Centrohermitian solutions of a factorization problem arising in vibration analysis. Part I: Lee’s transformation
Elisa Hubert, Yacine Bouzidi, Roudy Dagher, Alban Quadrat

To cite this version:
Elisa Hubert, Yacine Bouzidi, Roudy Dagher, Alban Quadrat. Centrohermitian solutions of a factorization problem arising in vibration analysis. Part I: Lee’s transformation. ECC 2021 - European Control Conference, Jun 2021, Delft (Virtual), Netherlands. 10.23919/ECC54610.2021.9655069 . hal-03530244

HAL Id: hal-03530244
https://inria.hal.science/hal-03530244
Submitted on 17 Jan 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
**Abstract**—Motivated by an application of vibration analysis to gearbox fault surveillance, a new demodulation approach for gearbox vibration signals has recently been developed. Within this approach, the demodulation problem yields the study of a rank factorization problem for centrohermitian matrices. In this paper, using the properties of centrohermitian matrices, we first show that the rank factorization problem for centrohermitian matrices can be transformed into a rank factorization problem for real matrices. Based on previous works, we then show how to parametrize a class of centrohermitian solutions of the rank factorization problem that is important in practice.

**Notation.** In what follows, let $k$ denote a field (e.g., $k = \mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$), $k^{n \times m}$ the $k$-vector space formed by all the $n \times m$ matrices with entries in $k$ and $I_n$ the identity matrix of $k^{n \times n}$. We also denote by $J_n$ the $n \times n$ exchange matrix, namely, the matrix formed by 1 on the second diagonal (i.e., the anti-diagonal) and 0 elsewhere. If $k = \mathbb{C}$, then $\overline{M}$ (resp., $M^*$) denotes the conjugate matrix (resp., the adjoint, namely, the conjugate transpose) of $M$. Finally, if $M \in k^{n \times m}$, then we can consider the following $k$-linear maps

$$M : k^{m \times 1} \rightarrow k^{n \times 1} \quad M : k^{1 \times n} \rightarrow k^{1 \times m}$$

and denote their kernels (resp., images) respectively by $\ker_k(M)$ and $\im_k(M)$ (resp., $\im_k(M)$ and $\ker_k(M)$).

**I. STATEMENT OF THE PROBLEM**

We first introduce the concept of a centrohermitian matrix.

**Definition 1** ([12], [5]): A matrix $M \in C^{n \times m}$ is called centrohermitian if $J_n \overline{M} J_n = M$. The set of all the centrohermitian matrices of $C^{n \times m}$ is denoted by $\text{CH}_{n,m}$.

Using $J_1 = 1$, a vector $u \in C^{1 \times 1}$ (resp., $v \in C^{1 \times m}$) is centrohermitian if $J_n \overline{u} J_n = u$ (resp., $\overline{v} J_m = v$).

The next examples show that centrohermitian vectors and matrices naturally appear while considering Fourier analysis of real signals (e.g., gearbox vibration signals) [6], [7].

**Example 1:** If $s : \mathbb{R} \rightarrow \mathbb{R}$ is a $T$-periodic integrable function, then $s$ can be expressed by its *Fourier series* [13]

$$s(t) = \sum_{j \in \mathbb{Z}} c_j(s) e^{2\pi i j t / T},$$

whose **Fourier coefficients** are defined by:

$$c_j(s) = \frac{1}{T} \int_0^T s(t) e^{-2\pi i j t / T} \, dt. \quad (1)$$

If $r \in \mathbb{Z} \setminus \{0\}$, then we can define the column vector

$$c = (c_r(s) \ldots c_0(s) \ldots c_{-r}(s))^T \in \mathbb{C}^{(2r+1) \times 1}$$

formed with the first Fourier coefficients $c_j(s)$ of $s$ centered around 0. Since $s$ is supposed to be real, using (1), we have

$$\forall j \in \mathbb{Z}, \quad c_j(s) = c_{-j}(\overline{s}) = c_{-j}(s), \quad (2)$$

which shows that $c$ is a centrohermitian column vector.

By Definition 1, a matrix $M$ is centrohermitian if $M$ is equal to the matrix obtained by reversing the rows and columns of $\overline{M}$. For instance, the following matrix

$$M = \begin{pmatrix} a & b & c \\ d & e & \overline{a} \\ \overline{c} & \overline{b} & \overline{a} \end{pmatrix},$$

where $a, b, c, d \in \mathbb{C}$ and $e \in \mathbb{R}$, is centrohermitian.

**Example 2:** We consider again Example 1. If we simply denote by $c_j$ the $j$th Fourier coefficient $c_j(s)$ of $s$ and fix $p, q \in \mathbb{Z}_{>0}$, then we can form the following complex matrix

$$M = \begin{pmatrix} c_q(2p+1)+p & \cdots & c_p & \cdots & c_{-q}(2p+1)+p \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_q(2p+1) & \cdots & c_0 & \cdots & c_{-q}(2p+1) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_q(2p+1)-p & \cdots & c_{-p} & \cdots & c_{-q}(2p+1)-p \end{pmatrix}$$

with the Fourier coefficients $c_j$ for $0 \leq |j| \leq q (2p+1)+p$.

Using (2), we can check that $M$ is a centrohermitian matrix.

In [6], [7], it was shown that the *gearbox spectrum* can be represented by a centrohermitian matrix $M$ and the problem of separating the time vibration signal into its main components amounts to estimating centrohermitian vectors $u$ and $v_1, \ldots, v_r$ that solve the following problem.

