



HAL
open science

PML applied to spacetime Trefftz-DG numerical formulation for the elastic wave equation

Vinduja Vasanthan, H el ene Barucq, Henri Calandra, Julien Diaz

► To cite this version:

Vinduja Vasanthan, H el ene Barucq, Henri Calandra, Julien Diaz. PML applied to spacetime Trefftz-DG numerical formulation for the elastic wave equation. Mathias Days 2021, TotalEnergies, Oct 2021, Paris, France. hal-03533362

HAL Id: hal-03533362

<https://inria.hal.science/hal-03533362v1>

Submitted on 18 Jan 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destin ee au d ep ot et  a la diffusion de documents scientifiques de niveau recherche, publi es ou non,  emanant des  tablissements d'enseignement et de recherche franais ou  trangers, des laboratoires publics ou priv es.



MATHIAS DAYS 2021 - OCTOBER 3-7

PML applied to spacetime Trefftz-DG numerical formulation for the elastic wave equation

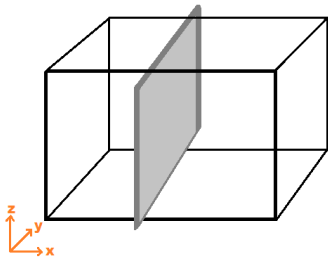
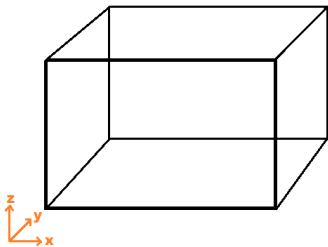
V. Vasanthan^{1,*}, H. Barucq¹, H. Calandra², J. Diaz¹

1. Project-Team Makutu, Inria, E2S UPPA, CNRS, France,
`firstname.lastname@inria.fr`
2. TotalEnergies, France, `firstname.lastname@totalenergies.com`

Introduction

- Aim of our work ?
 - ↪ investigate new solution methodologies which combine accuracy with fast resolution for seismic wave problems
- Our candidate :
 - ↪ Trefftz method in time domain with the Tent-Pitcher algorithm

Introduction



Introduction

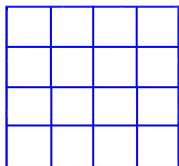


Figure: Quadrilateral mesh

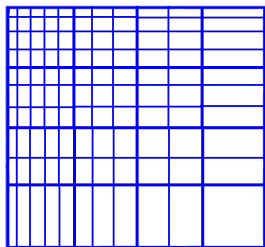


Figure: Adaptive refinement

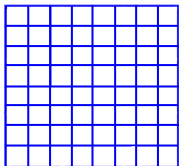


Figure: Refined quadrilateral mesh

Introduction



Figure: Different topology

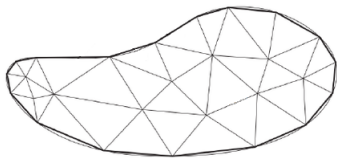


Figure: Triangular meshing

Introduction

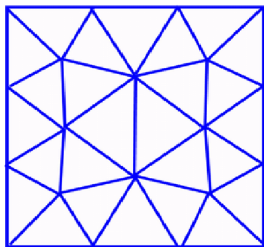


Figure: Triangular mesh

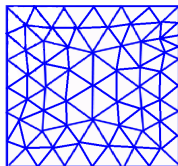


Figure: Refined mesh

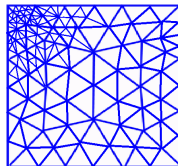


Figure: Adaptive refinement

Introduction

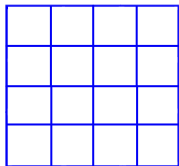


Figure: Grid for FD

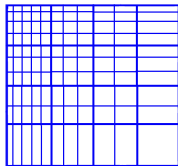


Figure: Mesh for SEM

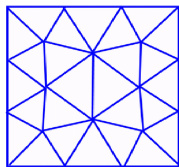


Figure: Mesh for FE

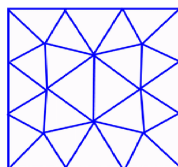


Figure: Mesh for DG

Introduction

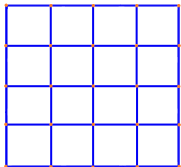


Figure: Degrees of Freedom for FD

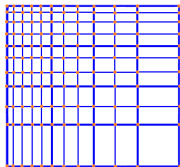


Figure: Degrees of Freedom for SEM

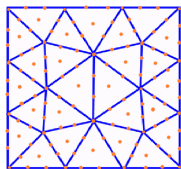


Figure: Degrees of Freedom for FE

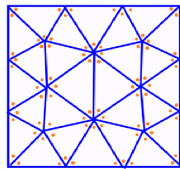


Figure: Degrees of Freedom for DG **7**



Trefftz approach

- Basis functions are local solutions of the initial PDEs
- We work on the skeleton of the mesh only
- Requires lower number of degrees of freedom
- Physical characteristics of the solution are included in the discrete solution

