# Convergence analysis of HDG formulations for the convected Helmholtz equation 

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## Convergence analysis of HDG formulations for the convected Helmholtz equation

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## Context of this work

Helioseismology in a nutshell:

- Aims at imaging the solar interior thanks to surface observations
- Surfacic «acoustic waves» can be measured thanks to relativistic Doppler effect
- The full models are too complicated for numerical simulation, two main approaches for computational heliosesimolgy:
- Aeroacoustics: magnetic effects are neglected
- Magnetoacoustics: hydrodynamics effects are neglected

(a) Dopplergram

(b) Power spectrum


## Outline

1. What is HDG ?
2. HDG methods for the convected Helmholtz equation
3. Numerical results

## What is HDG ?

## From CG... to DG... to HDG

From CG to DG:

- Continuity of the solution between two elements is not enforced by the choice of approximation spaces anymore
- Continuity and stability come from the choice of numerical flux between two elements (similar to the finite volume method)
- Advantages
- hp-adaptativity
- high-order
- parallelization
- Disadvantage
- large number of degrees of freedom

(c) $C G$

(d) $D G$

Figure: Degrees of freedom for degree 3

## From CG... to DG... to HDG

From CG to DG:

- Continuity of the solution between two elements is not enforced by the choice of approximation spaces anymore
- Continuity and stability come from the choice of numerical flux between two elements (similar to the finite volume method)
From DG to HDG:
- Introduce a new unknown on the skeleton of the mesh
- Interior degrees of freedom can be eliminated through static condensation
- Keep the advantages of DG for a reduced cost

(a) CG

(b) DG

(c) HDG

Figure: Degrees of freedom for degree 3

## From CG... to DG... to HDG

From CG to DG:

- Continuity of the solution between two elements is not enforced by the choice of approximation spaces anymore
- Continuity and stability come from the choice of numerical flux between two elements (similar to the finite volume method)
From DG to HDG:
- Introduce a new unknown on the skeleton of the mesh
- Interior degrees of freedom can be eliminated through static condensation
- Keep the advantages of DG for a reduced cost

HDG methods have a dual citizenship:

- Static condensation of DG methods
- Also related to the hybridization of mixed FEM (eg Raviart-Thomas or Brezzi-Douglas-Marini methods)

HDG methods for the convected Helmholtz equation

## Convected Helmholtz equation

Model problem :

$$
\rho_{0}\left(-\omega^{2} \boldsymbol{p}-2 i \omega \boldsymbol{v}_{0} \cdot \nabla p+\boldsymbol{v}_{0} \cdot \nabla\left(\boldsymbol{v}_{0} \cdot \nabla p\right)\right)-\operatorname{div}\left(\rho_{0} c_{0}^{2} \nabla p\right)=s
$$

where

- p: acoustic potential ( $\sim$ pressure perturbation)
- $\rho_{0}$ : mass density
- $v_{0}$ : velocity field
- $c_{0}$ : sound speed
- $s$ : acoustic source


## Convected Helmholtz equation

Model problem :

$$
\rho_{0}\left(-\omega^{2} p-2 i \omega \boldsymbol{v}_{0} \cdot \nabla p\right)-\operatorname{div}\left(\boldsymbol{M}_{0} \nabla p\right)=s
$$

where $\boldsymbol{M}_{\mathbf{0}}=\rho_{0} \boldsymbol{c}_{0}{ }^{2} \mathrm{ld}-\rho_{0} \boldsymbol{V}_{0} \boldsymbol{V}_{\mathbf{0}}{ }^{\top}$.

## Convected Helmholtz equation

Model problem :

- Choice 1: diffusive flux

$$
\begin{aligned}
W_{0} \boldsymbol{q}+\nabla p & =0 \\
-\rho_{0} \omega^{2} p-2 i \omega \rho_{0} \boldsymbol{v}_{0} \cdot \nabla p+\operatorname{div}(\boldsymbol{q}) & =s
\end{aligned}
$$

where $\boldsymbol{q}=-\boldsymbol{M}_{\mathbf{0}} \nabla p$ and $\boldsymbol{W}_{\mathbf{0}}=\boldsymbol{M}_{\mathbf{0}}{ }^{-1}$.

