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MATHIAS DAYS 2021 - OCTOBER 3-7

# PML applied to spacetime Trefftz-DG numerical formulation for the elastic wave equation

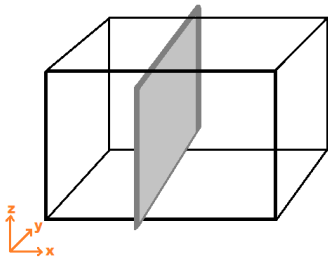
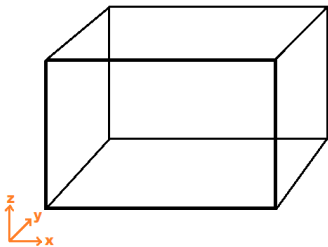
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2. TotalEnergies, France, `firstname.lastname@totalenergies.com`

# Introduction

- Aim of our work ?
  - ↪ investigate new solution methodologies which combine accuracy with fast resolution for seismic wave problems
- Our candidate :
  - ↪ Trefftz method in time domain with the Tent-Pitcher algorithm

# Introduction



# Introduction

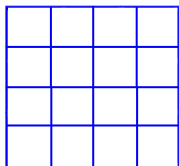


Figure: Quadrilateral mesh

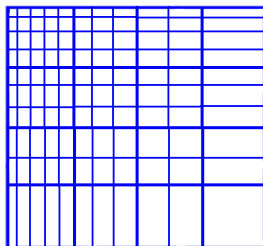


Figure: Adaptive refinement

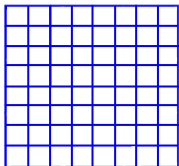


Figure: Refined quadrilateral mesh

# Introduction

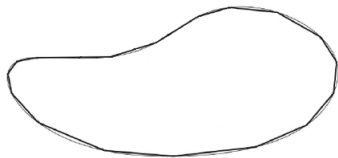


Figure: Different topology

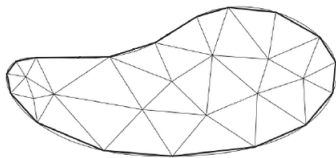


Figure: Triangular meshing

# Introduction

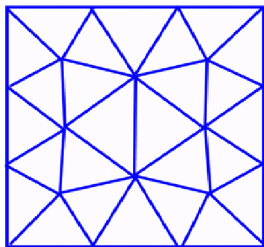


Figure: Triangular mesh

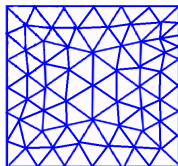


Figure: Refined mesh

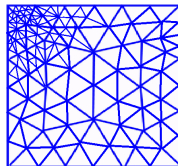


Figure: Adaptive refinement

# Introduction

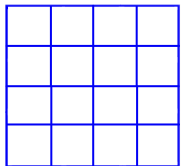


Figure: Grid for FD

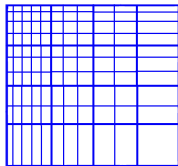


Figure: Mesh for SEM

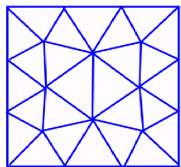


Figure: Mesh for FE

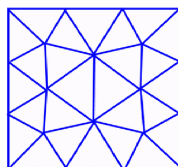


Figure: Mesh for DG



# Introduction

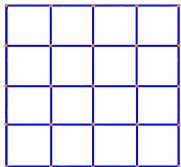


Figure: Degrees of Freedom for FD

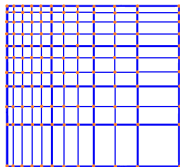


Figure: Degrees of Freedom for SEM

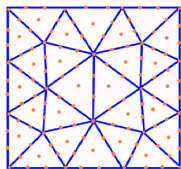


Figure: Degrees of Freedom for FE

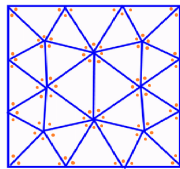


Figure: Degrees of Freedom for DG **7**



# Trefftz approach

- Basis functions are local solutions of the initial PDEs
- We work on the skeleton of the mesh only
- Requires lower number of degrees of freedom
- Physical characteristics of the solution are included in the discrete solution
  
- Basis functions are discontinuous  $\rightarrow$  Discontinuous Galerkin discretization
- Flexibility in choice of basis functions (plane waves, Green's function...)
  
