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# PML applied to spacetime Trefftz-DG numerical formulation for the acoustic wave equation

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- Aim of our work ?
  - ↪ to develop a Trefftz method in time domain with the Tent-Pitcher algorithm, for wave problems.
- Why ?
  - ↪ gain precision,
  - ↪ reduce computational time.
- What kind of problems ?
  - ↪ acoustic,
  - ↪ elastodynamic,
  - ↪ elasto-acoustic wave propagation systems.
- What kind of methods ?
  - ↪ Trefftz-Discontinuous Galerkin methods with Tent-Pitching.
- What kind of basis ?
  - ↪ polynomial basis functions,
  - ↪ Green's functions.

# Acoustic Wave equation

First order acoustic wave equation :

$$\left\{ \begin{array}{ll} \frac{1}{c^2 \rho} \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{v} = f & \text{in } \Omega, \\ \rho \frac{\partial \mathbf{v}}{\partial t} + \nabla p = 0 & \text{in } \Omega, \\ \mathbf{v}(\cdot, 0) = \mathbf{v}_0, p(\cdot, 0) = p_0 & \text{in } \mathcal{D}, \\ \mathbf{v} \cdot \mathbf{n}_{\mathbf{x}} = 0 & \text{in } \partial \mathcal{D} \times \mathcal{I}. \end{array} \right. \quad \begin{array}{l} \Omega = \mathcal{D} \times \mathcal{I} \\ \mathcal{I} = [0, T] \end{array}$$

with

$$\mathbf{x} = (x, y),$$

$$p = p(\mathbf{x}, t), \text{ pressure,}$$

$$\mathbf{v} = \mathbf{v}(\mathbf{x}, t) = (u, v), \text{ velocity,}$$

$$f = f(\mathbf{x}, t), \text{ source term,}$$

$$c = c(\mathbf{x}), \text{ wave speed,}$$

$$\rho = \rho(\mathbf{x}), \text{ density.}$$

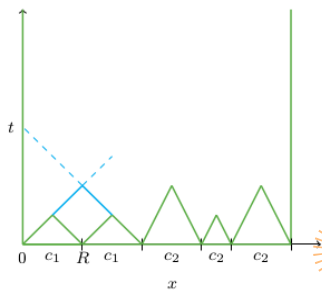
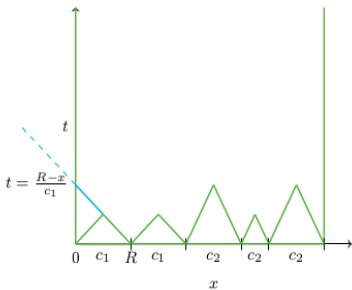
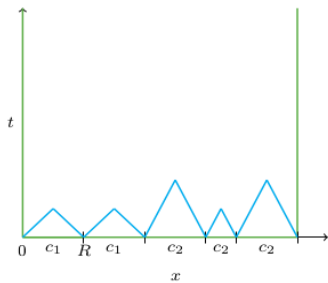
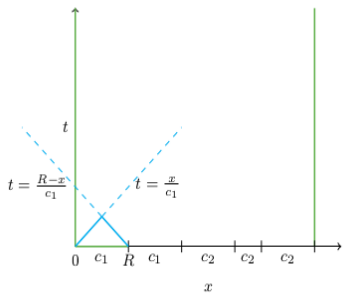
- Basis functions are local solutions of the initial PDEs
- We work on the skeleton of the mesh only
- Requires lower number of degrees of freedom
- Physical characteristics of the solution are included in the discrete solution
  
- Basis functions are discontinuous  $\rightarrow$  Discontinuous Galerkin discretization
- Flexibility in choice of basis functions (plane waves, Green's function...)
  
- Implicit scheme

How to reduce computational costs ?

- computations on time slabs
- approximate inverse matrix
- solve element-by-element if mesh respects constraints
  - ↪ Tent-Pitcher algorithm
    - introduced for hyperbolic problems in spacetime domain,
    - consists in building a causal mesh, which respects the wave propagation speed,
    - no approximation of the inverse matrix,
    - conducive to parallel computing.



