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Exponential stability of linear periodic difference-delay equations

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Abstract

This paper deals with the stability of linear time-periodic difference delay systems, where the value at time t of a solution is a linear combination with periodic coefficients of its value at finitely many delayed instants $t - \tau_1, \dots, t - \tau_N$. We give a necessary and sufficient condition for exponential stability of such systems when the coefficients have Hölder-continuous derivative, that generalizes the one obtained for constant coefficients by Henry and Hale in the 1970s. This condition has a control-theoretic interpretation in terms of analyticity in a half plane of the harmonic transfer function (which is operator-valued) of an associated linear control system.

Keywords: linear periodic systems, difference delay systems, exponential stability, harmonic transfer function, Henry-Hale theorem.

1. Introduction

The main result of the present paper is a necessary and sufficient condition for exponential stability of periodic *difference-delay systems*²; *i.e.*, linear dynamical systems of the form

$$y(t) = \sum_{j=1}^N D_j(t) y(t - \tau_j), \quad t > s, \quad (1.1)$$

where $\tau_1 < \dots < \tau_N$ are positive delays and $D_1(t), \dots, D_N(t)$ are complex $d \times d$ matrices, depending periodically on time t . Specifically, the condition applies when the maps $t \mapsto D_j(t)$ are $C^{1,\alpha}$ (*i.e.* differentiable with Hölder continuous derivatives of exponent α), $0 < \alpha < 1$, and periodic with some period $T > 0$. Periodicity is here essential, whereas $C^{1,\alpha}$ -regularity is technical. Our result supersedes the by now classical “Henry-Hale theorem” given in [8, 5] (see also the textbook [7]), that yields such a necessary and sufficient condition in the simpler case where the matrices D_j are constant.

Proper statements are made in Section 2; let us however briefly explain exponential stability. A *solution* is a map $y : [s - \tau_N, +\infty) \rightarrow \mathbb{C}^d$ that satisfies (1.1); it is uniquely determined by its restriction to $[s - \tau_N, s]$. To a solution y , we associate, for each $t \geq s$, the function $y_t : [-\tau_N, 0] \rightarrow \mathbb{C}^d$ defined by $y_t(\theta) = y(t + \theta)$. After fixing a normed space of \mathbb{C}^d -valued functions on $[-\tau_N, 0]$ to which y_t belongs for all t (for instance $L^p([-\tau_N, 0], \mathbb{C}^d)$, $1 \leq p \leq \infty$, endowed with its usual norm), we

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²There seems to be no general agreement on terminology here. The name “difference-delay system” is used in [1] as well as in [2, 3], but such systems are called *time-delay systems* in [4] and *difference equations* in [5, 6], while no specific name is coined in [7, 8] where (1.1) is written $Ly(\cdot) = 0$, and L is referred to as a *difference operator*.

say that the system is *exponentially stable* if, along any solution $y(\cdot)$, the norm of y_t decays exponentially as $t \rightarrow +\infty$.

A noticeable feature of the present work is the use of control-theoretic ideas, based on the introduction of the control system (2.18) whose exponential stability is equivalent to the one of (1.1). *In the time-invariant case* (that is: when the D_j are constant), the transfer function of that control system is a matrix-valued function of one complex variable, say $H(p)$, and the Henry-Hale theorem (Theorem 2 below) amounts to say that system (1.1) is exponentially stable if and only if H is holomorphic in a half plane $\{p \in \mathbb{C}, \Re(p) > \alpha\}$ for some $\alpha < 0$. In fact, due to the particular structure of (1.1), the Henry-Hale theorem is equivalent to the seemingly stronger statement: system (1.1) is exponentially stable if and only if H is holomorphic and bounded in some half plane $\{p \in \mathbb{C}, \Re(p) > \alpha\}$ with $\alpha < 0$; see discussion after Theorem 3. *For the case addressed in this paper, namely the periodic case*, we shall introduce for periodic difference-delay systems (whose state space is infinite-dimensional) a so-called *Harmonic Transfer Function* (HTF), generalizing the one introduced in control theory for periodic linear differential equations in some \mathbb{R}^n or \mathbb{C}^n (whose state space is finite dimensional); cf. [9, 10]. The HTF will be defined as a holomorphic map of one complex variable, valued in the space of linear operators on square summable \mathbb{C}^d -valued periodic functions, that reduces to the function mapping $p \in \mathbb{C}$ to multiplication by the ordinary transfer function evaluated at p when the system is time-invariant. Our main result has the same interpretation in terms of HTF as the Henry-Hale theorem in terms of classical transfer functions. Namely, a periodic difference-delay control system is exponentially stable if and only if its HTF is holomorphic and bounded in a half-plane $\{p \in \mathbb{C}, \Re(p) > \alpha\}$, for some $\alpha < 0$. This control-theoretic viewpoint not only provides one with an elegant formulation of the result, but also plays an essential role in the proof of the “only if” part. We believe that introducing the harmonic transfer function in the present context is another contribution of the paper, and that our analysis lays ground for a systematic study of the HTF for more general, infinite-dimensional periodic systems.

Motivations. Apart from their own interest and their natural character, systems (1.1) also arise naturally in connection with other specific classes of dynamical systems, let us mention three.

a) *Neutral functional differential equations* are functional differential equations of the form

$$\frac{d}{dt} \left(y(t) - \sum_{j=1}^N D_j(t) y(t - \tau_j) \right) = B_0(t)y(t) + \sum_{j=1}^N B_j(t) y(t - \tau_j), \quad t \geq s, \quad (1.2)$$

where the maps $B_i(\cdot)$ and $D_i(\cdot)$ are assumed regular enough. It is naturally related to (1.2). Indeed, the solution operator of (1.2), that maps the state at time s to the state at time t , is a compact perturbation of the one of (1.1), see [7, thm 7.3 Section 3.7]. Roughly speaking, this makes the exponential stability properties of (1.2) similar to those of (1.1), up to finitely many Jordan blocks of the monodromy operator whose eigenvalues might lie outside the unit disk. References [8, 5] handle (1.1) as a means to study exponential stability of neutral differential equation in the time invariant case; similarly, characterizing the exponential stability of (1.1) appears to be a substantial step towards analyzing the one of (1.2) in the periodic case. We take this step below, but discussing the exponential stability of (1.2) along these lines is beyond the scope of the present paper; see the discussion in Section 6.

b) Some 1-D hyperbolic systems are equivalent to a subclass of difference difference-delay systems. Indeed, after explicit integration in terms of arbitrary functions of one variable (backward and forward waves) using the method of characteristics, their evolution is governed by

equations like (1.1), see for example [4, 11, 2]. In particular, Theorem 3 can be applied to such systems in the case of linear periodic boundary conditions.

- c) An initial motivation of the authors for this study lies with the stability of active microwave circuits. These can be regarded as nonlinear systems with infinite dimensional state space, due to transmission lines that are modelled either by 1-D hyperbolic systems or by delays. The response to a periodic signal in the so-called high-gain regime is typically a periodic solution, and linearizing along this trajectory provides one with a linear periodic time-varying dynamical system. The latter is governed by a functional differential equation slightly more general than (1.2), which has a high frequency limit system of the form (1.1). The solution operator of the linearized system along the periodic trajectory turns out to be a compact perturbation of the solution operator of the high frequency limit system, and if the latter is exponentially stable then the stability of the periodic solution to the initial, nonlinear system can be investigated by chasing polar singularities of the harmonic transfer function of the linearized system; see [3] and [12] for a discussion of such methods in Engineering, based on frequency-wise numerical simulations obtained by so-called *harmonic-balance* techniques.

State of the art. In the *time-invariant* case, the stability of functional differential equations like (1.2) has been studied from the fifties on, either using Laplace transform and semi-group theory to reduce issues to the location of zeros of almost periodic holomorphic functions as in [13, 5, 8, 7], or else seeking Lyapunov–Krasovskii functionals to prove stability in specific cases as in [7, 14, 15]. For the special case of difference-delay systems, the Henry-Hale theorem stated in Section 2 as Theorem 2 gives a definite answer, theoretically at least, to exponential stability issues. It was initially established in [8] for finitely many commensurate delays, and later carried over to the case of countably many, not necessarily commensurate delays [5]. Generalizations to distributed delay systems may be found in [7, Chapter 9]. Of course the condition of the theorem is often difficult to check in practice.

Apparently, the stability of *time-varying* linear difference-delay systems was not studied nearly as extensively: literature we know of can be recapped quickly as follows. First, a (fairly restrictive) sufficient condition for exponential stability based on the Perron-Frobenius theorem can be found in [1]. Second, if a linear time-varying system of the form (1.1) arises from a network of lossless telegrapher’s equations (for instance as the high frequency limit of an electrical network), then a sufficient condition for exponential stability can be based on dissipativity at the nodes of the network [2]; related, somewhat specific criteria for hyperbolic 1-D systems may be found in [4]. Third, a representation formula for solutions of (1.1) in the general time-varying case is carried out in [6]. That paper contains interesting results on the insensitivity to p of L^p -exponential stability, and how stability can be preserved under perturbation of the delays. However, the characterization of exponential stability stated in [6] is intractable, as it involves the spectrum of sums of products of matrices with indefinitely growing number of terms and factors when t goes large, indexed according to combinatorial rules involving the lattice generated by the τ_j .

Organization of the paper. Section 2 introduces some notation and states our main result. Section 3 makes more definitions and notation, while Section 4 is devoted to preliminary steps like a variation-of-constants formula for solutions to difference-delay systems. We prove Theorem 3 in Section 5, and Section 6 concludes the paper with a conjecture on neutral periodic delay equations.

2. Statement of the Main Result

2.1. Notations

2.1.1. The real and complex fields are denoted by \mathbb{R} and \mathbb{C} . We denote by $\|\cdot\|$ the canonical Euclidean norm on \mathbb{C}^d and by $\|\cdot\|$ the norm of a matrix or operator on such a Euclidean (finite dimensional) space: $\|M\| = \sup_{\|x\|=1} \|Mx\|$. We write I_d for the $d \times d$ identity matrix or the identity operator on a vector space of dimension d , and by I_∞ for the ‘‘doubly infinite’’ identity matrix $[\delta_{i,j}]_{(i,j) \in \mathbb{Z}^2}$ or more specifically the identity operator on the Hilbert space $\ell^2(\mathbb{Z}, \mathbb{C}^d)$, see below.

2.1.2. For $E \subset \mathbb{R}$ a Lebesgue-measurable set, we write $L^q(E)$ for the space of (equivalence classes of a.e. coinciding) complex valued measurable functions on E with norm $\|g\|_{L^q(E)} = (\int_E |g(y)|^q dy)^{1/q}$ (ess. sup $_E |g|$ if $q = \infty$). The space $L^q_{\text{loc}}(E)$ consists of functions whose restriction to any compact $K \subset E$ lies in $L^q(K)$. We write $C^0(E)$ for the space of \mathbb{R} -valued or \mathbb{C} -valued continuous functions, and if E is compact we endow it with the *sup* norm denoted by $\|\cdot\|_{C^0}$. For $\alpha \in (0, 1)$, we designate with $C^\alpha(E)$ the space of Hölder continuous functions with exponent α ; *i.e.*, $f \in C^\alpha(E)$ if and only if $|f(x) - f(y)| \leq C|x - y|^\alpha$ for some constant C and all $x, y \in E$, the smallest C being the Hölder constant of f . When E is open, $C^1(E)$ indicates the space of complex functions with the first derivative of which lies in $C^0(E)$, and $C^{1,\alpha}(E)$ stands for functions whose first derivative belongs to $C^\alpha(E)$. When dealing with vector-valued functions, we indicate the target space as in $L^q(E, \mathbb{C}^{d \times d})$ or $C^{1,\alpha}(E, \mathbb{C}^d)$, while replacing in the definition the modulus by the norm in that target space.

2.1.3. Stating the main results makes use of doubly infinite matrices defined by $d \times d$ blocks, that define operators on the Hilbert space:

$$\ell^2(\mathbb{Z}, \mathbb{C}^d) := \{z = (z_j)_{j \in \mathbb{Z}} : z_j \in \mathbb{C}^d, \sum_{j=-\infty}^{+\infty} \|z_j\|^2 < +\infty\}, \quad (2.1)$$

equipped with the norm

$$\|z\|_{\ell^2} := \left(\sum_{j=-\infty}^{+\infty} \|z_j\|^2 \right)^{1/2}. \quad (2.2)$$

The norm of bounded operators $\ell^2(\mathbb{Z}, \mathbb{C}^d) \rightarrow \ell^2(\mathbb{Z}, \mathbb{C}^d)$ (or infinite matrices representing them) is denoted by $\|\cdot\|_2$: $\|L\|_2 = \sup_{\|z\|_{\ell^2}=1} \|Lz\|_{\ell^2}$.

2.1.4. If a function f is defined on a set E and $E' \subset E$, we put $f|_{E'}$ to mean the restriction of f to E' .

2.1.5. The Hardy space \mathcal{H}^2 of the right half-plane is comprised of those holomorphic functions f in $\{z \in \mathbb{C} : \Re(z) > 0\}$ satisfying

$$\|f\|_{\mathcal{H}^2}^2 := \sup_{x>0} \int_{-\infty}^{+\infty} |f(x + iy)|^2 dy < +\infty.$$

Such functions are $\sqrt{2}$ -metrically the Laplace transforms of square integrable functions on $[0, +\infty)$ [21, Ch. 8, p. 131]. That is, $\mathcal{H}^2 = \{f : f(z) = \int_0^{+\infty} e^{-zt} u(t) dt, u \in L^2([0, +\infty), \mathbb{C}), \Re(z) > 0\}$, and it holds that $\|f\|_{\mathcal{H}^2}^2 = 2\|u\|_{L^2([0, +\infty), \mathbb{C})}^2$. Hardy spaces are used in the proofs only.

2.2. Problem statement

Consider a periodic difference-delay system:

$$y(t) = \sum_{j=1}^N D_j(t)y(t - \tau_j), \quad t > s, \quad y(s + \theta) = \phi(\theta) \text{ for } -\tau_N \leq \theta \leq 0, \quad (2.3)$$

where $s \in \mathbb{R}$ is the initial time, d and N are positive integers, $\tau_1 < \dots < \tau_N$ are strictly positive real numbers (the delays) and the $D_j : \mathbb{R} \rightarrow \mathbb{C}^{d \times d}$ are continuous T -periodic matrix-valued functions:

$$D_j(t + T) = D_j(t), \quad 1 \leq j \leq N, \quad (2.4)$$

while $\phi : [-\tau_N, 0] \rightarrow \mathbb{C}^d$ is the initial condition and the solutions to (2.3) are \mathbb{C}^d -valued functions $y(t)$ of the time $t \in [s - \tau_N, +\infty)$. When the D_j are real-valued, all the results below specialize to real solutions, obtained by restricting to real initial conditions. Note that delays are assumed to be strictly positive. We assume that T is strictly larger than any of them: $\tau_N < T$, which is no loss of generality for we may replace T by kT with $k \in \mathbb{N}$.

One may seek solutions in different functional spaces, with different topologies, yielding a priori different notions of exponential stability. We investigate measurable or continuous solutions, both being possible because the maps $D_j(\cdot)$ are continuous. When seeking for continuous solutions, a compatibility condition is required for a continuous map $\phi : [-\tau_N, 0] \rightarrow \mathbb{C}^d$ to be an initial condition: ϕ must belong to the space

$$C_s := \left\{ \phi \in C^0([-\tau_N, 0], \mathbb{C}^d) : \phi(0) = \sum_{j=1}^N D_j(s)\phi(-\tau_j) \right\}. \quad (2.5)$$

Given $\phi \in C_s$, an easy recursion shows that system (2.3) has a unique continuous solution $y \in C^0([-\tau_N + s], \mathbb{C}^d)$ with initial condition ϕ . Thus, we can define for $t \geq s$ the *solution operator* $U(t, s) : C_s \rightarrow C_t$, mapping $\phi \in C_s$ to $U(t, s)\phi$ defined by

$$(U(t, s)\phi)(\theta) = y(t + \theta), \quad \theta \in [-\tau_N, 0]. \quad (2.6)$$

Note that $C_{s+T} = C_s$ and $U(t, s) = U(t + T, s + T)$, by the T -periodicity of the D_j . One may also search solutions of (2.3) in $L^q_{\text{loc}}([s - \tau_N, +\infty), \mathbb{C}^d)$, $1 \leq q \leq \infty$, requiring that (2.3) holds for almost every $t \in [s, +\infty)$; no compatibility condition on ϕ is needed in that case, an easy recursion shows that for each $\phi \in L^q([-\tau_N, 0], \mathbb{C}^d)$ the system (2.3) admits a unique solution $y \in L^q_{\text{loc}}([s, +\infty), \mathbb{C}^d)$ with initial condition ϕ . Consequently, one can define for $t \geq s$ the solution operator $U_q(t, s) : L^q([-\tau_N, 0], \mathbb{C}^d) \rightarrow L^q([-\tau_N, 0], \mathbb{C}^d)$ that maps ϕ to $U_q(t, s)\phi$ given by a relation analogous to (2.6):

$$(U_q(t, s)\phi)(\theta) = y(t + \theta), \quad \text{a.e. } \theta \in [-\tau_N, 0]. \quad (2.7)$$

These choices yield a priori different notions of exponential stability:

Definition 1. System (2.3) is called C^0 -exponentially stable if there exist $\gamma, K > 0$ such that

$$\|U(t, s)\phi\|_{C^0} \leq K e^{-\gamma(t-s)} \|\phi\|_{C^0}, \quad \text{for all } s \in \mathbb{R}, \text{ all } t \geq s \text{ and all } \phi \in C_s. \quad (2.8)$$

System (2.3) is called L^q -exponentially stable, $q \in [1, \infty]$, if there exist $\gamma, K > 0$ such that

$$\|U_q(t, s)\phi\|_{L^q} \leq K e^{-\gamma(t-s)} \|\phi\|_{L^q}, \quad \text{for all } s \in \mathbb{R}, \text{ all } t \geq s \text{ and all } \phi \in L^q([-\tau_N, 0], \mathbb{C}^d). \quad (2.9)$$

These notions in fact turn out to be equivalent, as shown by the following result contained in [2, Theorem 3.4] which is also a consequence of [6, Corollary 3.29].

Proposition 1. *For each $q \in [1, \infty]$, System (2.3) is L^q -exponentially stable if and only if it is C^0 -exponentially stable.*

Proposition 1 plays an essential role in the proof of Theorem 3 to come, because the sufficiency part establishes C^0 -exponential stability while the necessity part assumes L^2 -exponential stability. In view of Proposition 1, we simplify terminology by making the following definition.

Definition 2 (Exponential stability). System (2.3) is called *exponentially stable* if and only if there exist $\gamma, K > 0$ such that one of the equivalent properties (2.8) or (2.9) holds.

2.3. The Henry-Hale Theorem for time-invariant systems

In the time-invariant case, the following characterisation of exponential stability is known.

Theorem 2 (Henry-Hale [5, Section 3], [7, Theorem 3.5]). *Assume that the maps $t \mapsto D_j(t)$ are constant, $1 \leq j \leq N$. Then, a necessary and sufficient condition for System (2.3) to be C^0 -exponentially stable is the existence of a real number $\beta < 0$ such that :*

$$I_d - \sum_{j=1}^N e^{-p\tau_j} D_j \text{ is invertible in } \mathbb{C}^{d \times d} \text{ for every } p \text{ in } \{z \in \mathbb{C} : \Re(z) \geq \beta\}. \quad (2.10)$$

The present paper is devoted “generalizing” Theorem 2 to the case where the maps $t \mapsto D_j(t)$ are *not* constant, but periodic. A naive generalization would consist in simply requiring that (2.10) holds for all t . The following example shows that such a result does not hold.

Let $N = 1$, $d = 2$, $T = 2\pi$, $\tau_1 = \pi/2$, $s = 0$ and consider (2.3) with $D_1(t) = \begin{pmatrix} 1/2 & \alpha \cos^2 t \\ \alpha \sin^2 t & 1/2 \end{pmatrix}$ for some $\alpha \in (0, 1)$. On the one hand, condition (2.10) is satisfied because the eigenvalues of $D_1(t)$ are $\frac{1}{2}(1 \pm \alpha |\sin 2t|)$. On the other hand, every solution must satisfy

$$y(t + 2k\tau_1) = (D_1(t + 2\tau_1)D_1(t + \tau_1))^k y(t), \quad t \geq 0, \quad k \in \mathbb{N}.$$

Since $D_1(t + 2\tau_1)D_1(t + \tau_1) = \begin{pmatrix} 1/4 + \alpha^2 \cos^4 t & \alpha/2 \\ \alpha/2 & 1/4 + \alpha^2 \sin^4 t \end{pmatrix}$ has eigenvalue $\frac{1}{4}(1 + \alpha^2 + \alpha^2 \cos^2 2t) + \frac{1}{2}\alpha\sqrt{1 + \alpha^2 \cos^2 2t}$, which is larger than 1 for α close enough to 1 and t close enough to 0, exponential stability cannot hold.

