

# Explicit expressions for $M^k$ for $k = 2, 3, 4$ , etc. and $\exp(Mt)$ for certain matrices $M$

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- 1 Different Ways to Find a  $k^{\text{th}}$  Power of a Matrix
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# Sylvester's Eigenvalue Expansion Formula for $M^k$

Assume  $M$  is a real  $n \times n$  matrix that has  $n$  known distinct eigenvalues  $\omega_1, \omega_2, \dots, \omega_n$ . Then

$$M^k = A_1 \omega_1^k + A_2 \omega_2^k + A_3 \omega_3^k + \dots + A_n \omega_n^k$$

where  $k = 0, 1, 2, \dots$  and  $A_1, A_2, \dots, A_n$  are constant  $n \times n$  coefficient matrices called spectral projectors (Meyer) or Frobenius covariants.

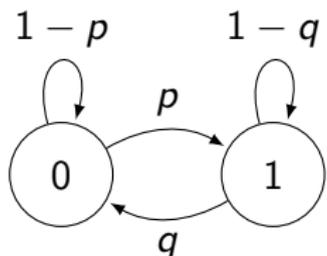
$$M^k(i, j) = A_1(i, j) \omega_1^k + A_2(i, j) \omega_2^k + A_3(i, j) \omega_3^k + \dots + A_n(i, j) \omega_n^k$$

where  $0 \leq i, j \leq n - 1$  for  $s = 1, 2, 3, \dots, n$ .

Note:  $A_s$  is the *matrix outer product*,  $\vec{R}_s \vec{L}_s$ , where  $\vec{R}_s$  and  $\vec{L}_s$  are right & left eigenvectors of matrix  $M$  and where  $\vec{L}_s$  and  $\vec{R}_s$  are scaled so that their dot product  $\vec{L}_s \cdot \vec{R}_s = 1$ .

## Example 1: General $2 \times 2$ Stochastic Matrix

Consider the following transition diagram and  $P$  matrix:



$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$$

where  $0 < p, q < 1$

Matrix  $P$  has eigenvalues  $\omega_1 = 1$  and  $\omega_2 = 1 - p - q$ . Then

$$P^k = A_1 \omega_1^k + A_2 \omega_2^k$$

$$= \begin{bmatrix} \frac{q}{p+q} & \frac{p}{p+q} \\ \frac{q}{p+q} & \frac{p}{p+q} \end{bmatrix} (1)^k + \begin{bmatrix} \frac{p}{p+q} & -\frac{p}{p+q} \\ -\frac{q}{p+q} & \frac{q}{p+q} \end{bmatrix} (1-p-q)^k$$

## Sylvester's Eigenvalue Expansion for $e^{Mt}$

Assume  $M$  is a real  $n \times n$  matrix that has  $n$  known distinct eigenvalues  $\omega_1, \omega_2, \dots, \omega_n$ . Then

$$e^{Mt} = A_1 e^{\omega_1 t} + A_2 e^{\omega_2 t} + A_3 e^{\omega_3 t} + \dots + A_n e^{\omega_n t}$$

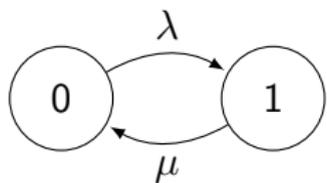
where  $k = 0, 1, 2, \dots$  and  $A_1, A_2, \dots, A_n$  are constant  $n \times n$  coefficient matrices called spectral projectors (Meyer) or Frobenius covariants.

$$e^{Mt}(i, j) = A_1(i, j)e^{\omega_1 t} + A_2(i, j)e^{\omega_2 t} + A_3(i, j)e^{\omega_3 t} + \dots + A_n(i, j)e^{\omega_n t}$$

where  $0 \leq i, j \leq n - 1$  for  $s = 1, 2, 3, \dots, n$  and  $0 \leq t$  is continuous time.

## Example 2: General $2 \times 2$ Exponential Matrix

Consider the following rate transition diagram and  $Q$  matrix:



$$Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

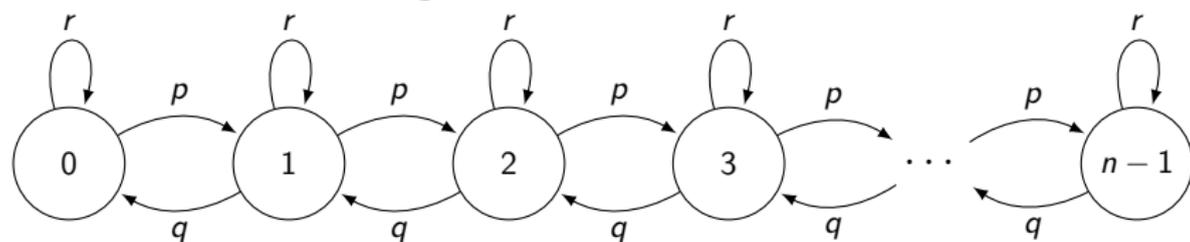
where  $\lambda, \mu > 0$

Matrix  $Q$  has eigenvalues  $\omega_1 = 0$  and  $\omega_2 = -(\lambda + \mu)$ . Then

$$\begin{aligned} e^{Qt} &= A_1 e^{\omega_1 t} + A_2 e^{\omega_2 t} \\ &= \begin{bmatrix} \frac{\mu}{\lambda + \mu} & \frac{\lambda}{\lambda + \mu} \\ \frac{\mu}{\lambda + \mu} & \frac{\lambda}{\lambda + \mu} \end{bmatrix} e^{(0)t} + \begin{bmatrix} \frac{\lambda}{\lambda + \mu} & -\frac{\lambda}{\lambda + \mu} \\ -\frac{\mu}{\lambda + \mu} & \frac{\mu}{\lambda + \mu} \end{bmatrix} e^{-(\lambda + \mu)t} \end{aligned}$$

# Probability of Paths in a Strip

Consider the transition diagram as follows:



where  $p, r, q$  are probabilities such that  $0 < p, q < 1$ ,  $p + r + q = 1$ .

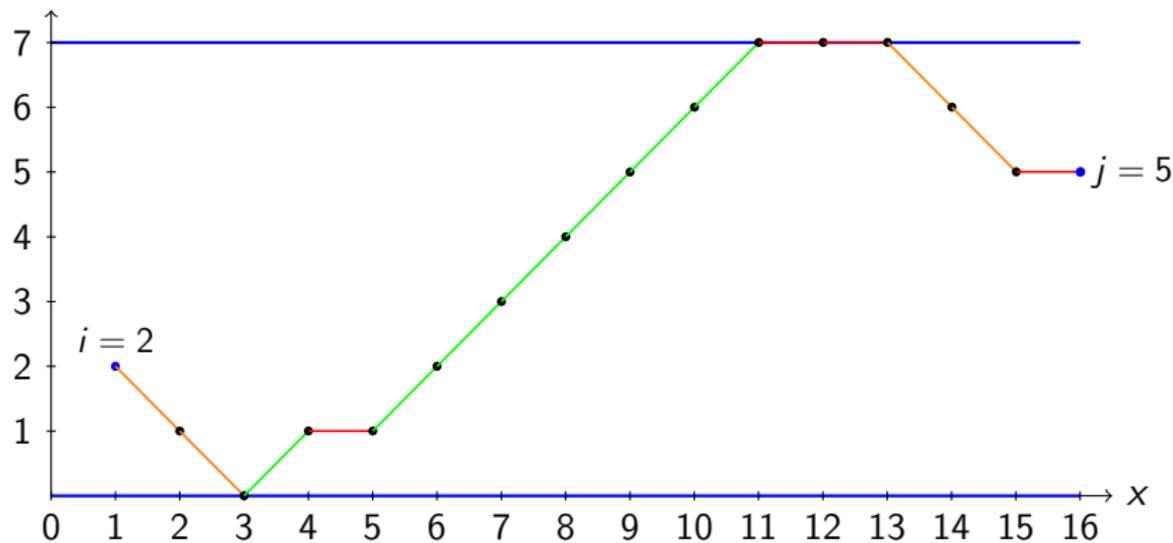
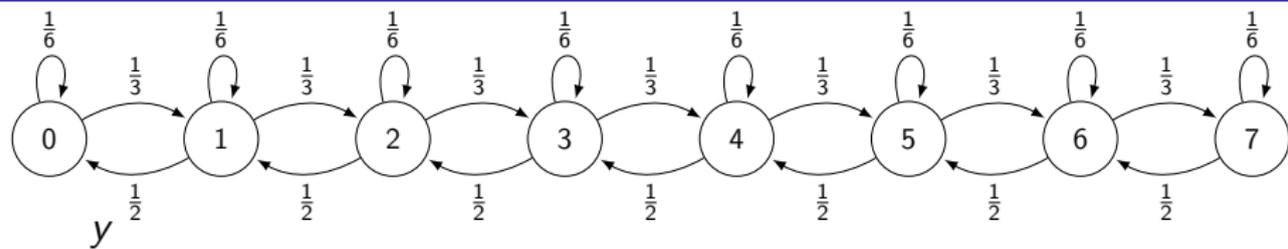
$$P = \begin{bmatrix} r & p & 0 & 0 & \dots & 0 & 0 \\ q & r & p & 0 & \dots & 0 & 0 \\ 0 & q & r & p & \dots & 0 & 0 \\ 0 & 0 & q & r & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & q & r \end{bmatrix}$$

$$P^k = \sum_{s=1}^n A_s \omega_s^k$$

$$\omega_s = r + 2\sqrt{pq} \cos(s\pi/(n+1))$$

$$A_s(i, j) = \frac{2}{n+1} \left(\frac{q}{p}\right)^{\frac{i-j}{2}} \sin\left(\frac{(i+1)\pi s}{n+1}\right) \sin\left(\frac{(j+1)\pi s}{n+1}\right)$$

# Example 3 Probability of All Paths from 2 to 5 in 15 Steps?



A sample path going from  $i = 1$  to  $j = 2$  in 15 steps staying within the strip

## Solution to Example 3

$$P^{15} = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{6} \end{pmatrix}^{15}$$

The solution is

$$P^{15}(2, 5) = \sum_{s=1}^8 \left[ \frac{2}{9} \left(\frac{3}{2}\right)^{-3} \sin\left(\frac{3\pi s}{9}\right) \sin\left(\frac{6\pi s}{9}\right) \left(\frac{1}{6} + 2\sqrt{\frac{1}{6}} \cos\left(\frac{s\pi}{9}\right)\right)^{15} \right] \approx 0.0298$$

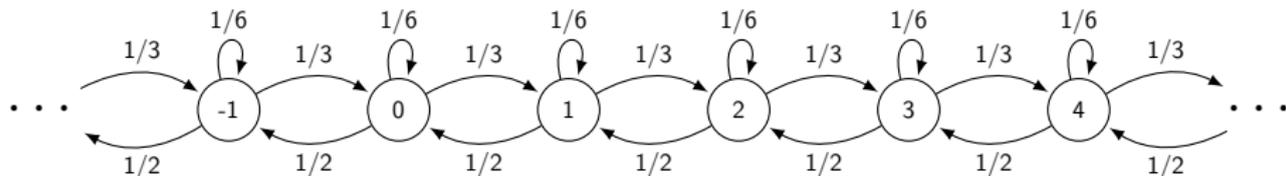
$$P_R^{15}(2, 5) = \frac{P^{15}(2, 5)}{\sum_{j=0}^7 P^{15}(2, j)} = 0.0925$$

## Ballot Box Problem

Suppose that in an election, candidate  $A$  receives  $a$  votes and candidate  $B$  receives  $b$  votes, where  $a > b$ . Compute the probability that  $A$  never falls behind  $B$  throughout the counting of the ballots.

