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Exponential stability of linear periodic difference-delay equations

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Abstract

This paper deals with the stability of linear periodic difference delay systems, where the value at time t of a solution is a linear combination with periodic coefficients of its values at finitely many delayed instants $t - \tau_1, \dots, t - \tau_N$. We establish a necessary and sufficient condition for exponential stability of such systems when the coefficients have Hölder-continuous derivative, that generalizes the one obtained for difference delay systems with constant coefficients by Henry and Hale in the 1970s. This condition may be construed as analyticity, in a half plane, of the (operator valued) harmonic transfer function of an associated linear control system.

Keywords: linear periodic systems, difference delay systems, exponential stability, harmonic transfer function, Henry-Hale theorem.

1. Introduction

In this paper we present a necessary and sufficient condition for exponential stability of periodic *difference-delay systems*²; *i.e.*, linear dynamical systems of the form

$$y(t) = \sum_{j=1}^N D_j(t) y(t - \tau_j), \quad t > s, \quad (1.1)$$

where $\tau_1 < \dots < \tau_N$ are positive delays and $D_1(t), \dots, D_N(t)$ complex $d \times d$ matrices, depending periodically on time t . This stability condition applies when the maps $t \mapsto D_j(t)$ are periodic and differentiable with Hölder continuous derivatives. Here, periodicity is essential but the Hölder-smoothness assumption is technical. Precise definitions of exponential stability are made in Section 2; for the time being, we simply describe it as the property that every *solution* $y : [s - \tau_N, +\infty) \rightarrow \mathbb{C}^d$ of (1.1) has a restriction to $[t - \tau_N, t]$ that decays exponentially fast with t as the latter goes to $+\infty$.

Dynamical systems like (1.1) are natural generalizations of time-invariant difference-delay systems, arising in various contexts of modeling and control. Let us mention three:

- a) The study of certain *1-D hyperbolic systems* reduces to the study of particular difference-delay systems: after integrating in terms of functions of one variable (backward and forward waves) using the method of characteristics, their evolution is governed by equations like (1.1),

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²There seems to be no general agreement on terminology. The name “difference-delay system” is used in [1, 2, 3], but these are called *time-delay systems* in [4] and *difference equations* in [5, 6], while no specific name is coined in [7, 8] where (1.1) is written $Ly(\cdot) = 0$ and L is referred to as a *difference operator*.

see [4, 9, 2]. In particular, results from the present paper apply to study the stability of such systems when subject to linear periodically time-varying boundary conditions.

b) *Neutral functional differential equations* are functional differential equations of the form

$$\frac{d}{dt} \left(y(t) - \sum_{j=1}^N D_j(t) y(t - \tau_j) \right) = B_0(t) y(t) + \sum_{j=1}^N B_j(t) y(t - \tau_j), \quad t \geq s, \quad (1.2)$$

where the maps $B_i(\cdot)$ and $D_i(\cdot)$ are regular enough. Such equations model quite general linear phenomena involving delays, and are naturally related to (1.1). Indeed, the solution operator of System (1.2) turns out to be a compact perturbation of the one governing System (1.1), which makes exponential stability properties of (1.2) close to those of (1.1): the spectra of their monodromy operators differ by at most finitely many eigenvalues [7, thm 7.3 Section 3.7]. References [8, 5] use this connection already in the time-invariant case, and one can surmise that characterizing the exponential stability of (1.1) is likewise a substantial step towards analyzing the one of (1.2) in the periodic case; see discussion in Section 6. In the language of electronic engineering, System (1.1) is called the *high frequency limit system* of (1.2) because it represents the limiting behaviour of the latter when y oscillates arbitrary fast (so that the right hand side of (1.2) goes to zero weakly).

c) The stability of *active microwave circuits* (like amplifiers), has been an initial motivation of the authors to undertake the present study. Circuits can be regarded as nonlinear systems with infinite dimensional state space, due to transmission lines that are modelled either by 1-D hyperbolic systems or by delays. Their response to a periodic signal is typically a periodic solution, and linearizing along this trajectory yields a linear periodic dynamical system, governed by a functional differential equation slightly more general than (1.2), whose operator solution is again a compact perturbation of a high-frequency limit system of the form (1.1). The same remarks as in b) now apply, to the effect that if the high frequency limit system is exponentially stable (this is the property we shall characterize), then the local stability of the periodic solution to the initial, nonlinear circuit depends only on whether the harmonic transfer function of the linearized system (HTF, to be defined later) has unstable *poles* [3]. Such methods are already instrumental in engineering (without much mathematical justification), based on frequency-wise simulation of the HTF obtained by so-called *harmonic-balance* techniques; see for example [10] for a discussion.

Our result supersedes the “Henry-Hale theorem”, that settles exponential stability issues for difference-delay systems in the *time-invariant case*; *i.e.*, when the matrices D_j in (1.1) are constant. This result, stated further below as Theorem 2 in Section 2, dates back to the 1970s; it was first established in [8] for finitely many commensurate delays and later carried over to countably many, not necessarily commensurate delays in [5]; generalizations to distributed delay systems may be found in [7, Chapter 9]. The stability of functional differential equations like (1.2) has been studied since the fifties in the time-invariant case, using either Laplace transform and semigroups to reduce the problem to localizing the zeros of almost periodic holomorphic functions [11, 5, 8, 7], or else devising Lyapunov–Krasovskii functionals to prove stability in special cases [7, 12, 13]. The stability of *time-varying* linear difference-delay systems was not investigated nearly as extensively, and the literature we know of can be summarized as follows. First, a (fairly restrictive) sufficient condition for exponential stability, based on the Perron-Frobenius theorem, can be found in [1]. Second, a formula representing the solutions of (1.1) in the general time-varying case is given in [6]. The latter paper contains interesting results on the insensitivity of L^p -exponential stability

to p , and on how stability is preserved under perturbation of the delays, but unfortunately the characterization of exponential stability proposed there is untractable for it involves the spectrum of sums of products of matrices with indefinitely growing number of terms and factors when t goes large, indexed according to combinatorial rules involving the lattice generated by the real numbers τ_j . Third, if a linear time-varying system of the form (1.1) arises from a network of lossless telegrapher's equations (for instance as the high frequency limit of an electrical network), then a sufficient condition for exponential stability can be based on dissipativity at the nodes of the network [2]. Related, somewhat specific criteria for hyperbolic 1-D systems may be found in [4]. When the maps $t \mapsto D_j(t)$ are periodic and Hölder-smooth, such results follow from the present ones [14].

In this paper we dwell on control-theoretic ideas, basing our approach on the introduction of an associated control system whose exponential stability is equivalent to the one of (1.1) (adding a virtual control in the simplest possible way, see (2.18)). This fresh point of view is suggestive of new tools to investigate stability of periodic systems, as we now explain. In the time-invariant case, the transfer function of the associated control system is a matrix-valued function $H(p)$ of a complex variable p , and the Henry-Hale theorem amounts to saying that system (1.1) is exponentially stable if and only if H is holomorphic in some half plane $\{p \in \mathbb{C}, \Re(p) > \alpha\}$, with $\alpha < 0$; this is equivalent to the seemingly stronger requirement that H be holomorphic *and bounded* in such a half-plane, see discussion after Theorem 3. To address the periodic case, we define a so-called *Harmonic Transfer Function* (HTF) for periodic difference-delay control systems (whose state space is infinite-dimensional), that generalizes the one introduced in [15] for periodic differential equations on \mathbb{R}^n (whose state space is finite-dimensional). Our HTF is a holomorphic map of a complex variable p , valued in the space of linear operators on $L^2([0, T], \mathbb{C}^d)$ with T the period of the system, that reduces to multiplication by the ordinary transfer function at p when the system is time-invariant. Then, our main result can be interpreted in terms of the HTF the same way as the Henry-Hale theorem does in terms of the classical transfer function. Namely, a periodic difference-delay system is exponentially stable if and only if the HTF of the associated control system is holomorphic and bounded (as an operator-valued map) in some half-plane $\{p \in \mathbb{C}, \Re(p) > \alpha\}$ with $\alpha < 0$. This is the content of Theorem 5, which is the main result of the paper; an equivalent formulation not mentioning the HTF (but featuring the latter in disguise) is given in Theorem 3.

Hence, just like transfer functions encode in their singularities the stability properties of time-invariant linear systems, harmonic transfer functions as defined in this paper reflect the stability of infinite-dimensional linear periodic systems. In this connection, we note that results from this paper were implicitly anticipated by the Engineering community when basing stable design of active circuits on the location of the singularities of certain analytic functions that are none but the first few Fourier coefficients of the HTF, see [10] and the bibliography therein. In turn, the present study questions Engineering practice; *e.g.*, asking when the singularities of the HTF coincide with those of finitely many such Fourier coefficients.

Besides classical tools from Fourier or complex analysis and periodic evolution families, the proofs appeal to further material like variation-of-constant formulas in the *BV*-setting and controlled inversion in Banach algebras³, along with the fact that L^p -exponential stability is independent of p for systems like (1.1). The most convoluted, and perhaps deeper part of this work is the connection, given by Lemma 14, between the HTF and the ordinary transfer function of the lifting

³Controlled inversion (bounding the norm of the inverse in a subalgebra in terms of the norm in this subalgebra and the norm of the inverse in the algebra) was pioneered in Baskakov [16] and Nikolski [17].

of the solution operator for (1.1) (which is a discrete time-invariant infinite-dimensional linear system). Because the dynamics of (a suitable realization of) this lifting is the monodromy operator (see Theorem 13), the above-mentioned connection is the ultimate reason why the singularities of the HTF reflect the spectrum of the monodromy operator and therefore also the exponential stability properties of the system.

The paper is organized as follows. In Section 2, we make pieces of notation and define exponential stability, before recalling the Henry-Hale theorem; we then state its generalization to the periodic case which constitutes our main result (Theorem 3), before defining the HTF and reformulating this main result in terms of control systems (Theorem 5). Section 3 contains basic facts on functions of bounded variation, while Section 4 introduces fundamental solutions and *a priori* estimates thereof, as well as variation-of-constant formulas, for systems of the form (1.1). The proof of Theorems 3 and 5 is given in Section 5, then Section 6 concludes with a discussion of a conjecture regarding neutral periodic delay equations.

2. Statement of the Main Result

2.1. Notations

2.1.1. The real and complex fields are denoted by \mathbb{R} and \mathbb{C} . We write $\|\cdot\|$ for Euclidean norm on \mathbb{C}^d and $\|\!\|M\!\|$ for the norm of an operator or a matrix $\mathbb{C}^d \rightarrow \mathbb{C}^d$: $\|\!\|M\!\| = \sup_{\|x\|=1} \|Mx\|$. We put I_d for the $d \times d$ identity matrix or operator on a vector space of dimension d , and I_∞ for the “doubly infinite” identity matrix $[\delta_{i,j}]_{(i,j) \in \mathbb{Z}^2}$ or the identity operator on the Hilbert space $\ell^2(\mathbb{Z}, \mathbb{C}^d)$.

2.1.2. For $E \subset \mathbb{R}$ a Lebesgue-measurable set, we write $L^q(E)$ for the space of (equivalence classes of a.e. coinciding) \mathbb{C} -valued measurable functions on E with norm $\|g\|_{L^q(E)} = (\int_E |g(y)|^q dy)^{1/q}$ (ess. sup $_E |g|$ if $q = \infty$). The space $L^q_{\text{loc}}(E)$ consists of functions whose restriction to any compact $K \subset E$ lies in $L^q(K)$. We write $C^0(E)$ for the space of \mathbb{C} -valued continuous functions on E , and if E is compact we endow it with the *sup* norm denoted by $\|\cdot\|_{C^0}$. For $\alpha \in (0, 1)$, we designate with $C^\alpha(E)$ the subspace of $C^0(E)$ consisting of Hölder continuous functions with exponent α ; *i.e.*, $f \in C^\alpha(E)$ if and only if $|f(x) - f(y)| \leq C|x - y|^\alpha$ for some constant C and all $x, y \in E$, the smallest C being the Hölder constant of f . When E is open, $C^1(E)$ indicates the space of complex functions whose first derivative lies in $C^0(E)$, and $C^{1,\alpha}(E)$ stands for functions whose first derivative belongs to $C^\alpha(E)$. When dealing with vector-valued functions, we indicate the target space as in $L^q(E, \mathbb{C}^{d \times d})$ or $C^{1,\alpha}(E, \mathbb{C}^d)$, while replacing in the definition the modulus by the corresponding norm.

2.1.3. We shall work in the Hilbert space:

$$\ell^2(\mathbb{Z}, \mathbb{C}^d) := \left\{ z = (z_j)_{j \in \mathbb{Z}} : z_j \in \mathbb{C}^d, \sum_{j=-\infty}^{+\infty} \|z_j\|^2 < +\infty \right\}, \quad (2.1)$$

equipped with the standard norm

$$\|z\|_{\ell^2} := \left(\sum_{j=-\infty}^{+\infty} \|z_j\|^2 \right)^{1/2}. \quad (2.2)$$

The norm of bounded operators $\ell^2(\mathbb{Z}, \mathbb{C}^d) \rightarrow \ell^2(\mathbb{Z}, \mathbb{C}^d)$ (often identified with doubly-infinite $d \times d$ block matrices representing them in a basis) is denoted by $\|\!\|\cdot\!\|_2$; *i.e.*, $\|\!\|L\!\|_2 = \sup_{\|z\|_{\ell^2}=1} \|Lz\|_{\ell^2}$.

2.1.4. If a function f is defined on E and $E' \subset E$, we put $f|_{E'}$ to mean the restriction of f to E' .

2.1.5. The Hardy space \mathcal{H}^2 of the right half-plane is comprised of those holomorphic functions f in $\{z \in \mathbb{C} : \Re(z) > 0\}$ satisfying

$$\|f\|_{\mathcal{H}^2}^2 := \sup_{x>0} \int_{-\infty}^{+\infty} |f(x+iy)|^2 dy < +\infty.$$

Such functions are $\sqrt{2}$ -metrically the Laplace transforms of square integrable functions on $[0, +\infty)$ [18, Ch. 8, p. 131]. That is, $\mathcal{H}^2 = \{f : f(z) = \int_0^{+\infty} e^{-zt}u(t)dt, u \in L^2([0, +\infty), \mathbb{C}), \Re(z) > 0\}$, and it holds that $\|f\|_{\mathcal{H}^2}^2 = 2\|u\|_{L^2([0, +\infty), \mathbb{C})}^2$.

2.2. Solution operators and exponential stability

Consider a periodic difference-delay system:

$$y(t) = \sum_{j=1}^N D_j(t)y(t - \tau_j), \quad t > s, \quad y(s + \theta) = \phi(\theta) \text{ for } -\tau_N \leq \theta \leq 0, \quad (2.3)$$

where $s \in \mathbb{R}$ is the initial time, d and N are positive integers, $\tau_1 < \dots < \tau_N$ are strictly positive real numbers (the delays) and the $D_j : \mathbb{R} \rightarrow \mathbb{C}^{d \times d}$ are continuous T -periodic matrix-valued functions:

$$D_j(t+T) = D_j(t), \quad 1 \leq j \leq N, \quad (2.4)$$

while $\phi : [-\tau_N, 0] \rightarrow \mathbb{C}^d$ is the initial condition and solutions to (2.3) are \mathbb{C}^d -valued functions $y(t)$ of the time $t \in [s - \tau_N, +\infty)$. When the D_j are real-valued, all results below specialize to real solutions obtained by restricting to real initial conditions. We shall assume that T is strictly larger than the delays: $\tau_N < T$, which is no loss of generality for we may replace T by kT with $k \in \mathbb{N}$.

One may seek solutions of (2.3) in various functional spaces. As the $D_j(\cdot)$ are continuous, we may for instance look for continuous solutions in which case a compatibility condition is required on the initial condition: it must belong to the space

$$C_s := \{\phi \in C^0([-\tau_N, 0], \mathbb{C}^d) : \phi(0) = \sum_{j=1}^N D_j(s)\phi(-\tau_j)\}. \quad (2.5)$$

Given $\phi \in C_s$, an easy recursion shows that system (2.3) has a unique continuous solution $y \in C^0([s - \tau_N, +\infty), \mathbb{C}^d)$ with initial condition ϕ . Thus, we can define for $t \geq s$ the *solution operator* $U(t, s) : C_s \rightarrow C_t$, mapping $\phi \in C_s$ to $U(t, s)\phi$ defined by

$$(U(t, s)\phi)(\theta) = y(t + \theta), \quad \theta \in [-\tau_N, 0]. \quad (2.6)$$

Note that $C_{s+T} = C_s$ and $U(t, s) = U(t+T, s+T)$, by the T -periodicity of the D_j .

One may also seek solutions of (2.3) in $L^q_{\text{loc}}([s - \tau_N, +\infty), \mathbb{C}^d)$ for $1 \leq q \leq \infty$, and then we require (2.3) to hold for almost every $t \in [s, +\infty)$ only; no compatibility condition on ϕ is needed anymore, and a recursive argument shows that for each $\phi \in L^q([-\tau_N, 0], \mathbb{C}^d)$ the system (2.3) admits a unique solution $y \in L^q_{\text{loc}}([s, +\infty), \mathbb{C}^d)$ with initial condition ϕ . Consequently, one can define for $t \geq s$ the solution operator $U_q(t, s) : L^q([-\tau_N, 0], \mathbb{C}^d) \rightarrow L^q([-\tau_N, 0], \mathbb{C}^d)$ that maps ϕ to $U_q(t, s)\phi$ given by a relation analogous to (2.6):

$$(U_q(t, s)\phi)(\theta) = y(t + \theta), \quad \text{a.e. } \theta \in [-\tau_N, 0]. \quad (2.7)$$

These different types of solutions *a priori* yield distinct notions of exponential stability defined as follows.

