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Convergence analysis of HDG formulations for the convected Helmholtz equation

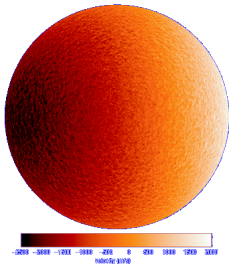
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Joint work with: H. Barucq, S. Tordeux

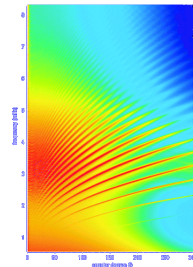
Inria Makutu – e2s-UPPA – LMAP

Helioseismology in a nutshell:

- ▶ Aims at imaging the solar interior thanks to surface observations
- ▶ Surfacic «acoustic waves» can be measured thanks to relativistic Doppler effect
- ▶ The full models are too complicated for numerical simulation, two main approaches for computational helioseismology:
 - ▶ Aeroacoustics: magnetic effects are neglected
 - ▶ Magnetoacoustics: hydrodynamics effects are neglected



(a) Dopplergram



(b) Power spectrum

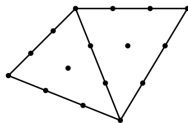


1. What is HDG ?
2. HDG methods for the convected Helmholtz equation
3. Numerical results

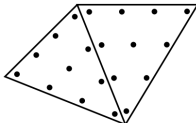
What is HDG ?

From CG to DG:

- ▶ Continuity of the solution between two elements is not enforced by the choice of approximation spaces anymore
- ▶ Continuity and stability come from the choice of numerical flux between two elements (similar to the finite volume method)
- ▶ Advantages
 - ▶ *hp*-adaptativity
 - ▶ high-order
 - ▶ parallelization
- ▶ Disadvantage
 - ▶ large number of degrees of freedom



(c) CG



(d) DG

Figure: Degrees of freedom for degree 3

From CG to DG:

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From DG to HDG:

- ▶ Introduce a new unknown on the skeleton of the mesh
- ▶ Interior degrees of freedom can be eliminated through static condensation
- ▶ Keep the advantages of DG for a reduced cost

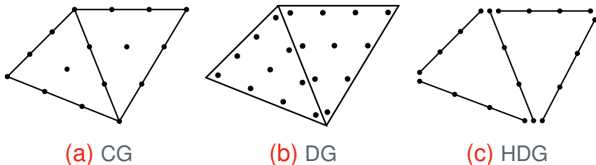


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From DG to HDG:

- ▶ Introduce a new unknown on the skeleton of the mesh
- ▶ Interior degrees of freedom can be eliminated through static condensation
- ▶ Keep the advantages of DG for a reduced cost

HDG methods have a dual citizenship:

- ▶ Static condensation of DG methods
- ▶ Also related to the hybridization of mixed FEM (eg Raviart-Thomas or Brezzi-Douglas-Marini methods)

HDG methods for the convected Helmholtz equation

Model problem :

$$\rho_0 (-\omega^2 p - 2i\omega \mathbf{v}_0 \cdot \nabla p + \mathbf{v}_0 \cdot \nabla(\mathbf{v}_0 \cdot \nabla p)) - \operatorname{div}(\rho_0 c_0^2 \nabla p) = s$$

where

- ▶ p : acoustic potential (\sim pressure perturbation)
- ▶ ρ_0 : mass density
- ▶ \mathbf{v}_0 : velocity field
- ▶ c_0 : sound speed
- ▶ s : acoustic source

Model problem :

$$\rho_0 (-\omega^2 \mathbf{p} - 2i\omega \mathbf{v}_0 \cdot \nabla \mathbf{p}) - \operatorname{div} (\mathbf{M}_0 \nabla \mathbf{p}) = \mathbf{s}$$

where $\mathbf{M}_0 = \rho_0 c_0^2 \operatorname{Id} - \rho_0 \mathbf{v}_0 \mathbf{v}_0^T$.

Model problem :

- ▶ Choice 1: *diffusive flux*

$$\begin{aligned}W_0 \mathbf{q} + \nabla p &= 0, \\ -\rho_0 \omega^2 p - 2i\omega \rho_0 \mathbf{v}_0 \cdot \nabla p + \operatorname{div}(\mathbf{q}) &= s\end{aligned}$$

where $\mathbf{q} = -\mathbf{M}_0 \nabla p$ and $W_0 = \mathbf{M}_0^{-1}$.

