Numerical simulation of aeroacoustic wave propagation in the Sun

Nathan Rouxelin, Hélène Barucq, Sébastien Tordeux

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January 2021
WCCM ECCOMAS

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Outline

1. An introduction to helioseismology

2. HDG methods for harmonic wave problems with convection

An introduction to helioseismology

Helioseismology in a nutshell

- Aims at imaging the solar interior thanks to surface observations

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- Surfacic «acoustic waves» can be measured thanks to relativistic Doppler effect

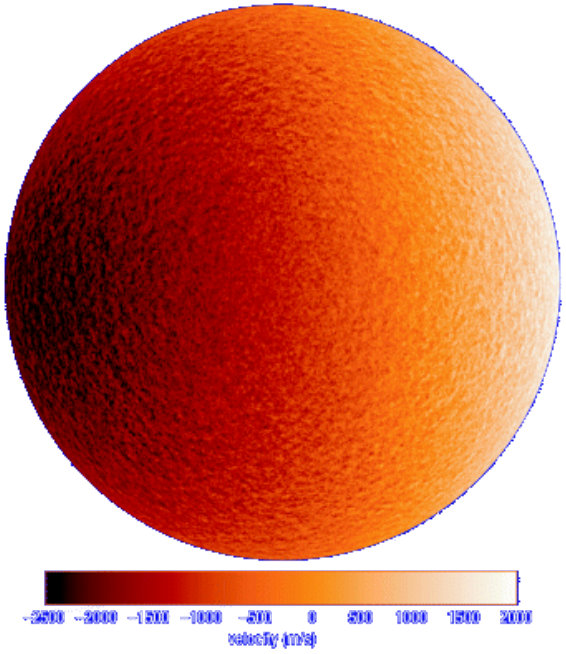


Figure 1: Doppler image of the Sun taken by the MDI instrument on board of the SOHO satellite (credit: NASA)

Helioseismology in a nutshell

- Aims at imaging the solar interior thanks to surface observations
- Surfacic «acoustic waves» can be measured thanks to relativistic Doppler effect
- Observations lead to the following power spectrum (used as reference for the inverse problem)

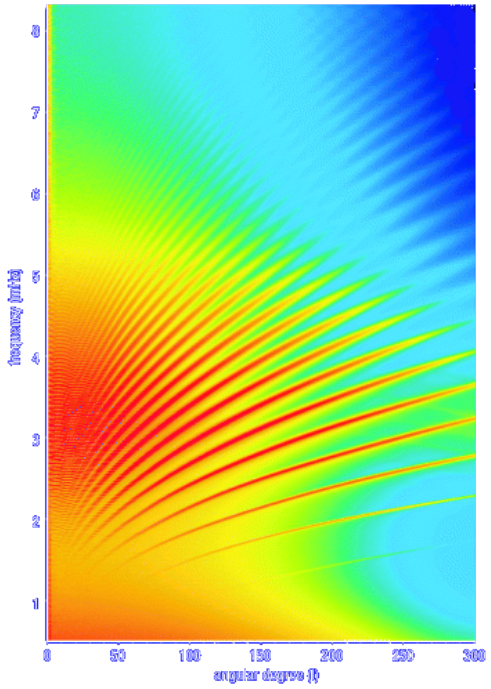


Figure 1: Power spectrum computed from MDI data (Credit: NASA)

Helioseismology in a nutshell

- Aims at imaging the solar interior thanks to surface observations
- Surfacic «acoustic waves» can be measured thanks to relativistic Doppler effect
- Observations lead to the following power spectrum (used as reference for the inverse problem)
- The full models are too complicated for numerical simulation, two main approaches for computational heliosesimolgy:
  - Aeroacoustics: magnetic effects are neglected (us)
  - Magnetoacoustics: hydrodynamics effects are neglected

Helioseismology in a nutshell

Victories of helioseismology:

- Solar neutrino flux
- Global structure of the Sun
- Equation of state

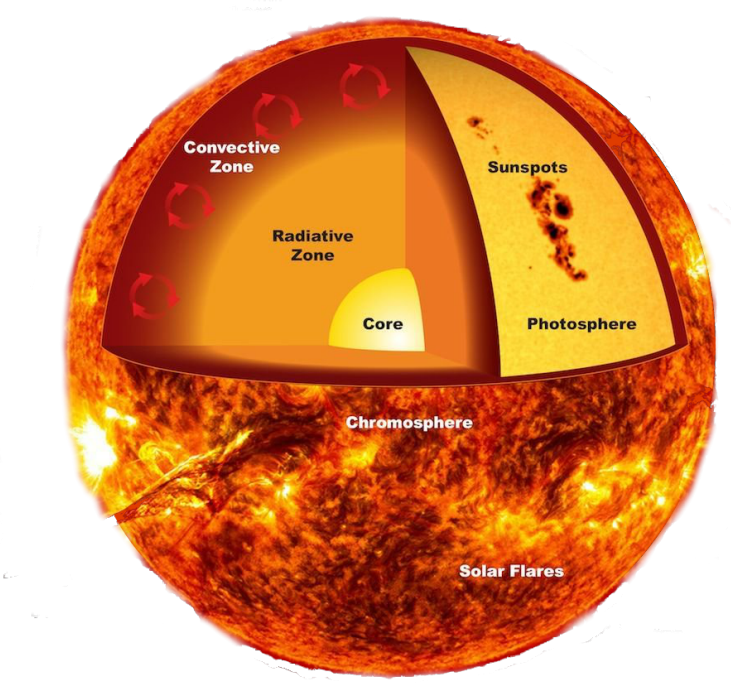


Figure 1: Global structure of the Sun

Helioseismology in a nutshell

Victories of helioseismology:

- Solar neutrino flux
- Global structure of the Sun
- Equation of state

Our goals:

- Use FEM instead of semi-analytic methods in the helioseismic inversion pipeline
- Take gravity into account

Further reading:

J. Christensen-Dalsgaard.
Lecture Notes on Stellar Oscillations.
http://astro.phys.au.dk/~jcd/oscilnotes/

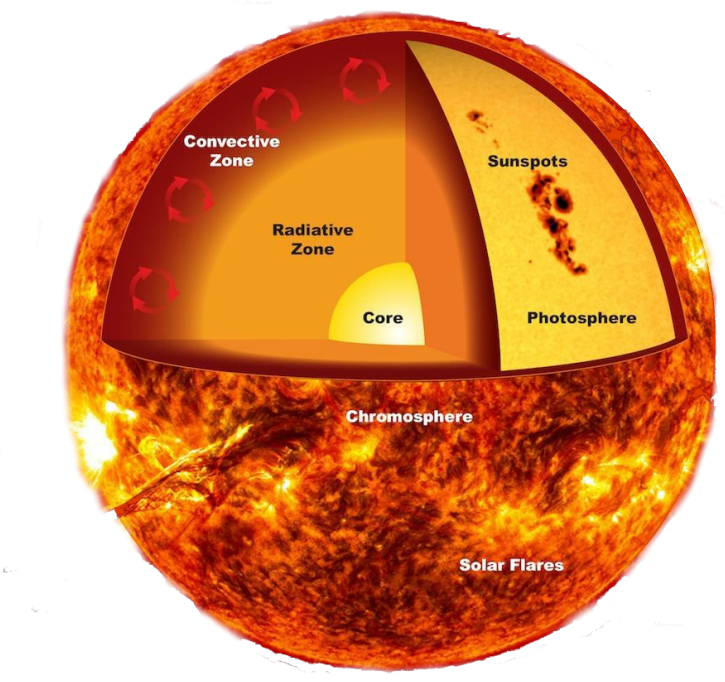


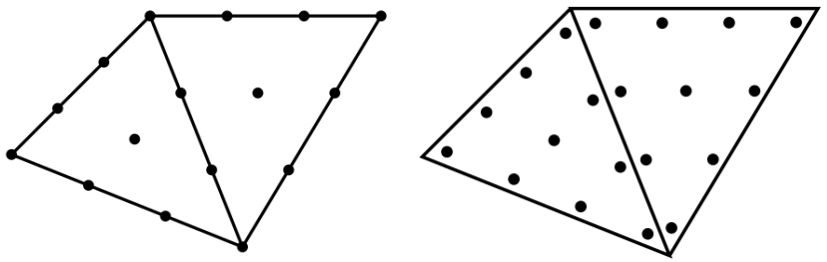
Figure 1: Global structure of the Sun

HDG methods for harmonic wave problems with convection

From CG... to DG... to HDG

From CG to DG:

- Continuity of the solution between two elements is not enforced by the choice of approximation spaces anymore
- Continuity and stability come from the choice of numerical flux between two elements (similar to the finite volume method)
- Advantages
  - *hp*-adaptativity
  - high-order
  - parallelization
- Disadvantage
  - large number of degrees of freedom



(a) CG (b) DG

Figure 2: Degrees of freedom for degree 3

From CG... to DG... to HDG

From CG to DG:

- Continuity of the solution between two elements is not enforced by the choice of approximation spaces anymore
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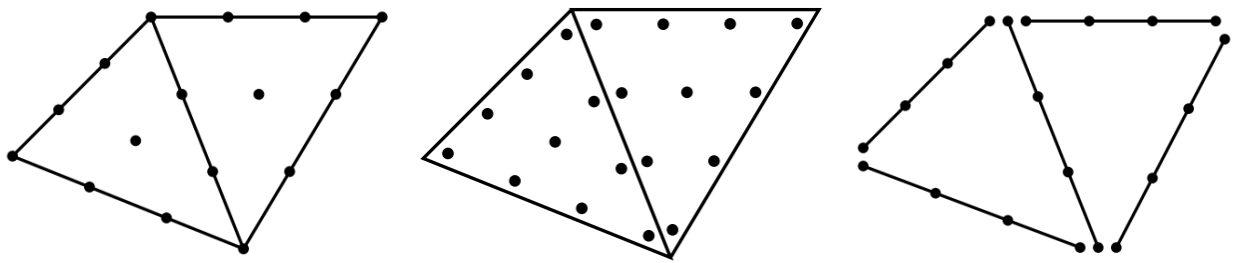
From DG to HDG:

- Introduce a new unknown on the skeleton of the mesh
- Interior degrees of freedom can be eliminated through static condensation
- Keep the advantages of DG for a reduced cost



(a) CG (b) DG (c) HDG

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HDG methods have a dual citizenship:

- Static condensation of DG methods
- Also related to the hybridization of mixed FEM (*eg* Raviart-Thomas or Brezzi-Douglas-Marini methods)

Convected Helmholtz equation

Model problem :

$$\rho_0 \left(-\omega^2 p - 2i\omega \boldsymbol{v_0} \cdot \nabla p + \boldsymbol{v_0} \cdot \nabla(\boldsymbol{v_0} \cdot \nabla p)\right) - \operatorname{div}\left(\rho_0 {c_0}^2 \nabla p\right) = s$$

where

- p: pressure perturbation
- ρ_0: mass density
- $\boldsymbol{v_0}$: velocity field
- c_0: sound speed
- s: acoustic source

Convected Helmholtz equation

Model problem :

$$\rho_0 \left(-\omega^2 p - 2i\omega \boldsymbol{v_0} \cdot \nabla p\right) - \operatorname{div}\left(\boldsymbol{M_0} \nabla p\right) = s$$

where $\boldsymbol{M_0} = \rho_0 {c_0}^2 \mathrm{Id} - \rho_0 \boldsymbol{v_0} {\boldsymbol{v_0}}^T$.