**The centrohermitian rank factorization problem:**

Let $D_1, \ldots, D_r \in \text{CH}_{n,n} \setminus \{0\}$ and $M \in \text{CH}_{n,m} \setminus \{0\}$. Determine $-$ when they exist $-$ vectors $u \in \text{CH}_{n,1}$ and $v_1, \ldots, v_r \in \text{CH}_{1,m}$ satisfying:

$$M = \sum_{i=1}^r D_i \, u \, v_i. \quad (3)$$

**Remark 1:** If a solution of (3) exists, then we have $M w = \sum_{i=1}^r D_i \, u \, (v_i \, w)$ for all $w \in k^{n \times 1}$, which implies that $\im_k(M) \subseteq \spn_k \{D_1 \, u, \ldots, D_r \, u\}$, and thus, we get:

$$\text{rank}_k(M) \leq \text{dim}_k(\spn_k \{D_1 \, u, \ldots, D_r \, u\}) \leq \min\{r, n\}.$$
The centrohermitian rank factorization problem can be generalized by considering general matrices \( M, D_1, \ldots, D_r \) and general vectors \( u \) and \( v_1, \ldots, v_r \) as follows.

**The rank factorization problem:**
Let \( D_1, \ldots, D_r \in \mathbb{C}^{n \times n} \setminus \{0\} \) and \( M \in \mathbb{C}^{n \times m} \setminus \{0\} \). Determine \( \lambda \) when they exist – vectors \( u \in \mathbb{C}^{n \times 1} \) and \( v_1, \ldots, v_r \in \mathbb{C}^{1 \times m} \) satisfying (3).

The rank factorization problem was solved for \( r = 1 \) and \( D_1 = I_n \) in [7], and for \( r = 2 \) and \( D_1 = I_n \) in [8]. In [9], the problem was solved with the assumption that the row vectors \( v_i \)’s are \( \mathbb{k} \)-linearly independent, i.e., the matrix \( v \) has full row rank. In our application, the vector \( v \) in \( \mathbb{C}^{1 \times n} \) contains the first \((n/2)\)th harmonics of a time signal (see Example 1). Hence, the assumption that \( v \) has full row rank means that the truncation of the underlying time signals at their first \((n/2)\)th harmonics are \( \mathbb{k} \)-linear independent. Based on module theory and computer algebra, local closed-form solutions were obtained in [10].

The goal of this paper is to show that the centrohermitian rank factorization problem can be reduced to the rank factorization for real matrices and real vectors. Hence, using the results obtained in [7], [9], [10], we can effectively solve the centrohermitian rank factorization problem when the row vectors \( v_i \)’s are supposed to be \( \mathbb{C} \)-linearly independent.

**II. CENTROHERMITIAN MATRICES**

If \( M_1, M_2 \in \mathbb{C}^{n \times n}, m_1 \in \mathbb{C}^{n \times 1}, m_2 \in \mathbb{C}^{1 \times n}, m_3 \in \mathbb{R} \), then we can check that a centrohermitian matrix \( M \in \mathbb{C}^{n \times m} \) has one of the following 4 forms [12]:

1) If \( n = 2t \) and \( m = 2s \), then:

\[
M = \begin{pmatrix}
M_1 & M_2 J_s \\
J_s M_2^T & J_s M_1^T
\end{pmatrix}.
\]

2) If \( n = 2t \) and \( m = 2s + 1 \), then:

\[
M = \begin{pmatrix}
M_1 & m_1 M_2 J_s \\
J_s M_2 & J_s m_1^T J_s M_1 J_s
\end{pmatrix}.
\]

3) If \( n = 2t + 1 \) and \( m = 2s \), then:

\[
M = \begin{pmatrix}
M_1 & M_2 J_s \\
m_2 & m_2 M_2^T J_s
\end{pmatrix}.
\]

4) If \( n = 2t + 1 \) and \( m = 2s + 1 \), then:

\[
M = \begin{pmatrix}
M_1 & m_1 M_2 J_s \\
m_2 & m_2 J_s m_3 \\
J_s M_2 & J_s m_1^T J_s M_1 J_s
\end{pmatrix}.
\]

We state again a few results on centrohermitian matrices. Clearly, if \( M \in \mathbb{C}^{n \times m} \) is a centrohermitian matrix, so are:

\( \overline{M}, M^T, M^*, J_n M, J_n M \).

Using \( J_n^2 = I_n \), the product of two centrohermitian matrices \( A \in \mathbb{C}^{n \times m} \) and \( B \in \mathbb{C}^{m \times p} \) is then also centrohermitian:

\[
\]

Hence, the set \( \text{CH}_{n,m} \) of all the centrohermitian matrices of \( \mathbb{C}^{n \times m} \) is stable under addition, multiplication by reals and multiplication, i.e., \( \text{CH}_{n,m} \) has an \( \mathbb{R} \)-algebra structure.

Finally, if \( M \in \text{CH}_{n,n} \) is non-singular, then \( M^{-1} \in \text{CH}_{n,n} \).

**Theorem 1 ([15], [12]):** If \( M \) is a centrohermitian, then so is its Moore-Penrose inverse \( M^+ \).

In what follows, we state Lee’s theorem [12] which explicitly characterizes \( \text{CH}_{n,n} \). To do that, we first introduce a few more definitions and simple results.

**Definition 2 ([12]):** The matrix \( Q \in \mathbb{C}^{n \times m} \) is called \( J \)-real if we have:

\[
J_n Q = Q.
\]

Hence, \( Q \) is \( J \)-real iff its columns are centrohermitian.