### Example 4 Ballot Box Problem

Suppose candidate  $A$  receives 3 votes and candidate  $B$  receives 2 votes. Calculate the probability of going from state  $i = 0$  to state  $j = 1$  in  $n = 5$  steps without going below the  $x$ -axis.



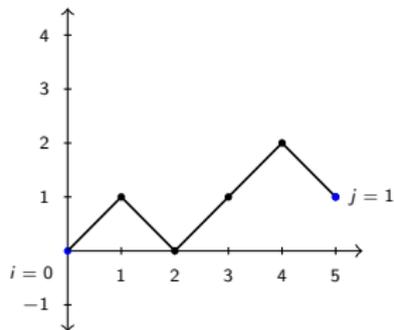
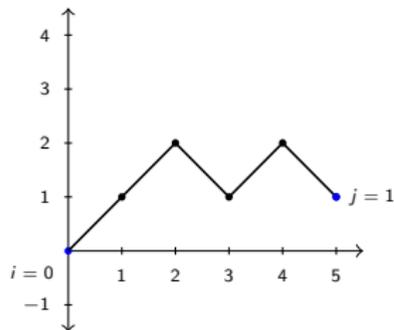
We will solve Example 3 in two different ways.

# Example 4 Ballot Box Problem

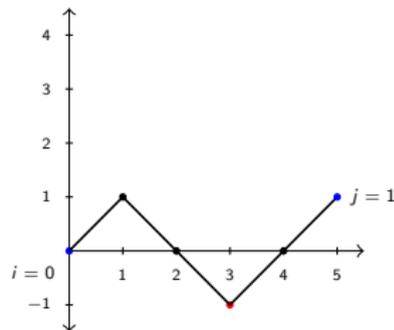
## Good and Bad Lattice Paths

A **good** lattice path from  $i$  to  $j$  in  $n$  steps never goes below the  $x$ -axis.

A **bad** lattice path from  $i$  to  $j$  in  $n$  steps goes below the  $x$ -axis somewhere along the way.

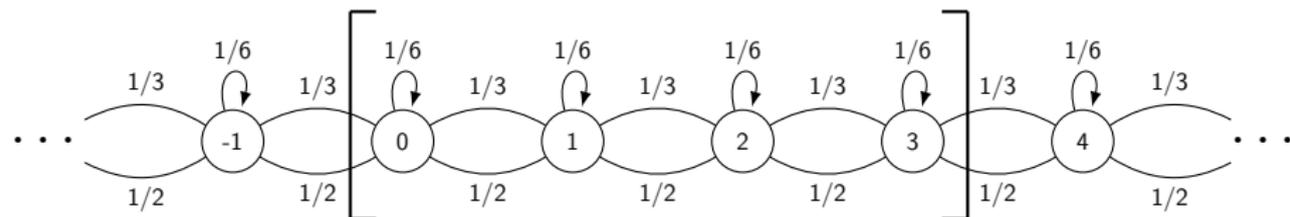


Examples of good lattice paths



Example of a bad lattice path

# Good Paths



Birth-death state transition diagram

$$\mathcal{G} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \left[ \begin{array}{cccc} \frac{1}{6} & \frac{1}{3} & 0 & 0 \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & \frac{1}{6} \end{array} \right] \end{matrix}$$

1-step transition probability matrix for **good paths**

## Matrix-Power Method to Solve Example 4

The probability of all paths starting at  $i = 0$ , ending at  $j = 1$ , in 5 steps & never going outside the boundaries of the solution strip is  $G_{0,1}^5 = \frac{305}{3888}$ .

$$G^5 = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & 0 & 0 \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & \frac{1}{6} \end{pmatrix}^5$$
$$= \begin{pmatrix} \frac{421}{7776} & \frac{305}{3888} & \frac{25}{486} & \frac{7}{243} \\ \frac{305}{2592} & \frac{1021}{7776} & \frac{473}{3888} & \frac{25}{486} \\ \frac{25}{216} & \frac{473}{2592} & \frac{1021}{7776} & \frac{305}{3888} \\ \frac{7}{72} & \frac{25}{216} & \frac{305}{2592} & \frac{219}{4045} \end{pmatrix}$$

The probability of starting at state  $i = 0$ , ending at state  $j = 1$  in 5 steps

with no restrictions is  $\mathcal{A}_{0,1}^5 = \frac{545}{3888}$  as shown below.

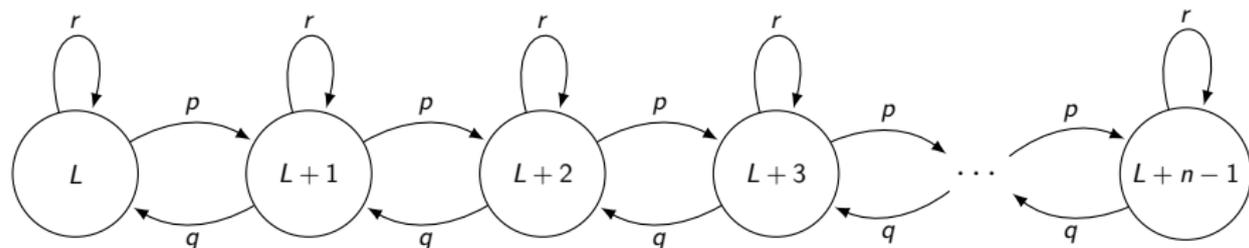
$$\mathcal{A}^5 = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{6} \end{pmatrix}^5$$

$$\begin{aligned} \text{So, } \frac{\mathcal{G}^5(0,1)}{\mathcal{A}^5(0,1)} &= \frac{\left(\frac{305}{3888}\right)}{\left(\frac{545}{3888}\right)} \\ &= \frac{305}{545} \\ &\approx 0.560 \end{aligned}$$

$$= \begin{pmatrix} \frac{421}{7776} & \frac{305}{3888} & \frac{25}{486} & \frac{17}{486} & \frac{5}{486} & \frac{1}{26} \\ \frac{305}{2592} & \frac{1021}{7776} & \frac{509}{3888} & \frac{65}{972} & \frac{10}{243} & \frac{5}{486} \\ \frac{25}{216} & \frac{509}{2592} & \frac{1201}{7776} & \frac{545}{3888} & \frac{65}{972} & \frac{17}{486} \\ \frac{17}{144} & \frac{65}{432} & \frac{545}{2592} & \frac{1201}{7776} & \frac{509}{3888} & \frac{25}{486} \\ \frac{5}{96} & \frac{5}{36} & \frac{65}{432} & \frac{509}{2592} & \frac{1021}{7776} & \frac{305}{3888} \\ \frac{1}{32} & \frac{5}{96} & \frac{17}{144} & \frac{25}{216} & \frac{305}{2592} & \frac{421}{7776} \end{pmatrix}$$

# Ballot Box Path Strips

The preceding ballot box transition diagrams can be generalized to



$n, L \in \mathbb{Z}$ , where  $n > 1$  and  $3 - n \leq L \leq 0$  and  $p, r, q$  are probabilities such that  $0 < p, q < 1$  and  $p + r + q = 1$ .

$$P = \begin{bmatrix} r & p & 0 & 0 & \dots & 0 & 0 \\ q & r & p & 0 & \dots & 0 & 0 \\ 0 & q & r & p & \dots & 0 & 0 \\ 0 & 0 & q & r & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & q & r \end{bmatrix}$$

$$P^k = \sum_{s=1}^n A_s \omega_s^k$$

$$\omega_s = r + 2\sqrt{pq} \cos(s\pi/(n+1))$$

$$A_s(i, j) = \frac{2}{n+1} \left(\frac{q}{p}\right)^{\frac{i-j}{2}} \sin\left(\frac{(i-j+k+2)\pi s}{2(n+1)}\right) \sin\left(\frac{(-i+j+k+2)\pi s}{2(n+1)}\right)$$

## Using the Sylvester Eigenvalue Expansion Formula to Solve Example 4

$$P^k(i, j) = \sum_{s=1}^n \frac{2}{n+1} \left(\frac{q}{p}\right)^{\frac{i-j}{2}} \sin\left(\frac{(i-j+k+2)\pi s}{2(n+1)}\right) \sin\left(\frac{(-i+j+k+2)\pi s}{2(n+1)}\right) \left(r + 2\sqrt{pq} \cos\left(\frac{s\pi}{n+1}\right)\right)^k$$

For **good** paths with  $i = 0$ ,  $j = 1$ ,  $k = 5$ ,  $n = 4$ ,  $q = \frac{1}{2}$ ,  $p = \frac{1}{3}$ ,  $r = \frac{1}{6}$ ,  $L = 0$

$$\begin{aligned} \mathcal{G}^5(0, 1) &= \sum_{s=1}^4 \frac{2}{5} \left(\frac{1/2}{1/3}\right)^{\frac{-1}{2}} \sin\left(\frac{3\pi s}{5}\right) \sin\left(\frac{4\pi s}{5}\right) \left(\frac{1}{6} + 2\sqrt{\frac{1}{3} \times \frac{1}{2}} \cos\left(\frac{s\pi}{5}\right)\right)^5 \\ &\approx \frac{179}{2531} + \frac{30}{12727} + \frac{1}{1188665} + \frac{66}{12301} \approx \frac{305}{3888} \end{aligned}$$

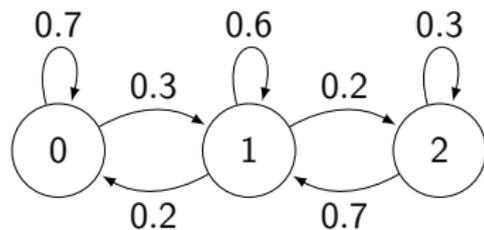
For **all** paths with  $i = 0$ ,  $j = 1$ ,  $k = 5$ ,  $n = 6$ ,  $q = \frac{1}{2}$ ,  $p = \frac{1}{3}$ ,  $r = \frac{1}{6}$ ,  $L = -2$

$$\begin{aligned} \mathcal{A}^5(0, 1) &= \sum_{s=1}^6 \frac{2}{7} \left(\frac{1/2}{1/3}\right)^{\frac{-1}{2}} \sin\left(\frac{3\pi s}{7}\right) \sin\left(\frac{4\pi s}{7}\right) \left(\frac{1}{6} + 2\sqrt{\frac{1}{2} \times \frac{1}{3}} \cos\left(\frac{s\pi}{7}\right)\right)^5 \\ &\approx \frac{37}{279} + \frac{-109}{17615} + \frac{24}{32809} + 0 + \frac{-23}{111265} + \frac{112}{8471} \approx \frac{545}{3888} \end{aligned}$$

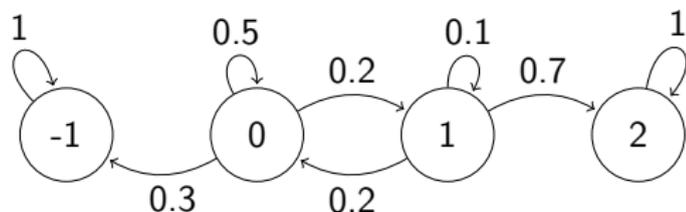
So  $\frac{\mathcal{G}^5(0,1)}{\mathcal{A}^5(0,1)} = \frac{\left(\frac{305}{3888}\right)}{\left(\frac{545}{3888}\right)} = \frac{305}{545} \approx 0.560$ , which agrees with our previous solution.