2.4. Main result: generalisation of the Henry-Hale theorem to the time-varying periodic case

From now on, we consider the case where the maps $t \mapsto D_j(t)$ are T -periodic but not necessarily constant. Let us introduce a few preliminaries before stating a proper generalization of Theorem 2 that is the core contribution of the paper. Defining

$$\omega := 2\pi/T, \quad (2.11)$$

we let $\check{D}_j(k)$, for $j \in \{1, \dots, n\}$ and $k \in \mathbb{Z}$, denote the k -th Fourier coefficient of D_j :

$$\check{D}_j(k) := \frac{1}{T} \int_0^T D_j(t) e^{-ik\omega t} dt. \quad (2.12)$$

The (doubly infinite) block Laurent matrix associated with $D_j(t)$ is then denoted by L_{D_j} ; its block entries are

$$(L_{D_j})_{k,n} := \check{D}_j(n - k), \quad n, k \in \mathbb{Z}. \quad (2.13)$$

We also let $\Delta_{\tau_j, \omega}$, $j \in \{1, \dots, N\}$ be the (doubly infinite) block diagonal matrix given by

$$\Delta_{\tau_j, \omega} := \text{diag}(\dots, e^{-2i\omega\tau_j} I_d, e^{-i\omega\tau_j} I_d, I_d, e^{+i\omega\tau_j} I_d, e^{+2i\omega\tau_j} I_d, \dots). \quad (2.14)$$

It is important to note that both L_{D_j} and $\Delta_{\tau_j, \omega}$ define, by matrix multiplication, bounded operators $\ell^2(\mathbb{Z}, \mathbb{C}^d) \rightarrow \ell^2(\mathbb{Z}, \mathbb{C}^d)$; let us briefly explain why (see Section 2.1.3 above for notations). On the one hand, $\Delta_{\tau_j, \omega}$ is even an isometry: obviously, $\|\Delta_{\tau_j, \omega}\|_2 = 1$. On the other hand, Fourier expansion defines, by the Parseval identity, an isometry between $\ell^2(\mathbb{Z}, \mathbb{C}^d)$ and the Hilbert space of square integrable \mathbb{C}^d -valued T -periodic functions of time that conjugates $L_{D_j} : \ell^2(\mathbb{Z}, \mathbb{C}^d) \rightarrow \ell^2(\mathbb{Z}, \mathbb{C}^d)$ to pointwise multiplication by D_j , a bounded operator with operator norm no larger than the sup norm of $D_j(\cdot)$; by isometry, one also has $\|L_{D_j}\|_2 \leq \|D_j\|_{C^0}$.

We can now define a function R of the complex variable p , valued in the space of (doubly infinite matrix representing) bounded operators $\ell^2(\mathbb{Z}, \mathbb{C}^d) \rightarrow \ell^2(\mathbb{Z}, \mathbb{C}^d)$, by the formula:

$$R(p) := I_\infty - \sum_{j=1}^N e^{-p\tau_j} L_{D_j} \Delta_{\tau_j, \omega}. \quad (2.15)$$

A simple computation also gives a description by block:

$$R(p) = I_\infty - \left(\sum_{j=1}^N e^{-(p-i\ell\omega)\tau_j} \check{D}_j(\ell - k) \right)_{(k, \ell) \in \mathbb{Z}^2}. \quad (2.16)$$

Our main result is the following.

Theorem 3 (Necessary and sufficient condition for exponential stability). *Assume that the $D_j : \mathbb{R} \rightarrow \mathbb{C}^{d \times d}$ are periodic and differentiable with Hölder continuous derivative for $1 \leq j \leq N$. Under this assumption, a necessary and sufficient condition for System (2.3) to be exponentially stable is the existence of a real number $\beta < 0$ such that :*

- (i) $R(p)$ is invertible $\ell^2(\mathbb{Z}, \mathbb{C}^d) \rightarrow \ell^2(\mathbb{Z}, \mathbb{C}^d)$ for each p in $\{z \in \mathbb{C} : \Re(z) \geq \beta\}$,
- (ii) there is a positive number M such that $\|R^{-1}(p)\|_2 \leq M$ for all p in $\{z \in \mathbb{C} : \Re(z) \geq \beta\}$.

When the D_j are constant matrices, we get from (2.13) that the operators $L_{D_j} \Delta_{\tau_j, \omega}$ are block diagonal with k -th diagonal block $e^{ik\omega\tau_j} D_j$, so condition (i) in Theorem 3 is equivalent to saying that $I_d - \sum_{j=1}^N e^{-p\tau_j} D_j$ is invertible for $\Re(p) \geq \beta$. In this case, since every analytic function in a vertical strip which is uniformly almost-periodic in the pure imaginary direction and has no zeros must be bounded below in modulus by a strictly positive constant [16, Ch. III, Sec. 2, Cor. 1], it holds that $|\det(I_d - \sum_{j=1}^N e^{-p\tau_j} D_j)| \geq c > 0$ when $\Re(p) \geq \beta$, whence condition (ii) is redundant. Thus, Theorem 3 yields back the Henry-Hale theorem for systems with constant coefficients.

For general periodic difference-delay systems, one can surmise that Condition (i) does not imply Condition (ii), but when all delays are commensurate it does. Indeed, $p \mapsto R(p)$ is then periodic of period $i\tau$ for some $\tau > 0$ and hence, since inversion is continuous on invertible elements of a Banach algebra while (ii) needs only be checked for p in a compact set by periodicity and a von Neumann series argument, we deduce that (i) implies (ii). This we record as a corollary:

Corollary 4. *If the delays are commensurate, then condition (ii) can be omitted in Theorem 3.*

Remark 1. In Theorem 3, we suppose that the periodic matrices D_i have Hölder-continuous derivatives. Actually, the sufficiency part remains true without any regularity hypothesis on the derivative of D_j if we strengthen assumption (ii) into:

$$\| \| R(p)^{-1} \| \|_{\mathfrak{W}} \text{ is uniformly bounded for all } p \text{ in } \{z \in \mathbb{C} : \Re(z) \geq \beta\}, \quad (2.17)$$

where the Wiener norm $\| \cdot \|_{\mathfrak{W}}$ of a doubly infinite block matrix is defined in (5.11).

2.5. A system-theoretic point of view; transfer functions and harmonic transfer functions.

To (1.1) one can associate the control system

$$y(t) = \sum_{j=1}^N D_j(t)y(t - \tau_j) + u(t), \quad t > s, \quad y(t) = u(t) = 0 \text{ for } t < s, \quad (2.18)$$

with control $u \in C^0([s, +\infty), \mathbb{C}^d)$ or $L_{loc}^p([s, +\infty), \mathbb{C}^d)$ and output y . System (2.18) is exponentially stable if, when driven by an input $u(t)$ vanishing for $t > t_0$ and generating an output y , the L^∞ (or L^q)-norm of the restriction $y|_{[t-\tau_N, t]}$ decays exponentially fast to zero as $t \rightarrow +\infty$.

Solving for u in (2.18), one sees that for any $t_0 > \tau_N$ and $\phi \in C_{t_0}$ (resp. $L^p([-\tau_N, 0], \mathbb{C}^d)$), there is $u \in C^0([s, t_0], \mathbb{C}^d)$ (resp. $L^p([s, t_0], \mathbb{C}^d)$) such that the corresponding output y satisfies $\phi(\theta) = y(t_0 + \theta)$ for $\theta \in [-\tau_N, 0]$. Consequently, outputs of (2.18) associated to controls that vanish for $t > t_0$ coincide with solutions to (1.1) where we put $s = t_0$. This explains why exponential stability of the control system (2.18) is equivalent to exponential stability of (1.1).

A linear time-invariant control system can be represented, under mild assumptions, as a convolution operator; *i.e.*, the output is obtained by convolving the input with some kernel [17]. Taking Fourier-Laplace transforms converts the latter into a multiplication operator in the Fourier-Laplace domain, the multiplier of which is called the *transfer function* of the system.

In the time-invariant case, the transfer function of (2.18) is the matrix-valued function of one complex variable, say p , given by $H(p) = \left(I_d - \sum_{j=1}^N e^{-p\tau_j} D_j \right)^{-1}$ (see *e.g.* [18]). The Henry-Hale theorem says that (2.3) is exponentially stable if and only if H is holomorphic in a half-plane $\{p \in \mathbb{C}, \Re(p) > \beta\}$ for some $\beta < 0$. In this case, as shown after Theorem 3, $H(p)$ is in fact bounded for $\Re(p) \geq \beta' > \beta$.

In the periodic case, the concept of transfer function generalizes into the one of *harmonic transfer function* (in short: HTF). It is again a function of one complex variable p , but instead of ranging in $\mathbb{C}^{d \times d}$ it takes values in the space of bounded operators $\ell^2(\mathbb{Z}, \mathbb{C}^d) \rightarrow \ell^2(\mathbb{Z}, \mathbb{C}^d)$. More precisely, one can check from (2.18) that $\|y\|_{L^2([s, \tau])} \leq C e^{\gamma(\tau-s)} \|u\|_{L^2([s, +\infty])}$ for appropriate constants C , γ and all $\tau > s$; see (5.51). Thus, when $u \in L^2([s, +\infty), \mathbb{C}^d)$ one can define for $\Re(p) > \gamma$ the Laplace transforms of $y(t)$ and $u(t)$ by the formulas

$$\hat{Y}(p) := \int_s^{+\infty} e^{-pt} y(t) dt \quad \text{and} \quad \hat{U}(p) := \int_s^{+\infty} e^{-pt} u(t) dt. \quad (2.19)$$

If \hat{Y} , \hat{U} get sampled at equally spaced points on vertical lines with mesh $i\omega$ (recall (2.11)), we

obtain infinite vectors of $d \times 1$ column vectors:

$$\hat{\mathbb{Y}}(p) := \begin{pmatrix} \vdots \\ \hat{Y}(p + i\omega) \\ \hat{Y}(p) \\ \hat{Y}(p - i\omega) \\ \vdots \end{pmatrix} \quad \text{and} \quad \hat{\mathbb{U}}(p) := \begin{pmatrix} \vdots \\ \hat{U}(p + i\omega) \\ \hat{U}(p) \\ \hat{U}(p - i\omega) \\ \vdots \end{pmatrix}, \quad \Re(p) > \gamma. \quad (2.20)$$

We choose to order the entries of doubly infinite vectors from top to bottom; *i.e.*, we could write

$$\hat{\mathbb{Y}}(p) = \left(\hat{Y}(p - ik\omega) \right)_{k \in \mathbb{Z}}^{\mathbf{t}}, \quad \text{and} \quad \hat{\mathbb{U}}(p) = \left(\hat{U}(p - ik\omega) \right)_{k \in \mathbb{Z}}^{\mathbf{t}}, \quad \Re(p) > \gamma, \quad (2.21)$$

where the superscript \mathbf{t} means transpose.

Since the $\hat{U}(p + ik\omega)/T$ (resp. $\hat{Y}(p + ik\omega)/T$) are the Fourier coefficients of the T -periodic function $x \mapsto \sum_{j=-\infty}^{+\infty} e^{-p(x+jT)}u(x+jT)$ (resp. $x \mapsto \sum_{j=-\infty}^{+\infty} e^{-p(x+jT)}y(x+jT)$), both $\hat{\mathbb{Y}}$ and $\hat{\mathbb{U}}$ lie in $\ell^2(\mathbb{Z}, \mathbb{C}^d)$. Now, for $\Re(p)$ large enough, there is a unique operator-valued holomorphic function $p \mapsto \mathbf{H}(p)$ such that $\mathbf{H}(p) : \ell^2(\mathbb{Z}, \mathbb{C}^d) \rightarrow \ell^2(\mathbb{Z}, \mathbb{C}^d)$ maps $\hat{\mathbb{Y}}$ to $\hat{\mathbb{U}}$. In fact, $\mathbf{H}(p)$ is none but the inverse of $R(p)$ defined in (2.15), that exists for $\Re(p)$ large enough by a von Neumann series argument; details are given in Section 5.2. Thus,

$$\hat{\mathbb{Y}}(p) = \mathbf{H}(p) \hat{\mathbb{U}}(p), \quad \Re(p) > \gamma'. \quad (2.22)$$

Definition 3 (Harmonic Transfer Function). The operator-valued holomorphic function $\mathbf{H}(\cdot)$ is called the harmonic transfer function (HTF).

We can now recast Theorem 3 as follows.

Corollary 5. *If system (2.3) has periodic coefficients of class $C^{1,\alpha}$, it is exponentially stable if and only if the control system (2.18) is exponentially stable, which is if and only if the harmonic transfer function of the latter is analytic and bounded in $\{p \in \mathbb{C} : \Re(p) > \beta\}$ for some $\beta < 0$.*

Equation (2.22) connects the inputs and outputs of a linear periodic control system in the frequency domain *via* the HTF. If the system is time-invariant with transfer function H , then $\hat{Y}(p) = H(p)\hat{U}(p)$ and the HTF reduces to the block diagonal $\mathbf{H}(p) = \text{diag}\{\dots, H, H, H, \dots\}$, which makes for a natural generalization. But the HTF generalizes the ordinary transfer function further, in a way which is worth explaining: if a stable, *time-invariant* control system is fed with a periodic input signal $e^{ikt}v$ with $v \in \mathbb{C}^d$, then the output is asymptotically $H(ik)v e^{ikt}$, with H the transfer function. If a stable *periodic* control system is fed with a periodic input signal $v e^{ikt}$, then the output is asymptotically $\mathbf{H}(ik)(\mathbf{v})e^{ikt}$, with \mathbf{H} the harmonic transfer function and \mathbf{v} the infinite column vector which is zero except for the element v on the block-line of index 0. Thus, while a periodic input is asymptotically mapped to a periodic output with the same period in the time-invariant case, it gets mapped in the periodic case to an oscillating signal with the same period but carried by the wave $\mathbf{H}(ik)(\mathbf{v})$ which has the period of the system. This follows from (5.69), (5.56), (5.55) and (5.47).

3. Reminder on functions with bounded variations and the Lebesgue-Stieltjes integral

For I a real interval and $f : I \rightarrow \mathbb{R}$ a function, the *total variation* of f on I is defined as

$$W_I(f) := \sup_{\substack{x_0 < x_1 < \dots < x_N \\ x_i \in I, N \in \mathbb{N}}} \sum_{i=1}^N |f(x_i) - f(x_{i-1})| < \infty. \quad (3.1)$$

The space $BV(I)$ of functions with *bounded variation* on I consists of those f such that $W_I(f) < \infty$, endowed with the norm $\|f\|_{BV(I)} = W_I(f) + |f(d)|$ where $d \in I$ is arbitrary but fixed. Different d give rise to equivalent norms for which $BV(I)$ is a Banach space, and $\|\cdot\|_{BV(I)}$ is stronger than the uniform norm. We let $BV_r(I)$ and $BV_l(I)$ be the closed subspaces of $BV(I)$ comprised of right and left continuous functions, respectively. We write $BV_{loc}(\mathbb{R})$ for the space of functions whose restriction to any bounded interval $I \subset \mathbb{R}$ lies in $BV(I)$. Observe that

$$W_I(fg) \leq W_I(f) \sup_{x \in I} |g(x)| + W_I(g) \sup_{x \in I} |f(x)|. \quad (3.2)$$

Each $f \in BV(I)$ has a limit $f(x^-)$ (resp. $f(x^+)$) from the left (resp. right) at every $x \in I$ where the limit applies [19, sec. 1.4]. Hence, one can associate to f a finite signed Borel measure ν_f on I such that $\nu_f((a, b)) = f(b^-) - f(a^+)$, and if I is bounded on the right (resp. left) and contains its endpoint b (resp. a), then $\nu_f(\{b\}) = f(b) - f(b^-)$ (resp. $f(a^+) - f(a)$) [19, ch. 7, pp. 185–189]. Note that different f may generate the same ν_f : for example if f and f_1 coincide except at isolated interior points of I , then $\nu_f = \nu_{f_1}$. For $g : I \rightarrow \mathbb{R}$ a measurable function, summable against ν_f , the Lebesgue-Stieltjes integral $\int gdf$ is defined as $\int g d\nu_f$, whence the differential element df identifies with $d\nu_f$ [19, ch. 7, pp. 190–191]. This type of integral is useful for it is suggestive of integration by parts, but caution must be used when integrating a function with respect to df over a subinterval $J \subset I$ because $\nu_{(f|_J)}$ needs *not* coincide with the restriction $(\nu_f)|_J$ of ν_f to J . More precisely, if the lower bound a (resp. the upper bound b) of J belongs to J and lies interior to I , then the two measures may differ by the weight they put on $\{a\}$ (resp. $\{b\}$), and they agree only when f is left (resp. right) continuous at a (resp. b). By $\int_J gdf$, we always mean that we integrate g against $\nu_{(f|_J)}$ and *not* against $(\nu_f)|_J$. We often trade the notation $\int_J gdf$ for one of the form $\int_{a^\pm}^{b^\pm} gdf$, where the interval of integration J is encoded in the bounds put on the integral sign: a lower bound a^- (resp. a^+) means that J contains (resp. does not contain) its lower bound a , while an upper bound b^+ (resp. b^-) means that J contains (resp. does not contain) its upper bound b . Then, the previous word of caution applies to additive rules: for example, when splitting $\int_{a^\pm}^{b^\pm} gdf$ into $\int_{a^\pm}^{c^\pm} gdf + \int_{c^\pm}^{b^\pm} gdf$ where $c \in (a, b)$, we must use c^+ (resp. c^-) if f is right (resp. left) continuous at c .

To a finite, signed or complex Borel measure μ on I , one can associate its *total variation measure* $|\mu|$, defined on a Borel set $B \subset I$ by $|\mu|(B) = \sup_{\mathcal{P}} \sum_{E \in \mathcal{P}} |\mu(E)|$ where \mathcal{P} ranges over all partitions of B into Borel sets, see [20, sec. 6.1]; its total mass $|\mu|(I)$ is called the total variation of μ , denoted as $\|\mu\|$. Thus, the total variation is defined both for functions of bounded variation and for measures, with different meanings. When $f \in BV(I)$ is monotonic then $W_I(f) = \|\nu_f\|$, but in general it only holds that $\|\nu_f\| \leq 2W_I(f)$; this follows from the Jordan decomposition of f as a difference of two increasing functions, each of which has variation at most $W_I(f)$ on I [19, Thm. 1.4.1]. In any case, it holds that $|\int gdf| \leq \int |g| d|\nu_f| \leq 2W_I(f) \sup_I |g|$.

The previous notations and definitions also apply to vector and matrix-valued functions BV -functions, replacing absolute values in (3.1) by Euclidean and operator norms, respectively.

4. Variation-of-constants formulas and a priori estimates

Below, we gather results of important use later on that we could not see proven in the literature.

