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Exponential stability of linear periodic difference-delay equations

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Abstract

This paper deals with stability of linear periodic time-varying difference delay systems, i.e. dynamical systems where a finite dimensional signal at time t is given as a linear time-varying function of its value at a finite number of delayed times $t - \tau_1, \dots, t - \tau_N$.

We give a necessary and sufficient condition for exponential stability, that is a generalisation of the one by Henry and Hale in the 1970s. It has a control theoretic interpretation in terms of the harmonic transfer function (HTF) of a corresponding linear control system.

Keywords: linear periodic systems, difference delay systems, exponential stability, harmonic transfer function, Henry-Hale theorem.

1. Introduction

This paper deals with stability of time-varying linear *difference-delay systems*²; i.e., linear dynamical systems of the form

$$y(t) = \sum_{j=1}^N D_j(t) y(t - \tau_j) \quad \text{for all } t \geq s, \quad (1.1)$$

where $\tau_1 < \dots < \tau_N$ are positive delays and $D_1(t), \dots, D_N(t)$ are complex $d \times d$ matrices depending on time (at least continuously all along this paper). A *solution* is a map $y(\cdot) : [s - \tau_N, +\infty) \rightarrow \mathbb{C}^d$ that satisfies (1.1). More general linear difference-delay systems are obtained by letting $y(t)$ be a function of the values of y at distributed time instants on the interval $[t - \tau_N, t)$; a general form drawn from [1] is

$$y(t) = \int_{\theta \in [-\tau_N, 0]} d\mu(t, \theta) y(t + \theta), \quad (1.2)$$

where $d\mu(t, \cdot)$ is a family of measures on the interval $[-\tau_N, 0]$ indexed by t , with values in $d \times d$ matrices (further assumptions are needed to guarantee well-posedness). One obtains (1.1) from (1.2) by taking $\mu(t, \cdot) = \sum_j D_j(t) \delta_{-\tau_j}(\cdot)$, a sum of N Dirac masses with coefficients the $D_j(t)$; in [4], the number of Dirac masses (and hence of delays) is countable but not necessarily finite. In

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²There seems to be no general agreement on terminology here. No specific name is coined in [1, 2], where (1.1) is written $Ly(\cdot) = 0$ with L a *difference operator*, and one speaks of exponential stability of the operator rather than the dynamical system. In [3], system (1.1) is termed a *time-delay system*. In [4, 5], the authors simply call (1.1) a *difference equation*; we adapted the latter into “difference-delay system” (cf. [6, 7]), to mean that it is a continuous-time dynamical system in which the value of signals at a given time is a function of their value at delayed instants; this name is in agreement with [8].

the present paper, we stick to systems of the form (1.1) but the techniques should be amenable to cover more general cases from (1.2). With an eye on this, we treat fundamental solutions and variation-of-constants formulas somewhat generally in Appendix A.

A solution to (1.1) is uniquely determined by its restriction to $[s - \tau_N, s]$. We say that system (1.1) is C^0 exponentially stable if all continuous solutions undergo uniform exponential decay of their restriction to $[t - \tau_N, t]$, with respect to the sup norm, as $t \rightarrow +\infty$. Exponential stability may be defined similarly with respect to the L^q norm, and it follows from [5, Corollary 3.29] or [6, Theorem 3.4] that for systems like (1.1), C^0 -exponential stability is equivalent to L^q -exponential stability, for any single $q \in [1, \infty]$.

The main result of the present paper is a necessary and sufficient condition for C^0 exponential stability of (1.1), when the maps $t \mapsto D_j(t)$ are smooth and periodic of period $T > 0$. By what we just said, the same condition is necessary and sufficient for L^q -exponential stability, regardless of $q \in [1, \infty]$. This condition is a natural generalisation of the one given in [2, 4], in the case where the D_j are *constant* matrices (the Henry-Hale theorem). The proof embeds system (1.1) into the control system (2.20) and leads us to introduce, in the present infinite-dimensional context, an analog of the so-called *harmonic transfer-function* introduced in Engineering to study finite-dimensional periodic systems; *cf.*, [9, 10]. Our analysis lays ground for a systematic study of the harmonic transfer function, defined here as a holomorphic operator-valued function, for the case of more general infinite-dimensional periodic systems. This is another contribution of the paper.

Some motivation. Dynamical systems of the form (1.1) or (1.2) are of interest in their own right, and characterizing their stability has stand-alone mathematical relevance. Besides, they arise naturally in connection with other, more specific classes of dynamical systems:

- a) Studying 1-D hyperbolic systems is almost equivalent to studying difference-delay systems. Indeed, after explicit integration in terms of arbitrary functions of one variable (backward and forward waves), the evolution of 1-D hyperbolic systems is governed by equations of the type (1.1), see for example [3] or the textbook [11], and also the paper [6] where the authors take advantage of this dual point of view.
- b) Difference-delay systems are also instrumental when studying the stability of functional *differential* equations, because they appear as “high frequency limits” of the latter. For instance, if we consider a so-called “neutral functional differential equation” of the form

$$\frac{d}{dt} \left(y(t) - \sum_{j=1}^N D_j(t) y(t - \tau_j) \right) = B_0(t) y(t) + \sum_{j=1}^N B_j(t) y(t - \tau_j), \quad t \geq s, \quad (1.3)$$

then its solution operator (mapping the state at time s to the state at time t) is a compact perturbation of the solution operator of (1.1) (see [1, thm 7.3 Section 3.7]); this makes the stability properties of (1.3) similar to those of (1.1), up to finitely many Jordan blocks.

- c) In fact, the initial motivation of the authors to undertake the present study was in stability of active microwave circuits. The reader is referred to [7, Chapter 2] and the introduction of [6] for details. Such circuits can be considered as nonlinear systems with infinite dimensional state, due to transmission lines that are modelled either by 1-D hyperbolic systems or by delays. The response to a periodic signal (for instance: the signal to be amplified in an amplifier) in the so-called high-gain regime is a periodic solution of this nonlinear system, and taking the linear approximation along this periodic trajectory provides one with a linear periodic time-varying dynamical system. The latter is governed by a functional differential

equation of more general type than (1.3), that mixes differential equations and pure delay equations, but still it has a high frequency limit system of the form (1.1). Subsequently, the solution operator of the linearized system along the periodic trajectory under examination turns out to be a compact perturbation of the solution operator of the high frequency limit system, and if the latter is stable then the stability of the periodic solution can be numerically investigated by chasing singular points of the harmonic transfer function; see [7], and also [12] for a discussion of such methods in Engineering, based on frequency-wise numerical simulations obtained *via* the so-called *harmonic-balance* techniques.

State of the art. Let us first point out that the simplest situation occurs when the delays and the period are commensurate; *i.e.*, when the ratio of any two numbers $\{\tau_1, \dots, \tau_N, T\}$ is rational (of course, we ignore T in the time-invariant case). Indeed, when this is the case, there exists an integer M such that $y(t)$ only depends on the values of y in at most M time instants from $[-\tau_N, 0]$, and then (1.1) can be viewed as a collection of infinitely many decoupled finite dimensional discrete-time systems. Without commensurability, the system is truly infinite-dimensional but the *time-invariant case*, where the matrices D_j in (1.1) are *constant* (or the matrices B_i, D_j in (1.3) if one considers the latter equation) is of course simpler.

In the time-invariant case, stability of functional differential equations like (1.3) has been studied from the fifties on. By and large, works on the subject fall into two classes: those relying on Laplace transform and the spectral theory of semi-groups to reduce issues of stability to the location of zeros of almost periodic holomorphic functions (see for instance [13, 4, 2, 1]), and those seeking to construct Lyapunov–Krasovskii functionals whose existence is sufficient for exponential stability to hold (for example as in [1, 14, 15]). For time invariant difference-delay systems of the form (1.1) (that is: with constant D_j), a well-known necessary and sufficient condition for exponential stability (either C^0 or L^q for any $q \in [1, \infty]$) can be given in terms of the matrix-valued function $H(p) = I - \sum_{j=1}^N e^{-p\tau_j} D_j$ of the complex variable p ; namely, exponential stability in this case is equivalent to the existence of $\alpha < 0$ such that $H(p)$ is invertible for $\Re(p) > \alpha$. This result, that we call the *Henry-Hale Theorem*, was initially established in [2] for finitely many commensurate delays and later generalized by Henry to the case of countably many, not necessarily commensurate delays [4]; it was subsequently carried over to distributed delay systems (1.2), see [1, Chapter 9]. Although this condition is often difficult to check and stability is frequently established in practice by exhibiting a Lyapunov-Krasovskii functional, the Henry-Hale theorem brings closure, theoretically at least, to exponential stability issues for systems like (1.1) in the time-invariant case.

The stability of *time-varying* linear difference-delay systems is much more delicate, and it has apparently not been studied nearly as extensively: literature we know of is scarce and can be recapped as follows. Firstly, a sufficient condition for exponential stability of linear time-varying delay systems, based on the Perron-Frobenius theorem, can be found in [8]; though interesting it is of rather special character. Secondly, when a linear time-varying system of the form (1.1) arises from a network of lossless telegrapher’s equations (for instance as the high frequency limit of an electrical network), then a sufficient condition for exponential stability can be given, based on dissipativity at the nodes of the network [6]; related, somewhat specific criteria for hyperbolic 1-D systems may be found in [3]. Thirdly, a detailed analysis of system (1.1) in the general time-varying case is carried out in [5]; this paper sets up a “representation formula” for the general solution and derives from it interesting results, for instance regarding how stability is preserved under perturbation of the delays. However, as it involves sums of products of matrices with indefinitely growing number of terms and factors when t goes large, indexed according to combinatorial rules involving the lattice generated by the τ_j , the characterization of exponential stability stated in [5]

can hardly be verified in practice and does not constitute an explicit criterion like the Henry-Hale theorem does in the time-invariant case.

Further below, we state as Theorem 3 a characterisation of exponential stability for periodic systems like (1.1) with Hölder-smooth coefficients, which seems to be a proper generalization of the Henry-Hale theorem. The matrix-valued function from the latter gets replaced by an operator valued one.

Control theoretic point of view. To the dynamical system (1.1), one may associate a control system by adding a term $u(t)$ (the control) in the right-hand side. *In the time-invariant case*, the transfer function H of that control system is the matrix-valued function $H : p \mapsto \left(I - \sum_{j=1}^N e^{-p\tau_j} D_j\right)^{-1}$ of the complex variable p (see *e.g.* [16]), and the Henry-Hale theorem (already mentioned and stated below as Theorem 2) can be rephrased by saying that system (1.1) is exponentially stable if and only if the transfer function is holomorphic in some half plane $\{p \in \mathbb{C}, \Re(p) > \alpha\}$, where α is a strictly negative real number. *In the periodic case*, the concept of transfer function generalizes to the one of *Harmonic Transfer Function* (HTF), and our main result (Theorem 3) has the same interpretation in terms of HTF and exponential stability of periodic difference-delay control systems as the Henry-Hale theorem has in terms of classical transfer functions and exponential stability of time-invariant difference-delay control systems. This is expounded in Section 2.4.

Organization of the paper. Section 2 describes the problem in greater detail than we just did, and contains statements of the main results. Section 3 makes the necessary definitions and notation, while Section 4 is devoted to preliminary results needed for our purposes, establishing in particular a variation-of-constants formula for difference-delay systems. We prove the necessity and sufficiency parts of Theorem 3 in Sections 5 and 6, respectively. The concluding remarks in Section 7 mostly consist in a conjecture about neutral periodic delay equations

2. Main results

2.1. Problem statement

We use complex coefficients and \mathbb{C}^k -valued functions throughout the paper to alleviate notation, but we do not use the complex structure in any meaningful way and everything goes through if one restricts to real coefficients and \mathbb{R}^k -valued functions.

Consider a periodic difference-delay system of the form (1.1):

$$y(t) = \sum_{j=1}^N D_j(t)y(t - \tau_j), \quad t \geq s, \quad y(s + \theta) = \phi(\theta) \text{ for } -\tau_N \leq \theta \leq 0, \quad (2.1)$$

where $s \in \mathbb{R}$ is the initial time, d and N are positive integers, $D_j : \mathbb{R} \rightarrow \mathbb{C}^{d \times d}$ is a continuous T -periodic matrix-valued function:

$$D_j(t + T) = D_j(t), \quad 1 \leq j \leq N, \quad (2.2)$$

$\phi : [-\tau_N, 0] \rightarrow \mathbb{C}^d$ is an initial condition and the solutions to (2.1) are \mathbb{C}^d valued functions $t \mapsto y(t)$ of the time t .

Note that delays are assumed to be strictly positive. Without loss of generality, we order them and assume that T is strictly larger than any of them:

$$0 < \tau_1 < \dots < \tau_N < T \quad (2.3)$$

(otherwise we may replace T by kT with $k > 0$ the smallest integer such that $kT > \tau_N$).

Since the $D_j(\cdot)$ are continuous, it is natural to seek continuous solutions $y \in C^0([s, +\infty), \mathbb{C}^d)$; see Section 3 for notation. A compatibility condition is required, however, for a continuous map $\phi : [-\tau_N, 0] \rightarrow \mathbb{C}^d$ to be the initial condition of a continuous solution to (2.1). Specifically, ϕ must belong to the space

$$C_s := \{\phi \in C^0([-\tau_N, 0], \mathbb{C}^d) \mid \phi(0) = \sum_{j=1}^N D_j(s)\phi(-\tau_j)\}. \quad (2.4)$$

Given $\phi \in C_s$, an easy recursion shows that the system (2.1) has a unique continuous solution y with initial condition ϕ . Thus, we can define for $t \geq s$ the solution operator $U(t, s) : C_s \rightarrow C_t$ that maps $\phi \in C_s$ to $U(t, s)\phi$ defined by

$$(U(t, s)\phi)(\theta) = y(t + \theta), \quad \theta \in [-\tau_N, 0]. \quad (2.5)$$

One obviously has $C_{s+T} = C_s$ and $U(t+T, s+T)$ for the maps D_j are T -periodic.

One may consider as well solutions in the local Lebesgue space $L_{\text{loc}}^q([s, +\infty), \mathbb{C}^d)$, $1 \leq q \leq \infty$. The latter comprises (equivalence classes of almost everywhere coinciding) functions whose norm to the q^{th} power is locally integrable (or whose norm is locally essentially bounded if $q = \infty$), see Section 3. In this case we only require (2.1) to hold for almost every $t \in \mathbb{R}$, and no compatibility condition on ϕ is needed anymore. More precisely, the same recursion as before shows that for each $\phi \in L^q([-\tau_N, 0], \mathbb{C}^d)$ the system (2.1) admits a unique solution $y \in L_{\text{loc}}^q([s, +\infty), \mathbb{C}^d)$. Consequently, we can define for $t \geq s$ the solution operator $U_q(t, s) : L^q([-\tau_N, 0], \mathbb{C}^d) \rightarrow L^q([-\tau_N, 0], \mathbb{C}^d)$ that maps any ϕ in $L^q([-\tau_N, 0], \mathbb{C}^d)$ to $U_q(t, s)\phi$ defined in a similar way to (2.5):

$$(U_q(t, s)\phi)(\theta) = y(t + \theta), \quad \text{a.e. } \theta \in [-\tau_N, 0]. \quad (2.6)$$

We endow C_s and $L^q([-\tau_N, 0], \mathbb{C}^d)$ with the classical C^0 (supremum) and L^q norms respectively (see Section 3), and define the corresponding notions of exponential stability as follows.

Definition 1. System (2.1) is called C^0 -exponentially stable if there exist $\gamma, K > 0$ such that

$$\|U(t, s)\phi\|_{C^0} \leq Ke^{-\gamma(t-s)}\|\phi\|_{C^0}, \quad \text{for all } s \in \mathbb{R}, \text{ all } t \geq s \text{ and all } \phi \in C_s. \quad (2.7)$$

It is called L^q -exponentially stable, $q \in [1, \infty]$, if there exist $\gamma, K > 0$ such that

$$\|U_q(t, s)\phi\|_{L^q} \leq Ke^{-\gamma(t-s)}\|\phi\|_{L^q}, \quad \text{for all } s \in \mathbb{R}, \text{ all } t \geq s \text{ and all } \phi \in L^q([-\tau_N, 0], \mathbb{C}^d). \quad (2.8)$$

It turns out that these properties are equivalent. Indeed, the following result is contained in [6, Theorem 3.4], and is also an easy consequence of [5, Corollary 3.29].

Proposition 1. *For any $q \in [1, \infty]$, system (2.1) is L^q -exponentially stable if and only if it is C^0 -exponentially stable.*

Proposition 1 is important to what follows, for the proof of sufficiency in Theorem 3 establishes C^0 -exponential stability assuming analyticity of a certain operator-valued function, while the proof of necessity establishes analyticity of the latter assuming L^2 -exponential stability. In view of Proposition 1, we no longer qualify the words “exponential stability” and we make the following definition.

Definition 2 (Exponential stability). System (2.1) is called *exponentially stable* if and only if there exist $\gamma, K > 0$ such that any of the equivalent properties (2.7) or (2.8) holds.

2.2. The Henry-Hale Theorem for time-invariant systems

Suppose for a while that the maps $t \mapsto D_j(t)$ are all constant. Then, assumption (2.2) holds for any $T > 0$, the sets C_s no longer depend on s , the operator $U(t, s)$ depends only on $t - s$ and the system clearly defines a semi-group; we do not elaborate on this point of view but let us recall the following characterisation of exponential stability that was obtained in this case.