- Choice 2: total flux

$$
\begin{array}{r}
W_{0} \sigma+\nabla p+2 i \omega p W_{0} \rho_{0} \boldsymbol{V}_{0}=0, \\
-\rho_{0} \omega^{2} p+\operatorname{div}(\boldsymbol{\sigma})=s,
\end{array}
$$

where $\sigma=-M_{0} \nabla p-2 i \omega p_{0} v_{0}$.
To use HDG methods, we need a first-order in space formulation.

## Convected Helmholtz equation

Model problem :
Seek $(\boldsymbol{q}, p) \in \boldsymbol{H}_{\text {div }}(\Omega) \times \boldsymbol{H}^{1}(\Omega)$ such that

$$
\begin{array}{r}
\int_{\Omega} w_{0} \boldsymbol{q} \cdot \boldsymbol{r}^{*} \mathrm{dx}-\int_{\Omega} \boldsymbol{p} \operatorname{div}\left(\boldsymbol{r}^{*}\right) \mathrm{dx}+\int_{\partial \Omega} \boldsymbol{p} \boldsymbol{r}^{*} \cdot \boldsymbol{n} \mathrm{~d} \sigma=0 \\
-\omega^{2} \int_{\Omega} \rho_{0} \boldsymbol{p} w^{*} \mathrm{dx}+2 i \omega \int_{\Omega} p \rho_{0} \boldsymbol{v}_{0} \cdot \nabla w^{*} \mathrm{dx} \\
-\int_{\Omega} \boldsymbol{q} \cdot \nabla w^{*} \mathrm{dx}+\int_{\partial \Omega} w^{*} \boldsymbol{q} \cdot \boldsymbol{n}-2 i \omega p w^{*} \rho_{0} \boldsymbol{v}_{0} \cdot \boldsymbol{n} \mathrm{~d} \sigma=\int_{\Omega} s w^{*} \mathrm{dx}
\end{array}
$$

for all $(\boldsymbol{r}, \boldsymbol{w}) \in \boldsymbol{H}_{\text {div }}(\Omega) \times H^{1}(\Omega)$.
Very similar if we use $\boldsymbol{\sigma}$ instead of $\boldsymbol{q}$.

## Convected Helmholtz equation

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-\omega^{2} \int_{\Omega} \rho_{0} p w^{*} \mathrm{dx}+2 i \omega \int_{\Omega} p_{\rho_{0}} v_{0} \cdot \nabla w^{*} \mathrm{dx} \\
-\int_{\Omega} \boldsymbol{q} \cdot \nabla w^{*} \mathrm{dx}+\int_{\partial \Omega} w^{*} \boldsymbol{q} \cdot \boldsymbol{n}-2 i \omega p w^{*} \rho_{0} \boldsymbol{v}_{0} \cdot \boldsymbol{n} \mathrm{~d} \sigma=\int_{\Omega} s w^{*} \mathrm{dx}
\end{array}
$$

for all $(\boldsymbol{r}, \boldsymbol{w}) \in \boldsymbol{H}_{\text {div }}(\Omega) \times \boldsymbol{H}^{1}(\Omega)$.
Well-posedness:

- if the carrier flow is subsonic

$$
\inf _{\Omega}\left(c_{0}^{2}-\left|\boldsymbol{v}_{\mathbf{0}}\right|^{2}\right)>0
$$

then the problem is of Fredholm type

- uniqueness of the solution comes from the boundary conditions.


## HDG(+) methods

Skeleton unknown: numerical trace $\lambda_{h}$ approximates $p$ on $\mathcal{E}_{h}$.

$$
\int_{\partial K} p \boldsymbol{r}^{*} \cdot \boldsymbol{n} \mathrm{~d} \sigma \xrightarrow{\text { discretization }} \int_{\partial K} \lambda_{h} \boldsymbol{r}^{*} \cdot \boldsymbol{n} \mathrm{~d} \sigma
$$

