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EIGENVALUE ASYMPTOTICS FOR CONFINING MAGNETIC SCHRÖDINGER OPERATORS WITH COMPLEX POTENTIALS

L. MORIN, N. RAYMOND, AND S. VŨ NGỌC

ABSTRACT. This article is devoted to the spectral analysis of the electro-magnetic Schrödinger operator on the Euclidean plane. In the semiclassical limit, we derive a pseudo-differential effective operator that allows us to describe the spectrum in various situations and appropriate regions of the complex plane. Not only results of the selfadjoint case are proved (or recovered) in the proposed unifying framework, but new results are established when the electric potential is complex-valued. In such situations, when the non-selfadjointness comes with its specific issues (lack of a "spectral theorem", resolvent estimates), the analogue of the "low-lying eigenvalues" of the selfadjoint case are still accurately described and the spectral gaps estimated.

1. INTRODUCTION

1.1. Context and motivation. In this article we study the spectrum of the non-selfadjoint electromagnetic Schrödinger operator:

$$(1.1) \quad \mathcal{L}_h = (-ih\nabla - \mathbf{A})^2 + hV(q_1, q_2),$$

which is an unbounded differential operator on $L^2(\mathbf{R}^2)$. We are particularly concerned by the semiclassical limit $h \rightarrow 0$. Here $\mathbf{A} : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ is a smooth vector potential and $V : \mathbf{R}^2 \rightarrow \mathbf{C}$ a smooth *complex* scalar potential. The associated magnetic field $B = \partial_1 A_2 - \partial_2 A_1$ is assumed to be positive and to belong to the class of *bounded symbols*

$$S_{\mathbf{R}^2}(1) = \{f \in \mathcal{C}^\infty(\mathbf{R}^2; \mathbf{C}) : \forall \alpha \in \mathbf{N}, \exists C > 0, |\partial^\alpha f| \leq C\}.$$

We also assume that the complex perturbation V belongs to this class, *i.e.*, $V \in S_{\mathbf{R}^2}(1)$, see Assumptions **I** and **II**. The chosen order of magnitude of the electric interaction hV is precisely when the magnetic and electric fields are in competition, as we will see in our results.

When $V = 0$, the low-lying spectrum of this operator has been studied in several papers, and summarized in the books [8, 18]. In particular, when the magnetic field has a unique minimum $b_0 > 0$, which is non-degenerate and not attained at infinity, it was proved in [11] that

$$\lambda_n(h) = b_0 h + ((2n - 1)c_0 + c_1)h^2 + o(h^2),$$

where $c_1 \in \mathbf{R}$ and $c_0 = b_0^{-1} |\frac{1}{2} \nabla^2 B(q_0)|^{1/2}$.

Such operators as (1.1) appear for instance in the context of the time-dependent Ginzburg-Landau equations, see [1]. These equations involve a damping term related to an induced current. Their linearization near a normal state gives rise to a propagation equation whose generator is an electromagnetic Schrödinger operator with complex electric potential. Its left-most eigenvalue governs the large time decay of the associated semi-group and thus the stability of the normal state.

Our analysis sets up a unifying (semiclassical) framework to study the "low-lying" eigenvalues of operators of the form \mathcal{L}_h , including the magnetic Laplacian itself [11, 19] and some of its selfadjoint perturbations (see the recent work [25]). But the most interesting novelty of our strategy is to cover also the case of *imaginary* electric potentials. More precisely, to the authors' knowledge, the present paper is the first to obtain precise eigenvalue asymptotics in the presence of strong¹ perturbations of the magnetic Schrödinger operator in the semiclassical limit. Note that, in non-asymptotic settings, complex perturbations of the magnetic Laplacian (and of the magnetic Dirac operator) have been considered in [7, 5], where it is proved that there are no eigenvalues when the electro-magnetic field is sufficiently decaying at infinity. For decaying electric potentials with a (quasi)constant magnetic field, Weyl estimates have also been established for the Pauli operator in [22, 21].

In general, it is well-known that even small non-selfadjoint perturbations of selfadjoint operators can have a dramatic effect on the spectrum. In the present context, this problem is all the more appealing that the magnetic Laplacian comes with its own issues (such as its lack of ellipticity). To overcome these combined difficulties, our approach is based on a microlocal dimensional reduction (involving operator-valued symbols, see for instance [16, 15], and also [2] for a recent application of the strategy in a self-adjoint context). It allows us to explore the spectral structure in a disk corresponding to the location of the low-lying eigenvalues in the selfadjoint case. More precisely, when the perturbation V is turned on, we describe how the spectrum moves, in a disc $D(\mu_0 h, Ch^2)$, where μ_0 depends on the electromagnetic field. In this disc, the fine structure of the spectrum is accurately described by estimating the splitting between the eigenvalues. Our main result is stated in Theorem 1.2, whereas its various (and sometimes non-trivial) applications are given in Section 1.3.

1.2. Main result. In this article we will make the following assumptions.

Assumption I. The magnetic field is non-vanishing: there exists $b_0 > 0$ such that

$$\forall q \in \mathbf{R}^2, \quad B(q) \geq b_0 > 0.$$

Assumption II. The functions B, V and

$$q \mapsto \int_0^{q_1} \frac{\partial B(s, q_2)}{\partial q_2} ds$$

are all in $S_{\mathbf{R}^2}(1)$.

Assumption III. There exist $u, v \in \mathbf{R}$, $u > 0$ such that the function

$$F = u(B + \operatorname{Re}(V)) + v\operatorname{Im}(V) = \operatorname{Re}((u - iv)(B + V))$$

admits a unique global minimum, not reached at infinity. We denote by $\mu_0 \in \mathbf{C}$ the value of $B + V$ at the minimum of F . It satisfies

$$\operatorname{Re}((u - iv)\mu_0) = u\operatorname{Re}\mu_0 + v\operatorname{Im}\mu_0 = \min_{q \in \mathbf{R}^2} F(q).$$

The function F should be interpreted as the localizing function for our operator: it gives information on where the spectrum should lie. These assumptions imply

¹The perturbation is not assumed to be small at infinity and plays at the same scale as the magnetic Laplacian.

discreteness of the spectrum in a disc $D(\mu_0 h, Ch^2)$ and localization of the associated eigenfunctions (Proposition 6.1 and Lemma 2.7). Here are some interesting particular cases where Assumption III holds:

1. $B + \operatorname{Re}(V)$ admits a unique global minimum, not reached at infinity, and $\operatorname{Im}(V)$ is arbitrary (take $u = 1$ and $v = 0$).
2. $\operatorname{Im}(V)$ admits a unique global minimum, not reached at infinity, and $B + \operatorname{Re}(V)$ is constant (take $u = 1$ and $v = 1$).

Remark 1.1. It may happen that there exist two different couples (u, v) for which Assumption III holds, and for which the corresponding minima of F are attained at two different locations q_0 and q'_0 . Then it follows from the assumptions that the values μ_0 and μ'_0 must be different as well. Hence our analysis will give the description of the spectrum of \mathcal{L}_h in two different regions in the complex plane. Here is an example of such a situation (see Figure 1). Assume that the electric field is purely imaginary, equal to $iV(q)$ and that magnetic field is $B = 1 - w(q)$, where w and V have disjoint compact supports, $0 \leq w < 1$ and $V \geq 0$. If w (resp. V) has a unique and non-degenerate maximum reached at q_0 (resp. reached at q'_0) then the functions $F = 1 - w(q)$ ($u = 1, v = 0$) and $F' = 1 - w(q) - V(q)$ ($u = 1, v = -1$) satisfy our assumption with respective minima $1 - w(q_0)$ and $1 - \max(w(q_0), V(q'_0))$. If $V(q'_0) > w(q_0)$ these minimas are reached at q_0 and q'_0 respectively and the corresponding values of $B + iV$ are

$$\mu_0 = 1 - w(q_0), \quad \mu'_0 = 1 + iV(q'_0).$$

Note that the value of F at its minimum plays no role in the spectral description. The interesting quantity is μ_0 .

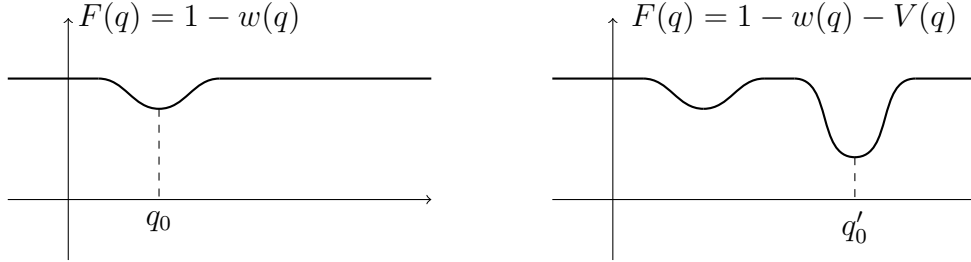


FIGURE 1. Two functions F

In this article, we compare the spectrum of \mathcal{L}_h in $D(\mu_0 h, Ch^2)$ with the spectrum of an effective operator $h\mathbf{P}_h^{\text{eff}}$ acting on $L^2(\mathbf{R})$, which has the following form. Let us denote by $\varphi : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ the diffeomorphism:

$$(1.2) \quad (\xi, x) = \varphi(q) = \left(\int_0^{q_1} B(s, q_2) ds, q_2 \right),$$

and $\mathring{B}(\xi, x) = B \circ \varphi^{-1}(\xi, x)$, $\mathring{V}(\xi, x) = V \circ \varphi^{-1}(\xi, x)$. Then $\mathbf{P}_h^{\text{eff}} = \mathbf{Op}_h^w \mu_h^{\text{eff}}$ is an h -pseudodifferential operator with a symbol in $S_{\mathbf{R}^2}(1)$ of the form

$$(1.3) \quad \mu_h^{\text{eff}}(x, \xi) = \mathring{B}(\xi, x) + \mathring{V}(\xi, x) + h\mu_1(x, \xi),$$

where the subprincipal term $\mu_1(x, \xi)$ has an explicit formula described in (5.2).

Our main result, Theorem 1.2, states that, provided that the eigenvalues and resolvent of $\mathbf{P}_h^{\text{eff}}$ are sufficiently well controlled near μ_0 , the spectrum of $h\mathbf{P}_h^{\text{eff}}$ in that region closely approximate the spectrum of \mathcal{L}_h .

Theorem 1.2. *Let Assumptions I, II, and III hold. Let $0 < C < C'$. Assume the following:*

- (a) *There exist $c, h_0 > 0$ such that, for $h \in (0, h_0)$, the spectrum of $\mathbf{P}_h^{\text{eff}}$ in $D(\mu_0, C'h)$ consists of a family of discrete eigenvalues $(\nu_j(h))_{1 \leq j \leq N}$ of algebraic multiplicity 1, such that*

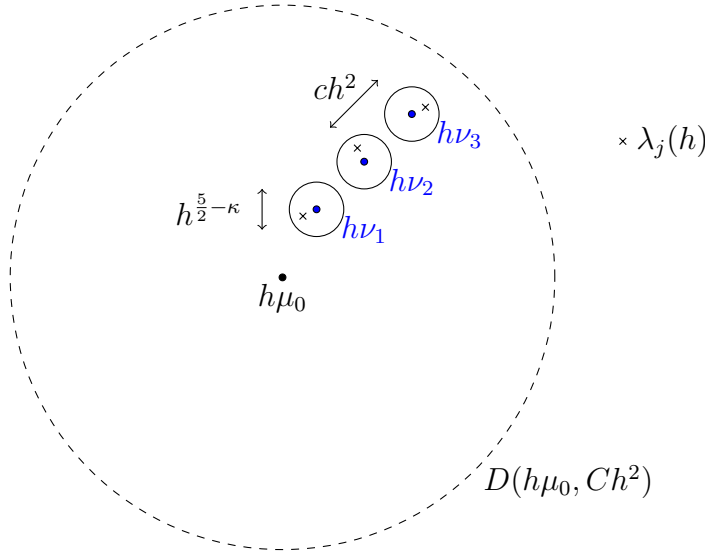
$$\forall (k, \ell) \in \{1, \dots, N\}^2, \quad k \neq \ell \Rightarrow |\nu_k(h) - \nu_\ell(h)| \geq ch.$$

- (b) *There exist $\kappa \in (0, \frac{1}{2})$, h_0 , and $C_0 > 0$ such that, for $h \in (0, h_0)$, for any $z \in D(\mu_0, Ch)$ satisfying $\text{dist}(z, \text{sp } \mathbf{P}_h^{\text{eff}}) \geq h^{\frac{3}{2}-\kappa}$,*

$$(1.4) \quad \|(z - \mathbf{P}_h^{\text{eff}})^{-1}\| \leq \frac{C_0}{\text{dist}(z, \text{sp } \mathbf{P}_h^{\text{eff}})}.$$

Then, for h small enough, the spectrum of \mathcal{L}_h in $D(\mu_0 h, Ch^2)$ consists of a family of discrete eigenvalues $(\lambda_j(h))_{1 \leq j \leq N}$ of algebraic multiplicity 1, such that

$$\lambda_j(h) = h\nu_j(h) + \mathcal{O}(h^{\frac{5}{2}-\kappa}).$$



Remark 1.3. This theorem holds for $N = 0$ as well, meaning that if $\mathbf{P}_h^{\text{eff}}$ has no spectrum in $D(\mu_0, Ch)$ then \mathcal{L}_h has no spectrum in $D(\mu_0 h, Ch^2)$.

1.3. Applications of the main theorem. Although the main incentive for Theorem 1.2 is to deal with non selfadjoint versions of the electromagnetic Schrödinger operator, it turns out that Theorem 1.2 also recovers and extends some recent results in the selfadjoint case.

Corollary 1.4 (Self-adjoint case). *Let Assumptions I, II hold, and assume moreover that $\text{Im}(V) = 0$, and $B + \text{Re}(V)$ admits a unique minimum, which is non-degenerate, $\mu_0 \in \mathbb{R}$ reached at 0 and not at infinity. Let $C > 0$. There exists $h_0 > 0$ such that, for $h \in (0, h_0)$ the spectrum of \mathcal{L}_h in $D(\mu_0 h, Ch^2)$ consists of a family of discrete eigenvalues $(\lambda_j(h))_{1 \leq j \leq N}$ of simple multiplicities such that*

$$\lambda_j(h) = \mu_0 h + ((2j - 1)c_0 + c_1)h^2 + o(h^2),$$

where $c_0 = B(0)^{-1}|\frac{1}{2}\nabla^2(B+V)(0)|^{1/2}$ and $c_1 \in \mathbf{R}$.

Proof. When $\text{Im}(V) = 0$, $\mathbf{P}_h^{\text{eff}}$ and \mathcal{L}_h are (essentially) selfadjoint. In particular, the resolvent bound (1.4) follows from the Spectral Theorem. The assumptions on $B+V$ imply that the spectrum of $\mathbf{P}_h^{\text{eff}}$ in $D(\mu_0, Ch)$ consists of discrete simple eigenvalues such that

$$\nu_j(h) = \mu_0 + \left((2j-1)|\frac{1}{2}\nabla^2(\hat{B} + \hat{V})(0)|^{1/2} + c_1 \right) h + o(h).$$

(See [23] for instance). The computation of c_0 follows from the link between $\hat{B} + \hat{V}$ and $B + V$. \square

In the case $V = 0$, the above result was proven in [19, 11]; the general case seems to be new.

We now turn to the non-selfadjoint case, where a remarkable consequence of Theorem 1.2 is the following result.