4.1. Fundamental solution and variation-of-constants formula

The *fundamental solution* $X : \mathbb{R}^2 \rightarrow \mathbb{R}^{d \times d}$ of (2.3) is defined by the following equation [7]:

$$X(t, s) = \begin{cases} 0 & \text{for } t < s, \\ I_d + \sum_{j=1}^N D_j(t) X(t - \tau_j, s) & \text{for } t \geq s. \end{cases} \quad (4.1)$$

Arguing inductively, it is easy to check that X uniquely exists and is continuous at each (t, s) such that $t - s \notin \mathcal{F}$, where \mathcal{F} is the positive lattice generated by the τ_ℓ :

$$\mathcal{F} := \left\{ \sum_{\ell=1}^N n_\ell \tau_\ell, (n_1, \dots, n_N) \in \mathbb{N}^N \right\}. \quad (4.2)$$

Clearly, X has a bounded jump across each line $t - s = \mathfrak{f}$ for $\mathfrak{f} \in \mathcal{F}$; in fact, a moment's thinking will convince the reader that

$$X(t, s) = - \sum_{\mathfrak{f} \in [0, t-s] \cap \mathcal{F}} \mathfrak{C}_\mathfrak{f}(t), \quad s \leq t, \quad (4.3)$$

where each $\mathfrak{C}_\mathfrak{f}(\cdot)$ lies in $C^{1,\delta}(\mathbb{R}, \mathbb{C}^{d \times d})$ if all the D_j do. One can see that $\mathfrak{C}_\mathfrak{f}(t)$ is a finite sum of products of $D_j(t - \mathfrak{f}')$, where \mathfrak{f}' ranges over the elements of \mathcal{F} whose defining integers n_ℓ in (4.2) do not exceed those defining \mathfrak{f} , the empty product being the identity matrix. A precise expression for $\mathfrak{C}_\mathfrak{f}$ can be obtained by reasoning as in [6, Sec. 3.2] or [2, Sec. 4.5], but we will not need it. Note that, for fixed t , the function $X(t, \cdot)$ is locally of bounded variation on \mathbb{R} . The importance of the fundamental solution stems from the following integral representation formula:

Proposition 6. *For $s \in \mathbb{R}$ and ϕ in C_s the solution $y \in C^0([s - \tau_N, +\infty), \mathbb{C}^{d \times d})$ to (2.3) is*

$$y(t) = - \sum_{j=1}^N \int_{s^-}^{(s+\tau_j)^-} d_\alpha X(t, \alpha) D_j(\alpha) \phi(\alpha - \tau_j - s), \quad t \geq s, \quad (4.4)$$

where X was defined in (4.1). Moreover, there exist $K > 0$ and $\lambda \in \mathbb{R}$ such that

$$\|X(t, s)\| \leq K e^{\lambda(t-s)}, \quad \text{for all } t \geq s. \quad (4.5)$$

PROOF. A proof is given in Appendix A; we provide one because the corresponding formulas for autonomous linear difference-delay systems in [5] and [22] seem to have issues. \square

4.2. The variation-of-constants formula in L^2

For the necessary part of Theorem 3, we need to deal with initial data in $L^2([s - \tau_N, s])$ and $L^2_{loc}([s, \infty))$ -solutions of (2.3). Let us consider the control system with input u :

$$y(t) = \sum_{j=1}^N D_j(t) y(t - \tau_j) + u(t), \quad u(t) \in \mathbb{C}^d, \quad t \geq s. \quad (4.6)$$

When $t \geq s$, we set $y_t(\theta) = y(t + \theta)$ for $-\tau_N \leq \theta \leq 0$. The function $y_t : [-\tau_N, 0] \rightarrow \mathbb{C}^d$ (more accurately: its equivalence class modulo coincidence a.e.) is the state at time t of the input-output system (4.6). In particular y_s is now the initial condition, previously denoted by ϕ when dealing with the homogeneous system (2.3) (for which $u \equiv 0$).

Recalling X from (4.1), we define for each $(t, \alpha) \in \mathbb{R}^2$ a map $K(t, \alpha) : [-\tau_N, 0] \rightarrow \mathbb{C}^{d \times d}$ by

$$K(t, \alpha)(\theta) = -X(t + \theta, \alpha), \quad \theta \in [-\tau_N, 0]. \quad (4.7)$$

For $t \in \mathbb{R}$, $\theta \in [-\tau_N, 0]$ and $I \subset \mathbb{R}$ an interval bounded on the left, the map $\alpha \mapsto X(t + \theta, \alpha)$ lies in $BV_l(I)$ and vanishes for $\alpha > t + \theta$. As indicated in Section 3, we associate to this map a $\mathbb{C}^{d \times d}$ -valued Borel measure $\nu_{(X(t+\theta, \cdot))|_I}$ on I . It follows from (4.1), (4.3) and (A.4) that for fixed t and θ , this measure is of the form

$$\nu_{(X(t+\theta, \cdot))|_I} = \sum_{\mathfrak{f} \in \mathcal{F}, t+\theta-\mathfrak{f} \in \tilde{I}} \mathfrak{C}_{\mathfrak{f}}(t + \theta) \delta_{t+\theta-\mathfrak{f}}, \quad (4.8)$$

where \tilde{I} denotes I deprived from its right endpoint (if contained in I , otherwise $\tilde{I} = I$) and $\delta_{t+\theta-\mathfrak{f}}$ is a Dirac mass at $t + \theta - \mathfrak{f}$. For fixed t , the boundedness of I from the left implies that the number of terms in (4.8) is majorized independently of $\theta \in [-\tau_N, 0]$, and clearly the map sending a Borel set $E \subset I$ to the function $\theta \mapsto -\nu_{(X(t+\theta, \cdot))|_I}(E)$ is a vector measure, valued in the space $\mathcal{B}([-\tau_N, 0], \mathbb{C}^{d \times d})$ of bounded measurable $\mathbb{C}^{d \times d}$ -valued functions on $[-\tau_N, 0]$ endowed with the sup norm. In view of (4.7) we denote this measure with $d_\alpha K(t, \alpha)$, and though it depends on I the latter will be understood from the context. Then, for $g \in C^0(I, \mathbb{C}^d)$, one can define the integral $\int_I dK(t, \alpha)g(\alpha)$ as a member of $\mathcal{B}([-\tau_N, 0], \mathbb{C}^d)$; see, e.g. [23] for a definition of integrals against a vector measure. To us, it is just a compact notation for the function

$$\left(\int_I dK(t, \alpha)g(\alpha) \right) (\theta) := - \int_I d_\alpha X(t + \theta, \alpha)g(\alpha) = - \sum_{\mathfrak{f} \in \mathcal{F}, t+\theta-\mathfrak{f} \in \tilde{I}} \mathfrak{C}_{\mathfrak{f}}(t + \theta)g(t + \theta - \mathfrak{f}). \quad (4.9)$$

The rightmost term in (4.9) can be rewritten as $\sum_{\mathfrak{f} \in \mathcal{F}} \mathfrak{C}_{\mathfrak{f}}(t + \theta)\tilde{g}(t + \theta - \mathfrak{f})$ where \tilde{g} is equal to g on \tilde{I} and to zero elsewhere. Since the $\mathfrak{C}_{\mathfrak{f}}$ are bounded (being periodic in $C^{1,\delta}(\mathbb{R}, \mathbb{C}^{d \times d})$) while those $\mathfrak{f} \in \mathcal{F}$ for which $t + \theta - \mathfrak{f} \in \tilde{I}$ for some $\theta \in [-\tau_N, 0]$ are finite in number because I is bounded on the left, one can check that $\|\int_I dK(t, \alpha)g(\alpha)\|_{L^2(I, \mathbb{C}^d)} \leq C\|g\|_{L^2(I, \mathbb{C}^d)}$ for some constant C depending on t , I , the D_j and the τ_j . Hence, (4.9) makes good sense for $g \in L^2(I, \mathbb{C}^d)$, but of course $(\int_I dK(t, \alpha)g(\alpha))(\theta)$ is only defined for a.e. $\theta \in [-\tau_N, 0]$ in this case. As indicated in Section (3), the interval of integration may alternatively be encoded in the bounds on the integral sign. It is immaterial whether I contains its right endpoint or not when writing $\int_I dK(t, \alpha)g(\alpha)$, for the rightmost term in (4.9) depends only on \tilde{I} . Moreover, $\int_I dK(t, \alpha)g(\alpha)$ is independent of $g(\alpha)$ for $\alpha > t$ as $K(t, \alpha) \equiv 0$ when $\alpha > t$. Hence, if $g \in L^2_{loc}([a, +\infty))$ then $\int_{a^-}^{b^\pm} dK(t, \alpha)g(\alpha)$ is independent of $b > t$ and of the choice of sign in b^\pm . Accordingly, we find it convenient to define

$$\int_{a^-}^{+\infty} dK(t, \alpha)g(\alpha) := \int_{a^-}^{b^\pm} dK(t, \alpha)g(\alpha) \quad \text{for any } b > t. \quad (4.10)$$

The variation-of-constants formula for L^2_{loc} -solutions to (4.6) now goes as follows.

Proposition 7. *For $s \in \mathbb{R}$, let $y(\cdot) \in L^2_{loc}([s - \tau_N, +\infty), \mathbb{C}^d)$ and $u(\cdot) \in L^2_{loc}([s, +\infty), \mathbb{C}^d)$ satisfy (4.6) for a.e. $t \geq s$. Then:*

$$y_t = U_2(t, s)y_s + \int_{s^-}^{+\infty} d_\alpha K(t, \alpha)u(\alpha), \quad t \geq s. \quad (4.11)$$

PROOF. Denote by η_t the right hand side of (4.11), and put $\eta(t + \theta) = \eta_t(\theta)$ for $t \geq s$ with $\theta \in [-\tau_N, 0]$. Write \tilde{y} for the solution of the homogeneous system (2.3) with initial condition y_s , so that $\tilde{y}_t = U_2(t, s)y_s$. Fix $t \geq s$; for a.e. θ in $[-\tau_N, 0]$ satisfying $t + \theta < s$ (such θ occur when $t < s + \tau_N$), it holds that $(\int_{s^-}^{+\infty} d_\alpha K(t, \alpha)u(\alpha))(\theta) = \int_{s^-}^{+\infty} d_\alpha X(t + \theta, \alpha)u(\alpha) = 0$ (because $X(t + \theta, \alpha) = 0$ when $t + \theta < s \leq \alpha$) and that $\tilde{y}_t(\theta) = \tilde{y}(t + \theta) = y_s(t + \theta) = y(t + \theta)$ (because \tilde{y} and y have the same initial condition y_s on $[-\tau_N, 0]$). Hence, we obtain indeed that $\eta_t(\theta) = y_t(\theta)$ for a.e. θ in $[-\tau_N, 0]$ such that $t + \theta < s$. Next, for a.e. $\theta \in [-\tau_N, 0]$ such that $t + \theta \geq s$, we deduce from the definition of η_t along with (2.7) (with $q = 2$), (2.3) (where y is set to \tilde{y}), (4.10) and (A.4) that

$$\begin{aligned} \eta_t(\theta) &= \tilde{y}(t + \theta) - \int_{s^-}^{+\infty} d_\alpha X(t + \theta, \alpha)u(\alpha) \\ &= \sum_{j=1}^N D_i(t + \theta)\tilde{y}(t + \theta - \tau_j) - \sum_{j=1}^N D_i(t + \theta) \int_{s^-}^{+\infty} d_\alpha X(t + \theta - \tau_j, \alpha)u(\alpha) + u(t + \theta) \\ &= \sum_{j=1}^N D_j(t + \theta)\eta(t + \theta - \tau_j) + u(t + \theta). \end{aligned}$$

Altogether, we proved that to each $t \geq 0$ there is a set $E_t \subset [-\tau_N, 0]$ of zero measure such that, for $\theta \in [-\tau_N, 0] \setminus E_t$, we have $\eta(t + \theta) = \sum_{j=1}^N D_j(t + \theta)\eta(t + \theta - \tau_j) + u(t + \theta)$ when $t + \theta \geq s$ and $\eta(t + \theta) = y_s(t + \theta)$ when $t + \theta < s$. Thus, letting \mathbb{Q} denote the rational numbers, the set $E := \cup_{q \in \mathbb{Q}} \{q + E_q\}$ has measure zero and since each $t \in \mathbb{R}$ can be written as $q + \theta$ with $q \in \mathbb{Q}$ and $\theta \in [-\tau_N, 0]$, we deduce that $\eta(t)$ satisfies (4.6) when $t \geq s$ and $t \notin E$, with initial condition $\eta = y_s$ a.e. on $[-\tau_N, 0]$. By uniqueness η must be equal to y a.e., as desired. \square

Let us now fix t and consider the function $s \mapsto X(t, t - s)$ from \mathbb{R} to $\mathbb{C}^{d \times d}$. It is right-continuous and lies in $BV_{loc}(\mathbb{R}, \mathbb{C}^{d \times d})$, moreover it is identically zero for $s < 0$. Thus, on each interval $I \subset \mathbb{R}$ which is bounded on the right, it generates a $\mathbb{C}^{d \times d}$ -valued measure $\nu_{(X(t, t - \cdot))_I}$ which is but the image of $d_s X(t, s)$ under the map $s \mapsto t - s$ from $\{t\} - I$ onto I . When I is understood, we denote this measure by $d_s X(t, t - s)$; we get from (4.8) that

$$d_s X(t, t - s) = - \sum_{\mathfrak{f} \in \mathcal{F} \cap \check{I}} \mathfrak{C}_{\mathfrak{f}}(t) \delta_{\mathfrak{f}}, \quad (4.12)$$

where \check{I} denotes I deprived from its left endpoint (if contained in I) and the $\mathfrak{C}_{\mathfrak{f}}$ are as in (4.8). By periodicity of the D_j , one sees that $X(t, t - s)$ is periodic in t and therefore the $\mathfrak{C}_{\mathfrak{f}}(t)$ are periodic as well. If the D_j belong to $C^{1, \delta}(\mathbb{R}, \mathbb{C}^{d \times d})$ then so do the $\mathfrak{C}_{\mathfrak{f}}$, and we may define the measure

$$\frac{\partial}{\partial t} d_s X(t, t - s) := - \sum_{\mathfrak{f} \in \mathcal{F} \cap \check{I}} \left(\frac{\partial}{\partial t} \mathfrak{C}_{\mathfrak{f}}(t) \right) \delta_{\mathfrak{f}} \quad (4.13)$$

with coefficients in $C^\delta(\mathbb{R}, \mathbb{C}^{d \times d})$. The number of terms in the right hand sides of (4.12), (4.13), and the number of sums of products of the $D_j(t - \mathfrak{f}')$ involved in these terms, tend to $+\infty$ with the length of I . Nevertheless, in the next setion we show that the growth of the corresponding left hand sides is at most exponential.

4.3. A priori estimates

For $I = [0, \tau]$ (say), we shall derive an *a priori* estimate, independent of t , for the quantities:

$$\|d_s X(t, t-s)\|_I := \sum_{\mathfrak{f} \in \mathcal{F} \cap \bar{I}} \|\mathfrak{c}_{\mathfrak{f}}(t)\|, \quad (4.14)$$

$$\left\| \frac{\partial}{\partial t} d_s X(t, t-s) \right\|_I := \sum_{\mathfrak{f} \in \mathcal{F} \cap \bar{I}} \left\| \frac{\partial}{\partial t} \mathfrak{c}_{\mathfrak{f}}(t) \right\|, \quad (4.15)$$

$$\Lambda_{I, \delta} \left(\frac{\partial}{\partial t} d_s X(t, t-s) \right) := \sum_{\mathfrak{f} \in \mathcal{F} \cap \bar{I}} \Lambda_{\delta} \left(\frac{\partial}{\partial t} \mathfrak{c}_{\mathfrak{f}}(t) \right), \quad (4.16)$$

where $\Lambda_{\delta}(g)$ indicates the Hölder constant of $g \in C^{\delta}(\mathbb{R}, \mathbb{C}^{d \times d})$. Note that (4.14) and (4.15) are just the total variations of the measures $d_s X(t, t-s)$ and $\frac{\partial}{\partial t} d_s X(t, t-s)$ on I , respectively.

Proposition 8. *Assume that $D_j \in C^{1, \delta}(\mathbb{R}, \mathbb{C}^{d \times d})$ is T -periodic for $1 \leq j \leq N$ and some $\delta \in (0, 1)$. Then, there exist $K, \gamma \geq 0$ such that, for all $\tau \geq 0$ and $t \in \mathbb{R}$:*

$$\max \left\{ \|d_s X(t, t-s)\|_{[0, \tau]}, \left\| \frac{\partial}{\partial t} d_s X(t, t-s) \right\|_{[0, \tau]}, \Lambda_{[0, \tau], \delta} \left(\frac{\partial}{\partial t} d_s X(t, t-s) \right) \right\} \leq K e^{\gamma \tau}. \quad (4.17)$$

PROOF. Observe that $d_s X(t, t-s)$ is the image of the measure $d_s X(t, s)$ under the map $s \mapsto t-s$ from $[t-\tau, t]$ onto $[0, \tau]$. Hence, it follows from (A.4) that

$$\begin{aligned} d_s X(t, t-s) &= \sum_{j=1}^N D_j(t) d_s X(t-\tau_j, t-s) \\ &= \sum_{j=1}^N D_j(t) d_s X(t-\tau_j, t-\tau_j-(s-\tau_j)) \quad \text{on } [0, \tau], \end{aligned}$$

as well as

$$\begin{aligned} \frac{\partial}{\partial t} d_s X(t, t-s) &= \sum_{j=1}^N \frac{\partial}{\partial t} D_j(t) d_s X(t-\tau_j, t-s) + \sum_{j=1}^N D_j(t) \frac{\partial}{\partial t} d_s X(t-\tau_j, t-s) \\ &= \sum_{j=1}^N \frac{\partial}{\partial t} D_j(t) d_s X(t-\tau_j, t-\tau_j-(s-\tau_j)) + \sum_{j=1}^N D_j(t) \frac{\partial}{\partial t} d_s X(t-\tau_j, t-\tau_j-(s-\tau_j)). \end{aligned}$$

Since $X(t-\tau_j, t-\tau_j-(s-\tau_j))$ is identically zero for $s < \tau_j$, we deduce from the previous identities that if K' is an upper bound (made independent of t by periodicity) for the $\|D_j\|$, the $\left\| \frac{\partial}{\partial t} D_j \right\|$, the $\Lambda_{\delta}(D_j)$ and the $\Lambda_{\delta}(\frac{\partial}{\partial t} D_j)$, then

$$\|d_s X(t, t-s)\|_{[0, \tau]} \leq K' \sum_{j=1}^N \|d_{\beta} X(t-\tau_j, t-\tau_j-\beta)\|_{[0, \tau-\tau_j]}, \quad (4.18)$$

$$\left\| \frac{\partial}{\partial t} d_s X(t, t-s) \right\|_{[0, \tau]} \leq K' \sum_{j=1}^N \|d_{\beta} X(t-\tau_j, t-\tau_j-\beta)\|_{[0, \tau-\tau_j]} + \left\| \frac{\partial}{\partial t} d_{\beta} X(t-\tau_j, t-\tau_j-\beta) \right\|_{[0, \tau-\tau_j]}, \quad (4.19)$$

$$\begin{aligned} \Lambda_{[0, \tau], \delta} \left(\frac{\partial}{\partial t} d_s X(t, t-s) \right) &\leq K' \sum_{j=1}^N \Lambda_{[0, \tau-\tau_j], \delta} (X(t-\tau_j, t-\tau_j-\beta)) + \|d_{\beta} X(t-\tau_j, t-\tau_j-\beta)\|_{[0, \tau-\tau_j]} \\ &\quad + \Lambda_{[0, \tau-\tau_j], \delta} \left(\frac{\partial}{\partial t} X(t-\tau_j, t-\tau_j-\beta) \right) + \left\| \frac{\partial}{\partial t} d_{\beta} X(t-\tau_j, t-\tau_j-\beta) \right\|_{[0, \tau-\tau_j]}, \quad (4.20) \end{aligned}$$

where we used that $\Lambda_\delta(AB) \leq \Lambda_\delta(A)\|B\| + \Lambda_\delta(B)\|A\|$. The proof may now be completed by an inductive step, similar to the one used to establish (4.5). More precisely, pick γ , K'' positive large enough that $K'Ne^{-\gamma\tau_1} < 1/4$ and

$$\max\{\|\mathfrak{C}_{\tau_1}(t)\|, \left\| \frac{\partial}{\partial t} \mathfrak{C}_{\tau_1}(t) \right\|, \Lambda_\delta(\mathfrak{C}_{\tau_1}), \Lambda_\delta\left(\frac{\partial}{\partial t} \mathfrak{C}_{\tau_1}\right)\} \leq K'', \quad t \in \mathbb{R}. \quad (4.21)$$

Note that such a K'' exists as \mathfrak{C}_{τ_1} is periodic and lies in $C^{1,\delta}(\mathbb{R})$. From (4.12) one sees that $\|d_s X(t, t-s)\|_{[0,\tau]}$ is equal to 0 when $\tau \in [0, \tau_1)$ and to $\|\mathfrak{C}_{\tau_1}(t)\|$ when $\tau = \tau_1$. *A fortiori* then, for $\tau \in [0, \tau_1]$ we have:

$$\max\left\{\|d_s X(t, t-s)\|_{[0,\tau]}, \left\| \frac{\partial}{\partial t} d_s X(t, t-s) \right\|_{[0,\tau]}\right\} \leq K'' e^{\gamma\tau}, \quad \Lambda_{[0,\tau],\delta}\left(\frac{\partial}{\partial t} d_s X(t, t-s)\right) \leq 2K'K'' e^{\gamma\tau},$$

so that (4.17) holds with $K := \max\{K'', 2K'K''\}$ for $0 \leq \tau \leq \tau_1$. Now, assume that (4.17) is true for all t and $\tau \in [0, k\tau_1)$, with $k > 0$ an integer. For $\tau \in (k\tau_1, (k+1)\tau_1]$, we get in view of this hypothesis and the definition of γ , that

$$\begin{aligned} \|d_s X(t, t-s)\|_{[0,\tau]} &\leq K'NK e^{\gamma(\tau-\tau_1)} \leq K e^{\gamma\tau}/4 && \text{by (4.18),} \\ \left\| \frac{\partial}{\partial t} d_s X(t, t-s) \right\|_{[0,\tau]} &\leq 2K'NK e^{\gamma(\tau-\tau_1)} \leq K e^{\gamma\tau}/2 && \text{by (4.19),} \\ \Lambda_{[0,\tau],\delta}\left(\frac{\partial}{\partial t} d_s X(t, t-s)\right) &\leq 4K'NK e^{\gamma(\tau-\tau_1)} \leq K e^{\gamma\tau} && \text{by (4.20).} \end{aligned}$$

By induction on k , this completes the proof. \square

5. Proof of Theorem 3

5.1. Sufficiency part of Theorem 3

The argument is essentially modeled after the one in [5] for the time-invariant case, but technicalities arise because when dealing with time-varying coefficients Laplace transformation yields infinite-dimensional functional equations. Assuming (i) and (ii) in Theorem 3, we shall prove exponential stability of System (2.3) in three steps.