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$$

Approximation spaces:

| Unknown | Space | HDG | HDG + |
| :--- | :---: | :---: | :---: |
| Pressure $p_{h}$ | $W_{h}(K)$ | $\mathcal{P}_{k}(K)$ | $\mathcal{P}_{k+1}(K)$ |
| Flux $\boldsymbol{q}_{h} / \sigma_{h}$ | $V_{h}(K)$ | $\mathcal{P}_{k}(K)$ |  |
| Trace $\lambda_{h}$ | $M_{h}(e)$ | $\mathcal{P}_{k}(e)$ |  |

Warning: HDG+ only works in $(\boldsymbol{q}, \boldsymbol{p})$ formulation !
We will compare the HDG- $\boldsymbol{q}_{h}, \mathrm{HDG}-\sigma_{h}$ and HDG+ methods for time-harmonic convected waves.

## HDG(+) methods

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$$
\int_{\partial K} p \boldsymbol{r}^{*} \cdot \boldsymbol{n} \mathrm{~d} \sigma \xrightarrow{\text { discretization }} \int_{\partial K} \lambda_{h} \boldsymbol{r}^{*} \cdot \boldsymbol{n} \mathrm{~d} \sigma
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| Flux $\boldsymbol{q}_{h} / \boldsymbol{\sigma}_{h}$ | $V_{h}(K)$ | $\mathcal{P}_{k}(K)$ |  |
| Trace $\lambda_{h}$ | $M_{h}(e)$ | $\mathcal{P}_{k}(e)$ |  |

Together with the weak formulation: leads to the discrete local problem (ie on an element $K \in \mathcal{T}_{h}$ ) :

$$
\mathbb{A}^{K}\left[\begin{array}{l}
p_{h}^{K} \\
\boldsymbol{q}_{h}^{K}
\end{array}\right]+\mathbb{C}^{K}\left[\lambda_{h}^{K}\right]=\mathbb{S}^{K}
$$

At this point:

- The approximation spaces are discontinuous.
- We need to enforce the continuity of the numerical solution between the elements.
- We need to ensure the stability of the method.

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All of this is done by choosing the numerical flux, so that

$$
\sum_{e \in \mathcal{E}_{h}} \int_{e} \llbracket \widehat{\boldsymbol{q}_{h}}-2 i \omega \boldsymbol{n} \cdot \widehat{\boldsymbol{v}_{0} \boldsymbol{p}_{h} \boldsymbol{n}} \rrbracket \mu^{*} \mathrm{~d} \sigma=0, \quad \text { or } \quad \sum_{e \in \mathcal{E}_{h}} \int_{e} \llbracket \widehat{\sigma_{h}} \rrbracket \mu^{*} \mathrm{~d} \sigma=0,
$$

for all $\mu \in M_{h}$ and where $\llbracket \rrbracket \rrbracket$ is the DG-jump operator.

## HDG(+) methods

Numerical flux and transmission condition

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$$
\sum_{e \in \mathcal{E}_{h}} \int_{e} \llbracket \widehat{\boldsymbol{q}_{h}}-2 i \omega \boldsymbol{n} \cdot \widehat{\boldsymbol{v}_{0} p_{h}} \boldsymbol{n} \rrbracket \mu^{*} \mathrm{~d} \sigma=0, \quad \text { or } \quad \sum_{e \in \mathcal{E}_{h}} \int_{e} \llbracket \widehat{\sigma_{h}} \rrbracket \mu^{*} \mathrm{~d} \sigma=0,
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Figure: Discontinuity between two elements

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$$

for all $\mu \in M_{h}$ and where $\llbracket \cdot \rrbracket$ is the DG-jump operator.
Numerical flux:

$$
\begin{aligned}
\widehat{\boldsymbol{q}}_{h}{ }^{K} \cdot \boldsymbol{n}-2 i \omega \widehat{{p_{h}}^{K} \boldsymbol{v}_{0}} \cdot \boldsymbol{n} & =\boldsymbol{q}_{h}{ }^{K} \cdot \boldsymbol{n}-2 i \omega\left(\boldsymbol{v}_{0} \cdot \boldsymbol{n}\right) \lambda_{h}{ }^{K}+2 i \omega \tau\left(P_{k} p_{h}{ }^{K}-\lambda_{h}{ }^{K}\right) \\
\widehat{\boldsymbol{\sigma}}_{h}{ }^{K} \cdot \boldsymbol{n} & =\boldsymbol{\sigma}_{h}{ }^{K} \cdot \boldsymbol{n}+i \omega \tau\left(p_{h}{ }^{K}-\lambda_{h}{ }^{K}\right)
\end{aligned}
$$

where

- $\tau$ : penalization parameter
- $P_{k}: L^{2}$-projection from $\mathcal{P}_{k+1}(e)$ to $\mathcal{P}_{k}(e)$