- Implicit scheme

# Tent Pitcher algorithm

How to reduce computational costs ?

- computations on time slabs
- approximate inverse matrix
- solve element-by-element if mesh respects constraints
  - ↪ Tent-Pitcher algorithm
    - introduced for hyperbolic problems in spacetime domain,
    - consists in building a causal mesh, which respects the wave propagation speed,
    - no approximation of the inverse matrix,
    - conducive to parallel computing.

# Tent-pitching 2d+t

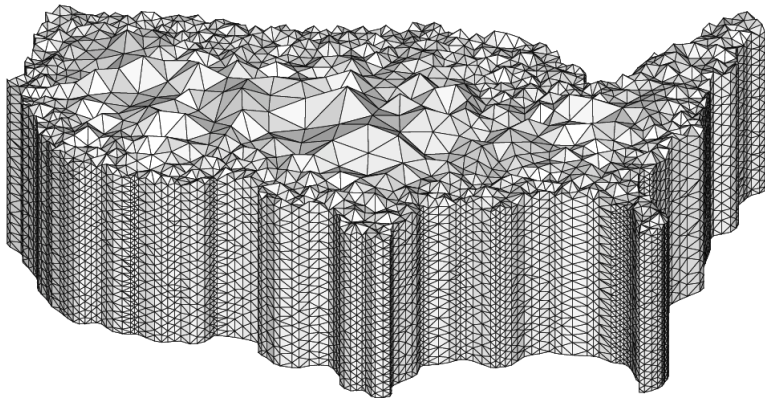


Figure: Spacetime mesh

Erickson, Guoy, Sullivan and Üngör [2005]

# Acoustic Wave equation

First order acoustic wave equation :

$$\left\{ \begin{array}{ll} \frac{1}{c^2 \rho} \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{v} = f & \text{in } \Omega, \\ \rho \frac{\partial \mathbf{v}}{\partial t} + \nabla p = 0 & \text{in } \Omega, \\ \mathbf{v}(\cdot, 0) = \mathbf{v}_0, p(\cdot, 0) = p_0 & \text{in } \mathcal{D}, \\ \mathbf{v} \cdot \mathbf{n}_{\mathbf{x}} = 0 & \text{in } \partial \mathcal{D} \times \mathcal{I}. \end{array} \right. \quad \begin{array}{l} \Omega = \mathcal{D} \times \mathcal{I} \\ \mathcal{I} = [0, T] \end{array}$$

with

$$\mathbf{x} = (x, y),$$

$$p = p(\mathbf{x}, t), \text{ pressure,}$$

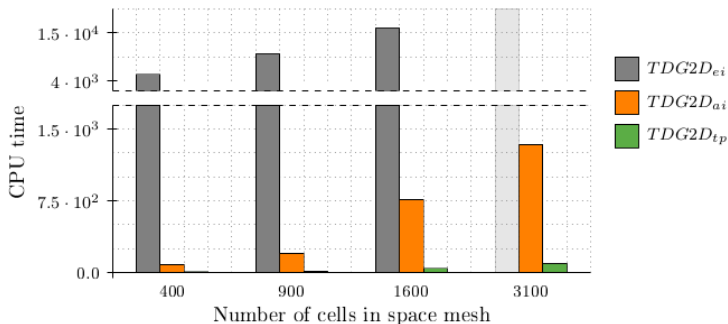
$$\mathbf{v} = \mathbf{v}(\mathbf{x}, t) = (u, v), \text{ velocity,}$$

$$f = f(\mathbf{x}, t), \text{ source term,}$$

$$c = c(\mathbf{x}), \text{ wave speed,}$$

$$\rho = \rho(\mathbf{x}), \text{ density.}$$

# Results



$$CPU_{exact} = 20 \times CPU_{approx}. \quad CPU_{approx} = 10 \times CPU_{tent-pitch}.$$

Thesis of E. Shishenina [2018]

# Absorbing boundary conditions

State of the art :

- not much work on absorbing boundary conditions with polynomials on spacetime Trefftz,
- no Perfectly Matched Layers for spacetime Trefftz.