- ▶ Choice 2: *total flux*

$$\begin{aligned}W_0 \boldsymbol{\sigma} + \nabla p + 2i\omega p W_0 \rho_0 \mathbf{v}_0 &= 0, \\ -\rho_0 \omega^2 p + \operatorname{div}(\boldsymbol{\sigma}) &= s,\end{aligned}$$

where $\boldsymbol{\sigma} = -\mathbf{M}_0 \nabla p - 2i\omega p \rho_0 \mathbf{v}_0$.

To use HDG methods, we need a first-order in space formulation.

Model problem :

Seek $(\mathbf{q}, p) \in \mathbf{H}_{\text{div}}(\Omega) \times H^1(\Omega)$ such that

$$\begin{aligned} \int_{\Omega} \mathbf{W}_0 \mathbf{q} \cdot \mathbf{r}^* \, dx - \int_{\Omega} p \operatorname{div}(\mathbf{r}^*) \, dx + \int_{\partial\Omega} p \mathbf{r}^* \cdot \mathbf{n} \, d\sigma = 0 \\ -\omega^2 \int_{\Omega} \rho_0 p w^* \, dx + 2i\omega \int_{\Omega} p \rho_0 \mathbf{v}_0 \cdot \nabla w^* \, dx \\ - \int_{\Omega} \mathbf{q} \cdot \nabla w^* \, dx + \int_{\partial\Omega} w^* \mathbf{q} \cdot \mathbf{n} - 2i\omega p w^* \rho_0 \mathbf{v}_0 \cdot \mathbf{n} \, d\sigma = \int_{\Omega} s w^* \, dx \end{aligned}$$

for all $(\mathbf{r}, w) \in \mathbf{H}_{\text{div}}(\Omega) \times H^1(\Omega)$.

Very similar if we use $\boldsymbol{\sigma}$ instead of \mathbf{q} .

Convected Helmholtz equation



Model problem :

Seek $(\mathbf{q}, p) \in \mathbf{H}_{\text{div}}(\Omega) \times H^1(\Omega)$ such that

$$\begin{aligned} \int_{\Omega} \mathbf{W}_0 \mathbf{q} \cdot \mathbf{r}^* \, dx - \int_{\Omega} p \operatorname{div}(\mathbf{r}^*) \, dx + \int_{\partial\Omega} p \mathbf{r}^* \cdot \mathbf{n} \, d\sigma = 0 \\ - \omega^2 \int_{\Omega} \rho_0 p w^* \, dx + 2i\omega \int_{\Omega} p \rho_0 \mathbf{v}_0 \cdot \nabla w^* \, dx \\ - \int_{\Omega} \mathbf{q} \cdot \nabla w^* \, dx + \int_{\partial\Omega} w^* \mathbf{q} \cdot \mathbf{n} - 2i\omega p w^* \rho_0 \mathbf{v}_0 \cdot \mathbf{n} \, d\sigma = \int_{\Omega} s w^* \, dx \end{aligned}$$

for all $(\mathbf{r}, w) \in \mathbf{H}_{\text{div}}(\Omega) \times H^1(\Omega)$.

Well-posedness:

- ▶ if the carrier flow is subsonic

$$\inf_{\Omega} (c_0^2 - |\mathbf{v}_0|^2) > 0$$

then the problem is of Fredholm type

- ▶ uniqueness of the solution comes from the boundary conditions.

Skeleton unknown: numerical trace λ_h approximates p on \mathcal{E}_h .

$$\int_{\partial K} p \mathbf{r}^* \cdot \mathbf{n} d\sigma \xrightarrow{\text{discretization}} \int_{\partial K} \lambda_h \mathbf{r}^* \cdot \mathbf{n} d\sigma$$

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Approximation spaces:

Unknown	Space	HDG	HDG+
Pressure p_h	$W_h(K)$	$\mathcal{P}_k(K)$	$\mathcal{P}_{k+1}(K)$
Flux \mathbf{q}_h/σ_h	$\mathbf{V}_h(K)$	$\mathcal{P}_k(K)$	
Trace λ_h	$M_h(e)$	$\mathcal{P}_k(e)$	

Warning: HDG+ only works in (\mathbf{q}, p) formulation !

We will compare the HDG- \mathbf{q}_h , HDG- σ_h and HDG+ methods for time-harmonic convected waves.