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Numerical flux:
$\widehat{\boldsymbol{q}}_{h}{ }^{K} \cdot \boldsymbol{n}-2 i \omega \widehat{\boldsymbol{p}_{h}{ }^{K} \boldsymbol{v}_{0}} \cdot \boldsymbol{n}=\boldsymbol{q}_{h}{ }^{K} \cdot \boldsymbol{n}-2 i \omega\left(\boldsymbol{v}_{0} \cdot \boldsymbol{n}\right) \lambda_{h}{ }^{K}+2 i \omega \tau\left(P_{k}{p_{h}}^{K}-\lambda_{h}{ }^{K}\right)$

$$
\widehat{\boldsymbol{\sigma}}_{h}{ }^{K} \cdot \boldsymbol{n}=\boldsymbol{\sigma}_{h}{ }^{K} \cdot \boldsymbol{n}+i \omega \tau\left(p_{h}{ }^{K}-\lambda_{h}{ }^{K}\right)
$$

This leads to the discrete transmission condition:

$$
\sum_{K \in \mathcal{T}_{h}}\left(\mathbb{B}^{K}\left[\begin{array}{l}
p_{h}{ }^{K} \\
\boldsymbol{q}_{h}^{K}
\end{array}\right]+\mathbb{L}^{K}\left[\lambda_{h}{ }^{K}\right]\right)=0
$$

## HDG(+) methods

Implementation by static condensation

Recall from the previous slides:

- Local problem on $K \in \mathcal{T}_{h}$

$$
\mathbb{A}^{K}\left[\begin{array}{l}
p_{h}^{K} \\
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- Transmission condition

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p_{h}{ }^{K} \\
\boldsymbol{q}_{h}{ }^{K}
\end{array}\right]=\left(\mathbb{A}^{K}\right)^{-1} \mathbb{S}^{K}-\left(\mathbb{A}^{K}\right)^{-1} \mathbb{C}^{K}\left[\lambda_{h}{ }^{K}\right]
$$

- Transmission condition

$$
\sum_{K \in \mathcal{T}_{h}}\left(\mathbb{B}^{K}\left[\begin{array}{l}
p_{h}{ }^{K} \\
\boldsymbol{q}_{h}^{K}
\end{array}\right]+\mathbb{L}^{K}\left[\lambda_{h}^{K}\right]\right)=0
$$

Dimensions of the unknowns: Notice that

$$
\left[\begin{array}{l}
p_{h}{ }^{K} \\
\boldsymbol{q}_{h}{ }^{K}
\end{array}\right] \text { is volumetric and }\left[\lambda_{h}{ }^{K}\right] \text { is surfacic. }
$$

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p_{h}{ }^{K} \\
\boldsymbol{q}_{h}{ }^{K}
\end{array}\right]+\mathbb{L}^{K}\left[\lambda_{h}{ }^{K}\right]\right)=0
$$

Static condensation:

$$
\mathbb{M}\left[\lambda_{h}\right]=\mathbb{S} \underset{\text { reconstruction }}{\stackrel{\text { elimination }}{\leftrightarrows}}\left[\begin{array}{l}
p_{h}{ }^{K} \\
\boldsymbol{q}_{h}^{K}
\end{array}\right]=\left(\mathbb{A}^{K}\right)^{-1} \mathbb{S}^{K}-\left(\mathbb{A}^{K}\right)^{-1} \mathbb{C}^{K}\left[\lambda_{h}{ }^{K}\right],
$$

with

$$
\mathbb{M}:=\sum_{K \in \mathcal{T}_{h}}\left(\mathbb{L}^{K}-\mathbb{B}^{K}\left(\mathbb{A}^{K}\right)^{-1} \mathbb{C}^{K}\right) \quad \text { and } \quad \mathbb{S}:=-\sum_{K \in \mathcal{T}_{h}} \mathbb{B}^{K}\left(\mathbb{A}^{K}\right)^{-1} \mathbb{S}^{K}
$$