**Example 3:** The following complex matrices are \( J \)-real:

\[
Q_{2t} = \frac{1}{\sqrt{2}} \begin{pmatrix} I_t & i I_t \\ J_t & -i J_t \end{pmatrix} \in \mathbb{C}^{2t \times 2t}, \quad Q_{1} = 1,
\]

\[
Q_{2t+1} = \frac{1}{\sqrt{2}} \begin{pmatrix} I_t & 0 \\ 0 & -i J_t \end{pmatrix} \in \mathbb{C}^{(2t+1) \times (2t+1)}.
\]

We note that both matrices \( Q_{2t} \) and \( Q_{2t+1} \) are also unitary:

\[
Q_{2t}^* Q_{2t} = Q_{2t+1}^* Q_{2t+1} = I_{2t}, \quad Q_{2t}^* Q_{2t+1} = Q_{2t+1}^* Q_{2t+1} = I_{2t+1}.
\]

Let \( U \in \mathbb{C}^{m \times m} \) and \( V \in \mathbb{C}^{n \times n} \) be two non-singular \( J \)-real matrices. Using \( J_n^2 = I_n \), then we have:

\[
V = (J_n V J_n) = J_n U,
\]

\[
J_n V = V \iff V^{-1} J_n V = I_n \iff V^{-1} = V^{-1} J_n.
\]

If \( M \in \text{CH}_{n,m} \) and \( \varphi(M) := V^{-1} M U \), then we have:

\[
\varphi(M) = V^{-1} M U = (V^{-1} J_n) (J_n M (J_n U)) \quad = V^{-1} M U = \varphi(M),
\]

which shows that \( \varphi(M) \in \mathbb{R}^{n \times m} \), i.e., \( \varphi(M) \) is a real matrix.

Now, if \( N \in \mathbb{R}^{n \times m} \), then we have:

\[
J_n (V N (V^{-1} J_n)) = J_n V N U^{-1} J_n = V N U^{-1},
\]

which shows that \( V N U^{-1} \) is a centrohermitian matrix.

These two results show the following important result.

**Theorem 2 ([12]):** Let \( U \in \mathbb{C}^{m \times m} \) and \( V \in \mathbb{C}^{n \times n} \) be two non-singular \( J \)-real matrices. Then, the \( \mathbb{R} \)-linear application

\[
\varphi: \mathbb{C}^{n \times m} \rightarrow \mathbb{C}^{n \times m}, \quad M \mapsto V^{-1} M U,
\]

bijectively maps \( \text{CH}_{n,m} \) onto \( \mathbb{R}^{n \times m} \), i.e.:

\[
\varphi(\text{CH}_{n,m}) = \mathbb{R}^{n \times m}.
\]

\[
\forall N \in \mathbb{R}^{n \times m}, \exists! M \in \text{CH}_{n,m} \subseteq \mathbb{C}^{n \times m} \text{ such that } N = V^{-1} M U.
\]

Hence, \( \text{CH}_{n,m} = \varphi^{-1}(\mathbb{R}^{n \times m}) \), i.e., \( M \in \text{CH}_{n,m} \) iff there exists a unique \( N \in \mathbb{R}^{n \times m} \) such that \( N = V^{-1} M U \).

Finally, if \( n = m \), then \( \varphi \) isomorphically maps the ring \( \text{CH}_{n,m} \) onto the ring \( \mathbb{R}^{n \times n} \) and \( \varphi \) is a ring automorphism.

**Example 4:** Let \( Q_n \) (resp., \( Q_m \)) be defined as in Example 3. Then, \( Q_n^{-1} = Q_n^* \) since \( Q_n \) is unitary. If \( u \in \mathbb{C}^{n \times 1} \) is a centrohermitian column vector, i.e., \( u \in \text{CH}_{n,1} \), then, by
Theorem 2. \(u_\varphi := Q_n^* u\) is a real vector, i.e., \(u_\varphi \in \mathbb{R}^{n \times 1}\). Similarly, if \(v_i \in \mathbb{C}^{1 \times m}\), then \(v_i \varphi := v_i Q_m \in \mathbb{R}^{1 \times m}\).

Example 5: Let us consider the following matrix:

\[
M = \begin{pmatrix}
\frac{45}{2} + 5i & 10 - 15i & 15i & \frac{45}{2} + 10i \\
\frac{5}{2} \sqrt{3} - \frac{15}{2} \sqrt{2} i & 10 \sqrt{2} & 10 \sqrt{2} & \frac{5}{2} \sqrt{2} + \frac{15}{2} \sqrt{2} i \\
\frac{45}{2} - 10 i & -15 i & 10 + 15i & \frac{45}{2} - 5i
\end{pmatrix}
\]

Clearly, \(M \in \mathbb{C}^{3 \times 4}\). With the notations of Example 3, i.e.,

\[
Q_3 = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & i \\
0 & \sqrt{2} & 0 \\
1 & 0 & -i
\end{pmatrix},
\]

\[
Q_4 = \frac{1}{\sqrt{2}} \begin{pmatrix}
I_2 & i I_2 \\
J_2 & -i J_2
\end{pmatrix},
\]

we then have \(Q_3^{-1} = Q_3^*\) and:

\[
\varphi(M) = Q_3^* M Q_4 = \begin{pmatrix}
45 & 10 & 5 & 30 \\
5 & 20 & 15 & 0 \\
15 & 0 & 0 & 10
\end{pmatrix} \in \mathbb{R}^{3 \times 4}.
\]

Note that Theorem 2 yields \(\text{dim}_\mathbb{R}(\mathbb{C}^{m \times n}) = n m\).

If \(\lambda\) is an eigenvalue of \(M \in \mathbb{C}^{n \times n}\) with \(x\) an associated eigenvector, then \(M x = \lambda x\) yields \(M \overline{x} = \overline{\lambda x}\), and thus, \(M (J_n \overline{x}) = \overline{\lambda} (J_n \overline{x})\), which shows that \(\lambda\) is also an eigenvalue of \(M^*\) with \(J_n \overline{x}\) as an associated eigenvector.