Skeleton unknown: numerical trace λ_h approximates p on \mathcal{E}_h .

$$\int_{\partial K} p \mathbf{r}^* \cdot \mathbf{n} d\sigma \xrightarrow{\text{discretization}} \int_{\partial K} \lambda_h \mathbf{r}^* \cdot \mathbf{n} d\sigma$$

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Trace λ_h	$M_h(e)$	$\mathcal{P}_k(e)$	

Together with the weak formulation: leads to the discrete **local** problem (ie on an element $K \in \mathcal{T}_h$) :

$$\mathbb{A}^K \begin{bmatrix} p_h^K \\ \mathbf{q}_h^K \end{bmatrix} + \mathbb{C}^K [\lambda_h^K] = \mathbb{S}^K$$



At this point:

- ▶ The approximation spaces are discontinuous.
- ▶ We need to enforce the continuity of the numerical solution between the elements.
- ▶ We need to ensure the stability of the method.

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All of this is done by choosing the numerical flux, so that

$$\sum_{e \in \mathcal{E}_h} \int_e \llbracket \widehat{\mathbf{q}}_h - 2i\omega \mathbf{n} \cdot \widehat{\mathbf{v}}_0 \widehat{\boldsymbol{\rho}}_h \mathbf{n} \rrbracket \mu^* d\sigma = 0, \quad \text{or} \quad \sum_{e \in \mathcal{E}_h} \int_e \llbracket \widehat{\boldsymbol{\sigma}}_h \rrbracket \mu^* d\sigma = 0,$$

for all $\mu \in M_h$ and where $\llbracket \cdot \rrbracket$ is the DG-jump operator.

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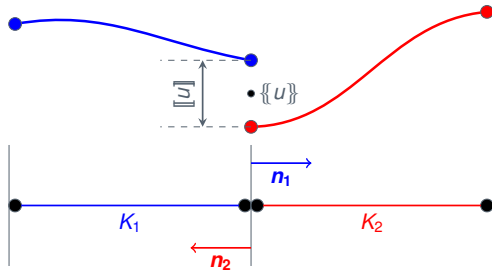


Figure: Discontinuity between two elements

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Numerical flux:

$$\begin{aligned} \widehat{\mathbf{q}}_h^K \cdot \mathbf{n} - 2i\omega \widehat{\rho}_h^K \widehat{\mathbf{v}}_0 \cdot \mathbf{n} &= \mathbf{q}_h^K \cdot \mathbf{n} - 2i\omega (\mathbf{v}_0 \cdot \mathbf{n}) \lambda_h^K + 2i\omega \tau (\mathcal{P}_k \rho_h^K - \lambda_h^K) \\ \widehat{\boldsymbol{\sigma}}_h^K \cdot \mathbf{n} &= \boldsymbol{\sigma}_h^K \cdot \mathbf{n} + i\omega \tau (\rho_h^K - \lambda_h^K) \end{aligned}$$

where

- ▶ τ : penalization parameter
- ▶ \mathcal{P}_k : L^2 -projection from $\mathcal{P}_{k+1}(e)$ to $\mathcal{P}_k(e)$

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$$\sum_{e \in \mathcal{E}_h} \int_e [\widehat{\mathbf{q}}_h - 2i\omega \mathbf{n} \cdot \widehat{\mathbf{v}}_0 \rho_h \mathbf{n}] \mu^* d\sigma = 0, \quad \text{or} \quad \sum_{e \in \mathcal{E}_h} \int_e [\widehat{\boldsymbol{\sigma}}_h] \mu^* d\sigma = 0,$$

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This leads to the discrete transmission condition:

$$\sum_{K \in \mathcal{T}_h} \left(\mathbb{B}^K \begin{bmatrix} \rho_h^K \\ \mathbf{q}_h^K \end{bmatrix} + \mathbb{L}^K [\lambda_h^K] \right) = 0$$

Recall from the previous slides:

- ▶ Local problem on $K \in \mathcal{T}_h$

$$\mathbb{A}^K \begin{bmatrix} p_h^K \\ \mathbf{q}_h^K \end{bmatrix} + \mathbb{C}^K [\lambda_h^K] = \mathbb{S}^K$$

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Dimensions of the unknowns: Notice that

$$\begin{bmatrix} \mathbf{p}_h^K \\ \mathbf{q}_h^K \end{bmatrix} \text{ is volumetric and } [\lambda_h^K] \text{ is surfacic.}$$