Theorem 1.5. *Let $p = B+V$; together with Assumptions I, II, and III, assume that $p^{-1}(\mu_0) \cap \mathbf{R}^2 = \{0\}$, with $\nabla p(0) = 0$ and that $\nabla^2 p(0)$ is non-degenerate. For $C > 0$, there exists $h_0 > 0$ such that, for $h \in (0, h_0)$ the spectrum of \mathcal{L}_h in $D(\mu_0 h, Ch^2)$ consists of a family of discrete eigenvalues $(\lambda_j(h))_{1 \leq j \leq N}$ with simple algebraic multiplicities such that*

$$\lambda_j(h) = \mu_0 h + ((2j-1)c_0 + c_1) h^2 + o(h^2),$$

where $c_0, c_1 \in \mathbf{C}$, $c_0 \neq 0$.

Proof. To apply Theorem 1.2, we need to check that $\mathbf{P}_h^{\text{eff}}$ has the expected spectral properties in $D(\mu_0, Ch)$. They are established in Section 7. \square

Remark 1.6. Operators like $\mathbf{P}_h^{\text{eff}}$ have been studied in [12], where a full asymptotic expansion was provided thanks to a Birkhoff normal form in a non-selfadjoint context. The method used there requires that the symbol be analytic in a tubular neighborhood of \mathbf{R}^2 . Our strategy to reduce the spectral analysis to the one of $\mathbf{P}_h^{\text{eff}}$ could actually give us an effective operator modulo $\mathcal{O}(h^\infty)$. Combined with Hitrik's result, one would get a full asymptotic expansion of the eigenvalues in $D(\mu_0, Ch)$.

Theorem 1.5 can also be applied to two other quite different interesting situations, where the confinement is given either by the imaginary part of V , or by the magnetic field B alone.

Corollary 1.7 (Constant real part). *Let assumptions I, II hold, and assume that $B + \text{Re}(V)$ is constant on \mathbf{R}^2 equal to μ_0 . Assume that $\text{Im}(V)$ admits a unique minimum, which is non-degenerate, reached at 0 and not at infinity, with $\text{Im}V(0) = 0$. For $C > 0$, there exists $h_0 > 0$ such that, for $h \in (0, h_0)$ the spectrum of \mathcal{L}_h in $D(\mu_0 h, Ch^2)$ consists of a family of discrete eigenvalues $(\lambda_j(h))_{1 \leq j \leq N}$ with simple algebraic multiplicities such that*

$$\lambda_j(h) = \mu_0 h + ((2j-1)ic_0 + c_1)h^2 + o(h^2),$$

where $c_0 > 0$ and $c_1 \in \mathbf{C}$.

Proof. In this case, Assumption III is valid with $u = v = 1$. We can apply Theorem 1.5. Indeed, we have $p = \mu_0 + i\text{Im}(V)$ so that $p(\mathbf{R}^2) \subset \mu_0 + i\mathbf{R}_+$ and 0 is the only point where $p = \mu_0$. The Hessian of p is $i\nabla^2 \text{Im}V(0)$, which is non degenerate. \square

Finally, we can also describe the spectrum of the following non-selfadjoint perturbation of the magnetic Laplacian:

$$\mathcal{L}_{h,\varepsilon} = (-ih\nabla - A)^2 + h\varepsilon V,$$

where the magnetic field B admits a unique minimum $b_0 > 0$ which is non-degenerate, reached at 0 and not at infinity. In this case, $B + \varepsilon V$ admits a unique critical point z_ε such that $z_\varepsilon = \mathcal{O}(\varepsilon)$, and we denote the critical value by $\mu_\varepsilon = B(z_\varepsilon) + \varepsilon V(z_\varepsilon)$.

Corollary 1.8 (Perturbation of a confining magnetic field). *Let Assumptions I, II hold, and assume that B has a unique minimum, which is non-degenerate, b_0 , reached at 0 and not at infinity. Let $C > 0$. There exist $\varepsilon_0, h_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ and $h \in (0, h_0)$ the spectrum of $\mathcal{L}_{h,\varepsilon}$ in $D(\mu_\varepsilon h, Ch^2)$ consists of a family of discrete eigenvalues $(\lambda_j(h, \varepsilon))_{1 \leq j \leq N}$ with algebraic multiplicities 1 such that*

$$\lambda_j(h, \varepsilon) = \mu_\varepsilon h + ((2j - 1)c_0(\varepsilon) + c_1(\varepsilon))h^2 + o(h^2),$$

where $c_0(\varepsilon), c_1(\varepsilon) \in \mathbf{C}$ satisfy

$$c_0(\varepsilon) = b_0^{-1} \left| \frac{1}{2} \nabla^2 B(0) \right|^{1/2} + \mathcal{O}(\varepsilon).$$

Proof. For $\varepsilon > 0$ small enough, $B + \varepsilon V$ admits a unique and non-degenerate real critical point z_ε close to 0, since it is a perturbation of B . Then $p = B + \varepsilon V$ satisfies the hypotheses of Theorem 1.5, as a perturbation of B , and Corollary 1.8 is just a reformulation of this result. Note that $h_0 > 0$ is independent of ε because, as one can check in our proof, the constants involved in our estimates can be chosen uniform in ε (in particular the distance ch between the eigenvalues in Theorem 1.2 (a)). \square

Remark 1.9. Our results are adapted to the ‘ground state’ of \mathcal{L}_h , i.e. eigenfunctions associated with eigenvalues whose location is dictated by the minimum of the function F . It would be interesting to try and adapt the method to treat excited states, for which we should be able to leverage Rouby’s results about 1D non-selfadjoint pseudodifferential operators [20].

1.4. Structure of the article. The article is organized as follows. In Section 2, using a phase space change of variables, we prove that \mathcal{L}_h is unitarily equivalent to an operator $h\widehat{\mathcal{L}}_h^0$ that can be seen as a perturbation of a harmonic oscillator. As a consequence, we obtain a microlocalization of eigenfunctions at distance $\mathcal{O}(h^\delta)$, $\delta \in [0, \frac{1}{2}]$, of the characteristic manifold. In Section 3, we introduce a slight modification of $\widehat{\mathcal{L}}_h^0$ by inserting microlocal cutoff functions in the symbol. In Section 3.2 we see $\widehat{\mathcal{L}}_h$ as a pseudodifferential operator with operator-valued symbol, and we expand its symbol in powers of $h^{1/2}$. The properties of its principal symbol P_0 , which is essentially a harmonic oscillator, are described in Section 3.3. In Section 4, we use a Grushin method to reduce the spectral analysis to the one of the effective operator P_h^{eff} , and in Section 5 we prove that the spectrum of $\widehat{\mathcal{L}}_h$ is approximated by the spectrum of P_h^{eff} . In Section 6 we ‘remove the cutoff functions’, proving that the spectrum of $\widehat{\mathcal{L}}_h^0$ is close to the spectrum of $\widehat{\mathcal{L}}_h$, thus concluding the proof of Theorem 1.2. Finally, in Section 7, we prove Theorem 1.5 by explaining how to describe the spectrum and the resolvent of P_h^{eff} .

2. A FIRST CONJUGATION

Using the semiclassical Weyl quantization, we may view \mathcal{L}_h as an h -pseudo-differential operator:

$$\mathcal{L}_h = \text{Op}_h^w(H), \quad H(q, p) = (p_1 - A_1(q_1, q_2))^2 + (p_2 - A_2(q_1, q_2))^2 + hV(q_1, q_2).$$

Microlocal analysis suggests that eigenfunctions of \mathcal{L}_h for eigenvalues of order $\mathcal{O}(h)$ should be localized near the characteristic manifold $H^{-1}(0)$. The aim of this section is to introduce new phase space coordinates (x_1, x_2, ξ_1, ξ_2) for which this characteristic manifold becomes linear, and such that the coordinates (x_1, ξ_1) represent the distance to the characteristic manifold. Scaling these coordinates by the natural factor \sqrt{h} will finally yield Proposition 2.1 below. A similar conjugation (without the scaling) was performed in [9].

Due to the gauge invariance, we may assume that \mathbf{A} has the form

$$\mathbf{A} = (0, A_2), \quad A_2(q) = \int_0^{q_1} B(s, q_2) ds.$$

The diffeomorphism $\varphi : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ defined in (1.2) now reads

$$\varphi(q) = (A_2(q), q_2).$$

For any function $f : \mathbf{R}^2 \rightarrow \mathbf{C}$, we shall denote $\mathring{f} = f \circ \varphi^{-1}$.

Proposition 2.1. *The operator \mathcal{L}_h is unitarily equivalent to*

$$h\widehat{\mathcal{L}}_h^0 := h\text{Op}_h^{w,2}\text{Op}_1^{w,1}\widehat{H}^0,$$

where

$$(2.1) \quad \begin{aligned} \widehat{H}^0(x, \xi) = & \mathring{B}(\xi_2 + h^{1/2}x_1, x_2 + h^{1/2}\xi_1)^2 \xi_1^2 + (x_1 + \mathring{\alpha}(\xi_2 + h^{1/2}x_1, x_2 + h^{1/2}\xi_1)\xi_1)^2 \\ & + \mathring{V}(\xi_2 + h^{1/2}x_1, x_2 + h^{1/2}\xi_1) + hW(\xi_2 + h^{1/2}x_1, x_2 + h^{1/2}\xi_1), \end{aligned}$$

and

- (i) $\alpha(q_1, q_2) = \partial_2 A_2(q_1, q_2)$,
- (ii) $W = \frac{1}{4}(\partial_1 \mathring{B})^2 + \frac{1}{4}(\partial_1 \mathring{\alpha})^2$.

Remark 2.2. Here $\text{Op}_h^{w,2}$ is the h -Weyl quantization with respect to (x_2, ξ_2) , and $\text{Op}_1^{w,1}$ is the non-semiclassical Weyl quantization with respect to (x_1, ξ_1) , which means

$$(2.2) \quad \text{Op}_h^{w,2}\text{Op}_1^{w,1}u(x_1, x_2) = \frac{1}{(2\pi)^2 h} \int_{\mathbf{R}^4} e^{\frac{i}{h}(x_2 - y_2)\xi_2 + i(x_1 - y_1)\xi_1} \widehat{H}^0\left(\frac{x_1 + y_1}{2}, \frac{x_2 + \xi_2}{2}\right) u(y_1, y_2) dy_1 dy_2 d\xi_1 d\xi_2.$$

2.1. Proof of Proposition 2.1. We split the proof into two steps : Lemma 2.3 and Lemma 2.4.

Lemma 2.3. *The operator \mathcal{L}_h is unitarily equivalent to $\widetilde{\mathcal{L}}_h = \text{Op}_h^w \widetilde{H}$, where*

$$(2.3) \quad \begin{aligned} \widetilde{H}(x, \xi) = & \mathring{B}(\xi_2 + x_1, x_2 + \xi_1)^2 \xi_1^2 + (x_1 + \mathring{\alpha}(\xi_2 + x_1, x_2 + \xi_1)\xi_1)^2 + h\mathring{V}(\xi_2 + x_1, x_2 + \xi_1) \\ & + \frac{h^2}{4}(\partial_1 \mathring{B}(\xi_2 + x_1, x_2 + \xi_1))^2 + \frac{h^2}{4}(\partial_1 \mathring{\alpha}(\xi_2 + x_1, x_2 + \xi_1))^2. \end{aligned}$$

Proof. First, let us rewrite the operator in the variables $(x_1, x_2) = \varphi(q_1, q_2)$. For any $u \in \mathcal{C}^\infty(\mathbf{R}^d)$ recall that we denote $\mathring{u} = u \circ \varphi^{-1}$, so that $\mathring{u}(x) = u(q)$. Then we have :

$$\begin{cases} \partial_{q_1} u = \mathring{B} \partial_{x_1} \mathring{u}, \\ \partial_{q_2} u = \mathring{\alpha} \partial_{x_1} \mathring{u} + \partial_{x_2} \mathring{u}, \end{cases}$$

with $\alpha(q) = \partial_2 A_2(q)$. Then \mathcal{L}_h is given in these variables by

$$\mathcal{L}_h u = -h^2 (\mathring{B} \partial_{x_1})^2 \mathring{u} + (-ih \mathring{\alpha} \partial_{x_1} - ih \partial_{x_2} - x_1)^2 \mathring{u} + h \mathring{V} \mathring{u},$$

because $x_1 = A_2(q)$. In other words, if U is the following unitary transformation:

$$U : \begin{cases} \mathbf{L}^2(\mathbf{R}^2) & \rightarrow & \mathbf{L}^2(\mathbf{R}^2) \\ u & \mapsto & |\mathrm{d}\varphi^{-1}|^{1/2} u \circ \varphi^{-1} \end{cases}$$

then, since the Jacobian $|\mathrm{d}\varphi|$ equals B ,

$$U \mathcal{L}_h U^* \mathring{u} = -h^2 \mathring{B}^{1/2} \partial_{x_1} \mathring{B} \partial_{x_1} (\mathring{B}^{1/2} \mathring{u}) + \mathring{B}^{-1/2} (-ih \mathring{\alpha} \partial_{x_1} - ih \partial_{x_2} - x_1)^2 (\mathring{B}^{1/2} \mathring{u}) + h \mathring{V} \mathring{u}.$$

With the notation $hD_j = -ih \partial_{x_j}$ we can rewrite it as

$$(2.4) \quad U \mathcal{L}_h U^* = \left(\mathring{B}^{1/2} h D_1 \mathring{B}^{1/2} \right)^2 + \left(\frac{1}{2} (\mathring{\alpha} D_1 + D_1 \mathring{\alpha}) + D_2 - x_1 \right)^2 + h \mathring{V}.$$

Indeed, this follows from

$$B^{-1/2} (\mathring{\alpha} D_1 + D_2 - x_1) B^{1/2} = \frac{1}{2} (\mathring{\alpha} D_1 + D_1 \mathring{\alpha}) + D_2 - x_1,$$

which one can get using $(D_1 \mathring{\alpha}) = \mathring{B}^{-1} (\mathring{\alpha} D_1 \mathring{B} + D_2 \mathring{B})$. Now, the Weyl-symbol of (2.4) is

$$\check{H}(x, \xi) = \mathring{B}(x)^2 \xi_1^2 + (\mathring{\alpha} \xi_1 + \xi_2 - x_1)^2 + h \mathring{V} + \frac{h^2}{4} (\partial_{x_1} \mathring{B})^2 + \frac{h^2}{2} (\partial_{x_1} \mathring{\alpha})^2.$$

Finally we do the following linear canonical change of variables,

$$\begin{cases} \tilde{x}_1 &= x_1 - \xi_2 \\ \tilde{\xi}_1 &= \xi_1 \\ \tilde{x}_2 &= x_2 - \xi_1 \\ \tilde{\xi}_2 &= \xi_2. \end{cases}$$

Using the linear Egorov Theorem (metaplectic representation [13, Theorem 18.5.9]), \mathcal{L}_h is unitarily equivalent to $\mathrm{Op}_h^w \tilde{H}$ with:

$$\tilde{H}(\tilde{x}_1, \tilde{x}_2, \tilde{\xi}_1, \tilde{\xi}_2) = H(\tilde{x}_1 + \tilde{\xi}_2, \tilde{x}_2 + \tilde{\xi}_1, \tilde{\xi}_1, \tilde{\xi}_2),$$

and the lemma is proved. \square

Lemma 2.4. *The operator $\widetilde{\mathcal{L}}_h = \mathrm{Op}_h^w \tilde{H}$ is unitary equivalent to $h \widetilde{\mathcal{L}}_h^0$.*

Proof. First we split the x_1 and x_2 quantizations:

$$\widetilde{\mathcal{L}}_h = \mathrm{Op}_h^{w,2} \mathrm{Op}_h^{w,1} \tilde{H}.$$

Then we can change the semiclassical quantization with respect to (x_1, ξ_1) into a non-semiclassical one. Removing the (x_2, ξ_2) -dependence in the notations, we have:

$$\left(\mathrm{Op}_h^{w,1} \tilde{H} \right) u(x_1) = \frac{1}{2\pi h} \int e^{\frac{i}{h}(x_1 - y_1)\xi_1} \tilde{H}\left(\frac{x_1 + y_1}{2}, \xi_1\right) u(y_1) \mathrm{d}y_1 \mathrm{d}\xi_1.$$

We do the following change of variables

$$(2.5) \quad x_1 = h^{1/2}\hat{x}_1, \quad y_1 = h^{1/2}\hat{y}_1, \quad \xi_1 = h^{1/2}\hat{\xi}_1,$$

$$\left(\mathrm{Op}_h^{w,1}\tilde{H}\right)u(x_1) = \frac{1}{2\pi} \int e^{(\hat{x}_1 - \hat{y}_1)\hat{\xi}_1} \tilde{H}\left(h^{1/2}\frac{\hat{x}_1 + \hat{y}_1}{2}, h^{1/2}\hat{\xi}_1\right) u(h^{1/2}\hat{y}_1) d\hat{y}_1 d\hat{\xi}_1.$$

If we denote by V the unitary transformation $Vu(\hat{x}_1) = u(h^{1/2}\hat{x}_1)h^{1/4}$ then we deduce

$$\left(\mathrm{Op}_h^{w,1}\tilde{H}\right)u = V^* \mathrm{Op}_1^{w,1}aVu$$

with $a(x_1, \xi_1, h) = \tilde{H}(h^{1/2}x_1, h^{1/2}\xi_1)$. Note that $a = h\hat{H}^0$ to conclude the proof. \square

Remark 2.5. In section 3.1 we show that $(x_2, \xi_2) \mapsto \mathrm{Op}_1^{w,1}\hat{H}^0$ belongs to a suitable class of operator-valued symbols, and hence can be seen as the operatorial symbol of $\widetilde{\mathcal{L}}_h$.