Convected Helmholtz equation

Model problem :

$$\boldsymbol{W_0 q} + \nabla p = 0$$
$$-\rho_0 \omega^2 p - 2i\omega\rho_0 \boldsymbol{v_0} \cdot \nabla p + \mathrm{div}\,(\boldsymbol{q}) = s$$

where $\boldsymbol{q} = -\boldsymbol{M_0}\nabla p$ and $\boldsymbol{W_0} = \boldsymbol{M_0}^{-1}$.
To use HDG methods, we need a first-order in space formulation.

Convected Helmholtz equation

Model problem :

Seek $(\boldsymbol{q}, p) \in \boldsymbol{H}_{\text{div}}(\Omega) \times H^1(\Omega)$ such that

$$\int_\Omega \boldsymbol{W_0 q} \cdot \boldsymbol{r}^* \mathrm{dx} - \int_\Omega p \mathrm{div}\,(\boldsymbol{r}^*)\,\mathrm{dx} + \int_{\partial\Omega} p \boldsymbol{r}^* \cdot \boldsymbol{n} \mathrm{d}\sigma = 0$$

$$-\omega^2 \int_\Omega \rho_0 p w^* \mathrm{dx} + 2i\omega \int_\Omega p \rho_0 \boldsymbol{v_0} \cdot \nabla w^* \mathrm{dx}$$

$$-\int_\Omega \boldsymbol{q} \cdot \nabla w^* \mathrm{dx} + \int_{\partial\Omega} w^* \boldsymbol{q} \cdot \boldsymbol{n} - 2i\omega p w^* \rho_0 \boldsymbol{v_0} \cdot \boldsymbol{n} \mathrm{d}\sigma = \int_\Omega s w^* \mathrm{dx}$$

for all $(\boldsymbol{r}, w) \in \boldsymbol{H}_{\text{div}}(\Omega) \times H^1(\Omega)$.

Convected Helmholtz equation

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for all $(\boldsymbol{r}, w) \in \boldsymbol{H}_{\text{div}}(\Omega) \times H^1(\Omega)$.
Well-posedness: if the carrier flow is subsonic

$$\inf_\Omega \left({c_0}^2 - |\boldsymbol{v_0}|^2 \right) > 0$$

then the problem is of Fredholm type with $(\boldsymbol{q}, p) \in \boldsymbol{H}_{\text{div}}(\Omega) \times H^1(\Omega)$.

Skeleton unknown: numerical trace $\lambda_h = p_h|_{\mathcal{E}_h}$.

$$\int_{\partial K} p_h \boldsymbol{r}^* \cdot \boldsymbol{n} \mathrm{d}\sigma \overset{\text{becomes}}{\longrightarrow} \int_{\partial K} \lambda_h \boldsymbol{r}^* \cdot \boldsymbol{n} \mathrm{d}\sigma$$

HDG(+) methods
Approximation spaces

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Approximation spaces:

Variable	Space	HDG	HDG+
Pressure p_h	$W_h(K)$	$\mathcal{P}_k(K)$	$\mathcal{P}_{k+1}(K)$
Flux $\boldsymbol{q}_h$	$\boldsymbol{V}_h(K)$	$\boldsymbol{\mathcal{P}}_k(K)$	
Trace λ_h	$M_h(e)$	$\mathcal{P}_k(e)$	

We will compare the HDG and HDG+ methods for time-harmonic convected waves.

HDG(+) methods
Approximation spaces

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Trace λ_h	$M_h(e)$	$\mathcal{P}_k(e)$	

Together with the weak formulation: leads to the discrete local problem (*ie* on an element $K \in \mathcal{T}_h$) :

$$\mathbb{A}^K \begin{bmatrix} {p_h}^K \\ {\boldsymbol{q}_h}^K \end{bmatrix} + \mathbb{C}^K \left[{\lambda_h}^K \right] = \mathbb{S}^K$$

HDG(+) methods
Numerical flux and transmission condition

At this point:

- The approximation spaces are discontinuous.
- We need to enforce the continuity of the numerical solution between the elements.
- We need to ensure the stability of the method.