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Static condensation:

$$\mathbb{M} [\lambda_h] = \mathbb{S} \begin{array}{c} \xleftarrow{\text{elimination}} \\ \xrightarrow{\text{reconstruction}} \end{array} \begin{bmatrix} p_h^K \\ \mathbf{q}_h^K \end{bmatrix} = (\mathbb{A}^K)^{-1} \mathbb{S}^K - (\mathbb{A}^K)^{-1} \mathbb{C}^K [\lambda_h^K],$$

with

$$\mathbb{M} := \sum_{K \in \mathcal{T}_h} (\mathbb{L}^K - \mathbb{B}^K (\mathbb{A}^K)^{-1} \mathbb{C}^K) \quad \text{and} \quad \mathbb{S} := - \sum_{K \in \mathcal{T}_h} \mathbb{B}^K (\mathbb{A}^K)^{-1} \mathbb{S}^K$$



We performed the numerical analysis of the method:

- ▶ **Local solvability:** \mathbb{A}^K is invertible if
 - ▶ $\omega h_K < C(\rho_0, \mathbf{v}_0, \mathbf{c}_0, \text{element geometry}),$
 - ▶ τ is well chosen

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 - ▶ τ is well chosen
 - for HDG- σ_h : optimal choice can be computed *a priori*,
 - for HDG+: large and not very sensitive,
 - for HDG- \mathbf{q}_h : small and quite sensitive.

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 - for HDG- σ_h : optimal choice can be computed *a priori*,
 - for HDG+: large and not very sensitive,
 - for HDG- \mathbf{q}_h : small and quite sensitive.
- ▶ **Global solvability:** \mathbb{M} is invertible if $\omega \notin \mathcal{R}$, where

$$\mathcal{R} := \{ \omega \mid \text{continuous problem not uniquely solvable} \}$$

is the set of *resonant frequencies*.



Error quantities:

$$\varepsilon_h^q := \mathbf{q}_h - \Pi \mathbf{q}, \quad \varepsilon_h^p := \mathbf{p}_h - \Pi \mathbf{p}, \quad \widehat{\varepsilon}_h^p := \lambda_h - \mathbf{P}_k \mathbf{p},$$

where Π , Π and \mathbf{P}_k are L^2 -projections onto the approximation spaces \mathbf{V}_h , W_h and M_h .

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Main steps of the analysis:

- ▶ **Step 1:** Energy-like estimate

$$\left| \|\varepsilon_h^{\mathbf{q}}\|^2 - 2i\omega \|\mathbf{P}_k \varepsilon_h^p - \widehat{\varepsilon}_h^p\|_{\partial\mathcal{T}_h}^2 \right| \lesssim F(\|\varepsilon_h^{\mathbf{q}}\|, \|\varepsilon_h^p\|, \|\mathbf{P}_k \varepsilon_h^p - \widehat{\varepsilon}_h^p\|_{\partial\mathcal{T}_h})$$

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- ▶ **Step 2:** Dual estimate

$$\|\varepsilon_h^p\| \lesssim \omega h \cdot G(\|\varepsilon_h^q\|, \|\mathbf{P}_k \varepsilon_h^p - \widehat{\varepsilon}_h^p\|_{\partial\mathcal{T}_h})$$

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- ▶ **Step 3:** Bootstrapping process

$$\|\varepsilon_h^q\| = \mathcal{O}(h^{k+1}), \quad \|\varepsilon_h^p\| = \mathcal{O}(h^{k+2})$$

Summary of convergence rates: (\mathbf{q}, p) : exact solution

- ▶ for HDG- \mathbf{q}_h : (optimal \times)

$$\|p_h - \Pi p\| = \mathcal{O}(h^{k+\frac{3}{2}}) \quad \text{and} \quad \|\mathbf{q}_h - \Pi \mathbf{q}\| = \mathcal{O}(h^{k+\frac{1}{2}})$$

- ▶ for HDG- σ_h : (optimal \checkmark)

$$\|p_h - \Pi p\| = \mathcal{O}(h^{k+1}) \quad \text{and} \quad \|\sigma_h - \Pi \sigma\| = \mathcal{O}(h^{k+1})$$

- ▶ for HDG+: (optimal \checkmark)

$$\|p_h - \Pi p\| = \mathcal{O}(h^{k+2}) \quad \text{and} \quad \|\mathbf{q}_h - \Pi \mathbf{q}\| = \mathcal{O}(h^{k+1})$$