We now check that the diffeomorphism φ behaves well with respect to the symbol classes.

Lemma 2.6. *For any order function m , and any function $f : \mathbf{R}^2 \rightarrow \mathbf{C}$,*

$$f \in S(m) \Rightarrow \mathring{f} \in S(\mathring{m}).$$

In particular, if $f \in S(1)$, then $\mathring{f} \in S(1)$.

Proof. The derivatives of \mathring{f} are related to the derivatives of f by:

$$\begin{cases} \partial_1 \mathring{f} = \frac{1}{B} \partial_1 f \\ \partial_2 \mathring{f} = -\frac{\alpha}{B} \partial_1 f + \partial_2 f. \end{cases}$$

Iterating this formula, and using $B \geq b_0$ for the denominators and $B \in S(1)$ for the numerators, we deduce that:

$$|\partial^\gamma \mathring{f}(\mathring{q})| \leq C_\gamma \sum_{|\ell| \leq |\gamma|} |\partial^\ell f(q)|,$$

and thus if $f \in S(m)$,

$$|\partial^\gamma \mathring{f}(\mathring{q})| \leq C_\gamma m(q) = C_\gamma \mathring{m}(\mathring{q}).$$

\square

2.2. Microlocalization of the eigenfunctions. Using the quadratic behaviour of the symbol \tilde{H} in the variable $X_1 := (x_1, \xi_1)$ (Equation (2.3)), we prove here that the eigenfunctions of $\widetilde{\mathcal{L}}_h$ corresponding to eigenvalues that are $\mathcal{O}(h^2)$ -close to $\mu_0 h$ are microlocalized in a band of width $(|X_1| \leq Ch^\delta)$, for all $\delta \in [0, \frac{1}{2}]$.

Lemma 2.7. *Let $\delta \in [0, \frac{1}{2})$ and $\chi \in \mathcal{C}_0^\infty(\mathbf{R}, \mathbf{R})$ equal to 1 on a neighborhood of 0. Then, for any normalized eigenpair (λ, ψ) of $\widetilde{\mathcal{L}}_h$ with $\lambda \in D(\mu_0 h, Ch^2)$:*

$$\psi = \mathrm{Op}_h^w \chi(h^{-\delta} x_1) \chi(h^{-\delta} \xi_1) \psi + \mathcal{O}_{L^2(\mathbf{R}^2)}(h^\infty).$$

Proof. Let us start by proving the result when $\delta = 0$, which is a crude microlocalisation. We let $\chi_0(x, \xi) = 1 - \chi(x_1)\chi(\xi_1) \in S_{\mathbf{R}^4}(1)$, and $\chi_0^w = \mathrm{Op}_h^w \chi_0$. We want to prove that $\|\chi_0^w \psi\| = \mathcal{O}(h^\infty)$. Consider

$$(2.6) \quad \widetilde{\mathcal{L}}_h \chi_0^w \psi = \chi_0^w \widetilde{\mathcal{L}}_h \psi + [\widetilde{\mathcal{L}}_h, \chi_0^w] \psi.$$

Since $\widetilde{\mathcal{L}}_h \psi = \lambda \psi$, we have

$$(2.7) \quad \|\chi_0^w \widetilde{\mathcal{L}}_h \psi\| \leq \tilde{C}h \|\chi_0^w \psi\|.$$

It follows from the symbolic calculus (see for instance [26, Theorem 4.18]) that $\left[\widetilde{\mathcal{L}}_h, \chi_0^w\right]$ is a pseudodifferential operator in $hS(1)$. By the Calderón-Vaillancourt Theorem, and taking $\underline{\chi}_0$ a cutoff function with the same properties of χ_0 , which is equal to 1 on $\text{supp} \chi_0$, we have

$$(2.8) \quad \left\| \left[\widetilde{\mathcal{L}}_h, \chi_0^w\right] \psi \right\| \leq Ch \|\underline{\chi}_0^w \psi\| + \mathcal{O}(h^\infty) \|\psi\|^2.$$

(In this text, we use the phrase “cutoff function” for smooth functions, independent of h , taking values in $[0, 1]$, whose support is not necessarily compact.) From (2.3), the symbol \tilde{H} of $\widetilde{\mathcal{L}}_h$ satisfies

$$\text{Re } \tilde{H} \geq c(\xi_1^2 + x_1^2) - Ch.$$

Let us consider a cutoff function $\chi_1 \in S_{\mathbf{R}^2}(1)$ equal to 1 in a neighborhood of the origin $X_1 = 0$, and such that, viewed as a function of $X \in \mathbf{R}^4$, we have $\text{supp} \chi_1 \cap \text{supp} \chi_0 = \emptyset$. We let $a_h(X) := \tilde{H}(X) + \chi_1(X_1)$. Then, for some $\tilde{c} > 0$ and h small enough,

$$\text{Re } a_h(X) \geq \tilde{c} \langle X_1 \rangle^2 \geq \tilde{c} > 0.$$

We have $\frac{1}{a_h} \in S(\langle X_1 \rangle^{-2})$. With the Calderón-Vaillancourt theorem, it follows that

$$\forall \psi \in L^2(\mathbf{R}^2), \quad \|[a_h^{-1}]^w \psi\| \leq C \|\psi\|.$$

Thus,

$$\|[a_h^{-1}]^w [a_h]^w \psi\| \leq C \|[a_h]^w \psi\|.$$

By using again the symbolic calculus,

$$\|\psi\| \leq \tilde{C} \|[a_h]^w \psi\|.$$

By using that the supports of χ_0 and χ_1 are disjoint, we deduce that

$$c \|\chi_0^w \psi\| \leq \|\widetilde{\mathcal{L}}_h \chi_0^w \psi\| + \mathcal{O}(h^\infty) \|\psi\|.$$

With (2.6), (2.7), and (2.8), we deduce that

$$c \|\chi_0^w \psi\| \leq Ch \|\underline{\chi}_0^w \psi\| + \mathcal{O}(h^\infty) \|\psi\|.$$

Iterating with $\underline{\chi}_0$ instead of χ_0 , we get $\|\chi_0^w \psi\| = \mathcal{O}(h^\infty)$ and this concludes the proof in the case when $\delta = 0$.

Let us now consider the case $\delta \in (0, \frac{1}{2})$. We write again

$$\tilde{H}^w \psi = \lambda \psi.$$

Thanks to the rough microlocalization of the eigenfunctions established when $\delta = 0$, up to a remainder $\mathcal{O}(h^\infty)$, we can replace \tilde{H} by a symbol \check{H} in $S(1)$ and that coincides with \tilde{H} on $\{|X_1| \leq M\}$ for some arbitrary $M > 0$ and that satisfies

$$(2.9) \quad \text{Re } \check{H} \geq cp(X_1) - Ch,$$

where $p(X_1)$ equals $|X_1|^2$ near $(0, 0)$ and is constant away from a neighborhood of $(0, 0)$.

We have

$$\check{H}^w \psi = \lambda \psi + R_h^w \psi, \quad R_h^w \psi = \mathcal{O}(h^\infty) \|\psi\|.$$

We let $\chi_{0,\delta}(x, \xi) = 1 - \chi(h^{-\delta}x_1)\chi(h^{-\delta}\xi_1)$. We write again

$$(2.10) \quad \check{H}^w \chi_{0,\delta}^w \psi = \chi_{0,\delta}^w \check{H}^w \psi + [\check{H}^w, \chi_{0,\delta}^w] \psi.$$

By the symbolic calculus in $S^\delta(1)$, we see that the symbol of $[\check{H}^w, \chi_{0,\delta}^w]$ belongs to $h^{1-2\delta}S^\delta(1)$ and, due to the quadratic behaviour in X_1 , actually belongs to $hS^\delta(1)$. Similarly to the case $\delta = 0$, we get

$$(2.11) \quad \|[\check{H}^w, \chi_{0,\delta}^w] \psi\| \leq Ch \|\chi_{0,\delta}^w \psi\| + \mathcal{O}(h^\infty) \|\psi\|^2.$$

It is known that, modulo $\mathcal{O}(h^\infty)$, the Weyl quantization is unitarily equivalent to a positive quantization, namely the Toeplitz quantization on the Bargmann space; we denote by Op_h^+ the corresponding positive quantization on $L^2(\mathbf{R}^2)$. Let q be the Toeplitz symbol of \check{H} , so that

$$\text{Op}_h^w \check{H} = \text{Op}_h^+ q + \mathcal{O}(h^\infty).$$

Since $\check{H} \in S(1)$, we have $q \in S(1)$ and $q = \check{H} + \mathcal{O}(h)$, see for instance [26, Theorem 13.10].

Now, we consider a smooth cutoff function $\tilde{\chi} \in \mathcal{C}_0^\infty(\mathbf{R}^2)$ equal to 1 in a neighborhood of the origin, whose support is disjoint from the support of χ_0 , viewed as a function of X_1 . Let $\tilde{\chi}_\delta(X_1) := \tilde{\chi}(X_1/h^\delta)$, and $\tilde{a}_{h,\delta}(X) := q(X) + h^{2\delta}\tilde{\chi}_\delta(X_1)$. For some $c > 0$ and h small enough, (2.9) gives

$$\text{Re } \tilde{a}_{h,\delta}(X) \geq cp(X_1) + ch^{2\delta} \geq ch^{2\delta}.$$

Hence $\text{Re } \text{Op}_h^+ \tilde{a}_{h,\delta} \geq ch^{2\delta}$, in the sense of selfadjoint operators. Let $\tilde{\chi}_\delta$ be the Weyl symbol of $\text{Op}_h^+ \tilde{a}_{h,\delta}$, so that $\text{Op}_h^+ \tilde{a}_{h,\delta} = \text{Op}_h^w(\check{H} + h^{2\delta}\tilde{\chi}_\delta) + \mathcal{O}(h^\infty)$. Using again [26, Theorem 13.10], we see that $\tilde{\chi}_\delta \in S^\delta(1)$ and takes real values. While $\tilde{\chi}_\delta$ cannot vanish on any open set (it is analytic), it admits as asymptotic expansion in powers of $h^{1-2\delta}$ in the topology of $S^\delta(1)$, whose support is, for all fixed h , contained in the support of $\tilde{\chi}_\delta$. In particular

$$(2.12) \quad \tilde{\chi}_\delta^w \chi_{0,\delta}^w = \mathcal{O}(h^\infty).$$

Since $\text{Im}(\check{H} + h^{2\delta}\tilde{\chi}_\delta) = \mathcal{O}(h)$, we have

$$ch^{2\delta} \|\varphi\| \leq \|(\check{H}^w + h^{2\delta}\tilde{\chi}_\delta^w)\varphi\| + \mathcal{O}(h^\infty) \|\varphi\|.$$

Taking $\varphi = \chi_{0,\delta}^w \psi$ and using (2.10), (2.11) and (2.12), we find that

$$ch^{2\delta} \|\chi_{0,\delta}^w \psi\| \leq Ch \|\chi_{0,\delta}^w \psi\| + \mathcal{O}(h^\infty) \|\psi\|.$$

Since $2\delta < 1$, an induction argument (on the size of the support of $\chi_{0,\delta}$) gives the result. \square

Remark 2.8. In this proof we have used a detour via the Toeplitz quantization because, using the standard S^δ symbolic calculus, if a symbol $a(x, \xi) \in S^\delta$ satisfies $a \geq ch^{2\delta}$, then the (precise) Gårding inequality only implies that

$$\text{Op}_h^w(a) \geq ch^{2\delta} - \mathcal{O}(h^{1-2\delta}).$$

Therefore, the proof of the lemma would require $\delta < \frac{1}{4}$. It turns out that one can get a better result while staying with the Weyl quantization, using the special form of the Fefferman-Phong inequality due to Bony [3], as follows.

We first write

$$\text{Op}_h^w a = \text{Op}_1^w a(x, h\xi),$$

which is unitarily equivalent to $\text{Op}_1^w a(h^{\frac{1}{2}}x, h^{\frac{1}{2}}\xi)$. We have

$$a(h^{\frac{1}{2}}x, h^{\frac{1}{2}}\xi) - ch^{2\delta} \geq 0.$$

Then, we notice that $a(h^{\frac{1}{2}}\cdot)$ belongs to $S(1)$ since $\delta \in [0, \frac{1}{2}[$. In fact, we even have, for all $\gamma \in \mathbf{N}^2$ with $|\gamma| \geq 4$,

$$|\partial^\gamma b_{h,\delta}(h^{\frac{1}{2}}\cdot)| \leq c_\gamma, \quad b_{h,\delta}(\cdot) = h^{-2+2\delta}(a(\cdot) - \tilde{c}h^{2\delta}).$$

Because of this, the Bony-Fefferman-Phong inequality states that, for all $\varphi \in L^2(\mathbf{R}^2)$,

$$\langle \text{Op}_1^w b_{h,\delta}(h^{\frac{1}{2}}\cdot)\varphi, \varphi \rangle \geq -C\|\varphi\|^2,$$

and thus, after rescaling,

$$\text{Re} \langle \text{Op}_h^w a \varphi, \varphi \rangle \geq (\tilde{c}h^{2\delta} - Ch^{2-2\delta})\|\varphi\|^2.$$

We now see that $\delta < \frac{1}{2}$ is enough to obtain

$$\langle \text{Op}_h^w a \varphi, \varphi \rangle \geq ch^{2\delta}\|\varphi\|^2,$$

3. THE TRUNCATED OPERATOR $\widehat{\mathcal{L}}_h$

It follows from Lemma 2.7 that eigenpairs (λ, φ) of $\widehat{\mathcal{L}}_h^0$ with $\lambda \in D(\mu_0, Ch)$ satisfy

$$\varphi = (\text{Op}_1^w \chi_\delta)\varphi + \mathcal{O}_{L^2(\mathbf{R}^2)}(h^\infty),$$

with $\chi_\delta(\hat{x}_1, \hat{\xi}_1) = \chi(h^{\frac{1}{2}-\delta}\hat{x}_1)\chi(h^{\frac{1}{2}-\delta}\hat{\xi}_1) \in S(1)$, see also (2.5). This motivates the introduction of a so-called *truncated* operator $\widehat{\mathcal{L}}_h$, whose spectrum, as we shall prove in Section 6, will be close to the spectrum of $\widehat{\mathcal{L}}_h^0$ in the desired region.