Definition 1. System (2.3) is called C^0 -exponentially stable if there exist $\gamma, K > 0$ such that

$$\|U(t, s)\phi\|_{C^0} \leq Ke^{-\gamma(t-s)}\|\phi\|_{C^0}, \text{ for all } s \in \mathbb{R}, \text{ all } t \geq s \text{ and all } \phi \in C_s. \quad (2.8)$$

System (2.3) is called L^q -exponentially stable, $q \in [1, \infty]$, if there exist $\gamma, K > 0$ such that

$$\|U_q(t, s)\phi\|_{L^q} \leq Ke^{-\gamma(t-s)}\|\phi\|_{L^q}, \text{ for all } s \in \mathbb{R}, \text{ all } t \geq s \text{ and all } \phi \in L^q([-\tau_N, 0], \mathbb{C}^d). \quad (2.9)$$

It is remarkable that these notions are in fact equivalent, as shown by the following result contained in [2, Theorem 3.4] which is also a consequence of [6, Corollary 3.29].

Proposition 1. *For each $q \in [1, \infty]$, System (2.3) is L^q -exponentially stable if and only if it is C^0 -exponentially stable.*

Proposition 1 plays an important role in the proof of Theorem 3, because the sufficiency part establishes C^0 -exponential stability while the necessity part assumes L^2 -exponential stability. In view of Proposition 1, hereafter we simplify terminology by making the following definition.

Definition 2 (Exponential stability). System (2.3) is called *exponentially stable* if and only if there exist $\gamma, K > 0$ such that one of the equivalent properties (2.8) or (2.9) holds.

2.3. The Henry-Hale Theorem

In the time-invariant case, the following characterization of exponential stability is known.

Theorem 2 (Henry-Hale [5, Section 3], [7, Theorem 3.5]). *Assume that the maps $t \mapsto D_j(t)$ are constant, $1 \leq j \leq N$. Then, a necessary and sufficient condition for System (2.3) to be C^0 -exponentially stable is the existence of a real number $\beta < 0$ such that :*

$$I_d - \sum_{j=1}^N e^{-p\tau_j} D_j \text{ is invertible in } \mathbb{C}^{d \times d} \text{ for every } p \text{ in } \{z \in \mathbb{C} : \Re(z) \geq \beta\}. \quad (2.10)$$

Our goal is to “generalize” Theorem 2 to the case where the maps $t \mapsto D_j(t)$ are *not* constant, but periodic. Let us first discard the naive attempt requiring that (2.10) holds for all t , for this is not enough to ensure stability as the following example shows.

Let $N = 1$, $d = 2$, $T = 2$, $\tau = 1$, and set $D_1(t) = \begin{pmatrix} 1/2 & a(t) \\ b(t) & 1/2 \end{pmatrix}$ in (2.3), with a a smooth function such that $a(t) \equiv 0$ if $t \in [1, 2]$, $a(t) \equiv 1$ if $t \in [\frac{1}{3}, \frac{2}{3}]$, and $a(t+2) = a(t)$ for all t , while $b(t) = a(t+1)$. On the one hand, condition (2.10) is satisfied because, since $a(t)b(t) = 0$ for all t , $D_1(t)$ always has $\frac{1}{2}$ as a double eigenvalue. On the other hand, for t in $[\frac{1}{3}, \frac{2}{3}]$, one has $y(t+2k) = (D_1(t+2)D_1(t+1))^k y(t)$ for all $k \in \mathbb{N}$, with $D_1(t+2)D_1(t+1) = \begin{pmatrix} 5/4 & 1/2 \\ 1/2 & 1/4 \end{pmatrix}$; since this matrix has one eigenvalue larger than 1 (namely $\frac{3}{4} + \frac{\sqrt{2}}{2}$), exponential stability cannot hold.

In the next section, we shall give a proper analog of Theorem 2 in the periodic case.

2.4. Main result: generalizing the Henry-Hale theorem to the time-varying periodic case.

From now on, we consider the case where the maps $t \mapsto D_j(t)$ are T -periodic. Let us define

$$\omega := 2\pi/T, \quad (2.11)$$

and put $\check{D}_j(k)$, for $j \in \{1, \dots, n\}$ and $k \in \mathbb{Z}$, to indicate the k -th Fourier coefficient of D_j :

$$\check{D}_j(k) := \frac{1}{T} \int_0^T D_j(t) e^{-ik\omega t} dt. \quad (2.12)$$

We denote by L_{D_j} the (doubly infinite) block Laurent matrix associated with $D_j(t)$; its block entries are:

$$(L_{D_j})_{k,\ell} := \check{D}_j(\ell - k), \quad \ell, k \in \mathbb{Z}. \quad (2.13)$$

We also put $\Delta_{\tau_j, \omega}$, $j \in \{1, \dots, N\}$, for the (doubly infinite) block diagonal matrix given by

$$\Delta_{\tau_j, \omega} := \text{diag}(\dots, e^{-2i\omega\tau_j} I_d, e^{-i\omega\tau_j} I_d, I_d, e^{+i\omega\tau_j} I_d, e^{+2i\omega\tau_j} I_d, \dots). \quad (2.14)$$

Both L_{D_j} and $\Delta_{\tau_j, \omega}$ define, by matrix multiplication, bounded operators $\ell^2(\mathbb{Z}, \mathbb{C}^d) \rightarrow \ell^2(\mathbb{Z}, \mathbb{C}^d)$. In fact, when $\ell^2(\mathbb{Z}, \mathbb{C}^d)$ gets identified with $L^2([0, T], \mathbb{C}^d)$ (the space of square integrable \mathbb{C}^d -valued functions on the circle of circumference T) *via* the Fourier coefficients (arranged columnwise so that indices increase from top to bottom), then $\Delta_{\tau_j, \omega}$ corresponds to the isometry $f(\xi) \mapsto f(e^{i\tau_j} \xi)$ while L_{D_j} becomes pointwise multiplication by $D_j(-t)$, whence $\|L_{D_j}\|_2 = \|D_j\|_{C^0}$. Hereafter, we often identify operators $\ell^2(\mathbb{Z}, \mathbb{C}^d) \rightarrow \ell^2(\mathbb{Z}, \mathbb{C}^d)$ and operators $L^2([0, T], \mathbb{C}^d) \rightarrow L^2([0, T], \mathbb{C}^d)$, *via* their matrix in the Fourier basis.

We define a function R of the complex variable p , valued in the space of bounded operators $\ell^2(\mathbb{Z}, \mathbb{C}^d) \rightarrow \ell^2(\mathbb{Z}, \mathbb{C}^d)$ by the formula

$$R(p) := I_\infty - \sum_{j=1}^N e^{-p\tau_j} L_{D_j} \Delta_{\tau_j, \omega}, \quad (2.15)$$

and the doubly infinite matrix representing $R(p)$ in the Fourier basis has the block description:

$$R(p) = I_\infty - \left(\sum_{j=1}^N e^{-(p-i\ell\omega)\tau_j} \check{D}_j(\ell - k) \right)_{(k,\ell) \in \mathbb{Z}^2}. \quad (2.16)$$

Our main result is now the following.

Theorem 3 (Necessary and sufficient condition for exponential stability). *Assume that the $D_j : \mathbb{R} \rightarrow \mathbb{C}^{d \times d}$ are periodic with Hölder continuous derivative for $1 \leq j \leq N$. Then, a necessary and sufficient condition for System (2.3) to be exponentially stable is the existence of a real number $\beta < 0$ such that :*

- (i) $R(p)$ is invertible $\ell^2(\mathbb{Z}, \mathbb{C}^d) \rightarrow \ell^2(\mathbb{Z}, \mathbb{C}^d)$ for each p in $\{z \in \mathbb{C} : \Re(z) \geq \beta\}$,
- (ii) there is a positive number M such that $\|R^{-1}(p)\|_2 \leq M$ for all p in $\{z \in \mathbb{C} : \Re(z) \geq \beta\}$.

When the D_j are constant matrices, the operators $L_{D_j} \Delta_{\tau_j, \omega}$ are block diagonal with k -th diagonal block $e^{ik\omega\tau_j} D_j$ and condition (i) in Theorem 3 is equivalent to saying that $I_d - \sum_{j=1}^N e^{-p\tau_j} D_j$ is invertible for $\Re(p) \geq \beta$. In this case, since a holomorphic function on a vertical strip which is uniformly almost-periodic in the pure imaginary direction and has no zeros must be bounded below in modulus by a strictly positive constant [19, Ch. III, Sec. 2, Cor. I], it holds that

$|\det(I_d - \sum_{j=1}^N e^{-p\tau_j} D_j)| \geq c > 0$ for $\Re(p) \geq \beta$, whence condition (ii) is redundant. Thus, Theorem 3 yields back the Henry-Hale theorem for systems with constant coefficients.

For periodic difference-delay systems, the authors doubt that Condition (i) implies Condition (ii) in general (though they know of no counterexample), but when all delays are commensurate it does. Indeed, $p \mapsto R(p)$ is then periodic of period $i\tau$ for some $\tau > 0$ and hence, since inversion is continuous on invertible elements of a Banach algebra while (ii) needs only be checked for p in a compact set by periodicity and a Neumann series argument, we deduce that (i) implies (ii) in this case. Thus, we obtain the following:

Corollary 4. *If the delays are commensurate, then condition (ii) can be omitted in Theorem 3.*

Remark 1. The proof of Theorem 3 will show that the sufficiency part remains true when the D_j are merely continuous, if only we strengthen assumption (ii) to:

$$\| \| R(p)^{-1} \| \|_{\mathfrak{W}} \text{ is uniformly bounded for all } p \text{ in } \{z \in \mathbb{C} : \Re(z) \geq \beta\}, \quad (2.17)$$

where the Wiener norm $\| \cdot \|_{\mathfrak{W}}$ of a doubly infinite block matrix is defined in (5.11).

2.5. A system-theoretic point of view; transfer functions and harmonic transfer functions.

Below we recast the previous considerations in system-theoretic language, introducing harmonic transfer functions and reformulating Theorem 3 as Theorem 5, which is more faithful to the version of our main result described in the introduction.

To (1.1) one can associate the control system:

$$y(t) = \sum_{j=1}^N D_j(t)y(t - \tau_j) + u(t), \quad t > s, \quad y(t) = u(t) = 0 \text{ for } t < s, \quad (2.18)$$

with control $u \in C^0([s, +\infty), \mathbb{C}^d)$ (or $u \in L^p_{loc}([s, +\infty), \mathbb{C}^d)$) and output y . System (2.18) is exponentially stable if, when driven by an input $u(t)$ vanishing for $t > t_0$ and generating an output y , the L^∞ (or L^q)-norm of the restriction $y|_{[t-\tau_N, t]}$ decays exponentially fast to zero as $t \rightarrow +\infty$.

Solving for u in (2.18), one sees that for any $t_0 > \tau_N$ and $\phi \in C_{t_0}$ (resp. $L^p([-\tau_N, 0], \mathbb{C}^d)$), there is $u \in C^0([s, t_0], \mathbb{C}^d)$ (resp. $L^p([s, t_0], \mathbb{C}^d)$) such that the corresponding output y satisfies $\phi(\theta) = y(t_0 + \theta)$ for $\theta \in [-\tau_N, 0]$. Consequently, outputs of (2.18) associated to controls that vanish for $t > t_0$ coincide with solutions to (1.1) where we put $s = t_0$. This explains why exponential stability of the control system (2.18) is equivalent to exponential stability of (1.1).

A linear time-invariant control system can be represented, under mild assumptions, as a convolution operator; *i.e.*, the output is obtained by convolving the input with some kernel [20]. Taking Fourier-Laplace transforms converts the latter into a multiplication operator in the Fourier-Laplace domain, and then the multiplier is called the *transfer function* of the system. Hereafter we specialize this to time-invariant delay difference control systems, before explaining the notion corresponding to the transfer-function in the periodic case.

In the time-invariant case, the transfer function of (2.18) is the matrix-valued function of one complex variable, say p , given by $H(p) = \left(I_d - \sum_{j=1}^N e^{-p\tau_j} D_j \right)^{-1}$ (see *e.g.* [21]). The Henry-Hale theorem says that (2.3) is exponentially stable if and only if H is holomorphic in a half-plane $\{p \in \mathbb{C}, \Re(p) > \beta\}$ for some $\beta < 0$. In this case, as explained after the statement of Theorem 3, $H(p)$ is in fact bounded for $\Re(p) \geq \beta' > \beta$.

In the periodic case, the concept of transfer function generalizes into the one of *harmonic transfer function* (in short: HTF). It is again a function of one complex variable p , but instead of ranging in $\mathbb{C}^{d \times d}$ it takes values in the space of bounded operators $\ell^2(\mathbb{Z}, \mathbb{C}^d) \rightarrow \ell^2(\mathbb{Z}, \mathbb{C}^d)$. More precisely, one can check from (2.18) that $\|y\|_{L^2([s, \tau])} \leq C e^{\gamma(\tau-s)} \|u\|_{L^2([s, +\infty])}$ for appropriate constants C , γ and all $\tau > s$; see (5.56). Thus, when $u \in L^2([s, +\infty), \mathbb{C}^d)$ one can define, for $\Re(p) > \gamma$, the Laplace transforms of $y(t)$ and $u(t)$ by the formulas

$$\hat{Y}(p) := \int_s^{+\infty} e^{-pt} y(t) dt \quad \text{and} \quad \hat{U}(p) := \int_s^{+\infty} e^{-pt} u(t) dt \quad (2.19)$$

(recall from (2.18) that u, y vanish on $(-\infty, s]$). Sampling \hat{Y}, \hat{U} at equally spaced points on vertical lines with mesh $i\omega$ (recall (2.11)), we construct infinite columns of $d \times 1$ vectors:

$$\hat{\mathbb{Y}}(p) := \begin{pmatrix} \vdots \\ \hat{Y}(p + i\omega) \\ \hat{Y}(p) \\ \hat{Y}(p - i\omega) \\ \vdots \end{pmatrix} \quad \text{and} \quad \hat{\mathbb{U}}(p) := \begin{pmatrix} \vdots \\ \hat{U}(p + i\omega) \\ \hat{U}(p) \\ \hat{U}(p - i\omega) \\ \vdots \end{pmatrix}, \quad \Re(p) > \gamma. \quad (2.20)$$

We choose to order the entries of doubly infinite vectors from top to bottom; *i.e.*, we write

$$\hat{\mathbb{Y}}(p) = \left(\hat{Y}(p - ik\omega) \right)_{k \in \mathbb{Z}}^{\mathbf{t}} \quad \text{and} \quad \hat{\mathbb{U}}(p) = \left(\hat{U}(p - ik\omega) \right)_{k \in \mathbb{Z}}^{\mathbf{t}}, \quad (2.21)$$

where the superscript \mathbf{t} means transpose.

Since $\hat{U}(p + ik\omega)/T$ (resp. $\hat{Y}(p + ik\omega)/T$) is the k -th Fourier coefficient of the T -periodic function $x \mapsto \sum_{j=-\infty}^{+\infty} e^{-p(x+jT)} u(x+jT)$ (resp. $x \mapsto \sum_{j=-\infty}^{+\infty} e^{-p(x+jT)} y(x+jT)$) that lies in $L^2([0, T])$ for $\Re(p) > \gamma$, both $\hat{\mathbb{Y}}$ and $\hat{\mathbb{U}}$ lie in $\ell^2(\mathbb{Z}, \mathbb{C}^d)$. If moreover $\Re(p) > \log(\sum_{j=1}^N \|D_j\|_{C^0})/\tau_1$, there is an operator-valued holomorphic function $p \mapsto \mathbf{H}(p)$ such that $\mathbf{H}(p) : \ell^2(\mathbb{Z}, \mathbb{C}^d) \rightarrow \ell^2(\mathbb{Z}, \mathbb{C}^d)$ maps $\hat{\mathbb{Y}}$ to $\hat{\mathbb{U}}$. Indeed, $\mathbf{H}(p)$ is none but the inverse of $R(p)$ defined in (2.15), computed *via* a Neumann series; *cf.* Section 5.2, Equation (5.58). Altogether, increasing γ if necessary, we get that

$$\hat{\mathbb{Y}}(p) = \mathbf{H}(p) \hat{\mathbb{U}}(p), \quad \Re(p) > \gamma. \quad (2.22)$$

Definition 3 (Harmonic Transfer Function). The operator-valued holomorphic function $\mathbf{H}(\cdot)$ is called the harmonic transfer function (HTF) of system (2.18).

We can now restate Theorem 3 as follows.

Theorem 5 (Reformulation of Theorem 3). *If system (2.3) has periodic coefficients of class $C^{1,\alpha}$, it is exponentially stable if and only if the control system (2.18) is in turn exponentially stable, and that is if and only if the harmonic transfer function of the latter is analytic and bounded in $\{p \in \mathbb{C} : \Re(p) > \beta\}$ for some $\beta < 0$.*

PROOF. Equivalence between the exponential stability of (2.3) and (2.18) was explained after Equation (2.18). The remaining assertion is formally equivalent to Theorem 3, granted the definition of the HTF.

