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# Analysis of a linearized poromechanics model for incompressible and nearly incompressible materials

Mathieu Barré<sup>1,2</sup>, Céline Grandmont<sup>3,4</sup>, Philippe Moireau<sup>1,2</sup>

<sup>1</sup> Inria, 1 Rue Honoré d’Estienne d’Orves, 91120 Palaiseau, France

<sup>2</sup> LMS, École Polytechnique, CNRS, Institut Polytechnique de Paris, 91120 Palaiseau, France

<sup>3</sup> Inria, 2 Rue Simone Iff, 75012 Paris, France

<sup>4</sup> LJLL, Sorbonne Université, CNRS, 4 Place Jussieu, 75005 Paris, France

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In this paper, we thoroughly analyze the linearized version of a poromechanics model developed to simulate biological tissues perfusion. This is a fully dynamical model in which the fluid and solid equations are strongly coupled through the interstitial pressure. As such, it generalizes Darcy, Brinkman and Biot equations of poroelasticity. The mathematical and numerical analysis of this model was first performed for a compressible porous material. Here, we focus on the nearly incompressible case with a semigroup approach that also enables to prove the existence of weak solutions. We show the existence and uniqueness of strong and weak solutions in the incompressible limit, for which a non-standard divergence constraint arises. Due to the special form of the coupling, the underlying problem is not coercive. Nevertheless, by using the notion of  $T$ -coercivity, we obtain stability estimates and well-posedness results. Our study also provides guidelines to propose a stable and robust approximation of the problem with mixed finite elements. In particular, we recover an inf-sup condition independent of the phase field. Finally, we investigate numerically the elliptic regularity of the associated steady-state problem and illustrate the sensitivity of the solution with respect to the different model parameters.

## Introduction

Poromechanical models aim at representing the mechanical behavior of saturated porous media with the interaction of a fluid flow within a deformable porous structure through the definition of a multi-phase continuum framework [26, 28]. The initial introduction of such models concerns geophysics [7, 43], but recent new applications of these models have been discovered in biomechanics, in particular to represent perfused living tissues. If the heart perfusion remains a leading example of application [45, 39, 20, 21], poroelastic models have also been considered to simulate lipid and drug transport in blood vessel walls [2, 35, 19], water transport and drug delivery in the brain [36, 44, 5, 41], fibrosis diagnosis in the lungs [33], or tissue degradation in cancer therapies [30].

Notably, in these biomedical applications, ingredients such as fluid inertia and solid quasi-incompressibility, generally neglected in civil engineering, are introduced for modeling purposes [22] but also for their mathematical implications [18]. In the non-linear context of large deformations, these terms allow to obtain a complete thermodynamical balance for the resulting model [17]. In [18, 4], this gives to the linearized formulation of the one proposed in [22] an energy balance of interest from a mathematical viewpoint, opening the way to well-posedness. Still, the general resulting formulation potentially shows a hyperbolic – parabolic coupling between the structure and the fluid with an additional incompressibility constraint involving a mixture velocity, and hence leads to challenging questions of analysis.

In the present paper, we propose a complete vision of well-posedness of the linearized formulation by unifying semigroup and variational approaches. Our result encompasses the variational case initially considered

in [18] and generalizes the results in [4] by relaxing the condition on the fluid source and fully analyzing the incompressible limit case, which corresponds to the physiological regime, when considering living tissues.

Moreover, our approach uses the recent parallel between inf-sup conditions and  $T$ -coercivity [23, 25, 12, 13] to prove the fundamental inf-sup conditions introduced with the mixture velocity in the incompressible case. It appears that the inf-sup condition is eventually independent of the porosity representing the fraction of fluid in the mixture. Such result, already conjectured or partially justified in [18, 4], is crucial to be able to use generic finite-element discretization. It would also be essential when considering the discretization of the non-linear model from [22], in which the porosity is an unknown of the system.

The paper is organized as follows. Section 1 presents the poromechanics model under study and general preliminaries such as the formal derivation of energy estimates on the system. In Section 2, we unify the semigroup and variational approaches from [18] and [4] by proving the existence and uniqueness of strong and weak solutions for a compressible porous material. We highlight the role of solid viscosity on the model by pointing out the differences that appear in the variational formulation when this coefficient vanishes. Section 3 is devoted to the incompressible regime and more specifically to the saddle-point structure of the problem arising in this case, with a particular attention dedicated to the existence and regularity of pressure. Next, in Section 4, we establish a link between the results of Sections 2 and 3 by passing to the incompressible limit obtained by letting the bulk modulus of the structure skeleton go to infinity. Finally, these theoretical results are complemented with numerical experiments exploring the regularity of solutions and the domain of the underlying semigroup operator.

## 1 Problem setting

We consider a porous media in a bounded domain  $\Omega \subset \mathbb{R}^d$  with Lipschitz boundary. The model we consider was introduced in [18] and further studied in [4], and is a variant of the well-known Biot system [7, 8, 9]. In each point of the domain  $\Omega$ , we consider a mixture of fluid and structure and we denote by  $\phi$  the porosity. Thus for all  $x \in \Omega$ ,  $\phi(x) \in (0, 1)$  represents the portion of fluid whereas  $1 - \phi(x)$  represents the portion of elastic media. The fluid phase is an homogeneous, viscous, Newtonian and incompressible fluid. We denote by  $v_f$  its velocity,  $\rho_f$  its density and  $\mu_f$  its viscosity. Since the fluid is assumed to be incompressible (resp. homogeneous),  $\rho_f$  is independent of time (resp. of space). We also assume that the structure is elastic and, to simplify, that its macroscopic behavior law is linear and isotropic and, thus, characterized by two Lamé constants  $\lambda$  and  $\mu$ . They stand for the elastic parameters characterizing the macroscopic behavior of the elastic part of the mixture (i.e. the homogenized behavior of a perforated elastic media with no fluid). We denote by  $u_s$  the structure displacement and by  $v_s = \partial_t u_s$  the structure velocity. The density of the structure is denoted by  $\rho_s$  and its viscosity by  $\eta$ . The fluid and the structure are coupled through a friction force that depends linearly on the relative velocity of the mixture  $v_f - v_s$  and reads  $\phi^2 k_f^{-1} (v_f - v_s)$ , where  $k_f$  is the permeability tensor. In addition, they are coupled through the interstitial pressure  $p$ , that is further linked to the incompressibility of the whole fluid-structure mixture.

The full coupled model then reads

$$\left\{ \begin{array}{ll} \rho_s(1 - \phi) \partial_{tt} u_s - \operatorname{div}(\sigma_s(u_s)) - \operatorname{div}(\sigma_s^{\text{vis}}(\partial_t u_s)) \\ \quad - \phi^2 k_f^{-1} (v_f - \partial_t u_s) + (1 - \phi) \nabla p = \rho_s(1 - \phi) f, & \text{in } \Omega \times (0, T), \quad (1a) \\ \rho_f \phi \partial_t v_f - \operatorname{div}(\phi \sigma_f(v_f)) + \phi^2 k_f^{-1} (v_f - \partial_t u_s) - \theta v_f + \phi \nabla p = \rho_f \phi f, & \text{in } \Omega \times (0, T), \quad (1b) \\ \frac{(1 - \phi)^2}{\kappa} \partial_t p + \operatorname{div}((1 - \phi) \partial_t u_s + \phi v_f) = \frac{\theta}{\rho_f}, & \text{in } \Omega \times (0, T), \quad (1c) \end{array} \right.$$

where the structure stress tensor is given by Hooke's law

$$\sigma_s(u) = \lambda \operatorname{Tr}(\varepsilon(u)) \mathcal{I} + 2\mu \varepsilon(u),$$

with  $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ , the structure additional viscosity is given by

$$\sigma_s^{\text{vis}}(v) = 2\eta \varepsilon(v),$$

and the fluid stress tensor reads

$$\sigma_f(v) = \lambda_f \operatorname{Tr}(\varepsilon(v)) \mathcal{I} + 2\mu_f \varepsilon(v).$$

In the above system, the data are the applied exterior force  $f$  and the additional fluid mass input  $\theta$ .

**Remark 1.1.** *The presence of the coefficient  $\lambda_f$  in the fluid stress tensor is explained by the fact that the fluid incompressibility is not associated with a macroscopic constraint, even if the fluid is actually incompressible. Indeed,  $\operatorname{div} v_f \neq 0$  in general because the variations of fluid volume are influenced by the volume occupied by the solid material within the porous matrix.*

The parameter  $\kappa$  represents the bulk modulus of the elastic media that constitutes the porous matrix. When  $\kappa < +\infty$  it corresponds to a compressible skeleton, whereas when  $\kappa = +\infty$  we have an incompressible elastic skeleton and since the fluid is assumed to be incompressible we deal with an incompressible porous media. This latter case is crucial when considering living tissues since they are nearly incompressible. Note that when  $\kappa = +\infty$  there is no dynamic of the pressure but the pressure is the Lagrange multiplier associated with the mixture constraint  $\rho_f \operatorname{div}((1 - \phi) \partial_t u_s + \phi v_f) = \theta$  involving the mixture velocity  $v_m = (1 - \phi) v_s + \phi v_f$ .

In the present work, we limit our analysis to the case of homogeneous Dirichlet boundary conditions for the structure and for the fluid:

$$\begin{cases} u_s = 0, & \text{on } \partial\Omega \times (0, T), \\ v_f = 0, & \text{on } \partial\Omega \times (0, T). \end{cases} \quad (2a)$$

$$(2b)$$

For a presentation of typical boundary conditions for such systems, we refer to [17, 18] and their analysis will imply further development. This coupled problem has to be completed with initial data:

$$\begin{cases} u_s(0) = u_{s0}, & \text{in } \Omega \\ \partial_t u_s(0) = v_{s0}, & \text{in } \Omega \\ v_f(0) = v_{f0}, & \text{in } \Omega \end{cases} \quad (3a)$$

$$(3b)$$

$$(3c)$$

and in the case  $\kappa < +\infty$

$$p(0) = p_0. \quad (4)$$

Existence of solutions of such a coupled system has been partially obtained in [18, 4]. More precisely, the case  $\kappa < +\infty$ ,  $\eta > 0$  and  $\theta = 0$  has already been studied in [18], where existence of strong solutions thanks to the semigroup formalism has been derived. The case  $\kappa < +\infty$ ,  $\eta = 0$  is treated in [4], where existence of variational solutions is obtained under a smallness hypothesis on  $\theta$ . Here, we consider all the different cases  $\kappa \leq +\infty$ ,  $\eta \geq 0$ , and any given  $\theta$  sufficiently smooth. We prove existence of unique strong and mild solutions using semigroup theory, from which we deduce existence of a unique variational solution. We eventually show that one can pass to the limit in the weak formulation as  $\kappa$  goes to infinity.

Before going through the proves, let us first derive formally some energy bounds satisfied by any smooth enough solutions of the coupled problem. We first derive them in the case  $\kappa < +\infty$  and then in the limit case  $\kappa = +\infty$ . Let us multiply (1a) by the structure velocity  $\partial_t u_s$ , integrate over  $\Omega$  and integrate by parts in space. No boundary terms appear thanks to the homogeneous Dirichlet boundary conditions (2a) and we obtain

$$\begin{aligned} & \frac{\rho_s}{2} \frac{d}{dt} \int_{\Omega} (1 - \phi) |\partial_t u_s|^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega} \sigma_s(u_s) : \varepsilon(u_s) dx + 2\eta \int_{\Omega} \varepsilon(\partial_t u_s) : \varepsilon(\partial_t u_s) dx \\ & - \int_{\Omega} \phi^2 k_f^{-1} (v_f - \partial_t u_s) \cdot \partial_t u_s dx - \int_{\Omega} p \operatorname{div}((1 - \phi) \partial_t u_s) dx = \int_{\Omega} \rho_s (1 - \phi) f \cdot \partial_t u_s dx. \end{aligned}$$

Let us also multiply (1b) by the fluid velocity  $v_f$ , integrate over  $\Omega$  and integrate by parts in space. No

boundary terms appear thanks to the homogeneous Dirichlet boundary conditions (2b) and we get

$$\begin{aligned} \frac{\rho_f}{2} \frac{d}{dt} \int_{\Omega} \phi |v_f|^2 dx + \int_{\Omega} \phi \sigma_f(v_f) : \varepsilon(v_f) dx \\ + \int_{\Omega} \phi^2 k_f^{-1} (v_f - \partial_t u_s) \cdot v_f dx - \int_{\Omega} \theta |v_f|^2 dx - \int_{\Omega} p \operatorname{div}(\phi v_f) dx = \int_{\Omega} \rho_f \phi f \cdot v_f dx. \end{aligned}$$

The last equation (1c) is multiplied by  $p$  and integrated over  $\Omega$ , which leads to

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \frac{(1-\phi)^2}{\kappa} |p|^2 dx + \int_{\Omega} \operatorname{div}((1-\phi) \partial_t u_s + \phi v_f) p dx = \int_{\Omega} \frac{\theta}{\rho_f} p dx. \quad (5)$$

Adding these three contributions, we see that the terms involving the divergence of the mixture velocity  $v_m$  cancels, and we have the following energy equality

$$\begin{aligned} \frac{\rho_s}{2} \frac{d}{dt} \int_{\Omega} (1-\phi) |\partial_t u_s|^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega} \sigma_s(u_s) : \varepsilon(u_s) dx + 2\eta \int_{\Omega} \varepsilon(\partial_t u_s) : \varepsilon(\partial_t u_s) dx + \frac{\rho_f}{2} \frac{d}{dt} \int_{\Omega} \phi |v_f|^2 dx \\ + \int_{\Omega} \phi \sigma_f(v_f) : \varepsilon(v_f) dx + \int_{\Omega} \phi^2 k_f^{-1} (v_f - \partial_t u_s) \cdot (v_f - \partial_t u_s) dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega} \frac{(1-\phi)^2}{\kappa} |p|^2 dx \\ = \int_{\Omega} \rho_s (1-\phi) f \cdot \partial_t u_s dx + \int_{\Omega} \rho_f \phi f \cdot v_f dx + \int_{\Omega} \frac{\theta}{\rho_f} p dx + \int_{\Omega} \theta |v_f|^2 dx. \quad (6) \end{aligned}$$

Consequently, in order to obtain an energy estimate, we impose the following assumptions on the data:

(h1) The constants  $\rho_s, \rho_f, \mu_f, \lambda, \mu$  are assumed to be strictly positive, whereas  $\eta \geq 0$  ;

(h2) The phase  $\phi \in H^{d/2+r}(\Omega)$  with  $r > 0$ , and is such that there exists  $(\phi_{\min}, \phi_{\max})$  satisfying

$$0 < \phi_{\min} \leq \phi(x) \leq \phi_{\max} < 1, \quad \forall x \in \Omega;$$

(h3) The friction tensor  $k_f$  is invertible and there exists  $k_0 > 0$  such that

$$k_f^{-1} v \cdot v \geq k_0 |v|^2, \quad \forall v \in \mathbb{R}^d;$$

(h4)  $f \in L^2((0, T) \times \Omega)$  ;

(h5)  $\theta \in C^0([0, T] \times \Omega)$ .

**Remark 1.2.** *The hypothesis (h2) implies that the phase  $\phi$  belongs to a multiplier space of  $H^1(\Omega)$ . This assumption is needed to define the term  $\operatorname{div}((1-\phi) \partial_t u_s + \phi v_f)$  in (5). Indeed, if  $\phi \in H^{d/2+r}(\Omega)$  with  $r > 0$ , then for any  $(w_s, w_f) \in [H_0^1(\Omega)]^d$  we have  $\operatorname{div}((1-\phi) w_s + \phi w_f) \in L^2(\Omega)$ .*

Under these assumptions, using Young inequality to bound the right-hand side of (6) by

$$\begin{aligned} \frac{1}{2} \int_{\Omega} \rho_s (1-\phi) |f|^2 dx + \frac{1}{2} \int_{\Omega} \rho_s (1-\phi) |\partial_t u_s|^2 dx + \frac{1}{2} \int_{\Omega} \rho_f \phi |f|^2 dx + \frac{1}{2} \int_{\Omega} \rho_f \phi |v_f|^2 dx \\ + \frac{1}{2} \int_{\Omega} \frac{\kappa}{\rho_f^2 (1-\phi_{\max})^2} |\theta|^2 dx + \frac{1}{2} \int_{\Omega} \frac{(1-\phi)^2}{\kappa} |p|^2 dx + \frac{2\|\theta\|_{C^0([0,T] \times \Omega)}}{\rho_f \phi_{\min}} \cdot \frac{1}{2} \int_{\Omega} \rho_f \phi |v_f|^2 dx, \end{aligned}$$

integrating in time from 0 to  $t$  and applying Grönwall Lemma, we obtain the following energy bound

$$\begin{aligned}
& \frac{\rho_s}{2} \int_{\Omega} (1-\phi) |\partial_t u_s(t)|^2 dx + \frac{1}{2} \int_{\Omega} \sigma_s(u_s(t)) : \varepsilon(u_s(t)) dx + 2\eta \int_0^t \int_{\Omega} \varepsilon(\partial_t u_s) : \varepsilon(\partial_t u_s) dx ds + \frac{\rho_f}{2} \int_{\Omega} \phi |v_f(t)|^2 dx \\
& + \int_0^t \int_{\Omega} \phi \sigma_f(v_f) : \varepsilon(v_f) dx ds + \int_0^t \int_{\Omega} \phi^2 k_f^{-1} (v_f - \partial_t u_s) \cdot (v_f - \partial_t u_s) dx ds + \frac{1}{2} \int_{\Omega} \frac{(1-\phi)^2}{\kappa} |p(t)|^2 dx \\
& \leq \exp \left( \max \left( 1, \frac{2\|\theta\|_{C^0([0,T] \times \Omega)}}{\rho_f \phi_{\min}} \right) t \right) \left( \left( \frac{\rho_s}{2} (1-\phi_{\min}) + \frac{\rho_f}{2} \phi_{\max} \right) \int_0^t \int_{\Omega} |f|^2 dx ds \right. \\
& \quad + \frac{\kappa}{2\rho_f^2 (1-\phi_{\max})^2} \int_0^t \int_{\Omega} |\theta|^2 dx ds + \frac{\rho_s}{2} \int_{\Omega} (1-\phi) |v_{s0}|^2 dx \\
& \quad \left. + \frac{1}{2} \int_{\Omega} \sigma_s(u_{s0}) : \varepsilon(u_{s0}) dx + \frac{\rho_f}{2} \int_{\Omega} \phi |v_{f0}|^2 dx + \frac{1}{2} \int_{\Omega} \frac{(1-\phi)^2}{\kappa} |p_0|^2 dx \right). \quad (7)
\end{aligned}$$

Note that the friction contribution induces dissipation in the system since

$$\int_{\Omega} \phi^2 k_f^{-1} (v_f - \partial_t u_s) \cdot (v_f - \partial_t u_s) dx \geq 0,$$

in virtue of (h3).

Moreover, Korn inequality [15, 31, 24] implies that the fluid and structure dissipative terms are coercive in  $[\mathbf{H}_0^1(\Omega)]^d$ . Namely, there exists  $C_d > 0$  such that

$$\forall v \in [\mathbf{H}_0^1(\Omega)]^d, \quad \int_{\Omega} \varepsilon(v) : \varepsilon(v) dx \geq C_d \|v\|_{[\mathbf{H}_0^1(\Omega)]^d}^2. \quad (8)$$

Furthermore the bilinear elastic and fluid forms are coercive in  $[\mathbf{H}_0^1(\Omega)]^d$  and verify

$$\forall v \in [\mathbf{H}_0^1(\Omega)]^d, \quad \int_{\Omega} \sigma_s(v) : \varepsilon(v) dx \geq 2\mu C_d \|v\|_{[\mathbf{H}_0^1(\Omega)]^d}^2, \quad (9)$$

and, thanks to assumption (h2),

$$\forall v \in [\mathbf{H}_0^1(\Omega)]^d, \quad \int_{\Omega} \phi \sigma_f(v) : \varepsilon(v) dx \geq 2\mu_f \phi_{\min} C_d \|v\|_{[\mathbf{H}_0^1(\Omega)]^d}^2. \quad (10)$$

Consequently, assuming that  $(u_{s0}, v_{s0}, v_{f0}, p_0) \in [\mathbf{H}_0^1(\Omega)]^d \times [\mathbf{L}^2(\Omega)]^d \times [\mathbf{L}^2(\Omega)]^d \times \mathbf{L}^2(\Omega)$ , it follows that  $u_s \in \mathbf{L}^\infty(0, T; [\mathbf{H}_0^1(\Omega)]^d)$ ,  $\partial_t u_s \in \mathbf{L}^\infty(0, T; [\mathbf{L}^2(\Omega)]^d)$ ,  $v_f \in \mathbf{L}^2(0, T; [\mathbf{H}_0^1(\Omega)]^d)$ ,  $p \in \mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega))$ , and that  $\partial_t u_s \in \mathbf{L}^2(0, T; [\mathbf{H}_0^1(\Omega)]^d)$  if  $\eta > 0$ .

Note that the energy bound (7) depends on the bulk modulus  $\kappa$ . Nonetheless, if  $\theta$  is regular enough, we can recover an energy estimate independent of  $\kappa$  by returning to the case where the right-hand side of (1c) is null, as we are now going to perform it in the incompressible case.

Let us now focus on the case  $\kappa_s = +\infty$  for which the equation (1c) reduces to

$$\operatorname{div} \left( (1-\phi) v_s + \phi v_f \right) = \frac{\theta}{\rho_f}, \quad \text{in } \Omega. \quad (11)$$

Without loss of generality we can assume that the right-hand side of (11) is null. Indeed by considering  $v_\theta$  such that  $\operatorname{div} v_\theta = \frac{\theta}{\rho_f}$  and considering the system satisfied by  $v_s - v_\theta$  and  $v_f - v_\theta$ , namely defining the new displacement

$$u_{s0} + \int_0^t (v_s - v_\theta) ds = u_s - \int_0^t v_\theta ds,$$

we end up with a system for which the constraint reads  $\operatorname{div}((1 - \phi)v_s + \phi v_f) = 0$ . To obtain the energy estimates we proceed as for the case  $\kappa < +\infty$  by multiplying (1a) by the structure velocity  $\partial_t u_s$ , and (1b) by the fluid velocity  $v_f$ . After integration over the domain and integration by parts, adding these two contributions and taking into account the mixture incompressibility constraint  $\operatorname{div}((1 - \phi)v_s + \phi v_f) = 0$  yields

$$\begin{aligned} & \frac{\rho_s}{2} \frac{d}{dt} \int_{\Omega} (1 - \phi) |\partial_t u_s|^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega} \sigma_s(u_s) : \varepsilon(u_s) dx + 2\eta \int_{\Omega} \varepsilon(\partial_t u_s) : \varepsilon(\partial_t u_s) dx \\ & \quad + \frac{\rho_f}{2} \frac{d}{dt} \int_{\Omega} \phi |v_f|^2 dx + \int_{\Omega} \phi \sigma_f(v_f) : \varepsilon(v_f) dx + \int_{\Omega} \phi^2 k_f^{-1} (v_f - \partial_t u_s) \cdot (v_f - \partial_t u_s) dx \\ & = \int_{\Omega} \rho_s (1 - \phi) f \cdot \partial_t u_s dx + \int_{\Omega} \rho_f \phi f \cdot v_f dx + \int_{\Omega} \theta |v_f|^2 dx. \end{aligned} \quad (12)$$

Grönwall Lemma implies

$$\begin{aligned} & \frac{\rho_s}{2} \int_{\Omega} (1 - \phi) |\partial_t u_s(t)|^2 dx + \frac{1}{2} \int_{\Omega} \sigma_s(u_s(t)) : \varepsilon(u_s(t)) dx + 2\eta \int_0^t \int_{\Omega} \varepsilon(\partial_t u_s) : \varepsilon(\partial_t u_s) dx ds \\ & \quad + \frac{\rho_f}{2} \int_{\Omega} \phi |v_f(t)|^2 dx + \int_0^t \int_{\Omega} \phi \sigma_f(v_f) : \varepsilon(v_f) dx ds + \int_0^t \int_{\Omega} \phi^2 k_f^{-1} (v_f - \partial_t u_s) \cdot (v_f - \partial_t u_s) dx ds \\ & \leq \exp\left(\max\left(1, \frac{2\|\theta\|_{C^0([0,T] \times \Omega)}}{\rho_f \phi_{\min}}\right)t\right) \left( \left( \frac{\rho_s}{2} (1 - \phi_{\min}) + \frac{\rho_f}{2} \phi_{\max} \right) \int_0^t \int_{\Omega} |f|^2 dx ds \right. \\ & \quad \left. + \frac{\rho_s}{2} \int_{\Omega} (1 - \phi) |v_{s0}|^2 dx + \frac{1}{2} \int_{\Omega} \sigma_s(u_{s0}) : \varepsilon(u_{s0}) dx + \frac{\rho_f}{2} \int_{\Omega} \phi |v_{f0}|^2 dx \right). \end{aligned} \quad (13)$$

Thanks to Korn inequality (8), coercivities (9), (10), assumptions (h1) – (h5) and assuming that  $(u_{s0}, v_{s0}, v_{f0}) \in [H_0^1(\Omega)]^d \times [L^2(\Omega)]^d \times [L^2(\Omega)]^d$ , we have  $u_s \in L^\infty(0, T; [H_0^1(\Omega)]^d)$ ,  $\partial_t u_s \in L^\infty(0, T; [L^2(\Omega)]^d)$ ,  $v_f \in L^2(0, T; [H_0^1(\Omega)]^d)$ , and if  $\eta > 0$ ,  $\partial_t u_s \in L^2(0, T; [H_0^1(\Omega)]^d)$ . Here the energy bounds does not give bounds on the pressure, which is the main difference between the cases  $\kappa < +\infty$  and  $\kappa = +\infty$ .

