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Spacetime Trefftz-DG formulation for Wave Propagation using Tent-Pitching Meshes

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- Aim of our work ?
 - ↪ to develop a spacetime method with the Tent-Pitcher algorithm, for wave problems.
- Why ?
 - ↪ gain precision,
 - ↪ reduce computational time.
- What kind of problems ?
 - ↪ acoustic,
 - ↪ elastodynamic,
 - ↪ elasto-acoustic wave propagation systems.
- What kind of methods ?
 - ↪ Trefftz-Discontinuous Galerkin methods with Tent-Pitching.
- What kind of basis ?
 - ↪ polynomial basis functions,
 - ↪ Green's functions.

Acoustic Wave equation

First order acoustic wave equation :

$$\left\{ \begin{array}{ll} \frac{1}{c^2 \rho} \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{v} = f & \text{in } \Omega, \\ \rho \frac{\partial \mathbf{v}}{\partial t} + \nabla p = 0 & \text{in } \Omega, \\ \mathbf{v}(\cdot, 0) = \mathbf{v}_0, p(\cdot, 0) = p_0 & \text{in } \mathcal{D}, \\ \mathbf{v} \cdot \mathbf{n}_{\mathbf{x}} = 0 & \text{in } \partial \mathcal{D} \times \mathcal{I}. \end{array} \right. \quad \begin{array}{l} \Omega = \mathcal{D} \times \mathcal{I} \\ \mathcal{I} = [0, T] \end{array}$$

with

$p = p(\mathbf{x}, t)$, pressure,

$\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$, velocity,

$f = f(\mathbf{x}, t)$, source term,

$c = c(\mathbf{x})$, acoustic wave
propagation speed,

$\rho = \rho(\mathbf{x})$, density.

- Basis functions are local solutions of the initial PDEs
- We work on the skeleton of the mesh only
- Requires low number of degrees of freedom,
- Characteristics of the solution are included in the discrete solution
- Basis functions are discontinuous → Discontinuous Galerkin discretization
- Flexibility in choice of basis functions (plane waves, Green's function...)
- Implicit scheme

Variational formulation

\mathcal{T}_h : non-overlapping space-time mesh composed of Lipschitz smooth elements $K \in \Omega$

$$V^h(\mathcal{T}_h) = \{\phi \in L^2(Q), \phi|_K \in H^1(K)\}$$

Trefftz space

$$\mathbf{T}(\mathcal{T}_h) = \{(\mathbf{w}, q) \in V^h(\mathcal{T}_h)^d \times V^h(\mathcal{T}_h) \mid \rho \frac{\partial \mathbf{w}}{\partial t} + \nabla q = 0 \text{ and } \frac{1}{c^2 \rho} \frac{\partial q}{\partial t} + \operatorname{div} \mathbf{w} = 0\}$$

Discrete Trefftz space :

$$\mathbf{T}^p(\mathcal{T}_h) = \{(\mathbf{w}, q) \in \mathbf{T}(\mathcal{T}_h), (\mathbf{w}, q)|_K \in \mathbb{P}^p(K)^d \times \mathbb{P}^p(K)\}$$

On \mathcal{F}_h^Q , the internal element faces, we have :

$$\begin{aligned}\{\{p\}\} &= \frac{1}{2}(p^- + p^+), \\ \llbracket p \rrbracket_t &= p^- \mathbf{n}_x^- + p^+ \mathbf{n}_x^+, \\ \llbracket p \rrbracket_x &= p^- n_t^- + p^+ n_t^+, \\ \{\{\mathbf{v}\}\} &= \frac{1}{2}(\mathbf{v}^- + \mathbf{v}^+), \\ \llbracket \mathbf{v} \rrbracket_t &= \mathbf{v}^- \cdot n_t^- + \mathbf{v}^+ \cdot n_t^+, \\ \llbracket \mathbf{v} \rrbracket_x &= \mathbf{v}^- \cdot \mathbf{n}_x^- + \mathbf{v}^+ \cdot \mathbf{n}_x^+.\end{aligned}$$