A couple of remarks are in order. First, the harmonic transfer function can be defined in the same manner for more general linear periodic systems than difference-delay ones, but we shall stick to the present setting. Second, Equation (2.22) connects inputs and outputs of a linear periodic control system in the frequency domain *via* the HTF, and if the system is time-invariant, then the HTF reduces to the block diagonal matrix $\mathbf{H}(p) = \text{diag}\{\dots, H(p+i\omega), H(p), H(p-i\omega), \dots\}$ where H is the ordinary transfer function, so that (2.22) is equivalent to the well-known input-output relation $\hat{Y}(p) = H(p)\hat{U}(p)$ for time-invariant systems. Third, the HTF further generalizes the ordinary transfer function in a way which is worth explaining: if a stable, *time-invariant* control system is fed with a periodic input signal $e^{i\nu t}v$ with $v \in \mathbb{C}^d$, then the output is asymptotically $H(i\nu)v e^{i\nu t}$, where H is the transfer function. If a stable *periodic* control system is fed with a periodic input signal $v e^{i\nu t}$, then the output is asymptotically $\mathbf{H}(i\nu)(\mathbf{v})e^{i\nu t}$, where \mathbf{H} is the harmonic transfer function and \mathbf{v} the constant function with value v in $L^2([0, T], \mathbb{C}^d)$ (so that $\mathbf{H}(i\nu)(\mathbf{v})$ is again an element of $L^2([0, T], \mathbb{C}^d)$). Thus, while in the time-invariant case a periodic input is asymptotically mapped to a periodic output with the same period, it gets asymptotically mapped in the periodic case to an oscillating signal with the same period but carried by the wave $\mathbf{H}(i\nu)(\mathbf{v})$ which has the period of the system. This attractive interpretation follows from (5.74), (5.61), (5.60) and (5.52).

3. Functions with bounded variation and Lebesgue-Stieltjes integrals

Hereafter we recall basic facts regarding functions of bounded variation and Stieltjes integrals, setting up some notation regarding bounds of integration that will be of use throughout.

For I a real interval and $f : I \rightarrow \mathbb{R}$ a function, the *total variation* of f on I is defined as

$$W_I(f) := \sup_{\substack{x_0 < x_1 < \dots < x_N \\ x_i \in I, N \in \mathbb{N}}} \sum_{i=1}^N |f(x_i) - f(x_{i-1})| < \infty. \quad (3.1)$$

The space $BV(I)$ of functions with *bounded variation* on I consists of those f such that $W_I(f) < \infty$, endowed with the norm $\|f\|_{BV(I)} = W_I(f) + |f(d)|$ where $d \in I$ is arbitrary but fixed. Different d give rise to equivalent norms for which $BV(I)$ is a Banach space, and $\|\cdot\|_{BV(I)}$ is stronger than the uniform norm. We let $BV_r(I)$ and $BV_l(I)$ be the closed subspaces of $BV(I)$ comprised of right and left continuous functions, respectively. We write $BV_{loc}(\mathbb{R})$ for the space of functions whose restriction to any bounded interval $I \subset \mathbb{R}$ lies in $BV(I)$. Observe that

$$W_I(fg) \leq W_I(f) \sup_{x \in I} |g(x)| + W_I(g) \sup_{x \in I} |f(x)|. \quad (3.2)$$

Each $f \in BV(I)$ has a limit $f(x^-)$ (resp. $f(x^+)$) from the left (resp. right) at every $x \in I$ where the limit applies [22, sec. 1.4]. Hence, one can associate to f a finite signed Borel measure ν_f on I such that $\nu_f((a, b)) = f(b^-) - f(a^+)$, and if I is bounded on the right (resp. left) and contains its endpoint b (resp. a), then $\nu_f(\{b\}) = f(b) - f(b^-)$ (resp. $f(a^+) - f(a)$) [22, ch. 7, pp. 185–189]. Note that different f may generate the same ν_f : for example if f and f_1 coincide except at isolated interior points of I , then $\nu_f = \nu_{f_1}$. For $g : I \rightarrow \mathbb{R}$ a measurable function, summable against ν_f , the Lebesgue-Stieltjes integral $\int gdf$ is defined as $\int g d\nu_f$, whence the differential element df identifies with $d\nu_f$ [22, ch. 7, pp. 190–191]. This type of integral is useful to integrate by parts, but caution must be used when integrating a function with respect to df over a subinterval $J \subset I$ because $\nu_{(f|_J)}$ needs *not* coincide with the restriction $(\nu_f)|_J$ of ν_f to J . More precisely, if the lower bound a (resp. the upper bound b) of J belongs to J and lies interior to I , then the two measures may

differ by the weight they put on $\{a\}$ (resp. $\{b\}$), and they agree only when f is left (resp. right) continuous at a (resp. b). By $\int_J gdf$, we always mean that we integrate g against $\nu_{(f,J)}$ and *not* against $(\nu_f)|_J$. We often trade the notation $\int_J gdf$ for one of the form $\int_{a^\pm}^{b^\pm} gdf$, where the interval of integration J is encoded in the bounds put on the integral sign: a lower bound a^- (resp. a^+) means that J contains (resp. does not contain) its lower bound a , while an upper bound b^+ (resp. b^-) means that J contains (resp. does not contain) its upper bound b . Then, the previous word of caution applies to additive rules: for example, when splitting $\int_{a^\pm}^{b^\pm} gdf$ into $\int_{a^\pm}^{c^\pm} gdf + \int_{c^\pm}^{b^\pm} gdf$ where $c \in (a, b)$, we must use c^+ (resp. c^-) if f is right (resp. left) continuous at c .

To a finite, signed or complex Borel measure μ on I , one can associate its *total variation measure* $|\mu|$, defined on a Borel set $B \subset I$ by $|\mu|(B) = \sup_{\mathcal{P}} \sum_{E \in \mathcal{P}} |\mu(E)|$ where \mathcal{P} ranges over all partitions of B into Borel sets, see [23, sec. 6.1]; its total mass $|\mu|(I)$ is called the total variation of μ , denoted as $\|\mu\|$. Thus, the total variation is defined both for functions of bounded variation and for measures, with different meanings. When $f \in BV(I)$ is monotonic then $W_I(f) = \|\nu_f\|$, but in general it only holds that $\|\nu_f\| \leq 2W_I(f)$; this follows from the Jordan decomposition of f as a difference of two increasing functions, each of which has variation at most $W_I(f)$ on I [22, Thm. 1.4.1]. In any case, it holds that $|\int gdf| \leq \int |g|d|\nu_f| \leq 2W_I(f) \sup_I |g|$.

The previous notation and definitions also apply to vector and matrix-valued BV -functions, replacing absolute values in (3.1) by Euclidean and operator norms, respectively.

4. Variation-of-constants formulas and a priori estimates

Below, we introduce fundamental solutions and variation-of-constants formulas for system (1.1), before deriving estimates thereof; this material will be of much use later in the paper, and we could not see it proven in the literature.

4.1. Fundamental solution and variation-of-constants formula for continuous solutions.

The *fundamental solution* $X : \mathbb{R}^2 \rightarrow \mathbb{C}^{d \times d}$ of (2.3) is defined by the following equation [7]:

$$X(t, s) = \begin{cases} 0 & \text{for } t < s, \\ I_d + \sum_{j=1}^N D_j(t)X(t - \tau_j, s) & \text{for } t \geq s. \end{cases} \quad (4.1)$$

Arguing inductively, it is easy to check that X uniquely exists. Moreover, it does not grow faster than exponential with respect to $t - s$, as the following proposition shows.

Proposition 6. *There exist $K > 0$ and $\lambda \in \mathbb{R}$ such that*

$$\|X(t, s)\| \leq Ke^{\lambda(t-s)}, \text{ for all } t \geq s. \quad (4.2)$$

PROOF. Since the maps D_i are continuous and periodic, there is a $K' > 0$ such that $\|D_i(t)\| \leq K'$ for $i \in \{1, \dots, N\}$ and $t \in \mathbb{R}$. Pick $K \geq 2$ and λ large enough that

$$K'Ne^{-\lambda\tau_1} < 1/2. \quad (4.3)$$

Let us prove by induction that, for any $k \in \mathbb{N}$, Equation (4.2) holds if $t - s < k\tau_1$; this claim clearly implies what we want. Now, this is obvious for $k = 0$, because $t - s < 0$ whence $X(t, s) = 0$ in this case. Next, assuming that the claim holds for a certain $k \geq 0$ and considering t, s such that $k\tau_1 \leq t - s < (k + 1)\tau_1$, we get from (4.1) that

$$\|X(t, s)\| \leq 1 + K'NK'e^{-\lambda\tau_1}e^{\lambda(t-s)},$$

which implies (4.2) in view of (4.3) and the inequality $1 \leq \frac{1}{2}Ke^{\lambda(t-s)}$. This concludes the induction. \square

By inspection, X is as smooth as the maps $D_j(\cdot)$. In particular, it is continuous about each (t, s) such that $t - s \notin \mathcal{F}$, where \mathcal{F} is the positive lattice generated by the τ_ℓ in \mathbb{R} :

$$\mathcal{F} := \left\{ \sum_{\ell=1}^N n_\ell \tau_\ell, (n_1, \dots, n_N) \in \mathbb{N}^N \right\}. \quad (4.4)$$

Clearly, X has a bounded jump across each line $t - s = \mathfrak{f}$ for $\mathfrak{f} \in \mathcal{F}$; in fact, a moment's thinking will convince the reader that it is of the form

$$X(t, s) = - \sum_{\mathfrak{f} \in [0, t-s] \cap \mathcal{F}} \mathfrak{C}_\mathfrak{f}(t), \quad s \leq t, \quad (4.5)$$

where each $\mathfrak{C}_\mathfrak{f}(\cdot)$ is differentiable with Hölder continuous derivative of the same exponent as the maps $D_j(\cdot)$. One can see also that $\mathfrak{C}_\mathfrak{f}(t)$ is a finite sum of products of matrix-valued functions of the form $D_j(t - \mathfrak{f}')$, where \mathfrak{f}' ranges over the elements of \mathcal{F} whose defining integers n_ℓ in (4.4) do not exceed those defining \mathfrak{f} , the empty product being the identity matrix. A precise expression for $\mathfrak{C}_\mathfrak{f}$ can be obtained by reasoning as in [6, Sec. 3.2] or [2, Sec. 4.5], but we will not need it. Note that, for fixed t , the function $X(t, \cdot)$ is locally of bounded variation on \mathbb{R} . More precisely, observe from (4.1) that $\alpha \mapsto X(t, \alpha)$ lies in $BV_{loc}(\mathbb{R})$ for all t , and that for $b > s$ we have:

$$d_\alpha X(t, \alpha) = \begin{cases} \sum_{j=1}^N D_j(t) d_\alpha X(t - \tau_j, \alpha) & \text{if } t \geq b \\ -I_d \delta_t + \sum_{j=1}^N D_j(t) d_\alpha X(t - \tau_j, \alpha) & \text{if } s \leq t < b \end{cases} \quad \text{on } [s, b], \quad (4.6)$$

where δ_t is the Dirac mass at t . The fundamental solution is quite important because it yields explicit integral formulas to express solutions of (2.3). For instance, Equation (4.7) below allows one to parametrize continuous solutions, and is similar in spirit to variation-of-constants formulas for autonomous time-invariant linear difference-delay systems given, say in [5] or [24, 7]. Proving it here would make the paper unbalanced; however, we provide an argument in the separate note [25]. We do so because, to the difference of [5, 24, 7], we deal with time-varying matrices $D_j(\cdot)$, and also because many such formulas in the literature seem to have issues. The integral $\int_{s^-}^{(s+\tau_j)^-}$ in Equation (4.7) must be understood as a Lebesgue-Stieltjes integral on the interval $[s, s + \tau_j]$; cf. Section 3. This type of integral is especially adequate in the present setting of jump singularities, as it cleanly accounts for excluding or including endpoints of intervals that may, or may not be charged by the integrand; everything is well defined here because $X(t, \cdot)$ is in $BV_{loc}(\mathbb{R})$.

Proposition 7. ([25]) For $s \in \mathbb{R}$ and ϕ in C_s the solution $y \in C^0([s - \tau_N, +\infty), \mathbb{C}^d)$ to (2.3) is

$$y(t) = - \sum_{j=1}^N \int_{s^-}^{(s+\tau_j)^-} d_\alpha X(t, \alpha) D_j(\alpha) \phi(\alpha - \tau_j - s), \quad t \geq s, \quad (4.7)$$

where X was defined in (4.1).

4.2. Variation-of-constants formula for L^2_{loc} solutions.

Proving necessity in Theorem 3 requires that we deal with $L^2([s - \tau_N, s])$ -initial data as well as $L^2_{loc}([s, \infty))$ -solutions for (2.3), and then the variation-of-constants formula features vector-valued integration. Specifically, let us consider the control system with input u :

$$y(t) = \sum_{j=1}^N D_j(t)y(t - \tau_j) + u(t), \quad u(t) \in \mathbb{C}^d, \quad t \geq s. \quad (4.8)$$

When $t \geq s$, we set $y_t(\theta) = y(t + \theta)$ for $-\tau_N \leq \theta \leq 0$. The function $y_t : [-\tau_N, 0] \rightarrow \mathbb{C}^d$ (more accurately: its equivalence class modulo coincidence almost everywhere) is the state at time t of the input-output system (4.8). In particular y_s is now the initial condition, previously denoted by ϕ when dealing with the homogeneous system (2.3) (for which $u \equiv 0$).

Recalling X from (4.1), we define for each $(t, \alpha) \in \mathbb{R}^2$ a map $K(t, \alpha) : [-\tau_N, 0] \rightarrow \mathbb{C}^{d \times d}$ by

$$K(t, \alpha)(\theta) = -X(t + \theta, \alpha), \quad \theta \in [-\tau_N, 0]. \quad (4.9)$$

For $t \in \mathbb{R}$, $\theta \in [-\tau_N, 0]$ and $I \subset \mathbb{R}$ an interval bounded on the left, the map $\alpha \mapsto X(t + \theta, \alpha)$ lies in $BV_I(I)$ and vanishes for $\alpha > t + \theta$. As indicated in Section 3, we associate to this map a $\mathbb{C}^{d \times d}$ -valued Borel measure $\nu_{(X(t+\theta, \cdot))|_I}$ on I . It follows from (4.1), (4.5) and (4.6) that for fixed t and θ , this measure is of the form

$$\nu_{(X(t+\theta, \cdot))|_I} = \sum_{\mathfrak{f} \in \mathcal{F}, t+\theta-\mathfrak{f} \in \tilde{I}} \mathfrak{C}_{\mathfrak{f}}(t + \theta) \delta_{t+\theta-\mathfrak{f}}, \quad (4.10)$$

where \tilde{I} denotes I deprived from its right endpoint (if contained in I , otherwise $\tilde{I} = I$) and $\delta_{t+\theta-\mathfrak{f}}$ is a Dirac mass at $t + \theta - \mathfrak{f}$. For fixed t , the boundedness of I from the left implies that the number of terms in (4.10) is majorized independently of $\theta \in [-\tau_N, 0]$, and clearly the map sending a Borel set $E \subset I$ to the function $\theta \mapsto -\nu_{(X(t+\theta, \cdot))|_I}(E)$ is a vector measure, valued in the space $\mathcal{B}([-\tau_N, 0], \mathbb{C}^{d \times d})$ of bounded measurable $\mathbb{C}^{d \times d}$ -valued functions on $[-\tau_N, 0]$ endowed with the sup norm. In view of (4.9) we denote this measure with $d_\alpha K(t, \alpha)$, and though it depends on I the latter will be understood from the context. Then, for $g \in C^0(I, \mathbb{C}^d)$, one can define the integral $\int_I dK(t, \alpha)g(\alpha)$ as a member of $\mathcal{B}([-\tau_N, 0], \mathbb{C}^d)$; see, *e.g.* [26] for a definition of integrals against a vector measure. To us, it is just a compact notation for the function

$$\left(\int_I dK(t, \alpha)g(\alpha) \right) (\theta) := - \int_I d_\alpha X(t + \theta, \alpha)g(\alpha) = - \sum_{\mathfrak{f} \in \mathcal{F}, t+\theta-\mathfrak{f} \in \tilde{I}} \mathfrak{C}_{\mathfrak{f}}(t + \theta)g(t + \theta - \mathfrak{f}). \quad (4.11)$$

The rightmost term in (4.11) can be rewritten as $\sum_{\mathfrak{f} \in \mathcal{F}} \mathfrak{C}_{\mathfrak{f}}(t + \theta)\tilde{g}(t + \theta - \mathfrak{f})$ where \tilde{g} is equal to g on \tilde{I} and to zero elsewhere. Since the $\mathfrak{C}_{\mathfrak{f}}$ are bounded (being periodic in $C^{1, \delta}(\mathbb{R}, \mathbb{C}^{d \times d})$) while those $\mathfrak{f} \in \mathcal{F}$ for which $t + \theta - \mathfrak{f} \in \tilde{I}$ for some $\theta \in [-\tau_N, 0]$ are finite in number because I is bounded on the left, one can check that $\|\int_I dK(t, \alpha)g(\alpha)\|_{L^2(I, \mathbb{C}^d)} \leq C\|g\|_{L^2(I, \mathbb{C}^d)}$ for some constant C depending on t , I , the D_j and the τ_j . Hence, (4.11) makes good sense for $g \in L^2(I, \mathbb{C}^d)$, but of course $(\int_I dK(t, \alpha)g(\alpha))(\theta)$ is only defined for a.e. $\theta \in [-\tau_N, 0]$ in this case. As indicated in Section 3, the interval of integration will be encoded in the bounds put on the integral sign. Note that it is immaterial here whether I contains its right endpoint or not when writing $\int_I dK(t, \alpha)g(\alpha)$, for the rightmost term in (4.11) depends only on \tilde{I} . Moreover, $\int_I dK(t, \alpha)g(\alpha)$ is independent of

$g(\alpha)$ for $\alpha > t$ as $K(t, \alpha) \equiv 0$ when $\alpha > t$. Hence, if $g \in L^2_{loc}([a, +\infty))$ then $\int_{a^-}^{b^\pm} dK(t, \alpha)g(\alpha)$ is independent of $b > t$ and of the choice of sign in b^\pm . In this case, we find it convenient to define

$$\int_{a^-}^{+\infty} dK(t, \alpha)g(\alpha) := \int_{a^-}^{b^\pm} dK(t, \alpha)g(\alpha) \quad \text{for any } b > t. \quad (4.12)$$

The variation-of-constants formula for L^2_{loc} -solutions to (4.8) now goes as follows, with $U_2(\cdot, \cdot)$ the solution operator defined in (2.7).