Definition 3.1. Fix $\delta \in]0, \frac{1}{2}[$ and let χ be a smooth cutoff function on \mathbf{R}^2 , supported in a small neighborhood of 0, and equal to 1 near 0. The truncated operator $\widehat{\mathcal{L}}_h$ is the pseudodifferential operator

$$\widehat{\mathcal{L}}_h = \text{Op}_h^{w,2} \text{Op}_1^{w,1} p_h$$

with symbol (see (2.1))

$$(3.1) \quad \begin{aligned} p_h(x, \xi) = & \mathring{B}^2(\xi_2 + h^{\frac{1}{2}}\chi_\delta x_1, x_2 + h^{\frac{1}{2}}\chi_\delta \xi_1)\xi_1^2 + \left(x_1 + \mathring{\alpha}(\xi_2 + h^{\frac{1}{2}}\chi_\delta x_1, x_2 + h^{\frac{1}{2}}\chi_\delta \xi_1)\xi_1\right)^2 \\ & + \mathring{V}(\xi_2 + h^{\frac{1}{2}}\chi_\delta x_1, x_2 + h^{1/2}\chi_\delta \xi_1) + hW(\xi_2 + h^{\frac{1}{2}}\chi_\delta x_1, x_2 + h^{\frac{1}{2}}\chi_\delta \xi_1), \end{aligned}$$

where $\chi_\delta = \chi(h^{\frac{1}{2}-\delta}(x_1, \xi_1))$.

Thanks to this cutoff function, we will expand \mathring{B} and $\mathring{\alpha}$ with respect to $h^{1/2}\chi_\delta x_1$ without increasing the powers of (x_1, ξ_1) at infinity, hence remaining in a suitable class of symbols. As in the previous section, we use the notation $X_j = (x_j, \xi_j)$, $j = 1, 2$, and $X = (X_1, X_2) \in \mathbf{R}^4$.

3.1. Operator-valued symbol of $\widehat{\mathcal{L}}_h$. We now focus on $\widehat{\mathcal{L}}_h = \text{Op}_h^{w,2} \text{Op}_1^{w,1}(p_h)$. Note that, due to our assumptions, we have for some $c_1, c_2 > 0$,

$$c_1(|X_1|^2 + 1) \leq p_h \leq c_2(|X_1| + 1)^2.$$

Notation 1. We consider the operator symbol of $\widehat{\mathcal{L}}_h$ defined by

$$\mathbf{P}_h(X_2) = \text{Op}_1^{w,1}(p_h),$$

which for each fixed $X_2 \in \mathbf{R}^2$ acts on the domain

$$B^2(\mathbf{R}) = \{\psi \in H^2(\mathbf{R}) : x^2\psi \in L^2(\mathbf{R})\}.$$

Lemma 3.2. *For all $X_2 \in \mathbf{R}^2$, the operator $(B^2(\mathbf{R}), \mathbf{P}_h(X_2))$ is closed. Its graph norm is equivalent to $\|\cdot\|_{B^2(\mathbf{R})}$ (uniformly in $X_2 \in \mathbf{R}^2$ and $h > 0$ small enough). In particular, $(B^2(\mathbf{R}), \mathbf{P}_h(X_2))$ has compact resolvent.*

Proof. It is enough to prove the following two inequalities,

$$(3.2) \quad \|\mathbf{P}_h(X_2)\psi\| \leq C\|(1 + |X_1|^2)^w\psi\|,$$

$$(3.3) \quad \|(\mathbf{P}_h(X_2) + 1)\psi\| \geq c\|(|X_1|^2)^w\psi\|,$$

for all $\psi \in \mathcal{S}(\mathbf{R})$, and for some positive constants C and c independent of h and X_2 . Note that

$$(3.4) \quad |\partial_{X_1}^\gamma p_h| \leq c_\gamma(|X_1|^2 + 1)$$

for some constant c_γ independent of (h, X_2) . In other words, p_h belongs to the symbol class $S(1 + |X_1|^2)$ uniformly with respect to (h, X_2) . Thus the Weyl product $p_h \star (1 + |X_1|^2)^{-1}$ belongs to $S(1)$ uniformly with respect to (h, X_2) , and by using Calderón-Vaillancourt theorem, we get

$$\|\mathbf{P}_h(X_2) [(1 + |X_1|^2)^{-1}]^w \varphi\| \leq C\|\varphi\|, \quad \forall \varphi \in \mathcal{S}(\mathbf{R}),$$

with $C > 0$ independent of (h, X_2) , and (3.2) follows. Actually, $p_h + 1$ is also elliptic in $S(1 + |X_1|^2)$ uniformly with respect to (X_2, h) :

$$\exists c_0 > 0, \quad |p_h + 1| \geq c_0(1 + |X_1|^2).$$

Hence $(p_h + 1) \star (1 + |X_1|^2)^{-1}$ is elliptic in $S(1)$ and the parametrix construction implies

$$\forall \varphi \in \mathcal{S}(\mathbf{R}), \quad \|(\mathbf{P}_h(X_2) + 1) [(1 + |X_1|^2)^{-1}]^w \varphi\| \geq c\|\varphi\|,$$

and (3.3) follows. \square

Let us consider the class of "bounded" operator-valued symbols (see, for instance, [15, Chapitre 2] or [16]):

$$\begin{aligned} & S(\mathbf{R}^2, \mathcal{L}(B^2(\mathbf{R}), L^2(\mathbf{R}))) \\ &= \left\{ \mathbf{P} \in \mathcal{C}^\infty(\mathbf{R}^2, \mathcal{L}(B^2(\mathbf{R}), L^2(\mathbf{R}))) : \forall \alpha \in \mathbf{N}^2, \exists C_\alpha > 0, \forall X_2 \in \mathbf{R}^2, \right. \\ & \quad \left. \forall \psi \in B^2(\mathbf{R}), \|\partial_{X_2}^\alpha \mathbf{P}(X_2)\psi\| \leq C_\alpha \|\psi\|_{B^2(\mathbf{R})} \right\}. \end{aligned}$$

Since the inequality (3.4) still holds when p_h is replaced by its X_2 -derivatives, we get the following.

Lemma 3.3. *The operator symbol $X_2 \mapsto \mathbf{P}_h(X_2)$ belongs to $S(\mathbf{R}^2, \mathcal{L}(B^2(\mathbf{R}), L^2(\mathbf{R})))$.*

3.2. Expansion of the symbol of $\widehat{\mathcal{L}}_h$. We now impose $\delta \in (\frac{1}{3}, \frac{1}{2})$. We prove the following expansion for the symbol $P_h = \text{Op}_1^{w,1} p_h$ of $\widehat{\mathcal{L}}_h$, where the condition $\delta > \frac{1}{3}$ ensures that the remainder $h^{3\delta} R_h$ is indeed negligible with respect to the other terms.

Lemma 3.4. *We have*

$$P_h = P_0 + h^{1/2} P_1 + h P_2 + h^{3\delta} R_h$$

for some symbols P_0, P_1, P_2 defined in (3.5), and $R_h \in S(\mathbf{R}^2, \mathcal{L}(B^2(\mathbf{R}), L^2(\mathbf{R})))$ uniformly bounded with respect to h . Moreover the principal symbol $P_0(X_2)$ is the following X_2 -dependent "harmonic oscillator":

$$P_0(X_2) = \text{Op}_1^{w,1} \left(\mathring{B}^2(\xi_2, x_2) \xi_1^2 + (x_1 + \mathring{\alpha}(\xi_2, x_2) \xi_1)^2 + \mathring{V}(\xi_2, x_2) \right).$$

Proof. From Formula (3.1) we notice that p_h can be seen as a smooth function $p_h = \check{p}(X_1, X_2, \hbar, \chi_\delta)$, where $\hbar = h^{1/2}$. We may Taylor expand the symbol \check{p} with respect to the third variable " \hbar ":

$$\check{p}(X_1, X_2, \hbar, \chi_\delta) = p_0 + \hbar p_1 + \hbar^2 p_2 + \hbar^3 r_h,$$

where

$$r_h = \frac{1}{2} \int_0^1 (1-t)^2 \partial_{\hbar}^3 \check{p}(X_1, X_2, t\hbar, \chi_\delta) dt,$$

and $p_j = \frac{1}{j!} \partial_{\hbar}^j \check{p}(X_1, X_2, 0, \chi_\delta)$. Note that the p_j still slightly depend on h (through the cutoff functions χ_δ). Explicitely, we have

$$\begin{aligned} p_0 &= \mathring{B}^2(\xi_2, x_2) \xi_1^2 + (x_1 + \mathring{\alpha}(\xi_2, x_2) \xi_1)^2 + \mathring{V}(\xi_2, x_2), \\ p_1 &= \chi_\delta \left[2\xi_1^2 \mathring{B} \nabla \mathring{B} \cdot X_1 + 2\xi_1 (x_1 + \mathring{\alpha} \xi_1) \nabla \mathring{\alpha} \cdot X_1 + \nabla \mathring{V} \cdot X_1 \right], \\ p_2 &= \chi_\delta^2 \left[\xi_1^2 (\nabla \mathring{B} \cdot X_1)^2 + \xi_1^2 \nabla^2 \mathring{B}(X_1, X_1) + \xi_1^2 (\nabla \mathring{\alpha} \cdot X_1)^2 + \xi_1 (x_1 + \mathring{\alpha} \xi_1) \nabla^2 \mathring{\alpha}(X_1, X_1) \right] \\ &\quad + \chi_\delta^2 \nabla^2 \mathring{V}(X_1, X_1) + W(\xi_2, x_2), \end{aligned}$$

where the functions $\mathring{B}, \mathring{\alpha}, \mathring{V}$ and their gradients are implicitly taken at (ξ_2, x_2) . Letting

$$(3.5) \quad P_j(X_2) = \text{Op}_1^{w,1} p_j,$$

we notice that

$$P_h(X_2) = P_0(X_2) + \hbar P_1(X_2) + \hbar^2 P_2(X_2) + \hbar^3 \text{Op}_1^{w,1} r_h.$$

By using the Calderón-Vaillancourt theorem, due to the cutoff functions we can check that

$$\forall \psi \in B^2(\mathbf{R}), \quad \|\partial_{X_2}^\alpha \text{Op}_1^{w,1} r_h \psi\| \leq C_\alpha h^{-3(\frac{1}{2}-\delta)} \|\psi\|_{B^2(\mathbf{R})}.$$

Therefore, we can write

$$P_h = P_0 + \hbar P_1 + \hbar^2 P_2 + \hbar^{6\delta} R_h$$

with $R_h \in S(\mathbf{R}^2, \mathcal{L}(B^2(\mathbf{R}), L^2(\mathbf{R})))$ uniformly bounded with respect to h . This leads to choosing $\delta \in (\frac{1}{3}, \frac{1}{2})$ and concludes the proof. \square

3.3. About the principal part $P_0(X_2)$. In this section we describe some important properties of $P_0(X_2)$. Since it is a ‘harmonic oscillator’, *i.e.* the quantization of a positive definite quadratic form in X_1 , we have the following spectral properties. If we let

$$f_{X_2}(x_1) = C(X_2) e^{-\frac{\mathring{B}}{2(\mathring{B}^2 + \mathring{\alpha}^2)} x_1^2} e^{i \frac{\mathring{\alpha}}{2(\mathring{B}^2 + \mathring{\alpha}^2)} x_1^2}, \quad C(X_2) = \left(\frac{\mathring{B}}{\pi(\mathring{B}^2 + \mathring{\alpha}^2)} \right)^{\frac{1}{4}}.$$

then we have

$$P_0(X_2) f_{X_2} = \mu(X_2) f_{X_2}, \quad \mu(X_2) = \mathring{B}(\xi_2, x_2) + \mathring{V}(\xi_2, x_2).$$

Moreover, the eigenvalues of $P_0(X_2)$ are in the form

$$(2n - 1) \mathring{B}(\xi_2, x_2) + \mathring{V}(\xi_2, x_2), \quad n \geq 1.$$

Thus f_{X_2} is the ground state of $P_0(X_2)$. When considering the restriction of $P_0(X_2)$ to $f_{X_2}^\perp$, which is a stable subspace, we get the following lemma.

Lemma 3.5. *The operator $(P_0(X_2) - z) : B^2(\mathbf{R}) \cap f_{X_2}^\perp \rightarrow f_{X_2}^\perp$ is bijective when*

$$u \operatorname{Re}(z - \mu_0) + v \operatorname{Im}(z - \mu_0) < 2ub_0,$$

and in particular when $z \in D(\mu_0, Ch)$ if h is small enough.

Proof. We notice that, for all $\psi \in B^2(\mathbf{R}) \cap f_{X_2}^\perp$,

$$\begin{aligned} \operatorname{Re} [(u - iv) \langle (P_0(X_2) - z)\psi, \psi \rangle] &\geq \left[u(3\mathring{B}(X_2) + \operatorname{Re}(\mathring{V}(X_2) - z)) + v \operatorname{Im}(\mathring{V}(X_2) - z) \right] \|\psi\|^2 \\ &\geq [2ub_0 + u \operatorname{Re}(\mu_0 - z) + v \operatorname{Im}(\mu_0 - z)] \|\psi\|^2, \end{aligned}$$

where we used that $u(\mathring{B}(X_2) + \operatorname{Re} \mathring{V}(X_2)) + v \operatorname{Im} \mathring{V}(X_2) \geq \mu_0$ (Assumption III). This shows that $P_0(X_2) - z$ is injective with closed range. But $P_0(X_2) - z$ is a selfadjoint harmonic oscillator (up to an additive constant), so the conclusion follows. \square

Proposition 3.6. *Let $z \in D(\mu_0, Ch)$. We consider the operator*

$$\mathcal{P}_0(X_2, z) = \begin{pmatrix} P_0(X_2) - z & \cdot f_{X_2} \\ \langle \cdot, f_{X_2} \rangle & 0 \end{pmatrix} \in \mathcal{L}(B^2(\mathbf{R}) \times \mathbf{C}, L^2(\mathbf{R}) \times \mathbf{C}).$$

We have

$$\mathcal{P}_0(\cdot, z) \in S(\mathbf{R}^2, \mathcal{L}(B^2(\mathbf{R}) \times \mathbf{C}, L^2(\mathbf{R}) \times \mathbf{C})).$$

Moreover, if h is small enough, $\mathcal{P}_0(X_2, z)$ is a bijection and

$$\mathcal{Q}_0 := \mathcal{P}_0^{-1} = \begin{pmatrix} (P_0 - z)^{-1} \Pi^\perp & \cdot f_{X_2} \\ \langle \cdot, f_{X_2} \rangle & z - [\mathring{B} + \mathring{V}] \end{pmatrix} \in S(\mathbf{R}^2, \mathcal{L}(L^2(\mathbf{R}) \times \mathbf{C}, B^2(\mathbf{R}) \times \mathbf{C})),$$

where $\Pi^\perp = \operatorname{Id} - \langle \cdot, f_{X_2} \rangle f_{X_2}$.