\mathcal{F}_h^D : boundary element faces $\partial D \times I$

\mathcal{F}_h^0 : initial time element faces $D \times \{0\}$

\mathcal{F}_h^T : final time element faces $D \times \{T\}$

Variational formulation

Seek $(\mathbf{v}_h, p_h) \in \mathbf{T}^p(\mathcal{T}_h)$ such that, $\forall(\mathbf{w}, q) \in \mathbf{T}^p(\mathcal{T}_h)$, it holds true :

$$\begin{aligned} & \int_{\mathcal{F}_h^Q} \left[\frac{1}{c^2 \rho} \{ \{ p_h \} \} [q]_t + \rho \{ \{ \mathbf{v}_h \} \} \cdot [\mathbf{w}]_t + \{ \{ p_h \} \} [\mathbf{w}]_x + \rho \{ \{ \mathbf{v}_h \} \} \cdot [q]_x \right] ds \\ & + \int_{\mathcal{F}_h^Q} \left[\alpha_1 [\mathbf{v}_h]_x [\mathbf{w}]_x + \beta_1 [p_h]_x [q]_x + \alpha_2 \rho [\mathbf{v}_h]_t \cdot [\mathbf{w}]_t + \frac{\beta_2}{c^2 \rho} [p_h]_t [q]_t \right] ds \\ & + \int_{\mathcal{F}_h^T} \left[\frac{1}{c^2 \rho} p_h q + \rho \mathbf{v}_h \cdot \mathbf{w} \right] ds + \int_{\mathcal{F}_h^D} \left[p_h \mathbf{w} \cdot \mathbf{n}_x + \alpha_1 (\mathbf{v}_h \cdot \mathbf{n}_x) (\mathbf{w} \cdot \mathbf{n}_x) \right] ds \\ & - \left(\frac{1}{2} - \beta_2 \right) \int_{\mathcal{F}_h^0} \frac{1}{c^2 \rho} p_h q ds - \left(\frac{1}{2} - \alpha_2 \right) \int_{\mathcal{F}_h^0} \rho \mathbf{v}_h \cdot \mathbf{w} ds = \\ & + \left(\frac{1}{2} + \beta_2 \right) \int_{\mathcal{F}_h^0} \frac{1}{c^2 \rho} p_0 q ds + \left(\frac{1}{2} + \alpha_2 \right) \int_{\mathcal{F}_h^0} \rho \mathbf{v}_0 \cdot \mathbf{w} ds \end{aligned}$$

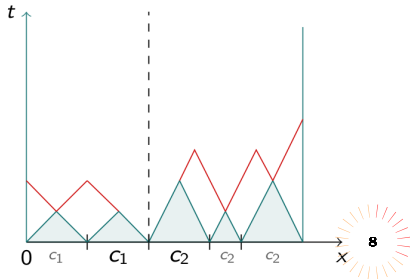
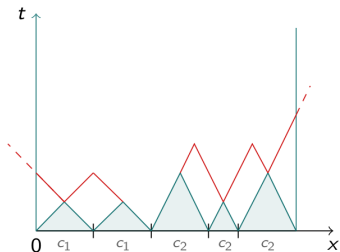
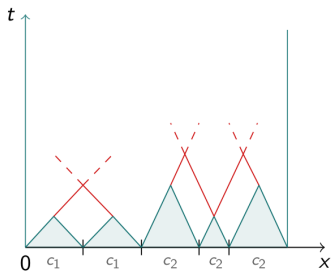
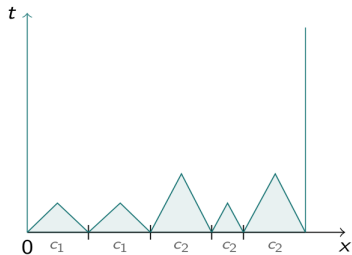
How to reduce computational costs ?

- solve element-by-element if mesh respects constraints

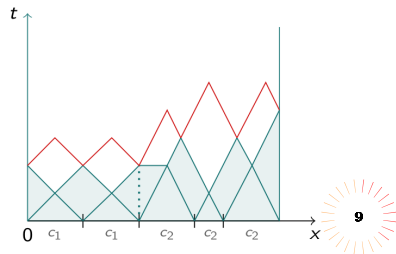
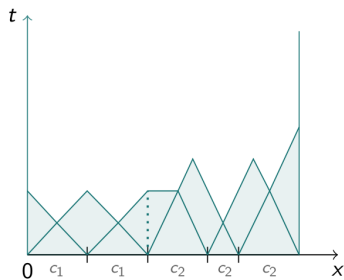
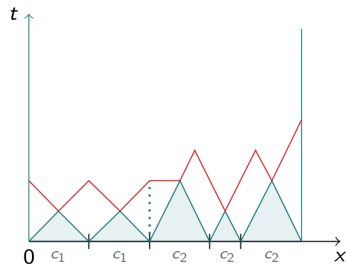
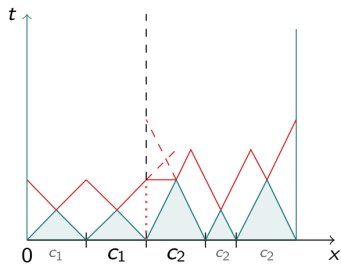
↳ Tent-Pitcher algorithm

- introduced for hyperbolic problems in spacetime domain,
- consists in building a causal mesh, which respects the wave propagation speed,
- conducive to parallel computing.

