PML applied to spacetime Trefftz-DG numerical formulation for the acoustic wave equation

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ICOSAHOM 2020 – JULY 12-16, 2021

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Introduction

- Aim of our work ?
  - ↪ to develop a Trefftz method in time domain with the Tent-Pitcher algorithm, for wave problems.
- Why ?
  - ↪ gain precision,
  - ↪ reduce computational time.
- What kind of problems ?
  - ↪ acoustic,
  - ↪ elastodynamic,
  - ↪ elasto-acoustic wave propagation systems.
- What kind of methods ?
  - ↪ Trefftz-Discontinuous Galerkin methods with Tent-Pitching.
- What kind of basis ?
  - ↪ polynomial basis functions,
  - ↪ Green's functions.

Acoustic Wave equation

First order acoustic wave equation :

$$
\begin{cases}
\dfrac{1}{c^2\rho}\dfrac{\partial p}{\partial t} + \nabla \cdot \mathbf{v} = f & \text{in } \Omega\,, \\[2ex]
\rho\dfrac{\partial \mathbf{v}}{\partial t} + \nabla p = 0 & \text{in } \Omega\,, \\[2ex]
\mathbf{v}(\cdot,0) = \mathbf{v}_0,\ p(\cdot,0) = p_0 & \text{in } \mathcal{D}\,, \\[2ex]
\mathbf{v} \cdot \mathbf{n}_{\mathbf{x}} = 0 & \text{in } \partial\mathcal{D} \times \mathcal{I}\,.
\end{cases}
\qquad
\begin{aligned}
\Omega &= \mathcal{D} \times \mathcal{I} \\
\mathcal{I} &= [0, T]
\end{aligned}
$$

with

$\mathbf{x} = (x, y)$,

$p = p(\mathbf{x}, t)$, pressure,

$\mathbf{v} = \mathbf{v}(\mathbf{x}, t) = (u, v)$, velocity,

$f = f(\mathbf{x}, t)$, source term,

$c = c(\mathbf{x})$, wave speed,

$\rho = \rho(\mathbf{x})$, density.

Trefftz approach

- Basis functions are local solutions of the initial PDEs
- We work on the skeleton of the mesh only
- Requires lower number of degrees of freedom
- Physical characteristics of the solution are included in the discrete solution

- Basis functions are discontinuous $\longrightarrow$ Discontinuous Galerkin discretization
- Flexibility in choice of basis functions (plane waves, Green's function...)

- Implicit scheme

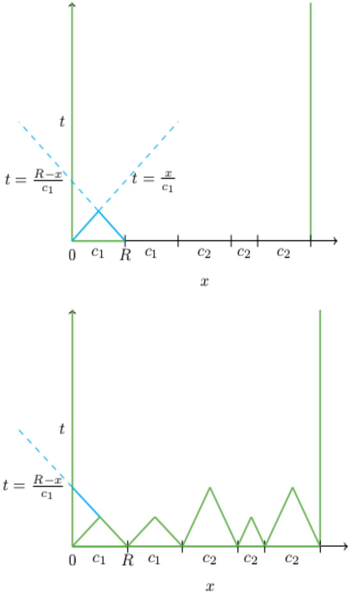
How to reduce computational costs ?

- computations on time slabs
- approximate inverse matrix
- solve element-by-element if mesh respects constraints

  $\hookrightarrow$ Tent-Pitcher algorithm

  - introduced for hyperbolic problems in spacetime domain,
  - consists in building a causal mesh, which respects the wave propagation speed,
  - no approximation of the inverse matrix,
  - conducive to parallel computing.