# Example: The Dual Matrix of a Birth-Death Matrix

Suppose  $P = \begin{bmatrix} 0.7 & 0.3 & 0 \\ 0.2 & 0.6 & 0.2 \\ 0 & 0.7 & 0.3 \end{bmatrix}$

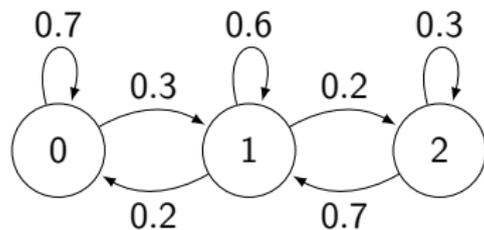


$$P^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.3 & 0.5 & 0.2 & 0 \\ 0 & 0.2 & 0.1 & 0.7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

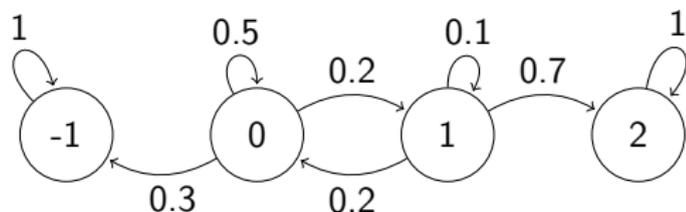


# Example: The Dual Matrix of a Birth-Death Matrix

Suppose  $P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.7 & 0.3 & 0 & 0 \\ 0.2 & 0.6 & 0.2 & 0 \\ 0 & 0.7 & 0.3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$



$P^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.3 & 0.5 & 0.2 & 0 \\ 0 & 0.2 & 0.1 & 0.7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$



# Further Explanation for How to Create the Dual Matrix

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.7 & 0.3 & 0 & 0 \\ 0.2 & 0.6 & 0.2 & 0 \\ 0 & 0.7 & 0.3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.3 & 0.5 & 0.2 & 0 \\ 0 & 0.2 & 0.1 & 0.7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.7 & 0.3 & 0 & 0 \\ 0.2 & 0.6 & 0.2 & 0 \\ 0 & 0.7 & 0.3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.3 & 0.5 & 0.2 & 0 \\ 0 & 0.2 & 0.1 & 0.7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.7 & 0.3 & 0 & 0 \\ 0.2 & 0.6 & 0.2 & 0 \\ 0 & 0.7 & 0.3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.3 & 0.5 & 0.2 & 0 \\ 0 & 0.2 & 0.1 & 0.7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

## Formal Definition of the Dual Matrix

Assume  $P = [P_{i,j}]$  for  $i, j$  going from  $0, 1, \dots, H$  is a  $(H + 1) \times (H + 1)$  stochastic matrix. Suppose that the  $(H + 2) \times (H + 2)$  matrix  $P^* = [P_{i,j}^*]$  where:

$$P_{i,j}^* = \sum_{s=i+1}^H [P_{j+1,s} - P_{j,s}]$$

is also a stochastic matrix under the following conventions:

- $P_{-1,s} = 0$  if  $-1 < s \leq H$
- $P_{H,H}^* = 1$
- $P_{i,H}^* = 1 - \sum_{s=0}^{H-1} P_{i,s}^*$

Then  $P^*$  is called the dual matrix of  $P$ . Note that  $P^*$  may not exist as a stochastic matrix.  $P$  and  $P^*$  have the same set of eigenvalues, see [7].

# Further Explanation for How to Create the Dual Matrix

- Applying the previously mentioned algorithm, we find the first 3 columns and rows of the dual matrix as
$$\text{sum of entries} - \text{sum of entries} = \text{entry}$$
- Using a property of stochastic matrices, we find the entries of the last column by subtracting the sum of the row from 1,
- The last row is found by the same property with  $P_{H,H}^* = 1$

# Duality Theorem

Suppose  $P$  is a stochastic  $(H + 1) \times (H + 1)$  matrix and its dual matrix  $P^*$  exists, then

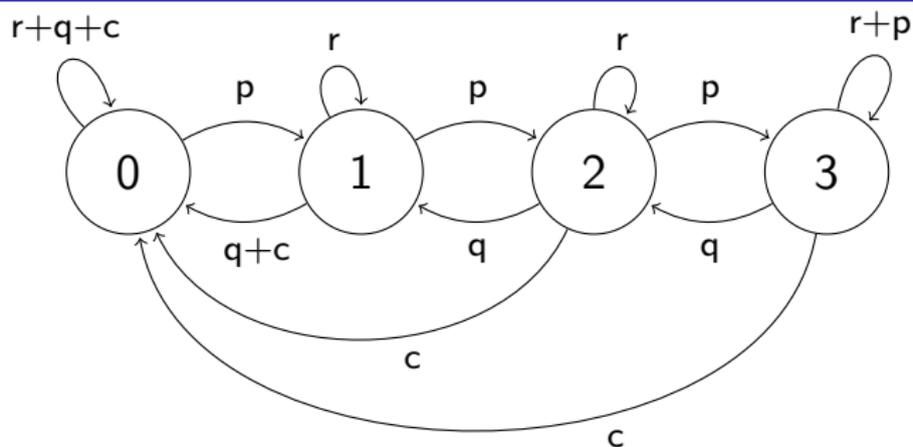
$$P_{i,j}^{*(n)} = \sum_{s=i+1}^H \left[ P_{j+1,s}^{(n)} - P_{j,s}^{(n)} \right] \quad \text{for } i, j = -1, 0, 1, \dots, H$$

$$P_{i,j}^{(n)} = \sum_{s=i}^H \left[ P_{j,k}^{*(n)} - P_{j-1,s}^{*(n)} \right] \quad \text{for } i, j = 0, 1, 2, \dots, H$$

hold for  $n = 1, 2, 3, \dots$  with the conventions:

- $P_{-1,s}^{(n)} = 0$  if  $-1 < s \leq H$
- $P_{H,H}^{*(n)} = 1$
- $P_{i,H}^{*(n)} = 1 - \sum_{s=0}^{H-1} P_{i,s}^{*(n)}$

# Birth-Death Chain with Catastrophes



$$P = \begin{bmatrix} r+q+c & p & 0 & 0 \\ q+c & r & p & 0 \\ c & q & r & p \\ c & 0 & q & r+p \end{bmatrix}$$

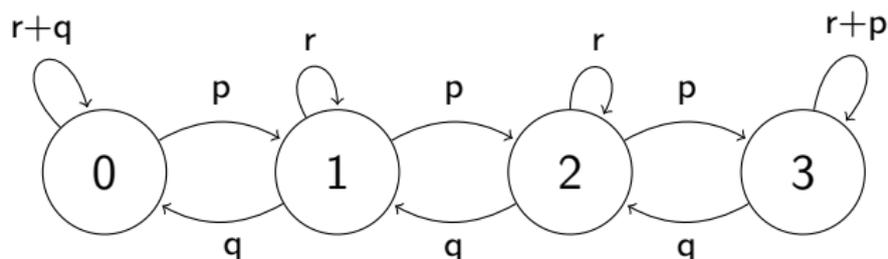
$$0 < p, q, c < 1$$

$$p + q + r + c = 1$$

$$0 \leq r < 1$$

# Birth-Death Chain

When  $c = 0$ , the birth-death chain with catastrophes become the following birth-death chain:



$$P = \begin{bmatrix} r+q & p & 0 & 0 \\ q & r & p & 0 \\ 0 & q & r & p \\ 0 & 0 & q & r+p \end{bmatrix}$$

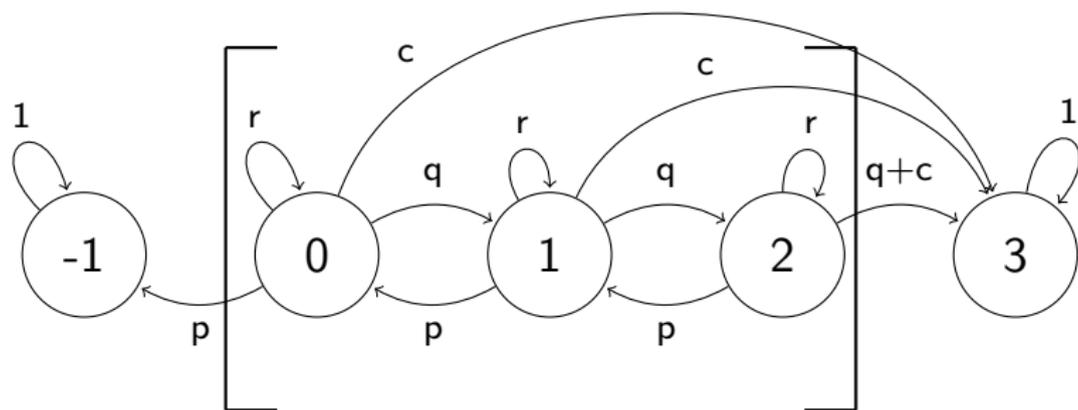
$$0 < p, q < 1$$

$$p + q + r = 1$$

$$0 \leq r < 1$$

# Finding the Dual

# Dual of the Birth-Death Chain with Catastrophes



$$P^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ p & \begin{bmatrix} r & q & 0 \\ p & r & q \\ 0 & p & r \end{bmatrix} & c \\ 0 & & c \\ 0 & & q+c \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} 0 < p, q, c < 1 \\ p + q + r + c &= 1 \\ 0 \leq r < 1 \end{aligned}$$

Solving for  $P_{i,j}^{*(k)}$  where  $i$  and  $j = 0, 1, 2$

$$P^{*(k)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ - & \begin{bmatrix} P_{0,0}^{*(k)} & P_{0,1}^{*(k)} & P_{0,2}^{*(k)} \\ P_{1,0}^{*(k)} & P_{1,1}^{*(k)} & P_{1,2}^{*(k)} \\ P_{2,0}^{*(k)} & P_{2,1}^{*(k)} & P_{2,2}^{*(k)} \end{bmatrix} & - & - & - \\ - & & - & - & - \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M^k(u, v) = \sum_{s=1}^n \left[ \frac{2}{n+1} \left( \frac{q}{p} \right)^{\frac{u-v}{2}} \sin \left( \frac{u\pi s}{n+1} \right) \sin \left( \frac{v\pi s}{n+1} \right) \left( r + 2\sqrt{pq} \cos \left( \frac{s\pi}{n+1} \right) \right)^k \right]$$

For  $u = i + 1$   $v = j + 1$  switch  $p$  for  $q$   $n = 3$

$$P_{i,j}^{*(k)} = \sum_{s=1}^n \left[ \frac{2}{n+1} \left( \frac{p}{q} \right)^{\frac{i-j}{2}} \sin \left( \frac{(i+1)\pi s}{n+1} \right) \sin \left( \frac{(j+1)\pi s}{n+1} \right) \left( r + 2\sqrt{pq} \cos \left( \frac{s\pi}{n+1} \right) \right)^k \right]$$

where  $i, j = 0, 1, 2$  and  $k \in \mathbb{N}$  and  $n$  in general is the dimension of the green matrix.