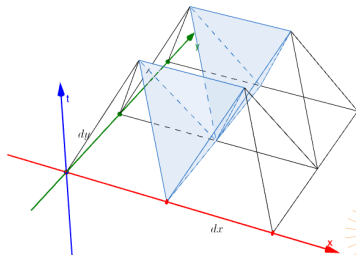
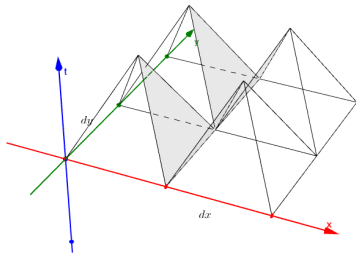
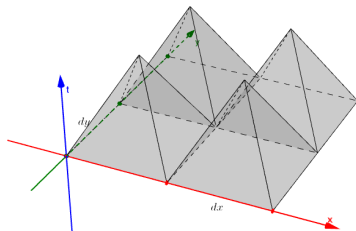
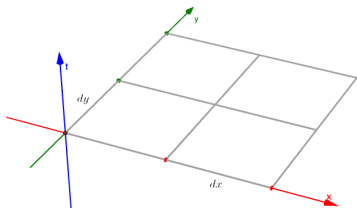
Tent-pitching process 1d+t



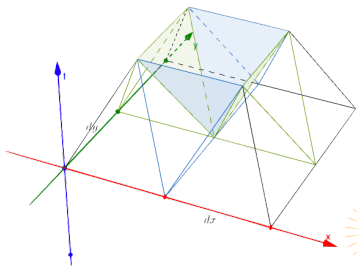
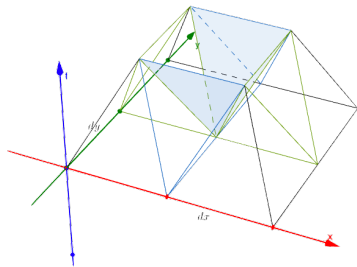
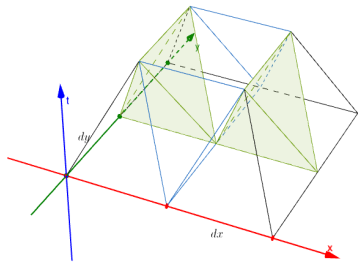
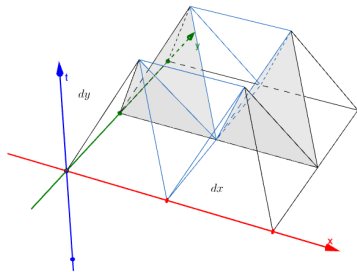
Tent-pitching process 1d+t



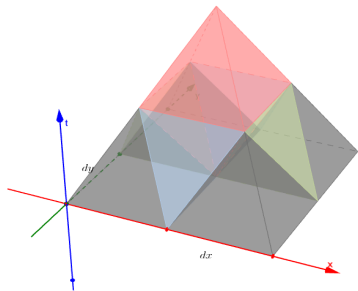
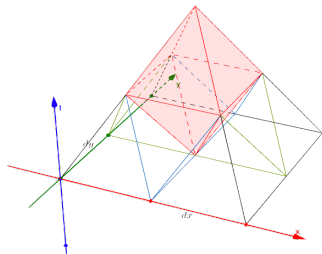
Tent-pitching process $2d+t$



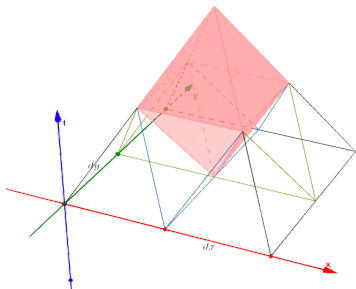
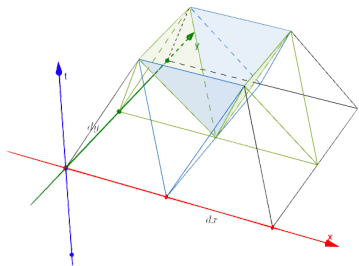
Tent-pitching process 2d+t