Theorem 2 shows that a centrohermitian matrix is similar to a real matrix. Hence, the spectral theory of centrohermitian matrices corresponds to the one of real matrices.

Theorem 3 ([12]): If \(M\) is centrohermitian and \(\lambda\) is an eigenvalue of \(M\) of algebraic multiplicity \(k\), then \(\lambda \overline{\lambda}\) is an eigenvalue of \(M^*\) of algebraic multiplicity \(k\). In particular, if \(M \in \mathbb{C}^{n \times n}\) with \(n\) odd, then \(M\) always has a real eigenvalue. The characteristic polynomial of \(M \in \mathbb{C}^{n \times n}\) has thus all real coefficients, and thus, \(\text{det}(M) \in \mathbb{R}\).

Example 6: Let us consider the centrohermitian matrix:

\[
M = \begin{pmatrix}
9 + 18 i & -225 & 9 + 198 i \\
0 & 0 & 0 \\
9 - 198 i & -225 & -9 + 18 i
\end{pmatrix} \in \mathbb{C}^{3 \times 3}.
\]

Let us consider the \(J\)-real matrix \(Q_3\) defined in Example 5. Then, we have \(Q_3^{-1} = Q_3^*\) and:

\[
\varphi(M) = Q_3^* M Q_3 = \begin{pmatrix}
18 & -225 & 216 \\
0 & 0 & 0 \\
180 & 0 & 0
\end{pmatrix} \in \mathbb{R}^{3 \times 3}.
\]

The characteristic polynomial \(p(\lambda) = \lambda (\lambda^2 - 18 \lambda - 38880)\) of \(M\) has real coefficients and \(\text{det}(M) = 0 \in \mathbb{R}\).

Using Theorem 2, we obtain the following consequence.

Proposition 1 ([4]): Let \(M \in \mathbb{C}^{m \times m}\), \(Q_m \in \mathbb{C}^{m \times m}\) and \(Q_n \in \mathbb{C}^{n \times n}\) be two \(J\)-real and unitary matrices (e.g., as defined in Example 3), and \(M_\varphi := Q_n^* M Q_m \in \mathbb{R}^{n \times m}\). Let \(M_\varphi = U_\varphi \Sigma V_\varphi^*\) be a Singular Value Decomposition (SVD) of \(M_\varphi\), where \(U_\varphi \in \mathbb{R}^{n \times n}\) and \(V_\varphi \in \mathbb{R}^{m \times m}\) are two orthogonal matrices and \(\Sigma\) is the diagonal matrix formed by the singular values of \(M_\varphi\). Then, a SVD of \(M\) is given by:

\[
M = (Q_n U_\varphi) \Sigma (Q_m V_\varphi)^*.
\]

Hence, the singular values of \(M\) are equal to those of \(M_\varphi\) and the left and right singular vectors of \(M\) are centrohermitian.

### III. Transforming the Centrohermitian Rank Factorization Problem into a Real Rank Factorization Problem

In this section, we use Theorem 2 to study the centrohermitian rank factorization problem stated in Section I.

Let \(Q_m \in \mathbb{C}^{m \times m}\) and \(Q_n \in \mathbb{C}^{n \times n}\) be two \(J\)-real and unitary matrices (e.g., as defined in Example 3). Since \(M\) and \(D_1, \ldots, D_r\) are centrohermitian matrices, using Theorem 2, we can define the following real matrices:

\[
M_\varphi := Q_n^* M Q_m \in \mathbb{R}^{n \times m},
\]

\[
D_i \varphi := Q_n^* D_i Q_n \in \mathbb{R}^{n \times n}, \quad i = 1, \ldots, r.
\]

Note that we have:

\[
\text{rank}_\mathbb{C}(M) = \text{rank}_\mathbb{C}(M_\varphi) = \text{rank}_\mathbb{R}(M_\varphi).
\]

Let us suppose that the centrohermitian rank factorization problem (3) admits a centrohermitian solution \(u, v_1, \ldots, v_r\). Again, by Theorem 2, we have the following real vectors:

\[
\begin{align*}
    u_\varphi & := Q_n^* u \in \mathbb{R}^{n \times 1}, \\
v_i \varphi & := v_i Q_m \in \mathbb{R}^{1 \times m}, \quad i = 1, \ldots, r.
\end{align*}
\]

See also Example 4. Substituting \(M = Q_n M_\varphi Q_m^*\) and \(D_i = Q_n D_i \varphi Q_m^*\) for \(i = 1, \ldots, r\) into (3), we then obtain:

\[
\begin{align*}
    Q_n M_\varphi Q_m^* & = \sum_{i=1}^r Q_n D_i \varphi Q_m^* v_i \\
    \Leftrightarrow M_\varphi & = \sum_{i=1}^r D_i \varphi v_i \varphi u_\varphi \\
    \Leftrightarrow M_\varphi & = \sum_{i=1}^r D_i \varphi \varphi v_i \varphi u_\varphi,
\end{align*}
\]

which shows that the rank factorization problem for the real matrices \(M_\varphi\) and \(D_i \varphi\), \(i = 1, \ldots, r\), then admits a real solution \((u_\varphi, v_1 \varphi, \ldots, v_r \varphi)\).

Conversely, if there exists a real solution \(u_\varphi \in \mathbb{R}^{n \times 1}\), \(v_i \varphi \in \mathbb{R}^{1 \times m}\), \(i = 1, \ldots, r\), of the rank factorization problem

\[
M_\varphi = \sum_{i=1}^r D_i \varphi u_\varphi v_i \varphi,
\]

then, by Theorem 2 (see also Example 4), the vectors

\[
\begin{align*}
    u & := Q_n u_\varphi \in \mathbb{C}^{n \times 1}, \\
v_i & := v_i \varphi Q_m^* \in \mathbb{C}^{1 \times m}, \quad i = 1, \ldots, r,
\end{align*}
\]

satisfy (3), i.e., they define a centrohermitian solution of the centrohermitian rank factorization problem (3).