Step 1

Since the D_j are assumed to have Hölder continuous derivatives, their Fourier coefficients (2.12) satisfy

$$\|\check{D}_j(k)\| \leq \frac{C}{1+|k|^{1+\delta}}, \quad j \in \{1, \dots, n\}, \quad (5.1)$$

where C is a positive constant and $\delta \in]0, 1[$ is the Hölder exponent of the derivative; indeed, (5.1) follows at once from the fact that the modulus of the k -th Fourier coefficient of $\frac{d}{dt} D_j$ is bounded by $C'/(1+|k|)^\delta$, see [24, Ch. 2, thm. 4.7]. Note, for later use, that the constant C in (5.1) depends affinely on the Hölder constant of $\frac{d}{dt} D_j$. Substituting the Fourier expansion of D_j in (4.1) yields:

$$X(t, s) = I_d + \sum_{k \in \mathbb{Z}} \sum_{j=1}^N e^{ik\omega t} \check{D}_j(k) X(t - \tau_j, s) \quad \text{if } t \geq s, \quad X(t, s) = 0 \quad \text{if } t < s, \quad (5.2)$$

where the right hand side of (5.2) is absolutely convergent, locally uniformly in (t, s) , thanks to (5.1) and the local boundedness of X . Hereafter, we shall denote with a hat the Laplace transform with respect to the first variable; *e.g.*, we put

$$\hat{X}(p, s) := \int_{-\infty}^{+\infty} e^{-pt} X(t, s) dt. \quad (5.3)$$

Taking the Laplace transform of both sides of (5.2) and interchanging the series and integral signs (this is possible thanks to (5.1)), we obtain that for $p \in \mathbb{C}$ with $\Re(p) > \lambda$ where λ is as in (4.5):

$$\int_{-\infty}^{+\infty} e^{-pt} X(t, s) dt = \int_{-\infty}^{+\infty} \mathbf{1}_{[s, +\infty)}(t) e^{-pt} I_d dt + \sum_{k \in \mathbb{Z}} \sum_{j=1}^N \int_{-\infty}^{+\infty} e^{(-p+ik\omega)t} \check{D}_j(k) X(t - \tau_j, s) dt, \quad (5.4)$$

where $\mathbf{1}_{[s, +\infty)}$ is the characteristic function equal to 1 on $[s, +\infty)$ and to 0 elsewhere; note that the factor $X(t - \tau_j, s)$ in the integrand on the right hand side of (5.4) is *a fortiori* zero for $t < s$. By an elementary change of variable, we deduce from (5.4) that

$$\hat{X}(p, s) = \frac{e^{-ps}}{p} I_d + \sum_{k \in \mathbb{Z}} \sum_{j=1}^N e^{(-p+ik\omega)\tau_j} \check{D}_j(k) \hat{X}(p - ik\omega, s), \quad \Re(p) > \lambda. \quad (5.5)$$

This can be rephrased as

$$\sum_{k \in \mathbb{Z}} (R(p))_{0,k} \hat{X}(p - ik\omega, s) = \frac{e^{-ps}}{p} I_d \quad (5.6)$$

with $R(p)$ defined in (2.15) and (2.16). Substituting $p + in\omega$ for p in (5.6) and letting n range over \mathbb{Z} , we obtain a system of countably many equations that may be written as

$$R(p) \hat{\mathbb{X}}(p, s) = \hat{e}(p, s), \quad (5.7)$$

with $\hat{\mathbb{X}}(p, s)$ and $\hat{e}(p, s)$ the infinite $d \times d$ block vectors:

$$\hat{\mathbb{X}}(p, s) := \begin{pmatrix} \vdots \\ \hat{X}(p + i\omega, s) \\ \hat{X}(p, s) \\ \hat{X}(p - i\omega, s) \\ \vdots \end{pmatrix} \quad \text{and} \quad \hat{e}(p, s) := \begin{pmatrix} \vdots \\ \frac{e^{-(p+i\omega)s}}{p+i\omega} I_d \\ \frac{e^{-ps}}{p} I_d \\ \frac{e^{-(p-i\omega)s}}{p-i\omega} I_d \\ \vdots \end{pmatrix}. \quad (5.8)$$

Clearly, these are the Laplace transforms of infinite $d \times d$ block vectors $\mathbb{X}(t, s)$ and $e(t, s)$:

$$\hat{\mathbb{X}}(p, s) = \int_{-\infty}^{+\infty} e^{-pt} \mathbb{X}(t, s) dt, \quad \hat{e}(p, s)(p, s) = \int_{-\infty}^{+\infty} e^{-pt} e(t, s) dt, \quad (5.9)$$

$$\text{with} \quad \mathbb{X}(t, s) := \begin{pmatrix} \vdots \\ e^{-\frac{2i\pi t}{T}} X(t, s) \\ X(t, s) \\ e^{\frac{2i\pi t}{T}} X(t, s) \\ \vdots \end{pmatrix}, \quad e(t, s) := \begin{pmatrix} \vdots \\ I_d e^{-\frac{2i\pi t}{T}} \mathbf{1}_{[s, +\infty)} \\ I_d \mathbf{1}_{[s, +\infty)} \\ I_d e^{\frac{2i\pi t}{T}} \mathbf{1}_{[s, +\infty)} \\ \vdots \end{pmatrix}. \quad (5.10)$$

The infinite dimensional linear constant system (5.7) recasts the finite-dimensional periodic time-varying delay system (4.1) in terms of Fourier series and Laplace transforms. In order to estimate $\mathbb{X}(p, s)$ -and eventually $X(t, s)$ - we shall invert the operator $R(p)$ in the next step.

Step 2

For $A := (a_{i,j})_{i,j \in \mathbb{Z}}$ a doubly infinite block matrix, where $a_{i,j}$ is a $d \times d$ complex matrix for each i and j in \mathbb{Z} , we say that A has off diagonal decay of order r if there is a constant C such that $\|a_{i,j}\| \leq C(1 + |i - j|)^{-r}$. We also define the Wiener norm of A to be

$$\|A\|_{\mathfrak{W}} := \sum_{k \in \mathbb{Z}} \sup_{|i-j|=k} \|a_{i,j}\|. \quad (5.11)$$

Note that $\|A\|_2 \leq \|A\|_{\mathfrak{W}}$, as follows immediately from the Schur test [25]. We denote by $B(\ell^2(\mathbb{Z}, \mathbb{C}^d), \ell^2(\mathbb{Z}, \mathbb{C}^d))$ the space of those doubly infinite block matrices A such that $\|A\|_2 < \infty$, and by $\mathfrak{W}(\ell^2(\mathbb{Z}, \mathbb{C}^d), \ell^2(\mathbb{Z}, \mathbb{C}^d))$ the subspace of those A satisfying $\|A\|_{\mathfrak{W}} < \infty$. It is easy to check that $\mathfrak{W}(\ell^2(\mathbb{Z}, \mathbb{C}^d), \ell^2(\mathbb{Z}, \mathbb{C}^d))$ is a subalgebra of the algebra $B(\ell^2(\mathbb{Z}, \mathbb{C}^d), \ell^2(\mathbb{Z}, \mathbb{C}^d))$.

One can see from (5.1) and (2.16) that $R(p)$ possesses off-diagonal decay of order $1 + \delta$, moreover the constant is uniform over any half-space $\Re(p) \geq a$. From this, it follows at once that $\|R(p)\|_{\mathfrak{W}}$ is uniformly bounded over such a half-space. Hence, $R(p) \in \mathfrak{W}(\ell^2(\mathbb{Z}, \mathbb{C}^d), \ell^2(\mathbb{Z}, \mathbb{C}^d))$ for all p and clearly, $R : \mathbb{C} \rightarrow B(\ell^2(\mathbb{Z}, \mathbb{C}^d), \ell^2(\mathbb{Z}, \mathbb{C}^d))$ is a Banach valued holomorphic function.

Now, assumption (i) of Theorem 3 tells us that $R(p)$ is invertible in $B(\ell^2(\mathbb{Z}, \mathbb{C}^d), \ell^2(\mathbb{Z}, \mathbb{C}^d))$ for $\Re(p) \geq \beta$, and assumption (ii) that the inverse operator $R^{-1}(p)$ has uniformly bounded $\|\cdot\|_2$ -norm. Therefore, as $R(p)$ has off-diagonal decay of order $1 + \delta$, we get from [26, thm. 1.2] and the boundedness of $\|R^{-1}(p)\|_2$ that $R^{-1}(p)$ also has off-diagonal decay of order $1 + \delta$, uniformly for $\Re(p) \geq \beta$. In particular, $\|R^{-1}(p)\|_{\mathfrak{W}}$ is uniformly bounded³ for $\Re(p) \geq \beta$:

$$\left\| \left[I_\infty - \sum_{i=j}^N e^{-p\tau_j} L_{D_j} \Delta_{\tau_j, \omega} \right]^{-1} \right\|_{\mathfrak{W}} \leq C_1, \quad p \in \{z \in \mathbb{C} \mid \Re(z) \geq \beta\}. \quad (5.12)$$

In view of (5.12), applying to (2.15) the result of [29, thm. 2.6] which is a generalization of the Wiener lemma to Banach algebra-valued almost periodic functions (the Banach algebra here being $\mathfrak{W}(\ell^2(\mathbb{Z}, \mathbb{C}^d), \ell^2(\mathbb{Z}, \mathbb{C}^d))$, see also [30] for a more general version of that theorem), we get for any α with $\beta < \alpha < 0$ that $R^{-1}(p)$ admits a Fourier expansion on the vertical line $p = \alpha + i\mathbb{R}$ of the form :

$$R^{-1}(\alpha + i\tilde{\omega}) = \sum_{k \in \mathbb{Z}} \tilde{R}^{\{k, \alpha\}} e^{i\beta_k \tilde{\omega}}, \quad \tilde{\omega} \in \mathbb{R}, \quad (5.13)$$

where β_k is real and $\tilde{R}^{\{k, \alpha\}} \in \mathfrak{W}(\ell^2(\mathbb{Z}, \mathbb{C}^d), \ell^2(\mathbb{Z}, \mathbb{C}^d))$ with

$$\sum_{k \in \mathbb{Z}} \|\tilde{R}^{\{k, \alpha\}}\|_{\mathfrak{W}} < +\infty. \quad (5.14)$$

Moreover, following the proof of the theorem in section 3 on Dirichlet's series page 147 of [16], which is possible because Cauchy's theorem holds for Banach valued holomorphic functions (one can apply all the arguments there in weak form by evaluating on a fixed vector in $\ell^2(\mathbb{Z}, \mathbb{C}^d)$), we deduce that $\tilde{R}^{\{k, \alpha\}} = R^{\{k\}} e^{\alpha\beta_k}$, where $R^{\{k\}} \in B(\ell^2(\mathbb{Z}, \mathbb{C}^d), \ell^2(\mathbb{Z}, \mathbb{C}^d))$ is independent of α . Thus, one can rewrite (5.13) and (5.14) as

$$R^{-1}(\alpha + i\tilde{\omega}) = \sum_{k \in \mathbb{Z}} R^{\{k\}} e^{\alpha\beta_k} e^{\beta_k \tilde{\omega}}, \quad \tilde{\omega} \in \mathbb{R}, \quad \alpha > \beta, \quad (5.15)$$

³Controlled inversion (bounding the norm of the inverse in a subalgebra in terms of the norm in this subalgebra and the norm of the inverse in the algebra) was pioneered in Baskakov [27] and Nikolski [28].

with

$$\sum_{k \in \mathbb{Z}} \left\| R^{\{k\}} \right\|_{\mathfrak{W}} e^{\alpha \beta_k} < +\infty. \quad (5.16)$$

In addition, observing from the von Neumann series that

$$R^{-1}(p) = \sum_{n=0}^{+\infty} \left(\sum_{j=1}^N L_{D_j} \Delta_{\tau_j, \omega} e^{-p\tau_j} \right)^n \quad (5.17)$$

whence

$$\left\| R^{-1}(p) \right\|_2 \leq \frac{1}{1 - e^{\Re(\tau_1)} \sum_{j=1}^N \left\| L_{D_j} \right\|_2 \left\| \Delta_{\tau_j, \omega} \right\|_2} \quad \text{for } \Re(p) > \gamma_1 := \frac{\log(\sum_{j=1}^N \left\| L_{D_j} \right\|_2 \left\| \Delta_{\tau_j, \omega} \right\|_2)}{\tau_1}, \quad (5.18)$$

we deduce that

$$\{\beta_k | k \in \mathbb{Z}\} \subset \left\{ \sum_{j=1}^N n_j \tau_j \mid n_j \text{ non-positive integers for } j = 1, \dots, N \right\} = -\mathcal{F} \quad (5.19)$$

where \mathcal{F} was defined in (4.2). We remark that (5.15), (5.16) and (5.19) are reminiscent of [13, p. 429, equations (12.15.12) and (12.15.13)], that deals with complex-valued functions.

Step 3

We now compute $\mathbb{X}(t, s)$. For this, we apply the Laplace inversion formula (see for example [7, Ch. 1, Lem. 5.2]). It gives us for $\Re(p) = c > \lambda$, with λ as in (4.5), that

$$\mathbb{X}(t, s) = \lim_{\tilde{\omega} \rightarrow +\infty} \frac{1}{2\pi i} \int_{c-i\tilde{\omega}}^{c+i\tilde{\omega}} R^{-1}(p) \hat{e}(p, s) e^{pt} dp. \quad (5.20)$$

We shall need two lemmas, the proof of which is postponed till the end of this section.

Lemma 9. *Let $\beta < \alpha < 0$ and $\rho_m := \frac{2\pi m}{T} + \frac{\pi}{T}$ for m a positive integer. Then, we have that*

$$\mathbb{X}(t, s) = \lim_{m \rightarrow +\infty} \frac{1}{2\pi i} \int_{\alpha - i\rho_m}^{\alpha + i\rho_m} R^{-1}(p) \hat{e}(p, s) e^{pt} dp + Q(t), \quad (5.21)$$

where

$$Q(t) := \dots + R_{-1}^{-1}(-i\omega) e^{-i\omega t} + R_0^{-1}(0) + R_1^{-1}(i\omega) e^{i\omega t} + \dots$$

with R_k^{-1} to mean the k -th column of R^{-1} .

Recall that $X(t, s)$ is the element of index 0 of the infinite column vector $\mathbb{X}(t, s)$. So, if for each $k \in \mathbb{Z}$ we let $(R_{0,n}^{\{k\}})_{n \in \mathbb{Z}}$ enumerate the line of index 0 of the matrix $R^{\{k\}}$ defined in (5.15), it is a consequence of (5.21) that

$$X(t, s) = \frac{1}{2\pi i} \lim_{m \rightarrow +\infty} \int_{\alpha - i\rho_m}^{\alpha + i\rho_m} \sum_{k, n \in \mathbb{Z}} R_{0,n}^{\{k\}} \frac{e^{-(p-in\omega)s}}{p - in\omega} e^{p(t+\beta_k)} dp + Q(t)_0, \quad (5.22)$$

where $Q(t)_0$ stands for the element of index 0 of the column vector $Q(t)$; note that the convergence of the series in (5.22) ensues from (5.16).

Lemma 10. For all t and s such that $t + \beta_k - s \neq 0$ for all $k \in \mathbb{Z}$, we have that

$$X(t, s) = \sum_{(t+\beta_k-s)<0, n \in \mathbb{Z}} R_{0,n}^{\{k\}} e^{in\omega(t+\beta_k)} + Q(t)_0. \quad (5.23)$$

Note that on a bounded interval there are only finitely many β_k , because the integer coefficients in (5.19) are non-positive. Observe also from (4.1) that $X(t, \cdot)$ is left continuous. Hence, Lemma 10 allows in fact to compute $X(t, s)$ for all (t, s) (the proof of the lemma will be given momentarily). Hereafter, we fix the symbol s to mean the initial time, and we shall evaluate the variation of $X(t, \tau)$ in its second argument when the latter ranges over $[s, s + \tau_N]$. To this effect, we substitute τ for s in (5.23) and observe, since $Q(t)_0$ is independent of τ while the sum in (5.23) is piecewise constant in τ with jumps at the $t + \beta_k$, caused by the constraint on the indices, where it is left continuous, that

$$W_{[s, s+\tau_N]} X(t, \cdot) \leq \sum_{0 \leq t+\beta_k-s \leq \tau_N, n \in \mathbb{Z}} \left\| R_{0,n}^{\{k\}} \right\| \quad (5.24)$$

$$\leq \left(\sum_{k, n \in \mathbb{Z}} \left\| R_{0,n}^{\{k\}} \right\| e^{\alpha\beta_k} e^{|\alpha|\tau_N} \right) e^{\alpha(t-s)}, \quad (5.25)$$

where we used in the second inequality that $\alpha(t + \beta_k - s - \tau_N) \geq 0$ because $\alpha < 0$. Thus, with K' an upper bound for the $\|D_i(t)\|$ as in (A.1), we deduce from (4.4) and (5.25) that

$$\|y(t)\| \leq K' N (W_{[s, s+\tau_N]} X(t, \cdot)) \|\phi\|_{C^0} \quad (5.26)$$

$$\leq K' N \left(\sum_{k, n \in \mathbb{Z}} \left\| R_{0,n}^{\{k\}} \right\| e^{\alpha\beta_k} e^{|\alpha|\tau_N} \right) e^{\alpha(t-s)} \|\phi\|_{C^0}, \quad (5.27)$$

thereby showing that System (2.3) is C^0 -exponentially stable. This achieves the proof of sufficiency in Theorem 3 granted Lemmas 9 and 10, that we now establish.