## Some results on the methods

We performed the numerical analysis of the method:

- Local solvability: $\mathbb{A}^{K}$ is invertible if
- $\omega h_{K}<C\left(\rho_{0}, v_{0}, c_{0}\right.$, element geometry $)$,
- $\tau$ is well chosen


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■ for HDG- $\sigma_{n}$ : optimal choice can be computed a priori,

- for HDG+: large and not very sensitive,
- for HDG- $\boldsymbol{q}_{h}$ : small and quite sensitive.


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- Local solvability: $\mathbb{A}^{K}$ is invertible if
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- $\tau$ is well chosen

■ for HDG- $\sigma_{h}$ : optimal choice can be computed a priori,

- for HDG+: large and not very sensitive,
- for HDG- $\boldsymbol{q}_{h}$ : small and quite sensitive.
- Global solvability: $\mathbb{M}$ is invertible if $\omega \notin \mathcal{R}$, where

$$
\mathcal{R}:=\{\omega \mid \text { continuous problem not uniquely solvable }\}
$$

is the set of resonant frequencies.

## Error analysis of the HDG+ method

## Error quantities:

$$
\varepsilon_{h}^{\boldsymbol{q}}:=\boldsymbol{q}_{h}-\Pi \boldsymbol{q}, \quad \varepsilon_{h}^{p}:=p_{h}-\Pi p, \widehat{\varepsilon}_{h}^{p}:=\lambda_{h}-\mathrm{P}_{k} p,
$$

where $\Pi, \Pi$ and $P_{k}$ are $L^{2}$-projections onto the approximation spaces $V_{h}, W_{h}$ and $M_{h}$.

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where $\Pi, \Pi$ and $P_{k}$ are $L^{2}$-projections onto the approximation spaces $V_{h}, W_{h}$ and $M_{h}$.
Main steps of the analysis:

- Step 1: Energy-like estimate

$$
\left|\left\|\varepsilon_{h}^{q}\right\|^{2}-2 i \omega\left\|\mathrm{P}_{k} \varepsilon_{h}^{p}-\widehat{\varepsilon}_{h}^{p}\right\|_{\partial \tau_{h}}^{2}\right| \lesssim F\left(\left\|\varepsilon_{h}^{q}\right\|,\left\|\varepsilon_{h}^{p}\right\|,\left\|\mathrm{P}_{k} \varepsilon_{h}^{p}-\widehat{\varepsilon}_{h}^{p}\right\|_{\partial \tau_{h}}\right)
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$$

- Step 2: Dual estimate

$$
\left\|\varepsilon_{h}^{p}\right\| \lesssim \omega h \cdot G\left(\left\|\varepsilon_{h}^{q}\right\|,\left\|P_{k} \varepsilon_{h}^{p}-\varepsilon_{h}^{p}\right\|_{\partial \tau_{h}}\right)
$$

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$$

- Step 3: Bootstrapping process

$$
\left\|\varepsilon_{h}^{\boldsymbol{q}}\right\|=\mathcal{O}\left(h^{k+1}\right), \quad\left\|\varepsilon_{h}^{p}\right\|=\mathcal{O}\left(h^{k+2}\right)
$$

## Error analysis of the HDG+ method

Summary of convergence rates: $(\boldsymbol{q}, p)$ : exact solution

- for HDG- $\boldsymbol{q}_{h}$ : (optimal $\boldsymbol{x}$ )

$$
\left\|p_{h}-\Pi p\right\|=\mathcal{O}\left(h^{k+\frac{3}{2}}\right) \text { and }\left\|\boldsymbol{q}_{h}-\Pi \boldsymbol{q}\right\|=\mathcal{O}\left(h^{k+\frac{1}{2}}\right)
$$