We then propose the following milestones for our analysis. We start by considering the compressible case for which  $\kappa < +\infty$ . In this case the pressure  $p$  has its own dynamic. Then the incompressible case, namely  $\kappa = +\infty$ , is treated and we have to deal with a divergence free constraint on the mixture velocity. Each case is split into two cases: the viscous one (namely  $\eta > 0$ ) for which we have a parabolic – parabolic coupling between the solid and fluid equations, and the inviscid one (namely  $\eta = 0$ ) for which we have a hyperbolic – parabolic coupling. For each four cases we prove existence of strong, mild and variational solutions and give the link between the three types of solutions. In particular existence of strong and mild solutions relies on the study of the first order system associated with (1) and the underlying operator using semigroup theory. Strong solution means that the equation is satisfied in  $L^2(\Omega)$  whereas mild solutions are solutions satisfying the Duhamel formula. Note that in the case  $\eta = 0$  in order to prove that the operator is maximal accretive we need to take care of the non coercivity of the associated bilinear form. This issue is solved thanks to the notion of  $T$ -coercivity introduced in [13, 25, 23, 12]. Next the variational solutions are obtained by an approximation strategy as the limit of a sequence of strong solutions. Note that, as we will see, the definition of the variational formulations is different when considering  $\eta > 0$  or  $\eta = 0$ . The main difference comes from the fact that, in the latter case, the structure velocity is not in  $[H_0^1(\Omega)]^d$  in space but only in  $[L^2(\Omega)]^d$ . We end up with the study of the incompressible limit, which allows to pass to the limit in the weak formulation for  $\kappa < +\infty$ , to recover the weak formulation associated with  $\kappa = +\infty$ . The theoretical results are further completed by numerical illustrations to investigate the regularity of the solutions.

## 2 Existence of solutions for a compressible skeleton $\kappa < +\infty$ .

In this section, we study the poromechanical problem for a compressible skeleton, that corresponds to  $\kappa < +\infty$ . First, we write (1) as a first-order evolution equation and we define the associated unbounded operator. Then, by investigating the properties of this operator, we use a semigroup approach to show existence and uniqueness of strong and mild solutions to the system. The existence of variational solutions is obtained by an approximation strategy. The cases  $\eta > 0$  and  $\eta = 0$  are treated separately in order to emphasize the influence of solid viscosity on the model.

### 2.1 Semigroup framework

The system (1) can be rewritten as a first order system as follows

$$\begin{cases} \partial_t u_s - v_s = 0, & \text{in } \Omega \times (0, T), & (14a) \\ \rho_s(1 - \phi) \partial_t v_s - \operatorname{div}(\sigma_s(u_s)) - \operatorname{div}(\sigma_s^{\text{vis}}(v_s)) \\ \quad - \phi^2 k_f^{-1}(v_f - v_s) + (1 - \phi) \nabla p = \rho_s(1 - \phi) f, & \text{in } \Omega \times (0, T), & (14b) \\ \rho_f \phi \partial_t v_f - \operatorname{div}(\phi \sigma_f(v_f)) \\ \quad + \phi^2 k_f^{-1}(v_f - v_s) - \theta v_f + \phi \nabla p = \rho_f \phi f, & \text{in } \Omega \times (0, T), & (14c) \\ \frac{(1 - \phi)^2}{\kappa} \partial_t p + \operatorname{div}((1 - \phi) v_s + \phi v_f) = \frac{\theta}{\rho_f}, & \text{in } \Omega \times (0, T). & (14d) \end{cases}$$

Let  $z = (u_s, v_s, v_f, p)$  and  $z_0 = (u_{s0}, v_{s0}, v_{f0}, p_0)$  denote respectively the unknown variable and the initial condition of (14). We formulate (14) as an abstract evolution problem

$$\begin{cases} \dot{z}(t) + A_\eta^\kappa z(t) + G(t)z(t) = g(t), & t \in [0, T], \\ z(0) = z_0, \end{cases} \quad (15)$$

where  $A_\eta^\kappa$  is an unbounded operator specified with respect to the solid viscosity  $\eta$  and the bulk modulus  $\kappa$ , and  $G(t)$  is a bounded perturbation defined below.

Let us first define the energy space

$$Z = [\mathbf{H}_0^1(\Omega)]^d \times [\mathbf{L}^2(\Omega)]^d \times [\mathbf{L}^2(\Omega)]^d \times \mathbf{L}^2(\Omega)$$

associated with (14). Since the functions  $\rho_s(1 - \phi)$ ,  $\rho_f \phi$  and  $\frac{(1 - \phi)^2}{\kappa}$  are bounded and bounded from below by strictly positive constants, the space  $Z$  can be endowed with the norm

$$\|z\|_Z^2 = \|u_s\|_s^2 + \int_\Omega \rho_s(1 - \phi) |v_s|^2 dx + \int_\Omega \rho_f \phi |v_f|^2 dx + \int_\Omega \frac{(1 - \phi)^2}{\kappa} p^2 dx, \quad (16)$$

where

$$\|u_s\|_s^2 = \int_\Omega \sigma_s(u_s) : \varepsilon(u_s) dx.$$

This norm is equivalent to the canonical norm on  $Z$  according to Korn inequality (9). We denote by  $(\cdot, \cdot)_Z$  the scalar product on  $Z$  associated with the norm (16).

Setting

$$Y = [\mathbf{H}_0^1(\Omega)]^d \times [\mathbf{H}_0^1(\Omega)]^d \times [\mathbf{H}_0^1(\Omega)]^d \times \mathbf{L}^2(\Omega),$$

as an intermediate space, we introduce the bilinear form  $a_\eta^\kappa$  defined for all  $z = (u_s, v_s, v_f, p) \in Y$  and



$y = (d_s, w_s, w_f, q) \in Y$  by

$$\begin{aligned} a_\eta^\kappa(z, y) = & - \int_\Omega \sigma_s(v_s) : \varepsilon(d_s) \, dx + \int_\Omega \sigma_s(u_s) : \varepsilon(w_s) \, dx + 2\eta \int_\Omega \varepsilon(v_s) : \varepsilon(w_s) \, dx \\ & + \int_\Omega \phi \sigma_f(v_f) : \varepsilon(w_f) \, dx + \int_\Omega \phi^2 k_f^{-1} (v_f - v_s) \cdot (w_f - w_s) \, dx \\ & + \int_\Omega \operatorname{div}((1 - \phi)v_s + \phi v_f) q \, dx - \int_\Omega p \operatorname{div}((1 - \phi)w_s + \phi w_f) \, dx. \end{aligned} \quad (17)$$

The bilinear form  $a_\eta^\kappa$  is continuous over  $Y \times Y$ .

Associated with this bilinear form, we introduce the unbounded operator  $(A_\eta^\kappa, D(A_\eta^\kappa))$  defined by

$$(A_\eta^\kappa z, y)_Z = a_\eta^\kappa(z, y), \quad \forall z \in D(A_\eta^\kappa), \forall y \in Y, \quad (18)$$

in the domain

$$D(A_\eta^\kappa) = \{z \in Y : \exists g \in Z, a_\eta^\kappa(z, y) = (g, y)_Z, \quad \forall y \in Y\}. \quad (19)$$

Finally, for all  $t \in [0, T]$ , we define the time-dependent operator

$$G(t) : z = (u_s, v_s, v_f, p) \in Z \longmapsto \left(0, 0, -\frac{\theta(t)}{\rho_f \phi} v_f, 0\right). \quad (20)$$

Taking  $g = \left(0, f, f, \frac{\kappa \theta}{(1-\phi)^2 \rho_f}\right)$  the formulation (15) is equivalent to the evolution equation (14).

**Remark 2.1.** Note that in the domain of operator the equation stating that the time derivative of the structure displacement is equal to the structure velocity (that comes from the first order rewriting of a second order in time problem) will hold true in  $[\mathbf{H}_0^1(\Omega)]^d$  in the space variable. This is the reason of the presence of the term  $-\int_\Omega \sigma_s(v_s) : \varepsilon(d_s) \, dx$  in (17). Yet, even if the solid velocity is considered in  $[\mathbf{H}_0^1(\Omega)]^d$  in the latter integral, we will see that when  $\eta = 0$  the resulting variational solution does not satisfy (14a) in  $[\mathbf{H}_0^1(\Omega)]^d$  but only in  $[\mathbf{L}^2(\Omega)]^d$ . The same issue appears when studying the wave equation.

**Remark 2.2.** When  $z \in D(A_\eta^\kappa)$ , we can write

$$A_\eta^\kappa z = N_0^{-1} \begin{pmatrix} 0 & -v_s & 0 & 0 \\ -\operatorname{div}(\sigma_s(u_s)) & -\operatorname{div}(\sigma_s^{vis}(v_s)) + \phi^2 k_f^{-1} v_s & -\phi^2 k_f^{-1} v_f & (1 - \phi) \nabla p \\ 0 & -\phi^2 k_f^{-1} v_s & -\operatorname{div}(\phi \sigma_f(v_f)) + \phi^2 k_f^{-1} v_f & \phi \nabla p \\ 0 & \operatorname{div}((1 - \phi)v_s) & \operatorname{div}(\phi v_f) & 0 \end{pmatrix}, \quad (21)$$

with

$$N_0 z = \begin{pmatrix} u_s & 0 & 0 & 0 \\ 0 & \rho_s(1 - \phi)v_s & 0 & 0 \\ 0 & 0 & \rho_f \phi v_f & 0 \\ 0 & 0 & 0 & \frac{(1-\phi)^2}{\kappa} p \end{pmatrix}.$$

Note that (21) is meaningful only for  $z \in D(A_\eta^\kappa)$  and specifying  $D(A_\eta^\kappa)$  in terms of classical Sobolev spaces requires to study the regularity of the solution to the static problem  $A_\eta^\kappa z = g$  with  $g \in Z$ , which is a coupled problem with a constraint on the divergence of the mixture. This issue will be explored more extensively in numerical experiments, see Section 5.

In what follows, we exploit the previous framework to prove that Problem (14) has a unique strong and mild solution for  $\kappa < +\infty$ . We also recover the existence of variational solutions as the limit of a sequence of strong solutions. If  $\eta > 0$ , the solid equation (14b) is parabolic, while it becomes hyperbolic when  $\eta = 0$ . For this reason, we distinguish the cases  $\eta > 0$  and  $\eta = 0$ .

## 2.2 The case $\eta > 0$

This case was treated in [18] for  $\theta = 0$ . Here, we propose a proof of existence and uniqueness which is valid for a time-dependent  $\theta$ .

**Theorem 2.3.** *Assume that (h1) – (h3) hold true and that  $\eta > 0$ .*

(i) *If  $\theta \in C^1([0, T]; L^\infty(\Omega))$ ,  $z_0 \in D(A_\eta^\kappa)$  and  $f \in H^1(0, T; [L^2(\Omega)]^d)$ , then there exists a unique strong solution  $z \in C^1([0, T]; Z) \cap C^0([0, T]; D(A_\eta^\kappa))$  satisfying (15).*

(ii) *If  $\theta \in C^0([0, T] \times \Omega)$ ,  $z_0 \in Z$  and  $f \in L^2(0, T; [L^2(\Omega)]^d)$ , then Problem (15) has a unique mild solution  $z \in C^0([0, T]; Z)$  such that  $z(0) = z_0$  and*

$$\int_0^T z(t)\psi(t) dt \in D(A_\eta^\kappa), \quad (22)$$

$$-\int_0^T z(t)\dot{\psi}(t) dt + A_\eta^\kappa \left( \int_0^T z(t)\psi(t) dt \right) + \int_0^T G(t)z(t)\psi(t) dt = \int_0^T g(t)\psi(t) dt, \quad (23)$$

for all  $\psi \in C_c^1([0, T]; \mathbb{R})$ . Moreover,  $z$  verifies the Duhamel formula

$$z(t) = \Phi_\eta^\kappa(t)z_0 + \int_0^t \Phi_\eta^\kappa(t-s)(-G(s)z(s) + g(s)) ds, \quad (24)$$

where  $\Phi_\eta^\kappa$  denotes the continuous semigroup generated by  $A_\eta^\kappa$  in the sense that

$$A_\eta^\kappa x = -\frac{d}{dt}(\Phi_\eta^\kappa(t)x)|_{t=0+}, \quad x \in Z. \quad (25)$$

*Proof.* Let us prove (ii). We shall first show that the operator  $A_\eta^\kappa$  defined by (18) is maximal-accretive, namely:

- $(A_\eta^\kappa z, z)_Z \geq 0, \quad \forall z \in D(A_\eta^\kappa)$  ;
- $A_\eta^\kappa + \lambda_0 I$  is surjective from  $D(A_\eta^\kappa)$  to  $Z$ .

For any  $z = (u_s, v_s, v_f, p) \in D(A_\eta^\kappa)$ , we have by definition of the bilinear form  $a_\eta^\kappa$  and the operator  $A_\eta^\kappa$

$$(A_\eta^\kappa z, z)_Z = a(z, z) = 2\eta \int_\Omega |\varepsilon(v_s)|^2 dx + \int_\Omega \phi^2 k_f^{-1}(v_f - v_s) \cdot (v_f - v_s) dx + \int_\Omega \phi \sigma_f(v_f) : \varepsilon(v_f) dx.$$

Since  $k_f^{-1}(v_f - v_s) \cdot (v_f - v_s) \geq 0$ , we find that  $(A_\eta^\kappa z, z)_Z \geq 0$ .

Let  $\lambda_0 > 0$  be a positive real number and let  $g$  be an element of  $Z$ . To prove that  $A_\eta^\kappa + \lambda_0 I$  is surjective from  $D(A_\eta^\kappa)$  to  $Z$ , we consider the variational problem

$$\begin{cases} \text{Find } z \in Y & \text{such that} \\ \forall y \in Y, & a_\eta^\kappa(z, y) + \lambda_0(z, y)_Z = (g, y)_Z. \end{cases} \quad (26)$$

Using Poincaré inequality, we see that the linear form  $y \mapsto (g, y)_Z$  is continuous over  $Y$  and that the bilinear form  $a_\eta^\kappa(\cdot, \cdot) + \lambda_0(\cdot, \cdot)_Z$  is continuous over  $Y \times Y$ . Moreover,

$$\begin{aligned} a_\eta^\kappa(z, z) + \lambda_0(z, z)_Z &= 2\eta \int_\Omega |\varepsilon(v_s)|^2 dx + \int_\Omega \phi^2 k_f^{-1}(v_f - v_s) \cdot (v_f - v_s) dx + \int_\Omega \phi \sigma_f(v_f) : \varepsilon(v_f) dx \\ &\quad + \lambda_0 \left( \|u_s\|_s^2 + \int_\Omega \rho_s(1 - \phi) |v_s|^2 dx + \int_\Omega \rho_f \phi |v_f|^2 dx + \int_\Omega \frac{(1 - \phi)^2}{\kappa} p^2 dx \right) \\ &\geq \lambda_0 \|u_s\|_s^2 + 2\eta \|\varepsilon(v_s)\|^2 + 2\mu_f \phi_{\min} \|\varepsilon(v_f)\|^2 + \lambda_0 \frac{(1 - \phi_{\max})^2}{\kappa} \|p\|^2, \end{aligned} \quad (27)$$

where  $\|\cdot\|$  denotes the  $L^2$  norm indifferently in  $[L^2(\Omega)]^d$  or  $L^2(\Omega)$ . Consequently, the bilinear form  $a_\eta^\kappa(\cdot, \cdot) + \lambda_0(\cdot, \cdot)_Z$  is coercive on  $Y$  thanks to Korn inequality (8).

From Lax-Milgram theorem, we deduce that there exists a unique  $z \in Y$  solution of (26). Since by construction  $a_\eta^\kappa(z, y) = (g - \lambda_0 z, y)_Z$  for all  $y \in Y$  and  $g - \lambda_0 z \in Z$ , we finally get that  $z \in D(A_\eta^\kappa)$  in view of (19).

Hence,  $A_\eta^\kappa$  is maximal-accretive and Lumer-Phillips theorem (see for instance [40, Chapter 1, Theorem 4.3]) implies that  $A_\eta^\kappa$  is the infinitesimal generator – in the sense of (25) – of a  $C^0$ -semigroup of contraction  $(\Phi_\eta^\kappa(t))_{t \geq 0}$ . In particular, we have

$$\|\Phi_\eta^\kappa(t)\|_{\mathcal{L}(Z)} \leq 1, \quad t \in [0, T]. \quad (28)$$

Then, we observe that  $G(t)$  is a bounded perturbation of  $A_\eta^\kappa$ . Indeed, for any  $z \in Z$ ,

$$\|G(t)z\|_Z^2 = \int_\Omega \rho_f \phi \left( \frac{\theta(t)}{\rho_f \phi} \right)^2 |v_f|^2 dx \leq \omega^2 \|z\|_Z^2,$$

with  $\omega = \|\theta\|_{L^\infty((0, T) \times \Omega)} (\rho_f \phi_{\min})^{-1}$ . Thus  $G \in C^0([0, T]; \mathcal{L}(Z))$  and

$$\|G(t)\|_{\mathcal{L}(Z)} \leq \omega, \quad t \in [0, T]. \quad (29)$$

Therefore, the assertion (ii) follows from [6, Part II, Chapter 1, Proposition 3.4] and [16, Corollary 2.19].

If  $\theta \in C^1([0, T]; L^\infty(\Omega))$  then  $G \in C^1([0, T]; \mathcal{L}(Z))$ , which proves (i) by an application of [6, Part II, Chapter 1, Proposition 3.5].  $\square$

**Remark 2.4.** *The bilinear form  $a_\eta^\kappa(\cdot, \cdot) + \lambda_0(\cdot, \cdot)_Z$  is coercive on  $Y$  precisely because  $\eta > 0$ . It will not be the case when  $\eta = 0$ . In particular, this implies that, here,  $(\Phi_\eta^\kappa(t))_{t \geq 0}$  is an analytic semigroup [6, Part II, Chapter 1, Theorem 2.12].*

Theorem 2.3 provides the existence and uniqueness of two types of solutions: the *strong* solution and the *mild* solution. The strong solution is regular but it requires high regularity assumptions on the source terms and on the initial conditions. The mild solution requires weaker assumptions, but the Duhamel formula (24) is quite abstract. The next theorem establishes the existence and uniqueness of a third notion of solution: the *variational* solution, that satisfies a weak formulation in the following sense.

**Theorem 2.5.** *Assume that (h1) – (h5) hold true and that  $\eta > 0$ . If  $z_0 = (u_{s0}, v_{s0}, v_{f0}, p_0) \in Z$ , then there exists a variational solution  $u_s \in C^0([0, T]; [H_0^1(\Omega)]^d)$ ,  $\partial_t u_s \in C^0([0, T]; [L^2(\Omega)]^d) \cap L^2(0, T; [H_0^1(\Omega)]^d)$ ,  $v_f \in C^0([0, T]; [L^2(\Omega)]^d) \cap L^2(0, T; [H_0^1(\Omega)]^d)$  and  $p \in C^0([0, T]; L^2(\Omega))$  such that*

$$(u_s(0), \partial_t u_s(0), v_f(0), p(0)) = (u_{s0}, v_{s0}, v_{f0}, p_0) \quad (30)$$

and such that the following equations hold true in  $\mathcal{D}'(0, T)$ :

$$\left\{ \begin{array}{l} \forall (w_s, w_f, q) \in [H_0^1(\Omega)]^d \times [H_0^1(\Omega)]^d \times L^2(\Omega), \\ \frac{d^2}{dt^2} \int_\Omega \rho_s (1 - \phi) u_s(t) \cdot w_s dx + \int_\Omega \sigma_s(u_s(t)) : \varepsilon(w_s) dx + 2\eta \int_\Omega \varepsilon(\partial_t u_s(t)) : \varepsilon(w_s) dx \\ \quad - \int_\Omega \phi^2 k_f^{-1} (v_f(t) - \partial_t u_s(t)) \cdot w_s dx - \int_\Omega p(t) \operatorname{div}((1 - \phi) w_s) dx = \int_\Omega \rho_s (1 - \phi) f(t) \cdot w_s dx, \quad (31a) \\ \frac{d}{dt} \int_\Omega \rho_f \phi v_f(t) \cdot w_f dx + \int_\Omega \phi \sigma_f(v_f(t)) : \varepsilon(w_f) dx + \int_\Omega \phi^2 k_f^{-1} (v_f(t) - \partial_t u_s(t)) \cdot w_f dx \\ \quad - \int_\Omega \theta(t) v_f(t) \cdot w_f dx - \int_\Omega p(t) \operatorname{div}(\phi w_f) dx = \int_\Omega \rho_f \phi f(t) \cdot w_f dx, \quad (31b) \\ \frac{d}{dt} \int_\Omega \frac{(1 - \phi)^2}{\kappa} p(t) q dx + \int_\Omega \operatorname{div}((1 - \phi) \partial_t u_s(t) + \phi v_f(t)) q dx = \int_\Omega \frac{\theta(t)}{\rho_f} q dx. \quad (31c) \end{array} \right.$$

Furthermore, the energy estimate (7) holds true and this solution is unique.

*Proof.* To show the existence of variational solutions verifying (31), we proceed as follows. First, we approximate the data by sequences of regular functions and we consider the sequence of strong solutions associated with these regular data. Then, we show that these strong solutions satisfy a variational formulation and we pass to the limit on this formulation after having established some *a priori* estimates and strong convergences of the sequences.