PROOF OF LEMMA 9. Considering the subsequence $\rho_m = \frac{2\pi m}{T} + \frac{\pi}{T}$, we deduce from the equation (5.20) that :

$$\mathbb{X}(t, s) = \lim_{m \rightarrow +\infty} \frac{1}{2\pi i} \int_{c-i\rho_m}^{c+i\rho_m} R^{-1}(p) \hat{e}(p, s) e^{pt} dp. \quad (5.28)$$

Note that each component of $R^{-1}(p) \hat{e}(p, s)$ is a meromorphic function in the half plane $\{p \in \mathbb{C} | \Re(p) > \beta\}$, with poles in the set $\{ik\pi/T, k \in \mathbb{Z}\}$. Thus, by (5.28) and the residue theorem, we get that

$$\mathbb{X}(t, s) = \lim_{m \rightarrow +\infty} \frac{1}{2\pi i} \left(\int_{\alpha-i\rho_m}^{\alpha+i\rho_m} + \int_{\alpha+i\rho_m}^{c+i\rho_m} - \int_{\alpha-i\rho_m}^{c-i\rho_m} \right) R^{-1}(p) \hat{e}(p, s) e^{pt} dp + Q(t) \quad (5.29)$$

where we recall that :

$$Q(t) := \dots + R_{-1}^{-1}(-i\omega) e^{-i\omega t} + R_0^{-1}(0) + R_1^{-1}(i\omega) e^{i\omega t} + \dots$$

with R_k^{-1} the k th column of R^{-1} . Observe that

$$\|Q(t)\|_2 \leq \left\| R^{-1}(0) \right\|_{\mathbb{W}} < +\infty, \quad (5.30)$$

which warrants taking the limit in (5.29). To establish the lemma, it remains to show that

$$\lim_{m \rightarrow +\infty} \int_{\alpha+i\rho_m}^{c+i\rho_m} R^{-1}(p) \hat{e}(p, s) e^{pt} dp = \lim_{m \rightarrow +\infty} \int_{\alpha-i\rho_m}^{c-i\rho_m} R^{-1}(p) \hat{e}(p, s) e^{pt} dp = 0. \quad (5.31)$$

For this, let us write the entry with index j of the vector $\int_{\alpha+i\rho_m}^{c+i\rho_m} R^{-1}(p) \hat{e}(p, s) e^{pt} dp$ as

$$\int_{\alpha+i\rho_m}^{c+i\rho_m} \sum_{n \in \mathbb{Z}} R_{j,n}^{-1}(p) \frac{e^{-(p-in\omega)s}}{p-in\omega} e^{pt} dp, \quad (5.32)$$

where $R_{j,n}^{-1}(p)$ denotes the entry with index (j, n) of $R^{-1}(p)$. As pointed out in step 2, $R^{-1}(p)$ has off-diagonal decay of order $1 + \delta$, uniformly over $\Re(p) \geq \beta$. Thus, if we pick $\varepsilon > 0$, there is $\mathbf{n} = \mathbf{n}(j)$ such that $\sum_{|n| \geq \mathbf{n}} \|R_{j,n}^{-1}(p)\| < \varepsilon$, uniformly with respect to $\Re(p) \in [\alpha, c]$, and since

$$|\rho_m - n\omega| \geq \frac{\pi}{T} \text{ for all } n \text{ and } m \quad (5.33)$$

we have that

$$\left\| \int_{\alpha+i\rho_m}^{c+i\rho_m} \sum_{|n| \geq \mathbf{n}} R_{j,n}^{-1}(p) \frac{e^{-(p-in\omega)s}}{p-in\omega} e^{pt} dp \right\| \leq \varepsilon e^{c(t-s)} \frac{(c-\alpha)T}{\pi} \quad (5.34)$$

which is arbitrary small with ε . In another connection, for fixed j and $n \in \mathbb{Z}$, we get since $|p-in\omega|$ becomes arbitrary large with m while the other terms in the integrand are uniformly bounded with respect to $\Re(p) \in [\alpha, c]$ that

$$\lim_{m \rightarrow \infty} \int_{\alpha+i\rho_m}^{c+i\rho_m} R_{j,n}^{-1}(p) \frac{e^{-(p-in\omega)s}}{p-in\omega} e^{pt} dp = 0. \quad (5.35)$$

In view of (5.34) and (5.35), we conclude that

$$\lim_{m \rightarrow +\infty} \int_{\alpha+i\rho_m}^{c+i\rho_m} R^{-1}(p) \hat{e}(p, s) e^{pt} dp = 0, \quad (5.36)$$

and in the same way one can prove that

$$\lim_{m \rightarrow +\infty} \int_{\alpha-i\rho_m}^{c-i\rho_m} R^{-1}(p) \hat{e}(p, s) e^{pt} dp = 0, \quad (5.37)$$

as wanted. \square

PROOF OF LEMMA 10. Let t and s be such that $t + \beta_k - s \neq 0$. By (5.33) and the absolute convergence in (5.16) that allow us to integrate termwise, we get by performing the change of variable $p \rightarrow p - in\omega$ in the integrals corresponding to index n that

$$\int_{\alpha-i\rho_m}^{\alpha+i\rho_m} \sum_{k,n \in \mathbb{Z}} R_{0,n}^{\{k\}} \frac{e^{-(p-in\omega)s}}{p-in\omega} e^{p(t+\beta_k)} dp = \sum_{k,n \in \mathbb{Z}} R_{0,n}^{\{k\}} e^{in\omega(t+\beta_k)} \int_{\alpha-in\omega-i\rho_m}^{\alpha-in\omega+i\rho_m} \frac{e^{p(t+\beta_k-s)}}{p} dp. \quad (5.38)$$

Integrating by parts, we get that

$$\int_{\alpha-in\omega-i\rho_m}^{\alpha-in\omega+i\rho_m} \frac{e^{p(t+\beta_k-s)}}{p} dp = \left[\frac{e^{p(t+\beta_k-s)}}{p(t+\beta_k-s)} \right]_{\alpha-in\omega-i\rho_m}^{\alpha-in\omega+i\rho_m} + \int_{\alpha-in\omega-i\rho_m}^{\alpha-in\omega+i\rho_m} \frac{e^{p(t+\beta_k-s)}}{(t+\beta_k-s)p^2} dp, \quad (5.39)$$

and by (5.19) there is $\delta > 0$ such that $|t + \beta_k - s| \geq \delta$ for all $k \in \mathbb{Z}$. Hence, using (5.33), we obtain:

$$\begin{aligned} \left| \left[\frac{e^{p(t+\beta_k-s)}}{p(t+\beta_k-s)} \right]_{\alpha-in\omega-i\rho_m}^{\alpha-in\omega+i\rho_m} \right| &\leq \frac{1}{|t+\beta_k-s|} \left(\frac{e^{\alpha(t+\beta_k-s)}}{\sqrt{\alpha^2 + (\rho_m - in\omega)^2}} + \frac{e^{\alpha(t+\beta_k-s)}}{\sqrt{\alpha^2 + (\rho_m + in\omega)^2}} \right) \\ &\leq \frac{2e^{\alpha(t+\beta_k-s)}}{\delta \sqrt{\alpha^2 + \left(\frac{\pi}{T}\right)^2}} \end{aligned} \quad (5.40)$$

and

$$\begin{aligned} \left| \int_{\alpha-in\omega-i\rho_m}^{\alpha-in\omega+i\rho_m} \frac{e^{p(t+\beta_k-s)}}{(t+\beta_k-s)p^2} dp \right| &\leq \frac{e^{\alpha(t+\beta_k-s)}}{|t+\beta_k-s|} \int_{-\rho_m-n\omega}^{\rho_m-n\omega} \frac{1}{\alpha^2 + p_2^2} dp_2 \\ &\leq \frac{e^{\alpha(t+\beta_k-s)}}{|t+\beta_k-s|} \left[\frac{1}{\alpha} \arctan \left(\frac{p_2}{\alpha} \right) \right]_{-\rho_m-n\omega}^{\rho_m-n\omega} \\ &\leq \frac{\pi}{|\alpha|\delta} e^{\alpha(t+\beta_k-s)}. \end{aligned} \quad (5.41)$$

Altogether, we deduce from (5.39), (5.40) and (5.41) that there exists a constant $K > 0$, independent of k , n and ρ_m , such that :

$$\left| \int_{\alpha-in\omega-i\rho_m}^{\alpha-in\omega+i\rho_m} \frac{e^{p(t+\beta_k-s)}}{p} dp \right| \leq K e^{\alpha(t+\beta_k-s)}. \quad (5.42)$$

Consequently, it holds that

$$\left\| R_{0,n}^{\{k\}} e^{in\omega(t+\beta_k)} \int_{\alpha-in\omega-i\rho_m}^{\alpha-in\omega+i\rho_m} \frac{e^{p(t+\beta_k-s)}}{p} dp \right\| \leq K e^{\alpha(t-s)} \left\| R_{0,n}^{\{k\}} \right\| e^{\alpha\beta_k}, \quad (5.43)$$

and since the right hand side of (5.43) is summable over k, n because of (5.16)), the dominated convergence theorem allows us to take the limit termwise in the right hand side of (5.38) as $m \rightarrow \infty$:

$$\begin{aligned} &\lim_{m \rightarrow +\infty} \frac{1}{2i\pi} \int_{\alpha-i\rho_m}^{\alpha+i\rho_m} \sum_{k,n \in \mathbb{Z}} R_{0,n}^{\{k\}} \frac{e^{-(p-in\omega)s}}{p-in\omega} e^{p(t+\beta_k)} dp \\ &= \sum_{k,n \in \mathbb{Z}} \frac{1}{2i\pi} R_{0,n}^{\{k\}} e^{in\omega(t+\beta_k)} \lim_{m \rightarrow +\infty} \int_{\alpha-in\omega-i\rho_m}^{\alpha-in\omega+i\rho_m} \frac{e^{p(t+\beta_k-s)}}{p} dp. \end{aligned} \quad (5.44)$$

Finally, the residue theorem gives us

$$\sum_{k,n \in \mathbb{Z}} \frac{1}{2i\pi} R_{0,n}^{\{k\}} e^{in\omega(t+\beta_k)} \lim_{m \rightarrow +\infty} \int_{\alpha-i\rho_m}^{\alpha+i\rho_m} \frac{e^{p(t+\beta_k-s)}}{p} dp = \sum_{\alpha(t+\beta_k-s) > 0, n \in \mathbb{Z}} R_{0,n}^{\{k\}} e^{in\omega(t+\beta_k)}. \quad (5.45)$$

Putting together (5.22), (5.44) and (5.45), this achieves the proof of Lemma 10. \square

5.2. Proof of the necessity part of Theorem 3

We now assume exponential stability, and prove that conditions (i) and (ii) of Theorem 3 are satisfied. In the time-invariant case, necessity is typically proven using the spectral theory of semigroups [5]. We cannot follow this path here, because there seems to be no adequate notion of semigroup for time-varying coefficients. However, according to Proposition 1), we may specifically assume L^2 exponential stability, and this will allow us to make contact with the spectral theory of monodromy operators thanks to the following, classical property.

Proposition 11. *System (2.3) is L^2 -exponentially stable if and only if the spectral radius of the monodromy operator $U_2(T, 0)$ is strictly less than one.*

Here, $U_2(\cdot, \cdot)$ refers to the solution operator (2.7). Proposition 11 (it even extends to much more general periodic evolution families than (2.3)) is formally a consequence of [31, lem. 4.2] if definition (2.9) of exponential stability (here $q = 2$) is restricted to $t \geq s \geq 0$. However, since the constants in that proof only depend on the $D_j(t)$ through $\sup\{\|D_i(t)\|, 1 \leq i \leq N, t \in \mathbb{R}\}$, the case of arbitrary s follows by periodicity. Choosing zero as initial time is arbitrary, and we might as well have put $U_2(T + s_0, s_0)$ for the monodromy operator, with arbitrary s_0 . Although different choices of s_0 lead to different monodromy operators, the relation of their spectrum to exponential stability is the same, and we stick to $s_0 = 0$ for definiteness.

In view of Proposition (11), we may assume that the spectral radius of the monodromy operator $U_2(T, 0)$ is less than 1. Then, we will resort to a classical construction from Control Theory (called *realization*) to connect the spectrum of the monodromy operator with the singular set of the harmonic transfer function. Specifically, we shall realize the periodic input-output system (4.6) as a discrete-time system in state space form, much like in the reference [10] except that the state is here infinite dimensional. Consider (4.6) with initial data 0 at time 0, and let $u \in L^2([0, +\infty), \mathbb{C}^d)$ be an input generating the output $y(t) \in \mathbb{C}^d$:

$$y(t) = \sum_{j=1}^N D_j(t)y(t - \tau_j) + u(t), \quad \text{a.e. } t \geq 0, \quad y(t) = 0 \text{ for } t \leq 0. \quad (5.46)$$

Using (4.11) with $s = 0$, $y_s = 0$, and performing the change of variables $\beta = t - \alpha$, we get that

$$y(t) = \int_{-\infty}^{t^+} d_\beta X(t, t - \beta)u(t - \beta), \quad \text{a.e. } t \geq 0. \quad (5.47)$$

Note, since $X(t, t - \beta)$ is identically zero when $\beta < 0$, that the integral in (5.47) can also be written as $\int_{c^-}^{t^+} d_\beta X(t, t - \beta)u(t - \beta)$ for any $c < 0$. From (5.47) and (4.12), it follows that

$$y(t) = \sum_{f \in \mathcal{F} \cap [0, t]} \mathfrak{E}_f(t)u(t - f), \quad \text{a.e. } t \geq 0. \quad (5.48)$$

Applying to (5.48) the Schwarz inequality and using (4.14), (4.17) together with the fact that $\mathfrak{E}_0 = -I_d$ (as follows readily from (4.8) and (4.1)), we obtain:

$$\begin{aligned} |y(t)|^2 &\leq \left(\sum_{f \in \mathcal{F} \cap [0, t]} \|\mathfrak{E}_f(t)\| \right)^2 \left(\sum_{f \in \mathcal{F} \cap [0, t]} \|u(t - f)\|^2 \right) \\ &\leq \left(\sum_{f \in \mathcal{F} \cap [0, t]} \|\mathfrak{E}_f(t)\| \right)^2 \left(\sum_{f \in \mathcal{F} \cap [0, t]} \|u(t - f)\|^2 \right) \\ &\leq (1 + Ke^{\gamma t})^2 \left(\sum_{f \in \mathcal{F} \cap [0, t]} \|u(t - f)\|^2 \right), \quad \text{a.e. } t \geq 0. \end{aligned} \quad (5.49)$$

In another connection, the cardinality of $\mathcal{F} \cap [0, t]$ is no larger than the number of N -tuples $(q_1, \dots, q_N) \in \mathbb{N}^N$ satisfying $\sum_{i=1}^N q_i \leq t/\tau_1$ (recall τ_1 is the smallest delay), and so

$$\text{Card} \{\mathcal{F} \cap [0, t]\} \leq \left(1 + \frac{t}{\tau_1}\right)^N, \quad t \in [0, +\infty), \quad (5.50)$$

where $\text{Card}\{E\}$ stands for the cardinality of E . Integrating (5.49) from 0 to $\tau > 0$ and taking (5.50) into account, we deduce that

$$\|y\|_{L^2([0, \tau])} \leq \tau^{1/2} (1 + Ke^{\gamma\tau}) \left(1 + \frac{\tau}{\tau_1}\right)^{N/2} \|u\|_{L^2([0, +\infty))}, \quad (5.51)$$

implying that to each $\gamma_1 > \gamma$ there is $K_1 > 0$ for which $\|y\|_{L^2([0, \tau])} \leq K_1 e^{\gamma_1 \tau} \|u\|_{L^2([0, +\infty))}$. Since $\|u\|_{L^2([0, +\infty))} < +\infty$, this warrants the existence, for $\Re(p) > \gamma$, of the Laplace transforms of $y(t)$ and $u(t)$:

$$\hat{Y}(p) := \int_0^{+\infty} e^{-pt} y(t) dt, \quad \hat{U}(p) := \int_0^{+\infty} e^{-pt} u(t) dt, \quad (5.52)$$

for if $\Re(p) > \gamma_1 > \gamma$ then $\int_k^{k+1} e^{-pt} |u|(t) dt$ and $\int_k^{k+1} e^{-pt} |y|(t) dt$ decay exponentially fast with $k \in \mathbb{N}$, by the Schwarz inequality. Proceeding on (5.46) like we did on (4.1) to obtain (5.7), namely expanding the $D_j(t)$ into Fourier series and taking Laplace transforms termwise using (5.1), we get on replacing the complex variable p by $p + in\omega$ and letting n range over \mathbb{Z} that

$$R(p) \hat{Y}(p) = \hat{U}(p), \quad \Re(p) > \gamma, \quad (5.53)$$

where $\hat{Y}(p)$ and $\hat{U}(p)$ are given by (2.21) and $R(p)$ by (2.15). Moreover, we see from (5.18) that $R(p)$ is continuously invertible for $\Re(p) > \gamma_1$, so we obtain with $\gamma_2 := \max\{\gamma, \gamma_1\}$ that

$$\hat{Y}(p) = R^{-1}(p) \hat{U}(p), \quad \Re(p) > \gamma_2. \quad (5.54)$$

Equation (5.54) identifies with Equation (2.22), showing that the operator-valued holomorphic function $p \mapsto R^{-1}(p)$ is the Harmonic Transfer Function of System (5.46); see Definition 3.

Next, we define the instantaneous transfer function:

$$G(t, p) = \int_{-\infty}^{+\infty} -d_\tau X(t, t - \tau) e^{-p\tau}, \quad t \in \mathbb{R}, \quad p \in \mathbb{C}, \quad \Re(p) > \gamma, \quad (5.55)$$

where the integral is understood as $-\lim_{b \rightarrow +\infty} \int_{c \pm}^{b \pm} d_\tau X(t, t - \tau) e^{-p\tau}$ for any $c < 0$. The existence of the limit is guaranteed by Proposition 8, and the fact that it does not depend on $c < 0$ can be argued similarly to the independence of (4.10) from the exact value of b . Observe from Proposition 8 that $t \mapsto G(t, p)$ lies in $C^{1+\delta}(\mathbb{R}, \mathbb{C}^{d \times d})$, and obviously it is T -periodic. Thus, we may expand this function in Fourier series as

$$G(t, p) = \sum_{k \in \mathbb{Z}} G_k(p) e^{i\omega kt}, \quad \Re(p) > \gamma, \quad (5.56)$$

where we recall that $\omega := 2\pi/T$, and the series converges absolutely for fixed p by a standard estimate used already in (5.1); more precisely, [24, Ch. 2, thm. 4.7] implies that

$$\|G_k(p)\| \leq \frac{\Re(\Re(p))}{1 + |k|^{1+\delta}}, \quad (5.57)$$

where $\Re(\Re(p)) > 0$ depends on $\Re(p)$, and also on K, γ in (4.17) though we do not show the latter dependence. Now, for $\mathfrak{f} \in \mathcal{F}$ and $k \in \mathbb{Z}$, let $c_{k,\mathfrak{f}}$ be the k -th Fourier coefficient of the T -periodic function $t \mapsto \mathfrak{C}_{\mathfrak{f}}(t)$. Since $\mathfrak{C}_{\mathfrak{f}} \in C^{1,\delta}(\mathbb{R})$, we have the estimate

$$\|c_{k,\mathfrak{f}}\| \leq \frac{K_{\mathfrak{f}}}{1 + |k|^{1+\delta}} \quad (5.58)$$

so that $\mathfrak{C}_{\mathfrak{f}}(t) = \sum_{k \in \mathbb{Z}} c_{k,\mathfrak{f}} e^{i\omega kt}$, where the series is uniformly absolutely convergent. Summing up over those $\mathfrak{f} \in \mathcal{F}$ such that $\mathfrak{f} \leq \tau$, we deduce from (4.12) that for any $\varepsilon > 0$:

$$d_s X(t, t-s) = \sum_{k \in \mathbb{Z}} d\mu_{k,\tau} e^{i\omega kt} \quad \text{on } [-\varepsilon, \tau], \quad (5.59)$$

where the measure $\mu_{k,\tau}$ is equal to $-\sum_{\mathfrak{f} \in \mathcal{F} \cap [0, \tau]} c_{k,\mathfrak{f}} \delta_{\mathfrak{f}}$ and absolute convergence with respect to the total variation holds in (5.59), because $\mathcal{F} \cap [0, \tau]$ is a finite set. If $\tau < 0$, then obviously $\mu_{k,\tau} = 0$. Moreover, as the constant $K_{\mathfrak{f}}$ in (5.58) is majorized by an affine function of the Hölder coefficient of $\frac{d}{dt} \mathfrak{C}_{\mathfrak{f}}$, we get from (4.17) that

$$\|\mu_{k,\tau}\| = \sum_{\mathfrak{f} \in \mathcal{F} \cap [0, \tau]} \|c_{k,\mathfrak{f}}\| \leq \frac{K_1 e^{\gamma\tau}}{1 + |k|^{1+\delta}}, \quad \tau \geq 0, \quad (5.60)$$

for some constant K_1 depending only on the D_j and the τ_j . Let us now define

$$\mu_k := \sum_{\mathfrak{f} \in \mathcal{F}} c_{k,\mathfrak{f}} \delta_{\mathfrak{f}}. \quad (5.61)$$

