,222,504	02:15	203	1.5%	1.5
3	46,930,868	02:18	253	2.3%	2

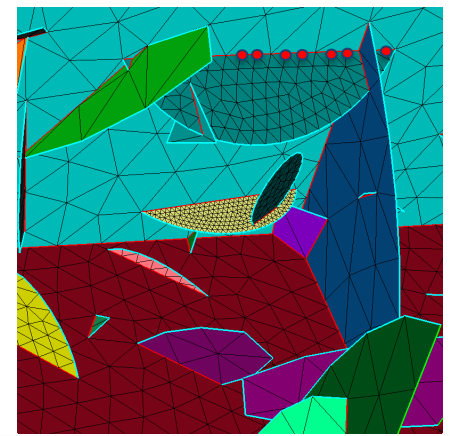
□ Comments

- ✓ As expected, time and RAM increase with k

How to take advantage of polygonal cells?



- Vertices of the triangles
- Extra vertices to create a polygon



20,522,575 triangles
All fractures → mesh step of 1.5

k	#dofs	Total Time (hh:m)	RAM (GiB)	Rel. Error K_x
0	31,711,430	1:21	128	10.7%
1	63,422,860	2:49	297	2.6%
2	95,134,290	04:33	551	0.9%
3	126,845,720	06:08	869	0.3%
4	158,557,150	08:48	1279	0.0%

18,648,084 triangles (-9%)
Most contributing fractures → step of 1.5
The others → mesh step of 2.

k	#dofs	Total Time (hh:m)	RAM (GiB)	Rel. Error K_x
0	28,685,198	01:11	119	11.7%
1	57,370,396	02:28	269	3.3%
2	86,055,594	03:39	498	1.5%
3	114,740,792	05:20	807	0.8%
4	143,425,990	07:41	1172	0.5%

Comparison with the fine mesh flow simulation: -12% to -20% time and -7% to -10% RAM

Conclusion

- ❑ We have **successfully** tested the **HHO method on large scale DFN** (more than **1 million of fractures**)
- ❑ The implementation of the method is **locally and globally conservative** (whatever $k \geq 0$) thanks to the basis functions
- ❑ The use of high order is an advantage to compute a **more accurate** flow in DFN
- ❑ We have shown that the use of non-matching meshes, the channelling effect of DFN flow and the polygonal feature of the **HHO method allows to reduce the number of #dofs and therefore the RAM requirements** in DFN flow simulations.

Perpectives

- ❑ Further reduction of the #dofs by using the **additional recent features of HHO (Unfitted HHO; merging triangles in polygons)**
- ❑ Studying **parallel direct or iterative (general and dedicated) solvers to reduce the RAM requirements**
- ❑ **PhD of Daniel Zegarra Vasquez, flow simulations in fractured porous media (2D/3D coupled flow problem)**
→ **Even larger linear systems to solve !**

E. Burman and A. Ern, An unfitted hybrid high-order method for elliptic interface problems, *SIAM J. Numer. Anal.*, 56(3), 1525-1546 (2018)

A. Miraçi, J. Papež, and M. Vohralík (2020)
A Multilevel Algebraic Error Estimator and the Corresponding Iterative Solver with p -Robust Behavior. *SIAM J. Numer. Anal.*, 58(5), 2856–2884.