Proof. Let us consider $(\psi, \beta) \in L^2(\mathbf{R}) \times \mathbf{C}$ and look for $(\varphi, \alpha) \in B^2(\mathbf{R}) \times \mathbf{C}$ such that

$$(P_0 - z)\varphi = \psi - \alpha f_{X_2}, \quad \langle \varphi, f_{X_2} \rangle = \beta.$$

The first equation has solutions only if

$$\langle \psi - \alpha f_{X_2}, f_{X_2} \rangle = (\mu - z) \langle \varphi, f_{X_2} \rangle = (\mu - z) \beta,$$

where $\mu = \mu(X_2) = \mathring{B}(X_2) + \mathring{V}(X_2)$; this is equivalent to $\alpha = \langle \psi, f_{X_2} \rangle + (z - \mu) \beta$. With this choice, we write

$$(P_0 - z)(\varphi - \beta f_{X_2}) = \psi - \alpha f_{X_2} + \beta(\mu - z) f_{X_2} \in f_{X_2}^\perp.$$

It remains to apply Lemma 3.5. □

4. PARAMETRIX CONSTRUCTION AND CONSEQUENCES

4.1. Parametrix construction. Let us now consider the “Grushin operator symbol”

$$X_2 \mapsto \mathcal{P}_h(X_2) = \begin{pmatrix} P_h(X_2) - z & \cdot f_{X_2} \\ \langle \cdot, f_{X_2} \rangle & 0 \end{pmatrix} \in S(\mathbf{R}^2, \mathcal{L}(B^2(\mathbf{R}) \times \mathbf{C}, L^2(\mathbf{R}) \times \mathbf{C})),$$

and notice that

$$(4.1) \quad \text{Op}_h^{w,2} \mathcal{P}_h = \begin{pmatrix} \widehat{\mathcal{L}}_h - z & \mathfrak{P}^* \\ \mathfrak{P} & 0 \end{pmatrix}, \quad \mathfrak{P} = \text{Op}_h^{w,2} \langle \cdot, f_{X_2} \rangle.$$

Note that for $\delta \in (\frac{1}{3}, \frac{1}{2})$,

$$\mathcal{P}_h(X_2) = \mathcal{P}_0 + h^{1/2} \mathcal{P}_1 + h \mathcal{P}_2 + \mathcal{O}(h^{3\delta}),$$

with \mathcal{P}_0 defined in Proposition 3.6, and for $j \geq 1$,

$$\mathcal{P}_j = \begin{pmatrix} P_j(X_2) & 0 \\ 0 & 0 \end{pmatrix},$$

and the remainder belongs to $S(\mathbf{R}^2, \mathcal{L}(B^2(\mathbf{R}) \times \mathbf{C}, L^2(\mathbf{R}) \times \mathbf{C}))$. The following proposition is an approximate parametrix construction.

Proposition 4.1. *For $z \in D(\mu_0, Ch)$ we consider*

$$\begin{cases} \mathcal{Q}_1 = -\mathcal{Q}_0 \mathcal{P}_1 \mathcal{Q}_0 \\ \mathcal{Q}_2 = -\mathcal{Q}_0 \mathcal{P}_2 \mathcal{Q}_0 - \mathcal{Q}_1 \mathcal{P}_1 \mathcal{Q}_0 - \frac{1}{2i} \{ \mathcal{Q}_0, \mathcal{P}_0 \} \mathcal{Q}_0, \end{cases}$$

and we let

$$\mathcal{Q}_h^{[2]} = \mathcal{Q}_0 + h^{1/2} \mathcal{Q}_1 + h \mathcal{Q}_2.$$

Then we have

$$\left(\text{Op}_h^{w,2} \mathcal{Q}_h^{[2]} \right) \left(\text{Op}_h^{w,2} \mathcal{P}_h \right) = \text{Id} + \mathcal{R}_{h,z}, \quad \mathcal{R}_{h,z} = \mathcal{O}(h^{3\delta}),$$

where the bounded operator $\mathcal{R}_{h,z}$ depends on z analytically. We also have

$$\left(\text{Op}_h^{w,2} \mathcal{P}_h \right) \left(\text{Op}_h^{w,2} \mathcal{Q}_h^{[2]} \right) = \text{Id} + \tilde{\mathcal{R}}_{h,z}, \quad \tilde{\mathcal{R}}_{h,z} = \mathcal{O}(h^{3\delta}),$$

Moreover, the operator $\mathcal{Q}_h^{[2]}$ has the form

$$\mathcal{Q}_h^{[2]}(X_2) = \begin{pmatrix} * & * \\ * & \mathcal{Q}_{h,\pm}^{[2]}(X_2) \end{pmatrix},$$

with the scalar function

$$\begin{aligned} \mathcal{Q}_{h,\pm}^{[2]} &= z - \mu(X_2) - h^{1/2} \langle P_1(X_2) f_{X_2}, f_{X_2} \rangle \\ &\quad + h \left(-\langle P_2(X_2) f_{X_2}, f_{X_2} \rangle + \langle P_1(X_2) (P_0(X_2) - z)^{-1} \Pi^\perp P_1(X_2) f_{X_2}, f_{X_2} \rangle \right). \end{aligned}$$

Proof. The composition $\text{Op}_h^{w,2} \mathcal{Q}_h^{[2]} \text{Op}_h^{w,2} \mathcal{P}_h$ gives a new pseudo-differential operator (with operator symbol). This symbol is given by the usual h -Moyal composition law (with $\hbar = h^{1/2}$) $\mathcal{Q}_h^{[2]} \star \mathcal{P}_h$, and we have

$$\begin{aligned} \mathcal{Q}_h^{[2]} \star \mathcal{P}_h &= (\mathcal{Q}_0 + \hbar \mathcal{Q}_1 + \hbar^2 \mathcal{Q}_2) \star (\mathcal{P}_0 + \hbar \mathcal{P}_1 + \hbar^2 \mathcal{P}_2 + \mathcal{O}(\hbar^{6\delta})) \\ &= \mathcal{Q}_0 \mathcal{P}_0 + \frac{\hbar^2}{2i} \{\mathcal{Q}_0, \mathcal{P}_0\} + \hbar (\mathcal{Q}_0 \mathcal{P}_1 + \mathcal{Q}_1 \mathcal{P}_0) \\ &\quad + \hbar^2 (\mathcal{Q}_2 \mathcal{P}_0 + \mathcal{Q}_0 \mathcal{P}_2 + \mathcal{Q}_1 \mathcal{P}_1) + \mathcal{O}(\hbar^{6\delta}). \end{aligned}$$

This leads to the formulas of the \mathcal{Q}_j . Let us now compute $\mathcal{Q}_{h,\pm}^{[2]}$ (slightly departing from the traditional $-+$ subscript in Grushin problems, we use the subscript \pm for the lower-right coefficient of block matrices). An easy product of operator matrices gives

$$\mathcal{Q}_{1,\pm} = -\langle \mathbf{P}_1(X_2) f_{X_2}, f_{X_2} \rangle.$$

In the same way, we get

$$(-\mathcal{Q}_0 \mathcal{P}_2 \mathcal{Q}_0)_{\pm} = -\langle \mathbf{P}_2(X_2) f_{X_2}, f_{X_2} \rangle.$$

From

$$-\mathcal{Q}_1 \mathcal{P}_1 \mathcal{Q}_0 = \mathcal{Q}_0 \mathcal{P}_1 \mathcal{Q}_0 \mathcal{P}_1 \mathcal{Q}_0,$$

we obtain

$$(-\mathcal{Q}_1 \mathcal{P}_1 \mathcal{Q}_0)_{\pm} = \langle \mathbf{P}_1(\mathbf{P}_0 - z)^{-1} \Pi^{\perp} \mathbf{P}_1 f_{X_2}, f_{X_2} \rangle.$$

Using

$$\{\mathcal{Q}_0, \mathcal{P}_0\} = \partial_{\xi} \mathcal{Q}_0 \partial_x \mathcal{P}_0 - \partial_x \mathcal{Q}_0 \partial_{\xi} \mathcal{P}_0,$$

another computation gives

$$\begin{aligned} (\{\mathcal{Q}_0, \mathcal{P}_0\} \mathcal{Q}_0)_{\pm} &= \langle (\partial_x \mathbf{P}_0) f_{X_2}, \partial_{\xi} f_{X_2} \rangle - \langle (\partial_{\xi} \mathbf{P}_0) f_{X_2}, \partial_x f_{X_2} \rangle \\ &\quad + (\mu - z) (\langle \partial_x f_{X_2}, \partial_{\xi} f_{X_2} \rangle - \langle \partial_{\xi} f_{X_2}, \partial_x f_{X_2} \rangle). \end{aligned}$$

Note that, since f_{X_2} is an L^2 -normalized eigenfunction,

$$\begin{aligned} 0 = \partial_x \langle (\mathbf{P}_0 - z) f_{X_2}, \partial_{\xi} f_{X_2} \rangle &= \langle (\partial_x \mathbf{P}_0) f_{X_2}, \partial_{\xi} f_{X_2} \rangle + \langle (\mathbf{P}_0 - z) \partial_x f_{X_2}, \partial_{\xi} f_{X_2} \rangle \\ &\quad + \langle (\mathbf{P}_0 - z) f_{X_2}, \partial_x \partial_{\xi} f_{X_2} \rangle, \end{aligned}$$

and

$$\begin{aligned} 0 = \partial_{\xi} \langle (\mathbf{P}_0 - z) f_{X_2}, \partial_x f_{X_2} \rangle &= \langle (\partial_{\xi} \mathbf{P}_0) f_{X_2}, \partial_x f_{X_2} \rangle + \langle (\mathbf{P}_0 - z) \partial_{\xi} f_{X_2}, \partial_x f_{X_2} \rangle \\ &\quad + \langle (\mathbf{P}_0 - z) f_{X_2}, \partial_x \partial_{\xi} f_{X_2} \rangle, \end{aligned}$$

Thus,

$$(\{\mathcal{Q}_0, \mathcal{P}_0\} \mathcal{Q}_0)_{\pm} = -\langle (\mathbf{P}_0 - \mu) \partial_x f, \partial_{\xi} f \rangle + \langle (\mathbf{P}_0 - \mu) \partial_{\xi} f, \partial_x f \rangle = 0,$$

where we used that $\mathbf{P}_0 - \mu$ is selfadjoint.

Therefore,

$$\mathcal{Q}_{2,\pm} = -\langle \mathbf{P}_2(X_2) f_{X_2}, f_{X_2} \rangle + \langle \mathbf{P}_1(X_2) (\mathbf{P}_0 - z)^{-1} \Pi^{\perp} \mathbf{P}_1(X_2) f_{X_2}, f_{X_2} \rangle.$$

□

Remark 4.2. The expression of the “Schur complement” $\mathcal{Q}_{h,\pm}^{[2]}$ has already appeared in previous works (see [15, Proposition 3.1.10]). Note however that the assumption of [15, Hypothèse 3.1.9] (*i.e.*, the principal operator symbol does not depend on ξ) is not satisfied in our context. It is also important to notice that our h -pseudo-differential operator is expanded according to the powers of $\hbar = h^{\frac{1}{2}}$ and not h . This avoids the nasty Poisson brackets computations of [15, Lemme 3.1.11].

The following lemma shows that the apparent subprincipal symbol of $\mathcal{Q}_{h,\pm}^{[2]}$ actually vanishes (modulo $\mathcal{O}(h^\infty)$). More precisely:

Lemma 4.3. *We have, in the symbol class $S(1)$,*

$$\langle \mathbf{P}_1(X_2)f_{X_2}, f_{X_2} \rangle = \mathcal{O}(h^\infty).$$

Proof. This follows from the fact that $\mathbf{P}_1(X_2) = \mathbf{Op}_1^{w,1}(\chi_\delta \tilde{p}_1)$ where

$$\tilde{p}_1 = 2\xi_1^2 \mathring{B} \nabla \mathring{B} \cdot X_1 + 2\xi_1(x_1 + \mathring{\alpha}\xi_1) \nabla \mathring{\alpha} \cdot X_1 + \nabla V \cdot X_1.$$

Indeed, \tilde{p}_1 is a homogeneous function of X_1 of odd order, and f_{X_2} is an even function of x_1 so that, for all $X_2 \in \mathbf{R}^2$,

$$\langle \mathbf{Op}_1^{w,1} \tilde{p}_1 f_{X_2}, f_{X_2} \rangle = 0.$$

From this, we see that the term χ_δ will only contribute to $\mathcal{O}(h^\infty)$ due to the exponential decay of f_{X_2} . The same argument applies to the derivatives with respect to X_2 . \square

Corollary 4.4. *For h small enough, $\text{Id} + \mathcal{R}_{h,z}$ and $\text{Id} + \tilde{\mathcal{R}}_{h,z}$ are bijective. If we let*

$$\mathcal{E}_h = (\text{Id} + \mathcal{R}_{h,z})^{-1} \mathbf{Op}_h^{w,2} \mathcal{Q}_h^{[2]}, \quad \tilde{\mathcal{E}}_h = \mathbf{Op}_h^{w,2} \mathcal{Q}_h^{[2]} (\text{Id} + \tilde{\mathcal{R}}_{h,z})^{-1},$$

we have

$$\mathcal{E}_h \cdot \mathbf{Op}_h^{w,2} \mathcal{P}_h = \text{Id}, \quad \mathbf{Op}_h^{w,2} \mathcal{P}_h \cdot \tilde{\mathcal{E}}_h = \text{Id}, \quad \tilde{\mathcal{E}}_h = \mathcal{E}_h.$$

4.2. From $\widehat{\mathcal{L}}_h$ to $\mathcal{E}_{h,\pm}$. According to Corollary 4.4, the operator $\mathcal{E}_h = \mathcal{E}_h(z)$ is the inverse of $\mathbf{Op}_h^{w,2} \mathcal{P}_h$. We can write it in the matrix form

$$\mathcal{E}_h = \begin{pmatrix} \mathcal{E}_{h,++} & \mathcal{E}_{h,+} \\ \mathcal{E}_{h,-} & \mathcal{E}_{h,\pm} \end{pmatrix}.$$

Then we have the following classical observation (see [24], for instance, for a review on Grushin methods).