Proposition 8. *For $s \in \mathbb{R}$, let $y(\cdot) \in L^2_{loc}([s - \tau_N, +\infty), \mathbb{C}^d)$ and $u(\cdot) \in L^2_{loc}([s, +\infty), \mathbb{C}^d)$ satisfy (4.8) for a.e. $t \geq s$. Then:*

$$y_t = U_2(t, s)y_s + \int_{s^-}^{+\infty} d_\alpha K(t, \alpha)u(\alpha), \quad t \geq s. \quad (4.13)$$

PROOF. Denote by η_t the right hand side of (4.13), and put $\eta(t + \theta) = \eta_t(\theta)$ for $t \geq s$ with $\theta \in [-\tau_N, 0]$. Write \tilde{y} for the solution of the homogeneous system (2.3) with initial condition y_s , so that $\tilde{y}_t = U_2(t, s)y_s$. Fix $t \geq s$; for a.e. θ in $[-\tau_N, 0]$ satisfying $t + \theta < s$ (such θ occur when $t < s + \tau_N$), it holds that $(\int_{s^-}^{+\infty} d_\alpha K(t, \alpha)u(\alpha))(\theta) = \int_{s^-}^{+\infty} d_\alpha X(t + \theta, \alpha)u(\alpha) = 0$ (because $X(t + \theta, \alpha) = 0$ when $t + \theta < s \leq \alpha$) and that $\tilde{y}_t(\theta) = \tilde{y}(t + \theta) = y_s(t + \theta) = y(t + \theta)$ (because \tilde{y} and y have the same initial condition y_s on $[-\tau_N, 0]$). Hence, we obtain indeed that $\eta_t(\theta) = y_t(\theta)$ for a.e. θ in $[-\tau_N, 0]$ such that $t + \theta < s$. Next, for a.e. $\theta \in [-\tau_N, 0]$ such that $t + \theta \geq s$, we deduce from the definition of η_t , (2.7) (with $q = 2$), (2.3) (where y is set to \tilde{y}), (4.12) and (4.6) that

$$\begin{aligned} \eta_t(\theta) &= \tilde{y}(t + \theta) - \int_{s^-}^{+\infty} d_\alpha X(t + \theta, \alpha)u(\alpha) \\ &= \sum_{j=1}^N D_i(t + \theta)\tilde{y}(t + \theta - \tau_j) - \sum_{j=1}^N D_i(t + \theta) \int_{s^-}^{+\infty} d_\alpha X(t + \theta - \tau_j, \alpha)u(\alpha) + u(t + \theta) \\ &= \sum_{j=1}^N D_j(t + \theta)\eta(t + \theta - \tau_j) + u(t + \theta). \end{aligned}$$

Altogether, we proved that to each $t \geq 0$ there is a set $E_t \subset [-\tau_N, 0]$ of zero measure such that, for $\theta \in [-\tau_N, 0] \setminus E_t$, we have $\eta(t + \theta) = \sum_{j=1}^N D_j(t + \theta)\eta(t + \theta - \tau_j) + u(t + \theta)$ when $t + \theta \geq s$ and $\eta(t + \theta) = y_s(t + \theta)$ when $t + \theta < s$. Thus, letting \mathbb{Q} denote the rational numbers, the set $E := \cup_{q \in \mathbb{Q}} \{q + E_q\}$ has measure zero and since each $t \in \mathbb{R}$ can be written as $q + \theta$ with $q \in \mathbb{Q}$ and $\theta \in [-\tau_N, 0]$, we deduce that $\eta(t)$ satisfies (4.8) when $t \geq s$ and $t \notin E$, with initial condition $\eta = y_s$ a.e. on $[-\tau_N, 0]$. Hence, by uniqueness of a solution with given initial condition, $\eta = y$ a.e. \square

Let us now fix t and consider the function $s \mapsto X(t, t - s)$ from \mathbb{R} to $\mathbb{C}^{d \times d}$. It is right-continuous and lies in $BV_{loc}(\mathbb{R}, \mathbb{C}^{d \times d})$, moreover it is identically zero for $s < 0$. Thus, on each interval $I \subset \mathbb{R}$ which is bounded on the right, it generates a $\mathbb{C}^{d \times d}$ -valued measure $\nu_{(X(t, t - \cdot))|_I}$ which is but the image of $d_s X(t, s)$ under the map $s \mapsto t - s$ from $t - I$ onto I (where $t - I$ indicates the set of time instants of the form $t - \tau$ for $\tau \in I$). When I is understood, we denote this measure by $d_s X(t, t - s)$ and we get from (4.10) that

$$d_s X(t, t - s) = - \sum_{\mathfrak{f} \in \mathcal{F} \cap \bar{I}} \mathfrak{e}_{\mathfrak{f}}(t) \delta_{\mathfrak{f}}, \quad (4.14)$$

where \check{I} denotes I deprived from its left endpoint (if contained in I) and the \mathfrak{C}_f are as in (4.10). By periodicity of the D_j , one sees that $X(t, t-s)$ is periodic in t and therefore the $\mathfrak{C}_f(t)$ are periodic as well. If the D_j belong to $C^{1,\delta}(\mathbb{R}, \mathbb{C}^{d \times d})$ then so do the \mathfrak{C}_f , and we may define the measure

$$\frac{\partial}{\partial t} d_s X(t, t-s) := - \sum_{f \in \mathcal{F} \cap \check{I}} \left(\frac{\partial}{\partial t} \mathfrak{C}_f(t) \right) \delta_f \quad (4.15)$$

with coefficients in $C^\delta(\mathbb{R}, \mathbb{C}^{d \times d})$. The number of terms in the right hand sides of (4.14), (4.15), and the number of sums of products of the $D_j(t-f')$ involved in these terms, tend to $+\infty$ with the length of I . Nevertheless, in the next section we show that the growth of these quantities is at most exponential with that length.

4.3. A priori estimates

For $I = [0, \tau]$, we shall need basic *a priori* estimates, independent of t , for the quantities:

$$\|d_s X(t, t-s)\|_I := \sum_{f \in \mathcal{F} \cap \check{I}} \|\mathfrak{C}_f(t)\|, \quad (4.16)$$

$$\left\| \frac{\partial}{\partial t} d_s X(t, t-s) \right\|_I := \sum_{f \in \mathcal{F} \cap \check{I}} \left\| \frac{\partial}{\partial t} \mathfrak{C}_f(t) \right\|, \quad (4.17)$$

$$\Lambda_{I,\delta} \left(\frac{\partial}{\partial t} d_s X(t, t-s) \right) := \sum_{f \in \mathcal{F} \cap \check{I}} \Lambda_\delta \left(\frac{\partial}{\partial t} \mathfrak{C}_f(t) \right), \quad (4.18)$$

where $\Lambda_\delta(g)$ indicates the Hölder constant of $g \in C^\delta(\mathbb{R}, \mathbb{C}^{d \times d})$. Note that (4.16) and (4.17) are just the total variations of the measures $d_s X(t, t-s)$ and $\frac{\partial}{\partial t} d_s X(t, t-s)$ on I , respectively.

Proposition 9. *Assume that $D_j \in C^{1,\delta}(\mathbb{R}, \mathbb{C}^{d \times d})$ is T -periodic for $1 \leq j \leq N$ and some $\delta \in (0, 1)$. Then, there exist $K, \gamma \geq 0$ such that, for all $\tau \geq 0$ and $t \in \mathbb{R}$:*

$$\max \left\{ \|d_s X(t, t-s)\|_{[0,\tau]}, \left\| \frac{\partial}{\partial t} d_s X(t, t-s) \right\|_{[0,\tau]}, \Lambda_{[0,\tau],\delta} \left(\frac{\partial}{\partial t} d_s X(t, t-s) \right) \right\} \leq K e^{\gamma\tau}. \quad (4.19)$$

PROOF. Observe that $d_s X(t, t-s)$ is the image of the measure $d_s X(t, s)$ under the map $s \mapsto t-s$ from $[t-\tau, t]$ onto $[0, \tau]$. Hence, it follows from (4.6) that

$$\begin{aligned} d_s X(t, t-s) &= \sum_{j=1}^N D_j(t) d_s X(t-\tau_j, t-s) \\ &= \sum_{j=1}^N D_j(t) d_s X(t-\tau_j, t-\tau_j-(s-\tau_j)) \quad \text{on } [0, \tau], \end{aligned}$$

as well as

$$\begin{aligned} \frac{\partial}{\partial t} d_s X(t, t-s) &= \sum_{j=1}^N \frac{\partial}{\partial t} D_j(t) d_s X(t-\tau_j, t-s) + \sum_{j=1}^N D_j(t) \frac{\partial}{\partial t} d_s X(t-\tau_j, t-s) \\ &= \sum_{j=1}^N \frac{\partial}{\partial t} D_j(t) d_s X(t-\tau_j, t-\tau_j-(s-\tau_j)) + \sum_{j=1}^N D_j(t) \frac{\partial}{\partial t} d_s X(t-\tau_j, t-\tau_j-(s-\tau_j)). \end{aligned}$$

Since $X(t - \tau_j, t - \tau_j - (s - \tau_j))$ is identically zero for $s < \tau_j$, we deduce from the previous identities that if K' is an upper bound (made independent of t by periodicity) for the $\|D_j\|$, the $\|\frac{\partial}{\partial t} D_j\|$, the $\Lambda_\delta(D_j)$ and the $\Lambda_\delta(\frac{\partial}{\partial t} D_j)$, then

$$\|d_s X(t, t - s)\|_{[0, \tau]} \leq K' \sum_{j=1}^N \|d_\beta X(t - \tau_j, t - \tau_j - \beta)\|_{[0, \tau - \tau_j]}, \quad (4.20)$$

$$\|\frac{\partial}{\partial t} d_s X(t, t - s)\|_{[0, \tau]} \leq K' \sum_{j=1}^N \|d_\beta X(t - \tau_j, t - \tau_j - \beta)\|_{[0, \tau - \tau_j]} + \|\frac{\partial}{\partial t} d_\beta X(t - \tau_j, t - \tau_j - \beta)\|_{[0, \tau - \tau_j]} \quad (4.21)$$

$$\begin{aligned} \Lambda_{[0, \tau], \delta}(\frac{\partial}{\partial t} d_s X(t, t - s)) &\leq K' \sum_{j=1}^N \Lambda_{[0, \tau - \tau_j], \delta}(X(t - \tau_j, t - \tau_j - \beta)) + \|d_\beta X(t - \tau_j, t - \tau_j - \beta)\|_{[0, \tau - \tau_j]} \\ &\quad + \Lambda_{[0, \tau - \tau_j], \delta}(\frac{\partial}{\partial t} X(t - \tau_j, t - \tau_j - \beta)) + \|\frac{\partial}{\partial t} d_\beta X(t - \tau_j, t - \tau_j - \beta)\|_{[0, \tau - \tau_j]}, \end{aligned} \quad (4.22)$$

where we used that $\Lambda_\delta(AB) \leq \Lambda_\delta(A)\|B\| + \Lambda_\delta(B)\|A\|$. The proof may now be completed by an inductive step, similar to the one used to establish (4.2). More precisely, pick γ , K'' positive large enough that $K'N e^{-\gamma\tau_1} < 1/4$ and

$$\max\{\|\mathfrak{C}_{\tau_1}(t)\|, \left\| \frac{\partial}{\partial t} \mathfrak{C}_{\tau_1}(t) \right\|, \Lambda_\delta(\mathfrak{C}_{\tau_1}), \Lambda_\delta(\frac{\partial}{\partial t} \mathfrak{C}_{\tau_1})\} \leq K'', \quad t \in \mathbb{R}. \quad (4.23)$$

Note that such a K'' indeed exists, as \mathfrak{C}_{τ_1} is periodic and lies in $C^{1, \delta}(\mathbb{R})$. From (4.14) one sees that $\|d_s X(t, t - s)\|_{[0, \tau]}$ is equal to 0 when $\tau \in [0, \tau_1)$ and to $\|\mathfrak{C}_{\tau_1}(t)\|$ when $\tau = \tau_1$. *A fortiori* then, for $\tau \in [0, \tau_1]$ we have:

$$\max\left\{\|d_s X(t, t - s)\|_{[0, \tau]}, \left\| \frac{\partial}{\partial t} d_s X(t, t - s) \right\|_{[0, \tau]}\right\} \leq K'' e^{\gamma\tau}, \quad \Lambda_{[0, \tau], \delta}(\frac{\partial}{\partial t} d_s X(t, t - s)) \leq 2K'K'' e^{\gamma\tau},$$

so that (4.19) holds with $K := \max\{K'', 2K'K''\}$ for $0 \leq \tau \leq \tau_1$. Now, assume that (4.19) is true for all t and $\tau \in [0, k\tau_1)$, with $k > 0$ an integer. For $\tau \in (k\tau_1, (k+1)\tau_1]$, we get in view of this hypothesis and the definition of γ , that

$$\begin{aligned} \|d_s X(t, t - s)\|_{[0, \tau]} &\leq K'NK e^{\gamma(\tau - \tau_1)} \leq K e^{\gamma\tau}/4 \quad \text{by (4.20),} \\ \|\frac{\partial}{\partial t} d_s X(t, t - s)\|_{[0, \tau]} &\leq 2K'NK e^{\gamma(\tau - \tau_1)} \leq K e^{\gamma\tau}/2 \quad \text{by (4.21),} \\ \Lambda_{[0, \tau], \delta}(\frac{\partial}{\partial t} d_s X(t, t - s)) &\leq 4K'NK e^{\gamma(\tau - \tau_1)} \leq K e^{\gamma\tau} \quad \text{by (4.22).} \end{aligned}$$

By induction on k , this completes the proof. \square

5. Proof of Theorem 3

5.1. Sufficiency

The proof of sufficiency can be modeled after the one in [5] for the time-invariant case, but new technicalities arise because applying Laplace transformation to linear relations with time-varying coefficients yields functional rather than algebraic equations. Assuming that both (i), (ii) hold in Theorem 3 we shall prove exponential stability of System (2.3) in three steps.