- Basis functions are discontinuous \rightarrow Discontinuous Galerkin discretization
- Flexibility in choice of basis functions (plane waves, Green's function...)

- Implicit scheme

Tent Pitcher algorithm

How to reduce computational costs ?

- computations on time slabs
- approximate inverse matrix
- solve element-by-element if mesh respects constraints
 - ↪ Tent-Pitcher algorithm
 - introduced for hyperbolic problems in spacetime domain,
 - consists in building a causal mesh, which respects the wave propagation speed,
 - no approximation of the inverse matrix,
 - conducive to parallel computing.

Tent-pitching 2d+t

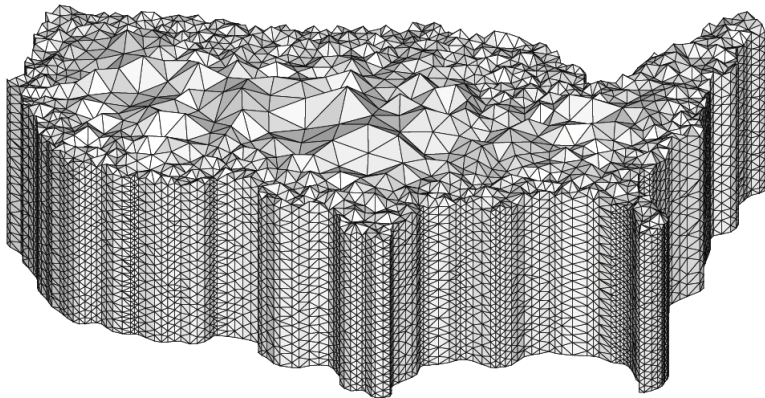


Figure: Spacetime mesh

Erickson, Guoy, Sullivan and Üngör [2005]

Acoustic Wave equation

First order acoustic wave equation :

$$\left\{ \begin{array}{ll} \frac{1}{c^2 \rho} \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{v} = f & \text{in } \Omega, \\ \rho \frac{\partial \mathbf{v}}{\partial t} + \nabla p = 0 & \text{in } \Omega, \\ \mathbf{v}(\cdot, 0) = \mathbf{v}_0, p(\cdot, 0) = p_0 & \text{in } \mathcal{D}, \\ \mathbf{v} \cdot \mathbf{n}_{\mathbf{x}} = 0 & \text{in } \partial \mathcal{D} \times \mathcal{I}. \end{array} \right. \quad \begin{array}{l} \Omega = \mathcal{D} \times \mathcal{I} \\ \mathcal{I} = [0, T] \end{array}$$

with

$$\mathbf{x} = (x, y),$$

$$p = p(\mathbf{x}, t), \text{ pressure,}$$

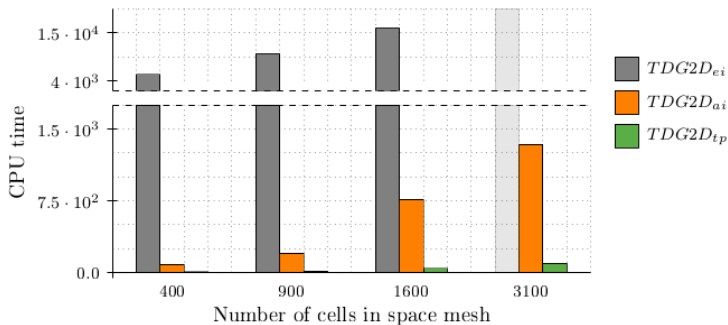
$$\mathbf{v} = \mathbf{v}(\mathbf{x}, t) = (u, v), \text{ velocity,}$$

$$f = f(\mathbf{x}, t), \text{ source term,}$$

$$c = c(\mathbf{x}), \text{ wave speed,}$$

$$\rho = \rho(\mathbf{x}), \text{ density.}$$

Results



$$CPU_{exact} = 20 \times CPU_{approx}. \quad CPU_{approx} = 10 \times CPU_{tent-pitch}.$$

Thesis of E. Shishenina [2018]

Absorbing boundary conditions

State of the art :

- not much work on absorbing boundary conditions with polynomials on spacetime Trefftz,
- no Perfectly Matched Layers for spacetime Trefftz.