The above results prove the following new result.

Theorem 4: Let \(M \in \mathbb{C}^{n \times m}\), \(D_i \in \mathbb{C}^{n \times n}\), \(i = 1, \ldots, r\), \(Q_m \in \mathbb{C}^{m \times m}\) and \(Q_n \in \mathbb{C}^{n \times n}\) be two \(J\)-real and unitary matrices (e.g., as defined in Example 3), and:

\[
\begin{align*}
    M_\varphi & := Q_n^* M Q_m \in \mathbb{R}^{n \times m}, \\
    D_i \varphi & := Q_n^* D_i Q_n \in \mathbb{R}^{n \times n}, \quad i = 1, \ldots, r.
\end{align*}
\]
Then, the centrohermitian rank factorization problem (3) admits a solution \((u, v_1, \ldots, v_r) \in \mathbb{C}_{n,1} \times \mathbb{C}_{r,m}^{*}\) if and only if the rank factorization problem (5) admits a solution \((u_{\psi}, v_1, \ldots, v_{r_{\psi}}) \in \mathbb{R}^{n \times 1} \times (\mathbb{R}^{1 \times m})^r\). Finally, the bijections between the two sets of solutions is given by:

\[
\begin{align*}
\mathbb{C}_{n,1} \times \mathbb{C}_{r,m}^{*} & \quad \mapsto \quad \mathbb{R}^{n \times 1} \times (\mathbb{R}^{1 \times m})^r \\
(u, v_1, \ldots, v_r) & \quad \mapsto \quad (Q_u u, v_i Q_m, \ldots, v_r Q_m), \\
\mathbb{R}^{n \times 1} \times (\mathbb{R}^{1 \times m})^r & \quad \mapsto \quad \mathbb{C}_{n,1} \times \mathbb{C}_{r,m}^{*} \\
(u_{\psi}, v_1, \ldots, v_{r_{\psi}}) & \quad \mapsto \quad (Q_u u_{\psi}, v_i Q_m^{*}, \ldots, v_{r_{\psi}} Q_m^{*}).
\end{align*}
\]

IV. A CLASS OF SOLUTIONS OF THE CENTROHERMITIAN RANK FACTORIZATION PROBLEM

Theorem 4 shows that centrohermitian solutions of the centrohermitian rank factorization problem are in a one-to-one correspondence with real solutions of the rank factorization problem (i.e., \(k = \mathbb{R}\)). As stated in Section I, a class of solutions of the rank factorization problem (namely, the solutions such that \(v = (v_1^T, \ldots, v_r^T)^T\) has full row rank) – important in practice – can be explicitly parametrized [9], [10]. Hence, we can parametrize the class of centrohermitian solutions of the centrohermitian rank factorization problem for which \(v\) has full row rank. Let us explain how to do it.

Let us state again results on the rank factorization problem (see Section I) obtained in [9], [10]. Let \(k\) be a field (e.g., \(k = \mathbb{Q}, \mathbb{R}, \mathbb{C}\)), \(M \in \mathbb{K}^{n \times m}\) and \(l := \operatorname{rank}_k(M)\). Stacking a basis of \(\operatorname{im}(M) := M \mathbb{K}^{m \times 1}\) into a full column rank matrix \(X \in \mathbb{K}^{n \times l}\), there exists a unique \(Y \in \mathbb{K}^{l \times m}\) such that:

\[
M = XY. \tag{7}
\]

Denoting

\[
\begin{align*}
A(u) := (D_1 u \ldots D_r u) & \in \mathbb{K}^{n \times r}, \\
v := (v_1^T \ldots v_r^T)^T & \in \mathbb{K}^{r \times m},
\end{align*}
\]

then (3) can be rewritten as follows:

\[
A(u) v = M. \tag{8}
\]

The bilinear structure of (8) implies that \((\lambda u, \lambda^{-1} v)\) is a solution for all solutions \((u, v)\) and for all \(\lambda \in \mathbb{K} \setminus \{0\}\). Hence, if a solution exists for the rank factorization problem, then it is not unique. Moreover, we note that (8) is equivalent to the existence of \(u \in \mathbb{K}^{n \times 1}\) such that \(\operatorname{im}_k(M) \subseteq \operatorname{im}_k(A(u))\).

In what follows, we shall investigate the class of solutions \((u, v)\) of (8) defined by a full row rank matrix \(v\). If such a solution of (8) exists, then \(v\) admits a right inverse since \(v\) has full row rank, i.e., there exists \(t \in \mathbb{K}^{m \times r}\) such that \(v t = I_r\). Hence, (8) yields \(A(u) = M t\), which shows that \(\operatorname{im}_k(A(u)) \subseteq \operatorname{im}_k(M)\), and thus, we obtain:

\[
\operatorname{im}_k(A(u)) \subseteq \operatorname{im}_k(M). \tag{9}
\]

Thus, the existence of a solution \((u, v)\) of (8), where \(v\) has full row rank, yields (9). Hence, let us focus on the existence of \(u \in \mathbb{K}^{n \times 1}\) satisfying (9). We note that (9) is equivalent to:

\[
\exists u \in \mathbb{K}^{n \times 1} \left\{ D_i u \in \operatorname{im}_k(M), \ i = 1, \ldots, r, \right.
\]

\[
\left. \operatorname{rank}_k(D_1 u \ldots D_r u) = \operatorname{rank}_k(M) = l. \right\} \tag{10}
\]

We note that \(v\) has been eliminated, i.e., we have transformed the polynomial problem (8) into a pure linear algebra problem (10). This last problem can easily be solved as follows.

