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Numerical simulation of aeroacoustic wave propagation in the Sun

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WCCM ECCOMAS

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Inria Magique-3D – e2s-UPPA – LMAP



1. An introduction to helioseismology
2. HDG methods for harmonic wave problems with convection

The background of the slide features a series of thin, light blue, wavy lines that create a sense of motion and depth, resembling ripples or seismic waves. These lines are arranged in a pattern that curves across the frame, with some lines being more prominent than others.

An introduction to helioseismology



- ▶ Aims at imaging the solar interior thanks to surface observations

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- ▶ Surfacic «acoustic waves» can be measured thanks to relativistic Doppler effect

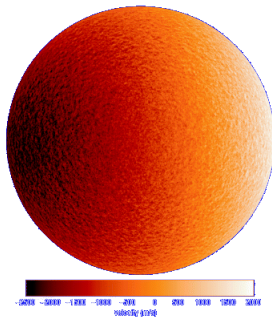


Figure 1: Doppler image of the Sun taken by the MDI instrument on board of the SOHO satellite (credit: NASA)

Helioseismology in a nutshell



- ▶ Aims at imaging the solar interior thanks to surface observations
- ▶ Surfacic «acoustic waves» can be measured thanks to relativistic Doppler effect
- ▶ Observations lead to the following power spectrum (used as reference for the inverse problem)

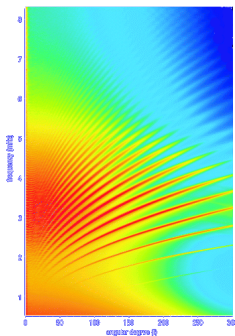


Figure 1: Power spectrum computed from MDI data (Credit: NASA)



- ▶ Aims at imaging the solar interior thanks to surface observations
- ▶ Surfacic «acoustic waves» can be measured thanks to relativistic Doppler effect
- ▶ Observations lead to the following power spectrum (used as reference for the inverse problem)
- ▶ The full models are too complicated for numerical simulation, two main approaches for computational heliosesimolgy:
 - ▶ Aeroacoustics: magnetic effects are neglected (us)
 - ▶ Magnetoacoustics: hydrodynamics effects are neglected

Victories of helioseismology:

- ▶ Solar neutrino flux
- ▶ Global structure of the Sun
- ▶ Equation of state

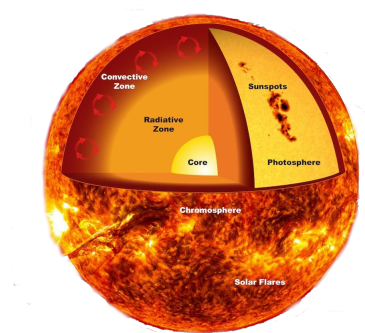


Figure 1: Global structure of the Sun

Victories of helioseismology:

- ▶ Solar neutrino flux
- ▶ Global structure of the Sun
- ▶ Equation of state

Our goals:

- ▶ Use FEM instead of semi-analytic methods in the helioseismic inversion pipeline
- ▶ Take gravity into account

Further reading:



J. Christensen-Dalsgaard.

Lecture Notes on Stellar Oscillations.

<http://astro.phys.au.dk/~jcd/oscilnotes/>

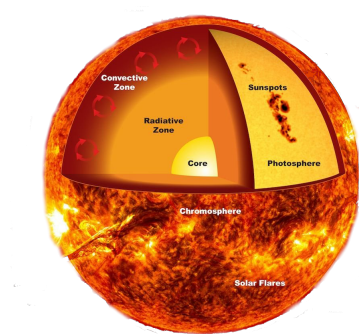


Figure 1: Global structure of the Sun

HDG methods for harmonic wave problems with
convection

From CG to DG:

- ▶ Continuity of the solution between two elements is not enforced by the choice of approximation spaces anymore
- ▶ Continuity and stability come from the choice of numerical flux between two elements (similar to the finite volume method)
- ▶ Advantages
 - ▶ hp -adaptativity
 - ▶ high-order
 - ▶ parallelization
- ▶ Disadvantage
 - ▶ large number of degrees of freedom

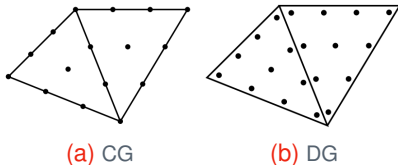


Figure 2: Degrees of freedom for degree 3

From CG to DG:

- ▶ Continuity of the solution between two elements is not enforced by the choice of approximation spaces anymore
- ▶ Continuity and stability come from the choice of numerical flux between two elements (similar to the finite volume method)

From DG to HDG:

- ▶ Introduce a new unknown on the skeleton of the mesh
- ▶ Interior degrees of freedom can be eliminated through static condensation
- ▶ Keep the advantages of DG for a reduced cost

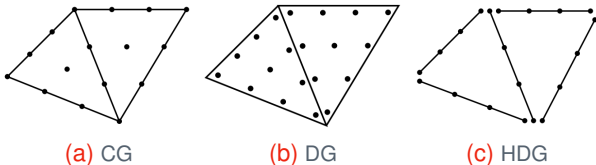


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- ▶ Keep the advantages of DG for a reduced cost

HDG methods have a dual citizenship:

- ▶ Static condensation of DG methods
- ▶ Also related to the hybridization of mixed FEM (eg Raviart-Thomas or Brezzi-Douglas-Marini methods)

Model problem :

$$\rho_0 (-\omega^2 p - 2i\omega \mathbf{v}_0 \cdot \nabla p + \mathbf{v}_0 \cdot \nabla (\mathbf{v}_0 \cdot \nabla p)) - \operatorname{div} (\rho_0 c_0^2 \nabla p) = s$$

where

- ▶ p : pressure perturbation
- ▶ ρ_0 : mass density
- ▶ \mathbf{v}_0 : velocity field
- ▶ c_0 : sound speed
- ▶ s : acoustic source

Model problem :

$$\rho_0 (-\omega^2 \mathbf{p} - 2i\omega \mathbf{v}_0 \cdot \nabla \mathbf{p}) - \operatorname{div} (\mathbf{M}_0 \nabla \mathbf{p}) = \mathbf{s}$$

where $\mathbf{M}_0 = \rho_0 c_0^2 \operatorname{Id} - \rho_0 \mathbf{v}_0 \mathbf{v}_0^T$.

Model problem :

$$\begin{aligned} \mathbf{W}_0 \mathbf{q} + \nabla p &= 0 \\ -\rho_0 \omega^2 p - 2i\omega \rho_0 \mathbf{v}_0 \cdot \nabla p + \operatorname{div}(\mathbf{q}) &= s \end{aligned}$$

where $\mathbf{q} = -\mathbf{M}_0 \nabla p$ and $\mathbf{W}_0 = \mathbf{M}_0^{-1}$.

To use HDG methods, we need a first-order in space formulation.

Model problem :

Seek $(\mathbf{q}, p) \in \mathbf{H}_{\text{div}}(\Omega) \times H^1(\Omega)$ such that

$$\begin{aligned} \int_{\Omega} \mathbf{w}_0 \mathbf{q} \cdot \mathbf{r}^* \, dx - \int_{\Omega} p \operatorname{div}(\mathbf{r}^*) \, dx + \int_{\partial\Omega} p \mathbf{r}^* \cdot \mathbf{n} \, d\sigma = 0 \\ -\omega^2 \int_{\Omega} \rho_0 p w^* \, dx + 2i\omega \int_{\Omega} p \rho_0 \mathbf{v}_0 \cdot \nabla w^* \, dx \\ - \int_{\Omega} \mathbf{q} \cdot \nabla w^* \, dx + \int_{\partial\Omega} w^* \mathbf{q} \cdot \mathbf{n} - 2i\omega p w^* \rho_0 \mathbf{v}_0 \cdot \mathbf{n} \, d\sigma = \int_{\Omega} s w^* \, dx \end{aligned}$$

for all $(\mathbf{r}, w) \in \mathbf{H}_{\text{div}}(\Omega) \times H^1(\Omega)$.

Convected Helmholtz equation



Model problem :

Seek $(\mathbf{q}, p) \in \mathbf{H}_{\text{div}}(\Omega) \times H^1(\Omega)$ such that

$$\begin{aligned} \int_{\Omega} W_0 \mathbf{q} \cdot \mathbf{r}^* \, dx - \int_{\Omega} p \operatorname{div}(\mathbf{r}^*) \, dx + \int_{\partial\Omega} p \mathbf{r}^* \cdot \mathbf{n} \, d\sigma = 0 \\ -\omega^2 \int_{\Omega} \rho_0 p w^* \, dx + 2i\omega \int_{\Omega} p \rho_0 \mathbf{v}_0 \cdot \nabla w^* \, dx \\ - \int_{\Omega} \mathbf{q} \cdot \nabla w^* \, dx + \int_{\partial\Omega} w^* \mathbf{q} \cdot \mathbf{n} - 2i\omega p w^* \rho_0 \mathbf{v}_0 \cdot \mathbf{n} \, d\sigma = \int_{\Omega} s w^* \, dx \end{aligned}$$

for all $(\mathbf{r}, w) \in \mathbf{H}_{\text{div}}(\Omega) \times H^1(\Omega)$.