As  $A_\eta^\kappa$  is maximal,  $D(A_\eta^\kappa)$  is dense in  $Z$ . Let  $z_0^n$  be a sequence of elements of  $D(A_\eta^\kappa)$  converging towards  $z_0$  strongly in  $Z$ . Let  $f^n$  denote a sequence of  $H^1(0, T; L^2(\Omega))$  converging towards  $f$  in  $L^2(0, T; L^2(\Omega))$  and  $\theta^n$  denote a sequence of  $C^1([0, T]; L^\infty(\Omega))$  converging towards  $\theta$  in  $C^0([0, T] \times \Omega)$ . From Theorem 2.3, we know that there exists a unique strong solution  $z^n = (u_s^n, v_s^n, v_f^n, p^n)$  to the problem

$$\begin{cases} \dot{z}^n(t) + A_\eta^\kappa z^n(t) + G^n(t)z^n(t) = g^n(t), & t \in [0, T], \\ z^n(0) = z_0^n. \end{cases} \quad (32)$$

In particular,  $z^n$  satisfies the following variational formulation: for all  $s \in [0, T]$ ,

$$(VF)^n \left\{ \begin{array}{l} \forall (d_s, w_s, w_f, p) \in Y = [H_0^1(\Omega)]^d \times [H_0^1(\Omega)]^d \times [H_0^1(\Omega)]^d \times L^2(\Omega), \\ \int_\Omega \sigma_s(\partial_t u_s^n(s)) : \varepsilon(d_s) \, dx = \int_\Omega \sigma_s(v_s^n(s)) : \varepsilon(d_s) \, dx, \\ \int_\Omega \rho_s(1 - \phi) \partial_t v_s^n(s) \cdot w_s \, dx + \int_\Omega \sigma_s(u_s^n(s)) : \varepsilon(w_s) \, dx + 2\eta \int_\Omega \varepsilon(v_s^n(s)) : \varepsilon(w_s) \, dx \\ \quad - \int_\Omega \phi^2 k_f^{-1}(v_f^n(s) - v_s^n(s)) \cdot w_s \, dx - \int_\Omega p^n(s) \operatorname{div}((1 - \phi)w_s) \, dx = \int_\Omega \rho_s(1 - \phi) f^n(s) \cdot w_s \, dx, \\ \int_\Omega \rho_f \phi \partial_t v_f^n(s) \cdot w_f \, dx + \int_\Omega \phi \sigma_f(v_f^n(s)) : \varepsilon(w_f) \, dx + \int_\Omega \phi^2 k_f^{-1}(v_f^n(s) - v_s^n(s)) \cdot w_f \, dx \\ \quad - \int_\Omega \theta^n(s) v_f^n(s) \cdot w_f \, dx - \int_\Omega p^n(s) \operatorname{div}(\phi w_f) \, dx = \int_\Omega \rho_f \phi f^n(s) \cdot w_f \, dx, \\ \int_\Omega \frac{(1 - \phi)^2}{\kappa} \partial_t p^n(s) q \, dx + \int_\Omega \operatorname{div}((1 - \phi)v_s^n(s) + \phi v_f^n(s)) q \, dx = \int_\Omega \frac{\theta^n(s)}{\rho_f} q \, dx. \end{array} \right.$$

Choosing  $d_s = u_s^n(s)$ ,  $w_s = v_s^n(s)$ ,  $w_f = v_f^n(s)$  and  $q = p^n(s)$  as test functions, integrating in time from 0 to  $t$  and applying Grönwall Lemma like in Section 1, we get the energy inequality

$$\begin{aligned} & \frac{1}{2} \int_\Omega \sigma_s(u_s^n(t)) : \varepsilon(u_s^n(t)) \, dx + \frac{\rho_s}{2} \int_\Omega (1 - \phi) |v_s^n(t)|^2 \, dx + 2\eta \int_0^t \int_\Omega \varepsilon(v_s^n) : \varepsilon(v_s^n) \, dx \, ds \\ & \quad + \frac{\rho_f}{2} \int_\Omega \phi |v_f^n(t)|^2 \, dx + \int_0^t \int_\Omega \phi \sigma_f(v_f^n) : \varepsilon(v_f^n) \, dx \, ds + \frac{1}{2} \int_\Omega \frac{(1 - \phi)^2}{\kappa} |p^n(t)|^2 \, dx \\ & \leq \exp\left(\max\left(1, \frac{2\|\theta^n\|_{C^0([0, T] \times \Omega)}}{\rho_f \phi_{\min}}\right)t\right) \left( C \int_\Omega |f^n|^2 \, dx \, ds + C \int_0^t \int_\Omega |\theta^n|^2 \, dx \, ds + \frac{1}{2} \int_\Omega \sigma_s(u_{s_0}^n) : \varepsilon(u_{s_0}^n) \, dx \right. \\ & \quad \left. + \frac{\rho_s}{2} \int_\Omega (1 - \phi) |v_{s_0}^n|^2 \, dx + \frac{\rho_f}{2} \int_\Omega \phi |v_{f_0}^n|^2 \, dx + \frac{1}{2} \int_\Omega \frac{(1 - \phi)^2}{\kappa} |p_0^n|^2 \, dx \right). \quad (33) \end{aligned}$$

Thanks to the assumptions done on the data, the right-hand side of the latter inequality is uniformly bounded with respect to  $n$ . Consequently, taking into account the assumptions (h2) on  $\phi$ , Korn inequality (8), the coercivity of the elastic and fluid forms (9) and (10), we deduce that  $u_s^n$  is uniformly bounded in  $C^0([0, T]; [H_0^1(\Omega)]^d)$ ,  $v_s^n$  in  $C^0([0, T]; [L^2(\Omega)]^d) \cap L^2(0, T; [H_0^1(\Omega)]^d)$ ,  $v_f^n$  in  $C^0([0, T]; [L^2(\Omega)]^d) \cap L^2(0, T; [H_0^1(\Omega)]^d)$  and  $p^n$  in  $C^0([0, T]; 0, T; L^2(\Omega))$ .

In the same way, one can show that  $z^n$  is a Cauchy sequence in  $C^0([0, T]; [H_0^1(\Omega)]^d) \times (C^0([0, T]; [L^2(\Omega)]^d) \cap L^2(0, T; [H_0^1(\Omega)]^d)) \times (C^0([0, T]; [L^2(\Omega)]^d) \cap L^2(0, T; [H_0^1(\Omega)]^d)) \times C^0([0, T]; 0, T; L^2(\Omega))$ . The only term that

requires a special treatment is the term  $\theta^n v_f^n$  in the fluid equation. Indeed, denoting  $z^{n,m} = z^n - z^m$ , the following equality holds for all  $t \in [0, T]$ :

$$\left\{ \begin{array}{l} \forall (d_s, w_s, w_f, p) \in Y, \\ \int_{\Omega} \sigma_s(\partial_t u_s^{n,m}(t)) : \varepsilon(d_s) \, dx = \int_{\Omega} \sigma_s(v_s^{n,m}(t)) : \varepsilon(d_s) \, dx, \\ \int_{\Omega} \rho_s(1-\phi) \partial_t v_s^{n,m}(t) \cdot w_s \, dx + \int_{\Omega} \sigma_s(u_s^{n,m}(t)) : \varepsilon(w_s) \, dx + 2\eta \int_{\Omega} \varepsilon(v_s^{n,m}(t)) : \varepsilon(w_s) \, dx \\ \quad - \int_{\Omega} \phi^2 k_f^{-1}(v_f^{n,m}(t) - v_s^{n,m}(t)) \cdot w_s \, dx - \int_{\Omega} p^{n,m}(t) \operatorname{div}((1-\phi)w_s) \, dx = \int_{\Omega} \rho_s(1-\phi) f^{n,m}(t) \cdot w_s \, dx, \\ \int_{\Omega} \rho_f \phi \partial_t v_f^{n,m}(t) \cdot w_f \, dx + \int_{\Omega} \phi \sigma_f(v_f^{n,m}(t)) : \varepsilon(w_f) \, dx \\ \quad - \int_{\Omega} \theta^{n,m}(t) v_f^n(t) \cdot w_f \, dx - \int_{\Omega} \theta^m(t) v_f^{n,m}(t) \cdot w_f \, dx \\ \quad + \int_{\Omega} \phi^2 k_f^{-1}(v_f^{n,m}(t) - v_s^{n,m}(t)) \cdot w_f \, dx - \int_{\Omega} p^{n,m}(t) \operatorname{div}(\phi w_f) \, dx = \int_{\Omega} \rho_f \phi f^{n,m}(t) \cdot w_f \, dx, \\ \int_{\Omega} \frac{(1-\phi)^2}{\kappa} \partial_t p^{n,m}(t) q \, dx + \int_{\Omega} \operatorname{div}((1-\phi)v_s^{n,m}(t) + \phi v_f^{n,m}(t)) q \, dx = \int_{\Omega} \frac{\theta^{n,m}(t)}{\rho_f} q \, dx. \end{array} \right.$$

This leads to

$$\begin{aligned} & \frac{\rho_s}{2} \frac{d}{dt} \int_{\Omega} (1-\phi) |v_s^{n,m}(t)|^2 \, dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega} \sigma_s(u_s^{n,m}(t)) : \varepsilon(u_s^{n,m}(t)) \, dx + 2\eta \int_{\Omega} \varepsilon(v_s^{n,m}(t)) : \varepsilon(v_s^{n,m}(t)) \, dx \\ & + \frac{\rho_f}{2} \frac{d}{dt} \int_{\Omega} \phi |v_f^{n,m}(t)|^2 \, dx + \int_{\Omega} \phi \sigma_f(v_f^{n,m}(t)) : \varepsilon(v_f^{n,m}(t)) \, dx + \int_{\Omega} \phi^2 k_f^{-1}(v_f^{n,m}(t) - v_s^{n,m}(t)) \cdot (v_f^{n,m}(t) - v_s^{n,m}(t)) \, dx \\ & = \int_{\Omega} \rho_s(1-\phi) f^{n,m}(t) \cdot v_s^{n,m}(t) \, dx + \int_{\Omega} \rho_f \phi f^{n,m}(t) \cdot v_f^{n,m}(t) \, dx + \int_{\Omega} \theta^{n,m}(t) v_f^n(t) \cdot v_f^{n,m}(t) \, dx + \int_{\Omega} \theta^m(t) |v_f^{n,m}(t)|^2 \, dx, \\ & \leq \frac{\rho_s}{2} \int_{\Omega} (1-\phi) |v_s^{n,m}(t)|^2 \, dx + \frac{\rho_f}{2} \int_{\Omega} \phi |v_f^{n,m}(t)|^2 \, dx + \frac{\rho_s}{2} \int_{\Omega} (1-\phi) |f^{n,m}(t)|^2 \, dx + \frac{\rho_f}{2} \int_{\Omega} \phi |f^{n,m}(t)|^2 \, dx \\ & \quad + \frac{1}{2} \|\theta^{n,m}\|_{L^\infty((0,T) \times \Omega)} \int_{\Omega} (|v_f^n(t)|^2 + |v_f^{n,m}(t)|^2) \, dx + \|\theta^m\|_{L^\infty((0,T) \times \Omega)} \int_{\Omega} |v_f^{n,m}(t)|^2 \, dx \end{aligned}$$

for all  $t \in [0, T]$ .

Thus, taking into account the energy estimate satisfied by  $z^n$  uniformly in  $n$ , the fact that  $\|\theta^m\|_{L^\infty((0,T) \times \Omega)}$  is bounded independently of  $m$ , we obtain

$$\frac{d}{dt} \chi^{n,m}(t) \leq C \chi^{n,m}(t) + \frac{\rho_s}{2} \int_{\Omega} (1-\phi) |f^{n,m}(t)|^2 \, dx + \frac{\rho_f}{2} \int_{\Omega} \phi |f^{n,m}(t)|^2 \, dx + C \|\theta^{n,m}\|_{L^\infty((0,T) \times \Omega)},$$

with

$$\begin{aligned} \chi^{n,m}(t) &= \frac{\rho_s}{2} \int_{\Omega} (1-\phi) |v_s^{n,m}(t)|^2 \, dx + \frac{1}{2} \int_{\Omega} \sigma_s(u_s^{n,m}(t)) : \varepsilon(u_s^{n,m}(t)) \, dx + \frac{\rho_f}{2} \int_{\Omega} \phi |v_f^{n,m}(t)|^2 \, dx \\ & \quad + 2\eta \int_0^t \int_{\Omega} \varepsilon(v_s^{n,m}) : \varepsilon(v_s^{n,m}) \, dx \, ds + \int_0^t \int_{\Omega} \phi \sigma_f(v_f^{n,m}) : \varepsilon(v_f^{n,m}) \, dx \, ds. \end{aligned}$$

Grönwall Lemma and the fact that the sequences associated with the data are Cauchy sequences imply that

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, \forall m \geq N, \quad \chi^{n,m}(t) \leq \varepsilon \exp(Ct).$$

As a consequence, there exists  $u_s \in C^0([0, T]; [\mathbf{H}_0^1(\Omega)]^d)$ ,  $v_s \in C^0([0, T]; [\mathbf{L}^2(\Omega)]^d) \cap L^2(0, T; [\mathbf{H}_0^1(\Omega)]^d)$ ,  $v_f \in C^0([0, T]; [\mathbf{L}^2(\Omega)]^d) \cap L^2(0, T; [\mathbf{H}_0^1(\Omega)]^d)$  and  $p \in C^0([0, T]; L^2(\Omega))$  such that

$$\begin{aligned} u_s^n &\longrightarrow u_s && \text{in } C^0([0, T]; [\mathbf{H}_0^1(\Omega)]^d), \\ v_s^n &\longrightarrow v_s && \text{in } C^0([0, T]; [\mathbf{L}^2(\Omega)]^d), \\ v_f^n &\longrightarrow v_f && \text{in } C^0([0, T]; [\mathbf{L}^2(\Omega)]^d), \\ p^n &\longrightarrow p && \text{in } C^0([0, T]; L^2(\Omega)), \\ v_s^n &\longrightarrow v_s && \text{in } L^2(0, T; [\mathbf{H}_0^1(\Omega)]^d), \\ v_f^n &\longrightarrow v_f && \text{in } L^2(0, T; [\mathbf{H}_0^1(\Omega)]^d). \end{aligned}$$

Note that since  $v_s^n = \partial_t u_s^n$ , it also holds true in the limit and  $\partial_t u_s = v_s \in C^0([0, T]; [\mathbf{L}^2(\Omega)]^d) \cap L^2(0, T; [\mathbf{H}_0^1(\Omega)]^d)$ . These convergences enable to pass to the limit in  $(VF)^n$ , except for the initial terms. Yet, these terms can be rewritten thanks to an integration by parts in time as follows: for  $\psi \in \mathcal{D}(0, T)$ , we have for example

$$\begin{aligned} \int_0^T \int_{\Omega} \rho_s(1 - \phi) \partial_t v_s^n(t) \cdot w_s \psi(t) \, dx \, dt &= - \int_0^T \int_{\Omega} \rho_s(1 - \phi) v_s^n(t) \cdot w_s \dot{\psi}(t) \, dx \, dt \\ &\xrightarrow{n \rightarrow \infty} - \int_0^T \int_{\Omega} \rho_s(1 - \phi) v_s(t) \cdot w_s \dot{\psi}(t) \, dx \, dt. \end{aligned}$$

By similar arguments and thanks to the strong convergences, we get, in  $\mathcal{D}'(0, T)$ ,

$$\left\{ \begin{array}{l} \forall (d_s, w_s, w_f, p) \in Y, \\ \frac{d}{dt} \int_{\Omega} \sigma_s(u_s(t)) : \varepsilon(d_s) \, dx = \int_{\Omega} \sigma_s(v_s(t)) : \varepsilon(d_s) \, dx, \quad (34a) \\ \frac{d}{dt} \int_{\Omega} \rho_s(1 - \phi) v_s(t) \cdot w_s \, dx + \int_{\Omega} \sigma_s(u_s(t)) : \varepsilon(w_s) \, dx + 2\eta \int_{\Omega} \varepsilon(v_s(t)) : \varepsilon(w_s) \, dx \\ \quad - \int_{\Omega} \phi^2 k_f^{-1} (v_f(t) - v_s(t)) \cdot w_s \, dx - \int_{\Omega} p(t) \operatorname{div}((1 - \phi) w_s) \, dx = \int_{\Omega} \rho_s(1 - \phi) f(t) \cdot w_s \, dx, \quad (34b) \\ \frac{d}{dt} \int_{\Omega} \rho_f \phi v_f(t) \cdot w_f \, dx + \int_{\Omega} \phi \sigma_f(v_f(t)) : \varepsilon(w_f) \, dx + \int_{\Omega} \phi^2 k_f^{-1} (v_f(t) - v_s(t)) \cdot w_f \, dx \\ \quad - \int_{\Omega} \theta(t) v_f(t) \cdot w_f \, dx - \int_{\Omega} p(t) \operatorname{div}(\phi w_f) \, dx = \int_{\Omega} \rho_f \phi f(t) \cdot w_f \, dx, \quad (34c) \\ \frac{d}{dt} \int_{\Omega} \frac{(1 - \phi)^2}{\kappa} p(t) q \, dx + \int_{\Omega} \operatorname{div}((1 - \phi) v_s(t) + \phi v_f(t)) q \, dx = \int_{\Omega} \frac{\theta(t)}{\rho_f} q \, dx. \quad (34d) \end{array} \right.$$

To obtain (31), it only remains to rewrite (34a) and (34b) as a second order equation in time, which holds true since  $v_s = \partial_t u_s$  in  $C^0([0, T]; [\mathbf{L}^2(\Omega)]^d) \cap L^2(0, T; [\mathbf{H}_0^1(\Omega)]^d)$ . Lastly, we recover the initial conditions (30) by simply passing to the limit in the second line of (32).

To ensure uniqueness, we observe that every variational solution satisfying the energy estimate is unique. Indeed, for  $f = 0$ ,  $\theta = 0$  and  $z_0 = 0$ , we obtain  $z = 0$  in virtue of (7). Therefore, it is sufficient to prove that every variational solution satisfying (31) verifies the energy identity (6). To do so, let us first derive a bound on  $(\partial_t u_s, \partial_t v_s, \partial_t v_f, \partial_t p)$ . From (34), we deduce

$$\begin{aligned} \forall y \in Y, \quad - \int_0^T (z(t), y)_Z \dot{\psi}(t) \, dt + \int_0^T a_{\eta}^{\kappa}(z(t), y) \psi(t) \, dt \\ + \int_0^T (G(t)z(t), y)_Z \psi(t) \, dt = \int_0^T (g(t), y)_Z \psi(t) \, dt. \quad (35) \end{aligned}$$

Since  $f \in L^2(0, T; [L^2(\Omega)]^d)$ ,  $\theta \in C^0((0, T) \times \Omega)$  and by continuity of the bilinear form  $a_\eta^\kappa$  over  $Y \times Y$ , we have

$$\forall y \in Y, \quad - \int_0^T (z(t), y)_Z \dot{\psi}(t) dt = \int_0^T (x(t), y)_Z \psi(t) dt \quad \text{with } x \in L^2(0, T; Y').$$

Thus  $\dot{z} \in L^2(0, T; Y')$ , namely

$$(\partial_t u_s, \partial_t v_s, \partial_t v_f, \partial_t p) \in L^2(0, T; [H^{-1}(\Omega)]^d) \times L^2(0, T; [H^{-1}(\Omega)]^d) \times L^2(0, T; [H^{-1}(\Omega)]^d) \times L^2(0, T; L^2(\Omega)).$$

Finally, it holds

$$\begin{aligned} \partial_t u_s &\in L^2(0, T; [H_0^1(\Omega)]^d) & \text{and} & \quad \partial_{tt} u_s \in L^2(0, T; [H^{-1}(\Omega)]^d), \\ v_f &\in L^2(0, T; [H_0^1(\Omega)]^d) & \text{and} & \quad \partial_t v_f \in L^2(0, T; [H^{-1}(\Omega)]^d), \\ p &\in L^2(0, T; L^2(\Omega)) & \text{and} & \quad \partial_t p \in L^2(0, T; L^2(\Omega)). \end{aligned} \quad (36)$$

Using a standard result of functional analysis (see for instance [27, Chapter XVIII, Proposition 7]), the previous regularities imply respectively that

$$\begin{aligned} \forall w_s \in [H_0^1(\Omega)]^d, \quad \frac{d^2}{dt^2} \int_\Omega \rho_s (1 - \phi) u_s(t) \cdot w_s dx &= \left\langle \rho_s (1 - \phi) \partial_{tt} u_s(t), w_s \right\rangle_{[H^{-1}(\Omega)]^d, [H_0^1(\Omega)]^d}, \\ \forall w_f \in [H_0^1(\Omega)]^d, \quad \frac{d}{dt} \int_\Omega \rho_f \phi v_f(t) \cdot w_f dx &= \left\langle \rho_f \phi \partial_t v_f(t), w_f \right\rangle_{[H^{-1}(\Omega)]^d, [H_0^1(\Omega)]^d}, \\ \forall q \in L^2(\Omega), \quad \frac{d}{dt} \int_\Omega \frac{(1 - \phi)^2}{\kappa} p(t) q dx &= \int_\Omega \frac{(1 - \phi)^2}{\kappa} \partial_t p(t) q dx. \end{aligned} \quad (37)$$

Now, for almost every  $t \in [0, T]$ , we know that  $\partial_t u_s(t) \in [H_0^1(\Omega)]^d$ ,  $v_f(t) \in [H_0^1(\Omega)]^d$  and  $p(t) \in L^2(\Omega)$ . Hence, we can choose  $(w_s, w_f, q) = (\partial_t u_s(t), v_f(t), p(t))$  as test functions in (31), which, together with (37), provides exactly (6).  $\square$

**Remark 2.6.** *The method used to prove Theorem 2.5 is standard and close to the Faedo-Galerkin method. The difference with Faedo-Galerkin method is that the approximated sequence is directly recovered from the existence of strong solutions instead of being constructed on a suitable finite dimensional space. This allows us to obtain strong convergence for the whole sequence, whereas Faedo-Galerkin method provides only weak convergence of subsequences. In addition, it directly provides the continuity with respect to time of the solution and the strong convergence of the initial condition  $z(0) = z_0$  in  $Z$ .*

**Remark 2.7.** *The variational solution could also be defined without assuming that it is continuous with respect to time, but only assuming that the regularities (36) are satisfied. The time continuity of the solution can then be recovered using the existence of a continuous and linear mapping of the space*

$$W(0, T) = \{u \in L^2(0, T; [H_0^1(\Omega)]^d) \text{ such that } \partial_t u \in L^2(0, T; [H^{-1}(\Omega)]^d)\}$$

into  $C^0([0, T]; [L^2(\Omega)]^d)$ , see [37, Chapter 1, Theorem 3.1].

The mild solution and the variational solution are two notions of solution whose existence and uniqueness here require the same hypotheses on the data. In fact, the following result states that these two types of solution are the same whenever  $f \in L^2((0, T) \times \Omega)$  and  $\theta \in C^0([0, T] \times \Omega)$ . Thus, they can be used indifferently depending on the context. For instance, the mild solution is widely used in control theory because of the practical aspects of Duhamel formula, whereas the variational solution formulation is usually the one implemented at the discrete level when considering finite element discretization.

**Proposition 2.8.** *If  $f \in L^2((0, T) \times \Omega)$  and  $\theta \in C^0([0, T] \times \Omega)$ , then the mild solution given by (24) and the variational solution satisfying (31) coincide.*

*Proof.* The mild solution and the variational solution are both unique. Hence, it is sufficient to show that the variational solution defined in Theorem 2.5 is also a mild solution, namely that it satisfies (22) and (23).