If \(\operatorname{im}_k(M) = \mathbb{K}^{n \times 1}\), then we let \(L := 0\). Else let \(L \in \mathbb{K}^{p \times n}\) be such that \(\ker_k(L) = \operatorname{im}_k(M)\). The first condition of (10) is equivalent to the following linear system on \(u\):

\[
\begin{align*}
(L D_1) u & = 0, \\
\vdots \\
(L D_r) u & = 0.
\end{align*} \tag{11}
\]

If the only solution of (11) is \(u = 0\), then (9) has no solution since \(A(0) = 0\) and \(M \neq 0\). Hence, let \(Z \in \mathbb{K}^{n \times d}\) be a full column matrix whose columns form a basis of the \(k\)-vector space (11). Note that if \(\operatorname{im}_k(M) = \mathbb{K}^{n \times 1}\), and thus, \(L = 0\), then we get \(Z = I_n\). Hence, the vectors \(u = Z \psi\) satisfy the first condition of (10) for all \(\psi \in \mathbb{K}^{d \times 1}\). Now, we note that \(D_i Z \psi \in \operatorname{im}_k(M) = \operatorname{im}_k(X)\) for all \(\psi \in \mathbb{K}^{d \times 1}\) and \(i = 1, \ldots, r\), which shows the existence of unique matrices \(W_i \in \mathbb{K}^{d \times d}\) satisfying the following identities:

\[
D_i Z = X W_i, \quad i = 1, \ldots, r.
\]

Let us note:

\[
\forall \psi \in \mathbb{K}^{d \times 1}, \quad (B(\psi) := (W_1 \psi \ldots W_r \psi) \in \mathbb{K}^{l \times r}).
\]

By assumption, \(l := \operatorname{rank}_k(M) \leq r\) (see Remark 1), which shows that \(B\) is a wide matrix. The second condition of (10) then becomes \(A(Z \psi) = (D_1 Z \psi \ldots D_r Z \psi) = X B(\psi)\) for all \(\psi \in \mathbb{K}^{d \times 1}\). Since \(X\) has full column rank, we get:

\[
\operatorname{rank}_k(A(Z \psi)) = \operatorname{rank}_k(B(\psi)).
\]

The second condition of (10) is equivalent to \(\psi \in \mathcal{P}\), where:

\[
\mathcal{P} := \{ \psi \in \mathbb{K}^{d \times 1} | \operatorname{rank}_k(B(\psi)) = l \}. \tag{12}
\]

If \(\psi \in \mathcal{P}\), then using (7) and the fact that \(X\) has full column rank, then we obtain:

\[
A(Z \psi) v = M \Leftrightarrow X B(\psi) v = X Y \Leftrightarrow B(\psi) v = Y. \tag{13}
\]

Moreover, since \(\operatorname{rank}_k B(\psi) = l, B(\psi) \in \mathbb{K}^{l \times r}\) and \(l \leq r, B(\psi)\) has full row rank, which shows that \(B(\psi)\) admits a right inverse \(E_\psi \in \mathbb{K}^{r \times l}\), i.e., \(B(\psi) E_\psi = I_l\). Hence, \(v = E_\psi Y\) is a particular solution of (13). If \(C_\psi \in \mathbb{K}^{l \times (r-l)}\) is a full column matrix whose columns form a basis of \(\ker_k(B(\psi))\), then the general solution of (13) is given by:

\[
\forall Y' \in \mathbb{K}^{(r-l) \times m}, \quad v = (E_\psi C_\psi) \left( \begin{array}{c} Y \\ Y' \end{array} \right).
\]

Finally, we note that \((E_\psi \ C_\psi)\) is invertible, i.e., is a nonsingular matrix, which shows that \(v\) has full row rank iff so has the matrix \((Y^T \ Y'^T)^T \in \mathbb{K}^{r \times m}\). We get the theorem.

**Theorem 5 (9):** With the above notations, (9) holds iff (12) is not empty. If so, then for every \(\psi \in \mathcal{P}\),

\[
\forall Y' \in \mathbb{K}^{(r-l) \times m}, \quad \left\{ \begin{array}{l}
u = Z \psi, \\
v = (E_\psi C_\psi) \left( \begin{array}{c} Y \\ Y' \end{array} \right).
\end{array} \right.
\]
are solutions of \((8)\), and thus, solutions of \((3)\). Finally, \(v\) has full row rank iff so has \((Y^T \ Y^{rT})^T \in \mathbb{R}^{r \times m}\).

Set \(C_r^l := r!/(l! \ (r-l)!l)\) and let \(\{m_k(\psi)\}_{k=1}^l \in C_r^l\) denote the set of all \(l \times l\)-minors of \(B(\psi)\). Note that the \(m_k\)'s are either 0 or homogeneous polynomials of degree \(l\), namely:

\[
\forall \lambda \in \mathbb{k} \setminus \{0\}, \quad m_k(\lambda x) = \lambda^l m_k(x), \quad k = 1, \ldots, C_r^l.
\]

Hence, we clearly have:

\[
P = \mathbb{k}^{d \times 1} \setminus \{\psi \in \mathbb{k}^{d \times 1} | m_k(\psi) = 0, \ k = 1, \ldots, C_r^l\}.
\]

Since \(C_r^l\) grows exponentially with \(r\), a more efficient characterization of \(P\) was obtained in [10] based on module theory and computer algebra (Gröbner bases [3], [10]).