Tent-pitching process 1d+t



Tent-pitching process 2d+t

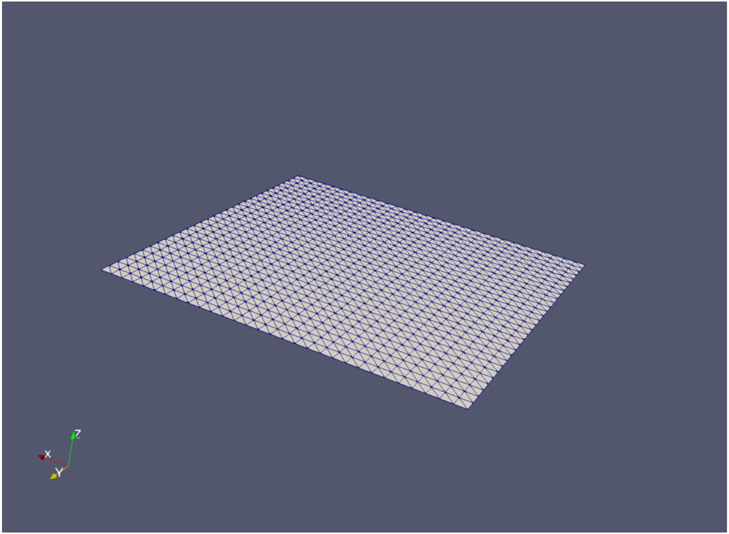


Figure: Initial spatial mesh

Tent-pitching process 2d+t

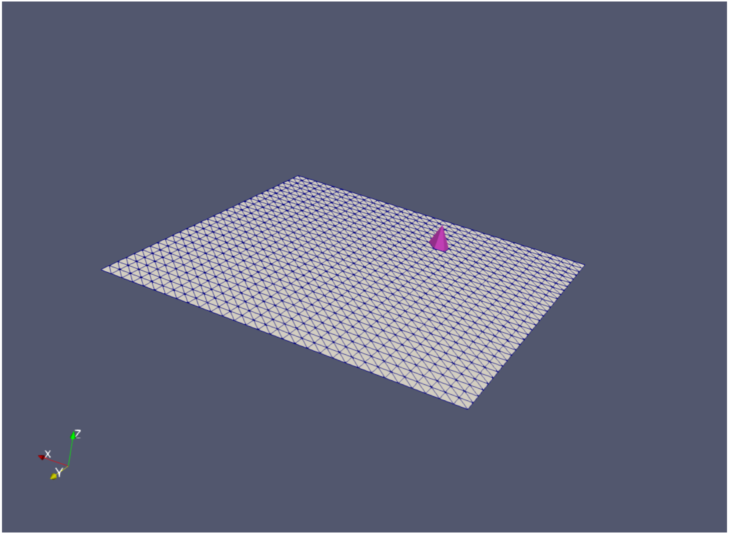


Figure: 1^{st} iteration

Tent-pitching process 2d+t

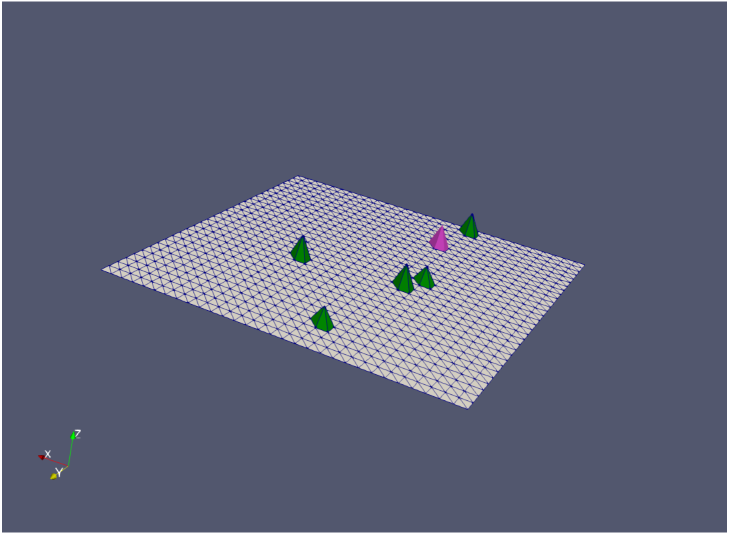


Figure: 5^{th} iteration

Tent-pitching process 2d+t

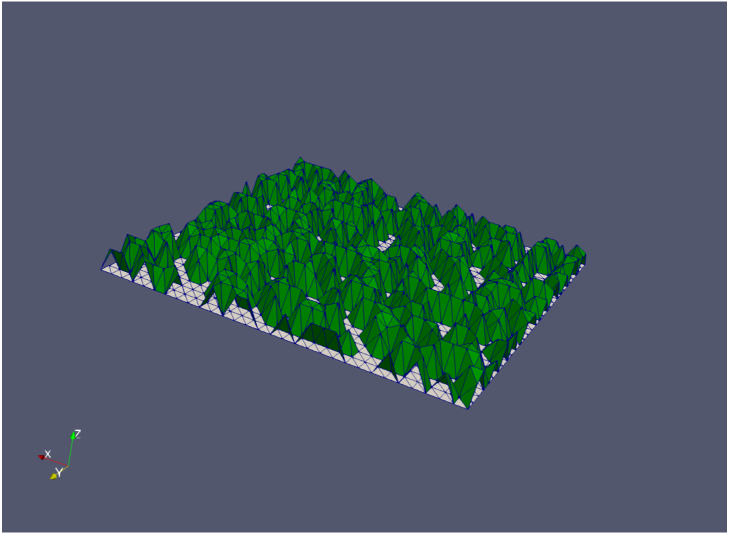


Figure: 450^{th} iteration

Tent-pitching process 2d+t

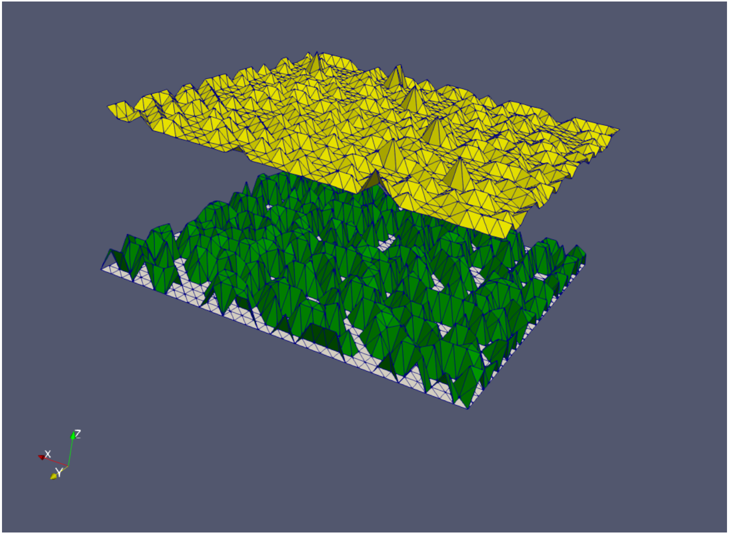


Figure: 6698^{th} iteration

Results