Theorem 2 (Henry-Hale [4, Section 3], [1, Theorem 3.5]). *Assume that the matrices D_j are constant, $1 \leq j \leq N$. Under this assumption, a necessary and sufficient condition for System (2.1) to be C^0 -exponentially stable is the existence of a real number $\beta < 0$ such that :*

$$I_d - \sum_{j=1}^N e^{-p\tau_j} D_j \text{ is invertible in } \mathbb{C}^{d \times d} \text{ for all } p \text{ in } \{z \in \mathbb{C} | \Re(z) \geq \beta\}. \quad (2.9)$$

2.3. Conditions for stability in the time-varying periodic case

We turn to the setting under investigation, where the maps $t \mapsto D_j(t)$ are not necessarily constant but still continuous and T -periodic, see (2.2). Further smoothness assumptions will be made shortly.

A naive generalization of Theorem 2 fails; *i.e.*, if one replaces the assumption that the D_j are constant by the periodicity requirement (2.2), then the conclusion of the theorem no longer holds, however smooth the D_j . For example, let $N = 1$, $d = 2$, $T = 2\pi$, $\tau_1 = \pi/2$, and consider (2.1) with $D_1(t) = \begin{pmatrix} 1/2 & \alpha \cos^2 t \\ \alpha \sin^2 t & 1/2 \end{pmatrix}$ for some $\alpha \in (0, 1)$. On the one hand, condition (2.9) is satisfied because the eigenvalues of $D_1(t)$ are $\frac{1}{2}(1 \pm \alpha |\sin 2t|) \in (0, \gamma)$ with $\gamma = \frac{1}{2}(1 + \alpha) < 1$, implying that (2.9) holds with $\beta = \log \gamma^{1/\tau_1}$. On the other hand, all solutions satisfy $y(t_o + 2k\tau_1) = (D_1(t_o + 2\tau_1)D_1(t_o + \tau_1))^k y(t_o)$ for any time t_o and any integer k , with $D_1(t_o + 2\tau_1)D_1(t_o + \tau_1) = \begin{pmatrix} 1/4 + \alpha^2 \cos^4 t_o & \alpha/2 \\ \alpha/2 & 1/4 + \alpha^2 \sin^4 t_o \end{pmatrix}$, whose largest eigenvalue is $\frac{1}{4}(1 + \alpha^2 + \alpha^2 \cos^2 2t_o) + \frac{1}{2}\alpha\sqrt{1 + \alpha^2 \cos^2 2t_o}$. Taking α close enough to 1, this is larger than 1 for t_o in an interval around 0, defeating exponential stability.

Let us now proceed with a proper generalization of Theorem 2. As an extra piece of notation, we introduce

$$\omega = 2\pi/T, \quad (2.10)$$

and we put $\check{D}_j(k)$, for $j \in \{1, \dots, n\}$ and $k \in \mathbb{Z}$, to denote the k -th Fourier coefficient of D_j ; *i.e.*,

$$\check{D}_j(k) := \frac{1}{T} \int_0^T D_j(t) e^{-ik\omega t} dt. \quad (2.11)$$

We let L_{D_j} be the (doubly infinite) block Laurent matrix associated with $D_j(t)$, whose entries are

$$(L_{D_j})_{k,n} := \check{D}_j(n - k), \quad n, k \in \mathbb{Z}. \quad (2.12)$$

Since the $D_j(t)$ are bounded periodic functions, the Laurent matrices L_{D_j} act on the Hilbert space:

$$\ell^2(\mathbb{Z}, \mathbb{C}^d) := \{z = (z_j)_{j \in \mathbb{Z}} | z_j \in \mathbb{C}^d, \sum_{j=-\infty}^{+\infty} \|z_j\|^2 < +\infty\}, \quad (2.13)$$

equipped with the norm $\|\cdot\|_{\ell^2}$ given by

$$\|z\|_{\ell^2} := \left(\sum_{j=-\infty}^{+\infty} \|z_j\|^2 \right)^{1/2}. \quad (2.14)$$

For $A : \ell^2(\mathbb{Z}, \mathbb{C}^d) \rightarrow \ell^2(\mathbb{Z}, \mathbb{C}^d)$ a linear operator, we denote by $\|A\|_2$ its operator norm:

$$\|A\|_2 := \sup_{z \neq 0} \frac{\|Az\|_{\ell^2}}{\|z\|_{\ell^2}}. \quad (2.15)$$

In fact, $\|L_{D_j} z\|_{\ell^2} = \|D_j Z\|_{L^2(\mathbb{T}_{1/\omega})}$ where \mathbb{T}_r indicates the circle of radius r , while $Z \in L^2(\mathbb{T}_{1/\omega})$ is the function whose Fourier coefficients are the z_k . In particular, this implies $\|L_{D_j}\|_2 \leq \|D_j\|_{C^0}$. Write I_d for the identity operator on \mathbb{C}^d (or the $d \times d$ identity matrix), $I_\infty : \ell^2(\mathbb{Z}, \mathbb{C}^d) \rightarrow \ell^2(\mathbb{Z}, \mathbb{C}^d)$ for the identity operator (or its doubly infinite matrix in any Hilbertian basis), and by $\Delta_{\tau_j, \omega}$, for $j \in \{1, \dots, N\}$, the (doubly infinite) block diagonal matrix given by

$$\Delta_{\tau_j, \omega} := \text{diag}\{\dots, e^{-2i\omega\tau_j} I_d, e^{-i\omega\tau_j} I_d, I_d, e^{+i\omega\tau_j} I_d, e^{+2i\omega\tau_j} I_d, \dots\}, \quad (2.16)$$

where ω is as in (2.10) (obviously, $\|\Delta_{\tau_j, \omega}\|_2 = 1$). We further define a function R of the complex variable p , valued in the space of bounded operators $\ell^2(\mathbb{Z}, \mathbb{C}^d) \rightarrow \ell^2(\mathbb{Z}, \mathbb{C}^d)$, by the formula:

$$R(p) := I_\infty - \sum_{j=1}^N e^{-p\tau_j} L_{D_j} \Delta_{\tau_j, \omega}. \quad (2.17)$$

We shall often identify $R(p)$ with the doubly infinite matrix representing this operator in the Fourier basis, that is:

$$I_\infty - \left(\sum_{j=1}^N e^{-(p - i\ell\omega)\tau_j} \check{D}_j(\ell - k) \right)_{(k, \ell) \in \mathbb{Z}^2}. \quad (2.18)$$

Of necessity, this matrix has square-summable rows and columns.

Our main result is the following characterization of exponential stability in the periodic case.

Theorem 3 (Necessary and sufficient condition for exponential stability). *Assume that the $D_j : \mathbb{R} \rightarrow \mathbb{C}^{d \times d}$ are periodic and differentiable with Hölder continuous derivative for $1 \leq j \leq N$. Under this assumption, a necessary and sufficient condition for System (2.1) to be exponentially stable is the existence of a real number $\beta < 0$ such that :*

- (i) $R(p)$ is invertible $\ell^2(\mathbb{Z}, \mathbb{C}^d) \rightarrow \ell^2(\mathbb{Z}, \mathbb{C}^d)$ for all p in $\{z \in \mathbb{C} | \Re(z) \geq \beta\}$,
- (ii) there is a positive number M such that $\|R^{-1}(p)\|_2 \leq M$ for all p in $\{z \in \mathbb{C} | \Re(z) \geq \beta\}$.

When the D_j are constant matrices, we see from (2.12) that the operators $L_{D_j} \Delta_{\tau_j, \omega}$ are block diagonal with k -th diagonal block equal to $e^{ik\omega\tau_j} D_j$, and condition (i) in Theorem 3 is easily seen to be equivalent to the invertibility of $I_d - \sum_{j=1}^N e^{-p\tau_j} D_j$ for $\Re(p) > \beta$. In this case, as every analytic function in a vertical strip which is uniformly almost-periodic in the pure imaginary direction and has no zeros must be bounded below in modulus by a strictly positive constant [17, Ch. III, Sec. 2, Cor. 1], the modulus of the determinant of $I_d - \sum_{j=1}^N e^{-p\tau_j} D_j$ is greater than some $c > 0$ and therefore condition (ii) of Theorem 3 is redundant. Thus, Theorem 3 yields back the Henry-Hale theorem (Theorem 2) for systems with constant coefficients.

For general periodic difference-delay systems, the question as to whether Condition (i) implies Condition (ii) remains unanswered here. The authors surmise that it is not the case, but have no counterexample yet: apparently, little is known about invertibility of analytic almost periodic

Banach valued functions. Still, in the special case where delays are commensurate, the operator $p \mapsto R(p)$ is periodic of period $i\tau$ for some $\tau > 0$. Hence, as inversion is continuous on invertible elements of a Banach algebra and since boundedness of the inverse in (ii) needs only be checked on a compact set of p by periodicity and a von Neumann series argument, we deduce that (i) implies (ii) in this case. This entails the following corollary.

Corollary 4. *If the delays are commensurate, then condition (ii) can be omitted in Theorem 3.*

Remark 1. In Theorem 3, we assume that the periodic matrices $D_i(\cdot)$ have Hölder-continuous derivatives. Actually, the sufficiency part only uses Hölder-smoothness to ensure that Fourier coefficients decay conveniently, and would remain true without regularity hypothesis on D_j but replacing assumption (ii) by:

$$\| \| R(p)^{-1} \| \|_{\mathfrak{W}} \text{ is uniformly bounded for all } p \text{ in } \{z \in \mathbb{C} | \Re(z) \geq \beta\}, \quad (2.19)$$

where the Wiener norm $\| \cdot \|_{\mathfrak{W}}$ of a doubly infinite block matrix, defined in (5.11), is the sum of the supremum norms of the diagonals. It would be interesting to know if the necessity part still holds in this case.

It is likely that Theorem 3 generalizes to more general delay equations. After all, in the time-invariant case, the Henry-Hale Theorem has extensions to distributed-delay systems [1, Ch. 9]. However, this matter is left here for further research. Another issue of interest is to generalize Theorem 3 to the case of periodic linear neutral differential equation; we formulate a conjecture in this direction; cf. Section 7.

2.4. A system-theoretic point of view; harmonic transfer functions.

Recall that a linear time-invariant system mapping a control $u(t)$ to an output $y(t)$ can be described, under mild assumptions, as a convolution operator, so that taking Fourier-Laplace transforms converts the latter into a multiplication operator in the Fourier-Laplace domain; the multiplier is called the *transfer function* of the system. Now, to the dynamical system (1.1) one may associate the control system

$$y(t) = \sum_{j=1}^N D_j(t)y(t - \tau_j) + u(t), \quad y(t) = 0 \text{ and } u(t) = 0 \text{ for } t < s, \quad (2.20)$$

with control $u \in C^0([s, +\infty), \mathbb{C}^d)$ or $L_{loc}^p([s, +\infty), \mathbb{C}^d)$. Solving for u in (2.20), it is easy to see that for any time instant $t > \tau_N$ and any $\phi \in C_t$, (resp. $L^p([- \tau_N, 0), \mathbb{C}^d)$), there is $u \in C^0([s, t], \mathbb{C}^d)$ (resp. $L^p([s, t], \mathbb{C}^d)$) for which the map $\psi : [- \tau_N, 0] \rightarrow \mathbb{C}^d$ given by $\psi(\theta) = y(t + \theta)$ coincides with ϕ . Consequently, outputs y of (2.20) associated to controls u that vanish for $t > t_1 > \tau_N$ coincide with solutions to (1.1) where we let $s = t_1$. This explains why exponential stability of the control system (2.20) (meaning that if the system is driven by some input $u(t)$ vanishing for $t > t_0$ then the C^0 (or L^q)-norm of the restriction $y|_{[t - \tau_N, t]}$ decays exponentially fast to zero as $t \rightarrow +\infty$) is equivalent to exponential stability of system (1.1), in the sense of Definition 2.

In the time-invariant case, the transfer function of (2.20) is the matrix-valued function of the complex variable p given by $p \mapsto \left(I_d - \sum_{j=1}^N e^{-p\tau_j} D_j \right)^{-1}$, and the Henry-Hale theorem says that (2.1) is exponentially stable if and only if the transfer function is holomorphic in the half-plane $\{p \in \mathbb{C}, \Re(p) > \alpha\}$ for some $\alpha < 0$.

In the periodic case, the concept of transfer function generalizes into the one of *harmonic transfer function* (in short: HTF). It is again a function of the complex variable p , but instead of being $\mathbb{C}^{d \times d}$ -valued it takes values in the space of bounded operators $\ell^2(\mathbb{Z}, \mathbb{C}^d) \rightarrow \ell^2(\mathbb{Z}, \mathbb{C}^d)$. More precisely, one can check that $\|y\|_{L^2([s, \tau])} \leq C e^{\gamma(\tau-s)} \|u\|_{L^2([s, +\infty])}$ for appropriate constants C , γ and all $\tau > s$; compare (6.6). Thus, when $u \in L^2([s, +\infty), \mathbb{C}^d)$ one can define for $\Re(p) > \gamma$ the Laplace transforms of $y(t)$ and $u(t)$ by the formulas

$$\hat{Y}(p) := \int_s^{+\infty} e^{-pt} y(t) dt \quad \text{and} \quad \hat{U}(p) := \int_s^{+\infty} e^{-pt} u(t) dt. \quad (2.21)$$

If \hat{Y} , \hat{U} get sampled at equally spaced points on vertical lines with mesh $i\omega$ (see (2.10)), we obtain infinite vectors of $d \times 1$ column vectors:

$$\hat{\mathbf{Y}}(p) := \begin{pmatrix} \vdots \\ \hat{Y}(p + i\omega) \\ \hat{Y}(p) \\ \hat{Y}(p - i\omega) \\ \vdots \end{pmatrix} \quad \text{and} \quad \hat{\mathbf{U}}(p) := \begin{pmatrix} \vdots \\ \hat{U}(p + i\omega) \\ \hat{U}(p) \\ \hat{U}(p - i\omega) \\ \vdots \end{pmatrix}, \quad \Re(p) > \gamma, \quad (2.22)$$

where the convention is that lines are numbered from top to bottom; *i.e.*, the above notations could also be written

$$\hat{\mathbf{Y}}(p) = \left(\hat{Y}(p - ik\omega) \right)_{k \in \mathbb{Z}}^{\mathbf{t}}, \quad \text{and} \quad \hat{\mathbf{U}}(p) = \left(\hat{U}(p - ik\omega) \right)_{k \in \mathbb{Z}}^{\mathbf{t}}, \quad \Re(p) > \gamma, \quad (2.23)$$

where \mathbb{Z} receives the natural ordering and superscript \mathbf{t} means transpose.

Since the $\hat{U}(p + ik\omega)/T$ (resp. $\hat{Y}(p + ik\omega)/T$) are the Fourier coefficients of the T -periodic function $x \mapsto \sum_{j=-\infty}^{+\infty} e^{-p(x+jT)} u(x+jT)$ (resp. $x \mapsto \sum_{j=-\infty}^{+\infty} e^{-p(x+jT)} y(x+jT)$), we conclude that $\hat{\mathbf{Y}}$, $\hat{\mathbf{U}}$ lie in $\ell^2(\mathbb{Z}, \mathbb{C}^d)$. It can be shown that, for $\Re(p)$ large enough, there is an unique operator-valued holomorphic function $p \mapsto \mathbf{H}(p)$ such that $\mathbf{H}(p) : \ell^2(\mathbb{Z}, \mathbb{C}^d) \rightarrow \ell^2(\mathbb{Z}, \mathbb{C}^d)$ maps $\hat{\mathbf{Y}}$ to $\hat{\mathbf{U}}$, and it is none but the inverse of (2.17) (that exists for $\Re(p)$ large enough by a von Neumann series argument); technical details are given in Section 6. Thus,

$$\hat{\mathbf{Y}}(p) = \mathbf{H}(p) \hat{\mathbf{U}}(p), \quad \Re(p) > \gamma'. \quad (2.24)$$

Definition 3 (Harmonic Transfer Function). The operator-valued holomorphic function $\mathbf{H}(\cdot)$ is called the harmonic transfer function (HTF).

Equation (2.24) connects the inputs and outputs of a linear periodic control system in the frequency domain *via* the harmonic transfer function. For periodic systems with finite-dimensional state-space (that is: generated by finitely many periodic differential equations), a similarly described harmonic transfer function has become a classical tool in Engineering; see, [9, 10, 18]. Actually, in the Engineering literature, the harmonic transfer function is commonly defined as the (doubly infinite) matrix of $\mathbf{H}(p)$ in the Fourier basis rather than an operator-valued analytic function.

We can now recast Theorem 3 as a criterion for exponential stability of System (2.20) in terms of its harmonic transfer function as follows:

Corollary 5. *System (2.1) with periodic Hölder-smooth derivative coefficients is exponentially stable if and only if the control system (2.20) is exponentially stable, and this is if and only if the Harmonic Transfer Function \mathbf{H} of the latter is analytic and bounded in $\{p \in \mathbb{C} \mid \Re(p) > \beta\}$ for some $\beta < 0$.*

Altogether, we gather that Theorem 3 has the same interpretation in terms of HTF and exponential stability of periodic control systems than the Henry-Hale theorem has in terms of classical transfer functions and exponential stability of time-invariant systems. Actually, there is more to it than a mere rephrasing for we actually use the control-theoretic point of view and the mathematical properties of the HTF to establish the result.

There is another, more subtle way in which the HTF generalizes the ordinary transfer function: if a stable, time-invariant control system of the form (4.6) is fed with a periodic input signal $e^{i\tilde{\omega}t}v$ with $v \in \mathbb{C}^d$, then the output is asymptotically $H(i\tilde{\omega})v e^{i\tilde{\omega}t}$, with H the transfer function. For instance, when $d = 1$ it means that a periodic input is asymptotically mapped to a periodic output with the same period, amplitude magnified by $|H(i\tilde{\omega})|$ and phase shift $\arg H(i\tilde{\omega})$. Now, if a stable periodic control system of the form (4.6) is fed with a periodic input signal $v e^{i\tilde{\omega}t}$, then the output is asymptotically $\mathbf{H}(i\tilde{\omega})(\mathbf{v})e^{i\tilde{\omega}t}$, with \mathbf{H} the harmonic transfer function and \mathbf{v} the sequence in $\ell^2(\mathbb{Z}, \mathbb{C}^d)$ whose only non zero coefficient is v at index 0; for instance, when $d = 1$ it means that a periodic input is asymptotically mapped to a periodic output with the same period, carried by the periodic wave $\mathbf{H}(i\tilde{\omega})(1)$ which has the period of the system. This follows from (6.24), (6.11), (6.10) and (6.2).