Lemma 4.5. *For $z \in D(\mu_0, Ch)$ we have, for h small enough,*

$$z \in \text{sp}(\widehat{\mathcal{L}}_h) \iff 0 \in \text{sp}(\mathcal{E}_{h,\pm}).$$

Moreover, when $z \notin \text{sp}(\widehat{\mathcal{L}}_h)$, the following formulas hold

$$(4.2) \quad \mathcal{E}_{h,\pm}^{-1} = -\mathfrak{P}(\widehat{\mathcal{L}}_h - z)^{-1} \mathfrak{P}^*,$$

and

$$(4.3) \quad (\widehat{\mathcal{L}}_h - z)^{-1} = \mathcal{E}_{h,++} - \mathcal{E}_{h,+} \mathcal{E}_{h,\pm}^{-1} \mathcal{E}_{h,-}.$$

Proof. From Corollary 4.4, and in view of (4.1), we have

$$(4.4) \quad (\widehat{\mathcal{L}}_h - z)\mathcal{E}_{h,+} + \mathfrak{P}^*\mathcal{E}_{h,\pm} = 0, \quad \mathfrak{P}\mathcal{E}_{h,+} = \text{Id},$$

and

$$(4.5) \quad \mathcal{E}_{h,-}(\widehat{\mathcal{L}}_h - z) + \mathcal{E}_{h,\pm}\mathfrak{P} = 0, \quad \mathcal{E}_{h,-}\mathfrak{P}^* = \text{Id}.$$

By using (4.4) and (4.5), we see that when $\widehat{\mathcal{L}}_h - z$ is bijective, so is $\mathcal{E}_{h,\pm}$. Then, assume that $\mathcal{E}_{h,\pm}$ is bijective and consider also

$$\mathcal{E}_{h,++}(\widehat{\mathcal{L}}_h - z) + \mathcal{E}_{h,+}\mathfrak{P} = \text{Id}.$$

With (4.5), we get

$$(\mathcal{E}_{h,++} - \mathcal{E}_{h,+}\mathcal{E}_{h,\pm}^{-1}\mathcal{E}_{h,-})(\widehat{\mathcal{L}}_h - z) = \text{Id}.$$

Using (4.4) and also

$$(\widehat{\mathcal{L}}_h - z)\mathcal{E}_{h,++} + \mathfrak{P}^*\mathcal{E}_{h,-} = \text{Id},$$

we get

$$(\widehat{\mathcal{L}}_h - z)(\mathcal{E}_{h,++} - \mathcal{E}_{h,+}\mathcal{E}_{h,\pm}^{-1}\mathcal{E}_{h,-}) = \text{Id}.$$

□

5. SPECTRAL REDUCTION

In Lemma 4.5 we proved that the spectrum of $\widehat{\mathcal{L}}_h$ in $D(\mu_0, Ch)$ is given by those z such that $\mathcal{E}_{h,\pm}(z)$ is not bijective. Moreover, according to Corollary 4.4 and Proposition 4.1, $\mathcal{E}_{h,\pm}(z) = \text{Op}_h^{w,2}\mathcal{Q}_{h,\pm}^{[2]} + \mathcal{O}(h^{3\delta})$, and hence

$$(5.1) \quad \mathcal{E}_{h,\pm}(z) = \text{Op}_h^w(z - \mu_h^{\text{eff}}(X_2) + \mathcal{O}(h^{3\delta})).$$

with the effective symbol, belonging to $S_{\mathbf{R}_{X_2}^2}(1)$, given by

$$(5.2) \quad \mu_h^{\text{eff}}(X_2) = \mathring{B}(X_2) + \mathring{V}(X_2) + h(\langle \text{P}_2(X_2)f_{X_2}, f_{X_2} \rangle - \langle \text{P}_1(X_2)(\text{P}_0(X_2) - z)^{-1}\Pi^\perp \text{P}_1(X_2)f_{X_2}, f_{X_2} \rangle).$$

Indeed, the $h^{1/2}$ -order term appears to be small by Lemma 4.3. We recall that P_j was defined in (3.5), f_{X_2} is the first eigenfunction of the harmonic oscillator $\text{P}_0(X_2)$ and Π^\perp is the orthogonal projection onto $f_{X_2}^\perp$ (see Lemma 3.5). We denote $\text{P}_h^{\text{eff}} = \text{Op}_h^w \mu_h^{\text{eff}}$. The aim of this section is to prove that the spectrum of $\widehat{\mathcal{L}}_h$ is given by the spectrum of P_h^{eff} up to a small error.

5.1. The spectrum of $\widehat{\mathcal{L}}_h$ is discrete.

Proposition 5.1. *The following families are analytic families of Fredholm operators of index 0:*

$$\begin{array}{cc} (z - \text{Op}_h^w(\mathring{B} + \mathring{V}))_{z \in D(\mu_0, Ch)} & (z - \text{Op}_h^w(\mu_h^{\text{eff}}))_{z \in D(\mu_0, Ch)} \\ (\mathcal{E}_{h,\pm}(z))_{z \in D(\mu_0, Ch)} & (\widehat{\mathcal{L}}_h - z)_{z \in D(\mu_0, Ch)} \end{array}$$

Proof. Let us consider the family $(z - \text{Op}_h^w(\mathring{B} + \mathring{V}))_{z \in D(\mu_0, Ch)}$. This is an analytic family of bounded operators. By perturbation, it is enough to prove that $\text{Op}_h^w(\mathring{B} + \mathring{V}) - \mu_0$ is a Fredholm operator with index 0 (since the set of Fredholm operators of index 0 is open). Let $u > 0$ and $v \in \mathbf{R}$ be given by Assumption III. The function $F = u(\mathring{B} + \text{Re } \mathring{V}) + v\text{Im } \mathring{V}$ admits a global minimum, and there exist a compact K and a constant $\gamma > \min F$ such that,

$$\forall X_2 \in \mathbf{R}^2 \setminus K, \quad F(X_2) \geq \gamma.$$

Thus we may consider a smooth cutoff function χ supported in a neighborhood of K such that

$$F + \chi \geq \gamma > \min F.$$

Define

$$P = \text{Op}_h^w(\mathring{B} + \mathring{V}) + (u - iv)^{-1}\chi.$$

Proving that $P - \mu_0$ is invertible is enough to conclude that $\text{Op}_h^w(\mathring{B} + \mathring{V}) - \mu_0$ is a Fredholm operator with index 0. Let $w = (u - iv)\mu_0$ and $Q = (u - iv)P$, so that

$$\begin{aligned} (u - iv)(P - \mu_0) &= Q - w \\ &= \text{Op}_h^w \left(u\text{Re}(\mathring{B} + \mathring{V} - \mu_0) + v\text{Im}(\mathring{V} - \mu_0) + \chi \right) \\ &\quad + i \text{Op}_h^w \left(u\text{Im}(\mathring{V} - \mu_0) - v\text{Re}(\mathring{V} + \mathring{B} - \mu_0) \right). \end{aligned}$$

Each parenthesis being selfadjoint, we deduce for all $\psi \in \mathbf{L}^2(\mathbf{R})$ that

$$\begin{aligned} |\langle (Q - w)\psi, \psi \rangle| &\geq \text{Re} \langle (Q - w)\psi, \psi \rangle \\ &\geq \left\langle \text{Op}_h^w \left(u\text{Re}(\mathring{B} + \mathring{V} - \mu_0) + v\text{Im}(\mathring{V} - \mu_0) + \chi \right) \psi, \psi \right\rangle. \end{aligned}$$

Using the Gårding inequality, and with $\min F = u\text{Re } \mu_0 + v\text{Im } \mu_0$, we get

$$|\langle (Q - w)\psi, \psi \rangle| \geq (\gamma - \min F) \|\psi\|^2.$$

Hence $Q - w$ is one-to-one with closed range. We can apply the same arguments for the adjoint of $Q - w$. We deduce that $Q - w$ is bijective, and so is $P - \mu_0$.

By (5.1) and (5.2), we have, for $z \in D(\mu_0, Ch)$,

$$\mathcal{E}_{h,\pm}(z) = \text{Op}_h^w \left(z - \mathring{B} - \mathring{V} \right) + \mathcal{O}(h).$$

Thus, $\mathcal{E}_{h,\pm}(z)$ is Fredholm of index 0, as soon as h is small enough (and by construction it is analytic with respect to z). The same perturbation argument hold for $z - \text{Op}_h^w \mu_h^{\text{eff}}$. Using again Corollary 4.4, this implies that $\widehat{\mathcal{L}}_h - z$ is also a Fredholm operator of index 0 (the Fredholmness of the Schur complement $\mathcal{E}_{h,\pm}$ is equivalent to that of $\widehat{\mathcal{L}}_h - z$). \square

Proposition 5.1 is not enough to establish that the spectrum of $\widehat{\mathcal{L}}_h$ is discrete in $D(\mu_0, Ch)$: We have to check that the resolvent set intersects $D(\mu_0, Ch)$.

Thanks to the assumptions on $\mathbf{P}_h^{\text{eff}}$, we can draw in the resolvent set of $\mathbf{P}_h^{\text{eff}}$ the circle $\Gamma_{j,h}$ of radius $h^{\frac{3}{2}-\kappa}$ and center $\nu_j(h)$ for h small enough.

Lemma 5.2. *Let us denote by $D_{j,h}$ the open disc of center $\nu_j(h)$ and radius $h^{\frac{3}{2}-\kappa}$ and let*

$$\mathcal{R}_h := D(\mu_0, Ch) \setminus \bigcup_{j \in \{1, \dots, N\}} D_{j,h}.$$

There exists $h_0 > 0$ such that, for all $h \in (0, h_0)$, we have

$$\mathcal{R}_h \subset \rho(\widehat{\mathcal{L}}_h).$$

Proof. We recall that, for $z \in D(\mu_0, Ch)$, we have

$$\mathcal{E}_{h,\pm}(z) = z - \mathbf{P}_h^{\text{eff}} + \mathcal{O}(h^{3\delta}).$$

By a classical perturbation argument using (1.4), we see that, for all $z \in \mathcal{R}_h$, $\mathcal{E}_{h,\pm}(z)$ is bijective. Indeed, it is bijective as soon as

$$\mathcal{O}(h^{3\delta}) \|(\mathbf{P}_h^{\text{eff}} - z)^{-1}\| < 1,$$

so it is sufficient that $h^{3\delta} < h^{\frac{3}{2}-\kappa}$, which we enforce by taking, as we may,

$$(5.3) \quad \delta > \max\left(\frac{1}{2} - \frac{\kappa}{3}, \frac{1}{3}\right)$$

(the lower bound $\delta > \frac{1}{3}$ comes from Section 3.2). Thanks to Lemma 4.5, we deduce that $\mathcal{R}_h \subset \rho(\widehat{\mathcal{L}}_h)$. \square

Lemma 5.2 and Proposition 5.1 imply that the spectrum of $\widehat{\mathcal{L}}_h$ in $D(\mu_0, Ch)$ is discrete (thanks to the analytic Fredholm theory). This also implies that, for $z \in D(\mu_0, Ch)$, $\mathcal{E}_{h,\pm}(z)$ is bijective except for discrete values of z .

5.2. The spectrum of $\widehat{\mathcal{L}}_h$ lies near the one of $\mathbf{P}_h^{\text{eff}}$. The following proposition states that the spectrum of $\widehat{\mathcal{L}}_h$ must be located near the spectrum of the effective operator.

Proposition 5.3. *There exist $h_0, \tilde{C} > 0$ such that, for all $h \in (0, h_0)$, if $\lambda \in D(\mu_0, Ch) \cap \text{sp}(\widehat{\mathcal{L}}_h)$, then*

$$\text{dist}(\lambda, \text{sp}(\mathbf{P}_h^{\text{eff}})) \leq \tilde{C} h^{\frac{3}{2}-\kappa}.$$

Proof. Since we know that the spectrum is discrete, we may consider an eigenpair (λ, ψ) . By Corollary 4.4, we have

$$\mathcal{Q}_{h,++}^{[2]}(\widehat{\mathcal{L}}_h - \lambda) + \mathcal{Q}_{h,+}^{[2]}\mathfrak{P} = \text{Id} + \mathcal{O}(h^{3\delta}),$$

and

$$\mathcal{Q}_{h,-}^{[2]}(\widehat{\mathcal{L}}_h - \lambda) + \mathcal{Q}_{h,\pm}^{[2]}\mathfrak{P} = \mathcal{O}(h^{3\delta}),$$

so that

$$\|\psi\| \leq C \|\mathfrak{P}\psi\|, \quad \|\mathcal{Q}_{h,\pm}^{[2]}\mathfrak{P}\psi\| \leq Ch^{3\delta} \|\psi\| \leq \tilde{C} h^{3\delta} \|\mathfrak{P}\psi\|.$$

The resolvent bound (1.4) provides us with

$$(5.4) \quad \text{dist}(\lambda, \text{sp}(\mathbf{P}_h^{\text{eff}})) \|\varphi\| \leq C \|(\lambda - \mathbf{P}_h^{\text{eff}}) \varphi\|,$$

and thus, we get

$$\text{dist}(\lambda, \text{sp}(\mathbf{P}_h^{\text{eff}})) \|\mathfrak{P}\psi\| \leq Ch^{3\delta} \|\mathfrak{P}\psi\|.$$

\square

5.3. The spectrum of $\mathbf{P}_h^{\text{eff}}$ lies near the one of $\widehat{\mathcal{L}}_h$.

Proposition 5.4. *Consider $j \in \{1, \dots, N\}$. There exists $h_0 > 0$ such that, for all $h \in (0, h_0)$, the circle of center $\nu_j(h)$ and radius $h^{\frac{3}{2}-\kappa}$ encircles at least one point in the spectrum of $\widehat{\mathcal{L}}_h$.*

Proof. We recall Lemma 5.2. For $z \in \Gamma_{j,h}$, thanks to a Neumann series we get

$$(5.5) \quad \mathcal{E}_{h,\pm,z}^{-1} = (z - \mathbf{P}_h^{\text{eff}})^{-1}(\text{Id} + \mathcal{N}_{h,z}),$$

where the bounded operator $\mathcal{N}_{h,z}$ satisfies

$$\|\mathcal{N}_{h,z}\| \leq Ch^{3\delta} \text{dist}(z, \text{sp}(\mathbf{P}_h^{\text{eff}}))^{-1} \leq \tilde{C}h^{3\delta-\frac{3}{2}+\kappa} < 1,$$

uniformly with respect to $z \in \Gamma_{j,h}$; the last inequality coming from (5.3). Therefore, we get

$$\|\mathcal{E}_{h,\pm,z}^{-1} - (z - \mathbf{P}_h^{\text{eff}})^{-1}\| \leq \tilde{C}h^{3\delta-3+2\kappa}.$$

Integrating over the contour (whose length is $2\pi h^{\frac{3}{2}-\kappa}$), we find that

$$\left\| \frac{1}{2i\pi} \int_{\Gamma_{j,h}} \mathcal{E}_{h,\pm,z}^{-1} dz - \frac{1}{2i\pi} \int_{\Gamma_{j,h}} (z - \mathbf{P}_h^{\text{eff}})^{-1} dz \right\| \leq \tilde{C}h^{3\delta-\frac{3}{2}+\kappa}.$$

We see that the right-hand-side goes to 0 when h goes to 0. We recall that $\frac{1}{2i\pi} \int_{\Gamma_{j,h}} (z - \mathbf{P}_h^{\text{eff}})^{-1} dz$ is the (Riesz) projection on the eigenspace of $\mathbf{P}_h^{\text{eff}}$ associated with $\nu_j(h)$. If $\Gamma_{j,h}$ does not encircle any element in the spectrum of $\widehat{\mathcal{L}}_h$, we see with Lemma 4.5 that $\frac{1}{2i\pi} \int_{\Gamma_{j,h}} \mathcal{E}_{h,\pm,z}^{-1} dz = 0$, and thus that the projection $\frac{1}{2i\pi} \int_{\Gamma_{j,h}} (z - \mathbf{P}_h^{\text{eff}})^{-1} dz$ must be zero, and this would be a contradiction. \square

In fact, we can prove slightly more.

Proposition 5.5. *Let us consider the spectral projector $\Pi_{j,h}$ of $\widehat{\mathcal{L}}_h$ associated with the contour $\Gamma_{j,h}$. Then,*

$$\dim \text{Ran } \Pi_{j,h} = 1.$$

In other words, there is exactly one eigenvalue of $\widehat{\mathcal{L}}_h$ encircled by $\Gamma_{j,h}$.