# Tent-pitching process 1d+t



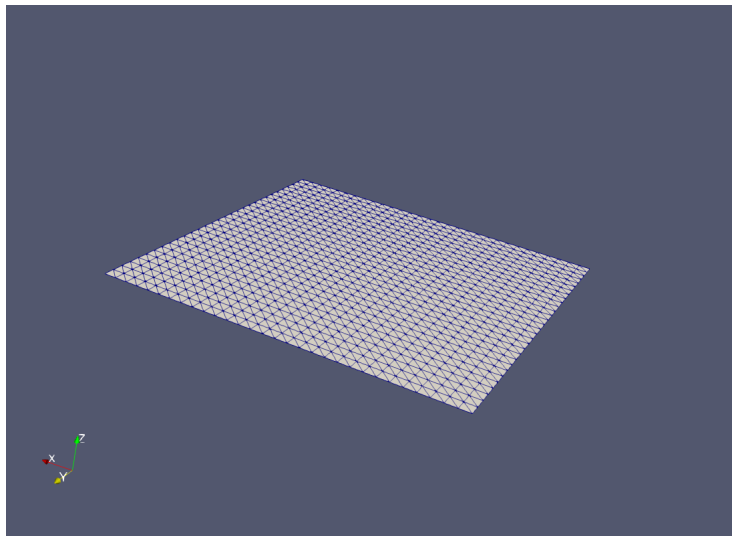


Figure: Initial spatial mesh



# Tent-pitching process $2d+t$

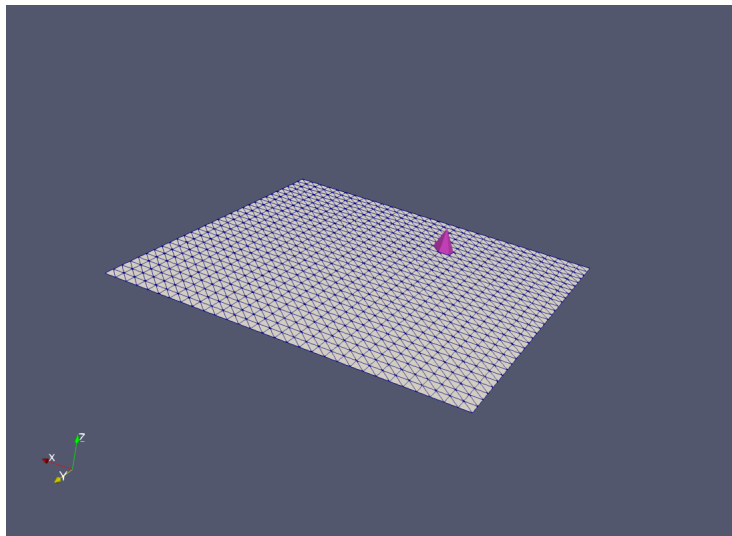


Figure: 1<sup>st</sup> iteration

# Tent-pitching process 2d+t

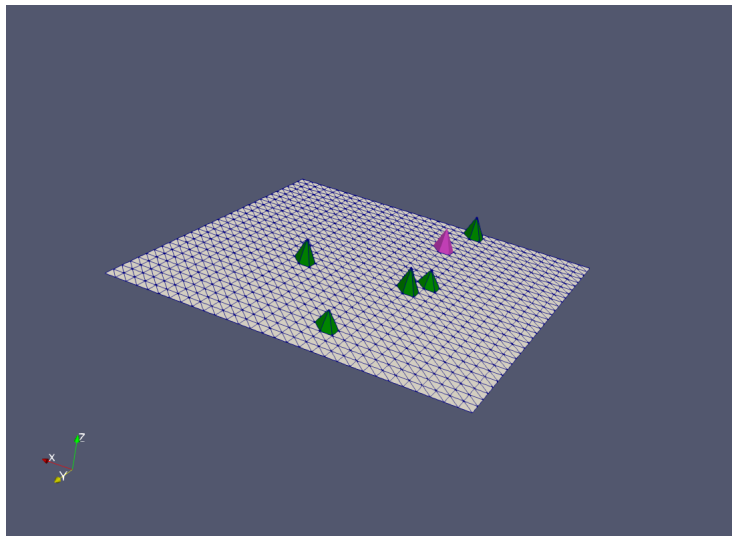


Figure: 5<sup>th</sup> iteration

# Tent-pitching process 2d+t

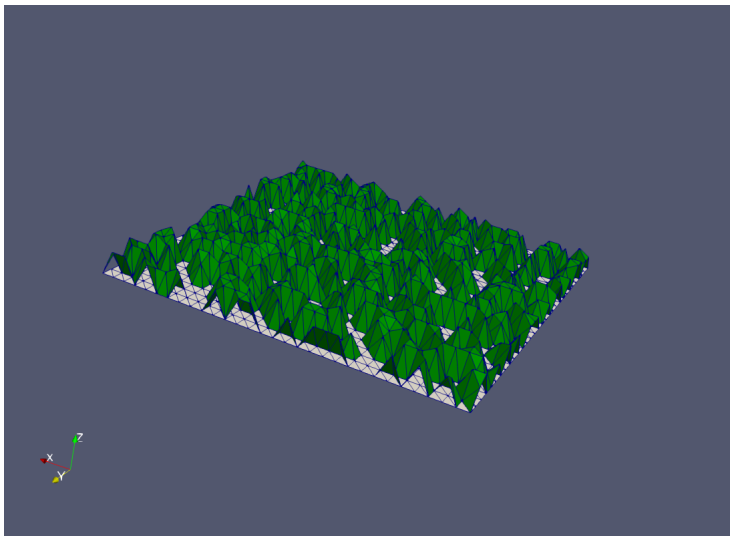