Tent-pitching process $2d+t$



Tent-pitching process



$$\begin{aligned}
 & -\left(\frac{1}{2} + \beta_2\right) \int_{\mathcal{F}_h^{in}} \frac{1}{c^2 \rho} p_h^{in} q n_t \\
 & -\left(\frac{1}{2} + \alpha_2\right) \int_{\mathcal{F}_h^{in}} \rho \mathbf{v}_h^{in} \cdot \mathbf{w} n_t \\
 & -\frac{1}{2} \int_{\mathcal{F}_h^{in}} p_h^{in} \mathbf{w} \cdot \mathbf{n}_x + q \mathbf{v}_h^{in} \cdot \mathbf{n}_x \\
 & = \int_{\mathcal{F}_h^{out}} \frac{1}{c^2 \rho} p_h q n_t + \rho \mathbf{v}_h \cdot \mathbf{w} n_t + p_h \mathbf{w} \cdot \mathbf{n}_x + q \mathbf{v}_h \cdot \mathbf{n}_x \\
 & + \left(\frac{1}{2} - \beta_2\right) \int_{\mathcal{F}_h^{in}} \frac{1}{c^2 \rho} p_h q n_t + \left(\frac{1}{2} - \alpha_2\right) \int_{\mathcal{F}_h^{in}} \rho \mathbf{v}_h \cdot \mathbf{w} n_t \\
 & + \frac{1}{2} \int_{\mathcal{F}_h^{in}} p_h \mathbf{w} \cdot \mathbf{n}_x + q \mathbf{v}_h \cdot \mathbf{n}_x
 \end{aligned}$$

- Neumann boundary condition

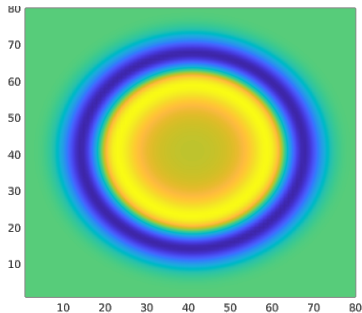


Figure: Pressure at time $t=0.308$ with Trefftz-DG+Tent-Pitching

$$CPU_{exact} = 20 \times CPU_{approx}.$$

- polynomial basis functions

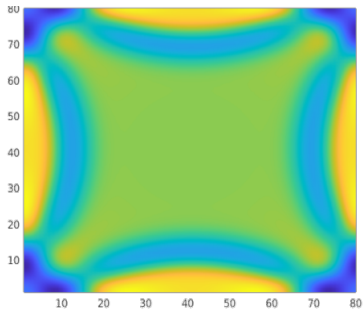


Figure: Pressure at time $t=0.616$ with Trefftz-DG+Tent-Pitching

$$CPU_{approx} = 10 \times CPU_{tent-pitch}.$$

- Boundary Elements Method (BEM) consists in finding the solution of a PDE, thanks to the solution on the boundary of the domain :
 - ✓ only the boundary has to be discretized,
 - ✗ need to determine fundamental solutions beforehand,
 - ✗ have to compute complex singular integrals,
 - ✗ meshes are hard to build.

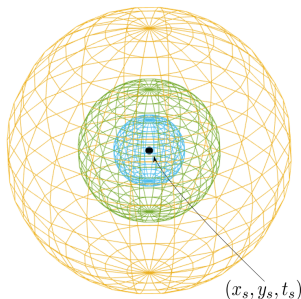
- Method of Fundamental Solutions (MFS) introduced as alternative to BEM :
 - ✓ no need to compute singular integrals,
 - ✓ meshfree method,
 - ✗ but we also need fundamental solutions, since the solution is approximated by a linear combination of these

Green's function as basis

$$G_p(x, y, t) = \frac{1}{\sqrt{(t - t_s)^2 - (x - x_s)^2 - (y - y_s)^2}},$$

$$G_{v_x}(x, y, t) = \frac{(t - t_s)(x - x_s)}{\sqrt{(t - t_s)^2 - (x - x_s)^2 - (y - y_s)^2}},$$

$$G_{v_y}(x, y, t) = \frac{(t - t_s)(y - y_s)}{\sqrt{(t - t_s)^2 - (x - x_s)^2 - (y - y_s)^2}}.$$