Proof. We already know from Proposition 5.4 that $\dim \text{Ran } \Pi_{j,h} \geq 1$. As in the proof of Proposition 5.3, we have

$$\|\psi\| \leq C\|\mathfrak{P}\psi\| + C\|(\widehat{\mathcal{L}}_h - \nu_j(h))\psi\| + Ch^{3\delta}\|\psi\|,$$

and

$$\|(\nu_j(h) - \text{Op}_h^w \mu_h^{\text{eff}})\mathfrak{P}\psi\| \leq Ch^{3\delta}\|\psi\| + C\|(\widehat{\mathcal{L}}_h - \nu_j(h))\psi\|.$$

Let us assume that ψ belong to the range of the projection

$$\Pi_{j,h} = \frac{1}{2i\pi} \int_{\Gamma_{j,h}} (\eta - \widehat{\mathcal{L}}_h)^{-1} d\eta.$$

We have

$$\begin{aligned} (\widehat{\mathcal{L}}_h - \nu_j(h))\psi &= \frac{1}{2i\pi} \int_{\Gamma_{j,h}} (\widehat{\mathcal{L}}_h - \nu_j(h))(\eta - \widehat{\mathcal{L}}_h)^{-1} \psi d\eta \\ &= \frac{1}{2i\pi} \int_{\Gamma_{j,h}} (\eta - \nu_j(h))(\eta - \widehat{\mathcal{L}}_h)^{-1} \psi d\eta, \end{aligned}$$

so that

$$\|(\widehat{\mathcal{L}}_h - \nu_j(h))\psi\| \leq h^{\frac{3}{2}-\kappa} h^{\frac{3}{2}-\kappa} \sup_{\eta \in \Gamma_{j,h}} \|(\eta - \widehat{\mathcal{L}}_h)^{-1}\| \|\psi\|.$$

We recall (5.5), and notice that, for all $\eta \in \mathcal{R}_h$, in view of (1.4),

$$\|\mathcal{E}_{h,\pm}^{-1}(\eta)\| \leq C \|(\eta - \mathbf{P}_h^{\text{eff}})^{-1}\| \leq C \text{dist}(\eta, \text{sp}(\mathbf{P}_h^{\text{eff}}))^{-1}.$$

With (4.3), this gives

$$(5.6) \quad \|(\eta - \widehat{\mathcal{L}}_h)^{-1}\| \leq C \text{dist}(\eta, \text{sp}(\mathbf{P}_h^{\text{eff}}))^{-1} \leq Ch^{\kappa-\frac{3}{2}}.$$

Thus, for all $\psi \in \text{Ran } \Pi_{j,h}$,

$$\|(\widehat{\mathcal{L}}_h - \nu_j(h))\psi\| \leq Ch^{\frac{3}{2}-\kappa} \|\psi\|.$$

It follows that

$$(5.7) \quad \|\psi\| \leq C \|\mathfrak{P}\psi\|,$$

and

$$\|(\nu_j(h) - \mathbf{P}_h^{\text{eff}})\mathfrak{P}\psi\| \leq Ch^{\frac{3}{2}-\kappa} \|\psi\| \leq \tilde{C}h^{\frac{3}{2}-\kappa} \|\mathfrak{P}\psi\|.$$

In particular, (5.7) implies that $\dim \mathfrak{P}(\text{Ran } \Pi_{j,h}) = \dim \text{Ran } \Pi_{j,h}$. Then, for all $\varphi \in \mathfrak{P}(\text{Ran } \Pi_{j,h})$,

$$\|(\nu_j(h) - \mathbf{P}_h^{\text{eff}})\varphi\| \leq Ch^{\frac{3}{2}-\kappa} \|\varphi\|.$$

Let us now consider the spectral projection $\Pi_{j,h}^{\text{eff}}$ associated with $\mathbf{P}_h^{\text{eff}}$ and the contour $\Gamma_{j,h}$:

$$\Pi_{j,h}^{\text{eff}} = \frac{1}{2i\pi} \int_{\Gamma_{j,h}} (\eta - \mathbf{P}_h^{\text{eff}})^{-1} d\eta.$$

In fact, since $\Gamma_{j,h}$ encircles only $\nu_j(h)$ as element of the spectrum of $\mathbf{P}_h^{\text{eff}}$ and due to the gap of order h between the eigenvalues of $\mathbf{P}_h^{\text{eff}}$, we have also

$$\Pi_{j,h}^{\text{eff}} = \frac{1}{2i\pi} \int_{\tilde{\Gamma}_{j,h}} (\eta - \mathbf{P}_h^{\text{eff}})^{-1} d\eta,$$

where $\tilde{\Gamma}_{j,h}$ is the circle of center $\nu_j(h)$ and radius $h^{\frac{3}{2}-\tilde{\kappa}}$ where $\tilde{\kappa} > \kappa$. We have

$$\begin{aligned} \Pi_{j,h}^{\text{eff}}\varphi &= \varphi + \frac{1}{2i\pi} \int_{\tilde{\Gamma}_{j,h}} [(\eta - \mathbf{P}_h^{\text{eff}})^{-1} - (\eta - \nu_j(h))^{-1}] \varphi d\eta \\ &= \varphi + \frac{1}{2i\pi} \int_{\tilde{\Gamma}_{j,h}} (\eta - \mathbf{P}_h^{\text{eff}})^{-1} (\eta - \nu_j(h))^{-1} (\mathbf{P}_h^{\text{eff}} - \nu_j(h)) \varphi d\eta. \end{aligned}$$

Since $\tilde{\Gamma}_{j,h} \subset \mathcal{R}_h$, we deduce that

$$\|\Pi_{j,h}^{\text{eff}}\varphi - \varphi\| \leq Ch^{\tilde{\kappa}-\frac{3}{2}} h^{\tilde{\kappa}-\frac{3}{2}} h^{\frac{3}{2}-\kappa} |\tilde{\Gamma}_{j,h}| \|\varphi\| \leq Ch^{\tilde{\kappa}-\kappa} \|\varphi\|.$$

In particular, for all $\varphi \in \mathfrak{P}(\text{Ran } \Pi_{j,h})$,

$$\|\varphi\| \leq C \|\Pi_{j,h}^{\text{eff}}\varphi\|.$$

This implies that

$$1 = \dim \text{Ran } \Pi_{j,h}^{\text{eff}} \geq \dim \text{Ran } \Pi_{j,h},$$

and the conclusion follows. \square

6. REMOVING THE CUTOFF FUNCTION

In the previous section, we proved that the spectrum of $\widehat{\mathcal{L}}_h$ is close to the spectrum of $\mathbf{P}_h^{\text{eff}}$. $\widehat{\mathcal{L}}_h$ was the operator in which we inserted cutoff functions χ_δ (see Definition 3.1). Let us now remove these cutoff functions and prove that the spectrum of the initial operator $\widehat{\mathcal{L}}_h^0$ (defined in (2.1)) is close to the spectrum of $\widehat{\mathcal{L}}_h$ (see Proposition 6.4).

Proposition 6.1. *The families $(\widehat{\mathcal{L}}_h^0 - z)_{z \in D(\mu_0, Ch)}$ and $(\mathcal{L}_h - z)_{z \in D(\mu_0 h, Ch^2)}$ are analytic families of Fredholm operators of index 0. In particular the spectrum of $\widehat{\mathcal{L}}_h^0$ in $D(\mu_0, Ch)$ and of \mathcal{L}_h in $D(\mu_0 h, Ch^2)$ are discrete.*

Proof. By using the unitary equivalence of $h\widehat{\mathcal{L}}_h^0$ and \mathcal{L}_h , we can focus on the family $(\mathcal{L}_h - z)_{z \in D(\mu_0 h, Ch^2)}$. Let $u > 0$ and $v \in \mathbf{R}$ given by Assumption III. The function $F = u(B + \text{Re } V) + v\text{Im } V$ admits a global minimum, and there exists a compact K and a constant $\gamma > \min F$ such that,

$$\forall q \in \mathbf{R}^2 \setminus K, \quad F(q) \geq \gamma.$$

Thus we may consider a smooth cutoff function χ supported near K such that

$$F + \chi \geq \gamma > \min F.$$

Define

$$P = \mathcal{L}_h + h(u - iv)^{-1}\chi.$$

Proving that $P - hz$ is invertible is enough to conclude that $\mathcal{L}_h - hz$ is a Fredholm operator of index 0. Let $w = (u - iv)z$ and $Q = (u - iv)P$, so that

$$\begin{aligned} (u - iv)(P - hz) &= Q - hw \\ &= (u((ih\nabla + \mathbf{A})^2 + h\text{Re}(V - z)) + vh\text{Im}(V - z) + h\chi) \\ &\quad + i(uh\text{Im}(V - z) - v((ih\nabla + \mathbf{A})^2 + h\text{Re}(V - z))) . \end{aligned}$$

Each parenthesis being selfadjoint, we deduce for $\psi \in \text{Dom}(\mathcal{L}_h)$ that

$$\begin{aligned} |\langle (Q - hw)\psi, \psi \rangle| &\geq \text{Re} \langle (Q - hw)\psi, \psi \rangle \\ &\geq \langle (u(ih\nabla + \mathbf{A})^2 + uh\text{Re}(V - z) + vh\text{Im}(V - z) + h\chi) \psi, \psi \rangle . \end{aligned}$$

Using the lower bound $(ih\nabla + \mathbf{A})^2 \geq hB$, we get

$$|\langle (Q - hw)\psi, \psi \rangle| \geq h \langle (u\text{Re}(B + V - z) + v\text{Im}(V - z) + \chi) \psi, \psi \rangle.$$

For $z \in \mathbf{C}$ such that $u\text{Re}(z - \mu_0) + v\text{Im}(z - \mu_0) < Ch$, since μ_0 satisfies

$$\min F = u\text{Re } \mu_0 + v\text{Im } \mu_0$$

we have

$$|\langle (Q - hw)\psi, \psi \rangle| \geq h(\gamma - \min F - Ch)\|\psi\|^2.$$

Hence $Q - hw$ is one-to-one with closed range. We can apply the same arguments for the adjoint of $Q - hw$. We deduce that $Q - w$ is bijective, and so is $P - hz$.

Thus $\mathcal{L}_h - hz$, for z in $\Omega = \{z \in \mathbf{C} : u\text{Re}(z - \mu_0) + v\text{Im}(z - \mu_0) < Ch\}$, is an analytic family of Fredholm operators with index 0. To conclude discreteness of the spectrum it remains to show that Ω intersects the resolvent set of \mathcal{L}_h . To see this, note that $\text{Re}(u - iv)\mathcal{L}_h \geq uh(B + \text{Re } V) + vh\text{Im } V \geq uh\text{Re } \mu_0 + vh\text{Im } \mu_0$, and thus

when $\operatorname{Re} z \rightarrow -\infty$ (in Ω) we must reach the resolvent set. The proposition follows since $D(\mu_0, Ch) \subset \Omega$. \square

Lemma 6.2. *There exists $h_0 > 0$ such that, for all $h \in (0, h_0)$ and all $\lambda \in D(\mu_0, Ch) \cap \operatorname{sp}(\widehat{\mathcal{L}}_h^0)$, we have $\lambda \in \cup_{j=1}^N D_{j,h}$. In particular, for all $j \in \{1, \dots, N\}$, $\Gamma_{j,h} \subset \rho(\widehat{\mathcal{L}}_h^0)$.*

Proof. Assume that it is not true. Then, for some h (as small as desired), we can find an element of the spectrum $\lambda \in D(\mu_0, Ch) \setminus \cup_{j=1}^N D_{j,h}$ and it is a discrete eigenvalue according to Proposition 6.1. Consider an associated normalized eigenfunction

$$\widehat{\mathcal{L}}_h^0 \psi = \lambda \psi.$$

Using the microlocalization Lemma 2.7 on the eigenfunctions of $\widehat{\mathcal{L}}_h^0$, we can add the cutoff functions χ_δ in the symbol to get

$$\widehat{\mathcal{L}}_h \psi = \lambda \psi + \mathcal{O}(h^\infty).$$

But we know from (5.6) that the spectrum of $\widehat{\mathcal{L}}_h$ inside $D(\mu_0, Ch)$ lies in $\cup_{j=1}^N D_{j,h}$ and that the resolvent is controlled by a negative power of h :

$$\|(\widehat{\mathcal{L}}_h - z)^{-1}\| \leq Ch^{-\frac{3}{2} + \kappa},$$

for $z \in \mathcal{R}_h$. This implies that $\psi = 0$, and this contradicts the normalization of ψ . \square

Proposition 6.3. *For each $j \in \{1, \dots, N\}$, the contour $\Gamma_{j,h}$ encircles at most one eigenvalue of $\widehat{\mathcal{L}}_h^0$ (with **geometric** multiplicity).*

Proof. If it is not the case, a contour $\Gamma_{j,h}$ encircles at least two eigenvalues λ and μ associated with normalized orthogonal eigenfunctions φ and ψ , respectively. We have $\hat{\Pi}_{j,h}^0 \varphi = \varphi$ and $\hat{\Pi}_{j,h}^0 \psi = \psi$. Then, the resolvent formula gives that

$$\begin{aligned} \hat{\Pi}_{j,h} - \hat{\Pi}_{j,h}^0 &= \frac{1}{2i\pi} \int_{\Gamma_{j,h}} \left((z - \widehat{\mathcal{L}}_h)^{-1} - (z - \widehat{\mathcal{L}}_h^0)^{-1} \right) dz \\ &= \frac{1}{2i\pi} \int_{\Gamma_{j,h}} (z - \widehat{\mathcal{L}}_h^0)^{-1} (\widehat{\mathcal{L}}_h - \widehat{\mathcal{L}}_h^0) (z - \widehat{\mathcal{L}}_h)^{-1} dz. \end{aligned}$$

By the microlocalization Lemma 2.7 on φ and ψ , we get:

$$\hat{\Pi}_{j,h} \varphi = \hat{\Pi}_{j,h}^0 \varphi + \mathcal{O}(h^\infty) = \varphi + \mathcal{O}(h^\infty), \quad \hat{\Pi}_{j,h} \psi = \psi + \mathcal{O}(h^\infty).$$

This implies that the range of $\hat{\Pi}_{j,h}$ is at least two, and this is a contradiction. \square

In fact, we can even prove that each $\Gamma_{j,h}$ encircles exactly one eigenvalue (with **algebraic** multiplicity).

Proposition 6.4. *For each $j \in \{1, \dots, N\}$, the contour $\Gamma_{j,h}$ encircles exactly one eigenvalue of $\widehat{\mathcal{L}}_h^0$ (with **algebraic** multiplicity).*

Proof. The proof uses the ellipticity at infinity with respect to X_1 . Let us consider a partition of the unity $\chi_{1,h}(X_1) + \chi_{2,h}(X_1) = 1$ with $\operatorname{supp} \chi_{2,h} \subset \{|X_1| \geq h^{-\delta}\}$ and such that the operator $\chi_{1,h}^w(\widehat{\mathcal{L}}_h - \widehat{\mathcal{L}}_h^0)$ is $\mathcal{O}(h^\infty)$ (which is possible by definition of $\widehat{\mathcal{L}}_h$ and $\widehat{\mathcal{L}}_h^0$).