3. Notation and function spaces

We let \mathbb{R} and \mathbb{C} be the real and complex fields, and \mathbb{R}^d (resp. \mathbb{C}^d) the Euclidean (resp. Hermitian) space of real (resp. complex) dimension d . When $x, y \in \mathbb{R}^d$ (resp. \mathbb{C}^d), we write $\langle x, y \rangle$ for their scalar (resp. Hermitian) product and $\|x\|$ for the norm of x . We often identify a linear operator $A : \mathbb{C}^d \rightarrow \mathbb{C}^m$ with its $m \times d$ matrix in the canonical bases. For A a linear map between Banach spaces, we let $\|A\|$ indicate its operator norm. The notation is the same for the real and complex case, but this causes no confusion.

For $E \subset \mathbb{R}^d$ (resp. \mathbb{C}^d) a Lebesgue-measurable set and $p \in [1, \infty]$, we write $L^p(E, \mathbb{R}^k)$ (resp. $L^p(E, \mathbb{C}^k)$) for the space of (equivalence classes of a.e. coinciding) \mathbb{R}^k (resp. \mathbb{C}^k)-valued measurable functions on E whose Euclidean (resp. Hermitian) norm to the p -th power is integrable, with norm $\|g\|_{L^p(E, \mathbb{R}^k)}$ (resp. $\|g\|_{L^p(E, \mathbb{C}^k)}$) given by $(\int_E \|g(y)\|^p dy)^{1/p}$ (ess. $\sup_E \|g\|$ if $p = \infty$). We set $L^p_{loc}(E, \mathbb{R}^k)$ (resp. $L^p_{loc}(E, \mathbb{C}^k)$) to consist of functions f whose restriction $f|_K$ lies in $L^p(K, \mathbb{R}^k)$ (resp. $L^p(K, \mathbb{C}^k)$) for every compact $K \subset E$. When $k = 1$, we simply write $L^p(E)$ or $L^p_{loc}(E)$ and the context makes it clear whether the functions are real or complex valued.

For $E \subset \mathbb{R}^d$ or \mathbb{C}^d , we let $C^0(E)$ be the space of continuous real or complex valued functions. Likewise, for \mathcal{B} a Banach space, we write $C^0(E, \mathcal{B})$ for continuous \mathcal{B} -valued maps on E . When E is compact, we endow $C^0(E)$ and $C^0(E, \mathcal{B})$ with sup norm. For $\alpha \in (0, 1)$ and I a real interval, we designate by $C^\alpha(I)$ the space of real or complex valued Hölder continuous functions with exponent α ; i.e., $f \in C^\alpha(I)$ if and only if $|f(x) - f(y)| \leq C\|x - y\|^\alpha$ for some constant C and all $x, y \in I$, the smallest C being the Hölder constant of f ; the space $C^\alpha(I, \mathcal{B})$ is defined analogously. When E is open, we let $C^1(E)$ (resp. $C^1(E, \mathcal{B})$) designate real or complex (resp. \mathcal{B} -valued) functions on E whose first derivatives lie in $C^0(E)$. The space $C^{1,\alpha}(E)$ is comprised of functions whose first derivatives belong to $C^\alpha(E)$, and similarly for $C^{1,\alpha}(E, \mathcal{B})$. The space of continuous periodic functions on \mathbb{R} with period $T > 0$ may be viewed as $C^0(\mathbb{T}_{T/2\pi})$; recall that \mathbb{T}_r indicates the circle of radius r .

For I a real interval, recall that a function $f : I \rightarrow \mathbb{R}$ is said to have bounded variation if

$$W_I(f) := \sup_{\substack{x_0 < x_1 < \dots < x_N \\ x_i \in I, N \in \mathbb{N}}} \sum_{i=1}^N |f(x_i) - f(x_{i-1})| < \infty. \quad (3.1)$$

In the degenerate cases where I is empty or reduces to a point, we set $W_I(f) = 0$. The number $W_I(f)$ is called the total variation of f on I , and we let $BV(I)$ indicate the space of functions of bounded variation on I endowed with the norm $\|f\|_{BV(I)} = W_I(f) + |f(d)|$, where d is arbitrary but fixed in I . Different d give rise to equivalent norms, for which $BV(I)$ is a Banach space. When I is bounded on the right and contains its endpoint b , we choose $d = b$ for definiteness. Note that functions in $BV(I)$ are bounded and $\|\cdot\|_{BV(I)}$ is stronger than the uniform norm, because evidently $|f(x)| \leq |f(d)| + W_I(f)$. Since a function of bounded variation is the difference of two nondecreasing functions, it has, at each $x \in I$, a limit $f(x^-)$ (resp. $f(x^+)$) from the left (resp. right) whenever the limit applies [19, sec. 1.4]. If $f, g \in BV(I)$, then clearly $fg \in BV(I)$ and

$$W_I(fg) \leq W_I(f) \sup_{x \in I} |g(x)| + W_I(g) \sup_{x \in I} |f(x)|. \quad (3.2)$$

To $f \in BV(I)$ one can associate a unique finite signed Borel measure ν_f on I such that, whenever $(a, b) \subset I$, we have $\nu_f((a, b)) = f(b^-) - f(a^+)$, while if I is bounded on the right (resp. left) and contains its endpoint b (resp. a), then $\nu_f(\{b\}) = f(b) - f(b^-)$ (resp. $f(a^+) - f(a)$) [19, ch. 7, pp. 185–189]. For $g : I \rightarrow \mathbb{R}$ a measurable function which is summable against ν_f , the Lebesgue-Stieltjes integral $\int gdf$ is defined to be $\int g d\nu_f$, whence the differential element df identifies with $d\nu_f$ [19, ch. 7, pp. 190–191]. Note that different f may generate the same ν_f : for example if f and f_1 coincide except at isolated interior points of I , then $\nu_f = \nu_{f_1}$.

A finite signed (or even complex) Borel measure μ on I also has a total variation, denoted as $\|\mu\|$: in fact, $\|\mu\|$ is the total mass of the so-called total variation measure of μ , written $|\mu|$, which is the positive measure defined on a Borel set $B \subset I$ by $|\mu|(B) = \sup_{\mathcal{P}} \sum_{E \in \mathcal{P}} |\mu(E)|$ where \mathcal{P} ranges over all partitions of B into Borel sets, see [20, sec. 6.1]. Thus, there are three distinct occurrences of the words “total variation”, which is traditional but unfortunate. We shall deal mainly with total variation of functions, but on occasions we need total variation of measures to derive estimates, as we now explain. Let $f \in BV(I)$, and ν_f indicate the associated measure on I described above. From (3.1) and the regularity of finite positive Borel measures [20, thm. 2.18], it follows that $W_I(f) = \|\nu_f\|$ when f is monotonic, but in general it only holds that $\|\nu_f\| \leq 2W_I(f)$; this follows from the Jordan decomposition of f as a difference of two increasing functions, each of which has variation at most $W_I(f)$ on I [19, Thm. 1.4.1]. In any case, we have the elementary inequality $|\int gdf| \leq \int |g|d\nu_f \leq 2W_I(f) \sup_I |g|$.

We let $BV_r(I) \subset BV(I)$ denote the subspace of functions that are right-continuous. Likewise, $BV_l(I) \subset BV(I)$ is the subspace of left-continuous functions. Because $\|\cdot\|_{BV(I)}$ is stronger than the uniform norm, $BV_r(I)$ and $BV_l(I)$ are Banach spaces in their own right.

Given $f \in BV(J)$, we often need to integrate a function g with respect to df over a subinterval $I \subset J$. The subinterval I is typically encoded by assigning bounds to the integral sign: a lower bound a^- (resp. a^+) means that I contains (resp. does not contain) its lower bound a , while an upper bound b^+ (resp. b^-) means that I contains (resp. does not contain) its upper bound b . Care must be used with this notion, for $\nu_{f|_I}$ needs not coincide with the restriction $\nu_f|_I$ of ν_f to I . More precisely, if the lower bound a (resp. the upper bound b) belongs to I and lies interior to J , then the two measures may differ by the weight they put on $\{a\}$ (resp. $\{b\}$): they agree only if f is left (resp. right) continuous at a (resp. b). In any case, the notation $\int_I gdf$ indicates that we integrate g against $\nu_{f|_I}$, but then the additivity of integrals needs attention. For example, in order to split an integral of the form $\int_{a^\pm}^{b^\pm} gdf$ as $\int_{a^\pm}^{c^\pm} gdf + \int_{c^\pm}^{b^\pm} gdf$ where $c \in (a, b)$, we must use c^+ (resp. c^-) if f is right (resp. left) continuous at c . We write $BV_{loc}(\mathbb{R})$ for the space of functions whose restriction to any bounded interval $I \subset \mathbb{R}$ lies in $BV(I)$.

The Hardy space \mathcal{H}^2 of the right half-plane is comprised of those holomorphic functions f in $\{z \in \mathbb{C} : \Re(z) > 0\}$ satisfying

$$\|f\|_{\mathcal{H}^2}^2 := \sup_{x>0} \int_{-\infty}^{+\infty} |f(x+iy)|^2 dy < +\infty.$$

By a Theorem of Paley and Wiener, such functions are exactly the Laplace transforms of square integrable functions on the positive real line: $f(z) = \int_0^{+\infty} e^{-zt} u(t) dt$ with $u \in L^2([0, +\infty), \mathbb{C})$, and it holds that $\|f\|_{\mathcal{H}^2}^2 = 2\|u\|_{L^2([0, +\infty), \mathbb{C})}^2$; see [21, Ch. 8, p. 131].

The above notation and definitions also apply to complex, vector and matrix-valued functions, replacing absolute values in (3.1) by moduli, Euclidean and operator norms, respectively. This of course means that the real and imaginary parts of the components lie in the corresponding spaces of real-valued functions.

4. Preliminary results and variation-of-constants formulas

We gather in this section some results of essential use to us, that we could not see proven in the literature in the form we need.

4.1. Representation formula for continuous solutions

It is traditional (see *e.g.* [1]) to define the so-called *fundamental solution* $X : \mathbb{R}^2 \rightarrow \mathbb{R}^{d \times d}$ of (2.1) through the following equation:

$$X(t, s) = \begin{cases} 0 & \text{for } t < s, \\ I_d + \sum_{j=1}^N D_j(t) X(t - \tau_j, s) & \text{for } t \geq s. \end{cases} \quad (4.1)$$

Arguing inductively, it is easy to check that X uniquely exists and is continuous at each (t, s) such that $t - s \notin \mathcal{F}$, where the set \mathcal{F} is defined as:

$$\mathcal{F} := \left\{ \sum_{\ell=1}^N n_\ell \tau_\ell, (n_1, \dots, n_N) \in \mathbb{N}^N \right\}. \quad (4.2)$$

Moreover, X has a bounded jump across each line $t - s = \mathfrak{f}$ as \mathfrak{f} ranges over \mathcal{F} ; more precisely, a moment's thinking will convince the reader that

$$X(t, s) = - \sum_{f \in [0, t-s] \cap \mathcal{F}} \mathfrak{C}_f(t), \quad \text{if } s \leq t, \quad (4.3)$$

where each $\mathfrak{C}_f(\cdot)$ lies in $C^{1,\delta}(\mathbb{R}, \mathbb{C}^{d \times d})$ when all the maps D_j are in $C^{1,\delta}(\mathbb{R}, \mathbb{C}^{d \times d})$. It is easy to see that each $\mathfrak{C}_f(\cdot)$ is a finite sum of products of maps $D_j(\cdot - f')$, where f' ranges over the members of \mathcal{F} whose defining numbers n_ℓ in (4.2) are no larger than those defining f , the empty product being the identity matrix; the precise expression of \mathfrak{C}_f can be derived by a reasoning resembling the one in [5, Sec. 3.2] or [6, Sec. 4.5], but we do not need it. Clearly, for fixed $t \in \mathbb{R}$, the function $X(t, \cdot)$ is locally of bounded variation on \mathbb{R} with respect to its second argument. The importance of the fundamental solution stems from the following integral representation formula, expressing the solution y of system (2.1) in terms of X and the initial data $\phi \in C_s$:

Proposition 6. *For any $s \in \mathbb{R}$ and any ϕ in C_s the unique solution $y \in C^0([s - \tau_N, +\infty), \mathbb{C}^{d \times d})$ to (2.1) is given by*

$$y(t) = - \sum_{j=1}^N \int_{s^-}^{(s+\tau_j)^-} d_\alpha X(t, \alpha) D_j(\alpha) \phi(\alpha - \tau_j - s), \quad t \geq s, \quad (4.4)$$

where X is the fundamental solution defined in (4.1). Moreover, X does not grow faster than exponential with respect to $t - s$; i.e., there exist $K > 0$ and $\lambda \in \mathbb{R}$ such that

$$\|X(t, s)\| \leq K e^{\lambda(t-s)}, \text{ for all } t \geq s. \quad (4.5)$$

PROOF. The proof is given in Appendix A. We give an extensive proof of (4.4), valid for BV -solutions, because we observe that the corresponding formulas for autonomous linear difference-delay systems which appear in [4] and [22] seem to have issues. \square

4.2. The variation-of-constants formula in L^2

Let us now switch from C^0 solutions to L^2 solutions. In the proof of necessity we need to handle square integrable initial data on the interval $[s - \tau_N, s]$, and locally square integrable solutions to (2.1). We also consider the control system obtained by forcing (2.1) with an additive input u :

$$y(t) = \sum_{j=1}^N D_j(t) y(t - \tau_j) + u(t), \quad u(t) \in \mathbb{C}^d, \quad t \geq s. \quad (4.6)$$

For $t \geq s$, we set $y_t(\theta) = y(t + \theta)$ for $\theta \in [-\tau_N, 0]$. The function $y_t : [-\tau_N, 0] \rightarrow \mathbb{C}^d$ (more accurately: its equivalence class modulo coincidence a.e. in our L^2 setting) will serve as a natural state for the input-output system (4.6) at time t . With this notation, y_s is the initial condition, previously denoted by ϕ in (2.1) when dealing with the homogeneous system (for which $u \equiv 0$).

Recalling from (4.1) the fundamental solution X , let us define for each $(t, \alpha) \in \mathbb{R}^2$ a map $K(t, \alpha) : [-\tau_N, 0] \rightarrow \mathbb{C}^{d \times d}$ by

$$K(t, \alpha)(\theta) = -X(t + \theta, \alpha), \quad \theta \in [-\tau_N, 0] \quad (4.7)$$

For $t \in \mathbb{R}$, $\theta \in [-\tau_N, 0]$ and $I \subset \mathbb{R}$ an interval bounded on the left, the map $\alpha \mapsto X(t + \theta, \alpha)$ lies in $BV_I(I)$ and is identically zero when $\alpha > t + \theta$. Thus, it is associated to a $\mathbb{C}^{d \times d}$ -valued Borel measure $\nu_{(X(t+\theta, \cdot))|_I}$ on I , as indicated in Section 3. Recall the discrete set $\mathcal{F} \subset \mathbb{R}^+$ introduced in (4.2). It follows from the definition of X (compare (4.1) and (A.4)) that for fixed t and θ , this measure is of the form

$$\nu_{(X(t+\theta, \cdot))|_I} = \sum_{f \in \mathcal{F}, t+\theta-f \in \tilde{I}} \mathfrak{C}_f(t + \theta) \delta_{t+\theta-f}, \quad (4.8)$$

where \tilde{I} denotes I deprived from its right endpoint (if contained in I), $\delta_{t+\theta-f}$ is a Dirac mass at $t + \theta - f$, and \mathfrak{C}_f were introduced in (4.3). For each fixed t , the boundedness of I from below implies that the number of terms in (4.8) is bounded independently of $\theta \in [-\tau_N, 0]$, and clearly the map sending a Borel set $E \subset I$ to the function $\theta \mapsto -\nu_{(X(t+\theta, \cdot))|_I}(E)$ is a vector measure, valued in the space $\mathcal{B}([-\tau_N, 0], \mathbb{C}^{d \times d})$ of bounded measurable $\mathbb{C}^{d \times d}$ -valued functions on $[-\tau_N, 0]$ endowed with the sup norm. In view of (4.7), we denote this measure by $d_\alpha K(t, \alpha)$ when I is understood. For $g \in C^0(I, \mathbb{C}^d)$ one can define the integral $\int_I dK(t, \alpha) g(\alpha)$ as a member of $\mathcal{B}([-\tau_N, 0], \mathbb{C}^d)$; see,

e.g. [23] for a definition of the integral against a vector measure. To us, it is just a compact notation for the function

$$\left(\int_I dK(t, \alpha) g(\alpha) \right) (\theta) := - \int_I d_\alpha X(t + \theta, \alpha) g(\alpha) = - \sum_{f \in \mathcal{F}, t + \theta - f \in \tilde{I}} \mathfrak{C}_f(t + \theta) g(t + \theta - f). \quad (4.9)$$