# Solving for $P_{i,-1}^{*(k)}$ where $i = 0, 1, 2$

$$P^{*(k)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ P_{0,-1}^{*(k)} & P_{0,0}^{*(k)} & P_{0,1}^{*(k)} & P_{0,2}^{*(k)} & P_{0,3}^{*(k)} \\ P_{1,-1}^{*(k)} & P_{1,0}^{*(k)} & P_{1,1}^{*(k)} & P_{1,2}^{*(k)} & P_{1,3}^{*(k)} \\ P_{2,-1}^{*(k)} & P_{2,0}^{*(k)} & P_{2,1}^{*(k)} & P_{2,2}^{*(k)} & P_{2,3}^{*(k)} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We can find the probability going from state  $i$  to state  $-1$  labelled by the highlighted column by using the formula given:

$$P_{i,-1}^{*(k)} = p \sum_{\alpha=i}^{k-1} P_{i,0}^{*(\alpha)} \text{ where } i = 0, 1, 2$$

Substituting  $P_{i,0}^{*(k)}$  from the previous slide and after some algebra we obtain:

# Solving for $P_{i,3}^{*(k)}$ where $i = 0, 1, 2$

$$P^{*(k)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ P_{0,-1}^{*(k)} & \begin{bmatrix} P_{0,0}^{*(k)} & P_{0,1}^{*(k)} & P_{0,2}^{*(k)} \end{bmatrix} & P_{0,3}^{*(k)} \\ P_{1,-1}^{*(k)} & \begin{bmatrix} P_{1,0}^{*(k)} & P_{1,1}^{*(k)} & P_{1,2}^{*(k)} \end{bmatrix} & P_{1,3}^{*(k)} \\ P_{2,-1}^{*(k)} & \begin{bmatrix} P_{2,0}^{*(k)} & P_{2,1}^{*(k)} & P_{2,2}^{*(k)} \end{bmatrix} & P_{2,3}^{*(k)} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We can then calculate the probability going from state  $i$  to 3, where  $i = 0, 1, 2$  represented by the blue column, by using the stochastic property of the matrix  $P_{i,3}^{*(k)} = 1 - \sum_{s=-1}^2 P_{i,s}^{*(k)}$

## Solving $P^{(k)}$

Once we solve the  $P^*$  we can use the duality theorem to calculate each element of our  $P$  matrix

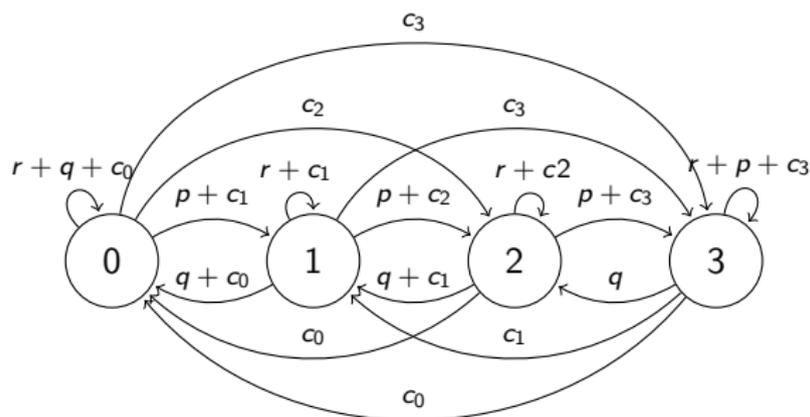
$$P_{i,j}^{*(k)} = \sum_{s=i+1}^H \left[ P_{j+1,s}^{(k)} - P_{j,s}^{(k)} \right]$$

$$P_{i,j}^{(k)} = \sum_{s=i}^H \left[ P_{j,s}^{*(k)} - P_{j-1,s}^{*(k)} \right]$$

for  $n \geq 0$  and for all states  $i, j = 0, 1, 2, \dots, H$  with the conventions:

- $P_{i,H}^{*(k)} = 1 - \sum_{s=0}^{H-1} P_{i,s}^{*(k)}$
- $P_{-1,s}^{(k)} = 0$  if  $s > -1$
- $P_{H,H}^{*(k)} = 1$

# Generalized Catastrophe Matrix

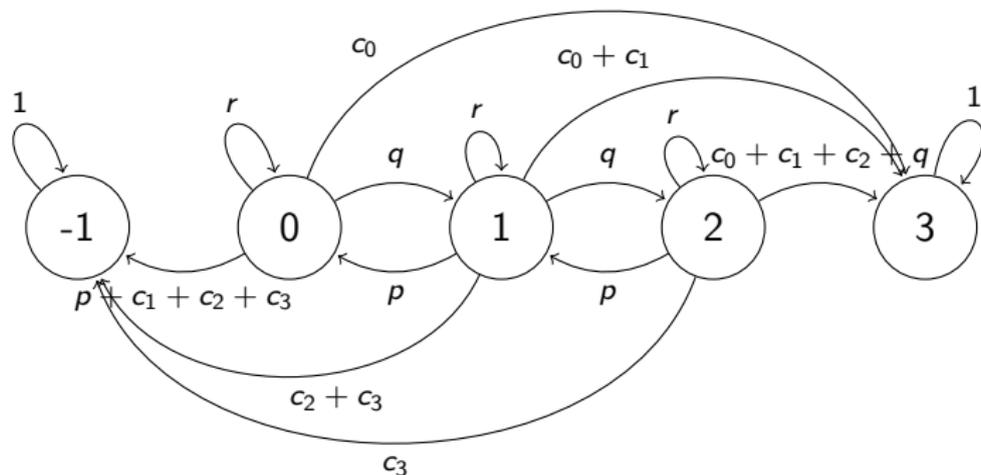


$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \left[ \begin{array}{cccc} r+q+c_0 & p+c_1 & c_2 & c_3 \\ q+c_0 & r+c_1 & p+c_2 & c_3 \\ c_0 & q+c_1 & r+c_2 & p+c_3 \\ c_0 & c_1 & q+c_2 & r+p+c_3 \end{array} \right] \end{matrix}$$

$$p+r+q+c_0+c_1+c_2+c_3 = 1, \quad 0 < p, q, c_0, c_1, c_2, c_3 < 1, \quad 0 \leq r < 1$$

# Finding the Dual Matrix

# Dual of the Generalized



$$P^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ p + c_1 + c_2 + c_3 & \begin{bmatrix} r & q & 0 \end{bmatrix} & 0 \\ c_2 + c_3 & \begin{bmatrix} p & r & q \end{bmatrix} & c_0 \\ c_3 & \begin{bmatrix} 0 & p & r \end{bmatrix} & c_0 + c_1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Solving for  $P_{i,j}^{*(k)}$  where  $i$  and  $j = 0, 1, 2$

$$P^{*(k)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ - & \begin{bmatrix} P_{0,0}^{*(k)} & P_{0,1}^{*(k)} & P_{0,2}^{*(k)} \\ P_{1,0}^{*(k)} & P_{1,1}^{*(k)} & P_{1,2}^{*(k)} \\ P_{2,0}^{*(k)} & P_{2,1}^{*(k)} & P_{2,2}^{*(k)} \end{bmatrix} & - & - & - \\ - & & - & - & - \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M^k(u, v) = \sum_{s=1}^n \left[ \frac{2}{n+1} \left( \frac{q}{p} \right)^{\frac{u-v}{2}} \sin \left( \frac{u\pi s}{n+1} \right) \sin \left( \frac{v\pi s}{n+1} \right) \left( r + 2\sqrt{pq} \cos \left( \frac{s\pi}{n+1} \right) \right) \right]$$

For  $u = i + 1$   $v = j + 1$  switch  $p$  for  $q$   $n = 3$

$$P_{i,j}^{*(k)} = \sum_{s=1}^n \left[ \frac{2}{n+1} \left( \frac{p}{q} \right)^{\frac{i-j}{2}} \sin \left( \frac{(i+1)\pi s}{n+1} \right) \sin \left( \frac{(j+1)\pi s}{n+1} \right) \left( r + 2\sqrt{pq} \cos \left( \frac{s\pi}{n+1} \right) \right)^k \right]$$

where  $i, j = 0, 1, 2$  and  $k \in \mathbb{N}$  and  $n$  in general is the dimension of the green matrix.

# Solving for $P_{i,-1}^{*(k)}$ where $i = 0, 1, 2$

Where  $P_{i,j}^{*(\alpha)}$ ,  $i, j = 0, 1, 2$  can be calculated by formula from slide 16 We can do a similar approach with what we have in slide 17 to solve for the highlighted column

$$P^{*(k)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ P_{0,-1}^{*(k)} & \begin{bmatrix} P_{0,0}^{*(k)} & P_{0,1}^{*(k)} & P_{0,2}^{*(k)} \end{bmatrix} & P_{0,3}^{*(k)} \\ P_{1,-1}^{*(k)} & \begin{bmatrix} P_{1,0}^{*(k)} & P_{1,1}^{*(k)} & P_{1,2}^{*(k)} \end{bmatrix} & P_{1,3}^{*(k)} \\ P_{2,-1}^{*(k)} & \begin{bmatrix} P_{2,0}^{*(k)} & P_{2,1}^{*(k)} & P_{2,2}^{*(k)} \end{bmatrix} & P_{2,3}^{*(k)} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P_{0,-1}^{*(k)} = (p + c_1 + c_2 + c_3) \sum_{\alpha=0}^{k-1} P_{0,0}^{*(\alpha)} + (c_2 + c_3) \sum_{\alpha=1}^{k-1} P_{0,1}^{*(\alpha)} + (c_3) \sum_{\alpha=2}^{k-1} P_{0,2}^{*(\alpha)}$$

similar approach can be used to solve  $P_{1,-1}^{*(k)}$   $P_{2,-1}^{*(k)}$

# Solving for $P_{i,3}^{*(k)}$ where $i = 0, 1, 2$

$$P^{*(k)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ P_{0,-1}^{*(k)} & \begin{bmatrix} P_{0,0}^{*(k)} & P_{0,1}^{*(k)} & P_{0,2}^{*(k)} \end{bmatrix} & P_{0,3}^{*(k)} \\ P_{1,-1}^{*(k)} & \begin{bmatrix} P_{1,0}^{*(k)} & P_{1,1}^{*(k)} & P_{1,2}^{*(k)} \end{bmatrix} & P_{1,3}^{*(k)} \\ P_{2,-1}^{*(k)} & \begin{bmatrix} P_{2,0}^{*(k)} & P_{2,1}^{*(k)} & P_{2,2}^{*(k)} \end{bmatrix} & P_{2,3}^{*(k)} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We can then calculate the probability going from state  $i$  to 3, where  $i = 0, 1, 2$  represented by the blue column, by using the stochastic property of the matrix  $P_{i,3}^{*(k)} = 1 - \sum_{s=-1}^2 P_{i,s}^{*(k)}$

## Solving $P^{(k)}$

Once we solve the  $P^*$  we can use the duality theorem to calculate each element of our  $P$  matrix

$$P_{i,j}^{*(k)} = \sum_{s=i+1}^H \left[ P_{j+1,s}^{(k)} - P_{j,s}^{(k)} \right]$$

$$P_{i,j}^{(k)} = \sum_{s=i}^H \left[ P_{j,s}^{*(k)} - P_{j-1,s}^{*(k)} \right]$$

for  $n \geq 0$  and for all states  $i, j = 0, 1, 2, \dots, H$  with the conventions:

- $P_{i,H}^{*(k)} = 1 - \sum_{s=0}^{H-1} P_{i,s}^{*(k)}$
- $P_{-1,s}^{(k)} = 0$  if  $s > -1$
- $P_{H,H}^{*(k)} = 1$

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Thank  
you!