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All of this is done by choosing the numerical flux, so that

$$\sum_{e\in\mathcal{E}_h}\int_e [\![\widehat{\boldsymbol{q}_h} - 2i\omega\boldsymbol{n}\cdot\widehat{\boldsymbol{v}_0 p_h}\boldsymbol{n}]\!]\mu^*\,\mathrm{d}\sigma = 0, \quad \forall\mu\in M_h$$

where $[\![\cdot]\!]$ is the DG-jump operator.

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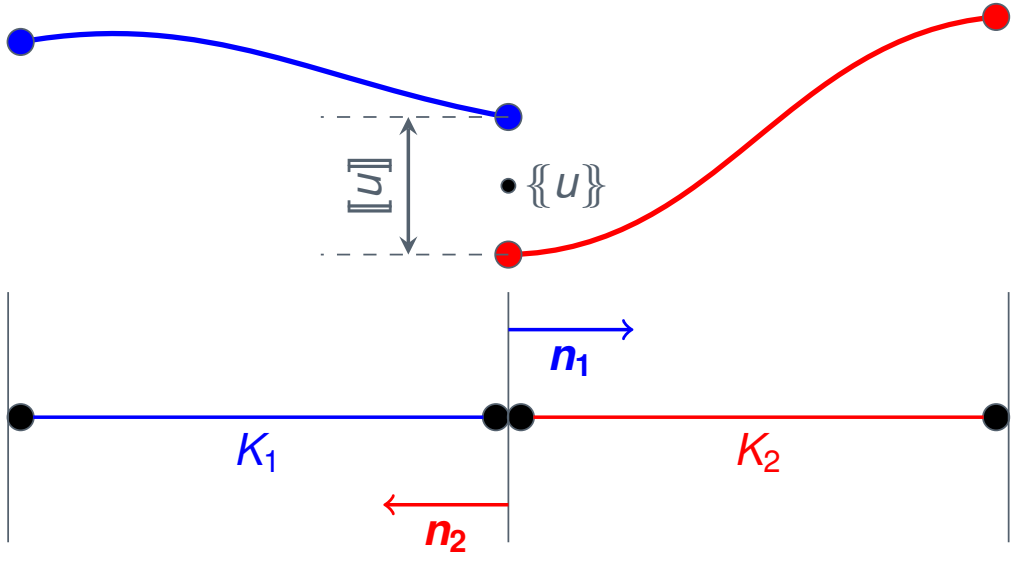

Figure 2: Discontinuity between two elements

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Numerical flux:

$$\widehat{\boldsymbol{q}_h}^K\cdot\boldsymbol{n} - 2i\omega\widehat{p_h{}^K\boldsymbol{v}_0}\cdot\boldsymbol{n} = \boldsymbol{q}_h{}^K\cdot\boldsymbol{n} - 2i\omega(\boldsymbol{v}_0\cdot\boldsymbol{n})\lambda_h{}^K + 2i\omega\tau\left(p_h{}^K - \lambda_h{}^K\right)$$

where

- τ: penalization parameter

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where

- τ: penalization parameter
- $\tau_{\mathrm{upw}} = \max(\boldsymbol{v}_0\cdot\boldsymbol{n}, 0)$: optional upwinding parameter

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This leads to the discrete transmission condition:

$$\sum_{K\in\mathcal{T}_h}\left(\mathbb{B}^K\begin{bmatrix}p_h{}^K\\ \boldsymbol{q}_h{}^K\end{bmatrix} + \mathbb{L}^K\left[\lambda_h{}^K\right]\right) = 0$$

Recall from the previous slides:

- Local problem

$$\mathbb{A}^K \begin{bmatrix} {p_h}^K \\ {\boldsymbol{q}_h}^K \end{bmatrix} + \mathbb{C}^K \left[{\lambda_h}^K \right] = \mathbb{S}^K$$

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HDG(+) methods
Implementation by static condensation

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Eliminate the interior nodes:

$$\begin{bmatrix} {p_h}^K \\ {\boldsymbol{q}_h}^K \end{bmatrix} = (\mathbb{A}^K)^{-1} \mathbb{S}^K - (\mathbb{A}^K)^{-1} \mathbb{C}^K \left[{\lambda_h}^K \right]$$

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Together with the transmission condition:

$$\sum_{K \in \mathcal{T}_h} \left(\mathbb{L}^K - \mathbb{B}^K (\mathbb{A}^K)^{-1} \mathbb{C}^K \right) \left[{\lambda_h}^K \right] = - \sum_{K \in \mathcal{T}_h} \mathbb{B}^K (\mathbb{A}^K)^{-1} \mathbb{S}^K$$

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Together with the transmission condition:

$$\mathbb{M} \left[\lambda_h \right] = \mathbb{S}$$

Some results on the methods

We performed the numerical analysis of the method:

- **Local solvability:** invertibility of $\mathbb{A}^K$, ✓ if

$$\omega h_K < C(\rho_0, \boldsymbol{v_0}, c_0, \text{element geometry})$$

τ well chosen

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$$\tau \text{ well chosen}$$

- **Convergence rate:** $(\boldsymbol{q}, p)$: exact solution
  - for HDG: (optimal ✗)

$$\|p_h - p\| = \mathcal{O}(h^{k+1}) \quad \text{and} \quad \|\boldsymbol{q}_h - \boldsymbol{q}\| = \mathcal{O}(h^{k+\frac{1}{2}})$$

  - for HDG+: (optimal ✓)

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  using:
  - L^2-orthogonal projections
  - a duality estimate only valid for regular solutions

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- **Global solvability:** invertibility of $\mathbb{M}$, ✓ if ω is not a resonant frequency.

Numerical results

Implementation of those methods is still a work in progress, but we have first promising results.

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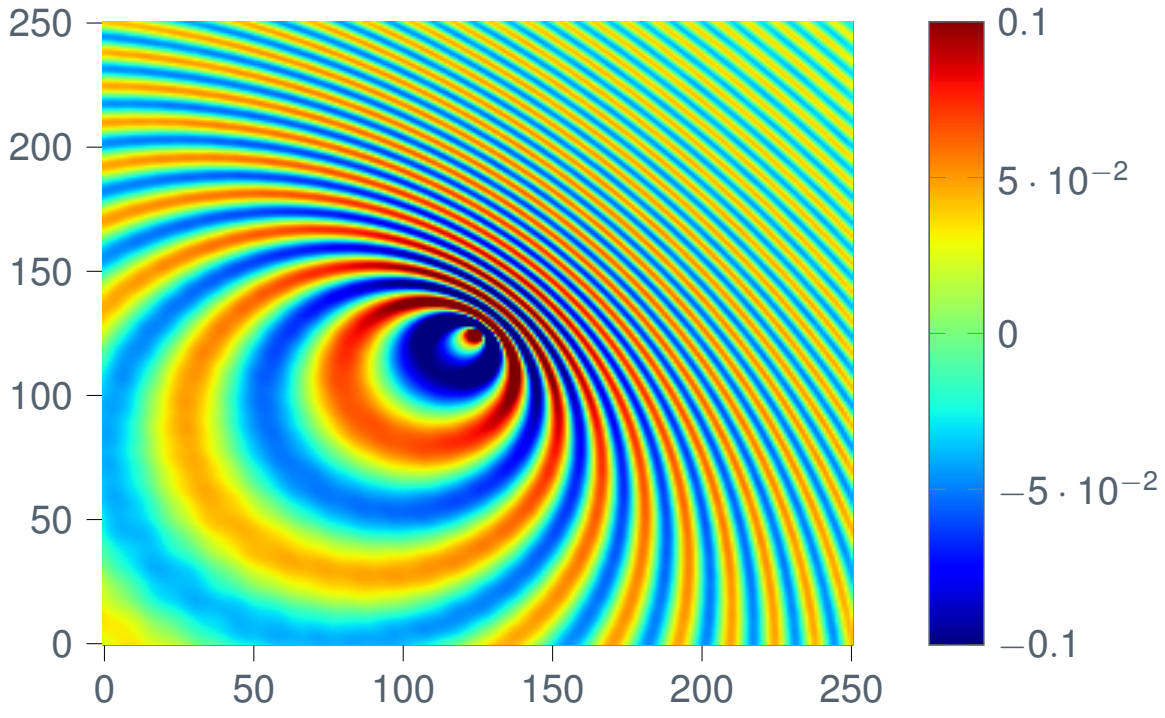