The rightmost term of (4.9) can be rewritten as $\sum_{f \in \mathcal{F}} \mathfrak{C}_f(t + \theta) \tilde{g}(t + \theta - f)$ where \tilde{g} is equal to g on \tilde{I} and to zero elsewhere, and since the \mathfrak{C}_f (being in $C^{1, \delta}(\mathbb{R}, \mathbb{C}^{d \times d})$) are locally bounded (in fact: bounded since they are periodic) while those $f \in \mathcal{F}$ for which $t + \theta - f \in \tilde{I}$ for some $\theta \in [-\tau_N, 0]$ are finite in number (because I is bounded on the left), one easily sees that $\|\int_I dK(t, \alpha) g(\alpha)\|_{L^2(I, \mathbb{C}^d)}$ is bounded above by $C\|g\|_{L^2(I, \mathbb{C}^d)}$ for some constant C depending on t and I , the D_j as well as the τ_j . Hence, (4.9) makes good sense for $g \in L^2(I, \mathbb{C}^d)$; of course, in this case $(\int_I dK(t, \alpha) g(\alpha))(\theta)$ is only defined for a.e. $\theta \in [-\tau_N, 0]$ in general. As for ordinary Lebesgue-Stieltjes integrals, the interval I will be encoded in the bounds we put on the integral. When writing $\int_I dK(t, \alpha) g(\alpha)$, it is irrelevant whether I contains its right endpoint or not, since the rightmost term in (4.9) depends only on \tilde{I} ; however, for easy quoting of formulas in the Appendix, we typically assume that I is closed on the right. Observe also that $K(t, \alpha) \equiv 0$ when $\alpha > t$, therefore $\int_I dK(t, \alpha) g(\alpha)$ does not depend on the values $g(\alpha)$ for $\alpha > t$. In particular, if $g \in L^2_{loc}([a, +\infty))$, then $\int_{a-}^{b\pm} dK(t, \alpha) g(\alpha)$ is independent of $b > t$ and of the choice of sign \pm . We find it convenient to choose $b = +\infty$ and record a convention that

$$\int_{a-}^{+\infty} dK(t, \alpha) g(\alpha) = \int_{a-}^{b\pm} dK(t, \alpha) g(\alpha) \quad \text{for any } b \text{ strictly larger than } t. \quad (4.10)$$

Proposition 7. *For $s \in \mathbb{R}$, let $y(\cdot) \in L^2_{loc}([s - \tau_N, +\infty), \mathbb{C}^d)$ and $u(\cdot) \in L^2_{loc}([s, +\infty), \mathbb{C}^d)$ satisfy (4.6) for almost all $t \geq s$. Then:*

$$y_t = U_2(t, s) y_s + \int_{s-}^{+\infty} d_\alpha K(t, \alpha) u(\alpha), \quad t \geq s. \quad (4.11)$$

PROOF. Denote by η_t the right hand side of (4.11), and set $\eta(t + \theta) = \eta_t(\theta)$ for $t \geq s$ and $\theta \in [-\tau_N, 0]$. Write \tilde{y} for the solution of the homogeneous system (2.1) with initial condition y_s , so that $\tilde{y}_t = U_2(t, s) y_s$. Now, fix $t \geq s$ and observe that for a.e. θ in $[-\tau_N, 0]$ satisfying $t + \theta < s$ (such θ occur when $t < s + \tau_N$), we have $(\int_{s-}^{+\infty} d_\alpha K(t, \alpha) u(\alpha))(\theta) = \int_{s-}^{+\infty} d_\alpha X(t + \theta, \alpha) u(\alpha) = 0$ (because $X(t + \theta, \alpha) = 0$ when $t + \theta < s \leq \alpha$) and also that $\tilde{y}_t(\theta) = \tilde{y}(t + \theta) = y_s(t + \theta) = y(t + \theta)$ (because \tilde{y} and y have the same initial condition y_s on $[-\tau_N, 0]$). Hence, we get indeed that $\eta_t(\theta) = y_t(\theta)$ for a.e. θ in $[-\tau_N, 0]$ such that $t + \theta < s$.

Next, for a.e. $\theta \in [-\tau_N, 0]$ such that $t + \theta \geq s$, we deduce from the definition of η_t , in view of (2.6) (with $q = 2$), (2.1) (where y is set to \tilde{y}), (4.10) and (A.4), that

$$\begin{aligned} \eta_t(\theta) &= \tilde{y}(t + \theta) - \int_{s-}^{+\infty} d_\alpha X(t + \theta, \alpha) u(\alpha) \\ &= \sum_{j=1}^N D_i(t + \theta) \tilde{y}(t + \theta - \tau_j) - \sum_{j=1}^N D_i(t + \theta) \int_{s-}^{+\infty} d_\alpha X(t + \theta - \tau_j, \alpha) u(\alpha) + u(t + \theta) \\ &= \sum_{j=1}^N D_j(t + \theta) \eta(t + \theta - \tau_j) + u(t + \theta). \end{aligned}$$

Altogether, we proved that to each $t \geq 0$ there is a set $E_t \subset [-\tau_N, 0]$ of zero measure such that, for $\theta \in [-\tau_N, 0] \setminus E_t$, we have $\eta(t + \theta) = \sum_{j=1}^N D_j(t + \theta)\eta(t + \theta - \tau_j) + u(t + \theta)$ when $t + \theta \geq s$ and $\eta(t + \theta) = y_s(t + \theta)$ when $t + \theta < s$. Thus, letting \mathbb{Q} denote the rational numbers, the set $E := \cup_{q \in \mathbb{Q}} \{q + E_q\}$ has measure zero and since each $t \in \mathbb{R}$ can be written as $q + \theta$ with $q \in \mathbb{Q}$ and $\theta \in [-\tau_N, 0]$, we deduce that $\eta(t)$ satisfies (4.6) when $t \geq s$ and $t \notin E$, with initial condition $\eta = y_s$ a.e. on $[-\tau_N, 0]$. In other words, η must be equal to y a.e., as desired. \square

For fixed $t \in \mathbb{R}$, let us now consider the function $s \mapsto X(t, t - s)$ from \mathbb{R} to $\mathbb{C}^{d \times d}$. It is right-continuous and lies in $BV_{loc}(\mathbb{R}, \mathbb{C}^{d \times d})$, moreover it is identically zero for $s < 0$. Thus, on each interval $I \subset \mathbb{R}$ which is bounded on the right, $X(t, t - \cdot)$ defines a $\mathbb{C}^{d \times d}$ -valued measure $\nu_{(X(t, t - \cdot))|_I}$ which is but the image of the measure $d_s X(t, s)$ under the map $s \mapsto t - s$ from $\{t\} - I$ onto I . When I is understood, we denote this measure by $d_s X(t, t - s)$ and we see from the discussion before (4.8) that

$$d_s X(t, t - s) = - \sum_{f \in \mathcal{F} \cap \tilde{I}} \mathfrak{C}_f(t) \delta_f, \quad (4.12)$$

where \tilde{I} denotes I deprived from its left endpoint (if contained in I) and the \mathfrak{C}_f are as in (4.8). By periodicity of $D_j(t)$, one sees that $X(t, t - s)$ is periodic in t , and therefore the $\mathfrak{C}_f(t)$ are likewise periodic. Moreover, if the $D_j(t)$ belong to $C^{1, \delta}(\mathbb{R})$ then the $\mathfrak{C}_f(t)$ also lie in $C^{1, \delta}(\mathbb{R})$, and we can further define the measure

$$\frac{\partial}{\partial t} d_s X(t, t - s) = - \sum_{f \in \mathcal{F} \cap \tilde{I}} \left(\frac{\partial}{\partial t} \mathfrak{C}_f(t) \right) \delta_f \quad (4.13)$$

whose coefficients $\left(\frac{\partial}{\partial t} \mathfrak{C}_f(t) \right)$ belong to $C^\delta(\mathbb{R}, \mathbb{C}^{d \times d})$.

For $I = [0, \tau]$, the number of terms in the right hand side of (4.12) tends to $+\infty$ with τ , and the terms themselves involve more and more sums of products of the $D_j(t - f')$ for $f' \leq f$.

4.3. Other estimates

We shall need an *a priori* estimate, independent of t , for the quantities:

$$\|d_s X(t, t - s)\|_I := \sum_{f \in \mathcal{F} \cap \tilde{I}} \|\mathfrak{C}_f(t)\|, \quad (4.14)$$

$$\left\| \frac{\partial}{\partial t} d_s X(t, t - s) \right\|_I := \sum_{f \in \mathcal{F} \cap \tilde{I}} \left\| \frac{\partial}{\partial t} \mathfrak{C}_f(t) \right\|, \quad (4.15)$$

$$\Lambda_{I, \delta} \left(\frac{\partial}{\partial t} d_s X(t, t - s) \right) := \sum_{f \in \mathcal{F} \cap \tilde{I}} \Lambda_\delta \left(\frac{\partial}{\partial t} \mathfrak{C}_f(t) \right), \quad (4.16)$$

where $\Lambda_\delta(g)$ indicates the Hölder constant of $g \in C^\delta(\mathbb{R}, \mathbb{C}^{d \times d})$. Note that (4.14) and (4.15) are but the total variation of the measures $d_s X(t, t - s)$ and $\frac{\partial}{\partial t} d_s X(t, t - s)$ on I , respectively.

Proposition 8. *Assume that the $D_j : \mathbb{R} \rightarrow \mathbb{C}^{d \times d}$ are periodic and differentiable with Hölder continuous derivative for $1 \leq j \leq N$; i.e. $D_j \in C^{1, \delta}(\mathbb{R}, \mathbb{C}^{d \times d})$ for some $\delta \in (0, 1)$. Then, there exist $K, \gamma \geq 0$ such that, for all $\tau \geq 0$ and $t \in \mathbb{R}$:*

$$\max \left\{ \|d_s X(t, t - s)\|_{[0, \tau]}, \left\| \frac{\partial}{\partial t} d_s X(t, t - s) \right\|_{[0, \tau]}, \Lambda_{[0, \tau], \delta} \left(\frac{\partial}{\partial t} d_s X(t, t - s) \right) \right\} \leq K e^{\gamma \tau}. \quad (4.17)$$

PROOF. Observe that $d_s X(t, t-s)$ is the image of the measure $d_s X(t, s)$ under the map $s \mapsto t-s$ from $[t-\tau, t]$ onto $[0, \tau]$. Hence, it follows from (A.4) that

$$\begin{aligned} d_s X(t, t-s) &= \sum_{j=1}^N D_j(t) d_s X(t-\tau_j, t-s) \\ &= \sum_{j=1}^N D_j(t) d_s X(t-\tau_j, t-\tau_j-(s-\tau_j)) \quad \text{on } [0, \tau], \end{aligned}$$

as well as

$$\begin{aligned} \frac{\partial}{\partial t} d_s X(t, t-s) &= \sum_{j=1}^N \frac{\partial}{\partial t} D_j(t) d_s X(t-\tau_j, t-s) + \sum_{j=1}^N D_j(t) \frac{\partial}{\partial t} d_s X(t-\tau_j, t-s) \\ &= \sum_{j=1}^N \frac{\partial}{\partial t} D_j(t) d_s X(t-\tau_j, t-\tau_j-(s-\tau_j)) + \sum_{j=1}^N D_j(t) \frac{\partial}{\partial t} d_s X(t-\tau_j, t-\tau_j-(s-\tau_j)). \end{aligned}$$

Since $X(t-\tau_j, t-\tau_j-(s-\tau_j))$ is identically zero for $s < \tau_j$, we deduce from the previous identities that if K' is an upper bound (made independent of t by periodicity) for the $\|D_j\|$, the $\|\frac{\partial}{\partial t} D_j\|$, the $\Lambda_\delta(D_j)$ and the $\Lambda_\delta(\frac{\partial}{\partial t} D_j)$, then

$$\|d_s X(t, t-s)\|_{[0, \tau]} \leq K' \sum_{j=1}^N \|d_\beta X(t-\tau_j, t-\tau_j-\beta)\|_{[0, \tau-\tau_j]}, \quad (4.18)$$

$$\|\frac{\partial}{\partial t} d_s X(t, t-s)\|_{[0, \tau]} \leq K' \sum_{j=1}^N \|d_\beta X(t-\tau_j, t-\tau_j-\beta)\|_{[0, \tau-\tau_j]} + \|\frac{\partial}{\partial t} d_\beta X(t-\tau_j, t-\tau_j-\beta)\|_{[0, \tau-\tau_j]}, \quad (4.19)$$

$$\begin{aligned} \Lambda_{[0, \tau], \delta}(\frac{\partial}{\partial t} d_s X(t, t-s)) &\leq K' \sum_{j=1}^N \Lambda_{[0, \tau-\tau_j], \delta}(X(t-\tau_j, t-\tau_j-\beta)) + \|d_\beta X(t-\tau_j, t-\tau_j-\beta)\|_{[0, \tau-\tau_j]} \\ &\quad + \Lambda_{[0, \tau-\tau_j], \delta}(\frac{\partial}{\partial t} X(t-\tau_j, t-\tau_j-\beta)) + \|\frac{\partial}{\partial t} d_\beta X(t-\tau_j, t-\tau_j-\beta)\|_{[0, \tau-\tau_j]}, \quad (4.20) \end{aligned}$$

where we used that $\Lambda_\delta(AB) \leq \Lambda_\delta(A)\|B\| + \Lambda_\delta(B)\|A\|$. The proof may now be completed by an inductive step, similar to the one used to establish (4.5). More precisely, pick γ and K'' positive and large enough that $K'Ne^{-\gamma\tau_1} < 1/4$ as well as

$$\max\{\|\mathfrak{C}_{\tau_1}(t)\|, \left\|\frac{\partial}{\partial t} \mathfrak{C}_{\tau_1}(t)\right\|, \Lambda_\delta(\mathfrak{C}_{\tau_1}), \Lambda_\delta(\frac{\partial}{\partial t} \mathfrak{C}_{\tau_1})\} \leq K'', \quad t \in \mathbb{R}. \quad (4.21)$$

Note that K'' exists as \mathfrak{C}_{τ_1} is periodic and lies in $C^{1, \delta}(\mathbb{R})$. From (4.12) one sees that $\|d_s X(t, t-s)\|_{[0, \tau]}$ is equal to 0 when $\tau \in [0, \tau_1)$ and to $\|\mathfrak{C}_{\tau_1}(t)\|$ when $\tau = \tau_1$. *A fortiori* then, for $\tau \in [0, \tau_1]$, we have:

$$\max\left\{\|d_s X(t, t-s)\|_{[0, \tau]}, \left\|\frac{\partial}{\partial t} d_s X(t, t-s)\right\|_{[0, \tau]}\right\} \leq K''e^{\gamma\tau}, \quad \Lambda_{[0, \tau], \delta}(\frac{\partial}{\partial t} d_s X(t, t-s)) \leq 2K'K''e^{\gamma\tau},$$

so that (4.17) holds with $K := \max\{K'', 2K'K''\}$ for $0 \leq \tau \leq \tau_1$. Now, assume that (4.17) is true for all t and $\tau \in [0, k\tau_1)$, with $k > 0$ an integer. For $\tau \in (k\tau_1, (k+1)\tau_1]$, we get in view of this hypothesis and the definition of γ , that

$$\|d_s X(t, t-s)\|_{[0, \tau]} \leq K'NKe^{\gamma(\tau-\tau_1)} \leq Ke^{\gamma\tau}/4 \quad \text{by (4.18),}$$

$$\left\| \frac{\partial}{\partial t} d_s X(t, t-s) \right\|_{[0, \tau]} \leq 2K'NK e^{\gamma(\tau-\tau_1)} \leq K e^{\gamma\tau}/2 \quad \text{by (4.19),}$$

$$\Lambda_{[0, \tau], \delta} \left(\frac{\partial}{\partial t} d_s X(t, t-s) \right) \leq 4K'NK e^{\gamma(\tau-\tau_1)} \leq K e^{\gamma\tau} \quad \text{by (4.20).}$$

By induction on k , this completes the proof. \square

5. Proof of the sufficiency part of Theorem 3

In essence, the proof of sufficiency is framed after the one in [4] for the time-invariant case, but new technicalities arise because, with time-varying coefficients $D_i(t)$, Laplace transformation yields infinite-dimensional functional equations.

Let us assume points (i) and (ii) in Theorem 3. We shall prove exponential convergence in three steps.