Green's functions :

- are exact local solutions,
- have interesting behaviour at infinity,
- can be computed analytically inside the PML.

# Perfectly Matched Layers

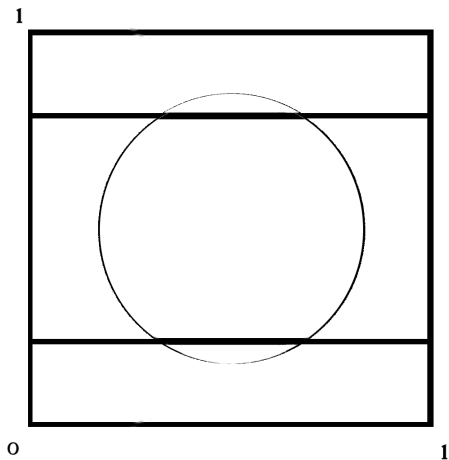


Figure: Domain with PML

J.-P. Bérenger [1994]



# Green's functions

Notations:  $\mathcal{X} = \mathbf{x} - \mathbf{x}_s$ ,  $r = \sqrt{\mathcal{X} \cdot \mathcal{X}}$   
 $\mathcal{X}_1 = x - x_s$ ,  $\mathcal{X}_2 = y - y_s$ ,  $\mathcal{T} = t - t_s$

Cagniard-De Hoop method :

$$G_p(\mathbf{x}, t, \mathbf{x}_s, t_s) = \frac{1}{2\pi \sqrt{\mathcal{T}^2 - \frac{r^2}{c^2}}},$$

$$G_u(\mathbf{x}, t, \mathbf{x}_s, t_s) = \frac{\mathcal{T} \mathcal{X}_1}{\rho r^2} G_p(\mathbf{x}, t, \mathbf{x}_s, t_s),$$

$$G_v(\mathbf{x}, t, \mathbf{x}_s, t_s) = \frac{\mathcal{T} \mathcal{X}_2}{\rho r^2} G_p(\mathbf{x}, t, \mathbf{x}_s, t_s).$$

## Green's functions inside of the PML

- $G_p^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, \sigma_x) = \exp^{-A} \cos(B) G_p(\mathbf{x}, t, \mathbf{x}_s, t_s)$
- $G_u^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, \sigma_x) = \frac{\exp^{-A} G_p(\mathbf{x}, t, \mathbf{x}_s, t_s)}{\rho r^2} \left( \mathcal{T} \mathcal{X}_1 \cos(B) + h \mathcal{X}_2 \sin(B) \right)$
- $G_v^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, \sigma_x) = \frac{\exp^{-A} G_p(\mathbf{x}, t, \mathbf{x}_s, t_s)}{\rho r^2} \left( \mathcal{T} \mathcal{X}_2 \cos(B) + h \mathcal{X}_1 \sin(B) \right)$

with

$$A = A(\mathbf{x}, t, \mathbf{x}_s, t_s) = \frac{\mathcal{X}_1}{r^2} \Sigma(x, x_s) \mathcal{T},$$

$$B = B(\mathbf{x}, t, \mathbf{x}_s, t_s) = \frac{\mathcal{X}_2}{r^2} \Sigma(x, x_s) h,$$

$$\Sigma(x, x_s) = \int_0^{\mathcal{X}_1} \sigma(\mathbf{x}, \mathbf{x}_s) d\mathcal{X},$$

$$h = h(\mathbf{x}, t, \mathbf{x}_s, t_s) = \sqrt{\mathcal{T}^2 - \frac{r^2}{c^2}}.$$