According to Theorem 4, a particular class of solutions of the centrohermitian rank factorization problem can be obtained by applying Theorem 5 to (6) with \(k = \mathbb{R}\). Hence, we obtain the following new result.

**Corollary 1:** Let \(Q_n\) and \(Q_m\) be two J-real unitary matrices (e.g., as defined in Example 3), \(M \in \mathbb{CH}_{m,n}\) and \(D_i \in \mathbb{CH}_{n,n}\) for \(i = 1, \ldots, r\), and \(M_\varphi \in \mathbb{R}^{n \times m}\) and \(D_{i\varphi} \in \mathbb{R}^{n \times n}\), \(i = 1, \ldots, r\), defined by (6). Moreover, if

\[
\left\{\begin{array}{l}
u = Q_n u_\varphi, \\
u = (E_\varphi \psi) \ C_\varphi\psi \ (Y \ Y^T),
\end{array}\right.
\]

denotes the solutions of the rank factorization problem (5) for \(k = \mathbb{R} - \) defined by Theorem 5 where \(Y^* \in \mathbb{R}^{(r-l) \times m}\) and \(\psi \in P := \{\psi \in \mathbb{k}^{d \times 1} | \text{rank}_k B_\varphi(\psi) = l\}\), then

\[
\left\{\begin{array}{l}
u = Q_n u_\varphi, \\
u = (E_\varphi \psi) \ C_\varphi\psi \ (Y \ Y^T)
\end{array}\right.,
\]

is a centrohermitian solution of \((3)\). Finally, \(v\) has full row rank iff so has \((Y^T \ Y^{rT})^T \in \mathbb{R}^{r \times m}\).

The last point of Corollary 1 comes from the fact that \(\lambda \nu = \lambda \nu_\varphi Q_m^\ast = 0\) iff \(\lambda \nu_\varphi = 0\), and thus, we get \(\lambda \nu_\varphi = 0\) iff \(v_\varphi = 0\) has full row rank, which is characterized in Theorem 5.

**Example 7:** We consider again the matrix \(M \in \mathbb{CH}_{3,4}\) defined in Example 5. Let us consider the following matrices:

\[
D_1 = \frac{1}{2} \begin{pmatrix}
-1 & 0 & i \\
0 & 0 & 0 \\
-1 & 0 & i
\end{pmatrix} \in \mathbb{CH}_{3,3},
\]

\[
D_2 = \frac{3i}{\sqrt{2}} \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & -1 & 0
\end{pmatrix} \in \mathbb{CH}_{3,3},
\]

\[
D_3 = \frac{i}{\sqrt{2}} \begin{pmatrix}
0 & 0 & 0 \\
-1 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix} \in \mathbb{CH}_{3,3},
\]

\[
D_4 = \frac{1}{2} \begin{pmatrix}
1 - 3i & 0 & -1 + 3i \\
0 & 0 & 0 \\
-1 - 3i & 0 & 1 + 3i
\end{pmatrix} \in \mathbb{CH}_{3,3}.
\]

Using (6), we have:

\[
\begin{align*}
D_{1\varphi} = Q_3^* \ D_1 \ Q_3 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{3 \times 3}, \\
D_{2\varphi} = Q_3^* \ D_2 \ Q_3 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \in \mathbb{R}^{3 \times 3}, \\
D_{3\varphi} = Q_3^* \ D_3 \ Q_3 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{3 \times 3}, \\
D_{4\varphi} = Q_3^* \ D_4 \ Q_3 &= \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{3 \times 3}.
\end{align*}
\]

Let \(M_\varphi\) denote the matrix \(\varphi(M)\) defined in Example 5. Then, we can solve (5) using Theorem 5. We can easily check that \(l := \text{rank}_k (M_\varphi) = \text{rank}_k (M) = 3 < r = 4\). We then get:

\[
X_\varphi = \begin{pmatrix}
45 & 10 & 5 \\
5 & 20 & 15 \\
15 & 0 & 0
\end{pmatrix} \in \mathbb{R}^{3 \times 3},
\]

\[
Y_\varphi = \frac{1}{3} \begin{pmatrix}
1 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & 1
\end{pmatrix} \in \mathbb{R}^{3 \times 3},
\]

\[
L_\varphi = 0, \quad Z_\varphi = I_3, \quad d_\varphi = 3, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \in \mathbb{R}^{3 \times 1},
\]

\[
W_{1\varphi} = \frac{1}{10} \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 3 \\
0 & 0 & -4
\end{pmatrix}, \quad W_{2\varphi} = \frac{1}{5} \begin{pmatrix}
0 & 1 & 0 \\
0 & -4 & 0 \\
17 & 0 & 0
\end{pmatrix},
\]

\[
W_{3\varphi} = \frac{1}{10} \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 0 & 0
\end{pmatrix}, \quad W_{4\varphi} = \frac{1}{30} \begin{pmatrix}
0 & 0 & 2 \\
0 & 0 & 1 \\
0 & 0 & -2
\end{pmatrix},
\]

\[
B_\varphi(\psi) = \frac{1}{30} \begin{pmatrix}
0 & 6 \psi_2 & 0 \\
-78 \psi_2 & -3 \psi_3 & \psi_3 \\
-12 \psi_3 & 102 \psi_2 & 6 \psi_3 & -2 \psi_3
\end{pmatrix},
\]

\[
P = \mathbb{R}^{3 \times 1} \setminus \{(\psi_1 \psi_2 \psi_3)^T | \psi_3 = 0\} = \mathbb{R} \times \mathbb{R} \times (\mathbb{R} \setminus \{0\}),
\]