Step 1

Since the D_j are assumed to have Hölder continuous derivatives, a classical estimate on their Fourier coefficients (2.11) is

$$\| \check{D}_j(k) \| \leq \frac{C}{1 + |k|^{1+\delta}}, \quad j \in \{1, \dots, n\}, \quad (5.1)$$

where C is a positive constant and $\delta \in]0, 1[$ is the Hölder exponent of the derivative; indeed, (5.1) follows at once from the fact that the modulus of the k -th Fourier coefficient of $\frac{d}{dt} D_j$ is bounded by $C'/(1+n)^\delta$, see [24, Ch. 2, thm. 4.7]. We note, for later use, that the constant C in (5.1) depends linearly on the Hölder constant of $\frac{d}{dt} D_j$. Substituting the Fourier expansion of D_j in the integral explicit expression (4.1) yields:

$$X(t, s) = I_d + \sum_{k \in \mathbb{Z}} \sum_{j=1}^N e^{ik\omega t} \check{D}_j(k) X(t - \tau_j, s) \quad \text{if } t \geq s, \quad X(t, s) = 0 \text{ if } t < s, \quad (5.2)$$

where the right hand side of (5.2) is absolutely convergent, locally uniformly in (t, s) , thanks to (5.1) and the local boundedness of X . Hereafter, we shall denote with a hat the Laplace transform with respect to the first variable; *e.g.*, we put

$$\hat{X}(p, s) := \int_{-\infty}^{+\infty} e^{-pt} X(t, s) dt. \quad (5.3)$$

Taking the Laplace transform of both sides of (5.2) and interchanging the series and integral signs (this is possible thanks to (5.1)), we obtain with λ as in (4.5) that for $p \in \mathbb{C}$ with $\Re(p) > \lambda$:

$$\int_{-\infty}^{+\infty} e^{-pt} X(t, s) dt = \int_{-\infty}^{+\infty} \mathbf{1}_{[s, +\infty)}(t) e^{-pt} I_d dt + \sum_{k \in \mathbb{Z}} \sum_{j=1}^N \int_{-\infty}^{+\infty} e^{(-p+ik\omega)t} \check{D}_j(k) X(t - \tau_j, s) dt, \quad (5.4)$$

where $\mathbf{1}_{[s, +\infty)}$ is the characteristic function equal to 1 on $[s, +\infty)$ and to 0 elsewhere; note that the factor $X(t - \tau_j, s)$ in the integrand on the right hand side of (5.4) is *a fortiori* zero for $t < s$. By an elementary change of variable, we deduce from (5.4) that

$$\hat{X}(p, s) = \frac{e^{-ps}}{p} I_d + \sum_{k \in \mathbb{Z}} \sum_{j=1}^N e^{(-p+ik\omega)\tau_j} \check{D}_j(k) \hat{X}(p - ik\omega, s), \quad \Re(p) > \lambda. \quad (5.5)$$

This can be rephrased as

$$\sum_{k \in \mathbb{Z}} (R(p))_{0,k} \hat{X}(p - ik\omega, s) = \frac{e^{-ps}}{p} I_d \quad (5.6)$$

with $R(p)$ defined in (2.17) and (2.18). Writing the above with p substituted by $p + i\omega$ and letting n range over \mathbb{Z} , we obtain a system of countably many equations, that may be written as

$$R(p) \hat{\mathbb{X}}(p, s) = \hat{e}(p, s), \quad (5.7)$$

with $\hat{\mathbb{X}}(p, s)$ and $\hat{e}(p, s)$ the infinite $d \times d$ block vectors:

$$\hat{\mathbb{X}}(p, s) := \begin{pmatrix} \vdots \\ \hat{X}(p + i\omega, s) \\ \hat{X}(p, s) \\ \hat{X}(p - i\omega, s) \\ \vdots \end{pmatrix} \quad \text{and} \quad \hat{e}(p, s) := \begin{pmatrix} \vdots \\ \frac{e^{-(p+i\omega)s}}{p+i\omega} I_d \\ \frac{e^{-ps}}{p} I_d \\ \frac{e^{-(p-i\omega)s}}{p-i\omega} I_d \\ \vdots \end{pmatrix}. \quad (5.8)$$

Clearly, these are the Laplace transforms of infinite $d \times d$ block vectors $\mathbb{X}(t, s)$ and $e(t, s)$:

$$\hat{\mathbb{X}}(p, s) = \int_{-\infty}^{+\infty} e^{-pt} \mathbb{X}(t, s) dt, \quad \hat{e}(p, s) = \int_{-\infty}^{+\infty} e^{-pt} e(t, s) dt, \quad (5.9)$$

$$\text{with } \mathbb{X}(t, s) := \begin{pmatrix} \vdots \\ e^{-\frac{2i\pi t}{T}} X(t, s) \\ X(t, s) \\ e^{\frac{2i\pi t}{T}} X(t, s) \\ \vdots \end{pmatrix}, \quad e(t, s) := \begin{pmatrix} \vdots \\ I_d e^{-\frac{2i\pi t}{T}} \mathbf{1}_{[s, +\infty)} \\ I_d \mathbf{1}_{[s, +\infty)} \\ I_d e^{\frac{2i\pi t}{T}} \mathbf{1}_{[s, +\infty)} \\ \vdots \end{pmatrix}. \quad (5.10)$$

The infinite dimensional linear constant system (5.7) recasts the finite-dimensional periodic time-varying delay system (4.1) through Fourier series and Laplace transforms. In order to estimate $\mathbb{X}(p, s)$ -and eventually $X(t, s)$ - we are going to invert the operator $R(p)$ in the next step.

Step 2

For $A := (a_{i,j})_{i,j \in \mathbb{Z}}$ a doubly infinite block matrix, where $a_{i,j}$ is a $d \times d$ complex matrix for each i and j in \mathbb{Z} , we say that A has off diagonal decay of order r if there is a constant C such that $\|a_{i,j}\| \leq C(1 + |i - j|)^{-r}$. We also define the Wiener norm of A to be

$$\|A\|_{\mathfrak{W}} := \sum_{k \in \mathbb{Z}} \sup_{|i-j|=k} \|a_{i,j}\|. \quad (5.11)$$

Note that $\|A\|_2 \leq \|A\|_{\mathfrak{W}}$, as follows immediately from the Schur test [25]. We denote by $B(\ell^2(\mathbb{Z}, \mathbb{C}^d), \ell^2(\mathbb{Z}, \mathbb{C}^d))$ the space of those doubly infinite block matrices A such that $\|A\|_2 < \infty$, and by $\mathfrak{W}(\ell^2(\mathbb{Z}, \mathbb{C}^d), \ell^2(\mathbb{Z}, \mathbb{C}^d))$ the subspace of those A satisfying $\|A\|_{\mathfrak{W}} < \infty$. It is easy to check that $\mathfrak{W}(\ell^2(\mathbb{Z}, \mathbb{C}^d), \ell^2(\mathbb{Z}, \mathbb{C}^d))$ is a subalgebra of the algebra $B(\ell^2(\mathbb{Z}, \mathbb{C}^d), \ell^2(\mathbb{Z}, \mathbb{C}^d))$.

One can see from (5.1) that $R(p)$ possesses off-diagonal decay of order $1 + \delta$, moreover the constant is uniform over any half-space $\Re(p) \geq a$. From this, it follows at once that $\|R(p)\|_{\mathfrak{W}}$ is uniformly bounded over such a half-space. Hence, $R(p) \in \mathfrak{W}(\ell^2(\mathbb{Z}, \mathbb{C}^d), \ell^2(\mathbb{Z}, \mathbb{C}^d))$ for all p and clearly, $R : \mathbb{C} \rightarrow B(\ell^2(\mathbb{Z}, \mathbb{C}^d), \ell^2(\mathbb{Z}, \mathbb{C}^d))$ is a Banach valued holomorphic function.

Now, assumption (i) of Theorem 3 tells us that $R(p)$ is invertible in $B(\ell^2(\mathbb{Z}, \mathbb{C}^d), \ell^2(\mathbb{Z}, \mathbb{C}^d))$ for $\Re(p) \geq \beta$, and assumption (ii) that the inverse operator $R^{-1}(p)$ has uniformly bounded $\|\cdot\|_2$ -norm. Therefore, as $R(p)$ has off-diagonal decay of order $1 + \delta$, we get from [26, thm. 1.2] and the boundedness of $\|R^{-1}(p)\|_2$ that $R^{-1}(p)$ also has off-diagonal decay of order $1 + \delta$, uniformly over $\Re(p) \geq \beta$. In particular, $\|R^{-1}(p)\|_{\mathfrak{W}}$ is uniformly bounded³ for $\Re(p) \geq \beta$:

$$\left\| \left[I_\infty - \sum_{i=j}^N e^{-p\tau_j} L_{D_j} \Delta_{\tau_j, \omega} \right]^{-1} \right\|_{\mathfrak{W}} \leq C_1, \quad p \in \{z \in \mathbb{C} | \Re(z) \geq \beta\}. \quad (5.12)$$

In view of (5.12), applying to (2.17) the result of [27, thm. 2.6] which is a generalization of the Wiener lemma to Banach algebra-valued almost periodic functions, the Banach algebra here being $\mathfrak{W}(\ell^2(\mathbb{Z}, \mathbb{C}^d), \ell^2(\mathbb{Z}, \mathbb{C}^d))$ (see also [28] for a more general version of that theorem), we get for any α with $\beta < \alpha < 0$ that $R^{-1}(p)$ admits a Fourier expansion on the vertical line $p = \alpha + i\mathbb{R}$ of the form :

$$R^{-1}(\alpha + i\tilde{\omega}) = \sum_{k \in \mathbb{Z}} \tilde{R}^{\{k, \alpha\}} e^{i\beta_k \tilde{\omega}}, \quad \tilde{\omega} \in \mathbb{R}, \quad (5.13)$$

where β_k is real and $\tilde{R}^{\{k, \alpha\}} \in \mathfrak{W}(\ell^2(\mathbb{Z}, \mathbb{C}^d), \ell^2(\mathbb{Z}, \mathbb{C}^d))$ with

$$\sum_{k \in \mathbb{Z}} \|\tilde{R}^{\{k, \alpha\}}\|_{\mathfrak{W}} < +\infty. \quad (5.14)$$

Moreover, following the proof of the theorem in section 3 on Dirichlet's series page 147 of [17], which is possible because Cauchy's theorem holds for Banach valued holomorphic functions (one can apply all the arguments there in weak form by evaluating on a fixed vector in $\ell^2(\mathbb{Z}, \mathbb{C}^d)$), we deduce that $\tilde{R}^{\{k, \alpha\}} = R^{\{k\}} e^{\alpha\beta_k}$, where $R^{\{k\}} \in B(\ell^2(\mathbb{Z}, \mathbb{C}^d), \ell^2(\mathbb{Z}, \mathbb{C}^d))$ is independent of α . Thus, one can rewrite (5.13) and (5.14) as

$$R^{-1}(\alpha + i\tilde{\omega}) = \sum_{k \in \mathbb{Z}} R^{\{k\}} e^{\alpha\beta_k} e^{\beta_k \tilde{\omega}}, \quad \tilde{\omega} \in \mathbb{R}, \quad \alpha > \beta, \quad (5.15)$$

with

$$\sum_{k \in \mathbb{Z}} \|R^{\{k\}}\|_{\mathfrak{W}} e^{\alpha\beta_k} < +\infty. \quad (5.16)$$

In addition, observing from the von Neumann series that

$$R^{-1}(p) = \sum_{n=0}^{+\infty} \left(\sum_{j=1}^N L_{D_j} \Delta_{\tau_j, \omega} e^{-p\tau_j} \right)^n \quad (5.17)$$

whence

$$\|R^{-1}(p)\|_2 \leq \frac{1}{1 - e^{\Re(\tau_1)} \sum_{j=1}^N \|L_{D_j}\|_2 \|\Delta_{\tau_j, \omega}\|_2} \quad \text{for } \Re(p) > \gamma_1 := \frac{\log(\sum_{j=1}^N \|L_{D_j}\|_2 \|\Delta_{\tau_j, \omega}\|_2)}{\tau_1}, \quad (5.18)$$

³Controlled inversion (bounding the norm of the inverse in a subalgebra in terms of the norm in this subalgebra and the norm of the inverse in the algebra) was pioneered in Baskakov [?] and Nikolski [?].

we deduce that

$$\{\beta_k | k \in \mathbb{Z}\} \subset \left\{ \sum_{j=1}^N n_j \tau_j \mid n_j \text{ non-positive integers for } j = 1, \dots, N \right\} = -\mathcal{F} \quad (5.19)$$

where \mathcal{F} was defined in (4.2). We remark that (5.15), (5.16) and (5.19) are reminiscent of [13, p. 429, equations (12.15.12) and (12.15.13)], that deals with complex-valued functions.

Step 3

We now compute $\mathbb{X}(t, s)$. For this, we apply the Laplace inversion formula (see for example [1, Ch. 1, Lem. 5.2]). It gives us for $\Re(p) = c > \lambda$, with λ as in (4.5), that

$$\mathbb{X}(t, s) = \lim_{\tilde{\omega} \rightarrow +\infty} \frac{1}{2\pi i} \int_{c-i\tilde{\omega}}^{c+i\tilde{\omega}} R^{-1}(p) \hat{e}(p, s) e^{pt} dp. \quad (5.20)$$

We shall need two lemmas, the proof of which is postponed till the end of this section.

Lemma 9. *Let $\beta < \alpha < 0$ and $\rho_m := \frac{2\pi m}{T} + \frac{\pi}{T}$ for m a positive integer. Then, we have that*

$$\mathbb{X}(t, s) = \lim_{m \rightarrow +\infty} \frac{1}{2\pi i} \int_{\alpha-i\rho_m}^{\alpha+i\rho_m} R^{-1}(p) \hat{e}(p, s) e^{pt} dp + Q(t), \quad (5.21)$$

where

$$Q(t) := \dots + R_{-1}^{-1}(-i\omega) e^{-i\omega t} + R_0^{-1}(0) + R_1^{-1}(i\omega) e^{i\omega t} + \dots$$

with R_k^{-1} to mean the k -th column of R^{-1} .

Recall that $X(t, s)$ is the element of index 0 of the infinite column vector $\mathbb{X}(t, s)$. So, if for each $k \in \mathbb{Z}$ we let $(R_{0,n}^{\{k\}})_{n \in \mathbb{Z}}$ enumerate the line of index 0 of the matrix $R^{\{k\}}$ defined in (5.15), it is a consequence of (5.21) that

$$X(t, s) = \frac{1}{2\pi i} \lim_{m \rightarrow +\infty} \int_{\alpha-i\rho_m}^{\alpha+i\rho_m} \sum_{k,n \in \mathbb{Z}} R_{0,n}^{\{k\}} \frac{e^{-(p-in\omega)s}}{p-in\omega} e^{p(t+\beta_k)} dp + Q(t)_0, \quad (5.22)$$

where $Q(t)_0$ stands for the element of index 0 of the column vector $Q(t)$; note that the convergence of the series in (5.22) ensues from (5.16).

Lemma 10. *For all t and s such that $t + \beta_k - s \neq 0$ for all $k \in \mathbb{Z}$, we have that*

$$X(t, s) = \sum_{(t+\beta_k-s) < 0, n \in \mathbb{Z}} R_{0,n}^{\{k\}} e^{in\omega(t+\beta_k)} + Q(t)_0. \quad (5.23)$$

Note that on a bounded interval there are only finitely many β_k , because the integer coefficients in (5.19) are non-positive. Observe also from (4.1) that $X(t, \cdot)$ is left continuous. Hence, Lemma 10 allows in fact to compute $X(t, s)$ for all (t, s) (the proof of the lemma will be given momentarily). Hereafter, we fix the symbol s to mean the initial time, and we shall evaluate the variation of $X(t, \tau)$ in its second argument when the latter ranges over $[s, s + \tau_N]$. To this effect, we substitute τ for s in (5.23) and observe, since $Q(t)_0$ is independent of τ while the sum in (5.23) is piecewise

constant in τ with jumps at the $t + \beta_k$, caused by the constraint on the indices, where it is left continuous, that

$$W_{[s, s+\tau_N]} X(t, \cdot) \leq \sum_{0 \leq t + \beta_k - s \leq \tau_N, n \in \mathbb{Z}} \left\| R_{0,n}^{\{k\}} \right\| \quad (5.24)$$

$$\leq \left(\sum_{k, n \in \mathbb{Z}} \left\| R_{0,n}^{\{k\}} \right\| e^{\alpha \beta_k} e^{|\alpha| \tau_N} \right) e^{\alpha(t-s)}, \quad (5.25)$$

where we used in the second inequality that $\alpha(t + \beta_k - s - \tau_N) \geq 0$ because $\alpha < 0$. Thus, with K' an upper bound for the $\|D_i(t)\|$ as in (A.1), we deduce from (4.4) and (5.25) that

$$\|y(t)\| \leq K' N (W_{[s, s+\tau_N]} X(t, \cdot)) \|\phi\|_{C^0} \quad (5.26)$$

$$\leq K' N \left(\sum_{k, n \in \mathbb{Z}} \left\| R_{0,n}^{\{k\}} \right\| e^{\alpha \beta_k} e^{|\alpha| \tau_N} \right) e^{\alpha(t-s)} \|\phi\|_{C^0}, \quad (5.27)$$

thereby showing that System (2.1) is C^0 -exponentially stable. This achieves the proof of the sufficiency part of Theorem 3.