- polynomial basis functions
- Neumann boundary condition

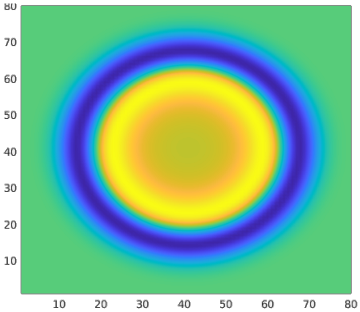


Figure: Pressure at time t=0.308 with Trefftz-DG+Tent-Pitching

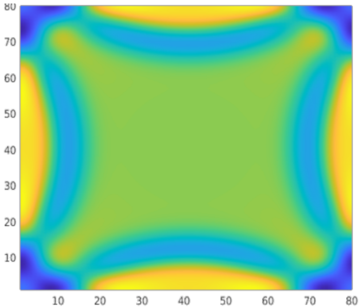
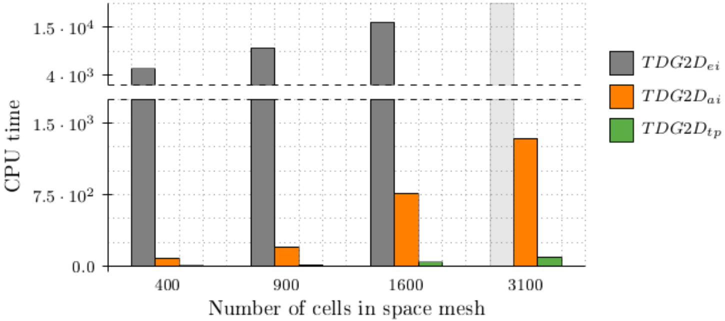


Figure: Pressure at time t=0.616 with Trefftz-DG+Tent-Pitching

Results



$$CPU_{exact} = 20 \times CPU_{approx}. \qquad CPU_{approx} = 10 \times CPU_{tent-pitch}.$$

Thesis of E. Shishenina [2018]

Green's function as basis

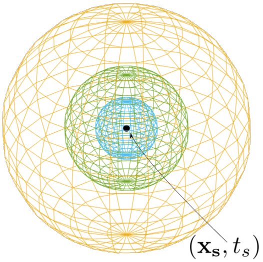
State of the art :