Clearly, μ_k is a distribution on \mathbb{R} valued in $\mathbb{C}^{d \times d}$ and supported on $[0, +\infty)$, but it is generally not a measure. However, its restriction to every interval bounded from above is a finite discrete measure. Hence, we can integrate against μ_k any $\boxed{\mathbb{C}^d\text{-valued}}$ bounded function which is zero for large arguments, and more generally any function that decays in norm as fast as $e^{-\gamma't}$ for some $\gamma' > \gamma$, by (5.60). Furthermore, when this function is supported on an interval of the form $[a, \tau]$ with $a < 0$ and $\tau < +\infty$, the integral against μ_k coincides with the integral against $\mu_{k,\tau}$. Using this remark, we can write for $\Re(p) > \gamma$:

$$\begin{aligned} G(t, p) &= - \lim_{b \rightarrow +\infty} \int_{-\infty}^{b^+} d_{\tau} X(t, t-\tau) e^{-p\tau} = - \lim_{b \rightarrow +\infty} \sum_{k \in \mathbb{Z}} e^{i\omega kt} \int_{-\infty}^{b^+} d\mu_{k,b}(\tau) e^{-p\tau} \\ &= - \sum_{k \in \mathbb{Z}} e^{i\omega kt} \lim_{b \rightarrow +\infty} \int_{-\infty}^{b^+} d\mu_k(\tau) e^{-p\tau} = - \sum_{k \in \mathbb{Z}} e^{i\omega kt} \int_{-\infty}^{+\infty} d\mu_k(\tau) e^{-p\tau}, \end{aligned}$$

where the second equality uses the absolute convergence in (5.59) and the third uses (5.60) and the fact that $\Re(p) > \gamma$. Considering (5.56), the previous chain of equalities yields that

$$G_k(p) = - \int_{0^-}^{+\infty} d\mu_k(\tau) e^{-p\tau}. \quad (5.62)$$

Next, assuming for a while that u is locally bounded, (5.47) and (5.59) imply

$$y(t) = \int_{-\infty}^{t^+} \sum_{k \in \mathbb{Z}} d\mu_{k,t}(\alpha) e^{i\omega kt} u(t-\alpha);$$

swapping the sum and integral sign thanks to (5.60) and using the above mentioned relation between $\mu_{k,t}$ and μ_k then yields $y(t) = \sum_{k \in \mathbb{Z}} \int_{-\infty}^{t^+} d\mu_k(\alpha) e^{i\omega kt} u(t - \alpha)$, and since $u(t - \alpha) = 0$ when $\alpha > t$ while μ_k is supported on $[0, \infty)$ we finally get

$$y(t) = \sum_{k \in \mathbb{Z}} A_k(t) \quad \text{with} \quad A_k(t) := \int_{0^-}^{+\infty} d\mu_k(\alpha) e^{i\omega k \alpha} u(t - \alpha) e^{i\omega k(t - \alpha)}. \quad (5.63)$$

Since for $\gamma' > \gamma$ we have that

$$\begin{aligned} \int_{0^-}^{+\infty} d|\mu_k|(\alpha) \int_0^{+\infty} |u(t - \alpha)| e^{-\gamma' t} dt &= \int_{0^-}^{+\infty} d|\mu_k|(\alpha) e^{-\gamma' \alpha} \int_0^{+\infty} |u(t - \alpha)| e^{-\gamma'(t - \alpha)} dt \\ &= \left(\int_{0^-}^{+\infty} d|\mu_k|(\alpha) e^{-\gamma' \alpha} \right) \left(\int_0^{+\infty} |u(\tau)| e^{-\gamma' \tau} d\tau \right) \leq \left(\frac{e^\gamma K_1}{1 + |k|^{1+\delta}} \sum_{j=0}^{+\infty} e^{(\gamma - \gamma')j} \right) \frac{\|u\|_{L^2(\mathbb{R})}}{\sqrt{2\gamma'}} < +\infty, \end{aligned} \quad (5.64)$$

where the first inequality uses (5.60) and the Schwarz inequality, we can use Fubini's theorem to compute the Laplace transform of $A_k(t)$ in the strip $\Re(p) > \gamma$. This gives us

$$\begin{aligned} \int_0^{+\infty} A_k(t) e^{-pt} dt &= \int_{0^-}^{+\infty} d\mu_k(\alpha) e^{(i\omega k - p)\alpha} \int_0^{+\infty} e^{(i\omega k - p)(t - \alpha)} u(t - \alpha) dt \\ &= \left(\int_{0^-}^{+\infty} d\mu_k(\alpha) e^{(i\omega k - p)\alpha} \right) \left(\int_0^{+\infty} e^{(i\omega k - p)\tau} u(\tau) d\tau \right) \\ &= G_k(p - i\omega k) \hat{U}(p - i\omega k) \end{aligned} \quad (5.65)$$

where the last equality uses (5.62). Summing over k , we find in view of (5.63) that

$$\hat{Y}(p) = \sum_{k \in \mathbb{Z}} G_k(p - i\omega k) \hat{U}(p - i\omega k), \quad \Re(p) > \gamma, \quad (5.66)$$

where the absolute convergence of the series follows from the estimates obtained in (5.64); *i.e.*,

$$|G_k(z)| \leq \frac{e^\gamma K_1 K_2 (\Re(z) - \gamma)}{1 + |k|^{1+\delta}} \quad \text{and} \quad |\hat{U}(z)| \leq \frac{\|u\|_{L^2(\mathbb{R})}}{\sqrt{2\Re(z)}}, \quad \Re(z) > \gamma,$$

with $K_2(x) := \sum_{j=0}^{+\infty} e^{-xj}$ for $x > 0$. Changing p into $p + i\omega n$ in (5.66) and renumbering, we get

$$\hat{Y}(p + i\omega n) = \sum_{m \in \mathbb{Z}} G_{n-m}(p + i\omega m) \hat{U}(p + i\omega m), \quad n \in \mathbb{Z}, \quad \Re(p) > \gamma. \quad (5.67)$$

Combining (5.54) and (5.67), we see that

$$\sum_{m \in \mathbb{Z}} (G_{n-m}(p + i\omega m) - R_{n,-m}^{-1}(p)) \hat{U}(p + i\omega m) = 0, \quad n \in \mathbb{Z}, \quad \Re(p) > \gamma, \quad (5.68)$$

where $R_{i,j}^{-1}(p)$ indicates, as in (5.32), the entry with index (i, j) of $R^{-1}(p)$. For each p with $\Re(p) > 0$, the sequence $\{z_m := p + i\omega m, m \in \mathbb{N}\}$ is hyperbolically separated in the right half-plane; *i.e.*,

$$\left| \frac{z_m - z_j}{z_m + \bar{z}_j} \right| \geq c > 0, \quad j, m \in \mathbb{N}, \quad j \neq m.$$

Indeed, we have that

$$\left| \frac{z_m - z_j}{z_m + \bar{z}_j} \right| = \left(\frac{4(\Re(p))^2}{\omega^2 |m - j|^2} + 1 \right)^{-1/2} \geq \left(\frac{4(\Re(p))^2}{\omega^2} + 1 \right)^{-1/2}.$$

Hence (see [32, Ch. VII, Thm 1.1] for an equivalent statement on the upper half-plane), $(z_m)_{m \in \mathbb{N}}$ is an interpolating sequence, meaning that to each bounded sequence a_m in \mathbb{C} there is a bounded analytic function F in the right half-plane with $F(z_m) = a_m$. In particular, to each $j \in \mathbb{N}$ and $v \in \mathbb{C}^d$, there is a \mathbb{C}^d -valued bounded analytic function $F_{j,v}$ such that $F_{j,v}(z_m) = 0$ for $m \neq j$ and $F_{j,v}(z_j) = v$. Multiplying $F_{j,v}$ by a function without zeros in the Hardy space \mathcal{H}^2 (for instance $p \mapsto (p+1)^{-1}$), we get a function $G_{j,v} \in (\mathcal{H}^2)^d$ which is zero at z_m when $m \neq j$ and a nonzero multiple of v at z_j . Now, by the Paley-Wiener theorem, there is $u \in L^2([0, \infty), \mathbb{C}^d)$ such that $\hat{U} = G_{j,v}$, and using this collection of u in (5.68) as j ranges over \mathbb{N} and v over \mathbb{C}^d provides us with

$$R_{n,-m}^{-1}(p) = G_{n-m}(p + im\omega), \quad n, m \in \mathbb{N}, \quad \Re(p) > \gamma. \quad (5.69)$$

Next, in order to link the monodromy operator to $R^{-1}(p)$ (this is achieved in Lemma 13 further below), we shall use periodicity to *realize* the dynamical system (5.46), that operates in continuous time, as a discrete time system in state space form; see Theorem 12. For this, we restrict the input and the output to intervals of length T by setting:

$$\left. \begin{aligned} \tilde{u}_k(t) &:= u(kT + t) \\ \tilde{y}_k(t) &:= y(kT + t) \end{aligned} \right\} \text{ for a.e. } t \in [0, T] \text{ and } k \in \mathbb{N}, \quad (5.70)$$

thereby giving rise to sequences \tilde{u}_k and \tilde{y}_k of functions in $L^2([0, T])$. We introduce another sequence of functions \tilde{z}_k , this time in $L^2([-\tau_N, 0])$, by putting $\tilde{z}_k := y_{kT}$, where the notation y_{kT} was defined after (4.6); *i.e.*, $\tilde{z}_k(\theta) := y(kT + \theta)$ for $k \in \mathbb{N}$ and a.e. $\theta \in [-\tau_N, 0]$. The function \tilde{z}_k will be the “state variable” of our discrete system at time k , initialized with $\tilde{z}_0 = y_0 = 0$. To realize this system, recall the monodromy operator $U_2(T, 0)$ from (2.9) (with $q = 2$) and the integration with respect to $K(t, \alpha)$ from (4.9). These allow us to define four operators as follows:

- $\tilde{A} : L^2([-\tau_N, 0], \mathbb{C}^d) \longrightarrow L^2([-\tau_N, 0], \mathbb{C}^d)$
 $v \longmapsto U_2(T, 0)v,$
- $\tilde{B} : L^2([0, T], \mathbb{R}^d) \longrightarrow L^2([-\tau_N, 0], \mathbb{C}^d)$
 $w \longmapsto \int_{0^-}^{+\infty} d_\alpha K(T, \alpha)w(\alpha),$
- $\tilde{C} : L^2([-\tau_N, 0], \mathbb{C}^d) \longrightarrow L^2([0, T], \mathbb{C}^d)$
 $v \longmapsto \{(U_2(t, 0)v)(0), t \in [0, T]\},$
- $\tilde{D} : L^2([0, T], \mathbb{C}^d) \longrightarrow L^2([0, T], \mathbb{C}^d)$
 $w \longmapsto \left(- \int_0^{+\infty} d_\alpha X(\cdot, \alpha)w(\alpha) \right) \Big|_{[0, T]}.$

Observe that \tilde{B} and \tilde{D} do not depend on the values of w outside $[0, T]$, and that $(U_2(t, 0)v)(0)$ exists for a.e. $t \in \mathbb{R}$, hence the operators \tilde{A} , \tilde{B} , \tilde{C} and \tilde{D} are well-defined and continuous.

Theorem 12. Let $u \in L^2([0, +\infty), \mathbb{C}^d)$ and $y \in L^2_{loc}([0, +\infty), \mathbb{C}^d)$ be, respectively, the input and output of System (5.46). For \tilde{u}_k, \tilde{y}_k and \tilde{z}_k as in (5.70), the following recursion holds:

$$\begin{cases} \tilde{z}_{k+1} &= \tilde{A}\tilde{z}_k + \tilde{B}\tilde{u}_k, \\ \tilde{y}_k &= \tilde{C}\tilde{z}_k + \tilde{D}\tilde{u}_k, \end{cases} \quad \tilde{z}_0 = 0, \quad k \in \mathbb{N}. \quad (5.71)$$

PROOF. Applying the variation-of-constants formula (4.11) to System (5.46) with s, t replaced by $kT, kT + t$ yields

$$y_{kT+t} = U_2(kT + t, kT)y_{kT} + \int_{(kT)^-}^{+\infty} d_\alpha K(kT + t, \alpha)u(\alpha),$$

and performing the change of variable $\alpha = kT + \beta$ while using T -periodicity we get that

$$y_{kT+t} = U_2(t, 0)y_{kT} + \int_{0^-}^{+\infty} d_\beta K(t, \beta)\tilde{u}_k(\beta). \quad (5.72)$$

When $t = T$, this gives us the first equation in (5.71). Next, evaluating each term of (5.72) at 0 (remember these terms are functions of $\theta \in [-\tau_N, 0]$ and that evaluation at a fixed point is possible for a.e. t) while using the definition of K (see (4.9)), we obtain:

$$y(t + kT) = (U_2(t, 0)\tilde{z}_k)(0) - \int_{0^-}^{+\infty} d_\beta X(t, \beta)\tilde{u}_k(\beta), \quad \text{a.e. } t \in [0, T],$$

which is the second equation in (5.71). □

For $a := (a_n)_{n \in \mathbb{N}}$ a sequence in a Banach space \mathfrak{X} , its z -transform is the formal series:

$$\mathcal{L}\{a\}(z) := \sum_{n \in \mathbb{N}} a_n z^{-n}. \quad (5.73)$$

We let $\mathfrak{X}[[z^{-1}]]$ denote the space of such power series. For \mathfrak{Y} a Banach space and $O := (O_n)_{n \in \mathbb{N}}$ a sequence of bounded operators from \mathfrak{X} to \mathfrak{Y} , the z -transform $\mathcal{L}\{O\}(z)$ acts naturally from $\mathfrak{X}[[z^{-1}]]$ into $\mathfrak{Y}[[z^{-1}]]$, since for each $k \in \mathbb{N}$ the number of terms involved in the coefficient of z^{-k} when computing the product $(\sum_{n \in \mathbb{N}} O_n z^{-n})(\sum_{n \in \mathbb{N}} a_n z^{-n})$ is finite. Now, if we put $\bar{y} := (\tilde{y}_n)_{n \in \mathbb{N}}, \bar{u} := (\tilde{u}_n)_{n \in \mathbb{N}}$ and $\bar{z} := (\tilde{z}_n)_{n \in \mathbb{N}}$, we get from (5.71) that

$$(zI_d - \tilde{A})\mathcal{L}\{\bar{z}\} = \tilde{B}\mathcal{L}\{\bar{u}\} \quad \text{and} \quad \mathcal{L}\{\bar{y}\} = \tilde{C}\mathcal{L}\{\bar{z}\} + \tilde{D}\mathcal{L}\{\bar{u}\}.$$

Note, since $\tilde{z}_0 = 0$, that $(zI_d - \tilde{A})\mathcal{L}\{\bar{z}\}$ indeed belongs to $L^2([-\tau_N, 0], \mathbb{C}^d)[[z^{-1}]]$. Observing that $(zI - \tilde{A})^{-1} = \sum_{n \in \mathbb{N}} \tilde{A}^n z^{-n-1}$ in the ring of formal Laurent series with coefficients the operators on $L^2([-\tau_N, 0], \mathbb{C}^d)$, we may write

$$\mathcal{L}\{\bar{y}\}(z) = \left[\tilde{C}(zId - \tilde{A})^{-1}\tilde{B} + \tilde{D} \right] \mathcal{L}\{\bar{u}\}(z). \quad (5.74)$$

Although (5.74) was derived formally, it becomes a valid relation between vector and operator valued analytic functions when $|z| > \max\{\|\tilde{A}\|, e^\gamma\}$ with γ as in Proposition 8, because all series involved are then normally convergent (compare (5.51)). In another connection, writing after a

change of variable the right hand side of (5.47) as a sum of integrals on subintervals while taking into account that $X(\tau, \cdot)$ is left continuous and using T -periodicity, we get that

$$\begin{aligned}
\tilde{y}_n(t) &= y(t + nT) = \sum_{k=0}^{n-1} \int_{(kT)^-}^{((k+1)T)^-} d_\alpha X(t + nT, \alpha) u(\alpha) + \int_{(nT)^-}^{+\infty} d_\alpha X(t + nT, \alpha) u(\alpha) \\
&= \sum_{k=0}^{n-1} \int_{0^-}^{T^-} d_\beta X(t + nT, kT + \beta) u(kT + \beta) + \int_{(nT)^-}^{((n+1)T)^+} d_\alpha X(t + nT, \alpha) u(\alpha) \\
&= \sum_{k=0}^{n-1} \int_{0^-}^{T^+} d_\beta X(t + nT, kT + \beta) u(kT + \beta) + \int_{0^-}^{T^+} d_\beta X(t + nT, nT + \beta) u(nT + \beta) \\
&= \sum_{k=0}^{n-1} \int_{0^-}^{T^+} d_\beta X(t + (n - k)T, \beta) \tilde{u}_k(\beta) + \int_{0^-}^{T^+} d_\beta X(t, \beta) \tilde{u}_n(\beta) \\
&= \sum_{k=0}^n H_{[n-k]} \tilde{u}_k(t), \quad \text{a.e. } t \in [0, T] \quad (5.75)
\end{aligned}$$

where, for $v \in L^2([0, T], \mathbb{C}^d)$, we have set:

$$H_{[k]} v(t) := \int_{0^-}^{T^+} d_\tau X(kT + t, \tau) v(\tau), \quad \text{a.e. } t \in [0, T]. \quad (5.76)$$

Observe that the equality $\int_{(nT)^-}^{+\infty} d_\alpha X(t + nT, \alpha) u(\alpha) = \int_{(nT)^-}^{((n+1)T)^+} d_\alpha X(t + nT, \alpha) u(\alpha)$ used in the third identity of (5.75) only holds for $t < T$ in general, because when $t = T$ and $(n+1)T \in \mathcal{F}$, then $\int_{(nT)^-}^{((n+1)T)^+} d_\alpha X(t + nT, \alpha) u(\alpha)$ will miss the jump that may occur at $(n+1)T$. However, since (5.75) is claimed for a.e. $t \in [0, T]$ only, this is unimportant. Note also that changing the upper bound from T^- to T^+ in the integrals summed over k after the fourth equal sign of (5.75) is permitted, because the left continuity of $X(\tau, \cdot)$ implies that $\{T\}$ carries no mass of $d_\beta X(t + nT, nT + \beta)$. If we take z -transforms in (5.75), we obtain

$$\mathcal{L}\{\bar{y}\}(z) = \mathcal{L}\{\bar{H}\}(z) \mathcal{L}\{\bar{u}\}(z) \quad (5.77)$$

where $\bar{H} := (H_{[n]})_{n \in \mathbb{N}}$, and since for each k we may pick \tilde{u}_k arbitrarily in $L^2([0, T])$ and $\tilde{u}_j \equiv 0$ for $j \neq k$, we deduce from (5.74) and (5.77) that

$$\mathcal{L}\{\bar{H}\}(z) = \tilde{C}(zId - \tilde{A})^{-1} \tilde{B} + \tilde{D} \quad (5.78)$$

The equality (5.78) *a priori* holds between formal power series in z^{-1} with coefficients in the ring $\mathcal{B}(L^2([0, T], \mathbb{C}^d))$ of bounded linear operators on $L^2([0, T], \mathbb{C}^d)$. However, if we let a denote the spectral radius of \tilde{A} and set $P_a := \{z \in \mathbb{C} : |z| > a\}$, then (5.78) holds in $\mathcal{B}(L^2([0, T], \mathbb{C}^d))$ when we substitute for z a value in P_a , for the series become normally convergent. Of course, by analytic continuation, the right hand-side extends to an operator-valued analytic function on the unbounded connected component of $\mathbb{C} \setminus \text{Spec } \tilde{A}$, where $\text{Spec } \tilde{A}$ stands for the spectrum of $\tilde{A} = U_2(T, 0)$, but this function may no longer be defined by a power series in z^{-1} . Since we assumed that System (2.3) is C^0 -exponentially stable, we know from Propositions 11 and 1 that $\text{Spec } \tilde{A}$ is compactly included the unit disk; that is to say: $a \in [0, 1)$. In particular, $\mathcal{L}\{\bar{H}\}(e^{pT})$ is a well-defined operator on $L^2([0, T], \mathbb{C}^d)$ as soon as $\Re(p) > \log a/T$, where $\log a$ is strictly negative or $-\infty$.

For $p \in \mathbb{C}$, consider the operator $E_p : L^2([0, T], \mathbb{C}^d) \rightarrow L^2([0, T], \mathbb{C}^d)$ of pointwise multiplication by the function $(t \mapsto e^{pt}) \in L^\infty([0, T])$. Clearly $E_p^{-1} = E_{-p}$, and it is easy to check that

$$\|E_p\|_2 = \max\{1, e^{\Re(p)T}\}. \quad (5.79)$$

When $\Re(p) > \log a/T$, we also define the operator

$$\begin{aligned} \Lambda(p) & : L^2([0, T], \mathbb{C}^d) \rightarrow L^2([0, T], \mathbb{C}^d), \\ \Lambda(p) & = E_{-p} \circ \mathcal{L}\{\overline{H}\}(e^{pT}) \circ E_p. \end{aligned} \quad (5.80)$$

In addition, for $\Re(p) > \gamma_2$ with γ_2 as in (5.54), we construe $R^{-1}(p) : l^2(\mathbb{Z}, \mathbb{C}^d) \rightarrow l^2(\mathbb{Z}, \mathbb{C}^d)$ as an operator from $L^2([0, T], \mathbb{C}^d)$ into itself by identifying a function $\phi \in L^2([0, T], \mathbb{C}^d)$ with $(\dots, a_2, a_1, a_0, a_{-1}, a_{-2}, \dots)$ where a_k is the k -th Fourier coefficient of ϕ .