Let  $\psi$  be given in  $\mathcal{D}(0, T)$ . Since  $v_s \in L^2(0, T; [\mathbf{H}_0^1(\Omega)]^d)$  and  $v_f \in L^2(0, T; [\mathbf{H}_0^1(\Omega)]^d)$ , it holds that

$$a_\eta^\kappa \left( \int_0^T z(t) \psi(t) dt, y \right) = \int_0^T a_\eta^\kappa(z(t), y) \psi(t) dt$$

for all  $y \in Y$ , so that we can rewrite (35) as

$$\forall y \in Y, \quad - \int_0^T (z(t), y)_Z \dot{\psi}(t) dt + a_\eta^\kappa \left( \int_0^T z(t) \psi(t) dt, y \right) + \int_0^T (G(t)z(t), y)_Z \psi(t) dt = \int_0^T (g(t), y)_Z \psi(t) dt.$$

From the definition of  $D(A_\eta^\kappa)$ , it follows that

$$\int_0^T z(t) \psi(t) dt \in D(A_\eta^\kappa)$$

and thus

$$- \int_0^T (z(t), y)_Z \dot{\psi}(t) dt + \left( A_\eta^\kappa \left( \int_0^T z(t) \psi(t) dt \right), y \right)_Z + \int_0^T (G(t)z(t), y)_Z \psi(t) dt = \int_0^T (g(t), y)_Z \psi(t) dt \quad (38)$$

for all  $y \in Y$ . As  $Y$  is dense in  $Z$ , (38) is also true for all  $y \in Z$ , which proves (23).  $\square$

**Remark 2.9.** Note that the mild and variational solutions coincide only under the assumption  $f \in L^2((0, T) \times \Omega)$  and  $\theta \in C^0([0, T] \times \Omega)$ . If  $f \in L^2(0, T; [\mathbf{H}^{-1}(\Omega)]^d)$ , we can readily extend the existence and uniqueness of variational solutions proved in Theorem 2.5, but the existence of a mild solution is not guaranteed.

### 2.3 The case $\eta = 0$

Without solid viscosity, the solid formulation becomes hyperbolic. This hyperbolic – parabolic coupling was studied in [4], where existence of variational solutions is derived. In [4], the fluid mass input  $\theta$  is supposed to be small enough, namely there exists  $C_f > 0$  such that

$$\forall v_f \in [\mathbf{H}_0^1(\Omega)]^d, \quad \int_\Omega \phi \sigma_f(v_f) : \varepsilon(v_f) dx - \int_\Omega \theta |v_f|^2 dx \geq C_f \|v_f\|_{[\mathbf{H}_0^1(\Omega)]^d}^2.$$

Here, we are going to prove existence results of strong and mild solutions thanks to semigroup theory and deduce existence of variational solutions directly.

The main issue in this case is that the underlying bilinear form is not coercive. Indeed if  $\eta = 0$ , then the bilinear form introduced in the proof of Theorem 2.3 is no more coercive on  $Y$  in view of (27). Despite this lack of coercivity, we are going to show that Problem (26) is still well-posed when  $\eta = 0$ . To do so, we use the  $T$ -coercivity approach [13, 25, 23, 12], which is an alternative to Banach-Nečas-Babuška theory. For the sake of completeness, the definition and properties of  $T$ -coercivity are recalled below.

**Definition 2.1.** Let  $V$  be an Hilbert space and let  $a(\cdot, \cdot)$  be a continuous bilinear form over  $V \times V$ . We say that  $a$  is  $T$ -coercive if there exists a bijective application  $T \in \mathcal{L}(V)$  and  $\underline{\alpha} > 0$  such that

$$|a(z, Tz)| \geq \underline{\alpha} \|z\|_V^2, \quad z \in V.$$

**Proposition 2.10.** Let  $V$  be an Hilbert space. Let  $\ell(\cdot)$  be a continuous linear form over  $V$  and  $a(\cdot, \cdot)$  be a continuous bilinear form over  $V \times V$ . The problem

$$\begin{cases} \text{Find } z \in V & \text{such that} \\ \forall y \in V, & a(z, y) = \ell(y) \end{cases}$$

is well-posed if and only if  $a$  is  $T$ -coercive.



The following theorem states existence and uniqueness of solutions to Problem (15) in the case  $\eta = 0$ .

**Theorem 2.11.** *If  $\eta = 0$ , then the conclusions of Theorem 2.3 remain true.*

*Proof.* Let us show that  $A_0^\kappa$  is maximal by proving that the variational problem

$$\begin{cases} \text{Find } z \in Y & \text{such that} \\ \forall y \in Y, & a_0^\kappa(z, y) + \lambda_0(z, y)_Z = (g, y)_Z \end{cases}$$

is well-posed, where

$$\begin{aligned} a_0^\kappa(z, y) + \lambda_0(z, y)_Z &= \int_{\Omega} \lambda_0 \sigma_s(u_s) : \varepsilon(d_s) \, dx - \int_{\Omega} \sigma_s(v_s) : \varepsilon(d_s) \, dx \\ &+ \int_{\Omega} \lambda_0 \rho_s (1 - \phi) v_s \cdot w_s \, dx + \int_{\Omega} \sigma_s(u_s) : \varepsilon(w_s) \, dx + \int_{\Omega} \phi^2 k_f^{-1} (v_f - v_s) \cdot (w_f - w_s) \, dx \\ &+ \int_{\Omega} \lambda_0 \rho_f \phi v_f \cdot w_f \, dx + \int_{\Omega} \phi \sigma_f(v_f) : \varepsilon(w_f) \, dx + \int_{\Omega} \lambda_0 \frac{(1 - \phi)^2}{\kappa} p q \, dx \\ &- \int_{\Omega} p \operatorname{div}((1 - \phi) w_s + \phi w_f) \, dx + \int_{\Omega} \operatorname{div}((1 - \phi) v_s + \phi v_f) q \, dx \end{aligned}$$

for any  $z = (u_s, v_s, v_f, p)$  and  $y = (d_s, w_s, w_f, q)$  in  $Y$ . From Proposition 2.10, it is sufficient to show that  $a_0^\kappa(\cdot, \cdot) + \lambda_0(\cdot, \cdot)_Z$  is  $T$ -coercive.

For a given  $z$ , we look for a  $y^*$  depending continuously on  $z$  such that  $a_0^\kappa(z, y^*) + \lambda_0(z, y^*)_Z \geq \underline{\alpha} \|z\|_Y^2$  for some constant  $\underline{\alpha} > 0$ . Choosing  $w_s^* = v_s$ ,  $w_f^* = v_f$ ,  $q^* = p$  and  $d_s^*$  in the form  $\beta u_s + \gamma v_s$  yields

$$\begin{aligned} a_0^\kappa(z, y^*) + \lambda_0(z, y^*)_Z &= \int_{\Omega} \lambda_0 \beta \sigma_s(u_s) : \varepsilon(u_s) \, dx + \int_{\Omega} \lambda_0 \gamma \sigma_s(u_s) : \varepsilon(v_s) \, dx \\ &- \int_{\Omega} \beta \sigma_s(v_s) : \varepsilon(u_s) \, dx - \int_{\Omega} \gamma \sigma_s(v_s) : \varepsilon(v_s) \, dx \\ &+ \int_{\Omega} \lambda_0 \rho_s (1 - \phi) |v_s|^2 \, dx + \int_{\Omega} \sigma_s(u_s) : \varepsilon(v_s) \, dx + \int_{\Omega} \phi^2 k_f^{-1} (v_f - v_s) \cdot (v_f - v_s) \, dx \\ &+ \int_{\Omega} \lambda_0 \rho_f \phi |v_f|^2 \, dx + \int_{\Omega} \phi \sigma_f(v_f) : \varepsilon(v_f) \, dx + \int_{\Omega} \lambda_0 \frac{(1 - \phi)^2}{\kappa} p^2 \, dx. \end{aligned}$$

By setting  $\beta = \frac{1}{2}$  and  $\gamma = -\frac{1}{2\lambda_0}$ , the terms of the form  $\int_{\Omega} \sigma_s(u_s) : \varepsilon(v_s) \, dx$  vanish so that

$$a_0^\kappa(z, y^*) + \lambda_0(z, y^*)_Z \geq \frac{\lambda_0}{2} \|u_s\|_s^2 + \frac{1}{2\lambda_0} \|v_s\|_s^2 + 2\mu_f \phi_{\min} \|\varepsilon(v_f)\|^2 + \lambda_0 \frac{(1 - \phi_{\max})^2}{\kappa} \|p\|^2.$$

Therefore,  $a_0^\kappa(\cdot, \cdot) + \lambda_0(\cdot, \cdot)_Z$  is  $T$ -coercive for the mapping  $T$  defined by

$$T : (u_s, v_s, v_f, p) \mapsto \left( \frac{1}{2} u_s - \frac{1}{2\lambda_0} v_s, v_s, v_f, p \right), \quad (39)$$

which is continuous and bijective on  $Y$ .

The remainder of the proof follows the very same lines as for the viscous case.  $\square$

Next we recover the existence of variational solutions from the existence of strong solutions. However, we obtain a variational formulation that slightly differs from (31) because of the hyperbolic – parabolic coupling between the solid and fluid equations.

**Theorem 2.12.** *Assume that (h1) – (h5) hold true and that  $\eta = 0$ . If  $z_0 = (u_{s0}, v_{s0}, v_{f0}, p_0) \in Z$ , then there exists a variational solution  $u_s \in C^0([0, T]; [\mathbf{H}_0^1(\Omega)]^d)$ ,  $\partial_t u_s \in C^0([0, T]; [\mathbf{L}^2(\Omega)]^d)$ ,  $v_f \in C^0([0, T]; [\mathbf{L}^2(\Omega)]^d) \cap L^2(0, T; [\mathbf{H}_0^1(\Omega)]^d)$  and  $p \in C^0([0, T]; L^2(\Omega))$  such that*

$$(u_s(0), \partial_t u_s(0), v_f(0), p(0)) = (u_{s0}, v_{s0}, v_{f0}, p_0) \quad (40)$$

and the following equations hold true, in  $\mathcal{D}'(0, T)$ ,

$$\left\{ \begin{array}{l} \forall (w_s, w_f, q) \in [\mathbf{H}_0^1(\Omega)]^d \times [\mathbf{H}_0^1(\Omega)]^d \times L^2(\Omega), \\ \frac{d^2}{dt^2} \int_{\Omega} \rho_s (1 - \phi) u_s(t) \cdot w_s \, dx + \int_{\Omega} \sigma_s(u_s(t)) : \varepsilon(w_s) \, dx \\ \quad - \int_{\Omega} \phi^2 k_f^{-1} (v_f(t) - \partial_t u_s(t)) \cdot w_s \, dx - \int_{\Omega} p(t) \operatorname{div}((1 - \phi) w_s) \, dx = \int_{\Omega} \rho_s (1 - \phi) f(t) \cdot w_s \, dx, \quad (41a) \\ \frac{d}{dt} \int_{\Omega} \rho_f \phi v_f(t) \cdot w_f \, dx + \int_{\Omega} \phi \sigma_f(v_f(t)) : \varepsilon(w_f) \, dx + \int_{\Omega} \phi^2 k_f^{-1} (v_f(t) - \partial_t u_s(t)) \cdot w_f \, dx \\ \quad - \int_{\Omega} \theta(t) v_f(t) \cdot w_f \, dx - \int_{\Omega} p(t) \operatorname{div}(\phi w_f) \, dx = \int_{\Omega} \rho_f \phi f(t) \cdot w_f \, dx, \quad (41b) \\ \frac{d}{dt} \int_{\Omega} \frac{(1 - \phi)^2}{\kappa} p(t) q \, dx + \frac{d}{dt} \int_{\Omega} \operatorname{div}((1 - \phi) u_s(t)) q \, dx + \int_{\Omega} \operatorname{div}(\phi v_f(t)) q \, dx = \int_{\Omega} \frac{\theta(t)}{\rho_f} q \, dx. \quad (41c) \end{array} \right.$$

This variational solution is unique and coincides with the mild solution. Furthermore, the energy estimate (7) with  $\eta = 0$  holds true .

**Remark 2.13.** *Theorem 2.12 sheds light on the influence of solid viscosity on the model. Since  $\eta = 0$ ,  $\partial_t u_s$  does not belong to  $L^2(0, T; [\mathbf{H}_0^1(\Omega)]^d)$  but only to  $C^0([0, T]; [\mathbf{L}^2(\Omega)]^d)$ . For this reason, equations (31c) and (41c) are not similar because, when  $\eta = 0$ , the term  $\operatorname{div}((1 - \phi) \partial_t u_s(t))$  does not belong to  $L^2(\Omega)$ .*

*Proof.* We follow the same steps as for the proof of Theorem 2.12. The input data are approximated by regular functions, *a priori* estimates are established for the approximated solutions and we pass to the limit on the variational formulation  $(VF)^n$  with  $\eta = 0$ .

As previously, the estimate (33) implies that  $z^n$  is a Cauchy sequence in  $C^0([0, T]; Z)$ . However, since  $\eta = 0$ , estimate (33) only implies that  $v_f^n$  is a Cauchy sequence in  $L^2(0, T; [\mathbf{H}_0^1(\Omega)]^d)$ . Hence, the convergence

$$v_f^n \longrightarrow v_f \quad \text{in } L^2(0, T; [\mathbf{H}_0^1(\Omega)]^d)$$

is still valid but now  $v_s^n = \partial_t u_s^n$  does not converge in  $L^2(0, T; [\mathbf{H}_0^1(\Omega)]^d)$  but only in  $C^0([0, T]; [\mathbf{L}^2(\Omega)]^d)$ . This changes the way to pass to the limit on  $(VF)^n$  and in particular in the first equation. But let  $\psi$  be an element of  $\mathcal{D}(0, T)$ . For any  $d_s \in [L^2(\Omega)]^d$ , we consider the unique solution  $\eta_s \in [\mathbf{H}_0^1(\Omega)]^d$  of  $-\operatorname{div}(\sigma_s(\eta_s)) = d_s$  as a test function, so that

$$\begin{aligned} \int_0^T \int_{\Omega} \sigma_s(v_s^n(t)) : \varepsilon(\eta_s) \psi(t) \, dx \, dt &= \int_0^T \int_{\Omega} \varepsilon(v_s^n(t)) : \sigma_s(\eta_s) \psi(t) \, dx \, dt \\ &= \int_0^T \int_{\Omega} v_s^n(t) \cdot d_s \psi(t) \, dx \, dt \xrightarrow{n \rightarrow \infty} \int_0^T \int_{\Omega} v_s(t) \cdot d_s \psi(t) \, dx \, dt. \end{aligned}$$

This proves that for all  $d_s \in [L^2(\Omega)]^d$ , we have

$$\frac{d}{dt} \int_{\Omega} u_s(t) \cdot d_s \, dx - \int_{\Omega} v_s(t) \cdot d_s \, dx = 0 \quad (42)$$

and we recover that  $v_s = \partial_t u_s$  in  $\mathcal{D}'((0, T) \times \Omega)$ . In particular, it holds that  $\partial_t u_s \in C^0([0, T]; [\mathbf{L}^2(\Omega)]^d)$ .

The obtention of (41a) and (41b) is not affected by the absence of solid viscosity. Finally, to get (41c), we observe that

$$\begin{aligned} \int_0^T \int_{\Omega} \operatorname{div}((1-\phi)v_s^n(t))q\psi(t) \, dx \, dt &= - \int_0^T \int_{\Omega} \operatorname{div}((1-\phi)u_s^n(t))q\dot{\psi}(t) \, dx \, dt \\ &\xrightarrow{n \rightarrow \infty} - \int_0^T \int_{\Omega} \operatorname{div}((1-\phi)u_s(t))q\dot{\psi}(t) \, dx \, dt, \end{aligned}$$

where we have integrated by parts in time and used that  $u_s^n$  converges in  $C^0(0, T; [H_0^1(\Omega)]^d)$ .

Now, let us prove the uniqueness of the variational solution. Contrary to the viscous case, we can not take  $(w_s, w_f, q) = (\partial_t u_s(t), v_f(t), p(t))$  as test functions in (41) because  $\partial_t u_s(t) \notin [H_0^1(\Omega)]^d$ . Nevertheless, we are going to obtain the uniqueness of the variational solution by showing that it coincides with the mild solution, which we know to be unique. To do so, we integrate by parts in space the tests functions of (41) in order to make appear the adjoint operator of  $A_0^{\kappa}$ . For the fluid terms, we write

$$\int_0^T \int_{\Omega} \phi \sigma_f(v_f(t)) : \varepsilon(w_f) \psi(t) \, dx \, dt = - \int_0^T \int_{\Omega} v_f(t) \cdot \operatorname{div}(\phi \sigma_f(w_f)) \psi(t) \, dx \, dt$$

and

$$\int_0^T \int_{\Omega} \operatorname{div}(\phi v_f(t))q\psi(t) \, dx \, dt = - \int_0^T \int_{\Omega} v_f(t) \cdot \phi \nabla q \psi(t) \, dx \, dt.$$

Choosing  $\eta_s \in [H_0^1(\Omega)]^d$  such that  $-\operatorname{div}(\sigma_s(\eta_s)) = d_s$ , (42) becomes

$$\frac{d}{dt} \int_{\Omega} \sigma_s(u_s) : \varepsilon(\eta_s) \, dx + \int_{\Omega} v_s \cdot \operatorname{div}(\sigma_s(\eta_s)) \, dx = 0.$$

Lastly, an integration by parts in space and in time yields

$$- \int_0^T \int_{\Omega} \operatorname{div}((1-\phi)u_s(t))q\dot{\psi}(t) \, dx \, dt = \int_0^T \int_{\Omega} u_s(t) \cdot (1-\phi) \nabla q \dot{\psi}(t) \, dx \, dt = - \int_0^T \int_{\Omega} v_s(t) \cdot (1-\phi) \nabla q \psi(t) \, dx \, dt.$$

Hence, for all  $y = (\eta_s, w_s, w_f, q) \in D(A_0^{\kappa*})$ , it holds that

$$\frac{d}{dt} (z(t), y)_Z + (z(t), A_0^{\kappa*} y)_Z + (G(t)z(t), y)_Z = (g(t), y)_Z, \quad (43)$$

where

$$A_0^{\kappa*} y = N_0^{-1} \begin{pmatrix} 0 & w_s & 0 & 0 \\ \operatorname{div}(\sigma_s(\eta_s)) & \phi^2 k_f^{-1} w_s & -\phi^2 k_f^{-1} w_f & -(1-\phi) \nabla q \\ 0 & -\phi^2 k_f^{-1} w_s & -\operatorname{div}(\phi \sigma_f(w_f)) + \phi^2 k_f^{-1} w_f & -\phi \nabla q \\ 0 & -\operatorname{div}((1-\phi)w_s) & -\operatorname{div}(\phi w_f) & 0 \end{pmatrix}.$$

Using a general result of Ball [3] that connects weak solutions in the sense of (43) to Duhamel formula, we conclude that  $z$  coincides with the mild solution of the problem, which is unique.

Now that we know that the variational solution is unique, it follows that it is necessarily the one obtained by the approximation process built from  $(VF)^n$ . Since this approximation process is based on the energy estimate (33) with  $\eta = 0$ , we can pass to the limit in this estimation to get (7).  $\square$

**Remark 2.14.** *In the previous proof, we took advantage of the semigroup framework to show the uniqueness of the variational solution and the energy equality. Note that it could also be shown by adapting regularization techniques (see for instance [37, Chapter 3, Theorem 8.1 and Lemma 8.3] for the wave equation).*

**Remark 2.15.** *As in the viscous case, we could also define the variational solution without assuming that it is continuous with respect to time, but rather by seeking for*

$$\begin{aligned} u_s &\in L^\infty(0, T; [\mathbf{H}_0^1(\Omega)]^d), & \partial_t u_s &\in L^\infty(0, T; [\mathbf{L}^2(\Omega)]^d) & \text{and} & \partial_{tt} u_s &\in L^2(0, T; [\mathbf{H}^{-1}(\Omega)]^d), \\ v_f &\in L^2(0, T; [\mathbf{H}_0^1(\Omega)]^d) \cap L^\infty(0, T; [\mathbf{L}^2(\Omega)]^d) & \text{and} & \partial_t v_f &\in L^2(0, T; [\mathbf{H}^{-1}(\Omega)]^d), \\ p &\in L^2(0, T; L^2(\Omega)) & \text{and} & \partial_t p &\in L^2(0, T; \mathbf{H}^{-1}(\Omega)), \end{aligned}$$

such that (40) and (41) are verified. With this definition, the continuity in time of the solution can then be recovered using, for instance, a parabolic regularization, while it is obtained directly in the above proof.

In the next section, we analyze the poromechanics problem for an incompressible elastic skeleton, modelled by the assumption  $\kappa = +\infty$ . This assumption is crucial for targeting biomedical applications since the tissues in our body are mostly composed of water, and thus are close to being incompressible.

### 3 Existence of solutions for an incompressible skeleton $\kappa = +\infty$ .

When  $\kappa = +\infty$ , the system of equations (14) reads

$$\begin{cases} \partial_t u_s - v_s = 0, & (44a) \\ \rho_s(1 - \phi) \partial_t v_s - \operatorname{div}(\sigma_s^{\text{vis}}(v_s)) - \operatorname{div}(\sigma_s(u_s)) - \phi^2 k_f^{-1}(v_f - v_s) + (1 - \phi) \nabla p = \rho_s(1 - \phi) f, & (44b) \\ \rho_f \phi \partial_t v_f - \operatorname{div}(\phi \sigma_f(v_f)) + \phi^2 k_f^{-1}(v_f - v_s) - \theta v_f + \phi \nabla p = \rho_f \phi f, & (44c) \\ \operatorname{div}((1 - \phi) v_s + \phi v_f) = \frac{\theta}{\rho_f}. & (44d) \end{cases}$$

It has to be completed with boundary conditions (2) and initial conditions (3). Note that, in the present case, there is no initial condition for the pressure anymore.

Equation (44d) traduces the mixture's incompressibility, which comes from the assumption that the solid and the fluid phases are both incompressible. It takes the form of a constraint on the divergence of the mixture's velocity.

But, as already noticed in Section 1, it is sufficient to consider the case

$$\operatorname{div}((1 - \phi) v_s + \phi v_f) = 0, \quad \text{in } \Omega \times (0, T). \quad (45)$$

Indeed, assuming for instance that  $\theta \in \mathbf{H}^1(0, T; L^2(\Omega)) \cap L^2(0, T; \mathbf{H}^1(\Omega))$  and that the compatibility condition  $\int_\Omega \theta(t) dx = 0$  is satisfied for all  $t \in [0, T]$ , we consider  $v_\theta \in \mathbf{H}^1(0, T; [\mathbf{H}_0^1(\Omega)]^d) \cap L^2(0, T; [\mathbf{H}^2(\Omega)]^d)$  such that

$$\operatorname{div} v_\theta = \frac{\theta}{\rho_f}. \quad (46)$$

The change of variables  $\hat{v}_s = v_s - v_\theta$  and  $\hat{v}_f = v_f - v_\theta$  gives  $\operatorname{div}((1 - \phi) \hat{v}_s + \phi \hat{v}_f) = 0$  by construction. Furthermore,  $(u_s - \int_0^t v_\theta(s) ds, \hat{v}_s, \hat{v}_f, p)$  verifies (44a), (44b) and (44c) with right-hand sides that are different but still regular since  $v_\theta \in \mathbf{H}^1(0, T; [\mathbf{H}_0^1(\Omega)]^d) \cap L^2(0, T; [\mathbf{H}^2(\Omega)]^d)$ .

The first part of this section is devoted to the functional analysis of the coupling constraint (45).

#### 3.1 Functional framework

We introduce the spaces

$$V_\phi = \left\{ (v_s, v_f) \in [\mathbf{H}_0^1(\Omega)]^d \times [\mathbf{H}_0^1(\Omega)]^d : \operatorname{div}((1 - \phi) v_s + \phi v_f) = 0 \quad \text{in } \Omega \right\},$$

and

$$H_\phi = \left\{ (v_s, v_f) \in [L^2(\Omega)]^d \times [L^2(\Omega)]^d : \operatorname{div}((1-\phi)v_s + \phi v_f) = 0 \quad \text{in } \Omega, \right. \\ \left. \text{and } ((1-\phi)v_s + \phi v_f) \cdot n = 0 \quad \text{on } \partial\Omega \right\},$$

of functions satisfying the constraint (45). The spaces  $V_\phi$  and  $H_\phi$  are respectively closed in  $[\mathbf{H}_0^1(\Omega)]^d \times [\mathbf{H}_0^1(\Omega)]^d$  and  $[L^2(\Omega)]^d \times [L^2(\Omega)]^d$ .