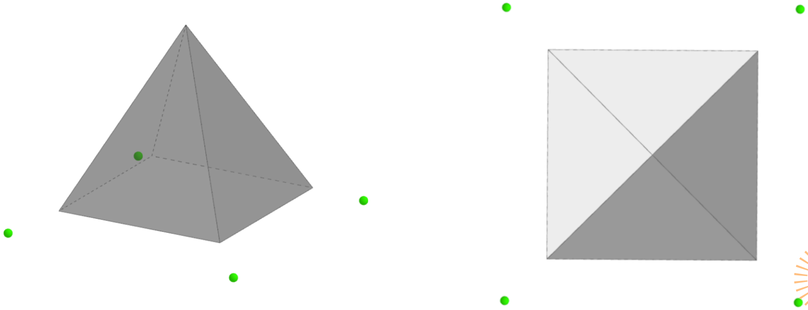
- not much work on absorbing boundary conditions with polynomials on spacetime Trefftz,
- no Perfectly Matched Layers for spacetime Trefftz.

Green's functions :

- are exact local solutions,
- have interesting behaviour at infinity,
- can be computed analytically inside the PML.

Source Points





Green's function as basis

Notations: $\mathcal{X} = \mathbf{x} - \mathbf{x}_s\,,\; r = \sqrt{\mathcal{X}.\mathcal{X}}$

$\mathcal{X}_1 = x - x_s\,,\; \mathcal{X}_2 = y - y_s\,,\; \mathcal{T} = t - t_s$

Cagniard-De Hoop method :

$$G_p(\mathbf{x}, t, \mathbf{x_s}, t_s) = \frac{1}{2\pi\sqrt{\mathcal{T}^2 - \frac{r^2}{c^2}}}\,,$$

$$G_u(\mathbf{x}, t, \mathbf{x_s}, t_s) = \frac{\mathcal{T}\mathcal{X}_1}{\rho r^2} G_p(\mathbf{x}, t, \mathbf{x_s}, t_s)\,,$$

$$G_v(\mathbf{x}, t, \mathbf{x_s}, t_s) = \frac{\mathcal{T}\mathcal{X}_2}{\rho r^2} G_p(\mathbf{x}, t, \mathbf{x_s}, t_s)\,.$$

Variational Formulation

$$\begin{cases} \rho \dfrac{\partial u}{\partial t} + \dfrac{\partial p}{\partial x} = 0\,, \; \color{red}{\times\, w} \\[2ex] \rho \dfrac{\partial v}{\partial t} + \dfrac{\partial p}{\partial y} = 0\,, \; \color{red}{\times\, z} \\[2ex] \dfrac{1}{c^2 \rho} \dfrac{\partial p}{\partial t} + \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0\,, \; \color{red}{\times\, q} \end{cases}$$

- w, z, q: test variables

Variational Formulation

$$
\begin{aligned}
&\int_{\mathcal{F}_h^{out}} \left(\frac{1}{c^2\rho} p_h q + \rho u_h w + \rho v_h z\right) n_t + (u_h q + p_h w) n_x + (v_h q + p_h z) n_y \\
&+ \int_{\mathcal{F}_h^{in}} (0.5-\alpha)(\rho u_h w + \rho v_h z) n_t + (0.5-\beta)\left(\frac{1}{c^2\rho} p_h q\right) n_t \\
&+0.5 \int_{\mathcal{F}_h^{in}} q(u_h n_x + v_h n_y) + p_h (w n_x + z n_y) \\
&= \int_{\mathcal{F}_h^{in}} -(0.5+\alpha)\left(\rho u^0 w + \rho v^0 z\right) n_t - (0.5+\beta)\left(\frac{1}{c^2\rho} p^0 q\right) n_t \\
&-0.5 \int_{\mathcal{F}_h^{in}} p^0 (w n_x + z n_y) + q(u^0 n_x + v^0 n_y)
\end{aligned}
$$