# Sylvester's Formula for Tridiagonal Toeplitz Matrices

For  $s = 1, 2, \dots, n$ , the eigenvalues of  $P$  are known to be

$$\omega_s = r + 2\sqrt{pq} \cos(s\pi/(n+1))$$

For tridiagonal Toeplitz matrices, the spectral projectors  $A_s$  are these

$$A_s(i, j) = \frac{2}{n+1} \left(\frac{q}{p}\right)^{\frac{i-j}{2}} \sin\left(\frac{(i+1)\pi s}{n+1}\right) \sin\left(\frac{(j+1)\pi s}{n+1}\right)$$

where  $i$  and  $j$  range from 0 to  $n-1$ . Knowing the eigenvalues and spectral projectors of  $P$ , we can write the entries of the  $k$ th power of  $P$ ,  $k \in \mathbb{N}$ , as

$$P^k(i, j) = \sum_{s=1}^n \frac{2}{n+1} \left(\frac{q}{p}\right)^{\frac{i-j}{2}} \sin\left(\frac{(i-j+k+2)\pi s}{2(n+1)}\right) \sin\left(\frac{(-i+j+k+2)\pi s}{2(n+1)}\right) \left(r + 2\sqrt{pq} \cos\left(\frac{s\pi}{n+1}\right)\right)^k$$

Note:  $A_s$  is the *matrix outer product*,  $\vec{R}_s \vec{L}_s$ , where  $\vec{R}_s$  and  $\vec{L}_s$  are right & left eigenvectors of matrix  $P$  and where  $\vec{L}_s$  and  $\vec{R}_s$  are scaled so that their *dot product*  $\vec{L}_s \cdot \vec{R}_s = 1$ .

Solving for  $P_{i,j}^{*(k)}$  where  $i$  and  $j = 0, 1, 2$

$$P^{*(k)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ - & \begin{bmatrix} P_{0,0}^{*(k)} & P_{0,1}^{*(k)} & P_{0,2}^{*(k)} \\ P_{1,0}^{*(k)} & P_{1,1}^{*(k)} & P_{1,2}^{*(k)} \\ P_{2,0}^{*(k)} & P_{2,1}^{*(k)} & P_{2,2}^{*(k)} \end{bmatrix} & - & - & - \\ - & & - & - & - \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

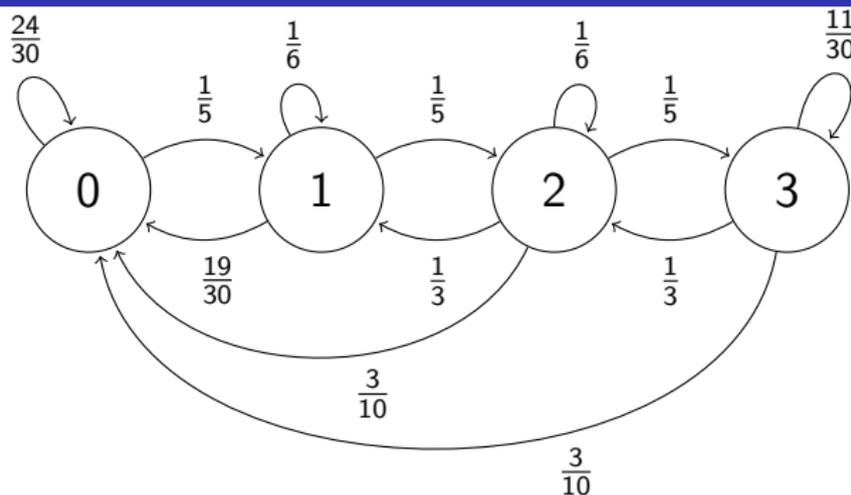
$$M^k(u, v) = \sum_{s=1}^n \left[ \frac{2}{n+1} \left( \frac{q}{p} \right)^{\frac{u-v}{2}} \sin \left( \frac{u\pi s}{n+1} \right) \sin \left( \frac{v\pi s}{n+1} \right) \left( r + 2\sqrt{pq} \cos \left( \frac{s\pi}{n+1} \right) \right) \right]$$

For  $u = i + 1$   $v = j + 1$  switch  $p$  for  $q$   $n = 3$

$$P_{i,j}^{*(k)} = \sum_{s=1}^n \left[ \frac{2}{n+1} \left( \frac{p}{q} \right)^{\frac{i-j}{2}} \sin \left( \frac{(i+1)\pi s}{n+1} \right) \sin \left( \frac{(j+1)\pi s}{n+1} \right) \left( r + 2\sqrt{pq} \cos \left( \frac{s\pi}{n+1} \right) \right)^k \right]$$

where  $i, j = 0, 1, 2$  and  $k \in \mathbb{N}$  and  $n$  in general is the dimension of the green matrix.

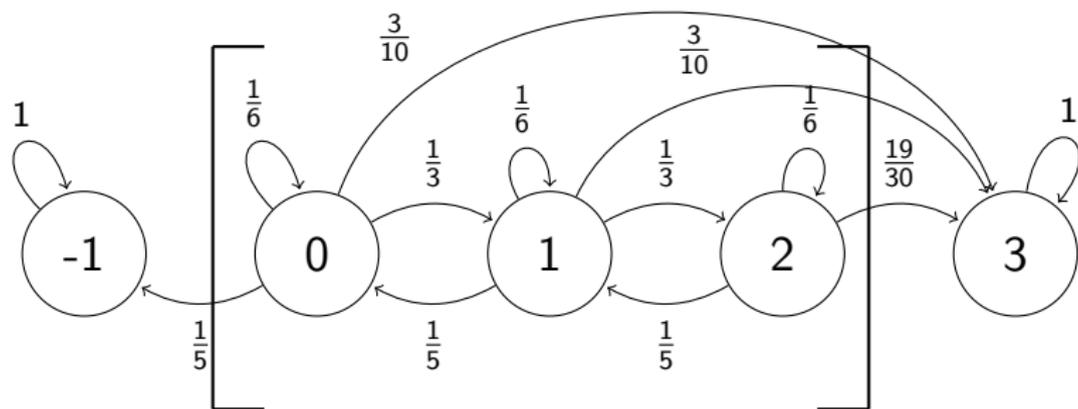
# Birth-Death Chain with Catastrophes



$$P = \begin{bmatrix} \frac{24}{30} & \frac{1}{5} & 0 & 0 \\ \frac{19}{30} & \frac{1}{6} & \frac{1}{5} & 0 \\ \frac{3}{10} & \frac{1}{3} & \frac{1}{6} & \frac{1}{5} \\ \frac{3}{10} & 0 & \frac{1}{3} & \frac{11}{30} \end{bmatrix}$$

$$P^5 = \begin{bmatrix} \frac{382}{519} & \frac{1461}{7391} & \frac{1159}{22333} & \frac{9}{625} \\ \frac{2087}{2877} & \frac{983}{5001} & \frac{188}{3279} & \frac{129}{6233} \\ \frac{1315}{1863} & \frac{476}{2441} & \frac{269}{4032} & \frac{224}{6907} \\ \frac{550}{799} & \frac{841}{4359} & \frac{270}{3569} & \frac{185}{42977} \end{bmatrix}$$

# Dual of the Birth-Death Chain with Catastrophes



$$P^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{5} & \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & 0 \\ 0 & \frac{1}{5} & \frac{1}{6} \end{bmatrix} & \frac{3}{10} \\ 0 & \begin{bmatrix} 0 & \frac{1}{5} & \frac{1}{6} \\ 0 & 0 & 0 \end{bmatrix} & \frac{3}{10} \\ 0 & 0 & \frac{19}{30} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

To find the probability of moving from state  $i = 2$  to  $j = -1$  in 5 step,

$$P_{i,j}^{*(k)} = \sum_{s=1}^3 \left[ \frac{2}{4} \left( \frac{p}{q} \right)^{\frac{i-j}{2}} \sin \left( \frac{(i+1)\pi s}{4} \right) \sin \left( \frac{(j+1)\pi s}{4} \right) \left( r + 2\sqrt{pq} \cos \left( \frac{s\pi}{4} \right) \right)^k \right]$$

$$n = 3, \quad p = \frac{1}{5}, \quad q = \frac{1}{3}, \quad r = \frac{1}{6}, \quad c = \frac{3}{10}$$

$$P_{2,0}^{*(k)} = \sum_{s=1}^3 \left[ \frac{2}{4} \left( \frac{3}{5} \right)^{\frac{2}{2}} \sin \left( \frac{(2)\pi s}{4} \right) \sin \left( \frac{\pi s}{4} \right) \left( 1/6 + 2\sqrt{\frac{1}{15}} \cos \left( \frac{s\pi}{4} \right) \right)^5 \right] \approx .0063$$

$$P^{*(5)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{137}{5195} & \begin{bmatrix} \frac{57}{5366} & \frac{84}{4295} & \frac{17}{972} \end{bmatrix} & \frac{550}{799} \\ \frac{179}{2700} & \begin{bmatrix} \frac{142}{1201} & \frac{269}{12739} & \frac{84}{4295} \end{bmatrix} & \frac{245}{278} \\ \frac{9}{625} & \begin{bmatrix} \frac{17}{2700} & \frac{142}{12101} & \frac{57}{5366} \end{bmatrix} & \frac{489}{511} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

we could also verified  
by looking at the 4,2  
entry of  $P^{*(5)}$

$$\frac{17}{2000} \approx .0063$$

To find the probability of moving from state  $i = 2$  to  $j = 0$  in 5 step,

$$P_{i,-1}^{*(k)} = P_{0,-1}^* \sum_{s=1}^3 \left[ \frac{2}{n+1} \left(\frac{p}{q}\right)^{\frac{i}{2}} \sin\left(\frac{(i+1)\pi s}{n+1}\right) \sin\left(\frac{\pi s}{n+1}\right) \frac{\lambda_s(1-\lambda_s^{k-1})}{1-\lambda_s} \right]$$

$$n = 3, \quad p = \frac{1}{5}, \quad q = \frac{1}{3}, \quad r = \frac{1}{6}, \quad c = \frac{3}{10}$$

$$P_{2,-1}^{*(5)} = P_{0,-1}^* \sum_{s=1}^3 \left[ \frac{2}{4} \left(\frac{3}{5}\right)^{\frac{2}{2}} \sin\left(\frac{3\pi s}{4}\right) \sin\left(\frac{\pi s}{4}\right) \frac{\lambda_s(1-\lambda_s^4)}{1-\lambda_s} \right]$$

$$P_{2,-1}^{*(5)} = \frac{1}{5} (.1568 - 0.0600 - 0.0248) \approx 0.0144$$

$$P^{*(5)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{137}{5195} & \begin{bmatrix} \frac{57}{5366} & \frac{84}{4295} & \frac{17}{972} \end{bmatrix} & \frac{550}{799} \\ \frac{179}{2700} & \begin{bmatrix} \frac{142}{1201} & \frac{269}{12739} & \frac{84}{4295} \end{bmatrix} & \frac{245}{278} \\ \frac{9}{625} & \begin{bmatrix} \frac{17}{2700} & \frac{142}{12101} & \frac{57}{5366} \end{bmatrix} & \frac{489}{511} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

we could also verified by looking at the 4,1 entry of  $P^{*(5)}$

$$\frac{9}{625} \approx .0144$$

# Duality Theorem

$$P_{i,j}^n = \sum_{s=i}^H \left( P_{j,s}^{*(n)} - P_{j-1,s}^{*(n)} \right) = \sum_{s=1}^3 \left( P_{2,s}^{*(n)} - P_{1,s}^{*(n)} \right)$$

$$P_{1,2}^5 = \left( P_{2,1}^{*(5)} + P_{2,2}^{*(5)} + P_{2,3}^{*(5)} - P_{1,1}^{*(5)} - P_{1,2}^{*(5)} - P_{1,3}^{*(5)} \right) \approx .057335$$

$$P^5 = \begin{bmatrix} \frac{382}{519} & \frac{1461}{7391} & \frac{1159}{22333} & \frac{9}{625} \\ \frac{2087}{2877} & \frac{983}{5001} & \frac{188}{3279} & \frac{129}{6233} \\ \frac{1315}{1863} & \frac{476}{2441} & \frac{269}{4032} & \frac{224}{6907} \\ \frac{550}{799} & \frac{841}{4359} & \frac{270}{3569} & \frac{185}{42977} \end{bmatrix}$$