- for HDG- $\sigma_{h}$ : (optimal $\sqrt{ }$ )

$$
\left\|p_{h}-\Pi p\right\|=\mathcal{O}\left(h^{k+1}\right) \text { and }\left\|\sigma_{h}-\Pi \boldsymbol{\sigma}\right\|=\mathcal{O}\left(h^{k+1}\right)
$$

- for HDG+: (optimal $\checkmark$ )

$$
\left\|p_{h}-\Pi p\right\|=\mathcal{O}\left(h^{k+2}\right) \quad \text { and } \quad\left\|\boldsymbol{q}_{h}-\Pi \boldsymbol{q}\right\|=\mathcal{O}\left(h^{k+1}\right)
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## Error analysis of the HDG+ method

Summary of convergence rates: $(\boldsymbol{q}, p)$ : exact solution

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$$

using:

- $L^{2}$-orthogonal projections for HDG- $\boldsymbol{q}_{h}$ and HDG+,
- tailored HDG projections for HDG- $\sigma_{h}$.


## Numerical results

## Convergence

We work with relative $L^{2}$-errors

$$
\mathcal{E}_{p}=\frac{\left\|p_{h}-\Pi p\right\|_{\mathcal{T}_{h}}}{\|\Pi p\|_{\mathcal{T}_{h}}}
$$



Figure: Convergence for $p_{h}$ at Mach 0.4

## Convergence

We work with relative $L^{2}$-errors

$$
\mathcal{E}_{p}=\frac{\left\|p_{h}-\Pi p\right\|_{\mathcal{T}_{h}}}{\|\Pi p\|_{\mathcal{T}_{h}}}
$$


(a) HDG+

Figure: Convergence for $p_{h}$ at Mach 0.4

## Illustrative examples

Uniform flow:

$$
v_{0}=0.6\left[\begin{array}{c}
\cos \frac{\pi}{4} \\
\sin \frac{\pi}{4}
\end{array}\right]
$$

Validation: analytic solution for (a).


Figure: Numerical experiments using HDG- $\sigma_{h}$

## Illustrative examples

Gaussian jet:

$$
v_{0}=M_{0}(y) \boldsymbol{e}_{x}, \quad \text { where } \quad M_{0}(y):=0.1+0.3 \exp \left(-\frac{y^{2}}{0.3^{2}}\right)
$$



Figure: Point source in a jet-flow using HDG+

## Illustrative examples

Potential flow around a circular obstacle:

$$
v_{0}=M_{\infty}\left[\left(1-\frac{R_{C}^{2}}{r^{2}}\right) \cos \theta \boldsymbol{e}_{r}+\left(1+\frac{R_{C}^{2}}{r^{2}}\right) \sin \theta \boldsymbol{e}_{\theta}\right],
$$

where $R_{C}$ is the radius of the obstacle and $M_{\infty}$ the Mach number at infinity.


Figure: Point source in a potential flow around an obstacle using HDG- $\sigma_{h}$

## Conclusion

Which method should we use ?

- HDG+
- HDG- $\sigma_{h}$
as they are
- less sensitive to the choice of $\tau$,
- more robust with high Mach numbers, than HDG- $\boldsymbol{q}_{h}$.

If you want to know more: a detailed pre-print is available
國 H. Barucq, N. Rouxelin, S. Tordeux.
HDG and HDG+ methods for harmonic wave problems with convection. https://hal.inria.fr/hal-03253415

## References

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Thank you for your attention! Any questions ?

## Large Mach convergence

We work with relative $L^{2}$-errors

$$
\mathcal{E}_{p}=\frac{\left\|p_{h}-\Pi p\right\|_{\mathcal{T}_{h}}}{\|\Pi p\|_{\mathcal{T}_{h}}}
$$



Figure: Large Mach convergence for $p_{h}$

## Large Mach convergence

We work with relative $L^{2}$-errors

$$
\mathcal{E}_{p}=\frac{\left\|p_{h}-\Pi p\right\|_{\mathcal{T}_{h}}}{\|\Pi p\|_{\mathcal{T}_{h}}}
$$


(a) HDG+

Figure: Large Mach convergence for $p_{h}$