Green's functions :

- are exact local solutions,
- have interesting behaviour at infinity,
- can be computed analytically inside the PML.

Perfectly Matched Layers

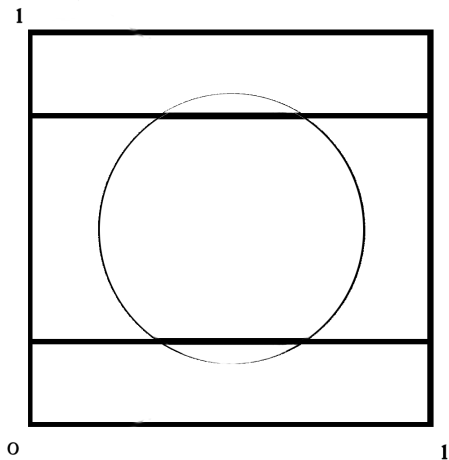


Figure: Domain with PML

J.-P. Bérenger [1994]

Green's functions

Notations: $\mathcal{X} = \mathbf{x} - \mathbf{x}_s$, $r = \sqrt{\mathcal{X} \cdot \mathcal{X}}$
 $\mathcal{X}_1 = x - x_s$, $\mathcal{X}_2 = y - y_s$, $\mathcal{T} = t - t_s$

Cagniard-De Hoop method :

$$G_p(\mathbf{x}, t, \mathbf{x}_s, t_s) = \frac{1}{2\pi \sqrt{\mathcal{T}^2 - \frac{r^2}{c^2}}},$$

$$G_u(\mathbf{x}, t, \mathbf{x}_s, t_s) = \frac{\mathcal{T} \mathcal{X}_1}{\rho r^2} G_p(\mathbf{x}, t, \mathbf{x}_s, t_s),$$

$$G_v(\mathbf{x}, t, \mathbf{x}_s, t_s) = \frac{\mathcal{T} \mathcal{X}_2}{\rho r^2} G_p(\mathbf{x}, t, \mathbf{x}_s, t_s).$$

Green's functions inside of the PML

- $G_p^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, \sigma_x) = \exp^{-A} \cos(B) G_p(\mathbf{x}, t, \mathbf{x}_s, t_s)$
- $G_u^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, \sigma_x) = \frac{\exp^{-A} G_p(\mathbf{x}, t, \mathbf{x}_s, t_s)}{\rho r^2} \left(\mathcal{T} \mathcal{X}_1 \cos(B) + h \mathcal{X}_2 \sin(B) \right)$
- $G_v^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, \sigma_x) = \frac{\exp^{-A} G_p(\mathbf{x}, t, \mathbf{x}_s, t_s)}{\rho r^2} \left(\mathcal{T} \mathcal{X}_2 \cos(B) + h \mathcal{X}_1 \sin(B) \right)$

with

$$A = A(\mathbf{x}, t, \mathbf{x}_s, t_s) = \frac{\mathcal{X}_1}{r^2} \Sigma(x, x_s) \mathcal{T},$$

$$B = B(\mathbf{x}, t, \mathbf{x}_s, t_s) = \frac{\mathcal{X}_2}{r^2} \Sigma(x, x_s) h,$$

$$\Sigma(x, x_s) = \int_0^{\mathcal{X}_1} \sigma(\mathbf{x}, \mathbf{x}_s) d\mathcal{X},$$

$$h = h(\mathbf{x}, t, \mathbf{x}_s, t_s) = \sqrt{\mathcal{T}^2 - \frac{r^2}{c^2}}.$$