# PML - Results

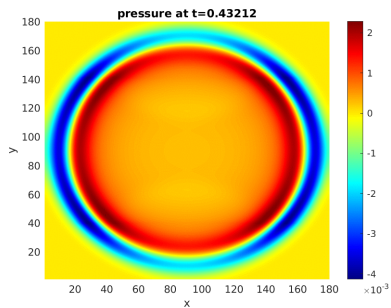


Figure: Pressure for  $\sigma = 5$

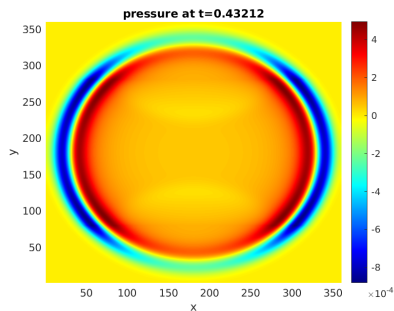


Figure: Pressure for  $\sigma = 10$

## Elastic Wave equation

$$\left\{ \begin{array}{ll} \frac{\partial \underline{\underline{\sigma}}}{\partial t} - \underline{\underline{C}} \underline{\underline{\epsilon}}(\mathbf{v}) = 0 & \text{in } \Omega, \\ \rho \frac{\partial \mathbf{v}}{\partial t} - \nabla \cdot \underline{\underline{\sigma}} = 0 & \text{in } \Omega, \\ \mathbf{v}(\cdot, 0) = \mathbf{v}_0, \underline{\underline{\sigma}}(\cdot, 0) = \underline{\underline{\sigma}}_0 & \text{in } \mathcal{D}, \\ \underline{\underline{\sigma}} \mathbf{n}_{\mathbf{x}} = g & \text{in } \partial \mathcal{D} \times \mathcal{I}. \end{array} \right. \quad (1)$$

with

$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}(\mathbf{x}, t)$ , stress tensor,

$\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ , velocity,

$\underline{\underline{C}}$ , elastic tensor,

$\underline{\underline{\epsilon}}$ , strain tensor,

$\rho$ , density,

$g$ , boundary value.

## Green's functions inside of the PML

$$\begin{cases} u^\Phi = \partial_x G_p^{pml} \\ v^\Phi = \partial_y G_p^{pml} \end{cases} \quad (2)$$

$$\begin{cases} u^\Psi = -\partial_y G_p^{pml} \\ v^\Psi = \partial_x G_p^{pml} \end{cases} \quad (3)$$

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix}^\Phi = - \begin{pmatrix} 2c_{13} & c_{12} & c_{11} \\ 2c_{23} & c_{22} & c_{21} \\ 2c_{33} & c_{32} & c_{31} \end{pmatrix} \begin{pmatrix} I \\ II \\ III \end{pmatrix}$$

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix}^\Psi = \begin{pmatrix} c_{11} - c_{12} & c_{13} & -c_{13} \\ c_{21} - c_{22} & c_{23} & -c_{23} \\ c_{31} - c_{32} & c_{33} & -c_{33} \end{pmatrix} \begin{pmatrix} I \\ II \\ III \end{pmatrix}$$

$$\begin{cases} I = \rho \partial_y G_u^{pml} \\ II = -\rho \partial_y G_v^{pml} \\ III = \left[ \left( \frac{t^2(y^2 - x^2)}{r^4} - \frac{y^2}{r^2 c^2} \right) \cos(B) - \frac{txy}{r^4} \sqrt{t^2 - \frac{r^2}{c^2}} \sin(B) \right] G_p^{pml} \end{cases}$$

# Prospects

- Implement PML for elastic case
- Parallelization
- Extend to unstructured meshes
- Extend 2D to 3D

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Thank you for your attention !