Step 1

By assumption the D_j have Hölder continuous derivatives, so their Fourier coefficients (2.12) satisfy

$$\| \check{D}_j(k) \| \leq \frac{C}{1 + |k|^{1+\delta}}, \quad j \in \{1, \dots, n\}, \quad (5.1)$$

where C is a positive constant and $\delta \in]0, 1[$ is the Hölder exponent of the derivative; indeed, (5.1) follows at once from the fact that the modulus of the k -th Fourier coefficient of $\frac{d}{dt}D_j$ is bounded by $C'/(1 + |k|)^\delta$, see [27, Ch. 2, thm. 4.7]. Note, for later use, that the constant C in (5.1) depends affinely on the Hölder constant of $\frac{d}{dt}D_j$. Substituting the Fourier expansion of D_j in (4.1) yields:

$$X(t, s) = I_d + \sum_{k \in \mathbb{Z}} \sum_{j=1}^N e^{ik\omega t} \check{D}_j(k) X(t - \tau_j, s) \quad \text{if } t \geq s, \quad X(t, s) = 0 \quad \text{if } t < s, \quad (5.2)$$

where the right hand side of (5.2) is absolutely convergent, locally uniformly in (t, s) , thanks to (5.1) and the local boundedness of X . Hereafter, we shall denote with a hat the Laplace transform with respect to the first variable; *e.g.*, we put

$$\hat{X}(p, s) := \int_{-\infty}^{+\infty} e^{-pt} X(t, s) dt. \quad (5.3)$$

Taking the Laplace transform of both sides of (5.2) and interchanging the series and integral signs (this is possible thanks to (5.1)), we obtain that for $p \in \mathbb{C}$ with $\Re(p) > \lambda$ where λ is as in (4.2):

$$\int_{-\infty}^{+\infty} e^{-pt} X(t, s) dt = \int_{-\infty}^{+\infty} \mathbf{1}_{[s, +\infty)}(t) e^{-pt} I_d dt + \sum_{k \in \mathbb{Z}} \sum_{j=1}^N \int_{-\infty}^{+\infty} e^{(-p+ik\omega)t} \check{D}_j(k) X(t - \tau_j, s) dt, \quad (5.4)$$

where $\mathbf{1}_{[s, +\infty)}$ is the characteristic function equal to 1 on $[s, +\infty)$ and to 0 elsewhere; note that the factor $X(t - \tau_j, s)$ in the integrand on the right hand side of (5.4) is zero for $t < s$. By an elementary change of variable, we deduce from (5.4) that

$$\hat{X}(p, s) = \frac{e^{-ps}}{p} I_d + \sum_{k \in \mathbb{Z}} \sum_{j=1}^N e^{(-p+ik\omega)\tau_j} \check{D}_j(k) \hat{X}(p - ik\omega, s), \quad \Re(p) > \lambda. \quad (5.5)$$

This can be rephrased as

$$\sum_{k \in \mathbb{Z}} (R(p))_{0,k} \hat{X}(p - ik\omega, s) = \frac{e^{-ps}}{p} I_d, \quad (5.6)$$

with $R(p)$ as in (2.15), (2.16) and where $(R(p))_{0,k}$ stands for the block at the intersection of block-line 0 and block-column k . Substituting $p + in\omega$ for p in (5.6) and letting n range over \mathbb{Z} , we obtain a system of countably many equations that may be written as

$$R(p) \hat{\mathbb{X}}(p, s) = \hat{e}(p, s), \quad (5.7)$$

where $\hat{\mathbb{X}}(p, s)$ and $\hat{e}(p, s)$ are the infinite $d \times d$ block vectors:

$$\hat{\mathbb{X}}(p, s) := \begin{pmatrix} \vdots \\ \hat{X}(p + i\omega, s) \\ \hat{X}(p, s) \\ \hat{X}(p - i\omega, s) \\ \vdots \end{pmatrix} \quad \text{and} \quad \hat{e}(p, s) := \begin{pmatrix} \vdots \\ \frac{e^{-(p+i\omega)s}}{p+i\omega} I_d \\ \frac{e^{-ps}}{p} I_d \\ \frac{e^{-(p-i\omega)s}}{p-i\omega} I_d \\ \vdots \end{pmatrix}. \quad (5.8)$$

Clearly, these are the Laplace transforms of infinite $d \times d$ block vectors $\mathbb{X}(t, s)$ and $e(t, s)$:

$$\hat{\mathbb{X}}(p, s) = \int_{-\infty}^{+\infty} e^{-pt} \mathbb{X}(t, s) dt, \quad \hat{e}(p, s)(p, s) = \int_{-\infty}^{+\infty} e^{-pt} e(t, s) dt, \quad (5.9)$$

$$\text{with } \mathbb{X}(t, s) := \begin{pmatrix} \vdots \\ e^{-i\omega t} X(t, s) \\ X(t, s) \\ e^{i\omega t} X(t, s) \\ \vdots \end{pmatrix}, \quad e(t, s) := \begin{pmatrix} \vdots \\ I_d e^{-i\omega t} \mathbf{1}_{[s, +\infty)} \\ I_d \mathbf{1}_{[s, +\infty)} \\ I_d e^{i\omega t} \mathbf{1}_{[s, +\infty)} \\ \vdots \end{pmatrix}. \quad (5.10)$$

The infinite dimensional linear constant system (5.7) recasts the finite-dimensional periodic time-varying delay system (4.1) in terms of Fourier series and Laplace transforms. In order to estimate $\mathbb{X}(p, s)$ –and eventually $X(t, s)$ – we shall invert the operator $R(p)$; this is our next step.

Step 2

For $A := (a_{i,j})_{i,j \in \mathbb{Z}}$ a doubly infinite block matrix, where $a_{i,j}$ is a $d \times d$ complex matrix for each i and j in \mathbb{Z} , we say that A has off diagonal decay of order r if there is a constant C such that $\|a_{i,j}\| \leq C(1 + |i - j|)^{-r}$. We also define the Wiener norm of A to be

$$\|A\|_{\mathfrak{W}} := \sum_{k \in \mathbb{Z}} \sup_{|i-j|=k} \|a_{i,j}\|. \quad (5.11)$$

Note that $\|A\|_2 \leq \|A\|_{\mathfrak{W}}$, as follows immediately from the Schur test [28]. We denote by $B(\ell^2(\mathbb{Z}, \mathbb{C}^d), \ell^2(\mathbb{Z}, \mathbb{C}^d))$ the space of doubly infinite block matrices A such that $\|A\|_2 < \infty$, and by $\mathfrak{W}(\ell^2(\mathbb{Z}, \mathbb{C}^d), \ell^2(\mathbb{Z}, \mathbb{C}^d))$ the subspace of those A satisfying $\|A\|_{\mathfrak{W}} < \infty$. It is easy to check that $\mathfrak{W}(\ell^2(\mathbb{Z}, \mathbb{C}^d), \ell^2(\mathbb{Z}, \mathbb{C}^d))$ is a subalgebra of $B(\ell^2(\mathbb{Z}, \mathbb{C}^d), \ell^2(\mathbb{Z}, \mathbb{C}^d))$.

One can see from (5.1) and (2.16) that $R(p)$ possesses off-diagonal decay of order $1 + \delta$, moreover the constant is uniform over any half-space $\Re(p) \geq a$. From this, it follows at once that $\|R(p)\|_{\mathfrak{W}}$ is uniformly bounded over such a half-space. Hence, $R(p) \in \mathfrak{W}(\ell^2(\mathbb{Z}, \mathbb{C}^d), \ell^2(\mathbb{Z}, \mathbb{C}^d))$ for all p and clearly, $R : \mathbb{C} \rightarrow B(\ell^2(\mathbb{Z}, \mathbb{C}^d), \ell^2(\mathbb{Z}, \mathbb{C}^d))$ is a Banach valued holomorphic function.

Now, assumption (i) of Theorem 3 tells us that $R(p)$ is invertible in $B(\ell^2(\mathbb{Z}, \mathbb{C}^d), \ell^2(\mathbb{Z}, \mathbb{C}^d))$ for $\Re(p) \geq \beta$, and assumption (ii) that the inverse operator $R^{-1}(p)$ has uniformly bounded $\|\cdot\|_2$ -norm. Therefore, as $R(p)$ has off-diagonal decay of order $1 + \delta$, we get from [29, thm. 1.2] and the boundedness of $\|R^{-1}(p)\|_2$ that $R^{-1}(p)$ also has off-diagonal decay of order $1 + \delta$, uniformly for $\Re(p) \geq \beta$. In particular, $\|R^{-1}(p)\|_{\mathfrak{W}}$ is uniformly bounded for $\Re(p) \geq \beta$:

$$\left\| \left[I_\infty - \sum_{i=j}^N e^{-p\tau_j} L_{D_j} \Delta_{\tau_j, \omega} \right]^{-1} \right\|_{\mathfrak{W}} \leq C_1, \quad p \in \{z \in \mathbb{C} \mid \Re(z) \geq \beta\}. \quad (5.12)$$

By (5.12), applying to (2.15) the result of [30, thm. 2.6] that generalizes the Wiener lemma to Banach algebra-valued almost periodic functions (the Banach algebra here is $\mathfrak{W}(\ell^2(\mathbb{Z}, \mathbb{C}^d), \ell^2(\mathbb{Z}, \mathbb{C}^d))$, see [31] for a still more general version of that theorem), we get for any α with $\beta < \alpha < 0$ that $R^{-1}(p)$ admits a generalized Fourier expansion on the vertical line $p = \alpha + i\mathbb{R}$ of the form :

$$R^{-1}(\alpha + i\tilde{\omega}) = \sum_{k \in \mathbb{Z}} \tilde{R}^{\{k, \alpha\}} e^{i\beta_k \tilde{\omega}}, \quad \tilde{\omega} \in \mathbb{R}, \quad (5.13)$$

where β_k is real and $\tilde{R}^{\{k,\alpha\}} \in \mathfrak{W}(\ell^2(\mathbb{Z}, \mathbb{C}^d), \ell^2(\mathbb{Z}, \mathbb{C}^d))$ with

$$\sum_{k \in \mathbb{Z}} \left\| \tilde{R}^{\{k,\alpha\}} \right\|_{\mathfrak{W}} < +\infty. \quad (5.14)$$

Moreover, following the proof of the theorem in section 3 on Dirichlet's series page 147 of [19], which is possible because Cauchy's theorem holds for Banach valued holomorphic functions (one can apply all the arguments there in weak form by evaluating on a fixed vector in $\ell^2(\mathbb{Z}, \mathbb{C}^d)$), we deduce that $\tilde{R}^{\{k,\alpha\}} = R^{\{k\}} e^{\alpha\beta_k}$, where $R^{\{k\}} \in B(\ell^2(\mathbb{Z}, \mathbb{C}^d), \ell^2(\mathbb{Z}, \mathbb{C}^d))$ is independent of α . Thus, one can rewrite (5.13) and (5.14) as

$$R^{-1}(\alpha + i\tilde{\omega}) = \sum_{k \in \mathbb{Z}} R^{\{k\}} e^{\alpha\beta_k} e^{\beta_k \tilde{\omega} i}, \quad \tilde{\omega} \in \mathbb{R}, \quad \alpha > \beta, \quad (5.15)$$

with

$$\sum_{k \in \mathbb{Z}} \left\| R^{\{k\}} \right\|_{\mathfrak{W}} e^{\alpha\beta_k} < +\infty. \quad (5.16)$$

In addition, observing from the Neumann series that for $\Re(p)$ large enough one has

$$R^{-1}(p) = \sum_{n=0}^{+\infty} \left(\sum_{j=1}^N L_{D_j} \Delta_{\tau_j, \omega} e^{-p\tau_j} \right)^n, \quad (5.17)$$

we deduce that

$$\{\beta_k | k \in \mathbb{Z}\} \subset \left\{ \sum_{j=1}^N n_j \tau_j \mid n_j \text{ non-positive integers for } j = 1, \dots, N \right\} = -\mathcal{F} \quad (5.18)$$

where \mathcal{F} was defined in (4.4). Note that (5.15), (5.16) and (5.18) are reminiscent of [11, p. 429, equations (12.15.12) and (12.15.13)], that deals with complex-valued functions. We also record for later use the following consequence of (5.17):

$$\left\| R^{-1}(p) \right\|_2 \leq \frac{1}{1 - e^{\Re(\tau_1)} \sum_{j=1}^N \left\| L_{D_j} \right\|_2 \left\| \Delta_{\tau_j, \omega} \right\|_2} \quad \text{for } \Re(p) > \gamma_1 := \frac{\log(\sum_{j=1}^N \left\| L_{D_j} \right\|_2 \left\| \Delta_{\tau_j, \omega} \right\|_2)}{\tau_1}. \quad (5.19)$$

Step 3

We now compute $\mathbb{X}(t, s)$. For this, we apply the Laplace inversion formula (see for example [7, Ch. 1, Lem. 5.2]). It gives us for $\Re(p) = c > \lambda$, with λ as in (4.2), that

$$\mathbb{X}(t, s) = \lim_{\tilde{\omega} \rightarrow +\infty} \frac{1}{2\pi i} \int_{c-i\tilde{\omega}}^{c+i\tilde{\omega}} R^{-1}(p) \hat{e}(p, s) e^{pt} dp. \quad (5.20)$$

We shall need two lemmas, the proof of which is postponed until the end of this section. The first one goes as follows.

Lemma 10. Let $\beta < \alpha < 0$ and $\rho_m := \frac{2\pi m}{T} + \frac{\pi}{T}$ for m a positive integer. Then, it holds that

$$\mathbb{X}(t, s) = \lim_{m \rightarrow +\infty} \frac{1}{2\pi i} \int_{\alpha - i\rho_m}^{\alpha + i\rho_m} R^{-1}(p) \hat{e}(p, s) e^{pt} dp + Q(t), \quad (5.21)$$

where

$$Q(t) := \cdots + R_{-1}^{-1}(-i\omega) e^{-i\omega t} + R_0^{-1}(0) + R_1^{-1}(i\omega) e^{i\omega t} + \cdots \quad (5.22)$$

with R_k^{-1} to mean the k -th column of R^{-1} .

Assume Lemma 10 for a while and recall that $X(t, s)$ is the element of index 0 of the infinite column vector $\mathbb{X}(t, s)$. Thus, if for each $k \in \mathbb{Z}$ we let $(R_{0,n}^{\{k\}})_{n \in \mathbb{Z}}$ enumerate the line of index 0 of the matrix $R^{\{k\}}$ defined in (5.15), it is a consequence of (5.21) that

$$X(t, s) = \frac{1}{2\pi i} \lim_{m \rightarrow +\infty} \int_{\alpha - i\rho_m}^{\alpha + i\rho_m} \sum_{k, n \in \mathbb{Z}} R_{0,n}^{\{k\}} \frac{e^{-(p - in\omega)s}}{p - in\omega} e^{p(t + \beta_k)} dp + Q(t)_0, \quad (5.23)$$

where $Q(t)_0$ stands for the element of index 0 of the column vector $Q(t)$; note that the convergence of the series in (5.23) ensues from (5.16). The second lemma we need is:

Lemma 11. For all t and s such that $t + \beta_k - s \neq 0$ for all $k \in \mathbb{Z}$, we have that

$$X(t, s) = \sum_{(t + \beta_k - s) < 0, n \in \mathbb{Z}} R_{0,n}^{\{k\}} e^{in\omega(t + \beta_k)} + Q(t)_0. \quad (5.24)$$

Assume Lemma 11 as well for the moment being, and note that on a bounded interval can lie only finitely many β_k , because the integer coefficients in (5.18) are non-positive. Observe also from (4.1) that $X(t, \cdot)$ is left continuous. Hence, Lemma 11 allows us to compute $X(t, s)$ for all (t, s) . Hereafter, we fix the symbol s to mean the initial time, and we evaluate the variation of $X(t, \tau)$ in its second argument when the latter ranges over $[s, s + \tau_N]$. To this effect, we substitute τ for s in (5.24) and observe, since $Q(t)_0$ is independent of τ while the sum in (5.24) is piecewise constant in τ with jumps (induced by the constraint on indices) at the $t + \beta_k$ where it is left continuous, that

$$W_{[s, s + \tau_N]} X(t, \cdot) \leq \sum_{0 \leq t + \beta_k - s \leq \tau_N, n \in \mathbb{Z}} \left\| R_{0,n}^{\{k\}} \right\| \quad (5.25)$$

$$\leq \left(\sum_{k, n \in \mathbb{Z}} \left\| R_{0,n}^{\{k\}} \right\| e^{\alpha \beta_k} e^{|\alpha| \tau_N} \right) e^{\alpha(t-s)}, \quad (5.26)$$

where we used in the second inequality that $\alpha(t + \beta_k - s - \tau_N) \geq 0$ because $\alpha < 0$. If $K' > 0$ is an upper bound for $\{\|D_i(t)\|, 1 \leq i \leq N, t \in \mathbb{R}\}$, then we deduce from (4.7) and (5.26) that

$$\|y(t)\| \leq K' N (W_{[s, s + \tau_N]} X(t, \cdot)) \|\phi\|_{C^0} \quad (5.27)$$

$$\leq K' N \left(\sum_{k, n \in \mathbb{Z}} \left\| R_{0,n}^{\{k\}} \right\| e^{\alpha \beta_k} e^{|\alpha| \tau_N} \right) e^{\alpha(t-s)} \|\phi\|_{C^0}, \quad (5.28)$$

thereby showing that System (2.3) is C^0 -exponentially stable. This achieves the proof of sufficiency in Theorem 3 granted Lemmas 10 and 11, that we now establish.