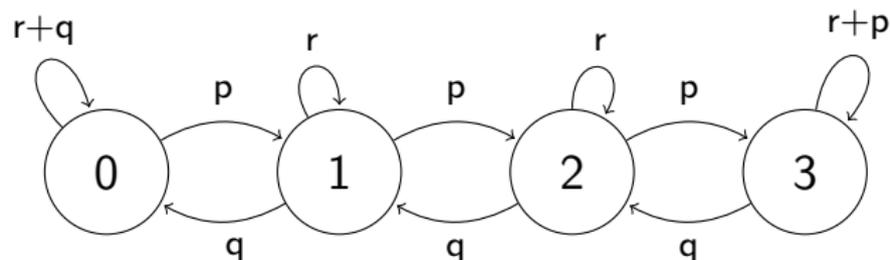
we could also verified by looking at the 2,3 entry of  $P^{*(5)}$

$$\frac{188}{3279} \approx .057335$$

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# Birth-Death Chain



$$P = \begin{bmatrix} r+q & p & 0 & 0 \\ q & r & p & 0 \\ 0 & q & r & p \\ 0 & 0 & q & r+p \end{bmatrix}$$

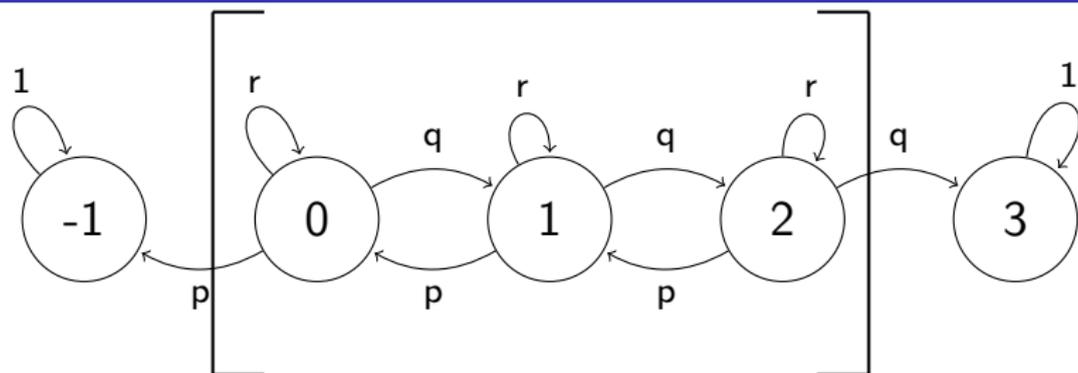
$$0 < p, q < 1$$

$$p + q + r = 1$$

$$0 \leq r < 1$$

# Find the dual

# Dual of the Birth-Death Chain



$$P^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ p & \begin{bmatrix} r & q & 0 \\ 0 & p & r \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 0 & p & r \end{bmatrix} & q \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$0 < p, q < 1$$

$$p + q + r = 1$$

$$0 \leq r < 1$$

# Dual of the Birth-Death Chain with Catastrophes

$$P^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ p & \begin{bmatrix} r & q & 0 \end{bmatrix} & c \\ 0 & \begin{bmatrix} p & r & q \end{bmatrix} & c \\ 0 & \begin{bmatrix} 0 & p & r \end{bmatrix} & q+c \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ r+q+c & p & 0 & 0 & 0 \\ q+c & r & p & 0 & 0 \\ c & q & r & p & 0 \\ c & 0 & q & r+p & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

# Duality Theorem

If  $P_{i,j}^{(n)}$  and  $P_{i,j}^{*(n)}$  are the  $n$ -step transient probabilities of the Markov Chains corresponding to the previous two slides, then

$$P_{r,s}^{*(n)} = \sum_{k=r+1}^H \left[ P_{s+1,k}^{(n)} - P_{s,k}^{(n)} \right]$$

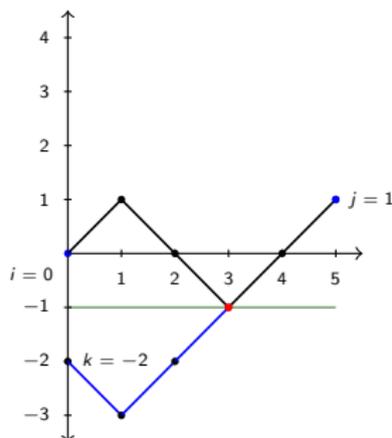
$$P_{r,s}^{(n)} = \sum_{k=r}^H \left[ P_{s,k}^{*(n)} - P_{s-1,k}^{*(n)} \right]$$

for  $n \geq 0$  and for all states  $r, s = 0, 1, 2, \dots, H$  with the convention  $P_{-1,k}^{(n)} = 0$  if  $k > -1$

# Counting Bad Lattice Paths

## The Reflection Principle

There is a bijection between bad lattice paths from state  $i$  to state  $j$  in  $n$ -steps and lattice paths from  $-i - 2$  to  $j$  in  $n$ -steps.



In our example, the bad paths from  $i = 0$  to  $j = 1$  correspond to the paths from  $k = -2$  to  $j = 1$ . Therefore, the number of bad paths is  $\binom{5}{4} = 5$ .

# Combinatorial Solution of Example 1

$$\left. \begin{array}{l} U - D = j - i = 1 \\ U + D = n = 5 \end{array} \right\} \implies \begin{cases} U = 3 \\ D = 2 \end{cases}$$

## Combinatorial Solution

$$\begin{aligned} \frac{|G|}{|A|} &= \frac{|A| - |B|}{|A|} = \frac{\text{all lattice paths} - \text{bad lattice paths}}{\text{all lattice paths}} \\ &= \frac{\binom{5}{3} - \binom{5}{4}}{\binom{5}{3}} \\ &= \frac{10 - 5}{10} = \frac{1}{2} \end{aligned}$$

# Vandermonde Matrix & Inverse

From Cayley-Hamilton/Linear Recurrence Relation Approach, we want to solve the following systems of linear equations for  $A$ 's.

$$\begin{aligned} I &= A_0 + A_1 + \dots + A_{n-1} \\ M^1 &= A_0 \lambda_0 + A_1 \lambda_1 + \dots + A_{n-1} \lambda_{n-1} \\ M^2 &= A_0 \lambda_0^2 + A_1 \lambda_1^2 + \dots + A_{n-1} \lambda_{n-1}^2 \\ &\vdots \\ M^{n-1} &= A_0 \lambda_0^{n-1} + A_1 \lambda_1^{n-1} + \dots + A_{n-1} \lambda_{n-1}^{n-1} \end{aligned}$$

$$\begin{bmatrix} \mathcal{M} \\ I \\ M^1 \\ M^2 \\ \vdots \\ M^{n-1} \end{bmatrix} = \begin{bmatrix} V \\ 1 & 1 & \dots & 1 \\ \lambda_0 & \lambda_1 & \dots & \lambda_{n-1} \\ \lambda_0^2 & \lambda_1^2 & \dots & \lambda_{n-1}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_0^{n-1} & \lambda_1^{n-1} & \dots & \lambda_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} \mathcal{A} \\ A_0 \\ A_1 \\ A_2 \\ \vdots \\ A_{n-1} \end{bmatrix}$$

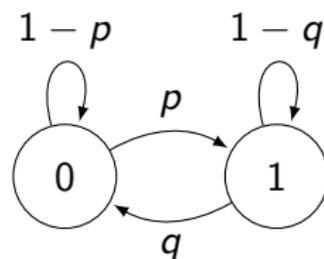
$$\mathcal{M} = V\mathcal{A}, \text{ then } \mathcal{A} = V^{-1}\mathcal{M}$$

To solve for  $A$ 's, we have to find the inverse of  $V$ , a Vandermonde Matrix. Each entry of the  $V^{-1}$  can be computed explicitly.

## Example 1: General $2 \times 2$ Stochastic Matrix

Consider the following diagram which represents a birth-death chain:

$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$$



where  $0 < p, q < 1$

### Eigenvalues

$$\lambda_0 = 1 - p - q \quad \text{and} \quad \lambda_1 = 1$$

## Example 1: General $2 \times 2$ Stochastic Matrix

Cayley-Hamilton/Linear Recurrence Relation Approach for  $k \in \mathbb{N}$

$$P^k = A_0 \lambda_0^k + A_1$$

$$P^k = \begin{bmatrix} \frac{p}{p+q} & -\frac{p}{p+q} \\ -\frac{q}{p+q} & \frac{q}{p+q} \end{bmatrix} (1-p-q)^k + \begin{bmatrix} \frac{q}{p+q} & \frac{p}{p+q} \\ \frac{q}{p+q} & \frac{p}{p+q} \end{bmatrix}$$

# Circulant Matrices

Consider the vector  $\vec{c} = \langle c_0 \ c_{n-1} \ \cdots \ c_2 \ c_1 \rangle$ .

An  $n \times n$  matrix  $C$  takes the form:

$$\begin{bmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{bmatrix}$$

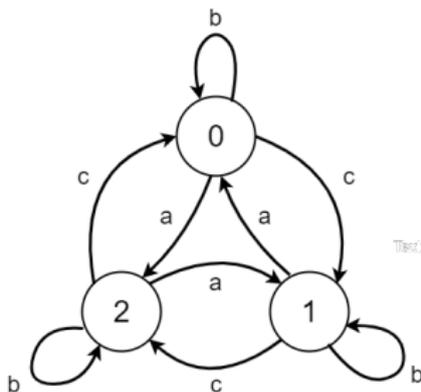
## Eigenvalues

$$\lambda_j = c_0 + c_{n-1}\omega_j + c_{n-2}\omega_j^2 + \cdots + c_1\omega_j^{n-1}, \quad j = 0, 1, \dots, n-1$$

where  $\omega_j = \exp\left(\frac{(2j\pi)i}{n}\right)$  are the  $n$ -th roots of unity and  $i$  is the imaginary unit.