Well-posedness: if the carrier flow is subsonic

$$\inf_{\Omega} (c_0^2 - |\mathbf{v}_0|^2) > 0$$

then the problem is of Fredholm type with $(\mathbf{q}, p) \in \mathbf{H}_{\text{div}}(\Omega) \times H^1(\Omega)$.

Skeleton unknown: numerical trace $\lambda_h = \mathbf{p}_h|_{\mathcal{E}_h}$.

$$\int_{\partial K} \mathbf{p}_h \mathbf{r}^* \cdot \mathbf{n} d\sigma \xrightarrow{\text{becomes}} \int_{\partial K} \lambda_h \mathbf{r}^* \cdot \mathbf{n} d\sigma$$

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Approximation spaces:

Variable	Space	HDG	HDG+
Pressure \mathbf{p}_h	$W_h(K)$	$\mathcal{P}_k(K)$	$\mathcal{P}_{k+1}(K)$
Flux \mathbf{q}_h	$V_h(K)$	$\mathcal{P}_k(K)$	
Trace λ_h	$M_h(e)$	$\mathcal{P}_k(e)$	

We will compare the HDG and HDG+ methods for time-harmonic convected waves.

Skeleton unknown: numerical trace $\lambda_h = \mathbf{p}_h|_{\mathcal{E}_h}$.

$$\int_{\partial K} \mathbf{p}_h \mathbf{r}^* \cdot \mathbf{n} d\sigma \xrightarrow{\text{becomes}} \int_{\partial K} \lambda_h \mathbf{r}^* \cdot \mathbf{n} d\sigma$$

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Trace λ_h	$M_h(e)$	$\mathcal{P}_k(e)$	

Together with the weak formulation: leads to the discrete **local** problem (i on an element $K \in \mathcal{T}_h$):

$$\mathbb{A}^K \begin{bmatrix} \mathbf{p}_h^K \\ \mathbf{q}_h^K \end{bmatrix} + \mathbb{C}^K [\lambda_h^K] = \mathbb{S}^K$$



At this point:

- ▶ The approximation spaces are discontinuous.
- ▶ We need to enforce the continuity of the numerical solution between the elements.
- ▶ We need to ensure the stability of the method.

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All of this is done by choosing the numerical flux, so that

$$\sum_{e \in \mathcal{E}_h} \int_e \llbracket \widehat{\mathbf{q}}_h - 2i\omega \mathbf{n} \cdot \widehat{\mathbf{v}}_0 \rho_h \mathbf{n} \rrbracket \mu^* d\sigma = 0, \quad \forall \mu \in M_h$$

where $\llbracket \cdot \rrbracket$ is the DG-jump operator.

All of this is done by choosing the numerical flux, so that

$$\sum_{e \in \mathcal{E}_h} \int_e [\widehat{\mathbf{q}}_h - 2i\omega \mathbf{n} \cdot \widehat{\mathbf{v}}_0 \rho_h \mathbf{n}] \mu^* d\sigma = 0, \quad \forall \mu \in M_h$$

where $[\cdot]$ is the DG-jump operator.

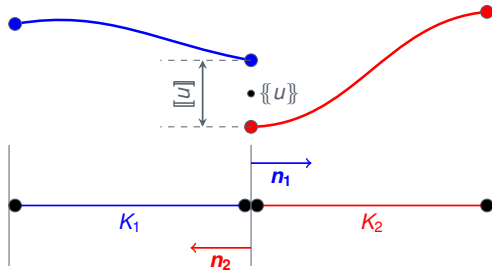


Figure 2: Discontinuity between two elements

All of this is done by choosing the numerical flux, so that

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where $\llbracket \cdot \rrbracket$ is the DG-jump operator.

Numerical flux:

$$\widehat{\mathbf{q}}_h^K \cdot \mathbf{n} - 2i\omega \widehat{\mathbf{p}}_h^K \widehat{\mathbf{v}}_0 \cdot \mathbf{n} = \mathbf{q}_h^K \cdot \mathbf{n} - 2i\omega (\mathbf{v}_0 \cdot \mathbf{n}) \lambda_h^K + 2i\omega \tau (\mathbf{p}_h^K - \lambda_h^K)$$

where

- ▶ τ : penalization parameter

All of this is done by choosing the numerical flux, so that

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$$\widehat{\mathbf{q}}_h^K \cdot \mathbf{n} - 2i\omega \widehat{\rho}_h^K \widehat{\mathbf{v}}_0 \cdot \mathbf{n} = \mathbf{q}_h^K \cdot \mathbf{n} - 2i\omega (\mathbf{v}_0 \cdot \mathbf{n}) \lambda_h^K + 2i\omega \tau (\rho_h^K - \lambda_h^K) + 2i\omega \tau_{\text{upw}} (\rho_h^K - \lambda_h^K)$$

where

- ▶ τ : penalization parameter
- ▶ $\tau_{\text{upw}} = \max(\mathbf{v}_0 \cdot \mathbf{n}, 0)$: optional upwinding parameter