PROOF OF LEMMA 10. Considering the subsequence $\rho_m = \frac{2\pi m}{T} + \frac{\pi}{T}$, we deduce from (5.20) that :

$$\mathbb{X}(t, s) = \lim_{m \rightarrow +\infty} \frac{1}{2\pi i} \int_{c-i\rho_m}^{c+i\rho_m} R^{-1}(p) \hat{e}(p, s) e^{pt} dp. \quad (5.29)$$

As each component of $R^{-1}(p) \hat{e}(p, s)$ is a meromorphic function in the half plane $\{p \in \mathbb{C} | \Re(p) > \beta\}$ with simple poles in the set $\{ik\pi/T : k \in \mathbb{Z}\}$, by (5.8) and the assumed conditions (i), (ii) of Theorem 3, we get from (5.29) together with the residue theorem that

$$\mathbb{X}(t, s) = \lim_{m \rightarrow +\infty} \frac{1}{2\pi i} \left(\int_{\alpha-i\rho_m}^{\alpha+i\rho_m} + \int_{\alpha+i\rho_m}^{c+i\rho_m} - \int_{\alpha-i\rho_m}^{c-i\rho_m} \right) R^{-1}(p) \hat{e}(p, s) e^{pt} dp + Q(t) \quad (5.30)$$

where Q is as in (5.22); observe that taking the limit in (5.30) is indeed permitted, since

$$\|Q(t)\|_2 \leq \|R^{-1}(0)\|_{\mathfrak{W}} < +\infty. \quad (5.31)$$

To establish the lemma, it remains to show that

$$\lim_{m \rightarrow +\infty} \int_{\alpha+i\rho_m}^{c+i\rho_m} R^{-1}(p) \hat{e}(p, s) e^{pt} dp = \lim_{m \rightarrow +\infty} \int_{\alpha-i\rho_m}^{c-i\rho_m} R^{-1}(p) \hat{e}(p, s) e^{pt} dp = 0. \quad (5.32)$$

For this, let us write the entry with index $j \in \mathbb{Z}$ of the block vector $\int_{\alpha+i\rho_m}^{c+i\rho_m} R^{-1}(p) \hat{e}(p, s) e^{pt} dp$ as

$$\int_{\alpha+i\rho_m}^{c+i\rho_m} \sum_{n \in \mathbb{Z}} R_{j,n}^{-1}(p) \frac{e^{-(p-in\omega)s}}{p-in\omega} e^{pt} dp, \quad (5.33)$$

where $R_{j,n}^{-1}(p)$ denotes the block entry with index (j, n) of $R^{-1}(p)$. As pointed out in step 2, $R^{-1}(p)$ has off-diagonal decay of order $1 + \delta$, uniformly for $\Re(p) \geq \beta$. Thus, if we pick $\varepsilon > 0$, there is $\mathbf{n} = \mathbf{n}(j)$ such that $\sum_{|n| \geq \mathbf{n}} \|R_{j,n}^{-1}(p)\| < \varepsilon$, uniformly with respect to $\Re(p) \in [\alpha, c]$, and since

$$|\rho_m - n\omega| \geq \frac{\pi}{T} \text{ for all } n \text{ and } m \quad (5.34)$$

we have that

$$\left\| \int_{\alpha+i\rho_m}^{c+i\rho_m} \sum_{|n| \geq \mathbf{n}} R_{j,n}^{-1}(p) \frac{e^{-(p-in\omega)s}}{p-in\omega} e^{pt} dp \right\| \leq \varepsilon e^{c(t-s)} \frac{(c-\alpha)T}{\pi} \quad (5.35)$$

which is arbitrary small with ε . In another connection, for fixed j and $n \in \mathbb{Z}$, we get since $|p-in\omega|$ becomes arbitrary large with m while the other terms in the integrand are uniformly bounded with respect to $\Re(p) \in [\alpha, c]$ that

$$\lim_{m \rightarrow \infty} \int_{\alpha+i\rho_m}^{c+i\rho_m} R_{j,n}^{-1}(p) \frac{e^{-(p-in\omega)s}}{p-in\omega} e^{pt} dp = 0. \quad (5.36)$$

In view of (5.35) and (5.36), we conclude that

$$\lim_{m \rightarrow +\infty} \int_{\alpha+i\rho_m}^{c+i\rho_m} R^{-1}(p) \hat{e}(p, s) e^{pt} dp = 0, \quad (5.37)$$

and in the same way one can prove that

$$\lim_{m \rightarrow +\infty} \int_{\alpha-i\rho_m}^{c-i\rho_m} R^{-1}(p) \hat{e}(p, s) e^{pt} dp = 0, \quad (5.38)$$

as wanted. \square

PROOF OF LEMMA 11. Let t and s be such that $t + \beta_k - s \neq 0$ for all $k \in \mathbb{Z}$. By (5.34) and (5.16) that allow us to integrate termwise the series below, we get on performing the change of variable $p \rightarrow p - in\omega$ in the integrals corresponding to index n that

$$\int_{\alpha - i\rho_m}^{\alpha + i\rho_m} \sum_{k, n \in \mathbb{Z}} R_{0, n}^{\{k\}} \frac{e^{-(p - in\omega)s}}{p - in\omega} e^{p(t + \beta_k)} dp = \sum_{k, n \in \mathbb{Z}} R_{0, n}^{\{k\}} e^{in\omega(t + \beta_k)} \int_{\alpha - in\omega - i\rho_m}^{\alpha - in\omega + i\rho_m} \frac{e^{p(t + \beta_k - s)}}{p} dp. \quad (5.39)$$

Integrating by parts, we obtain:

$$\int_{\alpha - in\omega - i\rho_m}^{\alpha - in\omega + i\rho_m} \frac{e^{p(t + \beta_k - s)}}{p} dp = \left[\frac{e^{p(t + \beta_k - s)}}{p(t + \beta_k - s)} \right]_{\alpha - in\omega - i\rho_m}^{\alpha - in\omega + i\rho_m} + \int_{\alpha - in\omega - i\rho_m}^{\alpha - in\omega + i\rho_m} \frac{e^{p(t + \beta_k - s)}}{(t + \beta_k - s)p^2} dp, \quad (5.40)$$

and by (5.18) there is $\delta > 0$ such that $|t + \beta_k - s| \geq \delta$ for all $k \in \mathbb{Z}$. Hence, using (5.34), we obtain:

$$\begin{aligned} \left| \left[\frac{e^{p(t + \beta_k - s)}}{p(t + \beta_k - s)} \right]_{\alpha - in\omega - i\rho_m}^{\alpha - in\omega + i\rho_m} \right| &\leq \frac{1}{|t + \beta_k - s|} \left(\frac{e^{\alpha(t + \beta_k - s)}}{\sqrt{\alpha^2 + (\rho_m - in\omega)^2}} + \frac{e^{\alpha(t + \beta_k - s)}}{\sqrt{\alpha^2 + (\rho_m + in\omega)^2}} \right) \\ &\leq \frac{2e^{\alpha(t + \beta_k - s)}}{\delta \sqrt{\alpha^2 + \left(\frac{\pi}{T}\right)^2}} \end{aligned} \quad (5.41)$$

and

$$\begin{aligned} \left| \int_{\alpha - in\omega - i\rho_m}^{\alpha - in\omega + i\rho_m} \frac{e^{p(t + \beta_k - s)}}{(t + \beta_k - s)p^2} dp \right| &\leq \frac{e^{\alpha(t + \beta_k - s)}}{|t + \beta_k - s|} \int_{-\rho_m - n\omega}^{\rho_m - n\omega} \frac{1}{\alpha^2 + p_2^2} dp_2 \\ &\leq \frac{e^{\alpha(t + \beta_k - s)}}{|t + \beta_k - s|} \left[\frac{1}{\alpha} \arctan \left(\frac{p_2}{\alpha} \right) \right]_{-\rho_m - n\omega}^{\rho_m - n\omega} \\ &\leq \frac{\pi}{|\alpha| \delta} e^{\alpha(t + \beta_k - s)}. \end{aligned} \quad (5.42)$$

Altogether, we deduce from (5.40), (5.41) and (5.42) that there exists a constant $K > 0$, independent of k , n and ρ_m , such that :

$$\left| \int_{\alpha - in\omega - i\rho_m}^{\alpha - in\omega + i\rho_m} \frac{e^{p(t + \beta_k - s)}}{p} dp \right| \leq K e^{\alpha(t + \beta_k - s)}. \quad (5.43)$$

Consequently, it holds that

$$\left\| \left\| R_{0, n}^{\{k\}} e^{in\omega(t + \beta_k)} \int_{\alpha - in\omega - i\rho_m}^{\alpha - in\omega + i\rho_m} \frac{e^{p(t + \beta_k - s)}}{p} dp \right\| \right\| \leq K e^{\alpha(t - s)} \left\| \left\| R_{0, n}^{\{k\}} \right\| \right\| e^{\alpha\beta_k}, \quad (5.44)$$

and since the right hand side of (5.44) is summable over k, n because of (5.16)), the dominated convergence theorem allows us to take the limit termwise in the right hand side of (5.39) as $m \rightarrow \infty$:

$$\begin{aligned} &\lim_{m \rightarrow +\infty} \frac{1}{2i\pi} \int_{\alpha - i\rho_m}^{\alpha + i\rho_m} \sum_{k, n \in \mathbb{Z}} R_{0, n}^{\{k\}} \frac{e^{-(p - in\omega)s}}{p - in\omega} e^{p(t + \beta_k)} dp \\ &= \sum_{k, n \in \mathbb{Z}} \frac{1}{2i\pi} R_{0, n}^{\{k\}} e^{in\omega(t + \beta_k)} \lim_{m \rightarrow +\infty} \int_{\alpha - in\omega - i\rho_m}^{\alpha - in\omega + i\rho_m} \frac{e^{p(t + \beta_k - s)}}{p} dp. \end{aligned} \quad (5.45)$$

Since $\alpha < 0$, we get on the one hand by Cauchy's theorem that for $\kappa < \alpha$:

$$\int_{\alpha-in\omega-i\rho_m}^{\alpha-in\omega+i\rho_m} \frac{e^{p(t+\beta_k-s)}}{p} dp = \left(\int_{\kappa-in\omega+i\rho_m}^{\alpha-in\omega+i\rho_m} + \int_{\kappa-in\omega-i\rho_m}^{\kappa-in\omega+i\rho_m} + \int_{\alpha-in\omega-i\rho_m}^{\kappa-in\omega-i\rho_m} \right) \frac{e^{p(t+\beta_k-s)}}{p} dp, \quad (5.46)$$

and on the other hand by the residue theorem that for $\kappa > 0$:

$$\int_{\alpha-in\omega-i\rho_m}^{\alpha-in\omega+i\rho_m} \frac{e^{p(t+\beta_k-s)}}{p} dp = 2i\pi + \left(\int_{\kappa-in\omega+i\rho_m}^{\alpha-in\omega+i\rho_m} + \int_{\kappa-in\omega-i\rho_m}^{\kappa-in\omega+i\rho_m} + \int_{\alpha-in\omega-i\rho_m}^{\kappa-in\omega-i\rho_m} \right) \frac{e^{p(t+\beta_k-s)}}{p} dp. \quad (5.47)$$

Since $|p|$ goes to ∞ with m while the integrands are uniformly bounded with $\Re(p)$, the first and third integrals in the right hand side of (5.46) and (5.47) tend to 0 as $m \rightarrow \infty$ for fixed κ . Hence,

$$\lim_{m \rightarrow +\infty} \int_{\alpha-in\omega-i\rho_m}^{\alpha-in\omega+i\rho_m} \frac{e^{p(t+\beta_k-s)}}{p} dp = \lim_{m \rightarrow +\infty} \int_{\kappa-in\omega-i\rho_m}^{\kappa-in\omega+i\rho_m} \frac{e^{p(t+\beta_k-s)}}{p} dp, \quad \kappa < \alpha, \quad (5.48)$$

$$\lim_{m \rightarrow +\infty} \int_{\alpha-in\omega-i\rho_m}^{\alpha-in\omega+i\rho_m} \frac{e^{p(t+\beta_k-s)}}{p} dp = 2i\pi + \lim_{m \rightarrow +\infty} \int_{\kappa-in\omega-i\rho_m}^{\kappa-in\omega+i\rho_m} \frac{e^{p(t+\beta_k-s)}}{p} dp, \quad 0 < \kappa. \quad (5.49)$$

Integrating by parts yields that

$$\int_{\kappa-in\omega-i\rho_m}^{\kappa-in\omega+i\rho_m} \frac{e^{p(t+\beta_k-s)}}{p} dp = \frac{e^{\kappa(t+\beta_k-s)}}{i(t+\beta_k-s)} \left[\frac{e^{i(t+\beta_k-s)y}}{\kappa+iy} \right]_{y=-n\omega-\rho_m}^{y=-n\omega+\rho_m} - \int_{\kappa-in\omega-i\rho_m}^{\kappa-in\omega+i\rho_m} \frac{e^{p(t+\beta_k-s)}}{i(t+\beta_k-s)p^2} dp$$

and therefore, if $(t+\beta_k-s) > 0$ (resp. $(t+\beta_k-s) < 0$), then the right hand side of (5.48) (resp. (5.49)) goes to 0 (resp. $2i\pi$) when $\kappa \rightarrow -\infty$ (resp. $+\infty$). Thus,

$$\sum_{k,n \in \mathbb{Z}} \frac{1}{2i\pi} R_{0,n}^{\{k\}} e^{in\omega(t+\beta_k)} \lim_{m \rightarrow +\infty} \int_{\alpha-in\omega-i\rho_m}^{\alpha-in\omega+i\rho_m} \frac{e^{p(t+\beta_k-s)}}{p} dp = \sum_{t+\beta_k-s < 0, n \in \mathbb{Z}} R_{0,n}^{\{k\}} e^{in\omega(t+\beta_k)}. \quad (5.50)$$

In view of (5.23), (5.45) and (5.50), this achieves the proof of Lemma 11. \square

5.2. Necessity

We now assume exponential stability of System (2.3), and prove that conditions (i) and (ii) of Theorem 3 are satisfied. In the time-invariant case, necessity can be proven using the spectral theory of semigroups [5]. Unfortunately, there seems to be no adequate notion of semigroup for time-varying coefficients, and we shall follow a different path: since exponential stability of System (2.3) implies in particular its L^2 exponential stability (see Proposition 1), we may resort to the spectral theory of monodromy operators for periodic evolution families [32]. These are just solution operators over a period; *i.e.*, if $U_2(\cdot, \cdot)$ refers to the solution operator (2.7) with $q = 2$, then the monodromy operator of system (2.3) is $U_2(T, 0)$. Choosing zero as initial time is arbitrary, and we might put $U_2(T + s_0, s_0)$ for the monodromy operator with arbitrary s_0 , but even though different choices of s_0 lead to different monodromy operators, the relation of their spectrum to exponential stability is the same, and we stick to $s_0 = 0$ for definiteness. The result we need is the following.

Proposition 12. *System (2.3) is L^2 -exponentially stable if and only if the spectral radius of the monodromy operator $U_2(T, 0)$ is strictly less than one.*

PROOF. This is formally a consequence of [32, lem. 4.2] if definition (2.9) of exponential stability (with $q = 2$) is restricted to $t \geq s \geq 0$. However, since the constants in that proof only depend on the $D_j(t)$ through $\sup\{\|D_i(t)\|, 1 \leq i \leq N, t \in \mathbb{R}\}$, the case of arbitrary s follows by periodicity.

We mention that [32] deals with much more general periodic evolution families than (2.3).

In view of Proposition (12), we may assume that the spectral radius of the monodromy operator $U_2(T, 0)$ is strictly less than 1. Then, we will resort to a classical construction from Control Theory (called *realization*) to connect the spectrum of the monodromy operator with the singular set of the harmonic transfer function. Specifically, we shall realize the periodic input-output system (4.8) as a discrete-time system in state space form, much like in the reference [33] except that the state is now infinite dimensional. Consider (4.8) with initial data 0 at time 0, and let $u \in L^2([0, +\infty), \mathbb{C}^d)$ be an input generating the output $y(t) \in \mathbb{C}^d$:

$$y(t) = \sum_{j=1}^N D_j(t)y(t - \tau_j) + u(t), \quad \text{a.e. } t \geq 0, \quad y(t) = 0 \text{ for } t \leq 0. \quad (5.51)$$

Using (4.13) with $s = 0$, $y_s = 0$, and performing the change of variables $\beta = t - \alpha$, we get that

$$y(t) = \int_{-\infty}^{t^+} d_\beta X(t, t - \beta)u(t - \beta), \quad \text{a.e. } t \geq 0. \quad (5.52)$$

Note, since $X(t, t - \beta)$ is identically zero when $\beta < 0$, that the integral in (5.52) can also be written as $\int_{c^\pm}^{t^+} d_\beta X(t, t - \beta)u(t - \beta)$ for any $c < 0$. From (5.52) and (4.14), it follows that

$$y(t) = \sum_{f \in \mathcal{F} \cap [0, t]} \mathfrak{C}_f(t)u(t - f), \quad \text{a.e. } t \geq 0. \quad (5.53)$$

Applying to (5.53) the Schwarz inequality and using (4.16), (4.19) together with the fact that $\mathfrak{C}_0 = -I_d$ (as follows readily from (4.10) and (4.1)), we obtain:

$$\begin{aligned} |y(t)|^2 &\leq \left(\sum_{f \in \mathcal{F} \cap [0, t]} \|\mathfrak{C}_f(t)\|^2 \right) \left(\sum_{f \in \mathcal{F} \cap [0, t]} \|u(t - f)\|^2 \right) \\ &\leq \left(\sum_{f \in \mathcal{F} \cap [0, t]} \|\mathfrak{C}_f(t)\|^2 \right)^2 \left(\sum_{f \in \mathcal{F} \cap [0, t]} \|u(t - f)\|^2 \right) \\ &\leq (1 + Ke^{\gamma t})^2 \left(\sum_{f \in \mathcal{F} \cap [0, t]} \|u(t - f)\|^2 \right), \quad \text{a.e. } t \geq 0. \end{aligned} \quad (5.54)$$

In another connection, the cardinality of $\mathcal{F} \cap [0, t]$ is no larger than the number of N -tuples $(q_1, \dots, q_N) \in \mathbb{N}^N$ satisfying $\sum_{i=1}^N q_i \leq t/\tau_1$ (recall τ_1 is the smallest delay), and so

$$\text{Card} \{ \mathcal{F} \cap [0, t] \} \leq \left(1 + \frac{t}{\tau_1} \right)^N, \quad t \in [0, +\infty), \quad (5.55)$$

where $\text{Card}\{E\}$ stands for the cardinality of E . Integrating (5.54) from 0 to $\tau > 0$ and taking (5.55) into account, we deduce that

$$\|y\|_{L^2([0, \tau])} \leq \tau^{1/2} (1 + Ke^{\gamma \tau}) \left(1 + \frac{\tau}{\tau_1} \right)^{N/2} \|u\|_{L^2([0, +\infty])}, \quad (5.56)$$

implying that to each $\gamma_1 > \gamma$ there is $K_1 > 0$ for which $\|y\|_{L^2([0, \tau])} \leq K_1 e^{\gamma_1 \tau} \|u\|_{L^2([0, +\infty])}$. Since $\|u\|_{L^2([0, +\infty])} < +\infty$, this warrants the existence, for $\Re(p) > \gamma$, of the Laplace transforms of $y(t)$ and $u(t)$:

$$\hat{Y}(p) := \int_0^{+\infty} e^{-pt} y(t) dt, \quad \hat{U}(p) := \int_0^{+\infty} e^{-pt} u(t) dt, \quad (5.57)$$

for if $\Re(p) > \gamma_1 > \gamma$ then $\int_k^{k+1} e^{-pt}|u|(t)dt$ and $\int_k^{k+1} e^{-pt}|y|(t)dt$ decay exponentially fast with $k \in \mathbb{N}$, by the Schwarz inequality. Proceeding on (5.51) like we did on (4.1) to obtain (5.7), namely expanding the $D_j(t)$ into Fourier series and taking Laplace transforms termwise using (5.1), we get on replacing the complex variable p by $p + in\omega$ and letting n range over \mathbb{Z} that

$$R(p)\hat{\mathbb{Y}}(p) = \hat{\mathbb{U}}(p), \quad \Re(p) > \gamma, \quad (5.58)$$

where $\hat{\mathbb{Y}}(p)$ and $\hat{\mathbb{U}}(p)$ are given by (2.21) and $R(p)$ by (2.15). Moreover, we see from (5.19) that $R(p)$ is continuously invertible for $\Re(p) > \gamma_1$, so we obtain with $\gamma_2 := \max\{\gamma, \gamma_1\}$ that

$$\hat{\mathbb{Y}}(p) = R^{-1}(p)\hat{\mathbb{U}}(p), \quad \Re(p) > \gamma_2. \quad (5.59)$$

Equation (5.59) identifies with Equation (2.22), showing that the operator-valued holomorphic function $p \mapsto R^{-1}(p)$ is the Harmonic Transfer Function of System (5.51); see Definition 3.