Figure: 450<sup>th</sup> iteration



# Tent-pitching process 2d+t

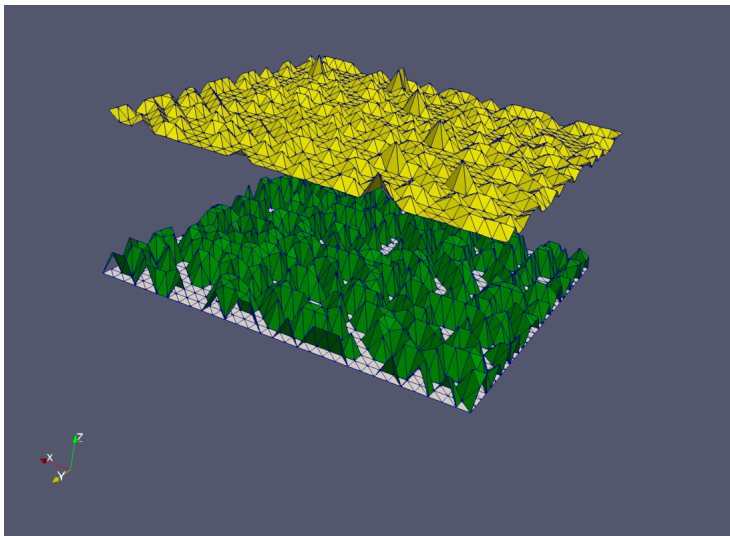


Figure: 6698<sup>th</sup> iteration

- Neumann boundary condition

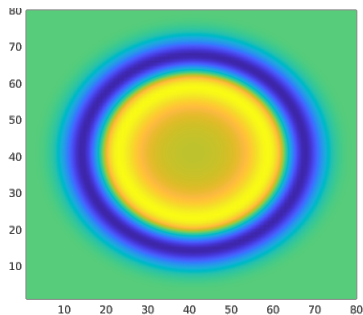


Figure: Pressure at time  $t=0.308$  with Trefftz-DG+Tent-Pitching

- polynomial basis functions

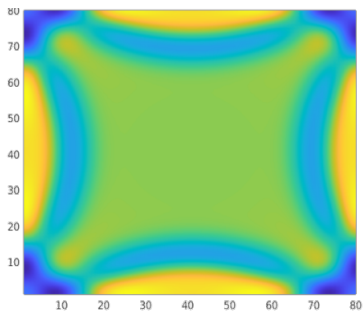
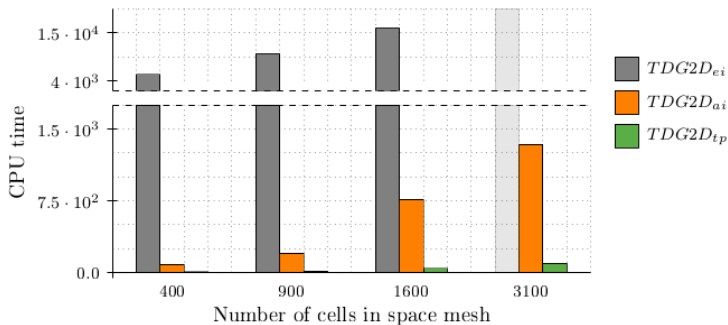