Proof of the two lemmas

PROOF OF LEMMA 9. Considering the subsequence $\rho_m = \frac{2\pi m}{T} + \frac{\pi}{T}$, we deduce from the equation (5.20) that :

$$\mathbb{X}(t, s) = \lim_{m \rightarrow +\infty} \frac{1}{2\pi i} \int_{c-i\rho_m}^{c+i\rho_m} R^{-1}(p) \hat{e}(p, s) e^{pt} dp. \quad (5.28)$$

Note that each component of $R^{-1}(p) \hat{e}(p, s)$ is a meromorphic function in the half plane $\{p \in \mathbb{C} | \Re(p) > \beta\}$, with poles in the set $\{p \in \mathbb{C} | p = \frac{ik\pi}{T}, k \in \mathbb{Z}\}$. Thus, by (5.28) and the residue theorem, we get that

$$\mathbb{X}(t, s) = \lim_{m \rightarrow +\infty} \frac{1}{2\pi i} \left(\int_{\alpha-i\rho_m}^{\alpha+i\rho_m} + \int_{\alpha+i\rho_m}^{c+i\rho_m} - \int_{\alpha-i\rho_m}^{c-i\rho_m} \right) R^{-1}(p) \hat{e}(p, s) e^{pt} dp + Q(t) \quad (5.29)$$

where we recall that :

$$Q(t) := \dots + R_{-1}^{-1}(-i\omega) e^{-i\omega t} + R_0^{-1}(0) + R_1^{-1}(i\omega) e^{i\omega t} + \dots$$

with R_k^{-1} the k th column of R^{-1} . Observe that

$$\|Q(t)\|_2 \leq \left\| R^{-1}(0) \right\|_{\mathfrak{W}} < +\infty, \quad (5.30)$$

which warrants taking the limit in (5.29). To establish the lemma, it remains to show that

$$\lim_{m \rightarrow +\infty} \int_{\alpha+i\rho_m}^{c+i\rho_m} R^{-1}(p) \hat{e}(p, s) e^{pt} dp = \lim_{m \rightarrow +\infty} \int_{\alpha-i\rho_m}^{c-i\rho_m} R^{-1}(p) \hat{e}(p, s) e^{pt} dp = 0. \quad (5.31)$$

For this, let us write the entry with index j of the vector $\int_{\alpha+i\rho_m}^{c+i\rho_m} R^{-1}(p) \hat{e}(p, s) e^{pt} dp$ as

$$\int_{\alpha+i\rho_m}^{c+i\rho_m} \sum_{n \in \mathbb{Z}} R_{j,n}^{-1}(p) \frac{e^{-(p-in\omega)s}}{p - in\omega} e^{pt} dp, \quad (5.32)$$

where $R_{j,n}^{-1}(p)$ denotes the entry with index (j, n) of $R^{-1}(p)$. As pointed out in step 2, $R^{-1}(p)$ has off-diagonal decay of order $1 + \delta$, uniformly over $\Re(p) \geq \beta$. Thus, if we pick $\varepsilon > 0$, there is $\mathbf{n} = \mathbf{n}(j)$ such that $\sum_{|n| \geq \mathbf{n}} \|R_{j,n}^{-1}(p)\| < \varepsilon$, uniformly with respect to $\Re(p) \in [\alpha, c]$, and since

$$|\rho_m - n\omega| \geq \frac{\pi}{T} \text{ for all } n \text{ and } m \quad (5.33)$$

we have that

$$\left\| \int_{\alpha + i\rho_m}^{c + i\rho_m} \sum_{|n| \geq \mathbf{n}} R_{j,n}^{-1}(p) \frac{e^{-(p - in\omega)s}}{p - in\omega} e^{pt} dp \right\| \leq \varepsilon e^{c(t-s)} \frac{(c - \alpha)T}{\pi} \quad (5.34)$$

which is arbitrary small with ε . In another connection, for fixed j and $n \in \mathbb{Z}$, we get since $|p - in\omega|$ becomes arbitrary large with m while the other terms in the integrand are uniformly bounded with respect to $\Re(p) \in [\alpha, c]$ that

$$\lim_{m \rightarrow \infty} \int_{\alpha + i\rho_m}^{c + i\rho_m} R_{j,n}^{-1}(p) \frac{e^{-(p - in\omega)s}}{p - in\omega} e^{pt} dp = 0. \quad (5.35)$$

In view of (5.34) and (5.35), we conclude that

$$\lim_{m \rightarrow +\infty} \int_{\alpha + i\rho_m}^{c + i\rho_m} R^{-1}(p) \hat{e}(p, s) e^{pt} dp = 0, \quad (5.36)$$

and in the same way one can prove that

$$\lim_{m \rightarrow +\infty} \int_{\alpha - i\rho_m}^{c - i\rho_m} R^{-1}(p) \hat{e}(p, s) e^{pt} dp = 0, \quad (5.37)$$

as wanted. \square

PROOF OF LEMMA 10. Let t and s be such that $t + \beta_k - s \neq 0$. By (5.33) and the absolute convergence in (5.16) that allow us to integrate termwise, we get by performing the change of variable $p \rightarrow p - in\omega$ in the integrals corresponding to index n that

$$\int_{\alpha - i\rho_m}^{\alpha + i\rho_m} \sum_{k, n \in \mathbb{Z}} R_{0,n}^{\{k\}} \frac{e^{-(p - in\omega)s}}{p - in\omega} e^{p(t + \beta_k)} dp = \sum_{k, n \in \mathbb{Z}} R_{0,n}^{\{k\}} e^{in\omega(t + \beta_k)} \int_{\alpha - in\omega - i\rho_m}^{\alpha - in\omega + i\rho_m} \frac{e^{p(t + \beta_k - s)}}{p} dp. \quad (5.38)$$

Integrating by parts, we get that

$$\int_{\alpha - in\omega - i\rho_m}^{\alpha - in\omega + i\rho_m} \frac{e^{p(t + \beta_k - s)}}{p} dp = \left[\frac{e^{p(t + \beta_k - s)}}{p(t + \beta_k - s)} \right]_{\alpha - in\omega - i\rho_m}^{\alpha - in\omega + i\rho_m} + \int_{\alpha - in\omega - i\rho_m}^{\alpha - in\omega + i\rho_m} \frac{e^{p(t + \beta_k - s)}}{(t + \beta_k - s)p^2} dp, \quad (5.39)$$

and by (5.19) there is $\delta > 0$ such that $|t + \beta_k - s| \geq \delta$ for all $k \in \mathbb{Z}$. Hence, using (5.33), we obtain:

$$\begin{aligned} \left| \left[\frac{e^{p(t + \beta_k - s)}}{p(t + \beta_k - s)} \right]_{\alpha - in\omega - i\rho_m}^{\alpha - in\omega + i\rho_m} \right| &\leq \frac{1}{|t + \beta_k - s|} \left(\frac{e^{\alpha(t + \beta_k - s)}}{\sqrt{\alpha^2 + (\rho_m - in\omega)^2}} + \frac{e^{\alpha(t + \beta_k - s)}}{\sqrt{\alpha^2 + (\rho_m + in\omega)^2}} \right) \\ &\leq \frac{2e^{\alpha(t + \beta_k - s)}}{\delta \sqrt{\alpha^2 + \left(\frac{\pi}{T}\right)^2}} \end{aligned} \quad (5.40)$$

and

$$\begin{aligned}
\left| \int_{\alpha-in\omega-i\rho_m}^{\alpha-in\omega+i\rho_m} \frac{e^{p(t+\beta_k-s)}}{(t+\beta_k-s)p^2} dp \right| &\leq \frac{e^{\alpha(t+\beta_k-s)}}{|t+\beta_k-s|} \int_{-\rho_m-n\omega}^{\rho_m-n\omega} \frac{1}{\alpha^2+p_2^2} dp_2 \\
&\leq \frac{e^{\alpha(t+\beta_k-s)}}{|t+\beta_k-s|} \left[\frac{1}{\alpha} \arctan\left(\frac{p_2}{\alpha}\right) \right]_{-\rho_m-n\omega}^{\rho_m-n\omega} \\
&\leq \frac{\pi}{|\alpha|\delta} e^{\alpha(t+\beta_k-s)}.
\end{aligned} \tag{5.41}$$

Altogether, we deduce from (5.39), (5.40) and (5.41) that there exists a constant $K > 0$, independent of k, n and ρ_m , such that :

$$\left| \int_{\alpha-in\omega-i\rho_m}^{\alpha-in\omega+i\rho_m} \frac{e^{p(t+\beta_k-s)}}{p} dp \right| \leq K e^{\alpha(t+\beta_k-s)}. \tag{5.42}$$

Consequently, it holds that

$$\left\| R_{0,n}^{\{k\}} e^{in\omega(t+\beta_k)} \int_{\alpha-in\omega-i\rho_m}^{\alpha-in\omega+i\rho_m} \frac{e^{p(t+\beta_k-s)}}{p} dp \right\| \leq K e^{\alpha(t-s)} \left\| R_{0,n}^{\{k\}} \right\| e^{\alpha\beta_k} dp, \tag{5.43}$$

and since the right hand side of (5.43) is summable over k, n because of (5.16)), the dominated convergence theorem allows us to take the limit termwise in the right hand side of (5.38) as $m \rightarrow \infty$:

$$\begin{aligned}
&\lim_{m \rightarrow +\infty} \frac{1}{2i\pi} \int_{\alpha-i\rho_m}^{\alpha+i\rho_m} \sum_{k,n \in \mathbb{Z}} R_{0,n}^{\{k\}} \frac{e^{-(p-in\omega)s}}{p-in\omega} e^{p(t+\beta_k)} dp \\
&= \sum_{k,n \in \mathbb{Z}} \frac{1}{2i\pi} R_{0,n}^{\{k\}} e^{in\omega(t+\beta_k)} \lim_{m \rightarrow +\infty} \int_{\alpha-in\omega-i\rho_m}^{\alpha-in\omega+i\rho_m} \frac{e^{p(t+\beta_k-s)}}{p} dp.
\end{aligned} \tag{5.44}$$

Finally, the residue theorem gives us

$$\sum_{k,n \in \mathbb{Z}} \frac{1}{2i\pi} R_{0,n}^{\{k\}} e^{in\omega(t+\beta_k)} \lim_{m \rightarrow +\infty} \int_{\alpha-i\rho_m}^{\alpha+i\rho_m} \frac{e^{p(t+\beta_k-s)}}{p} dp = \sum_{\alpha(t+\beta_k-s) > 0, n \in \mathbb{Z}} R_{0,n}^{\{k\}} e^{in\omega(t+\beta_k)}. \tag{5.45}$$

Putting together (5.22), (5.44) and (5.45), we achieve the proof of Lemma 10. \square

6. Proof of the necessity part of Theorem 3

In the time-invariant case, *i.e.* when proving the classical Henry-Hale theorem, stated in this paper as Theorem 2, necessity is proven using the spectral theory of semigroups [4]. We cannot follow this path here, with time-varying periodic coefficients, because there seems to be no adequate notion of semigroup to fit the situation.

We now assume exponential stability, and aim at proving conditions (i) and (ii) of Theorem 3. We chose to assume specifically L^2 exponential stability, anyway equivalent to C^0 stability or L^q stability, $q \neq 2$ according to Proposition 1). We start with the following classical property.

Proposition 11. *System (2.1) is L^2 -exponentially stable if and only if the spectral radius of the monodromy operator $U_2(T, 0)$ is strictly less than one.*

Here $U_2(.,.)$ refers to the solution operator (2.6). Proposition 11 — or even an extension for any periodic evolution family of operators — is a consequence of [29, lem. 4.2]. Strictly speaking, this reference yields Proposition 11 where the equation (2.8) is just true for $q = 2$ and $t \geq s \geq 0$ only, but since the constants merely depend on the $D_j(t)$ through $\sup\{\|D_i(t)\|, 1 \leq i \leq N, t \in \mathbb{R}\}$, the case of arbitrary s follows by periodicity. Choosing zero as initial time for the monodromy operator is quite arbitrary, and we may as well have termed $U_2(T+s_0, s_0)$ the monodromy operator, for some $s_0 \neq 0$. Although different choices of s_0 lead to different monodromy operators, the relation of their spectrum to exponential stability is the same, and we stick to the choice $s_0 = 0$ for definiteness.

Thanks to this proposition, we now specifically assume that the spectral radius of $U_2(T, 0)$ is less than 1, and resort to a classical construction from Control Theory allowing us to realize the periodic input-output system (4.6) as a discrete-time system in state space form (the state is here infinite dimensional), see [10]. More precisely, consider (4.6) with initial data 0 at time 0, and let $u \in L^2([0, +\infty), \mathbb{C}^d)$ be an input with $y(t) \in \mathbb{C}^d$ the corresponding output, given by

$$y(t) = \sum_{j=1}^N D_j(t)y(t - \tau_j) + u(t), \quad \text{a.e. } t \geq 0, \quad y(t) = 0 \text{ for } t \leq 0. \quad (6.1)$$

Using (4.11) with $s = 0$, $y_s = 0$, and performing the change of variables $\beta = t - \alpha$ imply at once

$$y(t) = \int_{-\infty}^{t^+} d_\beta X(t, t - \beta)u(t - \beta), \quad \text{a.e. } t \geq 0, \quad (6.2)$$

where, since $X(t, t - \beta)$ is identically zero when $\beta < 0$, this integral can also be written as $\int_{c^+}^{t^+} d_\beta X(t, t - \beta)u(t - \beta)$ for any $c < 0$. From (6.2) and (4.12), it follows that

$$y(t) = \sum_{f \in \mathcal{F} \cap [0, t]} \mathfrak{C}_f(t)u(t - f), \quad \text{a.e. } t \geq 0. \quad (6.3)$$

Applying to (6.3) the Schwarz inequality and using (4.14), (4.17) together with the fact that $\mathfrak{C}_0 = -I_d$ (as follows readily from (4.8) and (4.1)), we get that

$$\begin{aligned} |y(t)|^2 &\leq \left(\sum_{f \in \mathcal{F} \cap [0, t]} \|\mathfrak{C}_f(t)\|^2 \right) \left(\sum_{f \in \mathcal{F} \cap [0, t]} \|u(t - f)\|^2 \right) \\ &\leq \left(\sum_{f \in \mathcal{F} \cap [0, t]} \|\mathfrak{C}_f(t)\| \right)^2 \left(\sum_{f \in \mathcal{F} \cap [0, t]} \|u(t - f)\|^2 \right) \\ &\leq (1 + Ke^{\gamma t})^2 \left(\sum_{f \in \mathcal{F} \cap [0, t]} \|u(t - f)\|^2 \right), \quad \text{a.e. } t \geq 0. \end{aligned} \quad (6.4)$$

In another connection, the cardinality of $\mathcal{F} \cap [0, t]$ is no larger than the number of N -tuples $(q_1, \dots, q_N) \in \mathbb{N}^N$ satisfying $\sum_{i=1}^N q_i \leq t/\tau_1$ (recall τ_1 is the smallest delay), and *a fortiori*:

$$\text{Card} \{ \mathcal{F} \cap [0, t] \} \leq \left(1 + \frac{t}{\tau_1} \right)^N, \quad t \in [0, +\infty), \quad (6.5)$$

where $\text{Card}\{E\}$ stands for the cardinality of E . Integrating (6.4) from 0 to $\tau > 0$ and taking (6.5) into account, we deduce that

$$\|y\|_{L^2([0, \tau])} \leq \tau^{1/2} (1 + Ke^{\gamma \tau}) \left(1 + \frac{\tau}{\tau_1} \right)^{N/2} \|u\|_{L^2([0, +\infty))}, \quad (6.6)$$

implying that to each $\gamma_1 > \gamma$ there is $K_1 > 0$ for which $\|y\|_{L^2([0,\tau])} \leq K_1 e^{\gamma_1 \tau} \|u\|_{L^2([0,+\infty])}$. Since $\|u\|_{L^2([0,+\infty])} < +\infty$, this warrants the existence, for $\Re(p) > \gamma$, of the Laplace transforms of $y(t)$ and $u(t)$:

$$\hat{Y}(p) := \int_0^{+\infty} e^{-pt} y(t) dt, \quad \hat{U}(p) := \int_0^{+\infty} e^{-pt} u(t) dt, \quad (6.7)$$

for if $\Re(p) > \gamma_1 > \gamma$ then $\int_k^{k+1} e^{-pt} |u|(t) dt$ and $\int_k^{k+1} e^{-pt} |y|(t) dt$ decay exponentially fast with $k \in \mathbb{N}$, by the Schwarz inequality. Proceeding on (6.1) like we did on (4.1) to obtain (5.7), namely expanding the $D_j(t)$ into Fourier series and taking Laplace transforms termwise using (5.1), next replacing the complex variable p by $p + in\omega$ and letting n range over \mathbb{Z} , we deduce that

$$R(p) \hat{Y}(p) = \hat{U}(p), \quad \Re(p) > \gamma, \quad (6.8)$$

where $\hat{Y}(p)$ and $\hat{U}(p)$ are given by (2.23) and $R(p)$ by (2.17). Moreover, we see from (5.18) that $R(p)$ is continuously invertible for $\Re(p) > \gamma_1$, so we obtain with $\gamma_2 := \max\{\gamma, \gamma_1\}$ that

$$\hat{Y}(p) = R^{-1}(p) \hat{U}(p), \quad \Re(p) > \gamma_2. \quad (6.9)$$

Equation (6.9) is exactly the equation (2.24) and shows that the operator-valued holomorphic function $p \mapsto R^{-1}(p)$ is the Harmonic Transfer Function (Definition 3) of System (6.1).