# Cayley-Hamilton Form: General 3 by 3 Circulant Matrix

Consider the following Circular Markov Chain:



$$P = \begin{bmatrix} b & c & a \\ a & b & c \\ c & a & b \end{bmatrix}$$

$P$  is a  $3 \times 3$  **circulant** matrix.

$$a, c > 0, b \geq 0, a + b + c = 1$$

Eigenvalues given by Davis [8]

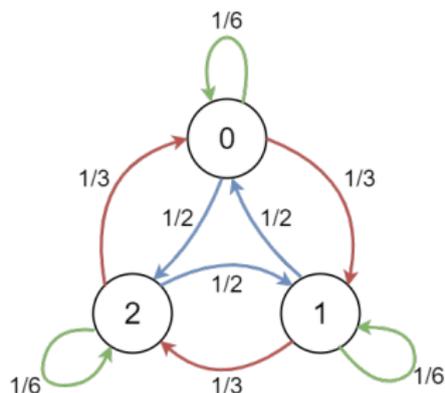
$$\lambda_1 = -\frac{a}{2} + b - \frac{c}{2} + i \left( -\frac{\sqrt{3}a}{2} + \frac{\sqrt{3}c}{2} \right)$$

$$\lambda_2 = -\frac{a}{2} + b - \frac{c}{2} + i \left( \frac{\sqrt{3}a}{2} - \frac{\sqrt{3}c}{2} \right)$$

$$\lambda_0 = 1$$

## Example 4A

Consider the following Circular Markov Chain:



$$P = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{bmatrix}$$

$P$  is a  $3 \times 3$  **circulant** matrix

Eigenvalues given by Davis [8]

$$\lambda_1 = -\frac{1}{4} - \frac{\sqrt{3}i}{12}, \quad \lambda_2 = -\frac{1}{4} + \frac{\sqrt{3}i}{12}$$

$$\lambda_0 = 1$$

Using Dave's Python program, we have :  $P^k = A_1\lambda_1^k + A_2\lambda_2^k + A_0$

$$\begin{aligned} \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{bmatrix}^k &= \begin{bmatrix} \frac{1}{3} & -\frac{1}{6} - \frac{\sqrt{3}i}{6} & -\frac{1}{6} + \frac{\sqrt{3}i}{6} \\ -\frac{1}{6} + \frac{\sqrt{3}i}{6} & \frac{1}{3} & -\frac{1}{6} - \frac{\sqrt{3}i}{6} \\ -\frac{1}{6} - \frac{\sqrt{3}i}{6} & -\frac{1}{6} + \frac{\sqrt{3}i}{6} & \frac{1}{3} \end{bmatrix} \left( -\frac{1}{4} - \frac{\sqrt{3}i}{12} \right)^k \\ &+ \begin{bmatrix} \frac{1}{3} & -\frac{1}{6} + \frac{\sqrt{3}i}{6} & -\frac{1}{6} - \frac{\sqrt{3}i}{6} \\ -\frac{1}{6} - \frac{\sqrt{3}i}{6} & \frac{1}{3} & -\frac{1}{6} + \frac{\sqrt{3}i}{6} \\ -\frac{1}{6} + \frac{\sqrt{3}i}{6} & -\frac{1}{6} - \frac{\sqrt{3}i}{6} & \frac{1}{3} \end{bmatrix} \left( -\frac{1}{4} + \frac{\sqrt{3}i}{12} \right)^k + \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \end{aligned}$$

**Proposition:** Consider the 4x4 stochastic matrix  $P$  below where  $a, b, c$  are fixed probabilities with  $0 \leq a+b+c \leq 1$  then:

$$P = \begin{pmatrix} b+c+q_1 & a+q_2-q_1 & -c+q_3-q_2 & 1-a-b-q_3 \\ c+q_1 & b+q_2-q_1 & a-c+q_3-q_2 & 1-a-b-q_3 \\ q_1 & c+q_2-q_1 & b-c+q_3-q_2 & 1-b-q_3 \\ q_1 & q_2-q_1 & q_3-q_2 & 1-q_3 \end{pmatrix}$$

has the same set of eigenvalues for different probabilities  $q_1, q_2, q_3$  satisfying

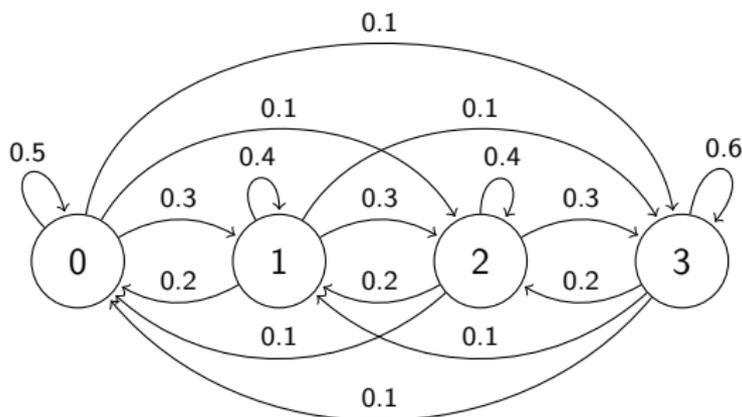
$$q_1 \leq q_2 \leq q_3 \leq 1-a-b \text{ and } c \leq q_3-q_2.$$

The eigenvalues are  $\{1, b, b+\sqrt{2ac}, b-\sqrt{2ac}\}$ . The Eigenvalues of the following matrices are:  $\{1, 0.3, 0.5, 0.1\}$ .

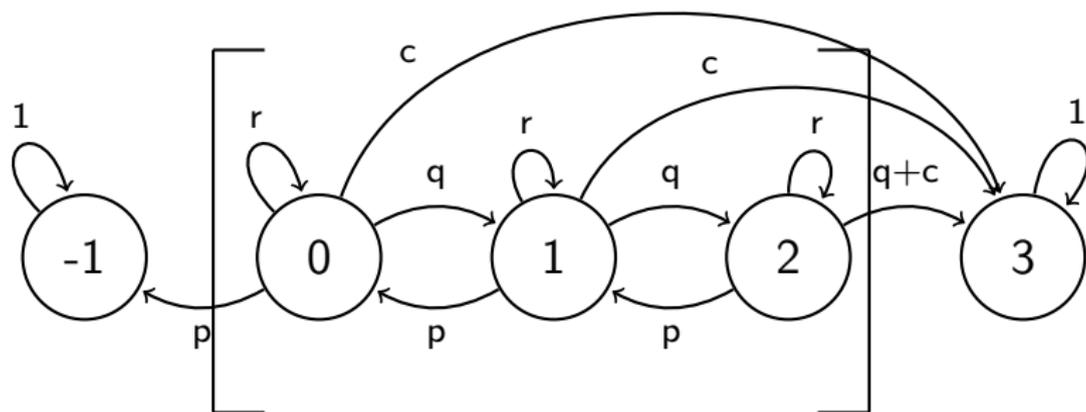
$$P_1 = \begin{pmatrix} .7 & .1 & 0 & .2 \\ .4 & .3 & .1 & .2 \\ .2 & .2 & .3 & .3 \\ .2 & 0 & .2 & .6 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} .7 & .1 & .1 & .1 \\ .4 & .3 & .2 & .1 \\ .2 & .2 & .4 & .2 \\ .2 & 0 & .3 & .5 \end{pmatrix}$$

# Non Birth-death Chain with Known Eigenvalues



# Dual of the Birth-Death Chain with Catastrophes



$$P^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ p & \begin{bmatrix} r & q & 0 \\ p & r & q \\ 0 & p & r \end{bmatrix} & c \\ 0 & 0 & 0 & q+c \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$0 < p, q, c < 1$$

$$p + q + r + c = 1$$

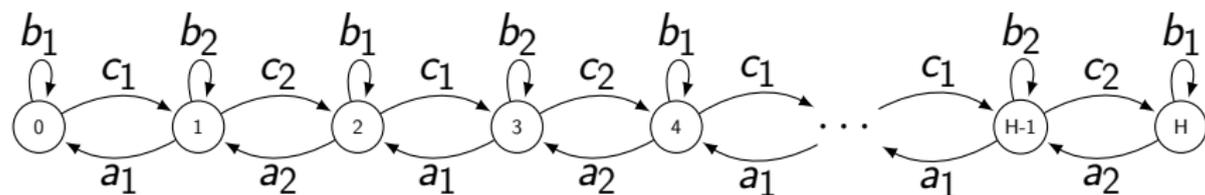
$$0 \leq r < 1$$

- The Cayley-Hamilton Recurrence method has been programmed in Python to calculate transition probabilities and transition probability functions in various cases.
  - With Vandermonde matrix, the inverse problem can be solved.
  - Constant matrices need to be computed only once for a given system.
- For circulant, birth-death matrices exact expressions for  $P^k$  are in terms of complex numbers. Results scale up for large  $n$ .
  - The probability of going from  $i$  to  $j$  in  $n$ -steps without going around the circle is calculated.
- Eigenvalue results can be generalized to a family of non birth-death Markov chains.

# References

- [1] Lilinoe Harbottle, Black Hunter and Alan Krinik, *The Irreducible Three and Four-State Markov process*, Integration, Mathematical Theory and Applications, Volume 2, Issue 3, pages 335-355, 2011.
- [2] Donald Knuth, *The Art of Computer Programming Vol 1: Fundamental Algorithms Third Edition*, Addison-Wesley 1997.
- [3] Said Kouachi, *Eigenvalues and Eigenvectors of Tridiagonal Matrices*, Electronic Journal of Linear Algebra, Vol. 15,(2006), pages 115-133.
- [4] Said Kouachi, *Eigenvalues and Eigenvectors of Some Tridiagonal Matrices with Non-Constant Diagonal Entries*, Applicationes Mathematicae 35, Vol. 1, (2008), pages 107-120.
- [5] Alan Krinik and Jeniffer Switkes, *An Element in the  $k$ th Power of an  $n \times n$  Matrix*, Preprint.
- [6] Clever Moler and Charles Van Loan, *Nineteen Dubious Ways to Compute the Exponential of a Matrix, Twenty-Five Years Later*, SIAM REVIEW Vol 45, NO.1, pp. 1-46, 2003.
- [7] Ben Noble and James W. Daniel, *Applied Linear Algebra*, Prentice-Hall, New Jersey 1988.
- [8] Philip J. Davis, *Circulant Matrices*, Wiley, New York, 1970

# Sub Birth-Death Chain I



$$P = \begin{pmatrix} b_1 & c_1 & 0 & 0 & \dots & \dots & \dots & 0 & 0 & 0 \\ a_1 & b_2 & c_2 & 0 & \dots & \dots & \dots & 0 & 0 & 0 \\ 0 & a_2 & b_1 & c_1 & \dots & \dots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & a_1 & b_2 & c_2 & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & a_1 & b_2 & c_2 \\ 0 & 0 & \dots & \dots & \dots & \dots & 0 & 0 & a_2 & b_1 \end{pmatrix}$$

## Constraints

$$d = a_1 c_1 = a_2 c_2$$

$$a_1 + b_2 + c_2 = 1$$

$$a_2 + b_1 + c_1 = 1$$

$$0 < a_1, a_2, c_1, c_2 < 1$$

$H$  is even

# Eigenvalues of $P$

Suppose  $P$  satisfies the preceding conditions.