Lemma 13. *For $p \in \mathbb{C}$ with $\Re(p) > \gamma_2$, it holds that*

$$\Lambda(p)\phi(t) = R^{-1}(p)\phi, \quad \phi \in L^2([0, T], \mathbb{C}^d). \quad (5.81)$$

PROOF. As $\gamma_2 > 0 > \log a/T$, we see from (5.76) that for any $v \in L^2([0, T], \mathbb{R})$ we have:

$$\mathcal{L}\{\overline{H}\}(z)v(t) = \sum_{k=0}^{+\infty} H_{[k]}v(t)z^{-k} = \sum_{k=-\infty}^{+\infty} z^{-k} \int_{0^-}^{T^+} d_\tau X(kT + t, \tau)v(\tau). \quad (5.82)$$

Performing the change of variable $\tau \rightarrow t - \tau$ in (5.55) and computing as in (5.75), we also obtain

$$\begin{aligned} G(t, p) & = \int_{-\infty}^{+\infty} d_\tau X(t, \tau)e^{p(\tau-t)} = \sum_{k=-\infty}^{+\infty} \int_{0^-}^{T^+} d_\tau X(t, \tau - kT)e^{p(\tau-t-kT)} \\ & = \sum_{k=-\infty}^{+\infty} e^{-kpT} \left(e^{-pt} \int_{0^-}^{T^+} d_\tau X(t + kT, \tau)e^{p\tau} \right). \end{aligned} \quad (5.83)$$

Multiplying (5.83) on the right by some arbitrary $V \in \mathbb{C}^d$ and comparing with (5.82) where we set $z = e^{pT}$ and $v(\tau) = e^{p\tau}V$ for $\tau \in [0, T]$, we get that

$$G(t, p)V = e^{-pt} [\mathcal{L}\{\overline{H}\}(e^{pT})(e^{p \cdot} V)](t), \quad \text{a.e. } t \in [0, T], \quad (5.84)$$

where $e^{p \cdot} V$ is just another way of writing v (the dot stands for a dummy argument). Now, assume for a while that $\phi \in C^{1,\alpha}([0, T], \mathbb{C}^d)$ with $\alpha \in (0, 1)$ and $\phi(0) = \phi(T)$. Write the Fourier expansion $\phi(t) = \sum_{k \in \mathbb{Z}} a_k e^{i\omega k t}$ with $a_k \in \mathbb{C}^d$. By normal convergence of the latter, we deduce from (5.84), (5.56) and (5.57) that

$$\begin{aligned} \Lambda(p)\phi(t) & = e^{-pt} [\mathcal{L}\{\overline{H}\}(e^{pT})e^{p \cdot} \sum_{k \in \mathbb{Z}} a_k e^{i\omega k \cdot}](t) = \sum_{k \in \mathbb{Z}} e^{-pt} [\mathcal{L}\{\overline{H}\}(e^{pT})e^{(p+i\omega k) \cdot} a_k](t) \\ & = \sum_{k \in \mathbb{Z}} e^{i\omega k t} e^{-(p+i\omega k)t} [\mathcal{L}\{\overline{H}\}(e^{(p+i\omega k)T})e^{(p+i\omega k) \cdot} a_k](t) = \sum_{k \in \mathbb{Z}} e^{i\omega k t} G(t, p + i\omega k) a_k \\ & = \sum_{k \in \mathbb{Z}} e^{i\omega k t} \left(\sum_{n \in \mathbb{Z}} e^{in\omega t} G_n(p + i\omega k) \right) a_k = \sum_{m \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} G_{m-k}(p + i\omega k) a_k \right) e^{im\omega t}, \end{aligned} \quad (5.85)$$

and so in view of (5.69) we get that

$$\Lambda(p)\phi(t) = \sum_{m \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} R_{m,-k}^{-1}(p) a_k \right) e^{im\omega t} = R^{-1}(p)\phi. \quad (5.86)$$

Equation (5.86) yields the desired conclusion, except for the assumption that $\phi \in C^{1,\alpha}([0, T], \mathbb{C}^d)$ with $\phi(0) = \phi(T)$. But since $\Lambda(p)$ and $R^{-1}(p)$ are continuous operators $L^2([0, T]) \rightarrow L^2([0, T])$ for $\Re(p) > \gamma_2$, we conclude by a density argument that (5.81) holds as stated. \square

To recap, we know from the discussion after (5.78) that $\mathcal{L}\{\overline{H}\}(e^{pT})$ is an analytic operator-valued function for $\Re(p) > \log a/T$, which is uniformly bounded in every half-plane $\{p : \Re(p) > \beta\}$ if $\beta > \log a/T$. Therefore $\Lambda(p)$ is, by (5.79) and (5.80), an analytic operator-valued function for $\Re(p) > \log a/T$, uniformly bounded in every strip $\{p : \beta_0 > \Re(p) > \beta\}$ if $\beta_0 > \beta > \log a/T$. Furthermore, in view of (2.15), $R(p)$ is an entire operator-valued function. As Lemma 13 tells us that $\Lambda(p)R(p) = I_\infty$ for $\Re(p) > \gamma_2$, this identity in fact holds on $\{p : \Re(p) > \log a/T\}$, by analytic continuation. So, if we let $\beta \in (\log a/T, 0)$ and $\beta_1 = \gamma_1 + 1$ with γ_1 as in (5.18), we get that $R^{-1}(p)$ exists and is equal to $\Lambda(p)$ for $\Re(p) > \log a/T$, and that $\|R^{-1}(p)\|_2 \leq C_1$ for $\beta < \Re(p) < \beta_1$ with a constant $C_1 > 0$ depending on β and β_1 but not on p . In another connection, we know from (5.18) that $\|R^{-1}(p)\|_2 \leq C_2$ for $\Re(p) > \gamma_1$, and altogether we conclude that

$$\|R^{-1}(p)\|_2 \leq C, \quad \Re(p) > \beta, \quad (5.87)$$

with $C = \max\{C_1, C_2\}$. This achieves the proof of the necessity part of Theorem 3, and thus of the theorem itself, considering Section 5.1.

6. Concluding remarks on neutral functional differential equations

Having proven a generalization of the Henry-Hale theorem to the periodic case, it is natural to ask if the result from [5] on the stability of neutral functional differential systems carries over to the periodic case as well. In other words, whether exponential stability of a periodic linear neutral differential system of the form (1.2) can be characterized through analyticity of some harmonic transfer function. It is transparent how to define the latter: simply put $\mathbf{H}(p) := \tilde{R}(p)^{-1}$ where

$$\tilde{R}(p) := D_\omega(p) \left[I_\infty - \sum_{j=1}^N e^{-p\tau_j} L_{D_j} \Delta_{\tau_j, \omega} \right] - \sum_{k=0}^N e^{-p\tau_k} L_{B_k} \tilde{D}_{\tau_k}, \quad (6.1)$$

with $\tau_0 := 0$ and

$$D_\omega(p) := \text{diag} \{ \dots, (p + i\omega)I_d, pI_d, (p - i\omega)I_d, \dots \} \quad (6.2)$$

while the D_j, B_j are as in (1.2). Then, with the natural definition of exponential stability paralleling Definition 1, we raise the following question:

Assume that the $D_j(\cdot)$ and $B_k(\cdot)$ are periodic and differentiable with Hölder continuous derivative for $1 \leq j \leq N$ and $0 \leq k \leq N$. Is it true that a necessary and sufficient condition for System (1.2) to be exponentially stable is the existence of a real number $\beta < 0$ satisfying

- (i) $\tilde{R}(p)$ is invertible $\ell^2(\mathbb{Z}, \mathbb{C}^d) \rightarrow \ell^2(\mathbb{Z}, \mathbb{C}^d)$ for all p in $\{z \in \mathbb{C} \mid \Re(z) \geq \beta\}$,
- (ii) there is $M > 0$ such that $\left\| \tilde{R}(p)^{-1} \right\|_2 \leq M$ for all p in $\{z \in \mathbb{C} \mid \Re(z) \geq \beta\}$?

We expect that the answer is yes, and that necessity can be proven as in theorem 3. The difficulty with the sufficiency part is that the matrix $D_\omega(p)$ now prevents us from deriving straightforward analogs of Lemmas 9 and 10.

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Appendix A. Proof of Proposition 6

Let us first prove (4.5) by induction, assuming (4.4). Let $K' > 0$ be an upper bound for $\{\|D_i(t)\|, 1 \leq i \leq N, t \in \mathbb{R}\}$; such a K' exists since the D_i are continuous and periodic. Pick λ large enough that

$$K'Ne^{-\lambda\tau_1} < \frac{1}{2}, \quad (\text{A.1})$$

and let us show (4.5) by induction on k such that $t \in [s+k\tau_1, s+(k+1)\tau_1)$. For $k = 0$, it is obvious that (4.5) holds with $K = 1$ because $X(t, s)$ is either 0 or I_d for $0 \leq t - s < \tau_1$. Next, assume that (4.5) hold for all t, s such that $0 \leq t - s < k\tau_1$. Then, for t, s verifying $k\tau_1 \leq t - s < (k+1)\tau_1$, we get from (4.1) that

$$\|X(t, s)\| \leq 1 + K'NK e^{-\lambda\tau_1} e^{\lambda(t-s)} \quad (\text{A.2})$$

$$\leq K e^{\lambda(t-s)}, \quad (\text{A.3})$$

which concludes the induction. To prove that $y(t)$ is C^0 -exponentially stable, in the forthcoming steps we will estimate the variation of $X(t, s)$ with respect in its second argument, using Laplace transforms.

We now turn to establishing formula (4.4). We make preliminaries about Volterra kernels in Sections A.1 and A.2, before giving the proof proper in Section A.3.

A.1. Reduction to a Volterra equation

Observe from (4.1) that $\alpha \mapsto X(t, \alpha)$ lies in $BV_{loc}(\mathbb{R})$ for all t , and that for $b > s$ we have:

$$d_\alpha X(t, \alpha) = \begin{cases} \sum_{j=1}^N D_j(t) d_\alpha X(t - \tau_j, \alpha) & \text{if } t \geq b \\ -I_d \delta_t + \sum_{j=1}^N D_j(t) d_\alpha X(t - \tau_j, \alpha) & \text{if } s \leq t < b \end{cases} \quad \text{on } [s, b], \quad (\text{A.4})$$

where δ_t is the Dirac delta at t . In particular, $\alpha \mapsto X(t, \alpha)$ has bounded variation on $[s, s + \tau_j]$ for each j , and for $t \geq s + \tau_N$ we have that

$$d_\alpha X(t, \alpha) = \sum_{j=1}^N D_j(t) d_\alpha X(t - \tau_j, \alpha) \quad \text{on } [s, s + \tau_j]. \quad (\text{A.5})$$

Then, substituting (A.5) in (4.4) formally yields (2.3), provided that $t \geq s + \tau_N$. Hence, by uniqueness of a solution y to (2.3) satisfying $y(s + \theta) = \phi(\theta)$ for $\theta \in [-\tau_N, 0]$, it is enough to check (4.4) for $s \leq t < s + \tau_N$. For this, we adopt the point of view of reference [7], which is to construe delay systems as Stieltjes-Volterra equations upon representing delays by measures. More precisely, we can rewrite (2.3) as a Lebesgue-Stieltjes integral:

$$y(t) = \int_{-\tau_N^-}^{0^-} d\mu(t, \theta) y(t + \theta), \quad t \geq s, \quad (\text{A.6})$$

with

$$\mu(t, \theta) = \sum_{j=1}^N D_j(t) \mathfrak{H}(\theta + \tau_j), \quad (\text{A.7})$$

where $y(\tau)$ is understood to be $\phi(\tau - s)$ when $s - \tau_N \leq \tau \leq s$ and $\mathfrak{H}(\tau)$ is the Heaviside function which is zero for $\tau \leq 0$ and 1 for $\tau > 0$, so that the associated measure on an interval of the form $[0, a]$ or $[0, a)$ is a Dirac delta at 0. Note that $\mathfrak{H}(0) = 0$, which is not the usual convention, but if we defined \mathfrak{H} so that $\mathfrak{H}(0) = 1$ then expanding (A.6) using (A.7) would not give us back (2.3) for the term $D_N(t)y(t - \tau_N)$ would be missing. Observe also, since $\tau_j > 0$ for all j , that the minus sign in the upper bound of the integral in (A.6) is immaterial and could be traded for a plus. For $s \leq t \leq s + \tau_N$, singling out the initial data in (A.6) yields

$$y(t) = \int_{(s-t)^-}^{0^-} d\mu(t, \theta) y(t + \theta) + f(t) \quad \text{with} \quad f(t) := \int_{-\tau_N^-}^{(s-t)^-} d\mu(t, \theta) \phi(t + \theta - s), \quad (\text{A.8})$$

where we took into account, when separating the integrals, that $\theta \mapsto \mu(t, \theta)$ is left continuous, while the integral over the empty interval is understood to be zero. It will be convenient to study (A.8) for $t \in [s, s + \tau_N]$, even though in the end the values of $y(t)$ only matter to us for $t \in [s, t + \tau_N)$. Define

$$k(t, \tau) := \begin{cases} \mu(t, \tau - t) - \sum_{j=1}^N D_j(t) & \text{for } \tau \in [s, t], \\ 0 & \text{for } \tau > t, \end{cases} \quad t, \tau \in [s, s + \tau_N]. \quad (\text{A.9})$$

Note that $k(t, \tau) = 0$ when $t - \tau < \tau_1$, and $d_\tau k(t, \tau) = d_\tau \mu(t, \tau - t)$ on $[s, s + \tau_N]$ for fixed t . Hence, (A.8) becomes

$$y(t) = \int_{s^-}^{t^-} dk(t, \tau) y(\tau) + f(t), \quad s \leq t \leq s + \tau_N. \quad (\text{A.10})$$

Now, (A.10) is the Stieltjes-Volterra equation we shall work with.

It suffices to prove (4.4) for $t \in [s, s + \tau_N)$ under the additional assumption that ϕ and the D_j , which are continuous by hypothesis, also have bounded and locally bounded variation, on $[-\tau_N, 0]$ and \mathbb{R} respectively. Indeed, functions of bounded variation are dense in C_s (resp. $C^0(\mathbb{T}, \mathbb{R}^{d \times d})$), for

instance because C^1 -functions are, and if ϕ_k converges uniformly to ϕ in C_s while $D_{j,k}$ converges uniformly to D_j in $[s, s + \tau_N]$ as $k \rightarrow \infty$, then the solution to (2.3) with initial condition ϕ_k and coefficients $D_{j,k}$ converges uniformly on $[s, s + \tau_N]$ to the solution with initial condition ϕ and coefficients D_j , as is obvious by inspection. Hence, we shall assume without loss of generality that ϕ has bounded variation and the D_j have locally bounded variation. Then, since it follows from (A.7) and (A.8) that

$$f(t) = \sum_{\tau_\ell \in (t-s, \tau_N]} D_\ell(t) \phi(t - s - \tau_\ell), \quad (\text{A.11})$$

it is clear from (A.11) and (3.2) that $f \in BV_l[s, s + \tau_N] \cap BV_r[s, s + \tau_N]$, since it is continuous.

A.2. Volterra kernels of type B^∞

Volterra equations for functions of a single variable have been studied extensively, see *e.g.* [33, 34]. However, the specific assumption that the kernel has bounded variation seems treated somewhat tangentially. On the one hand, it is subsumed in the measure-valued case presented in [34, Ch. 10], but no convenient criterion is given there for the existence of a resolvent kernel. On the other hand, [7, Ch. 9, Sec. 1] sketches the main arguments needed to handle kernels of bounded variation, but the exposition has issues⁴. This is why we felt a need for the present appendix.

We define a Stieltjes-Volterra kernel of type B^∞ on $[a, b] \times [a, b]$ as a measurable function $\kappa : [a, b] \times [a, b] \rightarrow \mathbb{R}^{d \times d}$, with $\kappa(t, \tau) = 0$ for $\tau \geq t$, such that the partial maps $\kappa(t, \cdot)$ lie in $BV_l[a, b]$ and $\|\kappa(t, \cdot)\|_{BV([a, b])}$ is uniformly bounded with respect to $t \in [a, b]$. In addition, we require that $\lim_{\tau \rightarrow t^-} W_{[\tau, t]}(\kappa(t, \cdot)) = 0$ uniformly with respect to t ; *i.e.*, to every $\varepsilon > 0$, there exists $\eta > 0$ such that $W_{[\tau, t]}(\kappa(t, \cdot)) < \varepsilon$ as soon as $0 < t - \tau < \eta$. Note that $W_{[\tau, t]}(\kappa(t, \cdot)) \rightarrow 0$ for fixed t as $\tau \rightarrow t^-$ whenever $\kappa(t, \cdot)$ has bounded variation on $[a, b]$, by the very definition (3.1); so, the assumption really is that the convergence is uniform with respect to t . Hereafter, we drop the qualifier “of type B^∞ ” and simply speak of Stieltjes-Volterra kernels on $[a, b]$. We endow the space $\mathcal{K}_{[a, b]}$ of such kernels with the norm $\|\kappa\|_{[a, b]} := \sup_{t \in [a, b]} \|\kappa(t, \cdot)\|_{BV[a, b]}$. If κ_k is a Cauchy sequence in $\mathcal{K}_{[a, b]}$, then κ_k converges uniformly on $[a, b] \times [a, b]$ to a $\mathbb{R}^{d \times d}$ -valued function κ because

$$\|\|\kappa_k(t, \tau) - \kappa_l(t, \tau)\|\| = \|\|(\kappa_k(t, \tau) - \kappa_l(t, \tau)) - (\kappa_k(t, t) - \kappa_l(t, t))\|\| \leq \|\kappa_k(t, \cdot) - \kappa_l(t, \cdot)\|_{BV[a, b]}.$$

Clearly, κ is measurable and left continuous for fixed t with $\kappa(t, \tau) = 0$ for $\tau \geq t$. Also, if m is so large that $\|\kappa_k - \kappa_l\|_{[a, b]} < \varepsilon$ for $k, l \geq m$ and $\eta > 0$ so small that $W_{[\tau, t]}(\kappa_m) < \varepsilon$ when $t - \tau < \eta$, we get that $W_{[\tau, t]}(\kappa_l) \leq W_{[\tau, t]}(\kappa_m) + W_{[\tau, t]}(\kappa_m - \kappa_l) < 2\varepsilon$, and letting $l \rightarrow \infty$ we get from [19, thm. 1.3.5] that $W_{[\tau, t]}(\kappa) \leq 2\varepsilon$. Furthermore, the same reference implies that $\|\kappa\|_{[a, b]} \leq \sup_k \|\kappa_k\|_{[a, b]}$ so that $\kappa \in \mathcal{K}_{[a, b]}$. Finally, writing that $W_{[a, b]}(\kappa_k(t, \cdot) - \kappa_l(t, \cdot)) < \varepsilon$ and passing to the limit as $l \rightarrow \infty$, we see that $\lim_k \|\kappa_k - \kappa\|_{[a, b]} = 0$, whence $\mathcal{K}_{[a, b]}$ is a Banach space. Note that a Stieltjes-Volterra kernel κ is necessarily bounded with $\sup_{[a, b] \times [a, b]} \|\|\kappa(t, \tau)\|\| \leq \|\kappa\|_{[a, b]}$. As \mathfrak{H} is left continuous and the D_j are bounded, on $[s, s + \tau_N]$, it is easy to check that $k(t, \tau)$ defined in (A.9) is a Stieltjes-Volterra kernel on $[s, s + \tau_N] \times [s, s + \tau_N]$.

A *resolvent* for the Stieltjes-Volterra kernel κ on $[a, b] \times [a, b]$ is a Stieltjes-Volterra kernel ρ on $[a, b] \times [a, b]$ satisfying

$$\rho(t, \beta) = -\kappa(t, \beta) + \int_{\beta^-}^{t^-} d\kappa(t, \tau) \rho(\tau, \beta), \quad a \leq t, \beta \leq b. \quad (\text{A.12})$$

⁴For example, the equation satisfied by $\tilde{\rho}(t, s)$ at top of page 258 is not right.