All of this is done by choosing the numerical flux, so that

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This leads to the discrete transmission condition:

$$\sum_{K \in \mathcal{T}_h} \left(\mathbb{B}^K \begin{bmatrix} \rho_h^K \\ \mathbf{q}_h^K \end{bmatrix} + \mathbb{L}^K [\lambda_h^K] \right) = 0$$

Recall from the previous slides:

- ▶ Local problem

$$\mathbb{A}^K \begin{bmatrix} p_h^K \\ \mathbf{q}_h^K \end{bmatrix} + \mathbb{C}^K [\lambda_h^K] = \mathbb{S}^K$$

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Eliminate the interior nodes:

$$\begin{bmatrix} p_h^K \\ \mathbf{q}_h^K \end{bmatrix} = (\mathbb{A}^K)^{-1} \mathbb{S}^K - (\mathbb{A}^K)^{-1} \mathbb{C}^K [\lambda_h^K]$$

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Together with the transmission condition:

$$\sum_{K \in \mathcal{T}_h} (\mathbb{L}^K - \mathbb{B}^K (\mathbb{A}^K)^{-1} \mathbb{C}^K) [\lambda_h^K] = - \sum_{K \in \mathcal{T}_h} \mathbb{B}^K (\mathbb{A}^K)^{-1} \mathbb{S}^K$$

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Together with the transmission condition:

$$\mathbb{M} [\lambda_h] = \mathbb{S}$$



We performed the numerical analysis of the method:

- ▶ **Local solvability:** invertibility of \mathbb{A}^K , ✓ if

$\omega h_K < C(\rho_0, \mathbf{v}_0, \mathbf{c}_0, \text{element geometry})$

τ well chosen

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τ well chosen

- ▶ **Convergence rate:** (\mathbf{q}, p) : exact solution
 - ▶ for HDG: (optimal ✗)

$$\|p_h - p\| = \mathcal{O}(h^{k+1}) \quad \text{and} \quad \|\mathbf{q}_h - \mathbf{q}\| = \mathcal{O}(h^{k+\frac{1}{2}})$$

- ▶ for HDG+: (optimal ✓)

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- ▶ for HDG: (optimal \times) **no super-convergence !**

$$\|\mathbf{p}_h - \mathbf{p}\| = \mathcal{O}(h^{k+1}) \quad \text{and} \quad \|\mathbf{q}_h - \mathbf{q}\| = \mathcal{O}(h^{k+\frac{1}{2}})$$

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using:

- ▶ L^2 -orthogonal projections
- ▶ a duality estimate only valid for regular solutions

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- ▶ for HDG+: (optimal ✓)

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- ▶ **Global solvability:** invertibility of \mathbb{M} , ✓ if ω is not a resonant frequency.



Implementation of those methods is still a work in progress, but we have first promising results.

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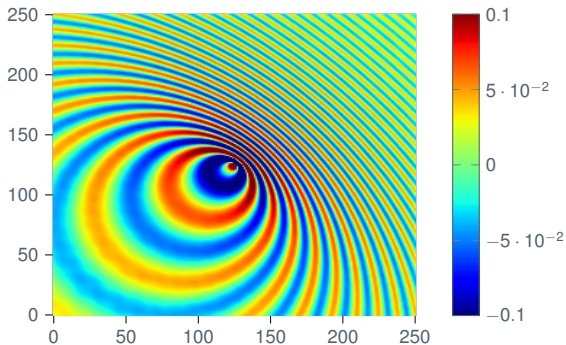


Figure 2: Green function

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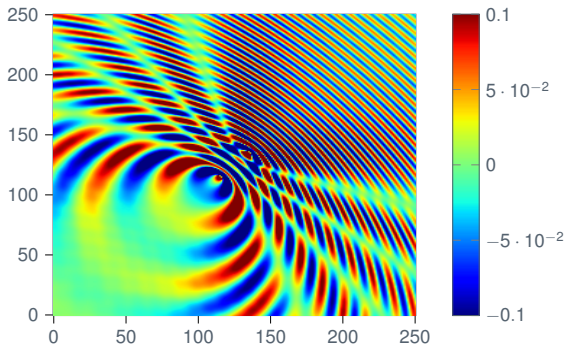


Figure 2: Interferences between two point sources

Thank you for your attention!
Any questions ?