Figure 2: Green function

Numerical results

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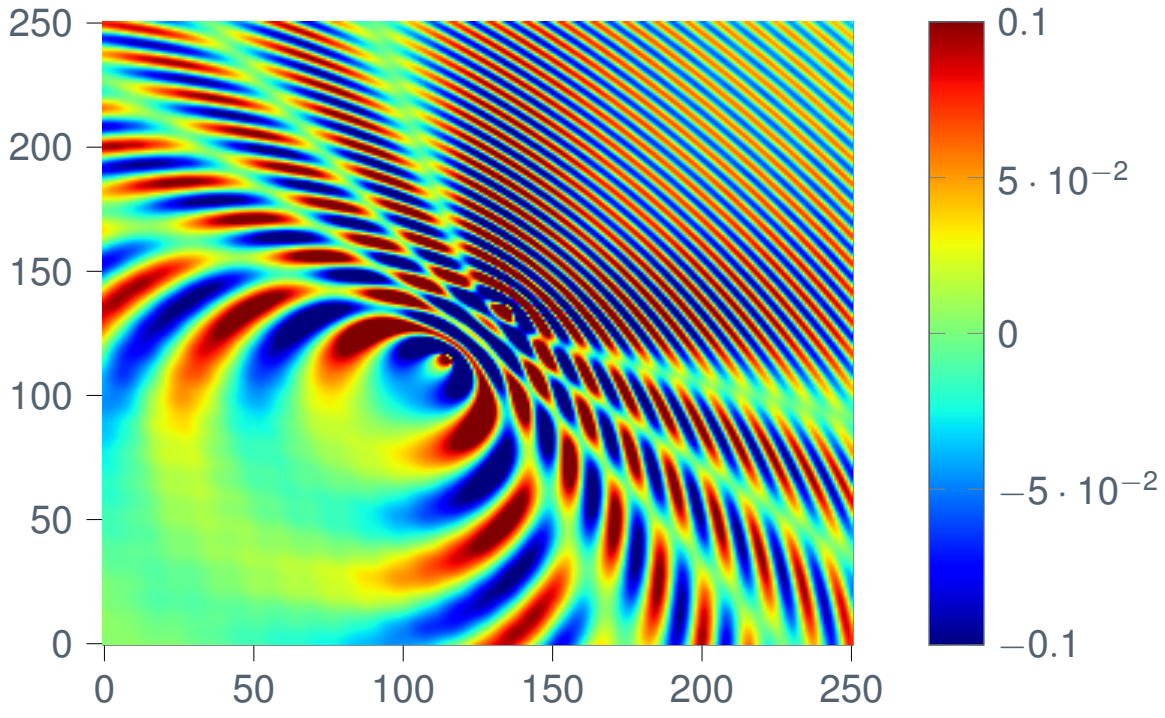

Figure 2: Interferences between two point sources

Thank you for your attention!
Any questions ?

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