PML - Results

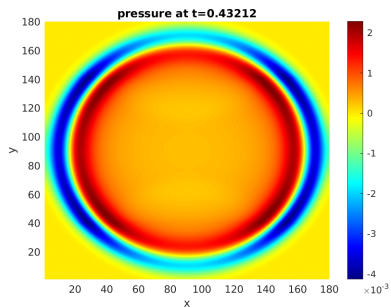


Figure: Pressure for $\sigma = 5$

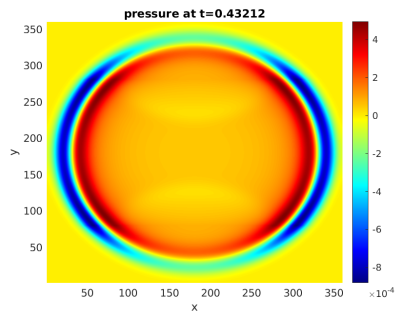


Figure: Pressure for $\sigma = 10$

Elastic Wave equation

$$\left\{ \begin{array}{ll} \frac{\partial \underline{\underline{\sigma}}}{\partial t} - \underline{\underline{C}} \underline{\underline{\epsilon}}(\mathbf{v}) = 0 & \text{in } \Omega, \\ \rho \frac{\partial \mathbf{v}}{\partial t} - \nabla \cdot \underline{\underline{\sigma}} = 0 & \text{in } \Omega, \\ \mathbf{v}(\cdot, 0) = \mathbf{v}_0, \underline{\underline{\sigma}}(\cdot, 0) = \underline{\underline{\sigma}}_0 & \text{in } \mathcal{D}, \\ \underline{\underline{\sigma}} \mathbf{n}_{\mathbf{x}} = g & \text{in } \partial \mathcal{D} \times \mathcal{I}. \end{array} \right. \quad (1)$$

with

$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}(\mathbf{x}, t)$, stress tensor,

$\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$, velocity,

$\underline{\underline{C}}$, elastic tensor,

$\underline{\underline{\epsilon}}$, strain tensor,

ρ , density,

g , boundary value.

Green's functions inside of the PML

$$\begin{cases} u^\Phi = \partial_x G_p^{pml} \\ v^\Phi = \partial_y G_p^{pml} \end{cases} \quad (2)$$

$$\begin{cases} u^\Psi = -\partial_y G_p^{pml} \\ v^\Psi = \partial_x G_p^{pml} \end{cases} \quad (3)$$

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix}^\Phi = - \begin{pmatrix} 2c_{13} & c_{12} & c_{11} \\ 2c_{23} & c_{22} & c_{21} \\ 2c_{33} & c_{32} & c_{31} \end{pmatrix} \begin{pmatrix} I \\ II \\ III \end{pmatrix}$$

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix}^\Psi = \begin{pmatrix} c_{11} - c_{12} & c_{13} & -c_{13} \\ c_{21} - c_{22} & c_{23} & -c_{23} \\ c_{31} - c_{32} & c_{33} & -c_{33} \end{pmatrix} \begin{pmatrix} I \\ II \\ III \end{pmatrix}$$

$$\begin{cases} I = \rho \partial_y G_u^{pml} \\ II = -\rho \partial_y G_v^{pml} \\ III = \left[\left(\frac{t^2(y^2 - x^2)}{r^4} - \frac{y^2}{r^2 c^2} \right) \cos(B) - \frac{txy}{r^4} \sqrt{t^2 - \frac{r^2}{c^2}} \sin(B) \right] G_p^{pml} \end{cases}$$

Prospects

- Implement PML for elastic case
- Parallelization
- Extend to unstructured meshes
- Extend 2D to 3D

References

Bérenger.

A perfectly matched layer for the absorption of electromagnetic waves.

Journal of Computational Physics, 114(2):185 – 200, 1994.

Jeff Erickson, Damrong Guoy, J. Sullivan, and Alper Üngör.

Building spacetime meshes over arbitrary spatial domains.

Engineering with Computers, 20:342–353, 2005.

I. Herrera.

Trefftz method: A general theory.

Eq., 16:561–580, 11 2000.

Ralf Hiptmair, Andrea Moiola, and Ilaria Perugia.

A survey of trefftz methods for the helmholtz equation.

06 2015.

Ilaria Perugia, Joachim Schöberl, Paul Stocker, and Christoph Wintersteiger.

Tent pitching and trefftz-dg method for the acoustic wave equation.

Computers & Mathematics with Applications, 2020.

Elvira Shishenina.

Space-Time Discretization of Elasto-Acoustic Wave Equation in Polynomial Trefftz-DG Bases.

PhD thesis, 2018.

Erich Trefftz.

Ein gegenstück zum ritzschen verfahren.

In *Proc. 2nd Int. Cong. Appl. Mech. Zurich*, pages 131–137, 1926.

Alper Üngör and Alla Sheffer.

Tent-pitcher: A meshing algorithm for space-time discontinuous galerkin methods.

In *In proc. 9th int'l. meshing roundtable*, pages 111–122, 01 2000.

Thank you for your attention !