Let us also define the mixture's divergence operator by

$$B : \begin{array}{ccc} [\mathbf{H}_0^1(\Omega)]^d \times [\mathbf{H}_0^1(\Omega)]^d & \longrightarrow & L_0^2(\Omega) \\ (v_s, v_f) & \longmapsto & \operatorname{div}((1-\phi)v_s + \phi v_f). \end{array}$$

The bounded operator  $B$  satisfies the following inf-sup condition.

**Proposition 3.1.** *Assume that  $\phi \in \mathbf{H}^{d/2+r}(\Omega)$  with  $r > 0$ . There exists  $\underline{\beta} > 0$  such that, for all  $p \in L_0^2(\Omega)$ ,*

$$\sup_{(v_s, v_f) \in [\mathbf{H}_0^1(\Omega)]^d \times [\mathbf{H}_0^1(\Omega)]^d} \frac{\int_\Omega \operatorname{div}((1-\phi)v_s + \phi v_f) p \, dx}{\|(v_s, v_f)\|_{[\mathbf{H}_0^1(\Omega)]^d \times [\mathbf{H}_0^1(\Omega)]^d}} \geq \underline{\beta} \|p\|. \quad (47)$$

*Proof.* There exists  $C_{\operatorname{div}} > 0$  such that for any  $p \in L_0^2(\Omega)$ , there exists  $v_p \in [\mathbf{H}_0^1(\Omega)]^d$  satisfying

$$\operatorname{div} v_p = p \quad \text{and} \quad \|\nabla v_p\| \leq C_{\operatorname{div}} \|p\|. \quad (48)$$

Setting  $v = (v_p, v_p)$ , we have  $Bv = p$  by construction and  $\|v\|_{[\mathbf{H}_0^1(\Omega)]^d \times [\mathbf{H}_0^1(\Omega)]^d} \leq C \|p\|$  from the above inequality.  $\square$

**Remark 3.2.** *Note that the constant  $\underline{\beta}$  of the above inf-sup condition does not depend on the phase  $\phi$  and is although valid for  $\phi = 0$  (resp.  $\phi = 1$ ) which are the limit cases for which there is no fluid (resp. no structure).*

This inf-sup condition allows us to state the following result, which is a generalization of De Rham Theorem [42, 34, 14]. It is a key ingredient to show the existence of pressure in the incompressible case.

**Theorem 3.3.** *Assume that  $\phi \in \mathbf{H}^{d/2+r}(\Omega)$  with  $r > 0$ . If  $f = (f_s, f_f) \in [\mathbf{H}^{-1}(\Omega)]^d \times [\mathbf{H}^{-1}(\Omega)]^d$  satisfies*

$$\langle f, w \rangle = \langle f_s, w_s \rangle_{[\mathbf{H}^{-1}(\Omega)]^d, [\mathbf{H}_0^1(\Omega)]^d} + \langle f_f, w_f \rangle_{[\mathbf{H}^{-1}(\Omega)]^d, [\mathbf{H}_0^1(\Omega)]^d} = 0, \quad \forall w = (w_s, w_f) \in V_\phi,$$

*then there exists a unique  $p \in L_0^2(\Omega)$  such that  $f_s = -(1-\phi)\nabla p$  and  $f_f = -\phi\nabla p$ .*

*Proof.* The proof follows standard arguments and is based on the Closed Range Theorem. Note that  $V_\phi = \operatorname{Ker} B$ . Let us characterize the adjoint of  $B$ . Since  $\phi \in \mathbf{H}^{d/2+r}(\Omega)$  with  $r > 0$ ,  $\phi$  is a multiplier of  $[\mathbf{H}_0^1(\Omega)]^d$ , namely

$$\forall v \in [\mathbf{H}_0^1(\Omega)]^d, \quad \phi v \in [\mathbf{H}_0^1(\Omega)]^d \quad \text{and} \quad \|\nabla(\phi v)\| \leq C_\phi \|\nabla v\|$$

for some positive constant  $C_\phi$ . Therefore, for all  $p \in L_0^2(\Omega)$ ,  $(1-\phi)\nabla p$  and  $\phi\nabla p$  belong to  $[\mathbf{H}^{-1}(\Omega)]^d$  so that we can define the adjoint operator as

$$B^* : \begin{array}{ccc} L_0^2(\Omega) & \longrightarrow & [\mathbf{H}^{-1}(\Omega)]^d \times [\mathbf{H}^{-1}(\Omega)]^d \\ p & \longmapsto & (-(1-\phi)\nabla p, -\phi\nabla p). \end{array}$$

Thanks to Proposition 3.1, the Closed Range Theorem implies that  $(\operatorname{Ker} B)^\circ = \operatorname{Rg} B^*$ . Consequently, for any  $f \in (\operatorname{Ker} B)^\circ = (V_\phi)^\circ$ , namely for any  $f = (f_s, f_f) \in [\mathbf{H}^{-1}(\Omega)]^d \times [\mathbf{H}^{-1}(\Omega)]^d$  satisfying

$$\langle f, w \rangle = 0, \quad \forall w \in V_\phi,$$

there exists a unique  $p \in L_0^2(\Omega)$  such that  $f_s = -(1-\phi)\nabla p$  and  $f_f = -\phi\nabla p$ .  $\square$

Theorem 3.3 is sufficient in our study. Nevertheless, we note that if  $\phi$  is more regular, it can be sharpened in the following way.

**Proposition 3.4.** *Assume that  $\phi \in C^k(\Omega)$  with  $k \geq 1$  and set*

$$\mathcal{V}_\phi^k = \{(v_s, v_f) \in [C_c^k(\Omega)]^d \times [C_c^k(\Omega)]^d : \operatorname{div}((1 - \phi)v_s + \phi v_f) = 0\}.$$

If  $f = (f_s, f_f) \in [H^{-1}(\Omega)]^d \times [H^{-1}(\Omega)]^d$  satisfies

$$\langle f, w \rangle = 0, \quad \forall w \in \mathcal{V}_\phi^k, \quad (49)$$

then there exists a unique  $p \in L_0^2(\Omega)$  such that  $f_s = -(1 - \phi)\nabla p$  and  $f_f = -\phi\nabla p$ .

*Proof.* Let  $v = (v_s, v_f)$  be an element of  $V_\phi$ . In order to apply Theorem 3.3, we need to build a sequence of elements of  $\mathcal{V}_\phi^k$  that converges towards  $v$  in  $[H_0^1(\Omega)]^d \times [H_0^1(\Omega)]^d$ . To do so, let  $(v_s^n, v_f^n)$  be a sequence of elements of  $[\mathcal{D}(\Omega)]^d \times [\mathcal{D}(\Omega)]^d$  converging towards  $v$  in  $[H_0^1(\Omega)]^d \times [H_0^1(\Omega)]^d$ .

The function  $q^n = \operatorname{div}((1 - \phi)v_s^n + \phi v_f^n)$  belongs to  $C_c^{k-1}(\Omega) \cap L_0^2(\Omega)$  and converges towards 0 in  $L^2(\Omega)$  since  $v \in V_\phi$ . Then, we aim at finding a lifting  $u^n \in [C_c^k(\Omega)]^d$  such that  $\operatorname{div} u^n = q^n$  and  $\|\nabla u^n\| \leq C\|q^n\|$ , so that

$$u^n \longrightarrow 0 \text{ in } [H_0^1(\Omega)]^d$$

and

$$\operatorname{div}((1 - \phi)(v_s^n - u^n) + \phi(v_f^n - u^n)) = 0.$$

For this purpose, we use an explicit construction which was first introduced by Bogovskii [11]. If  $\Omega$  is starlike with respect to some open ball  $\mathcal{B} \subset \Omega$ , let us denote by  $h$  a smoothing function such that

$$h \in \mathcal{D}(\mathbb{R}^d), \quad 0 \leq h \leq 1, \quad \operatorname{supp} h \subset \mathcal{B} \quad \text{and} \quad \int_{\mathbb{R}^d} h \, dx = 1.$$

It is shown in [11] that the function defined by

$$\forall x \in \Omega, \quad u^n(x) = \int_{\Omega} q^n(y) \frac{x - y}{|x - y|^d} \int_{|x - y|}^{\infty} h\left(y + \xi \frac{x - y}{|x - y|}\right) \xi^{d-1} \, d\xi \, dy$$

satisfies  $\operatorname{div} u^n = q^n$  and  $\|\nabla u^n\| \leq C\|q^n\|$  with  $C > 0$  independent of  $n$ . Moreover,  $u^n$  can be written as

$$\forall x \in \Omega, \quad u^n(x) = \int_{\Omega} q^n(y) K(x, y) \, dy, \quad \text{with } K(\cdot, y) \in \mathcal{D}(\Omega),$$

and thus  $u^n \in [C_c^k(\Omega)]^d$  since  $q^n \in C_c^{k-1}(\Omega)$ . If  $\Omega$  is a bounded Lipschitz domain, the previous construction can be extended by writing  $\Omega$  as the union of a finite number of bounded domains, each of which is starlike with respect to an open ball, and using a partition of unity subordinated to this covering.

Hence  $v^n = ((v_s^n - u^n), (v_f^n - u^n)) \in \mathcal{V}_\phi^k$  and converges towards  $v = (v_s, v_f)$  in  $[H_0^1(\Omega)]^d \times [H_0^1(\Omega)]^d$ . Consequently

$$\langle f, v \rangle = \lim_{n \rightarrow \infty} \langle f, v^n \rangle = 0 \quad \text{in virtue of (49).}$$

This equality being true for all  $v \in V_\phi$ , the conclusion follows from Theorem 3.3.  $\square$

As an immediate consequence, we get the following result.

**Corollary 3.5.** *Assume that  $\phi \in C^\infty(\Omega)$  and set*

$$\mathcal{V}_\phi = \{(v_s, v_f) \in [\mathcal{D}(\Omega)]^d \times [\mathcal{D}(\Omega)]^d : \operatorname{div}((1 - \phi)v_s + \phi v_f) = 0\}.$$

If  $f = (f_s, f_f) \in [H^{-1}(\Omega)]^d \times [H^{-1}(\Omega)]^d$  satisfies

$$\langle f, w \rangle = 0, \quad \forall w \in \mathcal{V}_\phi,$$

then there exists a unique  $p \in L_0^2(\Omega)$  such that  $f_s = -(1 - \phi)\nabla p$  and  $f_f = -\phi\nabla p$ .

**Remark 3.6.** *Corollary 3.5 extends the strong version of De Rham Theorem, which originates from differential geometry [29]. Note that it requires the porosity to be very regular. Otherwise, the manifold associated with the space  $\mathcal{V}_\phi$  would not be smooth enough and it is not clear whether the result holds true or not.*

When  $\phi \in C^\infty(\Omega)$ , Corollary 3.5 allows us to characterize  $V_\phi$  and  $H_\phi$  from the space  $\mathcal{V}_\phi$ .

**Proposition 3.7.** *If  $\phi \in C^\infty(\Omega)$ , then the space  $V_\phi$  is the closure of  $\mathcal{V}_\phi$  in  $[\mathbf{H}_0^1(\Omega)]^d \times [\mathbf{H}_0^1(\Omega)]^d$ .*

*Proof.* Let  $\tilde{V}$  denote the closure of  $\mathcal{V}_\phi$  in  $[\mathbf{H}_0^1(\Omega)]^d \times [\mathbf{H}_0^1(\Omega)]^d$ . By definition  $\mathcal{V}_\phi \subset V_\phi$  and  $V_\phi$  is closed in  $[\mathbf{H}_0^1(\Omega)]^d \times [\mathbf{H}_0^1(\Omega)]^d$ , so  $\tilde{V} \subset V_\phi$ .

To prove that  $V_\phi \subset \tilde{V}$ , we show that any linear form on  $V_\phi$  which vanishes on  $\mathcal{V}_\phi$  also vanishes on  $V_\phi$ . If  $f = (f_s, f_f) \in ([\mathbf{H}^{-1}(\Omega)]^d \times [\mathbf{H}^{-1}(\Omega)]^d) \cap \mathcal{V}_\phi^\perp$ , then Corollary 3.5 ensures the existence of a pressure  $p \in L_0^2(\Omega)$  such that  $f_s = -(1 - \phi) \nabla p$  and  $f_f = -\phi \nabla p$ . Hence, we can write

$$\langle f, w \rangle = -\langle (1 - \phi) \nabla p, w_s \rangle_{[\mathbf{H}^{-1}(\Omega)]^d, [\mathbf{H}_0^1(\Omega)]^d} - \langle \phi \nabla p, w_f \rangle_{[\mathbf{H}^{-1}(\Omega)]^d, [\mathbf{H}_0^1(\Omega)]^d} = \int_\Omega p \operatorname{div}((1 - \phi) w_s + \phi w_f) \, dx = 0$$

for all  $w = (w_s, w_f) \in V_\phi$ . □

**Proposition 3.8.** *If  $\phi \in C^\infty(\Omega)$ , then the space  $H_\phi$  is the closure of  $\mathcal{V}$  in  $[\mathbf{L}^2(\Omega)]^d \times [\mathbf{L}^2(\Omega)]^d$ .*

*Proof.* We denote by  $\tilde{H}$  the closure of  $\mathcal{V}_\phi$  in  $[\mathbf{L}^2(\Omega)]^d \times [\mathbf{L}^2(\Omega)]^d$ . The continuity of the normal trace operator implies that  $\tilde{H} \subset H_\phi$ .

Now, let  $H_\phi^*$  be the orthogonal complement of  $\tilde{H}$  into  $H_\phi$  and let  $f = (f_s, f_f)$  be an element of  $H_\phi^*$ . Noting that  $H_\phi^* \subset \tilde{H}^\perp$  and  $\mathcal{V}_\phi \subset \tilde{H}$ , it follows from Corollary 3.5 that there exists a pressure  $p \in L_0^2(\Omega)$  such that  $f_s = -(1 - \phi) \nabla p$  and  $f_f = -\phi \nabla p$ . Moreover since  $\nabla p = -(f_s + f_f)$  and  $(f_s, f_f) \in [\mathbf{L}^2(\Omega)]^d \times [\mathbf{L}^2(\Omega)]^d$ , we have  $p \in \mathbf{H}^1(\Omega)$ . Since  $f$  belongs to  $H_\phi$ , we have  $\operatorname{div}(((1 - \phi)^2 + \phi^2) \nabla p) = \operatorname{div}((1 - \phi)^2 \nabla p + \phi^2 \nabla p) = 0$  in  $\mathcal{D}'(\Omega)$  and  $((1 - \phi)^2 \nabla p + \phi^2 \nabla p) \cdot n = 0$ . Thus  $p$  is equal to zero (up to a constant) as the unique solution of an elliptic Neumann problem, so  $\nabla p = 0$  and  $f = 0$ . In conclusion,  $H_\phi^* = \{0\}$ , which proves that  $\tilde{H} = H_\phi$ . □

**Remark 3.9.** *If  $\phi \notin C^\infty(\Omega)$ , we can not apply Corollary 3.5 and it is not clear whether  $V_\phi$  and  $H_\phi$  can be identified as the closures of  $\mathcal{V}$  in  $[\mathbf{H}_0^1(\Omega)]^d \times [\mathbf{H}_0^1(\Omega)]^d$  and  $[\mathbf{L}^2(\Omega)]^d \times [\mathbf{L}^2(\Omega)]^d$ . Still, the space  $V_\phi$  remains dense in  $H_\phi$ . Indeed, if  $v = (v_s, v_f)$  is an element of  $H_\phi$ , it can be approximated by  $(v_s^n - u^n, v_f^n - u^n) \in V_\phi$ , where  $(v_s^n, v_f^n)$  is a sequence of elements of  $[\mathcal{D}(\Omega)]^d \times [\mathcal{D}(\Omega)]^d$  converging towards  $v$  in  $[\mathbf{L}^2(\Omega)]^d \times [\mathbf{L}^2(\Omega)]^d$  and  $u^n \in [\mathbf{H}_0^1(\Omega)]^d$  is a continuous lifting such that  $\operatorname{div} u^n = \operatorname{div}((1 - \phi) v_s^n + \phi v_f^n)$ .*

We are now going to combine this functional framework adapted to the constraint (45) with the semigroup approach in order to study Problem (44). Here again, we investigate the cases  $\eta > 0$  and  $\eta = 0$  separately. To simplify the proof we consider that  $\theta$ , which appears now only in the term  $-\theta v_f$  in the fluid equation, does not depend on time:  $\theta \in L^\infty(\Omega)$ . It simplifies the proof, but it could be easily modified to include the time-dependent case as in the proofs of Section 2.

### 3.2 The case $\eta > 0$

We formulate the problem in the functional framework established previously. We seek for a solution  $z = (u_s, v_s, v_f)$  in the energy space  $H = [\mathbf{H}_0^1(\Omega)]^d \times H_\phi$  endowed with the norm

$$\|z\|_H^2 = \|u_s\|_s^2 + \int_\Omega \rho_s (1 - \phi) |v_s|^2 \, dx + \int_\Omega \rho_f \phi |v_f|^2 \, dx.$$

Setting  $V = [\mathbf{H}_0^1(\Omega)]^d \times V_\phi$ , we consider the bilinear form

$$\begin{aligned} a_\eta^\infty(z, y) = & - \int_\Omega \sigma_s(v_s) : \varepsilon(d_s) \, dx + \int_\Omega \sigma_s(u_s) : \varepsilon(w_s) \, dx + 2\eta \int_\Omega \varepsilon(v_s) : \varepsilon(w_s) \, dx \\ & + \int_\Omega \phi^2 k_f^{-1}(v_f - v_s) \cdot (w_f - w_s) \, dx + \int_\Omega \phi \sigma_f(v_f) : \varepsilon(w_f) \, dx - \int_\Omega \theta v_f \cdot w_f \, dx \end{aligned} \quad (50)$$

defined for all  $z = (u_s, v_s, v_f)$  and  $y = (d_s, w_s, w_f)$  in  $V$ . This bilinear form is the same as (17) but without the terms involving the pressure because of the test functions in  $V$ . Note that here, since we have assumed that  $\theta$  does not depend on time, we can include the term  $\int_\Omega \theta v_f \cdot w_f$  in the definition of the bilinear form associated with our coupled problem. When  $\theta$  depends on time, one can not and we have to introduce the operator  $G(t)$  which is a bounded perturbation, see (20). As in (18) and (19), we define the unbounded operator  $(A_\eta^\infty, D(A_\eta^\infty))$  associated with the bilinear form (50) by

$$(A_\eta^\infty z, y)_H = a_\eta^\infty(z, y), \quad \forall z \in D(A_\eta^\infty), \forall y \in V$$

in the domain

$$D(A_\eta^\infty) = \{z \in V : \exists g \in H, a_\eta^\infty(z, y) = (g, y)_H, \quad y \in V\}.$$

Lastly, we set  $g = (0, f, f)$  and we denote by  $\Pi$  the Leray projection operator from  $[\mathbf{H}_0^1(\Omega)]^d \times [\mathbf{L}^2(\Omega)]^d \times [\mathbf{L}^2(\Omega)]^d$  into  $H$ . We are now ready to state the following existence and uniqueness result.

**Theorem 3.10.** *Assume that (h1) – (h3) hold true and that  $\eta > 0$ .*

(i) *If  $z_0 \in D(A_\eta^\infty)$  and  $f \in \mathbf{H}^1(0, T; [\mathbf{L}^2(\Omega)]^d)$  so that  $\Pi g \in \mathbf{H}^1(0, T; H)$ , then there exists a unique strong solution  $z \in \mathbf{C}^1([0, T]; H) \cap \mathbf{C}^0([0, T]; D(A_\eta^\infty))$  satisfying*

$$\begin{cases} \dot{z}(t) + A_\eta^\infty z(t) = \Pi g(t), & t \in [0, T], \\ z(0) = z_0. \end{cases} \quad (51)$$

(ii) *If  $z_0 \in H$  and  $f \in \mathbf{L}^2(0, T; [\mathbf{L}^2(\Omega)]^d)$  so that  $\Pi g \in \mathbf{L}^2(0, T; H)$ , then Problem (51) has a unique mild solution  $z \in \mathbf{C}^0([0, T]; H)$  such that  $z(0) = z_0$  and*

$$\begin{aligned} & \int_0^T z(t) \psi(t) \, dt \in D(A_\eta^\infty), \\ & - \int_0^T z(t) \dot{\psi}(t) \, dt + A_\eta^\infty \left( \int_0^T z(t) \psi(t) \, dt \right) = \int_0^T \Pi g(t) \psi(t) \, dt, \end{aligned}$$

for all  $\psi \in \mathbf{C}_c^1([0, T]; \mathbb{R})$ . Moreover,  $z$  is given by the Duhamel formula

$$z(t) = \Phi_\eta^\infty(t) z_0 + \int_0^t \Phi_\eta^\infty(t-s) \Pi g(s) \, ds, \quad (52)$$

where  $\Phi_\eta^\infty$  denotes the continuous semigroup generated by  $A_\eta^\infty$  in the sense that

$$A_\eta^\infty x = - \frac{d}{dt} (\Phi_\eta^\infty(t)x) \Big|_{t=0^+}, \quad x \in H. \quad (53)$$

*Proof.* The proof of this result is almost similar to the proof of Theorem 2.3, replacing  $Z$  by  $H$  and  $Y$  by  $V$ . The only difference is that the term  $-\theta v_f$  is treated within the operator  $A_\eta^\infty$  instead of being considered as a perturbation.

For any  $z = (u_s, v_s, v_f) \in D(A_\eta^\infty)$ , we observe that

$$a_\eta^\infty(z, z) = 2\eta \int_\Omega |\varepsilon(v_s)|^2 \, dx + \int_\Omega \phi^2 k_f^{-1}(v_f - v_s) \cdot (v_f - v_s) \, dx + \int_\Omega \phi \sigma_f(v_f) : \varepsilon(v_f) \, dx - \int_\Omega \theta |v_f|^2 \, dx.$$



Thus  $(A_\eta^\infty z, z)_H \geq -\omega \|z\|_H^2$  with  $\omega = \|\theta\|_{L^\infty(\Omega)}(\rho_f \phi_{\min})^{-1}$ .

Moreover, for all  $\lambda_0 > \omega$ , the operator  $A_\eta^\infty + \lambda_0 I$  is surjective from  $D(A_\eta^\infty)$  to  $H$  because

$$a_\eta^\infty(z, z) + \lambda_0(z, z)_H \geq \lambda_0 \|u_s\|_s^2 + 2\eta \|\varepsilon(v_s)\|^2 + 2\mu_f \phi_{\min} \|\varepsilon(v_f)\|^2 + (\lambda_0 \rho_f \phi_{\min} - \|\theta\|_{L^\infty(\Omega)}) \|v_f\|^2.$$

From Lumer-Phillips theorem, we deduce that  $A_\eta^\infty$  is the generator – in the sense of (53) – of a strongly continuous semigroup and the conclusion follows from [6, Part II, Chapter 1, Propositions 3.1–3.3] and [16, Corollary 2.25].  $\square$

**Remark 3.11.** *By reproducing the proof of Theorem 2.3, we can extend the result of Theorem 3.10 for a time-dependent  $\theta$  satisfying the compatibility condition  $\int_\Omega \theta(t) dx = 0$  for all  $t \in [0, T]$ . The existence of a mild solution then requires that  $\theta \in C^0([0, T] \times \Omega)$  and the existence of a strong solution is guaranteed under the assumption  $\theta \in C^1([0, T]; L^\infty(\Omega))$ . However, because of the lifting (46), more regularity on  $\theta$  is needed for the original problem to be well-posed. More precisely, when performing the change of variables  $(u_s, v_f, p) \mapsto (u_s - \int_0^t v_\theta(s) ds, v_f - v_\theta, p)$ , the right-hand sides of (44b) and (44c) become respectively*

$$\rho_s(1-\phi)f - \rho_s(1-\phi)\partial_t v_\theta + \operatorname{div} \left( \sigma_s \left( \int_0^t v_\theta ds \right) \right) + \operatorname{div} (\sigma_s^{\text{vis}}(v_\theta)) \quad \text{and} \quad \rho_f \phi f - \rho_f \phi \partial_t v_\theta + \operatorname{div} (\phi \sigma_f(v_\theta)) + \theta v_\theta.$$

The existence of a mild solution requires that all these terms belong to  $L^2(0, T; [L^2(\Omega)]^d)$  and thus that the lifting  $v_\theta \in H^1(0, T; [L^2(\Omega)]^d) \cap L^2(0, T; [H^2(\Omega)]^d)$ , which is ensured if  $\theta \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ . Similarly, the existence of a strong solution is guaranteed under the assumption  $\theta \in H^2(0, T; L^2(\Omega)) \cap H^1(0, T; H^1(\Omega))$ .