Next, we define the instantaneous transfer function:

$$G(t, p) = \int_{-\infty}^{+\infty} -d_\tau X(t, t - \tau) e^{-p\tau}, \quad t \in \mathbb{R}, \quad p \in \mathbb{C}, \quad \Re(p) > \gamma, \quad (5.60)$$

where the integral is understood as $-\lim_{b \rightarrow +\infty} \int_{c^-}^{c^+} d_\tau X(t, t - \tau) e^{-p\tau}$ for any $c < 0$. The existence of the limit is guaranteed by Proposition 9, and the fact that it does not depend on $c < 0$ can be argued similarly to the independence of (4.12) from the exact value of b . Observe from Proposition 9 that $t \mapsto G(t, p)$ lies in $C^{1+\delta}(\mathbb{R}, \mathbb{C}^{d \times d})$, and obviously it is T -periodic. Thus, we may expand this function in Fourier series as

$$G(t, p) = \sum_{k \in \mathbb{Z}} G_k(p) e^{i\omega kt}, \quad \Re(p) > \gamma, \quad (5.61)$$

where we recall that $\omega := 2\pi/T$, and the series converges absolutely for fixed p by a standard estimate used already in (5.1); more precisely, [27, Ch. 2, thm. 4.7] implies that

$$\|G_k(p)\| \leq \frac{\mathfrak{K}(\Re(p))}{1 + |k|^{1+\delta}}, \quad (5.62)$$

where $\mathfrak{K}(\Re(p)) > 0$ depends on $\Re(p)$, and also on K, γ in (4.19) though we do not show the latter dependence. Now, for $\mathfrak{f} \in \mathcal{F}$ and $k \in \mathbb{Z}$, let $c_{k,\mathfrak{f}}$ be the k -th Fourier coefficient of the T -periodic function $t \mapsto \mathfrak{C}_\mathfrak{f}(t)$. Since $\mathfrak{C}_\mathfrak{f} \in C^{1,\delta}(\mathbb{R})$, we have the estimate

$$\|c_{k,\mathfrak{f}}\| \leq \frac{K_\mathfrak{f}}{1 + |k|^{1+\delta}} \quad (5.63)$$

so that $\mathfrak{C}_\mathfrak{f}(t) = \sum_{k \in \mathbb{Z}} c_{k,\mathfrak{f}} e^{i\omega kt}$, where the series is uniformly absolutely convergent. Summing up over those $\mathfrak{f} \in \mathcal{F}$ such that $\mathfrak{f} \leq \tau$, we deduce from (4.14) that for any $\varepsilon > 0$:

$$d_s X(t, t - s) = \sum_{k \in \mathbb{Z}} d\mu_{k,\tau} e^{i\omega kt} \quad \text{on } [-\varepsilon, \tau], \quad (5.64)$$

where the measure $\mu_{k,\tau}$ is equal to $-\sum_{\mathfrak{f} \in \mathcal{F} \cap [0, \tau]} c_{k,\mathfrak{f}} \delta_\mathfrak{f}$ and absolute convergence with respect to the total variation holds in (5.64), because $\mathcal{F} \cap [0, \tau]$ is a finite set. If $\tau < 0$, then obviously $\mu_{k,\tau} = 0$.

Moreover, as the constant K_j in (5.63) is majorized by an affine function of the Hölder coefficient of $\frac{d}{dt}\mathfrak{C}_j$, we get from (4.19) that

$$\|\mu_{k,\tau}\| = \sum_{f \in \mathcal{F} \cap [0,\tau]} \|c_{k,f}\| \leq \frac{K_1 e^{\gamma\tau}}{1 + |k|^{1+\delta}}, \quad \tau \geq 0, \quad (5.65)$$

for some constant K_1 depending only on the D_j and the τ_j . Let us now define

$$\mu_k := \sum_{f \in \mathcal{F}} c_{k,f} \delta_f. \quad (5.66)$$

Clearly, μ_k is a distribution on \mathbb{R} valued in $\mathbb{C}^{d \times d}$ and supported on $[0, +\infty)$, but it is generally not a measure. However, its restriction to every interval bounded from above is a finite discrete measure. Hence, we can integrate against μ_k any \mathbb{C}^d -valued bounded function which is zero for large arguments, and more generally any function that decays in norm as fast as $e^{-\gamma't}$ for some $\gamma' > \gamma$, by (5.65). Furthermore, when this function is supported on an interval of the form $[a, \tau]$ with $a < 0$ and $\tau < +\infty$, the integral against μ_k coincides with the integral against $\mu_{k,\tau}$. Using this remark, we can write for $\Re(p) > \gamma$:

$$\begin{aligned} G(t, p) &= - \lim_{b \rightarrow +\infty} \int_{-\infty}^{b^+} d_\tau X(t, t - \tau) e^{-p\tau} = - \lim_{b \rightarrow +\infty} \sum_{k \in \mathbb{Z}} e^{i\omega kt} \int_{-\infty}^{b^+} d\mu_{k,b}(\tau) e^{-p\tau} \\ &= - \sum_{k \in \mathbb{Z}} e^{i\omega kt} \lim_{b \rightarrow +\infty} \int_{-\infty}^{b^+} d\mu_k(\tau) e^{-p\tau} = - \sum_{k \in \mathbb{Z}} e^{i\omega kt} \int_{-\infty}^{+\infty} d\mu_k(\tau) e^{-p\tau}, \end{aligned}$$

where the second equality uses the absolute convergence in (5.64) and the third uses (5.65) and the fact that $\Re(p) > \gamma$. Considering (5.61), the previous chain of equalities yields that

$$G_k(p) = - \int_{0^-}^{+\infty} d\mu_k(\tau) e^{-p\tau}. \quad (5.67)$$

Next, assuming for a while that u is locally bounded, (5.52) and (5.64) imply

$$y(t) = \int_{-\infty}^{t^+} \sum_{k \in \mathbb{Z}} d\mu_{k,t}(\alpha) e^{i\omega kt} u(t - \alpha);$$

swapping the sum and integral sign thanks to (5.65) and using the above mentioned relation between $\mu_{k,t}$ and μ_k yields that $y(t) = \sum_{k \in \mathbb{Z}} \int_{-\infty}^{t^+} d\mu_k(\alpha) e^{i\omega kt} u(t - \alpha)$, and since $u(t - \alpha) = 0$ when $\alpha > t$ while μ_k is supported on $[0, \infty)$ we finally get

$$y(t) = \sum_{k \in \mathbb{Z}} A_k(t) \quad \text{with} \quad A_k(t) := \int_{0^-}^{+\infty} d\mu_k(\alpha) e^{i\omega k\alpha} u(t - \alpha) e^{i\omega k(t-\alpha)}. \quad (5.68)$$

Since for $\gamma' > \gamma$ we have that

$$\begin{aligned} \int_{0^-}^{+\infty} d|\mu_k|(\alpha) \int_0^{+\infty} |u(t - \alpha)| e^{-\gamma't} dt &= \left(\int_{0^-}^{+\infty} d|\mu_k|(\alpha) e^{-\gamma'\alpha} \right) \left(\int_0^{+\infty} |u(\tau)| e^{-\gamma'\tau} d\tau \right) \\ &\leq \left(\frac{e^\gamma K_1}{1 + |k|^{1+\delta}} \sum_{j=0}^{+\infty} e^{(\gamma-\gamma')j} \right) \frac{\|u\|_{L^2(\mathbb{R})}}{\sqrt{2\gamma'}} < +\infty \end{aligned} \quad (5.69)$$

where the inequality uses (5.65) and the Schwarz inequality, we can use Fubini's theorem to compute the Laplace transform of $A_k(t)$ in the strip $\Re(p) > \gamma$. This gives us

$$\begin{aligned} \int_0^{+\infty} A_k(t) e^{-pt} dt &= \int_{0^-}^{+\infty} d\mu_k(\alpha) e^{(i\omega k - p)\alpha} \int_0^{+\infty} e^{(i\omega k - p)(t - \alpha)} u(t - \alpha) dt \\ &= \left(\int_{0^-}^{+\infty} d\mu_k(\alpha) e^{(i\omega k - p)\alpha} \right) \left(\int_0^{+\infty} e^{(i\omega k - p)\tau} u(\tau) d\tau \right) \\ &= G_k(p - i\omega k) \hat{U}(p - i\omega k) \end{aligned} \quad (5.70)$$

where the last equality uses (5.67). Summing over k , we find in view of (5.68) that

$$\hat{Y}(p) = \sum_{k \in \mathbb{Z}} G_k(p - i\omega k) \hat{U}(p - i\omega k), \quad \Re(p) > \gamma, \quad (5.71)$$

where the absolute convergence of the series follows from the estimates obtained in (5.69); *i.e.*,

$$|G_k(z)| \leq \frac{e^\gamma K_1 K_2 (\Re(z) - \gamma)}{1 + |k|^{1+\delta}} \quad \text{and} \quad |\hat{U}(z)| \leq \frac{\|u\|_{L^2(\mathbb{R})}}{\sqrt{2\Re(z)}}, \quad \Re(z) > \gamma,$$

with $K_2(x) := \sum_{j=0}^{+\infty} e^{-xj}$ for $x > 0$. Changing p into $p + i\omega n$ in (5.71) and renumbering, we get

$$\hat{Y}(p + i\omega n) = \sum_{m \in \mathbb{Z}} G_{n-m}(p + i\omega m) \hat{U}(p + i\omega m), \quad n \in \mathbb{Z}, \quad \Re(p) > \gamma. \quad (5.72)$$

Combining (5.59) and (5.72), we see that

$$\sum_{m \in \mathbb{Z}} (G_{n-m}(p + i\omega m) - R_{n,-m}^{-1}(p)) \hat{U}(p + i\omega m) = 0, \quad n \in \mathbb{Z}, \quad \Re(p) > \gamma, \quad (5.73)$$

where $R_{i,j}^{-1}(p)$ indicates, as in (5.33), the entry with index (i, j) of $R^{-1}(p)$. For each p with $\Re(p) > 0$, the sequence $\{z_m := p + i\omega m, m \in \mathbb{N}\}$ is hyperbolically separated in the right half-plane; *i.e.*,

$$\left| \frac{z_m - z_j}{z_m + \bar{z}_j} \right| \geq c > 0, \quad j, m \in \mathbb{N}, \quad j \neq m.$$

Indeed, we have that

$$\left| \frac{z_m - z_j}{z_m + \bar{z}_j} \right| = \left(\frac{4(\Re(p))^2}{\omega^2 |m - j|^2} + 1 \right)^{-1/2} \geq \left(\frac{4(\Re(p))^2}{\omega^2} + 1 \right)^{-1/2}.$$

Hence (see [34, Ch. VII, Thm 1.1] for an equivalent statement on the upper half-plane), $(z_m)_{m \in \mathbb{N}}$ is an interpolating sequence, meaning that to each bounded sequence a_m in \mathbb{C} there is a bounded analytic function F in the right half-plane with $F(z_m) = a_m$. In particular, to each $j \in \mathbb{N}$ and $v \in \mathbb{C}^d$, there is a \mathbb{C}^d -valued bounded analytic function $F_{j,v}$ such that $F_{j,v}(z_m) = 0$ for $m \neq j$ and $F_{j,v}(z_j) = v$. Multiplying $F_{j,v}$ by a function without zeros in the Hardy space \mathcal{H}^2 (for instance $p \mapsto (p+1)^{-1}$), we get a function $G_{j,v} \in (\mathcal{H}^2)^d$ which is zero at z_m when $m \neq j$ and a nonzero multiple of v at z_j . Now, by the Paley-Wiener theorem, there is $u \in L^2([0, \infty), \mathbb{C}^d)$ such that $\hat{U} = G_{j,v}$, and using this collection of u in (5.73) as j ranges over \mathbb{N} and v over \mathbb{C}^d provides us with

$$R_{n,-m}^{-1}(p) = G_{n-m}(p + i\omega m), \quad n, m \in \mathbb{N}, \quad \Re(p) > \gamma. \quad (5.74)$$

Next, in order to link the monodromy operator to $R^{-1}(p)$ (this is achieved in Lemma 14 further below), we shall use periodicity to *realize* the dynamical system (5.51), that operates in continuous time, as a discrete time system in state space form; see Theorem 13. For this, we restrict the input and the output to intervals of length T by setting:

$$\left. \begin{aligned} \tilde{u}_k(t) &:= u(kT + t) \\ \tilde{y}_k(t) &:= y(kT + t) \end{aligned} \right\} \text{ for a.e. } t \in [0, T] \text{ and } k \in \mathbb{N}, \quad (5.75)$$

thereby giving rise to sequences \tilde{u}_k and \tilde{y}_k of functions in $L^2([0, T])$. We introduce another sequence of functions \tilde{z}_k , this time in $L^2([-\tau_N, 0])$, by putting $\tilde{z}_k := y_{kT}$, where the notation y_{kT} was defined after (4.8); *i.e.*, $\tilde{z}_k(\theta) := y(kT + \theta)$ for $k \in \mathbb{N}$ and a.e. $\theta \in [-\tau_N, 0]$. The function \tilde{z}_k will be the “state variable” of our discrete system at time k , initialized with $\tilde{z}_0 = y_0 = 0$. To realize this system, recall the monodromy operator $U_2(T, 0)$ from (2.9) (with $q = 2$) and the integration with respect to $K(t, \alpha)$ from (4.11). These allow us to define four operators as follows:

- $\tilde{A} : L^2([-\tau_N, 0], \mathbb{C}^d) \longrightarrow L^2([-\tau_N, 0], \mathbb{C}^d)$
 $v \longmapsto U_2(T, 0)v,$
- $\tilde{B} : L^2([0, T], \mathbb{R}^d) \longrightarrow L^2([-\tau_N, 0], \mathbb{C}^d)$
 $w \longmapsto \int_{0^-}^{+\infty} d_\alpha K(T, \alpha)w(\alpha),$
- $\tilde{C} : L^2([-\tau_N, 0], \mathbb{C}^d) \longrightarrow L^2([0, T], \mathbb{C}^d)$
 $v \longmapsto \{(U_2(t, 0)v)(0), t \in [0, T]\},$
- $\tilde{D} : L^2([0, T], \mathbb{C}^d) \longrightarrow L^2([0, T], \mathbb{C}^d)$
 $w \longmapsto \left(- \int_{0^-}^{+\infty} d_\alpha X(\cdot, \alpha)w(\alpha) \right) \Big|_{[0, T]}.$

Observe that \tilde{B} and \tilde{D} do not depend on the values of w outside $[0, T]$, and that $(U_2(t, 0)v)(0)$ exists for a.e. $t \in \mathbb{R}$, hence the operators \tilde{A} , \tilde{B} , \tilde{C} and \tilde{D} are well-defined and continuous.