**Case 1** The eigenvalues of  $\lambda_k$  of  $P$  are:

**(A)**  $H = 2m$  where  $H \in \mathbb{N}$

$$\lambda_k = \begin{cases} \frac{(b_1 + b_2) - \sqrt{(b_1 - b_2)^2 + 16d^2 \cos^2 \theta_k}}{2}, & k = 0, \dots, m-1 \\ \frac{(b_1 + b_2) + \sqrt{(b_1 - b_2)^2 + 16d^2 \cos^2 \theta_k}}{2}, & k = m, \dots, 2m-1 \\ b_1, & k = H \end{cases}$$

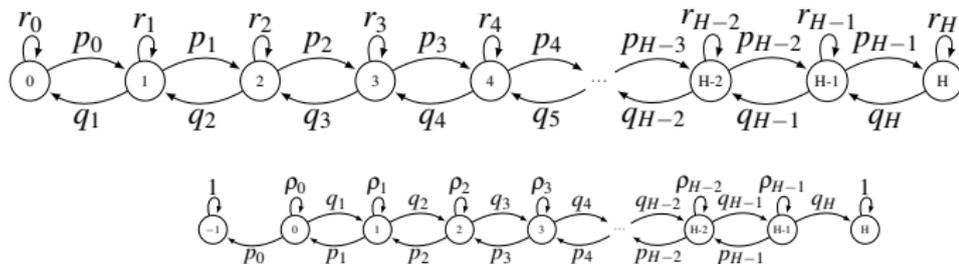
$$\text{where } \theta_k = \begin{cases} \frac{(k+1)\pi}{H+2}, & k = 0, 1, \dots, m-1 \\ \frac{(k+1-m)\pi}{H+2}, & k = m, m+1, \dots, 2m-1 \end{cases}$$

# Eigenvalues of $P$

**(B)**  $H = 2m - 1$  where  $m \in \mathbb{N}$

$$\lambda_k = \begin{cases} \frac{(b_1 + b_2) - \sqrt{(b_1 - b_2)^2 + 16d^2 \cos^2 \theta_k}}{2}, & k = 0, \dots, m - 1 \\ \frac{(b_1 + b_2) + \sqrt{(b_1 - b_2)^2 + 16d^2 \cos^2 \theta_k}}{2}, & k = m, \dots, H \end{cases}$$

$$\text{where } \theta_k = \begin{cases} \frac{(k + 1)\pi}{H + 2}, & k = 0, 1, \dots, m - 1 \\ \frac{(k + 1 - m)\pi}{H + 2}, & k = m, m + 1, \dots, H \end{cases}$$



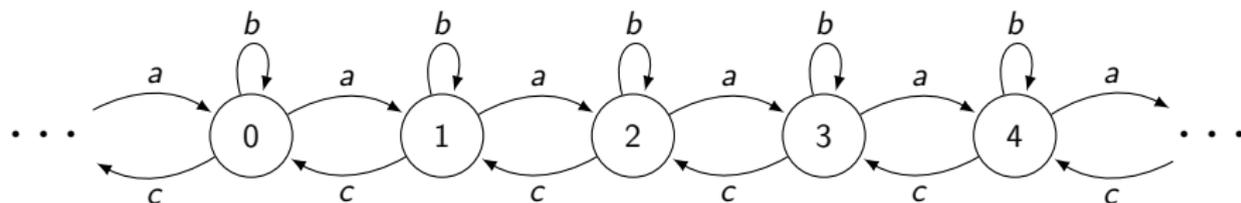
**Duality Theorem** If  $P_{i,j}^{(n)}$  and  $P_{i,j}^{*(n)}$  are the  $n$ -step transient probabilities of the birth-death chains corresponding to Figure 3 and 4 respectively, then

$$P_{r,s}^{(n)} = \sum_{k=r}^H [P_{s,k}^{*(n)} - P_{s-1,k}^{*(n)}] \quad \text{and} \quad P_{r,s}^{*(n)} = \sum_{k=0}^r [P_{s,k}^{(n)} - P_{s+1,k}^{(n)}] \quad (1.6)$$

for  $n \geq 0$  and for all states  $r, s = 0, 1, 2, 3, \dots, H$  with the convention  $P_{-1,k}^{(n)} = 0$  if  $k > -1$ .

# Concluding Remarks

- 1 The preceding ballot box example can be generalized to



where  $a, b, c$  are probabilities such that  $a + b + c = 1$

with answers:

$$\begin{aligned} G_{1,2}^5 &= \tilde{c}_0 \tilde{\lambda}_0^5 + \tilde{c}_1 \tilde{\lambda}_1^5 + \tilde{c}_2 \tilde{\lambda}_2^5 + \tilde{c}_3 \tilde{\lambda}_3^5 + \tilde{c}_4 \tilde{\lambda}_4^5 \\ &= \frac{\sqrt{ac}}{\sqrt{3}} (\sqrt{3ac})^5 - \frac{\sqrt{ac}}{\sqrt{3}} (-\sqrt{3ac})^5 \end{aligned}$$

$$\tilde{\lambda}_0 = \sqrt{ac}, \quad \tilde{\lambda}_1 = -\sqrt{ac}, \quad \tilde{\lambda}_2 = \sqrt{3ac}, \quad \tilde{\lambda}_3 = -\sqrt{3ac}, \quad \tilde{\lambda}_4 = 0$$

$$\tilde{c}_0 = 0, \quad \tilde{c}_1 = 0, \quad \tilde{c}_2 = \frac{\sqrt{ac}}{\sqrt{3}}, \quad \tilde{c}_3 = -\frac{\sqrt{ac}}{\sqrt{3}}, \quad \tilde{c}_4 = 0$$

# Concluding Remarks

$$\lambda_0 = b + 2\sqrt{ac} \cos\left(\frac{\pi}{7}\right)$$

$$c_0 = \frac{\frac{3}{2}c\lambda_4 + 4c\lambda_2 + 6c\lambda_0 - \left(\frac{23}{2}\right)(c - ac - c^2) + 7c\sqrt{ac}}{\sqrt{ac}(7\lambda_4 + 14\lambda_2 + 21\lambda_0 - 42(1 - a - c) + 28\sqrt{ac})}$$

$$\lambda_1 = b - 2\sqrt{ac} \cos\left(\frac{\pi}{7}\right)$$

$$c_1 = -c_0$$

$$\lambda_2 = b + 2\sqrt{ac} \cos\left(\frac{2\pi}{7}\right)$$

$$c_2 = \frac{6c\lambda_2 + 4c\lambda_4 - \frac{3}{2}c\lambda_0 - \frac{17}{2}(c - ac - c^2) - 7c\sqrt{ac}}{\sqrt{ac}(-21\lambda_2 - 14\lambda_4 + 7\lambda_0 + 28(1 - a - c) + 28\sqrt{ac})}$$

$$\lambda_3 = b - 2\sqrt{ac} \cos\left(\frac{2\pi}{7}\right)$$

$$c_3 = -c_2$$

$$\lambda_4 = b + 2\sqrt{ac} \cos\left(\frac{3\pi}{7}\right)$$

$$c_4 = \frac{-6c\lambda_4 + \frac{3}{2}c\lambda_2 + 4c\lambda_0 + \frac{1}{2}(c - ac - c^2) - 7c\sqrt{ac}}{\sqrt{ac}(-21\lambda_4 + 7\lambda_2 + 14\lambda_0 - 28\sqrt{ac})}$$

$$\lambda_5 = b - 2\sqrt{ac} \cos\left(\frac{3\pi}{7}\right)$$

$$c_5 = -c_4$$

## Concluding Remarks

Rewriting the following expression from slide 26, we get:

$$G_{1,2}^5 = \frac{\sqrt{ac}}{\sqrt{3}}(\sqrt{3ac})^5 - \frac{\sqrt{ac}}{\sqrt{3}}(-\sqrt{3ac})^5$$

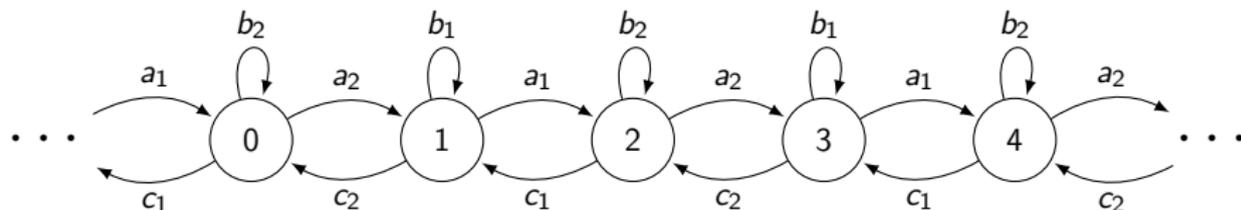
And calculating  $A_{1,2}^5$  with the values given on the previous slide gives:

$$A_{1,2}^5 = c_0\lambda_0^5 + c_1\lambda_1^5 + c_2\lambda_2^5 + c_3\lambda_3^5 + c_4\lambda_4^5 + c_5\lambda_5^5. \text{ So again,}$$

$$\frac{G_{1,2}^5}{A_{1,2}^5} = \frac{\frac{\sqrt{ac}}{\sqrt{3}}(\sqrt{3ac})^5 - \frac{\sqrt{ac}}{\sqrt{3}}(-\sqrt{3ac})^5}{c_0\lambda_0^5 + c_1\lambda_1^5 + c_2\lambda_2^5 + c_3\lambda_3^5 + c_4\lambda_4^5 + c_5\lambda_5^5}$$

# Concluding Remarks

- 2 The Cayley-Hamilton Linear Recurrence Eigenvalue approach can be used to solve the generalized ballot box problem for the model given in remark 1, for any natural number  $n$  since the formulas for eigenvalues scale up. In fact, eigenvalues are known for alternating diagram state.



## Concluding Remarks

- 3 An advantage of the Cayley-Hamilton Linear Recurrence Eigenvalue approach is that it solves related problems.

For example, suppose  $n = 9$ , then in **Example 1**  $\frac{G_{1,2}^9}{A_{1,2}^9} \approx 0.75$  is the probability of going from  $i = 1$  to  $j = 2$  in 9-steps and never leaving the strip 0 to 4, assuming we are restricted to the -1 to 4 strip.

And suppose  $n = 9$ , then in **Example 2**  $\frac{G_{1,2}^9}{A_{1,2}^9} \approx 0.8$  is the probability of going from  $i = 1$  to  $j = 2$  in 9-steps and never leaving the strip 0 to 4, assuming we are restricted to the -1 to 4 strip.

This is no longer the ballot box problem because the strip would need to be adjusted.

- 4 The eigenvalue approach is promising for families of matrices having distinct, known, eigenvalues whose expressions scale up e.g. circulant matrices.

# References

- [1] Lilinoe Harbottle, Black Hunter and Alan Krinik, *The Irreducible Three and Four-State Markov process*, Integration, Mathematical Theory and Applications, Volume 2, Issue 3, pages 335-355, 2011.
- [2] Donald Knuth, *The Art of Computer Programming Vol 1: Fundamental Algorithms Third Edition*, Addison-Wesley 1997.
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- [4] Said Kouachi, *Eigenvalues and Eigenvectors of Some Tridiagonal Matrices with Non-Constant Diagonal Entries*, Applicationes Mathematicae 35, Vol. 1, (2008), pages 107-120.
- [5] Alan Krinik and Jeniffer Switkes, *An Element in the  $k$ th Power of an  $n \times n$  Matrix*, Preprint.
- [6] Clever Moler and Charles Van Loan, *Nineteen Dubious Ways to Compute the Exponential of a Matrix, Twenty-Five Years Later*, SIAM REVIEW Vol 45, NO.1, pp. 1-46, 2003.
- [7] Ben Noble and James W. Daniel, *Applied Linear Algebra*, Prentice-Hall, New Jersey 1988.
- [8] Philip J. Davis, *Circulant Matrices*, Wiley, New York, 1970