The previous theorem only involves the displacement and velocity fields. The existence of pressure and the relation between (51) and the original system (44) are precised below.

**Theorem 3.12.** *Assume that  $z_0 \in D(A_\eta^\infty)$ ,  $f \in H^1(0, T; [L^2(\Omega)]^d)$  and let  $z = (u_s, v_s, v_f) \in C^1([0, T]; H) \cap C^0([0, T]; D(A_\eta^\infty))$  be the strong solution of (51). There exists a unique pressure  $p \in C^0([0, T]; L_0^2(\Omega))$  such that  $(z, p)$  satisfies (44b) and (44c).*

*Proof.* Let  $z = (u_s, v_s, v_f)$  be the solution of (51). Since  $z \in C^0([0, T]; D(A_\eta^\infty))$  and  $D(A_\eta^\infty) \subset V$ , the constraint (45) is fulfilled in  $C^0([0, T]; L^2(\Omega))$ .

Taking the scalar product of (51) with  $\varphi \in V$ , we see that

$$\left( \dot{z}(t) + A_\eta^\infty z(t) - \Pi g(t), \varphi \right)_H = 0, \quad \forall \varphi = (d_s, w_s, w_f) \in V, \quad \forall t \in [0, T]. \quad (54)$$

Thus  $(\partial_t u_s - v_s, d_s)_s = 0$  for any  $d_s \in [H_0^1(\Omega)]^d$  and (44a) is satisfied in  $C^0([0, T]; [H_0^1(\Omega)]^d)$ .

The equality (54) being true for all  $(w_s, w_f) \in V_\phi$ , Theorem 3.3 ensures, for almost every  $t \in (0, T)$ , the existence of a pressure  $p(t) \in L_0^2(\Omega)$  such that

$$\begin{aligned} \rho_s(1-\phi)\partial_t v_s - \operatorname{div}(\sigma_s^{\text{vis}}(v_s)) - \operatorname{div}(\sigma_s(u_s)) - \phi^2 k_f^{-1}(v_f - v_s) - \rho_s(1-\phi)f &= -(1-\phi)\nabla p, \quad \text{in } [H^{-1}(\Omega)]^d, \\ \rho_f \phi \partial_t v_f - \operatorname{div}(\phi \sigma_f(v_f)) + \phi^2 k_f^{-1}(v_f - v_s) - \rho_f \phi f &= -\phi \nabla p, \quad \text{in } [H^{-1}(\Omega)]^d. \end{aligned}$$

Summing these two equalities to remove  $\phi$ , we obtain that  $\nabla p \in C^0([0, T]; H^{-1}(\Omega))$ . By an application of Nečas Lemma, we conclude that  $p \in C^0([0, T]; L_0^2(\Omega))$ , which shows that (44b) and (44c) are verified in  $C^0([0, T]; [L^2(\Omega)]^d)$ .  $\square$

If the input data are less regular, we get the existence of displacement, velocities and pressure in the following weak sense.

**Theorem 3.13.** *Assume that (h1)–(h4) are satisfied,  $\eta > 0$  and  $z_0 = (u_{s0}, v_{s0}, v_{f0}) \in H$ . Then there exists a unique variational solution  $u_s \in C^0([0, T]; [\mathbf{H}_0^1(\Omega)]^d)$ ,  $\partial_t u_s \in C^0([0, T]; [\mathbf{L}^2(\Omega)]^d)$  and  $v_f \in C^0([0, T]; [\mathbf{L}^2(\Omega)]^d)$  with  $(\partial_t u_s, v_f) \in \mathbf{L}^2(0, T; V_\phi)$  such that*

$$(u_s(0), \partial_t u_s(0), v_f(0)) = (u_{s0}, v_{s0}, v_{f0})$$

and, for all  $(w_s, w_f) \in V_\phi$ , the following equation holds in  $\mathcal{D}'(0, T)$ :

$$\begin{aligned} & \frac{d^2}{dt^2} \int_{\Omega} \rho_s (1 - \phi) u_s(t) \cdot w_s \, dx + \int_{\Omega} \sigma_s(u_s(t)) : \varepsilon(w_s) \, dx + 2\eta \int_{\Omega} \varepsilon(\partial_t u_s(t)) : \varepsilon(w_s) \, dx \\ & + \frac{d}{dt} \int_{\Omega} \rho_f \phi v_f(t) \cdot w_f \, dx + \int_{\Omega} \phi \sigma_f(v_f(t)) : \varepsilon(w_f) \, dx + \int_{\Omega} \phi^2 k_f^{-1} (v_f(t) - \partial_t u_s(t)) \cdot (w_f - w_s) \, dx \\ & - \int_{\Omega} \theta v_f(t) \cdot w_f \, dx = \int_{\Omega} \rho_s (1 - \phi) f(t) \cdot w_s \, dx + \int_{\Omega} \rho_f \phi f(t) \cdot w_f \, dx. \end{aligned} \quad (55)$$

The energy estimate (13) holds true and the variational solution coincides with the mild solution given by (52). Furthermore, there exists a unique pressure  $p$  such that  $(u_s, v_s, v_f, p)$  satisfies (44) in the distribution sense, with  $v_s = \partial_t u_s$ .

**Remark 3.14.** *The incompressibility constraint is contained in the fact that  $(\partial_t u_s, v_f)$  belongs to  $\mathbf{L}^2(0, T; V_\phi)$ . It can be written in variational form as*

$$\forall q \in \mathbf{L}^2(\Omega), \quad \int_{\Omega} \operatorname{div}((1 - \phi) \partial_t u_s(t) + \phi v_f(t)) q \, dx = 0. \quad (56)$$

*Proof.* The proof of existence of  $(u_s, v_f)$  follows exactly the same lines as in the compressible case. We build a sequence of strong solutions for smooth data. These solutions  $(u_s^n, v_s^n, v_f^n)_n$  satisfy the energy estimate (13) and constitute a Cauchy sequence in  $C^0([0, T]; H)$ . Moreover  $(v_s^n, v_f^n)$  is a Cauchy sequence in  $\mathbf{L}^2(0, T; V_\phi)$  and we can pass to the limit as we did in the proof of Theorem 2.5. We get the first order system:

$$\left\{ \begin{array}{l} \forall (d_s, w_s, w_f) \in V = [\mathbf{H}_0^1(\Omega)]^d \times V_\phi, \\ \frac{d}{dt} \int_{\Omega} \sigma_s(u_s(s)) : \varepsilon(d_s) \, dx - \int_{\Omega} \sigma_s(v_s(s)) : \varepsilon(d_s) \, dx = 0, \\ \frac{d}{dt} \int_{\Omega} \rho_s (1 - \phi) v_s(s) \cdot w_s \, dx + \int_{\Omega} \sigma_s(u_s(s)) : \varepsilon(w_s) \, dx + 2\eta \int_{\Omega} \varepsilon(v_s(s)) : \varepsilon(w_s) \, dx \\ + \frac{d}{dt} \int_{\Omega} \rho_f \phi v_f(s) \cdot w_f \, dx + \int_{\Omega} \phi \sigma_f(v_f(s)) : \varepsilon(w_f) \, dx + \int_{\Omega} \phi^2 k_f^{-1} (v_f(s) - v_s(s)) \cdot (w_f - w_s) \, dx \\ - \int_{\Omega} \theta v_f(s) \cdot w_f \, dx = \int_{\Omega} \rho_s (1 - \phi) f(s) \cdot w_s \, dx + \int_{\Omega} \rho_f \phi f(s) \cdot w_f \, dx, \end{array} \right. \quad (57a)$$

which can be rewritten in second order to obtain (55).

Like in the compressible regime, the previous system provides some regularity on the time derivatives of the solution, namely

$$(\partial_t u_s, v_f) \in \mathbf{L}^2(0, T; V_\phi) \quad \text{and} \quad (\partial_{tt} u_s, \partial_t v_f) \in \mathbf{L}^2(0, T; V'_\phi).$$

These regularities, together with the density of  $V_\phi$  in  $H_\phi$ , yield

$$\begin{aligned} & \frac{d^2}{dt^2} \int_{\Omega} \rho_s (1 - \phi) u_s(t) \cdot w_s \, dx + \frac{d}{dt} \int_{\Omega} \rho_f \phi v_f(t) \cdot w_f \, dx \\ & = \left\langle \rho_s (1 - \phi) \partial_{tt} u_s(t), w_s \right\rangle_{[\mathbf{H}^{-1}(\Omega)]^d, [\mathbf{H}_0^1(\Omega)]^d} + \left\langle \rho_f \phi \partial_t v_f(t), w_f \right\rangle_{[\mathbf{H}^{-1}(\Omega)]^d, [\mathbf{H}_0^1(\Omega)]^d}. \end{aligned}$$

The energy estimate (13) follows by choosing  $(w_s, w_f) = (\partial_t u_s(t), v_f(t))$  in (55) and guarantees uniqueness.

The equivalence between the variational and mild solutions can be proved in the same way as in the compressible case (see Proposition 2.8). We only need to notice that  $V$  is dense in  $H$  (see Remark 3.9).

To show the existence of pressure, we integrate (57b) in time over  $(0, t)$  (see for instance [42] for a similar argument for the Stokes system). Let us define

$$U_s(t) = \int_0^t u_s(s) ds, \quad V_s(t) = \int_0^t v_s(s) ds, \quad V_f(t) = \int_0^t v_f(s) ds \quad \text{and} \quad F(t) = \int_0^t f(s) ds,$$

it follows that

$$\begin{aligned} & \left\langle \rho_s(1 - \phi)(v_s(t) - v_{s0}) - \operatorname{div}(\sigma_s^{\text{vis}}(V_s(t))) - \operatorname{div}(\sigma_s(U_s(t))) - \phi^2 k_f^{-1}(V_f(t) - V_s(t)) - \rho_s(1 - \phi)F(t), w_s \right\rangle \\ & + \left\langle \rho_f \phi(v_f(t) - v_{f0}) - \operatorname{div}(\phi \sigma_f(V_f(t))) + \phi^2 k_f^{-1}(V_f(t) - V_s(t)) - \rho_f \phi F(t), w_f \right\rangle = 0 \end{aligned}$$

for all  $(w_s, w_f) \in V_\phi$ . Combining Theorem 3.3 and Nečas Lemma furnishes the existence of  $P \in C^0([0, T]; L_0^2(\Omega))$  such that

$$\begin{aligned} \rho_s(1 - \phi)(v_s - v_{s0}) - \operatorname{div}(\sigma_s^{\text{vis}}(V_s)) - \operatorname{div}(\sigma_s(U_s)) - \phi^2 k_f^{-1}(V_f - V_s) - \rho_s(1 - \phi)F &= -(1 - \phi)\nabla P, \\ \rho_f \phi(v_f - v_{f0}) - \operatorname{div}(\phi \sigma_f(V_f)) + \phi^2 k_f^{-1}(V_f - V_s) - \rho_f \phi F &= -\phi \nabla P. \end{aligned}$$

As a consequence,  $p = \partial_t P$  satisfies (44) in the distribution sense.  $\square$

**Remark 3.15.** We could also define the variational solution without assuming time continuity. Time continuity would then follow from the continuous injection of the space

$$W_\phi(0, T) = \left\{ (v_s, v_f) \in L^2(0, T; V_\phi) \text{ such that } (\partial_t v_s, \partial_t v_f) \in L^2(0, T; V'_\phi) \right\}$$

into  $C^0([0, T]; H_\phi)$ .

### 3.3 The case $\eta = 0$

This case combines the two difficulties encountered earlier: the incompressibility constraint and the absence of solid viscosity. To handle this case, an option is to combine the functional framework adapted to the incompressibility constraint with the  $T$ -coercivity approach used in Section 2.3. This method provides the following result.

**Theorem 3.16.** *If  $\eta = 0$ , then the conclusions of Theorem 3.10 and Theorem 3.12 remain true.*

*Proof.* To prove that the operator  $A_0^\infty + \lambda_0 I$  is surjective from  $D(A_0^\infty)$  to  $H$  for all  $\lambda_0 > \omega$ , we show that  $a_0^\infty(\cdot, \cdot) + \lambda_0(\cdot, \cdot)_H$  is  $T$ -coercive for the mapping  $T : (u_s, v_s, v_f) \mapsto (\frac{1}{2}u_s - \frac{1}{2\lambda_0}v_s, v_s, v_f)$  defined by (39). This mapping is bijective from  $V$  into itself because it does not affect the velocity components and thus the mixture's divergence constraint. The rest of the proof is the same as in Theorem 3.10 and Theorem 3.12.  $\square$

Yet, we present here another approach to prove that  $A_0^\infty + \lambda_0 I$  is surjective. This proof is based on a mixed formulation of the problem and is more suitable for numerical approximation. Indeed, the formulation (50) involves a constrained space  $V_\phi$  that we would like to relax for numerical purpose, most numerical strategies relying on mixed formulations. Note moreover that the space  $V_\phi$  depends on the phase field and thus on a specific data set. The mixed problem we would like to solve writes

$$\begin{cases} \text{Find } z \in Y & \text{such that} \\ \forall y \in Y, & a_{\lambda_0}(z, y) = (g, y)_Z, \end{cases} \quad (58)$$

with

$$\begin{aligned}
a_{\lambda_0}(z, y) = & \lambda_0 \int_{\Omega} \sigma_s(u_s) : \varepsilon(d_s) \, dx - \int_{\Omega} \sigma_s(v_s) : \varepsilon(d_s) \, dx + \lambda_0 \int_{\Omega} \rho_s(1 - \phi) v_s \cdot w_s \, dx + \int_{\Omega} \sigma_s(u_s) : \varepsilon(w_s) \, dx \\
& + \int_{\Omega} \phi^2 k_f^{-1} (v_f - v_s) \cdot (w_f - w_s) \, dx + \lambda_0 \int_{\Omega} \rho_f \phi v_f \cdot w_f \, dx + \int_{\Omega} \phi \sigma_f(v_f) : \varepsilon(w_f) \, dx - \int_{\Omega} \theta v_f \cdot w_f \, dx \\
& - \int_{\Omega} p \operatorname{div} ((1 - \phi) w_s + \phi w_f) \, dx + \int_{\Omega} \operatorname{div} ((1 - \phi) v_s + \phi v_f) q \, dx
\end{aligned}$$

for any  $z = (u_s, v_s, v_f, p)$  and  $y = (d_s, w_s, w_f, q)$  in  $Y$ , where we recall that  $Z = [\mathbf{H}_0^1(\Omega)]^d \times [\mathbf{L}^2(\Omega)]^d \times [\mathbf{L}^2(\Omega)]^d \times \mathbf{L}^2(\Omega)$  and  $Y = [\mathbf{H}_0^1(\Omega)]^d \times [\mathbf{H}_0^1(\Omega)]^d \times [\mathbf{H}_0^1(\Omega)]^d \times \mathbf{L}^2(\Omega)$ .

**Proposition 3.17.** *Let  $g \in Z$ . If  $\lambda_0 > \|\theta\|_{\mathbf{L}^\infty(\Omega)} (\rho_f \phi_{\min})^{-1}$ , then Problem (58) is well-posed in  $Y$ .*

*Proof.* According to Proposition 2.10, it is sufficient to find  $y^*$  depending continuously on  $z$  such that the inequality  $a_{\lambda_0}(z, y^*) \geq \underline{\alpha} \|z\|_Y^2$  is satisfied for any  $z \in Z$ , with  $\underline{\alpha} > 0$ .

From the properties of the divergence operator, we know, as already stated in (48), that for any  $p \in \mathbf{L}^2(\Omega)$ , there exists  $v_p \in [\mathbf{H}_0^1(\Omega)]^d$  and  $C_{\operatorname{div}} > 0$  such that

$$\operatorname{div} v_p = p \quad \text{and} \quad \|\nabla v_p\|^2 \leq C_{\operatorname{div}} \|p\|^2.$$

For some constants  $\alpha$ ,  $\beta$  and  $\gamma$  to be adjusted, we choose  $d_s^* = \beta u_s + \gamma v_s$ ,  $w_s^* = \alpha v_s - v_p$ ,  $w_f^* = \alpha v_f - v_p$  and  $q^* = \alpha p$ . Thus

$$\begin{aligned}
a_{\lambda_0}(z, y^*) = & \lambda_0 \int_{\Omega} \beta \sigma_s(u_s) : \varepsilon(u_s) \, dx + \lambda_0 \int_{\Omega} \gamma \sigma_s(u_s) : \varepsilon(v_s) \, dx - \int_{\Omega} \beta \sigma_s(v_s) : \varepsilon(u_s) \, dx - \int_{\Omega} \gamma \sigma_s(v_s) : \varepsilon(v_s) \, dx \\
& + \lambda_0 \int_{\Omega} \rho_s(1 - \phi) (\alpha |v_s|^2 - v_s \cdot v_p) \, dx + \int_{\Omega} \alpha \sigma_s(u_s) : \varepsilon(v_s) \, dx - \int_{\Omega} \sigma_s(u_s) : \varepsilon(v_p) \, dx \\
& + \int_{\Omega} \alpha \phi^2 k_f^{-1} (v_f - v_s) \cdot (v_f - v_s) \, dx + \int_{\Omega} (\lambda_0 \rho_f \phi - \theta) (\alpha |v_f|^2 - v_f \cdot v_p) \, dx \\
& + \int_{\Omega} \phi (\alpha \sigma_f(v_f) : \varepsilon(v_f) - \sigma_f(v_f) : \varepsilon(v_p)) \, dx + \int_{\Omega} p \operatorname{div} ((1 - \phi) v_p + \phi v_p) \, dx.
\end{aligned}$$

Note that the term  $\int_{\Omega} p \operatorname{div} ((1 - \phi) v_p + \phi v_p) \, dx$  is equal to  $\|p\|^2$  thanks to the choice of  $v_p$ . As in the proof of Theorem 2.11, we set  $\beta = \frac{\alpha}{2}$  and  $\gamma = -\frac{\alpha}{2\lambda_0}$  in order to remove the terms in the form  $\int_{\Omega} \sigma_s(u_s) : \varepsilon(v_s) \, dx$ . Consequently, we have

$$\begin{aligned}
a_{\lambda_0}(z, y^*) \geq & \frac{\lambda_0 \alpha}{2} \int_{\Omega} \sigma_s(u_s) : \varepsilon(u_s) \, dx - \int_{\Omega} \sigma_s(u_s) : \varepsilon(v_p) \, dx + \frac{\alpha}{2\lambda_0} \int_{\Omega} \sigma_s(v_s) : \varepsilon(v_s) \, dx \\
& + \lambda_0 \rho_s(1 - \phi_{\max}) \int_{\Omega} (\alpha |v_s|^2 - v_s \cdot v_p) \, dx + (\lambda_0 \rho_f \phi_{\min} - \|\theta\|_{\mathbf{L}^\infty(\Omega)}) \int_{\Omega} (\alpha |v_f|^2 - v_f \cdot v_p) \, dx \\
& + \phi_{\min} \int_{\Omega} (\alpha \sigma_f(v_f) : \varepsilon(v_f) - \sigma_f(v_f) : \varepsilon(v_p)) \, dx + \int_{\Omega} p^2 \, dx. \tag{59}
\end{aligned}$$

We choose  $\lambda_0$  such that  $\lambda_0 \rho_f \phi_{\min} - \|\theta\|_{\mathbf{L}^\infty(\Omega)} > 0$ . Next, for all  $\delta > 0$ , Young inequality yields

$$\begin{aligned}
- \int_{\Omega} \sigma_s(u_s) : \varepsilon(v_p) \, dx & \geq -\frac{\delta}{2} \int_{\Omega} \sigma_s(u_s) : \varepsilon(u_s) \, dx - \frac{1}{2\delta} \int_{\Omega} \sigma_s(v_p) : \varepsilon(v_p) \, dx, \\
- \int_{\Omega} \sigma_f(v_f) : \varepsilon(v_p) \, dx & \geq -\frac{\delta}{2} \int_{\Omega} \sigma_f(v_f) : \varepsilon(v_f) \, dx - \frac{1}{2\delta} \int_{\Omega} \sigma_f(v_p) : \varepsilon(v_p) \, dx, \\
- \int_{\Omega} v_s \cdot v_p \, dx & \geq -\frac{\delta}{2} \int_{\Omega} |v_s|^2 \, dx - \frac{1}{2\delta} \int_{\Omega} |v_p|^2 \, dx, \\
- \int_{\Omega} v_f \cdot v_p \, dx & \geq -\frac{\delta}{2} \int_{\Omega} |v_f|^2 \, dx - \frac{1}{2\delta} \int_{\Omega} |v_p|^2 \, dx. \tag{60}
\end{aligned}$$

Furthermore, it holds

$$\begin{aligned} \|v_p\|^2 &\leq C_p \|\nabla v_p\|^2 \leq C_p C_{\text{div}} \|p\|^2, \\ \int_{\Omega} \sigma_f(v_p) : \varepsilon(v_p) \, dx &= \lambda_f \|\text{div } v_p\|^2 + 2\mu_f \|\varepsilon(v_p)\|^2 \leq (\lambda_f + 2\mu_f C_{\text{div}}) \|p\|^2, \\ \int_{\Omega} \sigma_s(v_p) : \varepsilon(v_p) \, dx &= \lambda \|\text{div } v_p\|^2 + 2\mu \|\varepsilon(v_p)\|^2 \leq (\lambda + 2\mu C_{\text{div}}) \|p\|^2, \end{aligned} \quad (61)$$

where  $C_p$  denotes the constant of Poincaré inequality.

Using (60) and (61) to bound from below the right-hand side of (59) and rearranging terms, we obtain

$$\begin{aligned} a_{\lambda_0}(z, y^*) &\geq \left( \frac{\lambda_0 \alpha}{2} - \frac{\delta}{2} \right) \|u_s\|_s^2 + \frac{\alpha}{2\lambda_0} \|v_s\|_s^2 + \lambda_0 \rho_s (1 - \phi_{\max}) \left( \alpha - \frac{\delta}{2} \right) \|v_s\|^2 \\ &\quad + (\lambda_0 \rho_f \phi_{\min} - \|\theta\|_{L^\infty(\Omega)}) \left( \alpha - \frac{\delta}{2} \right) \|v_f\|^2 + 2\mu_f \phi_{\min} \left( \alpha - \frac{\delta}{2} \right) \|\varepsilon(v_f)\|^2 + \left( 1 - \frac{\delta^*}{2\delta} \right) \|p\|^2, \end{aligned}$$

where  $\delta^* = \lambda + 2\mu C_{\text{div}} + \lambda_0 \rho_s (1 - \phi_{\max}) C_p C_{\text{div}} + (\lambda_0 \rho_f \phi_{\min} - \|\theta\|_{L^\infty(\Omega)}) C_p C_{\text{div}} + \phi_{\min} (\lambda_f + 2\mu_f C_{\text{div}})$ ,  $\delta^* > 0$ .