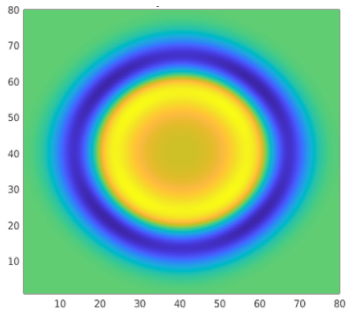


Figure: Pressure at time $t=0.308$
with Green's functions

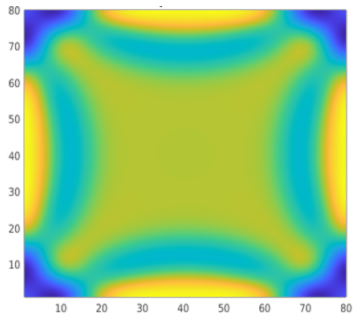
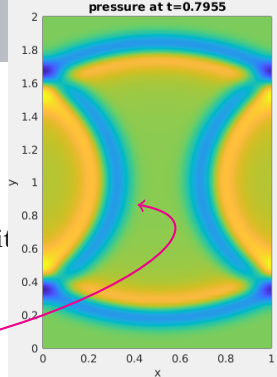


Figure: Pressure at time $t=0.616$
with Green's functions

Boundary Conditions

- In numerical simulations, cannot work in infinity
 - ↳ we truncate the domain
 - ↳ it creates reflections !



- Solution : introduce appropriate boundary conditions
 - ↳ Absorbing Boundary Conditions
 - ↳ Perfectly Matched Layers

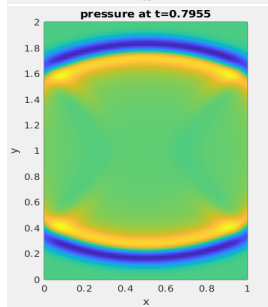
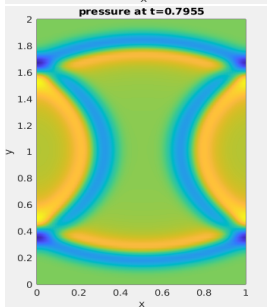
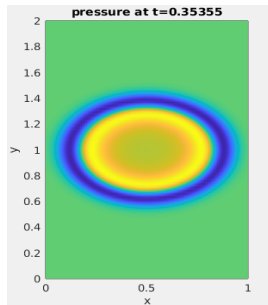
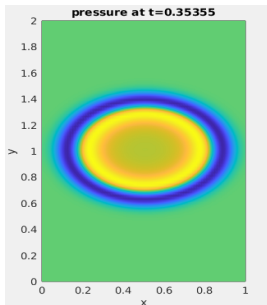
1st order Absorbing Boundary Conditions

- First order ABC for acoustic wave equation on absorbing boundary :

$$\frac{\partial p}{\partial n} = \frac{1}{c} \frac{\partial p}{\partial t} \text{ on } \Gamma .$$

- Here, use of Green's functions as ABC.

Results



Perfectly Matched Layers

$$\begin{cases} \rho \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} - \frac{\sigma}{i\omega + \sigma} \frac{\partial p}{\partial x} = 0, \\ \rho \frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} = 0, \\ \frac{1}{c^2 \rho} \frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \frac{\sigma}{i\omega + \sigma} \frac{\partial u}{\partial x} = 0. \end{cases}$$

Modified Green's functions :

$$G_p^{pml}(\mathbf{x}, t) = \exp(-A(\mathbf{x}, t)) \cos(B(\mathbf{x}, t)) G(\mathbf{x}, t)$$

$$A(\mathbf{x}, t) = \cos(\theta(\mathbf{x})) I(\mathbf{x}) \frac{t}{r(\mathbf{x})}$$

$$B(\mathbf{x}, t) = \sin(\theta(\mathbf{x})) I(\mathbf{x}) \sqrt{\frac{t^2}{r(\mathbf{x})^2} - \frac{1}{c^2}}$$

$$I(\mathbf{x}) = \int_0^x \sigma(\mathbf{x}) dx$$

Prospects

- Parallelization
- Extend to unstructured meshes
- Extend 2D to 3D
- Consider adaptive meshes & local-time stepping

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Thank you for your attention !
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