Let $N \in \mathbf{N}$. We have, for all $z \in \Gamma_{j,h}$, and all v ,

$$(6.1) \quad \|\chi_{2,h}^w(\widehat{\mathcal{L}}_h^0 - z)^{-1}v\| \leq Ch^{2\delta}\|v\| + Ch^N\|(\widehat{\mathcal{L}}_h^0 - z)^{-1}v\|.$$

The estimate (6.1) follows by considering the equation

$$(\widehat{\mathcal{L}}_h^0 - z)u = v,$$

and writing for instance that

$$(\widehat{\mathcal{L}}_h^0 - z)\chi_{2,h}^w u = \chi_{2,h}^w v + [\chi_{2,h}^w, \widehat{\mathcal{L}}_h^0]u,$$

so that

$$Ch^{-2\delta}\|\chi_{2,h}^w u\| \leq C\|v\| + Ch^\delta\|\chi_{2,h}^w u\| + Ch^N\|u\|,$$

where the support of $\chi_{2,h}$ is slightly larger than the one of $\chi_{2,h}$. By induction, we get (6.1). At this stage, we still do not control the whole resolvent $(\widehat{\mathcal{L}}_h^0 - z)^{-1}$. By the resolvent formula, and the symbolic calculus, we see that

$$\chi_{1,h}^w \left[(\widehat{\mathcal{L}}_h^0 - z)^{-1} - (\widehat{\mathcal{L}}_h - z)^{-1} \right] = \underbrace{\chi_{1,h}^w (\widehat{\mathcal{L}}_h - z)^{-1} [\widehat{\mathcal{L}}_h - \widehat{\mathcal{L}}_h^0]}_{=\mathcal{O}(h^\infty)} (\widehat{\mathcal{L}}_h^0 - z)^{-1}.$$

Therefore,

$$(6.2) \quad \|\chi_{1,h}^w (\widehat{\mathcal{L}}_h^0 - z)^{-1}v\| \leq \|(\widehat{\mathcal{L}}_h - z)^{-1}v\| + Ch^N\|(\widehat{\mathcal{L}}_h^0 - z)^{-1}v\|.$$

Combining (6.1) and (6.2), we get

$$\|(\widehat{\mathcal{L}}_h^0 - z)^{-1}v\| \leq C\|v\| + \|(\widehat{\mathcal{L}}_h - z)^{-1}v\| \leq \tilde{C} \left(1 + \|(\widehat{\mathcal{L}}_h - z)^{-1}\| \right) \|v\|.$$

In particular, for all $z \in \Gamma_{j,h}$,

$$\|(\widehat{\mathcal{L}}_h^0 - z)^{-1}\| \leq Ch^{-\frac{3}{2}+\kappa}.$$

Coming back to (6.1), we deduce that

$$(6.3) \quad \|\chi_{2,h}^w (\widehat{\mathcal{L}}_h^0 - z)^{-1}\| \leq \tilde{C}h^{2\delta}.$$

Let us now estimate the difference of the spectral projections by using the microlocal partition of the unity:

$$\begin{aligned} \hat{\Pi}_{j,h} - \hat{\Pi}_{j,h}^0 &= \frac{1}{2i\pi} \int_{\Gamma_{j,h}} \chi_{2,h}^w \left((z - \widehat{\mathcal{L}}_h)^{-1} - (z - \widehat{\mathcal{L}}_h^0)^{-1} \right) dz \\ &\quad + \frac{1}{2i\pi} \int_{\Gamma_{j,h}} \chi_{1,h}^w \left((z - \widehat{\mathcal{L}}_h)^{-1} - (z - \widehat{\mathcal{L}}_h^0)^{-1} \right) dz. \end{aligned}$$

We get

$$\|\hat{\Pi}_{j,h} - \hat{\Pi}_{j,h}^0\| \leq C|\Gamma_{j,h}|h^{2\delta} + \frac{1}{2\pi} \left\| \int_{\Gamma_{j,h}} \underbrace{\chi_{1,h}^w (z - \widehat{\mathcal{L}}_h)^{-1} (\widehat{\mathcal{L}}_h^0 - \widehat{\mathcal{L}}_h)}_{=\mathcal{O}(h^\infty)} \underbrace{(z - \widehat{\mathcal{L}}_h^0)^{-1}}_{=\mathcal{O}(h^{-\frac{3}{2}+\kappa})} dz \right\|.$$

Thus, for h small enough, we get

$$\|\hat{\Pi}_{j,h} - \hat{\Pi}_{j,h}^0\| < 1,$$

and these projections have the same rank. In particular, the contour $\Gamma_{j,h}$ encircles as many eigenvalues (with algebraic multiplicity) of $\widehat{\mathcal{L}}_h^0$ as of $\widehat{\mathcal{L}}_h$ (i.e., exactly one). \square

7. ON THE SPECTRUM OF $\mathbf{P}_h^{\text{eff}}$

In this section, we give a description of the spectrum in the disc $D(\mu_0, Ch)$ of $\mathbf{P}_h^{\text{eff}}$ whose symbol is

$$\mu_h^{\text{eff}}(X) = \mathring{p}(X) + h\mathring{p}_1(X), \quad X \in \mathbf{R}^2,$$

where

$$\mathring{p}(X) = \mathring{B}(X) + \mathring{V}(X),$$

and

$$\mathring{p}_1(X) = \langle \mathbf{P}_2(X)u_X, u_X \rangle - \langle \mathbf{P}_1(X)(\mathbf{P}_0(X) - z)^{-1}\Pi^\perp \mathbf{P}_1(X)u_X, u_X \rangle.$$

We work under the assumptions of Theorem 1.5. By using Assumption III, we may assume without loss of generality that $(u, v) = (1, 0)$ and that $\text{Re } \mathring{p}$ has its unique minimum at 0. In particular, we may write

$$(7.1) \quad \mu_h^{\text{eff}}(X) = \mu_0 + h\mathring{p}_1(0) + Q_0(X) + \underbrace{R_3(X) + hR_1(X)}_{=R_h(X)},$$

with $R_1(X) = \mathcal{O}(|X|)$, $R_3(X) = \mathcal{O}(|X|^3)$, and

$$Q_0(X) = \frac{1}{2} \text{Hess } \mathring{p}(X, X).$$

Under our assumptions, $\text{Re } Q_0$ is positive. By translation, we may assume that $\mu_0 = 0$ and $\mathring{p}_1(0) = 0$.

7.1. On the spectrum and resolvent of Q_0^w . The spectrum and the resolvent of Q_0^w are easy to describe in $D(0, Ch)$. We recall below these properties for the convenience of the reader. Some of the considerations below may be found in [6], [10, Chapter 14], [12], [4], or [17].

Proposition 7.1. *There exists $c_0 \in \mathbf{C}^*$ such that for all $h > 0$,*

$$\text{sp}(Q_0^w) = \{(2n-1)c_0h, n \geq 1\}.$$

The spectrum is made of eigenvalues of algebraic multiplicity one. Moreover there exists $D > 0$ such that, for all $h > 0$ and all $z \in D(0, Ch) \setminus \text{sp}(Q_0^w)$,

$$(7.2) \quad \|(z - Q_0^w)^{-1}\| \leq \frac{D}{\text{dist}(z, \text{sp}(Q_0^w))}.$$

Proof. By using the homogeneity of Q_0^w and the rescaling $x = h^{\frac{1}{2}}y$, we may assume that $h = 1$. Then, we write

$$Q_0 = \text{Re } Q_0 + i\text{Im } Q_0.$$

Since $\text{Re } Q_0$ is positive, up to a linear symplectic transformation, we may assume (thanks to the metaplectic representation) that

$$Q_0(X) = c(x^2 + \xi^2) + iQ_1(X),$$

where Q_1 is a real quadratic form and $c > 0$. Up to a Euclidean rotation, we may assume that $Q_1(X) = ax^2 + b\xi^2$ with $(a, b) \in \mathbf{R}^2$. Thus,

$$Q_0(X) = (c + ia)x^2 + (c + ib)\xi^2.$$

After dividing by $c + ib$ and rescaling, we are reduced to

$$Q_0(X) = \xi^2 + e^{i\alpha}x^2, \quad \alpha \in [0, \pi).$$

The complex harmonic oscillator Q_0^w has non-empty resolvent set (since it is a sectorial operator) and compact resolvent² (and thus its spectrum is discrete). Considering the classical Hermite functions $f_n = e^{-\frac{x^2}{2}} P_n$, we see that the functions $g_n(x) = f_n(e^{i\alpha/4}x)$ are eigenfunctions of Q_0^w associated with the eigenvalues $(2n - 1)e^{i\frac{\alpha}{2}}$. Moreover, the closure of $\text{span}(\overline{g_n}, n \in \mathbf{N}^*)$ being $L^2(\mathbf{R})$, we easily see that, if λ is eigenvalue, it must be in the form $\lambda = (2n - 1)e^{i\frac{\alpha}{2}}$. This gives the announced description of the spectrum.

We can also check that the eigenvalues are geometrically simple by using Wronskian considerations. In fact, we can see that they are algebraically simple by using the analytic perturbation theory of Kato (see [14, Chapter VII, § 2]) with respect to $\alpha \in [0, \pi)$. Indeed, the family $-\partial_x^2 + e^{i\alpha}x^2$ is analytic of type (A) (its domain is $B^2(\mathbf{R})$ and thus it does not depend on $\alpha \in [0, \pi)$) and the eigenvalues (which are explicit!) continuously move with respect to α . The rank of the associated Riesz projection is then constant, equal to 1 (the rank when $\alpha = 0$).

Let us now discuss the resolvent estimate. Since each eigenvalue is simple and isolated, the resolvent has a simple pole there, which gives the required estimate (7.2) in a small neighborhood of the eigenvalue. Since the disc of radius C contains only a finite number of eigenvalues, the result follows. \square

Let us now explain why the spectrum of Q_0^w gives an approximation of the spectrum of $\mathbf{P}_h^{\text{eff}}$ in $D(0, Ch)$.

7.2. Locating the spectrum of $\mathbf{P}_h^{\text{eff}}$. The spectrum of $\mathbf{P}_h^{\text{eff}}$ in $D(0, Ch)$ is close to the one of Q_0^w . Let us consider $\lambda \in \text{sp}(\mathbf{P}_h^{\text{eff}}) \cap D(0, Ch)$ and ψ be a corresponding eigenfunction. We have

$$\mathbf{P}_h^{\text{eff}} \psi = \lambda \psi.$$

Similarly to Lemma 2.7, one can check that ψ is microlocalized near $X = 0$ at a scale h^δ with $\delta \in (0, \frac{1}{2})$. We infer that

$$\|(Q_0^w - \lambda)\psi\| \leq \tilde{C}h^{3\delta}\|\psi\|.$$

With (7.2), we get

$$\text{dist}(\lambda, \text{sp}(Q_0^w)) \leq \tilde{C}h^{3\delta}.$$

Therefore, when h is small enough, the spectrum of $\mathbf{P}_h^{\text{eff}}$ is close to the one of Q_0^w at a distance bounded by $h^{\frac{3}{2}-\kappa}$ for all $\kappa > 0$.

7.3. Comparison of the spectral projections and resolvent bound. Let us explain why there is exactly one simple eigenvalue of $\mathbf{P}_h^{\text{eff}}$ in each disc $D(\mu, h^{\frac{3}{2}-\kappa})$ with $\mu \in D(0, Ch) \cap \text{sp}(Q_0^w)$.

It is enough to prove that the Riesz projections associated with $\mathbf{P}_h^{\text{eff}}$ and Q_0^w have the same rank. More precisely, we let

$$\Pi_h^{\text{eff}} = -\frac{1}{2i\pi} \int_{\mathcal{C}_h} (\mathbf{P}_h^{\text{eff}} - z)^{-1} dz, \quad \Pi_h^0 = -\frac{1}{2i\pi} \int_{\mathcal{C}_h} (Q_0^w - z)^{-1} dz,$$

²These elementary properties follow from the inequality

$$\text{Re} \left(e^{-i\frac{\alpha}{2}} \langle Q_0^w \psi, \psi \rangle \right) \geq \cos \left(\frac{\alpha}{2} \right) (\|\psi'\|^2 + \|x\psi\|^2) \geq \cos \left(\frac{\alpha}{2} \right) \|\psi\|^2.$$

where \mathcal{C}_h is the circle of center μ and radius $h^{\frac{3}{2}-\kappa}$. We can estimate the norm $\|\Pi_h^{\text{eff}} - \Pi_h^0\|$ by using the same method as in Proposition 6.4.

Let $\chi_1, \chi_2 \in \mathcal{C}^\infty(\mathbf{R}^2)$ be cutoff functions such that $\chi_1 + \chi_2 = 1$ and χ_2 supported in $\{|X| \leq h^\delta\}$. Then, we use the resolvent formula to get

$$\chi_2^w(\Pi_h^{\text{eff}} - \Pi_h^0) = \frac{1}{2i\pi} \int_{\mathcal{C}_h} \underbrace{\chi_2^w(Q_0^w - z)^{-1} R_h^w}_{\mathcal{O}(h^{-\frac{3}{2}+\kappa}(h^{3\delta} + h^{1+\delta}))} (\mathbf{P}_h^{\text{eff}} - z)^{-1} dz,$$

where R_h is defined in (7.1). By choosing δ close enough to $\frac{1}{2}$, we get, for some $\alpha > 0$,

$$\|\chi_2^w(\Pi_h^{\text{eff}} - \Pi_h^0)\| \leq Ch^\alpha \int_{\mathcal{C}_h} \|(\mathbf{P}_h^{\text{eff}} - z)^{-1}\| dz.$$

Since χ_1 is supported in $\{|X| \geq ch^\delta\}$, we have, for all $z \in D(0, Ch)$,

$$(7.3) \quad \|\chi_1^w(\mathbf{P}_h^{\text{eff}} - z)^{-1}\| \leq Ch^{-2\delta}, \quad \|\chi_1^w(Q_0^w - z)^{-1}\| \leq Ch^{-2\delta}.$$

We deduce that

$$\|\chi_1^w(\Pi_h^{\text{eff}} - \Pi_h^0)\| \leq Ch^{\frac{3}{2}-\kappa} h^{-2\delta}.$$

Summing up the χ_2 and the χ_1 parts, we deduce that

$$(7.4) \quad \|\Pi_h^{\text{eff}} - \Pi_h^0\| \leq Ch^{\frac{3}{2}-\kappa-2\delta} + Ch^\alpha \int_{\mathcal{C}_h} \|(\mathbf{P}_h^{\text{eff}} - z)^{-1}\| dz.$$

We must estimate the resolvent appearing in the right-hand-side.

For all $z \in D(0, Ch)$ such that $\text{dist}(z, \text{sp}(Q_0^w)) \geq h^{\frac{3}{2}-\kappa}$,

$$\|\chi_2^w((\mathbf{P}_h^{\text{eff}} - z)^{-1} - (Q_0^w - z)^{-1})\| \leq Ch^{-\frac{3}{2}+\kappa}(h^{3\delta} + h^{1+\delta}) \|(\mathbf{P}_h^{\text{eff}} - z)^{-1}\|.$$

Thus, with (7.3),

$$\|(\mathbf{P}_h^{\text{eff}} - z)^{-1}\| \leq \|(Q_0^w - z)^{-1}\| + Ch^{-2\delta} + C \underbrace{h^{-\frac{3}{2}+\kappa}(h^{3\delta} + h^{1+\delta})}_{=o(1)} \|(\mathbf{P}_h^{\text{eff}} - z)^{-1}\|,$$

which yields

$$\|(\mathbf{P}_h^{\text{eff}} - z)^{-1}\| \leq C\|(Q_0^w - z)^{-1}\| + Ch^{-2\delta}.$$

With (7.2), we deduce that, for all $z \in D(0, Ch)$ such that $\text{dist}(z, \text{sp}(Q_0^w)) \geq h^{\frac{3}{2}-\kappa}$,

$$(7.5) \quad \begin{aligned} \|(\mathbf{P}_h^{\text{eff}} - z)^{-1}\| &\leq \frac{C}{\text{dist}(z, \text{sp}(Q_0^w))} + Ch^{-2\delta} = \frac{C + Ch^{-2\delta} \text{dist}(z, \text{sp}(Q_0^w))}{\text{dist}(z, \text{sp}(Q_0^w))} \\ &\leq \frac{\tilde{C}}{\text{dist}(z, \text{sp}(Q_0^w))} \leq \tilde{C} h^{-\frac{3}{2}+\kappa}. \end{aligned}$$

With (7.4), this provides us with

$$\|\Pi_h^{\text{eff}} - \Pi_h^0\| \leq Ch^{\frac{3}{2}-\kappa-2\delta} + 2\pi C \tilde{C} h^\alpha = o(1).$$

Therefore, for h small enough, $\|\Pi_h^{\text{eff}} - \Pi_h^0\| < 1$, and the spectral projections have the same rank (that is rank one). We deduce that there is exactly one eigenvalue of $\mathbf{P}_h^{\text{eff}}$ at a distance of $h^{\frac{3}{2}-\kappa}$ near the spectrum of Q_0^w in the disc $D(0, Ch)$. With (7.5), this implies that, for all $z \in D(0, Ch)$ such that $\text{dist}(z, \text{sp}(Q_0^w)) \geq h^{\frac{3}{2}-\kappa}$,

$$\|(\mathbf{P}_h^{\text{eff}} - z)^{-1}\| \leq \frac{C}{\text{dist}(z, \text{sp}(\mathbf{P}_h^{\text{eff}}))}.$$

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