Figure: Pressure at time  $t=0.616$  with Trefftz-DG+Tent-Pitching



$$CPU_{exact} = 20 \times CPU_{approx}. \quad CPU_{approx} = 10 \times CPU_{tent-pitch}.$$

Thesis of E. Shishenina [2018]

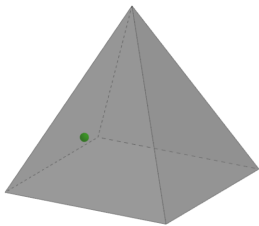
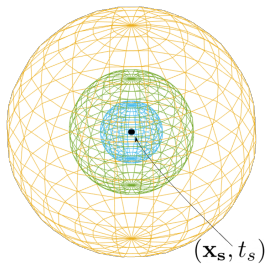
State of the art :

- not much work on absorbing boundary conditions with polynomials on spacetime Trefftz,
- no Perfectly Matched Layers for spacetime Trefftz.

Green's functions :

- are exact local solutions,
- have interesting behaviour at infinity,
- can be computed analytically inside the PML.

# Source Points





Notations:  $\mathcal{X} = \mathbf{x} - \mathbf{x}_s$ ,  $r = \sqrt{\mathcal{X} \cdot \mathcal{X}}$   
 $\mathcal{X}_1 = x - x_s$ ,  $\mathcal{X}_2 = y - y_s$ ,  $\mathcal{T} = t - t_s$

Cagniard-De Hoop method :

$$G_p(\mathbf{x}, t, \mathbf{x}_s, t_s) = \frac{1}{2\pi \sqrt{\mathcal{T}^2 - \frac{r^2}{c^2}}},$$

$$G_u(\mathbf{x}, t, \mathbf{x}_s, t_s) = \frac{\mathcal{T} \mathcal{X}_1}{\rho r^2} G_p(\mathbf{x}, t, \mathbf{x}_s, t_s),$$

$$G_v(\mathbf{x}, t, \mathbf{x}_s, t_s) = \frac{\mathcal{T} \mathcal{X}_2}{\rho r^2} G_p(\mathbf{x}, t, \mathbf{x}_s, t_s).$$

$$\left\{ \begin{array}{l} \rho \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = 0, \times w \\ \rho \frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} = 0, \times z \\ \frac{1}{c^2 \rho} \frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \times q \end{array} \right.$$

□  $w, z, q$ : test variables

$$\begin{aligned} & \int_{\mathcal{F}_h^{\text{out}}} \left( \frac{1}{c^2 \rho} p_h q + \rho u_h w + \rho v_h z \right) n_t + (u_h q + p_h w) n_x + (v_h q + p_h z) n_y \\ & + \int_{\mathcal{F}_h^{\text{in}}} (0.5 - \alpha) (\rho u_h w + \rho v_h z) n_t + (0.5 - \beta) \left( \frac{1}{c^2 \rho} p_h q \right) n_t \\ & + 0.5 \int_{\mathcal{F}_h^{\text{in}}} q (u_h n_x + v_h n_y) + p_h (w n_x + z n_y) \\ & = \int_{\mathcal{F}_h^{\text{in}}} - (0.5 + \alpha) (\rho u^0 w + \rho v^0 z) n_t - (0.5 + \beta) \left( \frac{1}{c^2 \rho} p^0 q \right) n_t \\ & - 0.5 \int_{\mathcal{F}_h^{\text{in}}} p^0 (w n_x + z n_y) + q (u^0 n_x + v^0 n_y) \end{aligned}$$