Hence, setting  $\delta = \delta^*$  and  $\alpha = \alpha^* = \max(\delta^*, \frac{2\delta^*}{\lambda_0})$ , we get

$$a_{\lambda_0}(z, y^*) \geq \frac{\delta^*}{2} \|u_s\|_s^2 + \frac{\alpha^*}{2\lambda_0} \|v_s\|_s^2 + \mu_f \phi_{\min} \delta^* \|\varepsilon(v_f)\|^2 + \frac{1}{2} \|p\|^2.$$

Finally, we infer that  $a_{\lambda_0}$  is  $T$ -coercive for the mapping

$$T : (u_s, v_s, v_f, p) \longmapsto \left( \frac{\alpha^*}{2} u_s - \frac{\alpha^*}{2\lambda_0} v_s, \alpha^* v_s - v_p, \alpha^* v_f - v_p, \alpha^* p \right), \quad (62)$$

which is bijective since  $p \longmapsto v_p$  is a bijection.  $\square$

**Remark 3.18.** *This mixed formulation is also applicable to the case  $\kappa = +\infty$  and  $\eta > 0$ . In that case, the proof can be simplified by considering the mapping  $T : (u_s, v_s, v_f, p) \longmapsto (u_s, \alpha^* v_s - v_p, \alpha^* v_f - v_p, \alpha^* p)$ .*

**Remark 3.19.** *The mixed formulation is equivalent to the constrained formulation thanks to the inf-sup property proved in Proposition 3.1. Note moreover that, as for the proof of Proposition 3.1, the  $T$ -coercivity only relies on the standard inf-sup condition for the divergence operator and therefore is independent of the phase field  $\phi$ .*

Finally, as for the compressible inviscid case, we can prove the existence of a variational solution.

**Theorem 3.20.** *Assume that (h1)–(h4) are satisfied,  $\eta = 0$  and  $z_0 = (u_{s0}, v_{s0}, v_{f0}) \in H$ . Then there exists a unique variational solution  $u_s \in C^0([0, T]; [H_0^1(\Omega)]^d)$ ,  $\partial_t u_s \in C^0([0, T]; [L^2(\Omega)]^d)$  and  $v_f \in C^0([0, T]; [L^2(\Omega)]^d)$  such that*

$$(u_s(0), \partial_t u_s(0), v_f(0)) = (u_{s0}, v_{s0}, v_{f0})$$

and the following equations hold in  $\mathcal{D}'(0, T)$

$$\left\{ \begin{aligned} &\forall (w_s, w_f, q) \in V_\phi \times L^2(\Omega), \\ &\frac{d^2}{dt^2} \int_{\Omega} \rho_s (1 - \phi) u_s(t) \cdot w_s \, dx + \int_{\Omega} \sigma_s(u_s(t)) : \varepsilon(w_s) \, dx \\ &\quad + \frac{d}{dt} \int_{\Omega} \rho_f \phi v_f(t) \cdot w_f \, dx + \int_{\Omega} \phi \sigma_f(v_f(t)) : \varepsilon(w_f) \, dx + \int_{\Omega} \phi^2 k_f^{-1} (v_f(t) - \partial_t u_s(t)) \cdot (w_f - w_s) \, dx \\ &\quad - \int_{\Omega} \theta v_f(t) \cdot w_f \, dx = \int_{\Omega} \rho_s (1 - \phi) f(t) \cdot w_s \, dx + \int_{\Omega} \rho_f \phi f(t) \cdot w_f \, dx, \end{aligned} \right. \quad (63a)$$

$$\left\{ \begin{aligned} &\frac{d}{dt} \int_{\Omega} \text{div}((1 - \phi) u_s(t)) q \, dx + \int_{\Omega} \text{div}(\phi v_f(t)) q \, dx = 0. \end{aligned} \right. \quad (63b)$$

The energy estimate (13) holds true (with  $\eta = 0$ ) and the variational solution coincides with the mild solution. Furthermore, there exists a unique pressure  $p$  such that  $(u_s, v_s, v_f, p)$  satisfies (44) in the distribution sense, with  $v_s = \partial_t u_s$ .

**Remark 3.21.** The constraint equation (63b) has been transformed compared to (56). Note nevertheless that  $\operatorname{div}((1 - \phi) \partial_t u_s)$  makes sense in  $L^2((0, T) \times \Omega)$  since it is equal to  $-\operatorname{div}(\phi v_f)$ .

## 4 Incompressible limit

In this section, we show how to pass to the limit in the weak formulation for  $\kappa < +\infty$  as  $\kappa$  goes to infinity and obtain the incompressible system.

For this purpose, we need to get an energy estimate independent of  $\kappa$  in the compressible case. This can be achieved by using the divergence lifting  $v_\theta$  defined in (46) and the change of variables  $(u_s, v_f, p) \mapsto (u_s - \int_0^t v_\theta(s) ds, v_f - v_\theta, p)$  so that the right-hand side of the pressure equation reduces to zero. As already noticed in Remark 3.11, such a lifting requires additional assumptions on the fluid mass input  $\theta$ , namely

$$(h5)_{\text{bis}} \begin{cases} \theta \in C^0([0, T] \times \Omega) \cap H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \\ \forall t \in [0, T], \quad \int_\Omega \theta(t) dx = 0. \end{cases} \quad (64)$$

In order to recover the initial conditions in the incompressible limit, we are not going to pass to the limit in (31) and (41) which are written in  $\mathcal{D}'(0, T)$ , but rather in the following weak formulation: for any  $(u_{s0}, v_{s0}, v_{f0}, p_0) \in Z$  and  $f \in L^2(0, T; [L^2(\Omega)]^d)$ , find  $u_s^\kappa \in C^0([0, T]; [H_0^1(\Omega)]^d)$ ,  $\partial_t u_s^\kappa \in C^0([0, T]; [L^2(\Omega)]^d) \cap L^2(0, T; [H_0^1(\Omega)]^d)$  if  $\eta > 0$  or  $\partial_t u_s^\kappa \in C^0([0, T]; [L^2(\Omega)]^d)$  if  $\eta = 0$ ,  $v_f^\kappa \in C^0([0, T]; [L^2(\Omega)]^d) \cap L^2(0, T; [H_0^1(\Omega)]^d)$  and  $p^\kappa \in C^0([0, T]; L^2(\Omega))$  such that

$$\left\{ \begin{array}{l} \int_0^T \int_\Omega \rho_s (1 - \phi) u_s^\kappa \cdot \partial_{tt} w_s dx dt + \int_0^T \int_\Omega \sigma_s(u_s^\kappa) : \varepsilon(w_s) dx dt + 2\eta \int_0^T \int_\Omega \varepsilon(\partial_t u_s^\kappa) : \varepsilon(w_s) dx dt \\ - \int_0^T \int_\Omega \phi^2 k_f^{-1} (v_f^\kappa - \partial_t u_s^\kappa) \cdot w_s dx dt - \int_0^T \int_\Omega p^\kappa \operatorname{div}((1 - \phi) w_s) dx dt \\ = \int_0^T \int_\Omega \rho_s (1 - \phi) f \cdot w_s dx dt + \int_\Omega \rho_s (1 - \phi) v_{s0} \cdot w_s(0) dx - \int_\Omega \rho_s (1 - \phi) u_{s0} \cdot \partial_t w_s(0) dx, \\ - \int_0^T \int_\Omega \rho_f \phi v_f^\kappa \cdot \partial_t w_f dx dt + \int_0^T \int_\Omega \phi \sigma_f(v_f^\kappa) : \varepsilon(w_f) dx dt \\ + \int_0^T \int_\Omega \phi^2 k_f^{-1} (v_f^\kappa - \partial_t u_s^\kappa) \cdot w_f dx dt - \int_0^T \int_\Omega \theta v_f^\kappa \cdot w_f dx dt - \int_0^T \int_\Omega p^\kappa \operatorname{div}(\phi w_f) dx dt \\ = \int_0^T \int_\Omega \rho_f \phi f \cdot w_f dx dt + \int_\Omega \rho_f \phi v_{f0} \cdot w_f(0) dx, \end{array} \right. \quad (65a)$$

and

$$\left\{ \begin{array}{l} - \int_0^T \int_\Omega \frac{(1 - \phi)^2}{\kappa} p^\kappa \partial_t q dx dt + \int_0^T \int_\Omega \operatorname{div}((1 - \phi) \partial_t u_s^\kappa) q dx dt \\ + \int_0^T \int_\Omega \operatorname{div}(\phi v_f^\kappa) q dx dt = \int_\Omega \frac{(1 - \phi)^2}{\kappa} p_0 q(0) dx \quad \text{if } \eta > 0, \quad (66a) \\ - \int_0^T \int_\Omega \frac{(1 - \phi)^2}{\kappa} p^\kappa \partial_t q dx dt - \int_0^T \int_\Omega \operatorname{div}((1 - \phi) u_s^\kappa) \partial_t q dx dt \\ + \int_0^T \int_\Omega \operatorname{div}(\phi v_f^\kappa) q dx dt = \int_\Omega \frac{(1 - \phi)^2}{\kappa} p_0 q(0) dx + \int_\Omega \operatorname{div}((1 - \phi) u_{s0}) q(0) dx \quad \text{if } \eta = 0, \quad (66b) \end{array} \right.$$

for all admissible test functions

$$\begin{cases} w_s \in C^2([0, T]; [\mathbf{L}^2(\Omega)]^d) \cap L^2(0, T; [\mathbf{H}_0^1(\Omega)]^d), \\ w_f \in C^1([0, T]; [\mathbf{L}^2(\Omega)]^d) \cap L^2(0, T; [\mathbf{H}_0^1(\Omega)]^d), \\ q \in C^1([0, T]; L^2(\Omega)), \\ w_s(T) = \partial_t w_s(T) = w_f(T) = q(T) = 0. \end{cases} \quad (67)$$

The main difference between the weak formulation (65) – (66) and (31) or (41) is that the test functions depend on space but also on time. Besides, the initial conditions are weakly imposed in (65) – (66), while they are strongly imposed in (30) or (40).

The weak formulation (65) – (66) can be obtained with the same approximation process used for the existence of variational solutions. We can show that this weak formulation is satisfied by the limit of the sequence  $(u_s^n, v_s^n, v_f^n, p^n)$ . As in Theorem 2.5 and Theorem 2.12, we show that  $u_s^n, v_s^n, v_f^n$  and  $p^n$  are Cauchy sequences and we pass to the limit in the approximated weak formulation. The terms appearing in the right-hand side of (65) – (66) involving the initial conditions arise when integrating by parts in time the mass terms and passing to the limit. For instance, taking into account (67), it holds

$$\begin{aligned} \int_0^T \int_{\Omega} \rho_f \phi \partial_t v_f^n \cdot w_f \, dx \, dt &= - \int_0^T \int_{\Omega} \rho_f \phi v_f^n \cdot \partial_t w_f \, dx \, dt - \int_{\Omega} \rho_f \phi v_{f0} \cdot w_f(0) \, dx \\ &\xrightarrow{n \rightarrow \infty} - \int_0^T \int_{\Omega} \rho_f \phi v_f^\kappa \cdot \partial_t w_f \, dx \, dt - \int_{\Omega} \rho_f \phi v_{f0} \cdot w_f(0) \, dx, \end{aligned}$$

and, if  $\eta = 0$ ,

$$\begin{aligned} \int_0^T \int_{\Omega} \operatorname{div}((1 - \phi) v_s^n) q \, dx \, dt &= - \int_0^T \int_{\Omega} \operatorname{div}((1 - \phi) u_s^n) \partial_t q \, dx \, dt - \int_{\Omega} \operatorname{div}((1 - \phi) u_{s0}^n) q(0) \, dx \\ &\xrightarrow{n \rightarrow \infty} - \int_0^T \int_{\Omega} \operatorname{div}((1 - \phi) u_s^\kappa) \partial_t q \, dx \, dt - \int_{\Omega} \operatorname{div}((1 - \phi) u_{s0}) q(0) \, dx. \end{aligned}$$

Furthermore, since we have lifted the mixture's divergence constraint, the right-hand side of the pressure equation is null and the resulting energy estimate reads

$$\begin{aligned} &\frac{\rho_s}{2} \int_{\Omega} (1 - \phi) |\partial_t u_s^\kappa(t)|^2 \, dx + \frac{1}{2} \int_{\Omega} \sigma_s(u_s^\kappa(t)) : \varepsilon(u_s^\kappa(t)) \, dx + 2\eta \int_0^t \int_{\Omega} \varepsilon(\partial_t u_s^\kappa) : \varepsilon(\partial_t u_s^\kappa) \, dx \, ds + \frac{\rho_f}{2} \int_{\Omega} \phi |v_f^\kappa(t)|^2 \, dx \\ &+ \int_0^t \int_{\Omega} \phi \sigma_f(v_f^\kappa) : \varepsilon(v_f^\kappa) \, dx \, ds + \int_0^t \int_{\Omega} \phi^2 k_f^{-1} (v_f^\kappa - \partial_t u_s^\kappa) \cdot (v_f^\kappa - \partial_t u_s^\kappa) \, dx \, ds + \frac{1}{2} \int_{\Omega} \frac{(1 - \phi)^2}{\kappa} |p^\kappa(t)|^2 \, dx \\ &\leq \exp\left(\max\left(1, \frac{2\|\theta\|_{C^0([0, T] \times \Omega)}}{\rho_f \phi_{\min}}\right)t\right) \left( \left( \frac{\rho_s}{2} (1 - \phi_{\min}) + \frac{\rho_f}{2} \phi_{\max} \right) \int_0^t \int_{\Omega} |f|^2 \, dx \, ds + \frac{\rho_s}{2} \int_{\Omega} (1 - \phi) |v_{s0}|^2 \, dx \right. \\ &\quad \left. + \frac{1}{2} \int_{\Omega} \sigma_s(u_{s0}) : \varepsilon(u_{s0}) \, dx + \frac{\rho_f}{2} \int_{\Omega} \phi |v_{f0}|^2 \, dx + \frac{1}{2} \int_{\Omega} \frac{(1 - \phi)^2}{\kappa} |p_0|^2 \, dx \right). \quad (68) \end{aligned}$$

**Remark 4.1.** *By choosing  $(w_s, w_f, q) = (\hat{w}_s(x), \hat{w}_f(x), \hat{q}(x)) \psi(t)$  with  $(\hat{w}_s, \hat{w}_f, \hat{q}) \in [\mathbf{H}_0^1(\Omega)]^d \times [\mathbf{H}_0^1(\Omega)]^d \times L^2(\Omega)$  and  $\psi \in \mathcal{D}(0, T)$ , we see that the weak formulation (65) – (66) implies the variational formulation (31) or (41). Hence, from the uniqueness of the variational solution, the solutions of these two formulations coincide.*

We are now ready to establish how the solution in the compressible case converges towards the solution in the incompressible regime as  $\kappa$  goes to infinity.

**Theorem 4.2.** *Assume that (h1) – (h4) and (h5)<sub>bis</sub> are satisfied. For  $z_0 = (u_{s0}, v_{s0}, v_{f0}) \in Z$ , let  $(u_s^\kappa, v_f^\kappa, p^\kappa)$  be the solution of (65) – (66). As  $\kappa$  goes to infinity,  $(u_s^\kappa, \partial_t u_s^\kappa, v_f^\kappa)$  converge weakly towards*

the solution of the following formulation: find  $u_s \in C^0([0, T]; [\mathbf{H}_0^1(\Omega)]^d)$ ,  $\partial_t u_s \in C^0([0, T]; [\mathbf{L}^2(\Omega)]^d)$  and  $v_f \in C^0([0, T]; [\mathbf{L}^2(\Omega)]^d) \cap \mathbf{L}^2(0, T; [\mathbf{H}_0^1(\Omega)]^d)$  or  $(\partial_t u_s, v_f) \in \mathbf{L}^2(0, T; V_\phi)$  if  $\eta > 0$ , such that

$$\left\{ \begin{array}{l} \int_0^T \int_\Omega \rho_s (1 - \phi) u_s \cdot \partial_{tt} w_s \, dx \, dt + \int_0^T \int_\Omega \sigma_s(u_s) : \varepsilon(w_s) \, dx \, dt + 2\eta \int_0^T \int_\Omega \varepsilon(\partial_t u_s) : \varepsilon(w_s) \, dx \, dt \\ - \int_0^T \int_\Omega \rho_f \phi v_f \cdot \partial_t w_f \, dx \, dt + \int_0^T \int_\Omega \phi \sigma_f(v_f) : \varepsilon(w_f) \, dx \, dt + \int_0^T \int_\Omega \phi^2 k_f^{-1} (v_f - \partial_t u_s) \cdot (w_f - w_s) \, dx \, dt \\ - \int_0^T \int_\Omega \theta v_f \cdot w_f \, dx \, dt = \int_0^T \int_\Omega \rho_s (1 - \phi) f \cdot w_s \, dx \, dt + \int_0^T \int_\Omega \rho_f \phi f \cdot w_f \, dx \, dt \\ + \int_\Omega \rho_s (1 - \phi) v_{s0} \cdot w_s(0) \, dx - \int_\Omega \rho_s (1 - \phi) u_{s0} \cdot \partial_t w_s(0) \, dx + \int_\Omega \rho_f \phi v_{f0} \cdot w_f(0) \, dx, \end{array} \right. \quad (69)$$

and

$$- \int_0^T \int_\Omega \operatorname{div}((1 - \phi) u_s) \partial_t q \, dx \, dt + \int_0^T \int_\Omega \operatorname{div}(\phi v_f) q \, dx \, dt = \int_\Omega \operatorname{div}((1 - \phi) u_{s0}) q(0) \, dx \quad \text{if } \eta = 0, \quad (70)$$

for all admissible test functions

$$\left\{ \begin{array}{l} w_s \in C^2([0, T]; [\mathbf{L}^2(\Omega)]^d), \\ w_f \in C^1([0, T]; [\mathbf{L}^2(\Omega)]^d), \\ (w_s, w_f) \in \mathbf{L}^2(0, T; V_\phi), \\ q \in C^1([0, T]; \mathbf{L}^2(\Omega)), \\ w_s(T) = \partial_t w_s(T) = w_f(T) = q(T) = 0. \end{array} \right. \quad (71)$$

*Proof.* Let us prove this result in the inviscid case  $\eta = 0$ , the viscous case being similar. From the energy bound (68), we deduce that, up to subsequences, the following weak convergences hold true as  $\kappa$  goes to infinity:

$$\begin{aligned} u_s^\kappa &\rightharpoonup u_s^\infty \text{ weakly star in } \mathbf{L}^\infty(0, T; [\mathbf{H}_0^1(\Omega)]^d), \\ \partial_t u_s^\kappa &\rightharpoonup \partial_t u_s^\infty \text{ weakly star in } \mathbf{L}^\infty(0, T; [\mathbf{L}^2(\Omega)]^d), \\ v_f^\kappa &\rightharpoonup v_f^\infty \text{ weakly star in } \mathbf{L}^\infty(0, T; [\mathbf{L}^2(\Omega)]^d), \\ v_f^\kappa &\rightharpoonup v_f^\infty \text{ weakly in } \mathbf{L}^2(0, T; [\mathbf{H}_0^1(\Omega)]^d), \end{aligned}$$

for some elements  $u_s^\infty \in \mathbf{L}^\infty(0, T; [\mathbf{H}_0^1(\Omega)]^d)$  with  $\partial_t u_s^\infty \in \mathbf{L}^\infty(0, T; [\mathbf{L}^2(\Omega)]^d)$  and  $v_f^\infty \in \mathbf{L}^\infty(0, T; [\mathbf{L}^2(\Omega)]^d) \cap \mathbf{L}^2(0, T; [\mathbf{H}_0^1(\Omega)]^d)$ .

We have no bound on the pressure  $p^\kappa$  but (68) implies that  $\frac{p^\kappa}{\sqrt{\kappa}}$  is bounded in  $\mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega))$ . Hence, we can select a subsequence (still denoted by  $p^\kappa$ ) such that  $\frac{p^\kappa}{\sqrt{\kappa}}$  converges in the weak-\* topology of  $\mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega))$ .

By adding (65a) to (65b) and by restricting the velocities test functions  $(w_s, w_f)$  to functions in  $\mathbf{L}^2(0, T; V_\phi)$ ,



it follows that  $(u_s^\kappa, v_f^\kappa, p^\kappa)$  satisfies

$$\left\{ \begin{array}{l} \int_0^T \int_\Omega \rho_s (1-\phi) u_s^\kappa \cdot \partial_{tt} w_s \, dx \, dt + \int_0^T \int_\Omega \sigma_s(u_s^\kappa) : \varepsilon(w_s) \, dx \, dt + 2\eta \int_0^T \int_\Omega \varepsilon(\partial_t u_s^\kappa) : \varepsilon(w_s) \, dx \, dt \\ - \int_0^T \int_\Omega \rho_f \phi v_f^\kappa \cdot \partial_t w_f \, dx \, dt + \int_0^T \int_\Omega \phi \sigma_f(v_f^\kappa) : \varepsilon(w_f) \, dx \, dt + \int_0^T \int_\Omega \phi^2 k_f^{-1} (v_f^\kappa - \partial_t u_s^\kappa) \cdot (w_f - w_s) \, dx \, dt \\ - \int_0^T \int_\Omega \theta v_f^\kappa \cdot w_f \, dx \, dt = \int_0^T \int_\Omega \rho_s (1-\phi) f \cdot w_s \, dx \, dt + \int_0^T \int_\Omega \rho_f \phi f \cdot w_f \, dx \, dt \\ + \int_\Omega \rho_s (1-\phi) v_{s0} \cdot w_s(0) \, dx - \int_\Omega \rho_s (1-\phi) u_{s0} \cdot \partial_t w_s(0) \, dx + \int_\Omega \rho_f \phi v_{f0} \cdot w_f(0) \, dx, \\ - \int_0^T \int_\Omega \frac{(1-\phi)^2}{\kappa} p^\kappa \partial_t q \, dx \, dt - \int_0^T \int_\Omega \operatorname{div}((1-\phi) u_s^\kappa) \partial_t q \, dx \, dt \\ + \int_0^T \int_\Omega \operatorname{div}(\phi v_f^\kappa) q \, dx \, dt = \int_\Omega \frac{(1-\phi)^2}{\kappa} p_0 q(0) \, dx + \int_\Omega \operatorname{div}((1-\phi) u_{s0}) q(0) \, dx, \end{array} \right.$$

for all admissible test functions verifying (71).

Thus we can pass to the weak limit in this formulation by noting that

$$\int_0^T \int_\Omega \frac{(1-\phi)^2}{\kappa} p^\kappa \partial_t q \, dx \, dt = \frac{1}{\sqrt{\kappa}} \int_0^T \int_\Omega (1-\phi)^2 \frac{p^\kappa}{\sqrt{\kappa}} \partial_t q \, dx \, dt \longrightarrow 0$$

and that

$$\int_\Omega \frac{(1-\phi)^2}{\kappa} p_0 q(0) \, dx \longrightarrow 0$$

as  $\kappa$  goes to infinity.

In conclusion,  $(u_s^\infty, v_f^\infty)$  satisfies exactly (69) and (70) in the incompressible limit. Moreover  $u_s^\infty$ ,  $\partial_t u_s^\infty$  and  $v_f^\infty$  are continuous functions in time because they also satisfy (63) and hence coincide with the mild solution. Indeed, (63) can be recovered from (69) – (70) by taking admissible test functions of the form  $(w_s, w_f, q) = (\hat{w}_s(x), \hat{w}_f(x), \hat{q}(x)) \psi(t)$  with  $(\hat{w}_s, \hat{w}_f, \hat{q}) \in V_\phi \times L^2(\Omega)$  and  $\psi \in \mathcal{D}(0, T)$ .  $\square$

**Remark 4.3.** Hypothesis (h5)<sub>bis</sub> is crucial to get the incompressible limit convergence. Without this assumption, we can not lift the mixture's divergence constraint and we only have an energy estimate that does not provide a uniform bound in  $\kappa$  as  $\kappa$  goes to infinity because of the coefficient  $\frac{\kappa}{2\rho_f^2(1-\phi_{\max})^2}$  appearing in the right-hand side of (7).