Summary of convergence rates: (\mathbf{q}, ρ) : exact solution

- ▶ for HDG- \mathbf{q}_h : (optimal \times) **no super-convergence !**

$$\|\rho_h - \Pi\rho\| = \mathcal{O}(h^{k+\frac{3}{2}}) \quad \text{and} \quad \|\mathbf{q}_h - \Pi\mathbf{q}\| = \mathcal{O}(h^{k+\frac{1}{2}})$$

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- ▶ for HDG+: (optimal \checkmark)

$$\|\rho_h - \Pi\rho\| = \mathcal{O}(h^{k+2}) \quad \text{and} \quad \|\mathbf{q}_h - \Pi\mathbf{q}\| = \mathcal{O}(h^{k+1})$$

using:

- ▶ L^2 -orthogonal projections for HDG- \mathbf{q}_h and HDG+,
- ▶ tailored HDG projections for HDG- σ_h .

Numerical results

Convergence



We work with relative L^2 -errors

$$\mathcal{E}_p = \frac{\|p_h - \Pi p\|_{\mathcal{T}_h}}{\|\Pi p\|_{\mathcal{T}_h}}.$$

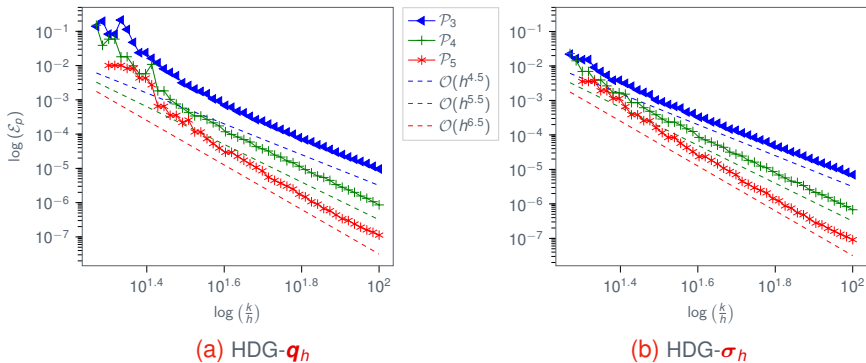
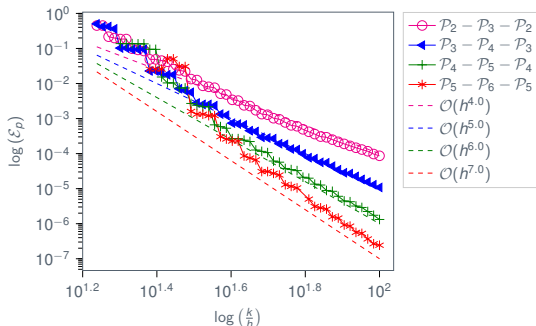


Figure: Convergence for p_h at Mach 0.4

Convergence

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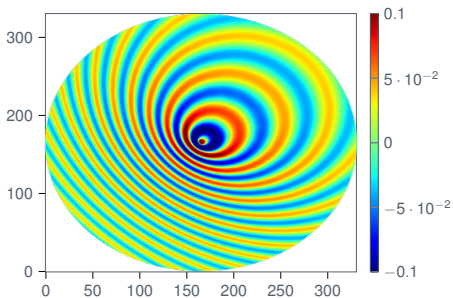
(a) HDG+

Figure: Convergence for p_h at Mach 0.4

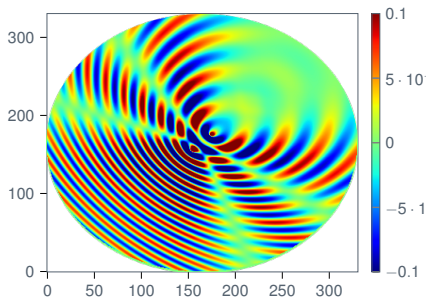
Uniform flow:

$$\mathbf{v}_0 = 0.6 \begin{bmatrix} \cos \frac{\pi}{4} \\ \sin \frac{\pi}{4} \end{bmatrix}$$

Validation: analytic solution for (a).