Next, we define the instantaneous transfer function:

$$G(t, p) = \int_{-\infty}^{+\infty} -d_\tau X(t, t - \tau) e^{-p\tau}, \quad t \in \mathbb{R}, \quad p \in \mathbb{C}, \quad \Re(p) > \gamma, \quad (6.10)$$

where the integral is understood as $-\lim_{b \rightarrow +\infty} \int_{c^\pm}^{b^\pm} d_\tau X(t, t - \tau) e^{-p\tau}$ for any $c < 0$. The existence of the limit is guaranteed by Proposition 8, and the fact that it does not depend on c provided that it is strictly negative is argued similarly to the independence of (4.10) from the exact value of b . Observe from Proposition 8 that $t \mapsto G(t, p)$ lies in $C^{1+\delta}(\mathbb{R}, \mathbb{C}^{d \times d})$, and obviously it is T -periodic. Thus, we may expand this function in Fourier series as

$$G(t, p) = \sum_{k \in \mathbb{Z}} G_k(p) e^{i\omega kt}, \quad \Re(p) > \gamma, \quad (6.11)$$

where we recall that $\omega := 2\pi/T$, and the series converges absolutely for fixed p by a standard estimate used already in (5.1); more precisely, [24, Ch. 2, thm. 4.7] implies that

$$\|G_k(p)\| \leq \frac{\Re(\Re(p))}{1 + |k|^{1+\delta}}, \quad (6.12)$$

where $\Re(\Re(p)) > 0$ depends on $\Re(p)$, and also on K, γ in (4.17) though we do not show the latter dependence. Now, for $f \in \mathcal{F}$ and $k \in \mathbb{Z}$, let $c_{k,f}$ be the k -th Fourier coefficient of the T -periodic function $t \mapsto \mathfrak{C}_f(t)$. Since $\mathfrak{C}_f \in C^{1,\delta}(\mathbb{R})$, we have the estimate

$$\|c_{k,f}\| \leq \frac{K_f}{1 + |k|^{1+\delta}} \quad (6.13)$$

so that $\mathfrak{C}_f(t) = \sum_{k \in \mathbb{Z}} c_{k,f} e^{i\omega kt}$, where the series is uniformly absolutely convergent. Summing up over those $f \in \mathcal{F}$ such that $f \leq \tau$, we deduce from (4.12) that for any $\varepsilon > 0$:

$$d_s X(t, t - s) = \sum_{k \in \mathbb{Z}} d\mu_{k,\tau} e^{i\omega kt} \quad \text{on } [-\varepsilon, \tau], \quad (6.14)$$

where the measure $\mu_{k,\tau}$ is equal to $-\sum_{f \in \mathcal{F} \cap [0,\tau]} c_{k,f} \delta_f$ and absolute convergence with respect to the total variation holds in (6.14), because $\mathcal{F} \cap [0,\tau]$ is a finite set. If $\tau < 0$, then obviously $\mu_{k,\tau} = 0$. Moreover, as the constant K_f in (6.13) is majorized by a linear function of the Hölder coefficient of $\frac{d}{dt} \mathfrak{C}_f$, we get from (4.17) that

$$\|\mu_{k,\tau}\| = \sum_{f \in \mathcal{F} \cap [0,\tau]} \|c_{k,f}\| \leq \frac{K_1 e^{\gamma\tau}}{1 + |k|^{1+\delta}}, \quad \tau \geq 0, \quad (6.15)$$

for some constant K_1 depending only on the D_j and the τ_j . Let us now define

$$\mu_k := \sum_{f \in \mathcal{F}} c_{k,f} \delta_f. \quad (6.16)$$

Clearly, μ_k is a distribution on \mathbb{R} valued in $\mathbb{C}^{d \times d}$ and supported on $[0, +\infty)$, but it is not a measure. However, its restriction to every interval bounded from above is a finite discrete measure. Hence, we can integrate against μ_k any \mathbb{R}^d -valued bounded function which is zero for large arguments, and more generally any function that decays in norm as fast as $e^{-\gamma' t}$ for some $\gamma' > \gamma$, by (6.15). Furthermore, when this function is supported on an interval of the form $[a, \tau]$ with $a < 0$ and $\tau < +\infty$, the integral against μ_k coincides with the integral against $\mu_{k,\tau}$. Using this remark, we can write for $\Re(p) > \gamma$:

$$\begin{aligned} G(t, p) &= - \lim_{b \rightarrow +\infty} \int_{-\infty}^{b^+} d_\tau X(t, t - \tau) e^{-p\tau} = - \lim_{b \rightarrow +\infty} \sum_{k \in \mathbb{Z}} e^{i\omega k t} \int_{-\infty}^{b^+} d\mu_{k,b}(\tau) e^{-p\tau} \\ &= - \sum_{k \in \mathbb{Z}} e^{i\omega k t} \lim_{b \rightarrow +\infty} \int_{-\infty}^{b^+} d\mu_k(\tau) e^{-p\tau} = - \sum_{k \in \mathbb{Z}} e^{i\omega k t} \int_{-\infty}^{+\infty} d\mu_k(\tau) e^{-p\tau}, \end{aligned}$$

where the second equality uses the absolute convergence in (6.14) and the third uses (6.15) and the fact that $\Re(p) > \gamma$. Considering (6.11), the previous chain of equalities yields that

$$G_k(p) = - \int_{0^-}^{+\infty} d\mu_k(\tau) e^{-p\tau}. \quad (6.17)$$

Next, assuming for a while that u is locally bounded, (6.2) and (6.14) imply

$$y(t) = \int_{-\infty}^{t^+} \sum_{k \in \mathbb{Z}} d\mu_{k,t}(\alpha) e^{i\omega k t} u(t - \alpha);$$

swapping the sum and integral sign thanks to (6.15) and using the above mentioned relation between $\mu_{k,t}$ and μ_k then yields $y(t) = \sum_{k \in \mathbb{Z}} \int_{-\infty}^{t^+} d\mu_k(\alpha) e^{i\omega k t} u(t - \alpha)$, and finally, since $u(t - \alpha) = 0$ when $\alpha > t$ and μ_k is supported on $[0, \infty)$,

$$y(t) = \sum_{k \in \mathbb{Z}} A_k(t) \quad \text{with} \quad A_k(t) := \int_{0^-}^{+\infty} d\mu_k(\alpha) e^{i\omega k \alpha} u(t - \alpha) e^{i\omega k (t - \alpha)}. \quad (6.18)$$

Since for $\gamma' > \gamma$ we have that

$$\begin{aligned} \int_{0^-}^{+\infty} d|\mu_k|(\alpha) \int_0^{+\infty} |u(t - \alpha)| e^{-\gamma' t} dt &= \int_{0^-}^{+\infty} d|\mu_k|(\alpha) e^{-\gamma' \alpha} \int_0^{+\infty} |u(t - \alpha)| e^{-\gamma' (t - \alpha)} dt \\ &= \left(\int_{0^-}^{+\infty} d|\mu_k|(\alpha) e^{-\gamma' \alpha} \right) \left(\int_0^{+\infty} |u(\tau)| e^{-\gamma' \tau} d\tau \right) \leq \left(\frac{e^\gamma K_1}{1 + |k|^{1+\delta}} \sum_{j=0}^{+\infty} e^{(\gamma - \gamma')j} \right) \frac{\|u\|_{L^2(\mathbb{R})}}{\sqrt{2\gamma'}} < +\infty, \end{aligned} \quad (6.19)$$

where the first inequality uses (6.15) and the Schwarz inequality, we can use Fubini's theorem to compute the Laplace transform of $A_k(t)$ in the strip $\Re(p) > \gamma$. This gives us

$$\begin{aligned} \int_0^{+\infty} A_k(t) e^{-pt} dt &= \int_{0^-}^{+\infty} d\mu_k(\alpha) e^{(i\omega k - p)\alpha} \int_0^{+\infty} e^{(i\omega k - p)(t - \alpha)} u(t - \alpha) dt \\ &= \left(\int_{0^-}^{+\infty} d\mu_k(\alpha) e^{(i\omega k - p)\alpha} \right) \left(\int_0^{+\infty} e^{(i\omega k - p)\tau} u(\tau) d\tau \right) \\ &= G_k(p - i\omega k) \hat{U}(p - i\omega k) \end{aligned} \quad (6.20)$$

where the last equality uses (6.17). Summing over k , we find in view of (6.18) that

$$\hat{Y}(p) = \sum_{k \in \mathbb{Z}} G_k(p - i\omega k) \hat{U}(p - i\omega k), \quad \Re(p) > \gamma, \quad (6.21)$$

where the absolute convergence of the series follows from the estimates obtained in (6.19); *i.e.*,

$$|G_k(z)| \leq \frac{e^\gamma K_1 K_2 (\Re(z) - \gamma)}{1 + |k|^{1+\delta}} \quad \text{and} \quad |\hat{U}(z)| \leq \frac{\|u\|_{L^2(\mathbb{R})}}{\sqrt{2\Re(z)}}, \quad \Re(z) > \gamma,$$

with $K_2(x) := \sum_{j=0}^{+\infty} e^{-xj}$ for $x > 0$. Changing p into $p + i\omega$ in (6.21) and renumbering, we get

$$\hat{Y}(p + i\omega) = \sum_{m \in \mathbb{Z}} G_{n-m}(p + i\omega m) \hat{U}(p + i\omega m), \quad n \in \mathbb{Z}, \quad \Re(p) > \gamma. \quad (6.22)$$

Combining (6.9) and (6.22), we see that

$$\sum_{m \in \mathbb{Z}} (G_{n-m}(p + i\omega m) - R_{n,-m}^{-1}(p)) \hat{U}(p + i\omega m) = 0, \quad n \in \mathbb{Z}, \quad \Re(p) > \gamma, \quad (6.23)$$

where $R_{i,j}^{-1}(p)$ indicates, as in (5.32), the entry with index (i, j) of $R^{-1}(p)$. For each p with $\Re(p) > 0$, the sequence $\{z_m := p + i\omega m, m \in \mathbb{N}\}$ is hyperbolically separated in the right half-plane; *i.e.*,

$$\left| \frac{z_m - z_j}{z_m + \bar{z}_j} \right| \geq c > 0, \quad j, m \in \mathbb{N}, \quad j \neq m.$$

Indeed, we have that

$$\left| \frac{z_m - z_j}{z_m + \bar{z}_j} \right| = \left(\frac{4(\Re(p))^2}{\omega^2 |m - j|^2} + 1 \right)^{-1/2} \geq \left(\frac{4(\Re(p))^2}{\omega^2} + 1 \right)^{-1/2}.$$

Hence, as is well-known (see [30, Ch. VII, Thm 1.1] for an equivalent on the upper half-plane), $(z_m)_{m \in \mathbb{N}}$ is an interpolating sequence, meaning that to each bounded sequence a_m in \mathbb{C} there is a bounded analytic function F in the right half-plane with $F(z_m) = a_m$. In particular, to each $j \in \mathbb{N}$ and $v \in \mathbb{C}^d$, there is a \mathbb{C}^d -valued bounded analytic function $F_{j,v}$ such that $F_{j,v}(z_m) = 0$ for $m \neq j$ and $F_{j,v}(z_j) = v$. Multiplying $F_{j,v}$ by a function without zeros in the Hardy space \mathcal{H}^2 (for instance $p \mapsto (p+1)^{-1}$), we get a function $G_{j,v} \in (\mathcal{H}^2)^d$ which is zero at z_m when $m \neq j$ and a nonzero multiple of v at z_j . Now, by the Paley-Wiener theorem, there is $u \in L^2([0, \infty), \mathbb{C}^d)$ such that $\hat{U} = G_{j,v}$, and using this collection of u in (6.23) as j ranges over \mathbb{N} and v over \mathbb{C}^d , we conclude that

$$R_{n,-m}^{-1}(p) = G_{n-m}(p + i\omega m), \quad n, m \in \mathbb{N}, \quad \Re(p) > \gamma. \quad (6.24)$$

Next, in order to link the monodromy operator to $R^{-1}(p)$, we shall discretize the continuous dynamical system (6.1) in state space form. For this, we restrict the input and the output to intervals of length T by setting:

$$\left. \begin{aligned} \tilde{u}_k(t) &:= u(kT + t) \\ \tilde{y}_k(t) &:= y(kT + t) \end{aligned} \right\} \text{ for a.e. } t \in [0, T] \text{ and } k \in \mathbb{N}, \quad (6.25)$$

thereby giving rise to sequences \tilde{u}_k and \tilde{y}_k of functions in $L^2([0, T])$. We introduce another sequence of functions \tilde{z}_k , this time in $L^2([-\tau_N, 0])$, by putting $\tilde{z}_k := y_{kT}$, where the notation y_{kT} was defined after (4.6); i.e., $\tilde{z}_k(\theta) := y(kT + \theta)$ for $k \in \mathbb{N}$ and a.e. $\theta \in [-\tau_N, 0]$. The function \tilde{z}_k will be the “state variable” of our discrete system at time k , initialized with $\tilde{z}_0 = y_0 = 0$. Recall also the monodromy operator $U_2(T, 0)$ from (2.8) (with $q = 2$) and the integration with respect to $K(t, \alpha)$ from (4.9). They allow us to define four operators as follows:

- $\tilde{A} : L^2([-\tau_N, 0], \mathbb{C}^d) \longrightarrow L^2([-\tau_N, 0], \mathbb{C}^d)$
 $v \longmapsto U_2(T, 0)v,$
- $\tilde{B} : L^2([0, T], \mathbb{R}^d) \longrightarrow L^2([-\tau_N, 0], \mathbb{C}^d)$
 $w \longmapsto \int_{0-}^{+\infty} d_\alpha K(T, \alpha)w(\alpha),$
- $\tilde{C} : L^2([-\tau_N, 0], \mathbb{C}^d) \longrightarrow L^2([0, T], \mathbb{C}^d)$
 $v \longmapsto \{(U_2(t, 0)v)(0), t \in [0, T]\},$
- $\tilde{D} : L^2([0, T], \mathbb{C}^d) \longrightarrow L^2([0, T], \mathbb{C}^d)$
 $w \longmapsto \left(-\int_0^{+\infty} d_\alpha X(\cdot, \alpha)w(\alpha)\right)\Big|_{[0, T]}.$

Observe that \tilde{B} and \tilde{D} do not depend on the values of w outside $[0, T]$, and that $(U_2(t, 0)v)(0)$ exists for a.e. $t \in \mathbb{R}$, hence the operators \tilde{A} , \tilde{B} , \tilde{C} and \tilde{D} are well-defined and continuous.

Theorem 12. *Let $u \in L^2([0, +\infty), \mathbb{C}^d)$ and $y \in L_{loc}^2([0, +\infty), \mathbb{C}^d)$ be, respectively, the input and output of System (6.1). For \tilde{u}_k , \tilde{y}_k and \tilde{z}_k as in (6.25), the following recursion holds:*

$$\begin{cases} \tilde{z}_{k+1} &= \tilde{A}\tilde{z}_k + \tilde{B}\tilde{u}_k, \\ \tilde{y}_k &= \tilde{C}\tilde{z}_k + \tilde{D}\tilde{u}_k, \end{cases} \quad \tilde{z}_0 = 0, \quad k \in \mathbb{N}. \quad (6.26)$$

PROOF. Applying the variation-of-constants formula (4.11) to System (6.1) with s, t replaced with $kT, kT + t$ yields

$$y_{kT+t} = U_2(kT + t, kT)y_{kT} + \int_{(kT)-}^{+\infty} d_\alpha K(kT + t, \alpha)u(\alpha),$$

i.e., performing the change of variable $\alpha = kT + \beta$ and using T -periodicity,

$$y_{kT+t} = U_2(T, 0)y_{kT} + \int_{0-}^{+\infty} d_\beta K(T, \beta)\tilde{u}_k(\beta). \quad (6.27)$$

This, with $t = T$, is the first equation in (6.26). Next, evaluating (6.27) at 0 (this is possible for a.e. t) while using the definition of K (see (4.9)) yields:

$$y(t + kT) = (U_2(t, 0)\tilde{z}_k)(0) - \int_{0^-}^{+\infty} d_\beta X(t, \beta)\tilde{u}_k(\beta), \quad \text{a.e. } t \in [0, T],$$

which is the second equation in (6.26). \square

For $a := (a_n)_{n \in \mathbb{N}}$ a sequence in a Banach space \mathfrak{X} , we define its z -transform as the formal series

$$\mathcal{L}\{a\}(z) := \sum_{n \in \mathbb{N}} a_n z^{-n}. \quad (6.28)$$

We denote by $\mathfrak{X}[[z^{-1}]]$ the space of such power series. Let \mathfrak{Y} a Banach space. For $O := (O_n)_{n \in \mathbb{N}}$ a sequence of operators from \mathfrak{X} to \mathfrak{Y} , the z -transform $\mathcal{L}\{O\}(z)$ acts naturally from $\mathfrak{X}[[z^{-1}]]$ to $\mathfrak{Y}[[z^{-1}]]$ since the number of terms involved in the coefficient of each power of z^{-1} is finite. If we let now $\bar{y} := (\tilde{y}_n)_{n \in \mathbb{N}}$, $\bar{u} := (\tilde{u}_n)_{n \in \mathbb{N}}$ and $\bar{z} := (\tilde{z}_n)_{n \in \mathbb{N}}$, we get from (6.26) that

$$(zI_d - \tilde{A})\mathcal{L}\{\bar{z}\} = \tilde{B}\mathcal{L}\{\bar{u}\} \quad \text{and} \quad \mathcal{L}\{\bar{y}\} = \tilde{C}\mathcal{L}\{\bar{z}\} + \tilde{D}\mathcal{L}\{\bar{u}\}.$$

Note, since $\tilde{z}_0 = 0$, that $(zI_d - \tilde{A})\mathcal{L}\{\bar{z}\}$ indeed belongs to $L^2([-\tau_N, 0], \mathbb{C}^d)[[z^{-1}]]$. Observing that $(zI - \tilde{A})^{-1} = \sum_{n \in \mathbb{N}} \tilde{A}^n z^{-n-1}$ in the ring of formal Laurent series with operator coefficients on $L^2([-\tau_N, 0], \mathbb{C}^d)$, we may write

$$\mathcal{L}\{\bar{y}\}(z) = \left[\tilde{C}(zI_d - \tilde{A})^{-1}\tilde{B} + \tilde{D} \right] \mathcal{L}\{\bar{u}\}(z). \quad (6.29)$$

Although (6.29) was derived formally, it becomes a valid relation between vector and operator valued analytic functions when $|z| > \max\{\|\tilde{A}\|, e^\gamma\}$, with γ as in Proposition 8, because all series involved are then normally convergent (compare (6.6)). In another connection, decomposing the first integral of (6.2) on subintervals taking into account that $X(\tau, \cdot)$ is left continuous, and using T -periodicity, we get that

$$\begin{aligned} \tilde{y}_n(t) &= y(t + nT) = \sum_{k=0}^{n-1} \int_{(kT)^-}^{((k+1)T)^-} d_\alpha X(t + nT, \alpha) u(\alpha) + \int_{(nT)^-}^{+\infty} d_\alpha X(t + nT, \alpha) u(\alpha) \\ &= \sum_{k=0}^{n-1} \int_{0^-}^{T^-} d_\beta X(t + nT, kT + \beta) u(kT + \beta) + \int_{(nT)^-}^{((n+1)T)^+} d_\alpha X(t + nT, \alpha) u(\alpha) \\ &= \sum_{k=0}^{n-1} \int_{0^-}^{T^+} d_\beta X(t + nT, kT + \beta) u(kT + \beta) + \int_{0^-}^{T^+} d_\beta X(t + nT, nT + \beta) u(nT + \beta) \\ &= \sum_{k=0}^{n-1} \int_{0^-}^{T^+} d_\beta X(t + (n - k)T, \beta) \tilde{u}_k(\beta) + \int_{0^-}^{T^+} d_\beta X(t, \beta) \tilde{u}_n(\beta) \\ &= \sum_{k=0}^n H_{[n-k]} \tilde{u}_k(t), \quad \text{a.e. } t \in [0, T] \end{aligned} \quad (6.30)$$

where, for $v \in L^2([0, T], \mathbb{C}^d)$, we have set:

$$H_{[k]}v(t) := \int_{0^-}^{T^+} d_\tau X(kT + t, \tau) v(\tau), \quad \text{a.e. } t \in [0, T]. \quad (6.31)$$