**Remark 4.4.** Theorem 4.2 provides the weak convergence of the displacement and velocities in the incompressible limit. If the incompressible regime solution is more regular, we can also obtain the pressure convergence and recover strong convergence for the displacement and velocities. More precisely, following the same guidelines as in [32, Lemma 75.1], we can show that

$$\begin{aligned} & \|u_s^\kappa - u_s^\infty\|_{L^\infty(0, T; [H_0^1(\Omega)]^d)}^2 + \|\partial_t u_s^\kappa - \partial_t u_s^\infty\|_{L^\infty(0, T; [L^2(\Omega)]^d)}^2 + \|v_f^\kappa - v_f^\infty\|_{L^\infty(0, T; [L^2(\Omega)]^d)}^2 + \frac{1}{\kappa} \|p^\kappa - p^\infty\|_{L^\infty(0, T; L^2(\Omega))}^2 \\ & + \eta \|\partial_t u_s^\kappa - \partial_t u_s^\infty\|_{L^2(0, T; [H_0^1(\Omega)]^d)}^2 + \|v_f^\kappa - v_f^\infty\|_{L^2(0, T; [H_0^1(\Omega)]^d)}^2 \lesssim \frac{1}{\kappa^2} \|\partial_t p^\infty\|_{H^1(0, T; L^2(\Omega))}^2. \end{aligned}$$

Thus, if  $\partial_t p^\infty \in H^1(0, T; L^2(\Omega))$ , the above error estimate specifies the convergence speed of  $(u_s^\kappa, \partial_t u_s^\kappa, v_f^\kappa, p^\kappa)$  towards  $(u_s^\infty, \partial_t u_s^\infty, v_f^\infty, p^\infty)$  as  $\kappa$  goes to infinity.

## 5 Numerical experiments

In this section, we present some numerical examples to illustrate the theoretical results presented previously. In particular, we numerically investigate the regularity of solutions to the static and time-dependent

problems. All the simulations of this section were implemented with the FEniCS finite element software [38, 1].

## 5.1 Spatial discretization

For small values of bulk modulus, our equations can be discretized with standard finite elements. However, when the coefficient  $\kappa$  becomes large, we have to take into account the saddle-point structure of the problem involving the mixture's divergence constraint and to chose finite element spaces that satisfy the inf-sup condition (47) at the discrete level.

In the incompressible or nearly incompressible case, the expression of the mapping

$$T : (u_s, v_s, v_f, p) \longmapsto \left( \frac{\alpha^*}{2} u_s - \frac{\alpha^*}{2\lambda_0} v_s, \alpha^* v_s - v_p, \alpha^* v_f - v_p, \alpha^* p \right)$$

defined in (62) suggests us how to select convenient finite element spaces in order to discretize the problem. Indeed, to get a stable discretization, it is sufficient to reproduce the construction of  $v_p$  at the discrete level. This is possible by choosing finite elements that are stable (in the Brezzi [10] sense) for Stokes equations.

More precisely, let us suppose that we use a *conforming* approximation of the space  $Y = [\mathbf{H}_0^1(\Omega)]^d \times [\mathbf{H}_0^1(\Omega)]^d \times [\mathbf{H}_0^1(\Omega)]^d \times L^2(\Omega)$  by a finite dimensional space

$$Y_h = V_{s,h} \times V_{s,h} \times V_{f,h} \times Q_h \subset Y,$$

where  $V_{s,h}$ ,  $V_{f,h}$  and  $Q_h$  denote respectively the finite element spaces chosen to discretize the solid part, the fluid part and the pressure of the mixture. Assume further that  $(V_{s,h}, Q_h)$  and  $(V_{f,h}, Q_h)$  are two inf-sup stable pairs associated to the standard Stokes problem and verifying Fortin Lemma [10, Proposition 5.4.2], i.e. there exists two operators  $\Pi_{s,h} : [\mathbf{H}_0^1(\Omega)]^d \mapsto V_{s,h}$  and  $\Pi_{f,h} : [\mathbf{H}_0^1(\Omega)]^d \mapsto V_{f,h}$  satisfying, for each  $v \in [\mathbf{H}_0^1(\Omega)]^d$ ,

- For all  $q_h \in Q_h$ ,

$$\int_{\Omega} \operatorname{div} v q_h \, dx = \int_{\Omega} \operatorname{div} (\Pi_{i,h}(v)) q_h \, dx, \quad i \in \{s, f\}; \quad (72)$$

- There exists a constant  $C_{i,\pi} > 0$  independent of  $h$  such that

$$\|\nabla(\Pi_{i,h}(v))\| \leq C_{i,\pi} \|\nabla v\|, \quad i \in \{s, f\}. \quad (73)$$

Under these hypotheses, we claim that the bilinear form  $a_{\lambda_0}$  defined in the mixed formulation (58) is uniformly  $T_h$ -coercive. Namely, it holds

$$\forall h > 0, \exists T_h \in \mathcal{L}(Y_h), \forall z_h \in Y_h, \quad |a_{\lambda_0}(z_h, T_h z_h)| \geq \underline{\alpha} \|z_h\|_{Y_h}^2 \quad \text{and} \quad \|T_h\| \leq C, \quad (74)$$

for some constants  $\underline{\alpha} > 0$  and  $C > 0$  independent of  $h$ .

Indeed, setting

$$T_h : (u_{s,h}, v_{s,h}, v_{f,h}, p_h) \longmapsto \left( \frac{\alpha^*}{2} u_{s,h} - \frac{\alpha^*}{2\lambda_0} v_{s,h}, \alpha^* v_{s,h} - \Pi_{s,h}(v_{p_h}), \alpha^* v_{f,h} - \Pi_{f,h}(v_{p_h}), \alpha^* p_h \right), \quad (75)$$

where  $v_{p_h}$  is defined by (48), the property (72) enables us to reproduce the calculations from the proof of Proposition 3.17 at the discrete level, so that the first condition of (74) holds true. The second condition then follows from (73) since

$$\|\nabla(\Pi_{i,h}(v_{p_h}))\| \leq C_{i,\pi} \|\nabla v_{p_h}\| \leq C_{i,\pi} C_{\operatorname{div}} \|p_h\|$$

for each  $i \in \{s, f\}$ .

Therefore, a stable discretization of the incompressible system is offered by standard inf-sup stable conforming finite elements associated to the Stokes system. For instance, as it was observed in [4], one

can use Taylor-Hood elements  $[\mathbb{P}_{k+1}]^d - \mathbb{P}_k$  ( $k \geq 1$ ) for the pairs  $(V_{s,h}, Q_h)$  and  $(V_{f,h}, Q_h)$ . More broadly, the previous  $T_h$ -coercivity argument implies the stability of the MINI element  $\mathbb{P}_1^b - \mathbb{P}_1$ , the  $\mathbb{P}_2 - \mathbb{P}_0$  element, or also Scott-Vogelius elements  $[\mathbb{P}_k]^d - \mathbb{P}_{k-1}^{-1}$  with  $k \geq 4$  and  $d = 2$ . Besides, we can select different finite element spaces for the solid and fluid parts, as long as each of them form a Stokes-stable pair with the space chosen for pressure.

Finally, note that the mapping (75) is independent of the phase field  $\phi$  and that the obtention of (74) does not require any assumption on the size of the permeability tensor  $k_f$ , as it was supposed in [4]. Hence, our approach furnishes a robust discretization regardless of porosity and permeability.

## 5.2 Regularity of the operator's domain

In both compressible and incompressible cases, we proved the existence and uniqueness of a strong solution in  $C^0([0, T]; D(A_\eta^\kappa))$ , with  $\eta \geq 0$  and  $0 < \kappa \leq +\infty$ . The operator's domain  $D(A_\eta^\kappa)$  was defined by extension from a continuous bilinear form (see e.g. (18) and (19)) but we did not express it as a standard Sobolev space. In what follows, we give some numerical evidences that the operator's domain is not regular, namely

$$D(A_{\eta \geq 0}^\kappa) \neq [H^2(\Omega)]^d \cap [H_0^1(\Omega)]^d \times [H^2(\Omega)]^d \cap [H_0^1(\Omega)]^d \times [H^2(\Omega)]^d \cap [H_0^1(\Omega)]^d \times H^1(\Omega)$$

and

$$D(A_{\eta > 0}^\infty) \neq [H^2(\Omega)]^d \cap [H_0^1(\Omega)]^d \times ([H^2(\Omega)]^d \times [H^2(\Omega)]^d \cap V_\phi) \times H^1(\Omega).$$

To do so, we compute numerically the solution of the static problem  $z + A_\eta^\kappa z = g$  with  $g \in Z$  or  $g \in H$ , viz.

$$\begin{cases} u_s - v_s = g_u, \\ \rho_s(1 - \phi)v_s - \operatorname{div}(\sigma_s(u_s)) - \operatorname{div}(\sigma_s^{\text{vis}}(v_s)) + (1 - \phi)\nabla p = \rho_s(1 - \phi)g_s, \\ \rho_f\phi v_f - \operatorname{div}(\phi\sigma_f(v_f)) + \phi\nabla p = \rho_f\phi g_f, \\ \frac{(1-\phi)^2}{\kappa}p + \operatorname{div}((1-\phi)v_s + \phi v_f) = \begin{cases} \frac{(1-\phi)^2}{\kappa}g_p & \text{if } \kappa < +\infty, \\ 0 & \text{if } \kappa = +\infty, \end{cases} \end{cases} \quad (76)$$

with  $(g_u, g_s, g_f, g_p) \in [L^2(\Omega)]^d \times [L^2(\Omega)]^d \times [L^2(\Omega)]^d \times L^2(\Omega)$  and where we have assumed that  $k_f = 0$  and  $\theta = 0$  without loss of generality.

We consider  $\Omega = \{x \in \mathbb{R}^2, |x| \leq 1\}$  a very smooth domain and  $(\mathcal{T}_h)_h$  a regular family of meshes of  $\bar{\Omega}$ , made of triangles. The coarsest mesh size  $H$  corresponds to a uniform mesh constructed with 8 subdivisions along each axis direction. Setting  $\rho_s = \rho_f = \lambda_f = \mu_f = \eta = \lambda = \mu = 1$  and taking a constant porosity  $\phi = 0.5$ , we compute the error in  $L^2$ -norm between the approximated solution  $(u_{s,h}, v_{s,h}, v_{f,h}, p_h)$  of (76) and a reference solution computed on a very refined mesh.

The resulting convergence graphs are presented in Figure 1 for  $\kappa = 1$  and smooth data  $g_s, g_f, g_p$ . The convergence rates depend on the regularity of the solid displacement data  $g_u$ . If  $g_u$  is smooth, we obtain optimal orders of convergence, as expected by the theory. However, if  $g_u \in [H_0^1(\Omega)]^d \setminus [H^2(\Omega)]^d$ , Figure 1 (right) exhibits suboptimal convergence rates, thus indicating that the solution of (76) is not in  $H^2$ . The same occurs in the incompressible case, as shown in Figure 2.

## 5.3 Regularity of solutions

Even if the operator's domain is not regular, we are now going to shed light on the regularizing effect for the solution of the unsteady problem. For this purpose, we use the time discretization introduced and fully analyzed in [18], which consists of a midpoint scheme for the solid fields and an implicit Backward Euler scheme for the fluid and the pressure. The major interest of this scheme is that it preserves energy balance at the discrete level. The perturbation term  $-\theta v_f$  is treated explicitly.

As before, we perform the simulation on the smooth domain  $\Omega = \{x \in \mathbb{R}^2, |x| \leq 1\}$  meshed by a regular family of triangulations  $(\mathcal{T}_h)_h$ . We set  $\rho_s = \rho_f = \lambda_f = \mu_f = \lambda = \mu = 1$ ,  $k_f = 0$ ,  $\theta = 0$  and  $\phi = 0.5$ . We take  $\kappa = 10^{10}$ , but smallest values of  $\kappa$  would lead to comparable results. All the simulations are run during a hundred of time iterations, with a time step  $\Delta t = 10^{-2}$  and up to the final time  $T = 1$ .

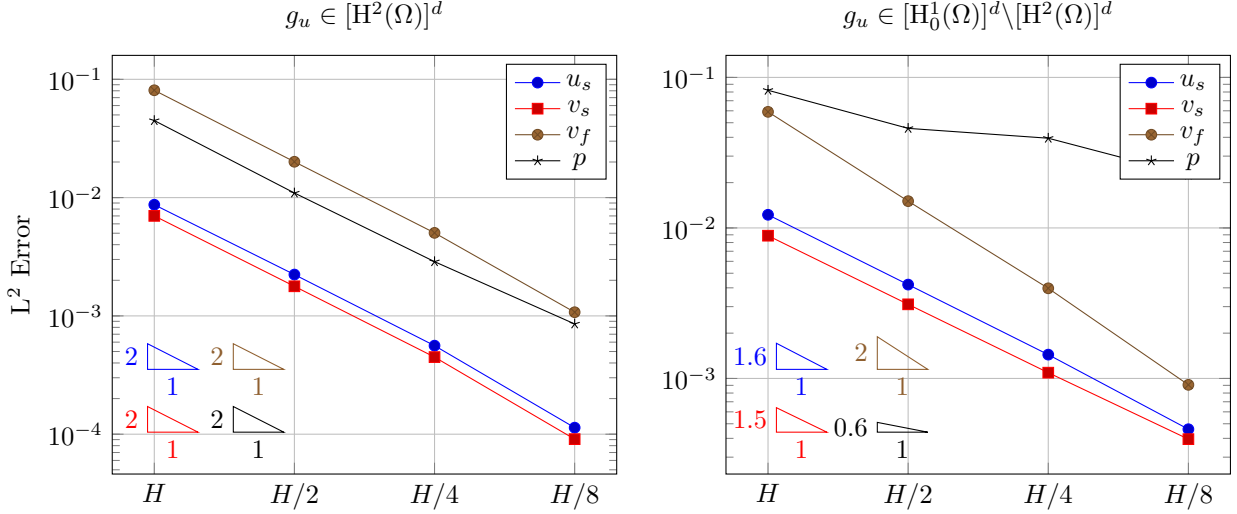


Figure 1: Approximation errors and computed convergence rates for the discretization of the compressible ( $\kappa = 1$ ) steady-state problem (76) with  $[\mathbb{P}_1]^2 \times [\mathbb{P}_1]^2 \times [\mathbb{P}_1]^2 \times \mathbb{P}_1$  elements.

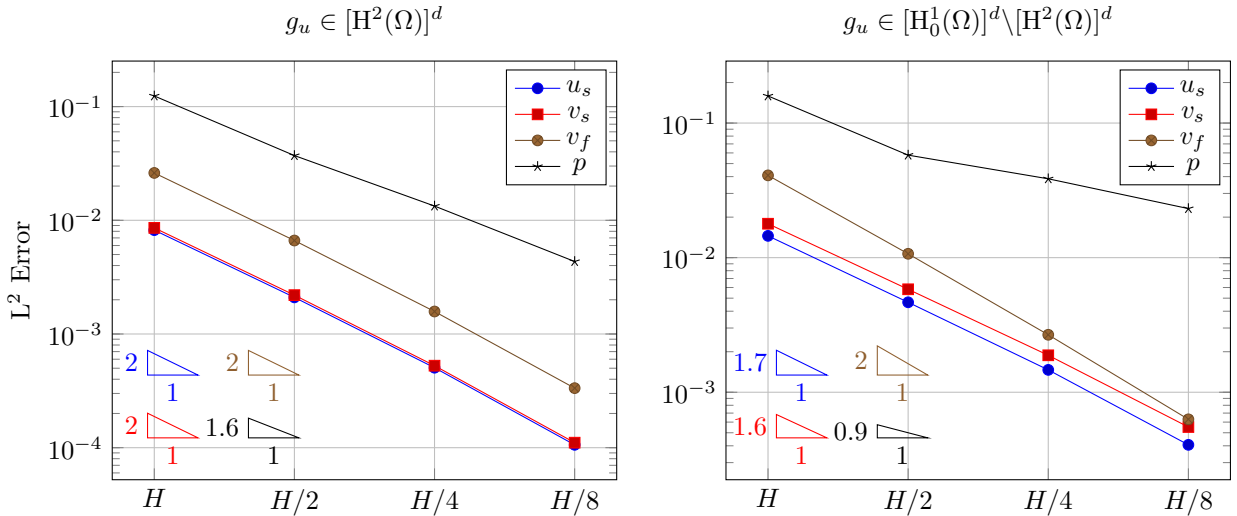


Figure 2: Approximation errors and computed convergence rates for the discretization of the incompressible ( $\kappa = +\infty$ ) steady-state problem (76) with  $\mathbb{P}_1^b \times \mathbb{P}_1^b \times \mathbb{P}_1^b \times \mathbb{P}_1$  elements.

Figure 3 illustrates the regularizing effect of time. Indeed, although the initial velocities and the applied exterior body force belong to  $[\mathbb{H}_0^1(\Omega)]^d \setminus [\mathbb{H}^2(\Omega)]^d$ , we recover optimal convergence rates in  $L^\infty(0, T; Z)$ -norm between the approximated and reference solutions. Note that putting  $\eta = 0$  slightly degrades the convergence order of the solid velocity, but does not affect the regularity of solid displacement, fluid velocity and pressure.

If the initial conditions and the right-hand side belong strictly to the energy space, Figure 4 highlights that the convergence rates are considerably diminished, which confirms the regularity found in Theorems 3.13 and 3.20. Moreover, the solution is less regular when  $\eta = 0$ , as predicted by our theoretical results.

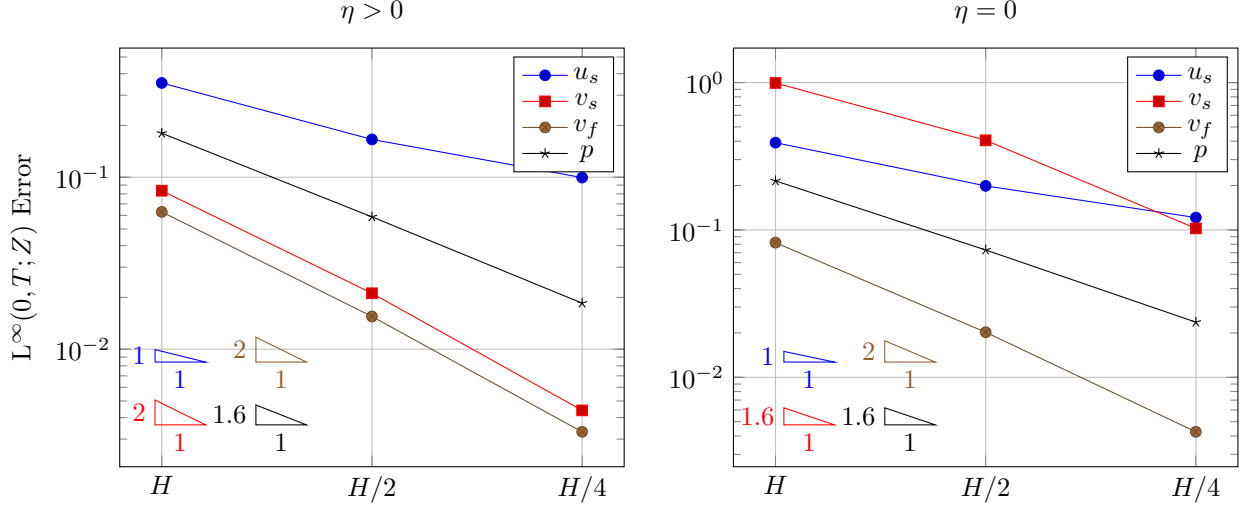


Figure 3: Approximation errors and convergence rates for the discretization of the time-dependent problem (44) with  $\mathbb{P}_1^b \times \mathbb{P}_1^b \times \mathbb{P}_1^b \times \mathbb{P}_1$  elements and  $v_{s0}, v_{f0}, f$  in  $[\mathbb{H}_0^1(\Omega)]^d \setminus [\mathbb{H}^2(\Omega)]^d$

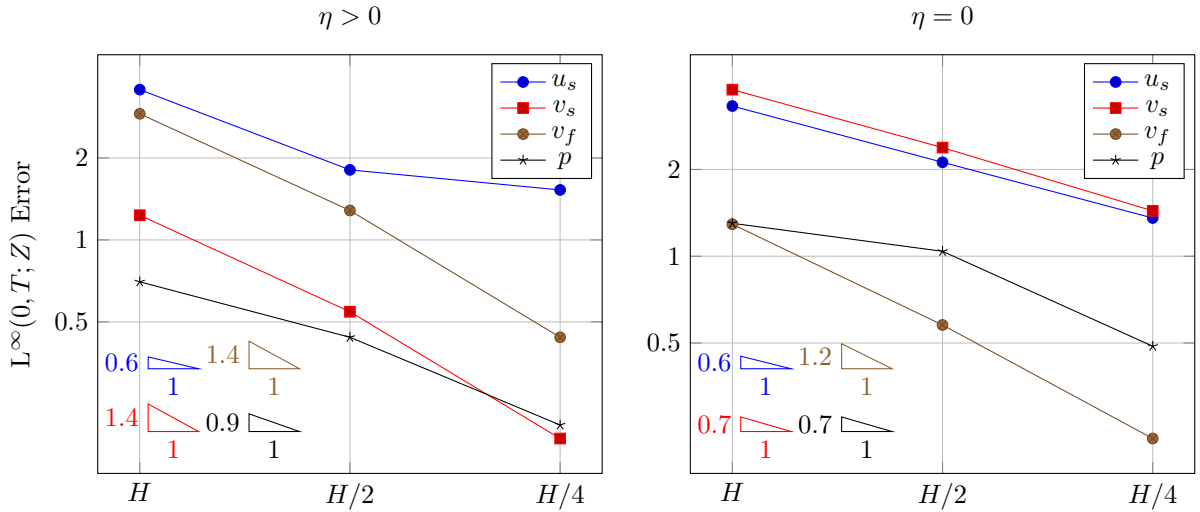


Figure 4: Approximation errors and convergence rates for the discretization of the time-dependent problem (44) with  $\mathbb{P}_1^b \times \mathbb{P}_1^b \times \mathbb{P}_1^b \times \mathbb{P}_1$  elements and  $v_{s0}, v_{f0}, f$  in  $[\mathbb{L}^2(\Omega)]^d \setminus [\mathbb{H}_0^1(\Omega)]^d$

## Conclusion

In this work, we proposed a complete vision of well-posedness for a fully dynamical and strongly coupled poromechanics model. Using an original combination of semigroup theory and  $T$ -coercivity, we demonstrated the existence and uniqueness of strong solutions in the compressible and incompressible cases, with or without solid viscosity. By unifying semigroup and variational techniques, we also recovered the existence and uniqueness of weak solutions. From the methodological point of view, this unified approach enabled us to take benefit of the best of both theories depending on the hyperbolic – parabolic or parabolic – parabolic nature of the coupling, in particular to prove uniqueness. To handle the incompressible case, we developed a functional framework and an extension of De Rham Theorem adapted to the mixture’s divergence constraint.

When the incompressible regime is reached, our analysis offers a spatial discretization of the problem with conforming finite elements, including Taylor-Hood elements but also any Stokes-stable elements such as the MINI element. Moreover, this choice of finite element discretization spaces is robust with respect to porosity and permeability. Finally, our theoretical results are corroborated by numerical experiments. We have shown numerically that the operator’s domain is not regular, and illustrate the possible regularizing effect on the unsteady problem, which remains an open question from the theoretical point of view.

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