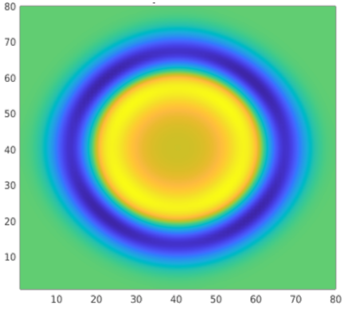


Figure: Pressure at time t=0.308 with Green's functions

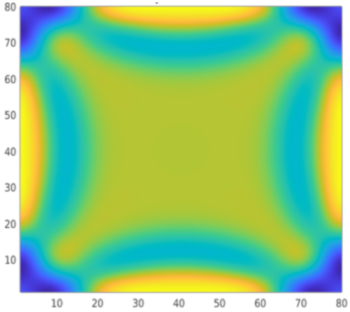
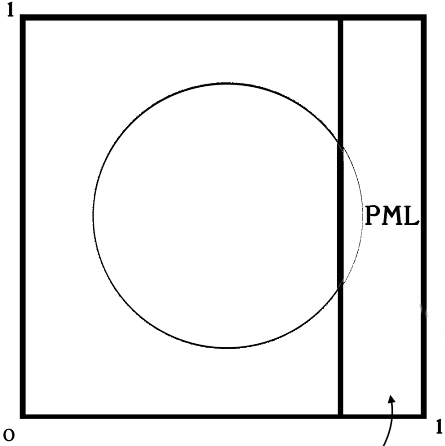


Figure: Pressure at time t=0.616 with Green's functions

Perfectly Matched Layers



$$\partial_x \longrightarrow \frac{\partial_t}{\partial_t + \sigma_x}\partial_x$$

$$\rho\frac{\partial u}{\partial t} + \sigma_x u + \frac{\partial p}{\partial x} = 0\,,$$

$$\rho(\frac{\partial}{\partial t} + \sigma_x)u + \frac{\partial p}{\partial x} = 0\,,$$

$$\rho(\frac{\partial}{\partial t} + \sigma_x)\frac{\partial u}{\partial t} + \frac{\partial}{\partial t}\frac{\partial p}{\partial x} = 0\,,$$

$$\rho\frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} - \frac{\sigma_x}{\frac{\partial}{\partial t} + \sigma_x}\frac{\partial p}{\partial x} = 0\,,$$

- σ_x: absorbing coefficient in the x-direction

J.-P. Bérenger [1994]

Variational Formulation

$$
\begin{cases}
\rho \dfrac{\partial u}{\partial t} + \dfrac{\partial p}{\partial x} - p_1 = 0\,, \\[2ex]
\rho \dfrac{\partial v}{\partial t} + \dfrac{\partial p}{\partial y} = 0\,, \\[2ex]
\dfrac{1}{c^2 \rho} \dfrac{\partial p}{\partial t} + \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} - u_1 = 0\,, \\[2ex]
\Big(\dfrac{\partial}{\partial t} + \sigma_x \Big) p_1 = \sigma_x \dfrac{\partial p}{\partial x}\,, \\[2ex]
\Big(\dfrac{\partial}{\partial t} + \sigma_x \Big) u_1 = \sigma_x \dfrac{\partial u}{\partial x}\,.
\end{cases}
$$

- u_1, p_1: auxiliary variables

Variational Formulation

$$
\begin{cases}
\rho \dfrac{\partial u}{\partial t} + \dfrac{\partial p}{\partial x} - p_1 = 0\,, \times w \\[2ex]
\rho \dfrac{\partial v}{\partial t} + \dfrac{\partial p}{\partial y} = 0\,, \times z \\[2ex]
\dfrac{1}{c^2 \rho} \dfrac{\partial p}{\partial t} + \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} - u_1 = 0\,, \times q \\[2ex]
\left(\dfrac{\partial}{\partial t} + \sigma_x \right) p_1 = \sigma_x \dfrac{\partial p}{\partial x}\,, \times w_1 \\[2ex]
\left(\dfrac{\partial}{\partial t} + \sigma_x \right) u_1 = \sigma_x \dfrac{\partial u}{\partial x}\,. \times q_1
\end{cases}
$$

- w, z, q: test variables
- w_1, q_1: auxiliary test variables

PML - Variational Formulation

$$\int_{\mathcal{F}_h^{out}} (\frac{1}{c^2\rho}p_h q + \rho u_h w + \rho v_h z)n_t + (u_h q + p_h w)n_x + (v_h q + p_h z)n_y$$
$$+(u_1 q_1 + p_1 w_1)n_t - \sigma_x(uq_1 + pw_1)n_x$$
$$+ \int_{\mathcal{F}_h^{in}} (0.5-\alpha)(\rho u_h w + \rho v_h z + u_1 q_1)n_t + (0.5-\beta)\left(\frac{1}{c^2\rho}p_h q + p_1 w_1\right)n_t$$
$$+0.5\int_{\mathcal{F}_h^{in}} q(u_h n_x + v_h n_y) + p_h(wn_x + zn_y) - \sigma_x(uq_1 + pw_1)n_x$$
$$= \int_{\mathcal{F}_h^{in}} -(0.5+\alpha)\left(\rho u^0 w + \rho v^0 z + u_1^0 q_1\right)n_t - (0.5+\beta)\left(\frac{1}{c^2\rho}p^0 q + p_1^0 w_1\right)n_t$$
$$-0.5\int_{\mathcal{F}_h^{in}} p^0(wn_x + zn_y) + q(u^0 n_x + v^0 n_y) - \sigma_x(u^0 q_1 + p^0 w_1)n_x$$