(a) One point-source



(b) Two point-sources

Figure: Numerical experiments using HDG- σ_h

Gaussian jet:

$$\mathbf{v}_0 = M_0(y)\mathbf{e}_x, \quad \text{where} \quad M_0(y) := 0.1 + 0.3 \exp\left(-\frac{y^2}{0.3^2}\right)$$

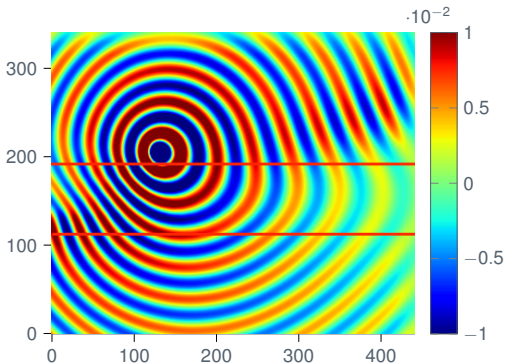


Figure: Point source in a jet-flow using HDG+

Potential flow around a circular obstacle:

$$\mathbf{v}_0 = M_\infty \left[\left(1 - \frac{R_C^2}{r^2} \right) \cos \theta \mathbf{e}_r + \left(1 + \frac{R_C^2}{r^2} \right) \sin \theta \mathbf{e}_\theta \right],$$

where R_C is the radius of the obstacle and M_∞ the Mach number at infinity.

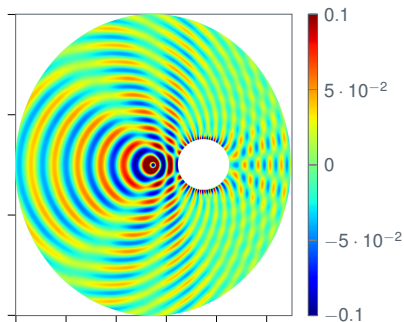


Figure: Point source in a potential flow around an obstacle using HDG- σ_h

Which method should we use ?

- ▶ HDG+
- ▶ HDG- σ_h

as they are

- ▶ less sensitive to the choice of τ ,
- ▶ more robust with high Mach numbers,

than HDG- q_h .

If you want to know more: a detailed pre-print is available



H. Barucq, N. Rouxelin, S. Tordeux.

HDG and HDG+ methods for harmonic wave problems with convection.

<https://hal.inria.fr/hal-03253415>



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HDG and HDG+ methods for harmonic wave problems with convection.

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An invitation to the theory of HDG methods.

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Using HDG+ to Compute Solutions of the 3D Linear Elastic and Poroelastic Wave Equations.

PhD Thesis - University of Delaware



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Hybrid Discontinuous Galerkin methods for solving incompressible flow problems.

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DOI:10.1090/S0025-5718-10-02334-3

Thank you for your attention!
Any questions ?

Large Mach convergence



We work with relative L^2 -errors

$$\mathcal{E}_p = \frac{\|p_h - \Pi p\|_{\mathcal{T}_h}}{\|\Pi p\|_{\mathcal{T}_h}}.$$

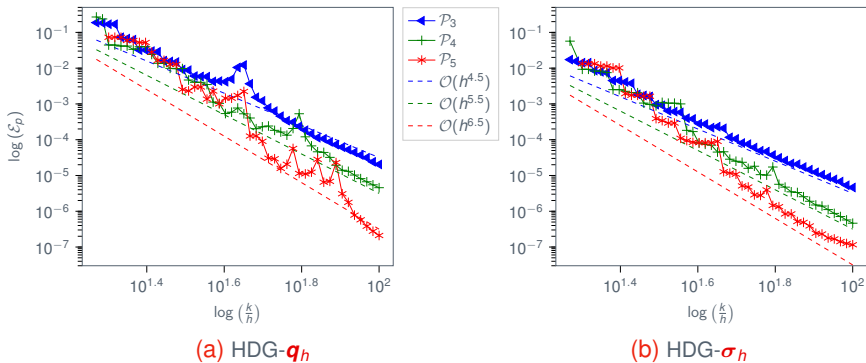


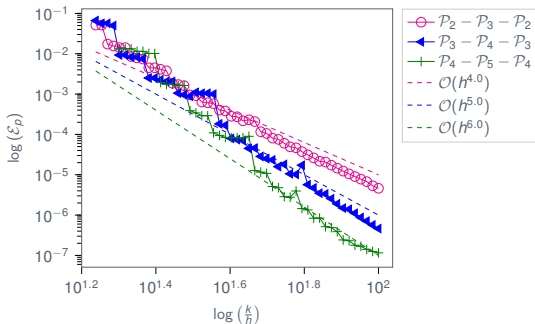
Figure: Large Mach convergence for p_h

Large Mach convergence



We work with relative L^2 -errors

$$\mathcal{E}_p = \frac{\|p_h - \Pi p\|_{\mathcal{T}_h}}{\|\Pi p\|_{\mathcal{T}_h}}.$$



(a) HDG+

Figure: Large Mach convergence for p_h