\[
E_{\psi_3} = \frac{1}{\psi_3} \begin{pmatrix}
0 & 10 & 5 \\
0 & 0 & 0 \\
15 & 0 & 0
\end{pmatrix}, \quad C = \begin{pmatrix} 9 \psi_2 \\ \psi_3 \\ 0 \end{pmatrix},
\]
Real solutions of $M_{\varphi} = \sum_{i=1}^{4} D_{i,\varphi} u_{i,\varphi} v_{i,\varphi}$ are then given by:

$$
\forall \psi \in \mathcal{P}, \forall Y' = (y_1', y_2', y_3', y_4') \in \mathbb{R}^{1 \times 4},
$$

$$
u_{\varphi} = Z_{\varphi} \psi = \psi,
$$

$$v_{\varphi} = E_{\varphi} Y + CSV' =
\begin{pmatrix}
9 \psi_3 y_1' & 10 \psi_3 y_1' + 9 \psi_2 y_2' / \psi_3 & 5 \psi_3 + 9 \psi_2 y_3' & 9 \psi_2 y_4' \\
\psi_3 y_1' & \psi_3 y_2' & \psi_3 y_3' & \psi_3 y_4' \\
5 / \psi_3 & 20 / \psi_3 & 15 / \psi_3 & 0 \\
15 / \psi_3 - 3 \psi_2 y_1' & -3 \psi_2 y_2' & -3 \psi_2 y_3' & 10 / \psi_3 - 3 \psi_2 y_4'
\end{pmatrix}
$$

Moreover, $v_{\varphi}$ has full row rank iff det $(Y'TY'T) \neq 0$, i.e., iff $Y' = (y_1', y_2', y_3', y_4') \in \mathbb{R}^{1 \times 4}$ is chosen such that:

$$2 y_1' + 2 y_2' - 3 y_3' - 3 y_4' \neq 0. \quad (15)$$

Now, using (14) (see Corollary 1), centrohermitian solutions of $M = \sum_{i=1}^{4} D_i u v_i$ are then defined by

$$\forall \psi \in \mathcal{P}, \forall Y' = (y_1', y_2', y_3', y_4') \in \mathbb{R}^{1 \times 4},$$

$$u = Q_3 \psi = \begin{pmatrix}
\sqrt{3} / 2 \\ \psi_2 \\
\sqrt{3} / 2 \\ -\sqrt{2} / 2 \\ \psi_3 \\
\end{pmatrix},
$$

$$v = \begin{pmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4 \\
\end{pmatrix} = v_{\varphi} Q_4^*,$$

$$v_1 = \frac{1}{\sqrt{2} \psi_3} (9 \psi_2 \psi_3 y_1' - 9 \psi_2 \psi_3 y_5' i - 5 i + 10 + 9 \psi_2 \psi_3 y_2' - 9 \psi_2 \psi_3 y_4' i + 10 + 9 \psi_2 \psi_3 y_3' + 9 \psi_2 \psi_3 y_4' i + 5 i),$$

$$v_2 = \frac{1}{2} \psi_3 (y_1' \sqrt{2} - y_3' \sqrt{2} i + y_2' \sqrt{2} - y_4' \sqrt{2} i),$$

$$v_3 = \frac{5}{2} \psi_3 (1 \psi_2 \sqrt{2} - 4 \sqrt{2} i + \psi_3 \sqrt{2} i),$$

$$v_4 = \frac{1}{\sqrt{2} \psi_3} (15 - 3 \psi_2 \psi_3 y_1' + 3 \psi_2 \psi_3 y_3' i - 3 \psi_2 \psi_3 y_2' + 3 \psi_2 \psi_3 y_3' i - 10 i - 3 \psi_2 \psi_3 y_2' - 3 \psi_2 \psi_3 y_3' i + 10 i - 15 - 3 \psi_2 \psi_3 y_1' - 3 \psi_2 \psi_3 y_3' i).$$

Finally, the matrix $v$ has full row rank iff the row vector $Y' = (y_1', y_2', y_3', y_4') \in \mathbb{R}^{1 \times 4}$ satisfies (15).

All the results developed in this paper can be easily implemented using, e.g., the OREMULDES package [3].

**V. CONCLUSION**

In this paper, using the structure of centrohermitian matrices and Lee’s theorem [12], we proved that the centrohermitian rank factorization problem can be reduced to a real rank factorization one. Using results of [9], [10], we then show how to parametrize a class of solutions which are important in practice. In [11], we develop an alternative approach to Lee’s transformation based on the so-called coinvolutary matrices. This alternative approach brings new insights.

An important issue that will be investigated in a future publication is the following minimization problem

$$\min_{u \in CH_{n,1}(C), v_i \in CH_{1,m}(C)} \sum_{i=1}^{4} D_i u v_i - M \|_{\text{Frob}},$$

where $\| A \|_{\text{Frob}} = \sqrt{\text{trace}(A^* A)}$ stands for the standard Frobenius norm. If $U$ and $V$ are two unitary matrices, using the cyclic property of the trace, then we have:

$$\| U A V \|_{\text{Frob}} = \sqrt{\text{trace}(V^* A^* U^* U A V)} = \sqrt{\text{trace}(V^* A^* A)} = \sqrt{\text{trace}(A^* A)} = \| A \|_{\text{Frob}}.$$

Hence, using the fact that the matrices $Q_n$ and $Q_m$ in Theorem 4 are unitary, we then obtain

$$16 \leftrightarrow \min_{u_{\varphi} \in \mathbb{R}^{n \times 1}, v_{\varphi} \in \mathbb{R}^{1 \times n}} \sum_{i=1}^{4} D_i u_{\varphi} v_{\varphi} - M_{\varphi} \|_{\text{Frob}},$$

which shows that (16) can be reduced to the search for real solutions to a polynomial optimization problem.

**REFERENCES**