PML - Basis functions

- $G_p^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, \sigma_x) = exp^{-A}\cos(B) G_p(\mathbf{x}, t, \mathbf{x}_s, t_s)$

- $G_u^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, \sigma_x) = \dfrac{exp^{-A} G_p(\mathbf{x}, t, \mathbf{x}_s, t_s)}{\rho r^2}\left(\mathcal{T}\mathcal{X}_1\cos(B) + h\mathcal{X}_2\sin(B)\right)$

- $G_v^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, \sigma_x) = \dfrac{exp^{-A} G_p(\mathbf{x}, t, \mathbf{x}_s, t_s)}{\rho r^2}\left(\mathcal{T}\mathcal{X}_2\cos(B) + h\mathcal{X}_1\sin(B)\right)$

with

$$A = A(\mathbf{x}, t, \mathbf{x}_s, t_s) = \frac{\mathcal{X}_1}{r^2}\Sigma(x, x_s)\mathcal{T}\,,$$

$$B = B(\mathbf{x}, t, \mathbf{x}_s, t_s) = \frac{\mathcal{X}_2}{r^2}\Sigma(x, x_s)h\,,$$

$$\Sigma(x, x_s) = \int_0^{\mathcal{X}_1}\sigma(\mathbf{x}, \mathbf{x}_s)d\mathcal{X}\,,$$

$$h = h(\mathbf{x}, t, \mathbf{x_s}, t_s) = \sqrt{\mathcal{T}^2 - \frac{r^2}{c^2}}\,.$$

PML - Test functions

$$\begin{cases} \rho \dfrac{\partial w}{\partial t} + \dfrac{\partial q}{\partial x} - \sigma_x \dfrac{\partial q_1}{\partial x} = 0\,, \\ \rho \dfrac{\partial z}{\partial t} + \dfrac{\partial q}{\partial y} = 0\,, \\ \dfrac{1}{c^2 \rho} \dfrac{\partial q}{\partial t} + \dfrac{\partial w}{\partial x} + \dfrac{\partial z}{\partial y} - \sigma_x \dfrac{\partial w_1}{\partial x} = 0\,, \\ \left(\dfrac{\partial}{\partial t} - \sigma_x \right) q_1 = -q\,, \\ \left(\dfrac{\partial}{\partial t} - \sigma_x \right) w_1 = -w\,. \end{cases}$$

- ${G_q}^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, \sigma) = {G_p}^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, -\sigma)$
- ${G_w}^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, \sigma) = {G_u}^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, -\sigma)$
- ${G_z}^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, \sigma) = {G_v}^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, -\sigma)$

PML - Results

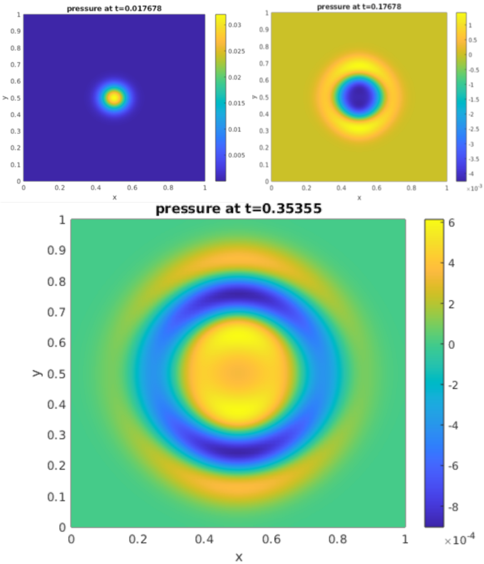


Figure: Pressure at different time steps with PML

Prospects

- Coupling PML with computational domain
- Parallelization
- Extend to unstructured meshes
- Extend 2D to 3D

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Thank you for your attention !