Lemma 14. *If κ is a Stieltjes-Volterra kernel on $[a, b] \times [a, b]$, a resolvent for κ uniquely exists.*

PROOF. Pick $r > 0$ to be adjusted later, and for $\Psi \in \mathcal{K}_{[a,b]}$ let us define

$$F_r(\Psi)(t, \beta) := \int_{\beta^-}^{t^-} e^{-r(t-\tau)} d\kappa(t, \tau) \Psi(\tau, \beta), \quad a \leq \beta, t \leq b.$$

Then, $F_r(\Psi)(t, \beta) = 0$ for $\beta \geq t$, and for $a \leq \beta_1 < \beta_2 < t$ we have that

$$F_r(\Psi)(t, \beta_2) - F_r(\Psi)(t, \beta_1) = \int_{\beta_2^-}^{t^-} e^{-r(t-\tau)} d\kappa(t, \tau) (\Psi(\tau, \beta_2) - \Psi(\tau, \beta_1)) - \int_{\beta_1^-}^{\beta_2^-} e^{-r(t-\tau)} d\kappa(t, \tau) \Psi(\tau, \beta_1),$$

where we used that $\kappa(t, \cdot)$ is left continuous to assign the lower (resp. upper) bound β_2^- to the first (resp. second) integral in the above right hand side. Now, the first integral goes to 0 as $\beta_1 \rightarrow \beta_2$ by dominated convergence, since $\Psi(t, \cdot)$ is left-continuous; the second integral also goes to 0, because $|\nu_{\kappa(t, \cdot)}|([\beta_1, \beta_2]) \rightarrow 0$ when $\beta_1 \rightarrow \beta_2$, by standard properties of finite measures. Altogether, we see that $F_r(\Psi)(t, \cdot)$ is left-continuous. Moreover, for $[c, d] \subset [a, t]$ and $c = \beta_0 < \beta_1 < \dots < \beta_N = d$,

$$\begin{aligned} & \sum_{i=1}^N \left\| F_r(\Psi)(t, \beta_i) - F_r(\Psi)(t, \beta_{i-1}) \right\| \\ & \leq \sum_{i=1}^N \left\| \int_{\beta_i^-}^{t^-} e^{-r(t-\tau)} d\kappa(t, \tau) (\Psi(\tau, \beta_i) - \Psi(\tau, \beta_{i-1})) \right\| + \sum_{i=1}^N \left\| \int_{\beta_{i-1}^-}^{\beta_i^-} e^{-r(t-\tau)} d\kappa(t, \tau) \Psi(\tau, \beta_{i-1}) \right\| \\ & \leq \sum_{i=1}^N \int_{\beta_i^-}^{t^-} e^{-r(t-\tau)} d|\nu_{\kappa(t, \cdot)}| \left\| (\Psi(\tau, \beta_i) - \Psi(\tau, \beta_{i-1})) \right\| + \sum_{i=1}^N \int_{\beta_{i-1}^-}^{\beta_i^-} e^{-r(t-\tau)} d|\nu_{\kappa(t, \cdot)}| \left\| \Psi(\tau, \beta_{i-1}) \right\| \\ & \leq \int_{d^-}^{t^-} d|\nu_{\kappa(t, \cdot)}| \sum_{i=1}^N \left\| (\Psi(\tau, \beta_i) - \Psi(\tau, \beta_{i-1})) \right\| \\ & \quad + e^{-r(t-d)} \int_{c^-}^{d^-} d|\nu_{\kappa(t, \cdot)}| \sum_{i=1}^N \left\| (\Psi(\tau, \beta_i) - \Psi(\tau, \beta_{i-1})) \right\| + \sup_{[a,t] \times [a,t]} \left\| \Psi \right\| \int_{c^-}^{d^-} e^{-r(t-\tau)} d|\nu_{\kappa(t, \cdot)}| \\ & \leq 2 W_{[d,t]}(\kappa(t, \cdot)) \sup_{\tau \in [d,t]} W_{[c,d]}(\Psi(\tau, \cdot)) \\ & \quad + 2 e^{-r(t-d)} W_{[c,d]}(\kappa(t, \cdot)) \sup_{\tau \in [c,d]} W_{[c,d]}(\Psi(\tau, \cdot)) + 2 e^{-r(t-d)} \sup_{[a,t] \times [a,t]} \left\| \Psi \right\| W_{[c,d]}(\kappa(t, \cdot)). \end{aligned}$$

When $d = t$, the same inequality holds but then $W_{[d,t]}(\kappa(t, \cdot))$ is zero. Setting $c = a$ and $d = t$, we get from the above majorization that $W_{[a,t]}(F_r(\Psi)(t, \cdot)) \leq 4 \|\kappa\|_{[a,b]} \|\Psi\|_{[a,b]}$, and since $F_r(\Psi)(t, \tau) = 0$ for $\tau \geq t$ we deduce that $W_{[a,b]}(F_r(\Psi)(t, \cdot)) = W_{[a,t]}(F_r(\Psi)(t, \cdot))$ is bounded, uniformly with respect to t . Next, if we fix $\varepsilon > 0$ and pick $\eta > 0$ so small that $W_{[\tau,t]}(\kappa(t, \cdot)) \leq \varepsilon$ as soon as $t - \tau \leq \eta$ (this is possible because $\kappa \in \mathcal{K}_{[a,b]}$), the same estimate yields

$$W_{[c,t]}(F_r(\Psi)(t, \cdot)) \leq 4 W_{[c,t]}(\kappa(t, \cdot)) \|\Psi\|_{[a,b]} \leq 4 \varepsilon \|\Psi\|_{[a,b]}, \quad t - c \leq \eta. \quad (\text{A.13})$$

Altogether, we just showed that $F_r(\Psi) \in \mathcal{K}_{[a,b]}$. Moreover, if we take r so large that $e^{-r\eta} < \varepsilon$, then either $t - a \leq \eta$ and then (A.13) with $c = a$ gives us $W_{[a,t]}(F_r(\Psi)(t, \cdot)) \leq 4\varepsilon \|\Psi\|_{[a,b]}$, or else $t - \eta > a$

in which case (A.13) with $c = t - \eta$, together with our initial estimate when $c = a$ and $d = t - \eta$, team up to produce:

$$\begin{aligned}
W_{[a,t]}(F_r(\Psi))(t, \cdot) &= W_{[a,t-\eta]}(F_r(\Psi)(t, \cdot)) + W_{[t-\eta,t]}(F_r(\Psi)(t, \cdot)) \\
&\leq 2\varepsilon \sup_{\tau \in [t-\eta,t]} W_{[a,t-\eta]}(\Psi(\tau, \cdot)) + 2\varepsilon W_{[a,t-\eta]}(\kappa(t, \cdot)) \sup_{\tau \in [a,t-\tau]} W_{[a,t-\eta]}(\Psi(\tau, \cdot)) \\
&\quad + 2\varepsilon \sup_{[a,t] \times [a,t]} \|\Psi\| W_{[a,t-\eta]}(\kappa(t, \cdot)) + 4\varepsilon \|\Psi\|_{[a,b]} \\
&\leq 2\varepsilon \|\Psi\|_{[a,b]} (3 + 2\|\kappa\|_{[a,b]}). \tag{A.14}
\end{aligned}$$

Consequently, as $W_{[a,t]}(F_r(\Psi)(t, \cdot)) = W_{[a,t]}(F_r(\Psi)(t, \cdot))$ by the left continuity of $F_r(\Psi)(t, \cdot)$, we can ensure upon choosing r sufficiently large that the operator $F_r : \mathcal{K}_{[a,b]} \rightarrow \mathcal{K}_{[a,b]}$ has arbitrary small norm. Hereafter, we fix r so that $\|F_r\| < \lambda < 1$.

Now, let $\tilde{\rho}_0 = 0$ and define inductively:

$$\tilde{\rho}_{k+1}(t, \beta) = -e^{-rt}\kappa(t, \beta) + F_r(\tilde{\rho}_k)(t, \beta).$$

Clearly $(t, \beta) \mapsto e^{-rt}\kappa(t, \beta)$ lies in $\mathcal{K}_{[a,b]}$, so that $\tilde{\rho}_k \in \mathcal{K}_{[a,b]}$ for all k . Moreover, we get from what precedes that $\|\tilde{\rho}_{k+1} - \tilde{\rho}_k\|_{[a,b]} \leq \lambda \|\tilde{\rho}_k - \tilde{\rho}_{k-1}\|_{[a,b]}$. Thus, by the shrinking lemma, $\tilde{\rho}_k$ converges in $\mathcal{K}_{[a,b]}$ to the unique $\tilde{\rho} \in \mathcal{K}_{[a,b]}$ such that

$$\begin{aligned}
\tilde{\rho}(t, \beta) &= -e^{-rt}\kappa(t, \beta) + F_r(\tilde{\rho})(t, \beta) \\
&= -e^{-rt}\kappa(t, \beta) + \int_{\beta^-}^{t^-} e^{-r(t-\tau)} d\kappa(t, \tau) \tilde{\rho}(\tau, \beta), \quad a \leq t, \beta \leq b. \tag{A.15}
\end{aligned}$$

Putting $\rho(t, \beta) := e^{rt}\tilde{\rho}(t, \beta)$, one can see that ρ lies in $\mathcal{K}_{[a,b]}$ if and only if $\tilde{\rho}$ does, and that (A.15) is equivalent to (A.12). This achieves the proof. \square

Lemma 15. *Let κ be a Stieltjes-Volterra kernel on $[a, b] \times [a, b]$ and ρ its resolvent. For each \mathbb{R}^d -valued function $g \in BV_r([a, b])$, the unique bounded measurable solution to the equation*

$$y(t) = \int_{a^-}^{t^-} d\kappa(t, \tau)y(\tau) + g(t), \quad a \leq t \leq b, \tag{A.16}$$

is given by

$$y(t) = g(t) - \int_{a^-}^{t^-} d\rho(t, \alpha)g(\alpha), \quad a \leq t \leq b. \tag{A.17}$$

PROOF. Define y through (A.17) so that $y(a) = g(a)$, by inspection. Since $g \in BV_r[a, b]$ and $\rho(t, \cdot), \kappa(t, \cdot)$ lie in $BV_i[a, b]$, integrating by parts [19, thm. 7.5.9]⁵ while using (A.12) along with

⁵This reference deals with open intervals only, and we stick to this case at the cost of a slightly lengthier computation

Fubini's theorem and the relations $\kappa(t, \alpha) = \rho(t, \alpha) = 0$ for $\alpha \geq t$, we get that

$$\begin{aligned}
\int_{a^-}^{t^-} d\kappa(t, \alpha)y(\alpha) &= (\kappa(t, a^+) - \kappa(t, a))y(a) + \int_{a^+}^{t^-} d\kappa(t, \alpha)y(\alpha) \\
&= (\kappa(t, a^+) - \kappa(t, a))g(a) + \int_{a^+}^{t^-} d\kappa(t, \alpha)g(\alpha) - \int_{a^+}^{t^-} d\kappa(t, \alpha) \int_{a^-}^{\alpha^-} d\rho(\alpha, \beta)g(\beta) \\
&= (\kappa(t, a^+) - \kappa(t, a))g(a) + \int_{a^+}^{t^-} d\kappa(t, \alpha)g(\alpha) - \int_{a^+}^{t^-} d\kappa(t, \alpha) \int_{a^+}^{\alpha^-} d\rho(\alpha, \beta)g(\beta) \\
&\quad - \int_{a^+}^{t^-} d\kappa(t, \alpha)(\rho(\alpha, a^+) - \rho(\alpha, a))g(a) \\
&= (\kappa(t, a^+) - \kappa(t, a))g(a) + \int_{a^+}^{t^-} d\kappa(t, \alpha)g(\alpha) + \int_{a^+}^{t^-} d\kappa(t, \alpha) \int_{a^+}^{\alpha^-} \rho(\alpha, \beta)dg(\beta) \\
&\quad - \int_{a^+}^{t^-} d\kappa(t, \alpha) [\rho(\alpha, \beta)g(\beta)]_{\beta=a^+}^{\beta=\alpha^-} - \int_{a^+}^{t^-} d\kappa(t, \alpha)(\rho(\alpha, a^+) - \rho(\alpha, a))g(a) \\
&= (\kappa(t, a^+) - \kappa(t, a))g(a) + \int_{a^+}^{t^-} d\kappa(t, \alpha)g(\alpha) + \int_{a^+}^{t^-} \left(\int_{\beta^-}^{t^-} d\kappa(t, \alpha)\rho(\alpha, \beta) \right) dg(\beta) \\
&\quad + \int_{a^+}^{t^-} d\kappa(t, \alpha)\rho(\alpha, a^+)g(a) - \int_{a^+}^{t^-} d\kappa(t, \alpha)(\rho(\alpha, a^+) - \rho(\alpha, a))g(a) \\
&= (\kappa(t, a^+) - \kappa(t, a))g(a) + \int_{a^+}^{t^-} d\kappa(t, \alpha)g(\alpha) \\
&\quad + \int_{a^+}^{t^-} (\rho(t, \beta) + \kappa(t, \beta))dg(\beta) + \int_{a^+}^{t^-} d\kappa(t, \alpha)\rho(\alpha, a)g(a) \\
&= (\kappa(t, a^+) - \kappa(t, a))g(a) + [\kappa(t, \alpha)g(\alpha)]_{\alpha=a^+}^{\alpha=t^-} + \int_{a^+}^{t^-} \rho(t, \beta)dg(\beta) + \int_{a^+}^{t^-} d\kappa(t, \alpha)\rho(\alpha, a)g(a) \\
&= -\kappa(t, a)g(a) + [\rho(t, \beta)g(\beta)]_{\beta=a^+}^{\beta=t^-} - \int_{a^+}^{t^-} d\rho(t, \beta)g(\beta) + \int_{a^-}^{t^-} d\kappa(t, \alpha)\rho(\alpha, a)g(a) \\
&= -\kappa(t, a)g(a) - \rho(t, a^+)g(a) - \int_{a^+}^{t^-} d\rho(t, \beta)g(\beta) + (\kappa(t, a) + \rho(t, a))g(a) \\
&= -\int_{a^-}^{t^-} d\rho(t, \beta)g(\beta) = y(t) - g(t).
\end{aligned}$$

Thus, y is a solution to (A.16). Clearly, it is measurable, and it is also bounded since $\|\rho(t, \cdot)\|_{BV([a,b])}$ is bounded independently of t and g is bounded. If \tilde{y} is another solution to (A.16) then $\tilde{y}(a) = y(a) = g(a)$ by inspection, so that $z := y - \tilde{y}$ is a bounded measurable solution to the homogeneous equation:

$$z(t) = \int_{a^+}^{t^-} d\kappa(t, \tau)z(\tau), \quad a \leq t \leq b.$$

Pick $r > 0$ to be adjusted momentarily, and set $\tilde{z}(t) := e^{-rt}z(t)$ so that

$$\tilde{z}(t) = \int_{a^+}^{t^-} e^{-r(t-\tau)} d\kappa(t, \tau)\tilde{z}(\tau). \quad (\text{A.18})$$

Let $\eta > 0$ be so small that $W_{[\tau,t]}(\kappa(t, \cdot)) \leq 1/4$ as soon as $t - \tau \leq \eta$; this is possible because $\kappa \in \mathcal{K}_{[a,b]}$. Then, it follows from (A.18) that for $t - \eta > a$:

$$\begin{aligned}
|\tilde{z}(t)| &\leq \left| \int_{a^+}^{(t-\eta)^+} e^{-r(t-\tau)^+} d\kappa(t, \tau)\tilde{z}(\tau) \right| + \left| \int_{(t-\eta)^+}^{t^-} e^{-r(t-\tau)^-} d\kappa(t, \tau)\tilde{z}(\tau) \right| \\
&\leq 2e^{-r\eta}W_{(a,t-\eta]}(\kappa(t, \cdot)) \sup_{(a,t-\eta]} |\tilde{z}| + \frac{1}{2} \sup_{(t-\eta,t)} |\tilde{z}| \leq \sup_{(a,t)} |\tilde{z}| (2e^{-r\eta}\|\kappa\|_{[a,b]} + \frac{1}{2}),
\end{aligned}$$

while for $t - \eta \leq a$ we simply get $|\tilde{z}(t)| \leq \sup_{(a,t)} |\tilde{z}|/2$. Hence, choosing r large enough, we may assume that $|\tilde{z}(t)| \leq \lambda \sup_{(a,t)} |\tilde{z}|$ for some $\lambda < 1$ and all $t \in [a, b]$. Thus, if we choose $\lambda' \in (\lambda, 1)$ and $t_0 \in (a, b]$, we can find $t_1 \in (a, t_0)$ such that $|\tilde{z}(t_1)| \geq (1/\lambda')|\tilde{z}(t_0)|$, and proceeding inductively we construct a sequence (t_n) in $(a, t_0]$ with $|\tilde{z}(t_n)| \geq (1/\lambda')^n |\tilde{z}(t_0)|$. If we had $|\tilde{z}(t_0)| > 0$, this would contradict the boundedness of \tilde{z} , therefore $\tilde{z} \equiv 0$ on $(a, b]$, whence $z \equiv 0$ so that $y = \tilde{y}$. \square

A.3. Proof of formula (4.4).

Let ρ denote the resolvent of the Stieltjes-Volterra kernel k on $[s, s + \tau_N] \times [s, s + \tau_N]$ defined in (A.9). As f defined in (A.8) lies in $BV_r[s, s + \tau_N]$ (it is even continuous), the solution y to (A.10) is given, in view of Lemma 15, by

$$y(t) = f(t) - \int_{s^-}^{t^-} d\rho(t, \alpha)f(\alpha), \quad s \leq t \leq s + \tau_N. \quad (\text{A.19})$$

Since $\rho(t, \alpha) = 0$ when $\alpha \geq t$, the integral $\int_{s^-}^{t^-}$ can be replaced by $\int_{s^-}^{(s+\tau_N)^+}$ in (A.19). Thus, setting $\tilde{X}(t, \alpha) := I_d \mathcal{H}(t - \alpha) + \rho(t, \alpha)$ with $\mathcal{H}(\tau)$ the ‘‘standard’’ Heaviside function which is 0 for $\tau < 0$ and 1 for $\tau \geq 0$, we deduce from (A.11), since $d_\alpha \mathcal{H}(t - \alpha) = -\delta_t$ on $[s, s + \tau_N]$ for $s \leq t < s + \tau_N$, that

$$y(t) = - \int_{s^-}^{(s+\tau_N)^+} d\tilde{X}(t, \alpha)f(\alpha) = - \int_{s^-}^{(s+\tau_N)^+} d\tilde{X}(t, \alpha) \left(\sum_{\tau_\ell \in (\alpha-s, \tau_N]} D_\ell(\alpha)\phi(\alpha - s - \tau_\ell) \right), \quad s \leq t < s + \tau_N.$$

Rearranging, we get that

$$y(t) = - \sum_{j=1}^N \int_{s^-}^{(s+\tau_j)^-} d\tilde{X}(t, \alpha) D_j(\alpha)\phi(\alpha - s - \tau_j), \quad s \leq t < s + \tau_N,$$

which is what we want (namely: formula (4.4) for $s \leq t < s + \tau_N$) if only we can show that $\tilde{X}(t, \alpha)$ coincides with $X(t, \alpha)$ when $\alpha \in [s, s + \tau_j)$ for each j and every $t \in [s, s + \tau_N]$; here, $X(t, \alpha)$ is defined by (4.1) where we set $s = \alpha$.

For this, we first observe that $X(t, \alpha) = \tilde{X}(t, \alpha) = 0$ when $\alpha > t$ and that $X(t, t) = \tilde{X}(t, t) = Id$. Hence, we need only consider the case $\alpha \in [s, t)$ with $s < t < s + \tau_N$. For $s \leq \alpha < t$, we get that

$$-k(t, \alpha) = k(t, t^-) - k(t, \alpha) = \int_{\alpha^-}^{t^-} dk(t, \tau).$$

Thus, (A.12) (where $\kappa = k$) in concert with the definition of $\tilde{X}(t, \alpha)$ imply that

$$\begin{aligned} \tilde{X}(t, \alpha) &= I_d \mathcal{H}(t - \alpha) - k(t, \alpha) + \int_{\alpha^-}^{t^-} dk(t, \tau)\rho(\tau, \alpha) = I_d + \int_{\alpha^-}^{t^-} dk(t, \tau)(I_d + \rho(\tau, \alpha)) \\ &= I_d + \int_{\alpha^-}^{t^-} dk(t, \tau)\tilde{X}(\tau, \alpha). \end{aligned}$$

Now, on $[\alpha, t)$, we compute from (A.7) and (A.9) that $d_\tau k(t, \tau) = \sum_{t-\tau_j \geq \alpha} D_j(t)\delta_{t-\tau_j}$ and hence, since $\tilde{X}(t - \tau_j, \alpha) = 0$ when $\alpha > t - \tau_j$, the previous equation becomes:

$$\tilde{X}(t, \alpha) = I_d + \sum_{j=1}^N D_j(t)\tilde{X}(t - \tau_j, \alpha) \quad \text{for } s \leq \alpha < t \text{ and } s \leq t < s + \tau_N. \quad (\text{A.20})$$

Comparing (A.20) and (4.1), we see that $\tilde{X}(t, \alpha)$ and $X(t, \alpha)$ coincide on $[s, s + \tau_N) \times [s, s + \tau_N)$, thereby ending the proof.