Theorem 13. *Let $u \in L^2([0, +\infty), \mathbb{C}^d)$ and $y \in L^2_{loc}([0, +\infty), \mathbb{C}^d)$ be, respectively, the input and output of System (5.51). For \tilde{u}_k , \tilde{y}_k and \tilde{z}_k as in (5.75), the following recursion holds:*

$$\begin{cases} \tilde{z}_{k+1} &= \tilde{A}\tilde{z}_k + \tilde{B}\tilde{u}_k, \\ \tilde{y}_k &= \tilde{C}\tilde{z}_k + \tilde{D}\tilde{u}_k, \end{cases} \quad \tilde{z}_0 = 0, \quad k \in \mathbb{N}. \quad (5.76)$$

PROOF. Applying the variation-of-constants formula (4.13) to System (5.51) with s, t replaced by $kT, kT + t$ yields

$$y_{kT+t} = U_2(kT + t, kT)y_{kT} + \int_{(kT)^-}^{+\infty} d_\alpha K(kT + t, \alpha)u(\alpha),$$

and performing the change of variable $\alpha = kT + \beta$ while using T -periodicity we get that

$$y_{kT+t} = U_2(t, 0)y_{kT} + \int_{0^-}^{+\infty} d_\beta K(t, \beta)\tilde{u}_k(\beta). \quad (5.77)$$

When $t = T$, this gives us the first equation in (5.76). Next, evaluating each term of (5.77) at 0 (remember these terms are functions of $\theta \in [-\tau_N, 0]$ and that evaluation at a fixed point is possible for a.e. t) while using the definition of K (see (4.11)), we obtain:

$$y(t + kT) = (U_2(t, 0)\tilde{z}_k)(0) - \int_{0^-}^{+\infty} d_\beta X(t, \beta)\tilde{u}_k(\beta), \quad \text{a.e. } t \in [0, T],$$

which is the second equation in (5.76). \square

For $a := (a_n)_{n \in \mathbb{N}}$ a sequence in a Banach space \mathfrak{X} , its z -transform is the formal series:

$$\mathcal{L}\{a\}(z) := \sum_{n \in \mathbb{N}} a_n z^{-n}. \quad (5.78)$$

We let $\mathfrak{X}[[z^{-1}]]$ denote the space of such power series. For \mathfrak{Y} a Banach space and $O := (O_n)_{n \in \mathbb{N}}$ a sequence of bounded operators from \mathfrak{X} to \mathfrak{Y} , the z -transform $\mathcal{L}\{O\}(z)$ acts naturally from $\mathfrak{X}[[z^{-1}]]$ into $\mathfrak{Y}[[z^{-1}]]$, since for each $k \in \mathbb{N}$ the number of terms involved in the coefficient of z^{-k} when computing the product $(\sum_{n \in \mathbb{N}} O_n z^{-n})(\sum_{n \in \mathbb{N}} a_n z^{-n})$ is finite. Now, if we put $\bar{y} := (\tilde{y}_n)_{n \in \mathbb{N}}$, $\bar{u} := (\tilde{u}_n)_{n \in \mathbb{N}}$ and $\bar{z} := (\tilde{z}_n)_{n \in \mathbb{N}}$, we get from (5.76) that

$$(zI_d - \tilde{A})\mathcal{L}\{\bar{z}\} = \tilde{B}\mathcal{L}\{\bar{u}\} \quad \text{and} \quad \mathcal{L}\{\bar{y}\} = \tilde{C}\mathcal{L}\{\bar{z}\} + \tilde{D}\mathcal{L}\{\bar{u}\}.$$

Note, since $\tilde{z}_0 = 0$, that $(zI_d - \tilde{A})\mathcal{L}\{\bar{z}\}$ indeed belongs to $L^2([-\tau_N, 0], \mathbb{C}^d)[[z^{-1}]]$. Observing that $(zI - \tilde{A})^{-1} = \sum_{n \in \mathbb{N}} \tilde{A}^n z^{-n-1}$ in the ring of formal Laurent series with coefficients the operators on $L^2([-\tau_N, 0], \mathbb{C}^d)$, we may write

$$\mathcal{L}\{\bar{y}\}(z) = \left[\tilde{C}(zI_d - \tilde{A})^{-1}\tilde{B} + \tilde{D} \right] \mathcal{L}\{\bar{u}\}(z). \quad (5.79)$$

Although (5.79) was derived only formally, it becomes a valid relation between vector and operator valued analytic functions when $|z| > \max\{\|\tilde{A}\|, e^\gamma\}$ with γ as in Proposition 9, because all series involved are then normally convergent (compare (5.56)). In another connection, writing after a change of variable the right hand side of (5.52) as a sum of integrals on subintervals while taking into account that $X(\tau, \cdot)$ is left continuous and using T -periodicity, we get that

$$\begin{aligned} \tilde{y}_n(t) = y(t + nT) &= \sum_{k=0}^{n-1} \int_{(kT)^-}^{((k+1)T)^-} d_\alpha X(t + nT, \alpha)u(\alpha) + \int_{(nT)^-}^{+\infty} d_\alpha X(t + nT, \alpha)u(\alpha) \\ &= \sum_{k=0}^{n-1} \int_{0^-}^{T^-} d_\beta X(t + nT, kT + \beta)u(kT + \beta) + \int_{(nT)^-}^{((n+1)T)^+} d_\alpha X(t + nT, \alpha)u(\alpha) \\ &= \sum_{k=0}^{n-1} \int_{0^-}^{T^+} d_\beta X(t + nT, kT + \beta)u(kT + \beta) + \int_{0^-}^{T^+} d_\beta X(t + nT, nT + \beta)u(nT + \beta) \\ &= \sum_{k=0}^{n-1} \int_{0^-}^{T^+} d_\beta X(t + (n-k)T, \beta)\tilde{u}_k(\beta) + \int_{0^-}^{T^+} d_\beta X(t, \beta)\tilde{u}_n(\beta) \\ &= \sum_{k=0}^n H_{[n-k]} \tilde{u}_k(t), \quad \text{a.e. } t \in [0, T] \end{aligned} \quad (5.80)$$

where, for $v \in L^2([0, T], \mathbb{C}^d)$, we have set:

$$H_{[k]}v(t) := \int_{0^-}^{T^+} d_\tau X(kT + t, \tau)v(\tau), \quad \text{a.e. } t \in [0, T]. \quad (5.81)$$

Observe that the equality $\int_{(nT)^-}^{+\infty} d_\alpha X(t + nT, \alpha)u(\alpha) = \int_{(nT)^-}^{((n+1)T)^+} d_\alpha X(t + nT, \alpha)u(\alpha)$ used in the third identity of (5.80) only holds for $t < T$ in general, because when $t = T$ and $(n+1)T \in \mathcal{F}$, then $\int_{(nT)^-}^{((n+1)T)^+} d_\alpha X(t + nT, \alpha)u(\alpha)$ will miss the jump that may occur at $(n+1)T$. However, since (5.80) is claimed for a.e. $t \in [0, T]$ only, this is unimportant. Note also that changing the upper bound from T^- to T^+ in the integrals summed over k after the fourth equal sign of (5.80) is permitted, because the left continuity of $X(\tau, \cdot)$ implies that $\{T\}$ carries no mass of $d_\beta X(t + nT, nT + \beta)$. If we take z -transforms in (5.80), we obtain that

$$\mathcal{L}\{\bar{y}\}(z) = \mathcal{L}\{\bar{H}\}(z)\mathcal{L}\{\bar{u}\}(z) \quad (5.82)$$

where $\bar{H} := (H_{[n]})_{n \in \mathbb{N}}$, and since for each k we may pick \tilde{u}_k arbitrarily in $L^2([0, T])$ and $\tilde{u}_j \equiv 0$ for $j \neq k$, we deduce from (5.79) and (5.82) that

$$\mathcal{L}\{\bar{H}\}(z) = \tilde{C}(zId - \tilde{A})^{-1}\tilde{B} + \tilde{D} \quad (5.83)$$

The equality (5.83) *a priori* holds between formal power series in z^{-1} with coefficients in the ring $\mathcal{B}(L^2([0, T], \mathbb{C}^d))$ of bounded linear operators on $L^2([0, T], \mathbb{C}^d)$. However, if we let a denote the spectral radius of \tilde{A} and set $P_a := \{z \in \mathbb{C} : |z| > a\}$, then (5.83) holds in $\mathcal{B}(L^2([0, T], \mathbb{C}^d))$ when we substitute for z a value in P_a , for the series become normally convergent. Of course, by analytic continuation, the right hand-side extends to an operator-valued analytic function on the unbounded connected component of $\mathbb{C} \setminus \text{Spec } \tilde{A}$, where $\text{Spec } \tilde{A}$ stands for the spectrum of $\tilde{A} = U_2(T, 0)$, but this function may no longer be defined by a power series in z^{-1} . Since we assumed that System (2.3) is exponentially stable, we know from Proposition 12 that $\text{Spec } \tilde{A}$ is compactly included the unit disk; that is to say: $a \in [0, 1)$. In particular, $\mathcal{L}\{\bar{H}\}(e^{pT})$ is a well-defined operator on $L^2([0, T], \mathbb{C}^d)$ as soon as $\Re(p) > \log a/T$, where $\log a$ is strictly negative or $-\infty$.

For $p \in \mathbb{C}$, consider the operator $E_p : L^2([0, T], \mathbb{C}^d) \rightarrow L^2([0, T], \mathbb{C}^d)$ of pointwise multiplication by the function $(t \mapsto e^{pt}) \in L^\infty([0, T])$. Clearly $E_p^{-1} = E_{-p}$, and it is easy to check that

$$\|E_p\|_2 = \max\{1, e^{\Re(p)T}\}. \quad (5.84)$$

When $\Re(p) > \log a/T$, we also define the operator

$$\begin{aligned} \Lambda(p) & : L^2([0, T], \mathbb{C}^d) \rightarrow L^2([0, T], \mathbb{C}^d), \\ \Lambda(p) & = E_{-p} \circ \mathcal{L}\{\bar{H}\}(e^{pT}) \circ E_p. \end{aligned} \quad (5.85)$$

In addition, for $\Re(p) > \gamma_2$ with γ_2 as in (5.59), we construe $R^{-1}(p) : l^2(\mathbb{Z}, \mathbb{C}^d) \rightarrow l^2(\mathbb{Z}, \mathbb{C}^d)$ as an operator from $L^2([0, T], \mathbb{C}^d)$ into itself by identifying a function $\phi \in L^2([0, T], \mathbb{C}^d)$ with $(\cdots, a_2, a_1, a_0, a_{-1}, a_{-2}, \cdots)$ where a_k is the k -th Fourier coefficient of ϕ .

Lemma 14. *For $p \in \mathbb{C}$ with $\Re(p) > \gamma_2$, it holds that*

$$\Lambda(p)\phi(t) = R^{-1}(p)\phi, \quad \phi \in L^2([0, T], \mathbb{C}^d). \quad (5.86)$$

PROOF. As $\gamma_2 > 0 > \log a/T$, we see from (5.81) that for any $v \in L^2([0, T], \mathbb{R})$ we have:

$$\mathcal{L}\{\overline{H}\}(z)v(t) = \sum_{k=0}^{+\infty} H_{[k]}v(t)z^{-k} = \sum_{k=-\infty}^{+\infty} z^{-k} \int_{0^-}^{T^+} d_\tau X(kT + t, \tau)v(\tau). \quad (5.87)$$

Performing the change of variable $\tau \rightarrow t - \tau$ in (5.60) and computing as in (5.80), we also obtain

$$\begin{aligned} G(t, p) &= \int_{-\infty}^{+\infty} d_\tau X(t, \tau)e^{p(\tau-t)} = \sum_{k=-\infty}^{+\infty} \int_{0^-}^{T^+} d_\tau X(t, \tau - kT)e^{p(\tau-t-kT)} \\ &= \sum_{k=-\infty}^{+\infty} e^{-kpT} \left(e^{-pt} \int_{0^-}^{T^+} d_\tau X(t + kT, \tau)e^{p\tau} \right). \end{aligned} \quad (5.88)$$

Multiplying (5.88) on the right by some arbitrary $V \in \mathbb{C}^d$ and comparing with (5.87) where we set $z = e^{pT}$ and $v(\tau) = e^{p\tau}V$ for $\tau \in [0, T]$, we get that

$$G(t, p)V = e^{-pt} [\mathcal{L}\{\overline{H}\}(e^{pT})(e^{p \cdot} V)](t), \quad \text{a.e. } t \in [0, T], \quad (5.89)$$

where $e^{p \cdot} V$ is another way of writing v (the dot stands for a dummy argument). Now, assume for a while that $\phi \in C^{1,\alpha}([0, T], \mathbb{C}^d)$ with $\alpha \in (0, 1)$ and $\phi(0) = \phi(T)$. Write the Fourier expansion $\phi(t) = \sum_{k \in \mathbb{Z}} a_k e^{i\omega kt}$ with $a_k \in \mathbb{C}^d$. By normal convergence of the latter, we deduce from (5.89), (5.61) and (5.62) that

$$\begin{aligned} \Lambda(p)\phi(t) &= e^{-pt} [\mathcal{L}\{\overline{H}\}(e^{pT})e^{p \cdot} \sum_{k \in \mathbb{Z}} a_k e^{i\omega k \cdot}](t) = \sum_{k \in \mathbb{Z}} e^{-pt} [\mathcal{L}\{\overline{H}\}(e^{pT})e^{(p+i\omega k) \cdot} a_k](t) \\ &= \sum_{k \in \mathbb{Z}} e^{i\omega kt} e^{-(p+i\omega k)t} [\mathcal{L}\{\overline{H}\}(e^{(p+i\omega k)T})e^{(p+i\omega k) \cdot} a_k](t) = \sum_{k \in \mathbb{Z}} e^{i\omega kt} G(t, p + i\omega k) a_k \\ &= \sum_{k \in \mathbb{Z}} e^{i\omega kt} \left(\sum_{n \in \mathbb{Z}} e^{in\omega t} G_n(p + i\omega k) \right) a_k = \sum_{m \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} G_{m-k}(p + i\omega k) a_k \right) e^{im\omega t}, \end{aligned} \quad (5.90)$$

and so in view of (5.74) we get that

$$\Lambda(p)\phi(t) = \sum_{m \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} R_{m,-k}^{-1}(p) a_k \right) e^{im\omega t} = R^{-1}(p)\phi. \quad (5.91)$$

Equation (5.91) yields the desired conclusion, except for the assumption that $\phi \in C^{1,\alpha}([0, T], \mathbb{C}^d)$ with $\phi(0) = \phi(T)$. But since $\Lambda(p)$ and $R^{-1}(p)$ are continuous operators $L^2([0, T]) \rightarrow L^2([0, T])$ for $\Re(p) > \gamma_2$, we conclude by a density argument that (5.86) holds as stated. \square

To recap, we know from the discussion after (5.83) that $\mathcal{L}\{\overline{H}\}(e^{pT})$ is an analytic operator-valued function for $\Re(p) > \log a/T$, which is uniformly bounded in every half-plane $\{p : \Re(p) > \beta\}$ if $\beta > \log a/T$. Therefore $\Lambda(p)$ is, by (5.84) and (5.85), an analytic operator-valued function for $\Re(p) > \log a/T$, uniformly bounded in every strip $\{p : \beta_0 > \Re(p) > \beta\}$ if $\beta_0 > \beta > \log a/T$. Furthermore, in view of (2.15), $R(p)$ is an entire operator-valued function. As Lemma 14 tells us that $\Lambda(p)R(p) = I_\infty$ for $\Re(p) > \gamma_2$, this identity in fact holds on $\{p : \Re(p) > \log a/T\}$, by analytic continuation. So, if we let $\beta \in (\log a/T, 0)$ and $\beta_1 = \gamma_1 + 1$ with γ_1 as in (5.19), we get that $R^{-1}(p)$

exists and is equal to $\Lambda(p)$ for $\Re(p) > \log a/T$, and that $\|R^{-1}(p)\|_2 \leq C_1$ for $\beta < \Re(p) < \beta_1$ with a constant $C_1 > 0$ depending on β and β_1 but not on p . In another connection, we know from (5.19) that $\|R^{-1}(p)\|_2 \leq C_2$ for $\Re(p) > \gamma_1$, and altogether we conclude that

$$\|R^{-1}(p)\|_2 \leq C, \quad \Re(p) > \beta, \quad (5.92)$$

with $C = \max\{C_1, C_2\}$. This achieves the proof of the necessity part of Theorem 3, and thus of the theorem itself in view of Section 5.1.

6. Concluding remarks on neutral functional differential equations

Having proven a generalization of the Henry-Hale theorem in the periodic case, it is natural to ask if the result from [5] on the stability of neutral functional differential systems carries over to the periodic case as well. In other words, whether exponential stability of a periodic linear neutral differential system of the form (1.2) can be characterized through analyticity of its harmonic transfer function. It is transparent how to define the latter: simply put $\mathbf{H}(p) := \tilde{R}(p)^{-1}$ where

$$\tilde{R}(p) := D_\omega(p) \left[I_\infty - \sum_{j=1}^N e^{-p\tau_j} L_{D_j} \Delta_{\tau_j, \omega} \right] - \sum_{k=0}^N e^{-p\tau_k} L_{B_k} \tilde{D}_{\tau_k}, \quad (6.1)$$

with $\tau_0 := 0$ and

$$D_\omega(p) := \text{diag} \{ \dots, (p + i\omega)I_d, pI_d, (p - i\omega)I_d, \dots \} \quad (6.2)$$

while the D_j, B_j are as in (1.2). Then, with the natural definition of exponential stability paralleling Definition 1, we raise the following question:

Assume that the $D_j(\cdot)$ and $B_k(\cdot)$ are periodic and differentiable with Hölder continuous derivative for $1 \leq j \leq N$ and $0 \leq k \leq N$. Is it true that a necessary and sufficient condition for System (1.2) to be exponentially stable is the existence of a real number $\beta < 0$ satisfying

- (i) $\tilde{R}(p)$ is invertible $\ell^2(\mathbb{Z}, \mathbb{C}^d) \rightarrow \ell^2(\mathbb{Z}, \mathbb{C}^d)$ for all p in $\{z \in \mathbb{C} | \Re(z) \geq \beta\}$,
- (ii) there is $M > 0$ such that $\|\tilde{R}(p)^{-1}\|_2 \leq M$ for all p in $\{z \in \mathbb{C} | \Re(z) \geq \beta\}$?

We expect the answer to be positive, and it should transpire already that necessity can be proven as in theorem 3. The difficulty with the sufficiency part is that the matrix $D_\omega(p)$ now prevents us from deriving straightforward analogs of Lemmas 10 and 11.

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