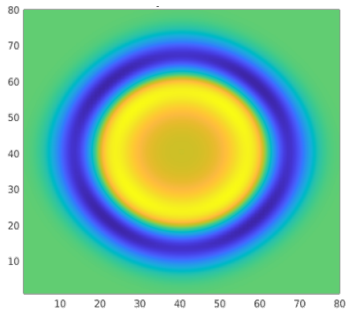


Figure: Pressure at time  $t=0.308$   
with Green's functions

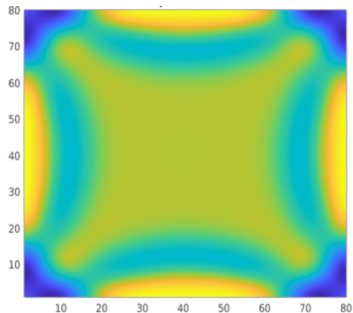
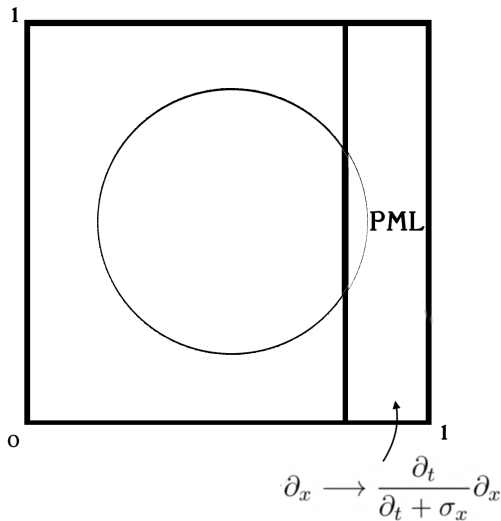


Figure: Pressure at time  $t=0.616$   
with Green's functions

# Perfectly Matched Layers



$$\rho \frac{\partial u}{\partial t} + \sigma_x u + \frac{\partial p}{\partial x} = 0,$$

$$\rho \left( \frac{\partial}{\partial t} + \sigma_x \right) u + \frac{\partial p}{\partial x} = 0,$$

$$\rho \left( \frac{\partial}{\partial t} + \sigma_x \right) \frac{\partial u}{\partial t} + \frac{\partial}{\partial t} \frac{\partial p}{\partial x} = 0,$$

$$\rho \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} - \frac{\sigma_x}{\frac{\partial}{\partial t} + \sigma_x} \frac{\partial p}{\partial x} = 0,$$

□  $\sigma_x$ : absorbing coefficient in the  $x$ -direction

$$\left\{ \begin{array}{l} \rho \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} - p_1 = 0, \\ \rho \frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} = 0, \\ \frac{1}{c^2 \rho} \frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - u_1 = 0, \\ \left( \frac{\partial}{\partial t} + \sigma_x \right) p_1 = \sigma_x \frac{\partial p}{\partial x}, \\ \left( \frac{\partial}{\partial t} + \sigma_x \right) u_1 = \sigma_x \frac{\partial u}{\partial x}. \end{array} \right.$$

□  $u_1, p_1$ : auxiliary variables

$$\left\{ \begin{array}{l} \rho \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} - p_1 = 0, \times w \\ \rho \frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} = 0, \times z \\ \frac{1}{c^2 \rho} \frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - u_1 = 0, \times q \\ \left( \frac{\partial}{\partial t} + \sigma_x \right) p_1 = \sigma_x \frac{\partial p}{\partial x}, \times w_1 \\ \left( \frac{\partial}{\partial t} + \sigma_x \right) u_1 = \sigma_x \frac{\partial u}{\partial x}, \times q_1 \end{array} \right.$$

- $w, z, q$ : test variables
- $w_1, q_1$ : auxiliary test variables

$$\begin{aligned}
 & \int_{\mathcal{F}_h^{\text{out}}} \left( \frac{1}{c^2 \rho} p_h q + \rho u_h w + \rho v_h z \right) n_t + (u_h q + p_h w) n_x + (v_h q + p_h z) n_y \\
 & \quad + (u_1 q_1 + p_1 w_1) n_t - \sigma_x (u q_1 + p w_1) n_x \\
 & + \int_{\mathcal{F}_h^{\text{in}}} (0.5 - \alpha) (\rho u_h w + \rho v_h z + u_1 q_1) n_t + (0.5 - \beta) \left( \frac{1}{c^2 \rho} p_h q + p_1 w_1 \right) n_t \\
 & + 0.5 \int_{\mathcal{F}_h^{\text{in}}} q (u_h n_x + v_h n_y) + p_h (w n_x + z n_y) - \sigma_x (u q_1 + p w_1) n_x \\
 & = \int_{\mathcal{F}_h^{\text{in}}} - (0.5 + \alpha) (\rho u^0 w + \rho v^0 z + u_1^0 q_1) n_t - (0.5 + \beta) \left( \frac{1}{c^2 \rho} p^0 q + p_1^0 w_1 \right) n_t \\
 & - 0.5 \int_{\mathcal{F}_h^{\text{in}}} p^0 (w n_x + z n_y) + q (u^0 n_x + v^0 n_y) - \sigma_x (u^0 q_1 + p^0 w_1) n_x
 \end{aligned}$$



# PML - Basis functions

- $G_p^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, \sigma_x) = \exp^{-A} \cos(B) G_p(\mathbf{x}, t, \mathbf{x}_s, t_s)$
- $G_u^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, \sigma_x) = \frac{\exp^{-A} G_p(\mathbf{x}, t, \mathbf{x}_s, t_s)}{\rho r^2} \left( \mathcal{T} \mathcal{X}_1 \cos(B) + h \mathcal{X}_2 \sin(B) \right)$
- $G_v^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, \sigma_x) = \frac{\exp^{-A} G_p(\mathbf{x}, t, \mathbf{x}_s, t_s)}{\rho r^2} \left( \mathcal{T} \mathcal{X}_2 \cos(B) + h \mathcal{X}_1 \sin(B) \right)$

with

$$A = A(\mathbf{x}, t, \mathbf{x}_s, t_s) = \frac{\mathcal{X}_1}{r^2} \Sigma(x, x_s) \mathcal{T},$$

$$B = B(\mathbf{x}, t, \mathbf{x}_s, t_s) = \frac{\mathcal{X}_2}{r^2} \Sigma(x, x_s) h,$$

$$\Sigma(x, x_s) = \int_0^{\mathcal{X}_1} \sigma(\mathbf{x}, \mathbf{x}_s) d\mathcal{X},$$

$$h = h(\mathbf{x}, t, \mathbf{x}_s, t_s) = \sqrt{\mathcal{T}^2 - \frac{r^2}{c^2}}.$$

$$\left\{ \begin{array}{l} \rho \frac{\partial w}{\partial t} + \frac{\partial q}{\partial x} - \sigma_x \frac{\partial q_1}{\partial x} = 0, \\ \rho \frac{\partial z}{\partial t} + \frac{\partial q}{\partial y} = 0, \\ \frac{1}{c^2 \rho} \frac{\partial q}{\partial t} + \frac{\partial w}{\partial x} + \frac{\partial z}{\partial y} - \sigma_x \frac{\partial w_1}{\partial x} = 0, \\ \left( \frac{\partial}{\partial t} - \sigma_x \right) q_1 = -q, \\ \left( \frac{\partial}{\partial t} - \sigma_x \right) w_1 = -w. \end{array} \right.$$

- $G_q^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, \sigma) = G_p^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, -\sigma)$
- $G_w^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, \sigma) = G_u^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, -\sigma)$
- $G_z^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, \sigma) = G_v^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, -\sigma)$

# PML - Results

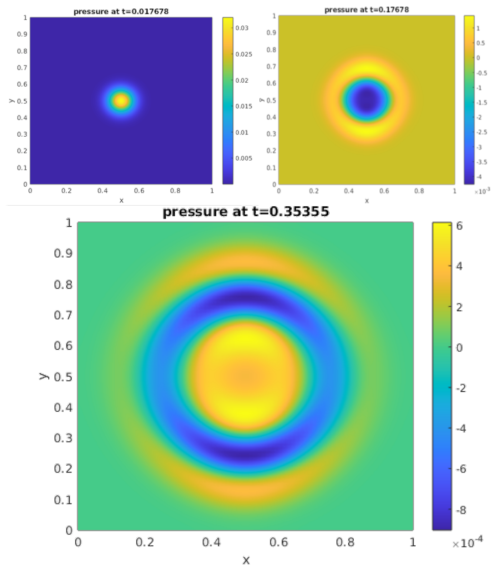


Figure: Pressure at different time steps with PML

- Coupling PML with computational domain
- Parallelization
- Extend to unstructured meshes
- Extend 2D to 3D

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The background of the slide features a series of wavy, concentric orange lines that create a sense of depth and movement, resembling a stylized fingerprint or a topographical map. The lines are thin and closely spaced, filling the entire frame.

Thank you for your attention !