Observe that the equality between $\int_{(nT)^-}^{+\infty} d_\alpha X(t+nT, \alpha)u(\alpha)$ and $\int_{(nT)^-}^{((n+1)T)^+} d_\alpha X(t+nT, \alpha)u(\alpha)$, used in the third identity of (6.30), only holds for $t < T$ in general because when $t = T$ and $(n+1)T \in \mathcal{F}$, then $\int_{(nT)^-}^{((n+1)T)^+} d_\alpha X(t+nT, \alpha)u(\alpha)$ will miss the jump that may occur at $(n+1)T$. However, since (6.30) is claimed for a.e. $t \in [0, T]$ only, this is unimportant. Note also that changing the upper bound from T^- to T^+ in the integrals summed over k after the fourth equal sign of (6.30) is permitted because the left continuity of $X(\tau, \cdot)$ implies that $\{T\}$ carries no mass of $d_\beta X(t+nT, nT+\beta)$. If we take z -transforms in (6.30), we obtain:

$$\mathcal{L}\{\bar{y}\}(z) = \mathcal{L}\{\bar{H}\}(z)\mathcal{L}\{\bar{u}\}(z), \quad (6.32)$$

where $\bar{H} := (H_{[n]})_{n \in \mathbb{N}}$, and since for each k we may pick \tilde{u}_k arbitrary in $L^2([0, T])$ and $\tilde{u}_j \equiv 0$ for $j \neq k$, we deduce from (6.29) and (6.32) that

$$\mathcal{L}\{\bar{H}\}(z) = \tilde{C}(zId - \tilde{A})^{-1}\tilde{B} + \tilde{D} \quad (6.33)$$

The equality (6.33) *a priori* holds between formal power series in z^{-1} with coefficients in the ring $\mathcal{B}(L^2([0, T], \mathbb{C}^d))$ of bounded linear operators on $L^2([0, T], \mathbb{C}^d)$. However, if we let a denote the spectral radius of \tilde{A} and set $P_a := \{z \in \mathbb{C} : |z| > a\}$, then (6.33) holds in $\mathcal{B}(L^2([0, T], \mathbb{C}^d))$ when we substitute for z a value in P_a , for the series become normally convergent. Of course, by analytic continuation, the right hand-side (therefore also the left hand-side) extends to an operator-valued analytic function on a connected component of $\mathbb{C} \setminus \text{Spec } \tilde{A}$ where $\text{Spec } \tilde{A}$ stands for the spectrum of $\tilde{A} = U_2(T, 0)$, but this function may no longer be defined by a power series in z^{-1} . Since we assumed that System (2.1) is C^0 -exponentially stable, we know by Propositions 11 and 1 that $\text{Spec } \tilde{A}$ is compactly included the unit disk; that is to say $a \in [0, 1)$. In particular, $\mathcal{L}\{\bar{H}\}(e^{pT})$ is a well-defined operator on $L^2([0, T], \mathbb{C}^d)$ as soon as $\Re(p) > \log a/T$, where $\log a$ is strictly negative or $-\infty$.

For $p \in \mathbb{C}$, consider the operator $E_p : L^2([0, T], \mathbb{C}^d) \rightarrow L^2([0, T], \mathbb{C}^d)$ consisting of pointwise multiplication by the function $t \mapsto e^{pt}$ that lies in $L^\infty([0, T])$. Clearly $E_p^{-1} = E_{-p}$, and it is easy to check that

$$\|E_p\| = \max\{1, e^{\Re(p)T}\}. \quad (6.34)$$

When $\Re(p) > \log a/T$, we also define the operator

$$\begin{aligned} \Lambda(p) & : L^2([0, T], \mathbb{C}^d) \rightarrow L^2([0, T], \mathbb{C}^d), \\ \Lambda(p) & = E_{-p} \circ \mathcal{L}\{\bar{H}\}(e^{pT}) \circ E_p. \end{aligned} \quad (6.35)$$

In addition, for $\Re(p) > \gamma_2$ with γ_2 as in (6.9), we construe $R^{-1}(p) : l^2(\mathbb{Z}, \mathbb{C}^d) \rightarrow l^2(\mathbb{Z}, \mathbb{C}^d)$ as an operator from $L^2([0, T], \mathbb{C}^d)$ into itself by identifying a function $\phi \in L^2([0, T], \mathbb{C}^d)$ with $(\cdots, a_2, a_1, a_0, a_{-1}, a_{-2}, \cdots)$ where a_k is the k -th Fourier coefficient of ϕ .

Lemma 13. *For $p \in \mathbb{C}$ with $\Re(p) > \gamma_2$, it holds that*

$$\Lambda(p)\phi(t) = R^{-1}(p)\phi, \quad \phi \in L^2([0, T], \mathbb{C}^d). \quad (6.36)$$

PROOF. As $\gamma_2 > 0 > \log a/T$, we see from (6.31) that for any $v \in L^2([0, T], \mathbb{R})$ we have:

$$\mathcal{L}\{\bar{H}\}(z)v(t) = \sum_{k=0}^{+\infty} H_{[k]}v(t)z^{-k} = \sum_{k=-\infty}^{+\infty} z^{-k} \int_0^{T^+} d_\tau X(kT+t, \tau)v(\tau). \quad (6.37)$$

Next, performing the change of variable $\tau \rightarrow t - \tau$ in (6.10) and computing as in (6.30), we obtain

$$\begin{aligned} G(t, p) &= \int_{-\infty}^{+\infty} d_\tau X(t, \tau) e^{p(\tau-t)} = \sum_{k=-\infty}^{+\infty} \int_{0^-}^{T^+} d_\tau X(t, \tau - kT) e^{p(\tau-t-kT)} \\ &= \sum_{k=-\infty}^{+\infty} e^{-kpT} \left(e^{-pt} \int_{0^-}^{T^+} d_\tau X(t + kT, \tau) e^{p\tau} \right). \end{aligned} \quad (6.38)$$

Multiplying (6.38) on the right by some arbitrary $V \in \mathbb{C}^d$ and comparing with (6.37), we obtain upon setting $z = e^{pT}$ and $v = e^p V$ that

$$G(t, p)V = e^{-pt} [\mathcal{L}\{\overline{H}\}(e^{pT})e^p V](t), \quad \text{a.e. } t \in [0, T]. \quad (6.39)$$

Now, assume for a while that $\phi \in C^{1,\alpha}([0, T], \mathbb{C}^d)$ with $\alpha \in (0, 1)$ and $\phi(0) = \phi(T)$, and write the Fourier expansion $\phi(t) = \sum_{k \in \mathbb{Z}} a_k e^{i\omega k t}$ where $a_k \in \mathbb{C}^d$. By normal convergence of the latter, we deduce from (6.39), (6.11) and (6.12) that

$$\begin{aligned} \Lambda(p)\phi(t) &= e^{-pt} [\mathcal{L}\{\overline{H}\}(e^{pT})e^p \sum_{k \in \mathbb{Z}} a_k e^{i\omega k \cdot}](t) = \sum_{k \in \mathbb{Z}} e^{-pt} [\mathcal{L}\{\overline{H}\}(e^{pT})e^{(p+i\omega k) \cdot} a_k](t) \\ &= \sum_{k \in \mathbb{Z}} e^{i\omega k t} e^{-(p+i\omega k)t} [\mathcal{L}\{\overline{H}\}(e^{(p+i\omega k)T})e^{(p+i\omega k) \cdot} a_k](t) = \sum_{k \in \mathbb{Z}} e^{i\omega k t} G(t, p + i\omega k) a_k \\ &= \sum_{k \in \mathbb{Z}} e^{i\omega k t} \left(\sum_{n \in \mathbb{Z}} e^{in\omega t} G_n(p + i\omega k) \right) a_k = \sum_{m \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} G_{m-k}(p + i\omega k) a_k \right) e^{im\omega t}. \end{aligned} \quad (6.40)$$

Thus, in view of (6.24), we get that

$$\Lambda(p)\phi(t) = \sum_{m \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} R_{m,-k}^{-1}(p) a_k \right) e^{im\omega t} = R^{-1}(p)\phi. \quad (6.41)$$

Equation (6.41) yields the desired conclusion, except for the assumption that $\phi \in C^{1,\alpha}([0, T], \mathbb{C}^d)$ with $\phi(0) = \phi(T)$. But since $\Lambda(p)$ and $R^{-1}(p)$ are continuous operators $L^2([0, T]) \rightarrow L^2([0, T])$ for $\Re(p) > \gamma_2$, we conclude by a density argument that (6.36) holds as stated. \square

To recap, we know from the discussion after (6.33) that $\mathcal{L}\{\overline{H}\}(e^{pT})$ is an analytic operator-valued function for $\Re(p) > \log a/T$, which is uniformly bounded in every half-plane $\{p : \Re(p) > \beta\}$ if $\beta > \log a/T$. Therefore, by (6.34) and the very definition (6.35), $\Lambda(p)$ is an analytic operator-valued function for $\Re(p) > \log a/T$, which is uniformly bounded in every strip $\{p : \beta_1 > \Re(p) > \beta\}$ if $\beta_1 > \beta > \log a/T$. Furthermore, in view of (2.17), $R(p)$ is an entire operator-valued function. As Lemma 13 tells us that $\Lambda(p)R(p) = I_\infty$ for $\Re(p) > \gamma_2$, this identity in fact holds on the larger domain $\{p : \Re(p) > \log a/T\}$, by analytic continuation. So, if we let $\beta \in (\log a/T, 0)$ and $\beta_1 = \gamma_1 + 1$ with γ_1 as in (5.18), we get that $R^{-1}(p)$ exists and is equal to $\Lambda(p)$ for $\Re(p) > \log a/T$, and that $\|R^{-1}(p)\|_2 \leq C_1$ for $\beta < \Re(p) < \beta_1$ for some constant $C_1 > 0$ depending on β and β_1 but not on p . In another connection, we know from (5.18) that $\|R^{-1}(p)\|_2 \leq C_2$ for $\Re(p) > \gamma_1$, and altogether we conclude that

$$\|R^{-1}(p)\|_2 \leq C, \quad \Re(p) > \beta, \quad (6.42)$$

with $C = \max\{C_1, C_2\}$. This ends the proof of the necessity part of Theorem 3, and thus of the theorem itself, considering Section 5.

7. Concluding remarks

Having proven a generalization of the Henry-Hale theorem on the stability of difference-delay systems to the periodic case, it is natural to ask whether the result from [4] on the stability of neutral functional differential systems carries over to the periodic case as well. In other words, whether the stability of a periodic linear neutral differential system of the form (1.3) can be characterized in terms of some harmonic transfer function. It is transparent how to define the latter: simply put $\mathbf{H}(p) := \tilde{R}(p)^{-1}$ where

$$\tilde{R}(p) := D_\omega(p) \left[I_\infty - \sum_{j=1}^N e^{-p\tau_j} L_{D_j} \Delta_{\tau_j, \omega} \right] - \sum_{k=0}^N e^{-p\tau_k} L_{B_k} \tilde{D}_{\tau_k}, \quad (7.1)$$

with

$$D_\omega(p) := \text{Diag}(\cdots, (p + i\omega)I_d, pI_d, (p - i\omega)I_d, \cdots) \quad (7.2)$$

and the D_j, B_j are as in (1.3). Then, with the natural definition of exponential stability paralleling Definition 1, we raise the following question:

Assume that the $D_j(\cdot)$ and $B_k(\cdot)$ are periodic and differentiable with Hölder continuous derivative for $1 \leq j \leq N$ and $0 \leq k \leq N$. Is it true that a necessary and sufficient condition for System (1.3) to be exponentially stable is the existence of a real number $\beta < 0$ satisfying

- (i) $\tilde{R}(p)$ is invertible $\ell^2(\mathbb{Z}, \mathbb{C}^d) \rightarrow \ell^2(\mathbb{Z}, \mathbb{C}^d)$ for all p in $\{z \in \mathbb{C} | \Re(z) \geq \beta\}$,
- (ii) there is $M > 0$ such that $\left\| \tilde{R}(p)^{-1} \right\|_2 \leq M$ for all p in $\{z \in \mathbb{C} | \Re(z) \geq \beta\}$?

We expect that the answer is yes, and that the necessity part can be proven in the same way as in theorem 3. The difficulty with the sufficiency part arises from the matrix $D_\omega(p)$, that prevents us from obtaining straightforward analogs of Lemmas 9 and 10.

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Appendix A. Proof of Proposition 6, and in particular formula (4.4)

Let us first quickly prove (4.5) by induction, assuming (4.4). Let $K' > 0$ be an upper bound for $\{\|D_i(t)\|, 1 \leq i \leq N, t \in \mathbb{R}\}$; such a K' exists since the D_i are continuous and periodic. Pick λ large enough that

$$K'Ne^{-\lambda\tau_1} < \frac{1}{2}, \tag{A.1}$$

and let us show (4.5) by induction on k such that $t \in [s + k\tau_1, s + (k+1)\tau_1)$. For $k = 0$, it is obvious that (4.5) holds with $K = 1$ because $X(t, s)$ is either 0 or I_d for $0 \leq t - s < \tau_1$. Next, assume that

(4.5) hold for all t, s such that $0 \leq t - s < k\tau_1$. Then, for t, s verifying $k\tau_1 \leq t - s < (k+1)\tau_1$, we get from (4.1) that

$$\|X(t, s)\| \leq 1 + K'NK e^{-\lambda\tau_1} e^{\lambda(t-s)} \quad (\text{A.2})$$

$$\leq K e^{\lambda(t-s)}, \quad (\text{A.3})$$

which concludes the induction. To prove that $y(t)$ is C^0 -exponentially stable, in the forthcoming steps we will estimate the variation of $X(t, s)$ with respect in its second argument, using Laplace transforms.

We now turn to proving formula (4.4). This requires preliminaries about Volterra kernels in Sections A.1 and A.2, before giving the proof proper in Section A.3.

A.1. Reduction to a Volterra equation

Observe from (4.1) that $\alpha \mapsto X(t, \alpha)$ lies in $BV_{loc}(\mathbb{R})$ for all t , and that for $b > s$ we have:

$$d_\alpha X(t, \alpha) = \begin{cases} \sum_{j=1}^N D_j(t) d_\alpha X(t - \tau_j, \alpha) & \text{if } t \geq b \\ -I_d \delta_t + \sum_{j=1}^N D_j(t) d_\alpha X(t - \tau_j, \alpha) & \text{if } s \leq t < b \end{cases} \quad \text{on } [s, b], \quad (\text{A.4})$$

where δ_t is the Dirac delta at t . In particular, $\alpha \mapsto X(t, \alpha)$ has bounded variation on $[s, s + \tau_j]$ for each j , and for $t \geq s + \tau_N$ we have that

$$d_\alpha X(t, \alpha) = \sum_{j=1}^N D_j(t) d_\alpha X(t - \tau_j, \alpha) \quad \text{on } [s, s + \tau_j]. \quad (\text{A.5})$$

Then, substituting (A.5) in (4.4) formally yields (2.1), provided that $t \geq s + \tau_N$. Hence, by uniqueness of a solution y to (2.1) satisfying $y(s + \theta) = \phi(\theta)$ for $\theta \in [-\tau_N, 0]$, it is enough to check (4.4) for $s \leq t < s + \tau_N$. For this, we adopt the point of view of reference [1], which is to construe delay systems as Stieltjes-Volterra equations upon representing delays by measures. More precisely, we can rewrite (2.1) as a Lebesgue-Stieltjes integral:

$$y(t) = \int_{-\tau_N}^{0^-} d\mu(t, \theta) y(t + \theta), \quad t \geq s, \quad (\text{A.6})$$

with

$$\mu(t, \theta) = \sum_{j=1}^N D_j(t) \mathfrak{H}(\theta + \tau_j), \quad (\text{A.7})$$

where $y(\tau)$ is understood to be $\phi(\tau - s)$ when $s - \tau_N \leq \tau \leq s$ and $\mathfrak{H}(\tau)$ is the Heaviside function which is zero for $\tau \leq 0$ and 1 for $\tau > 0$, so that the associated measure on an interval of the form $[0, a]$ or $[0, a)$ is a Dirac delta at 0. Note that $\mathfrak{H}(0) = 0$, which is not the usual convention, but if we defined \mathfrak{H} so that $\mathfrak{H}(0) = 1$ then expanding (A.6) using (A.7) would not give us back (2.1) for the term $D_N(t)y(t - \tau_N)$ would be missing. Observe also, since $\tau_j > 0$ for all j , that the minus sign in the upper bound of the integral in (A.6) is immaterial and could be traded for a plus. For $s \leq t \leq s + \tau_N$, singling out the initial data in (A.6) yields

$$y(t) = \int_{(s-t)^-}^{0^-} d\mu(t, \theta) y(t + \theta) + f(t) \quad \text{with} \quad f(t) := \int_{-\tau_N}^{(s-t)^-} d\mu(t, \theta) \phi(t + \theta - s), \quad (\text{A.8})$$

where we took into account, when separating the integrals, that $\theta \mapsto \mu(t, \theta)$ is left continuous, while the integral over the empty interval is understood to be zero. It will be convenient to study (A.8) for $t \in [s, s + \tau_N]$, even though in the end the values of $y(t)$ only matter to us for $t \in [s, t + \tau_N]$. Define

$$k(t, \tau) := \begin{cases} \mu(t, \tau - t) - \sum_{j=1}^N D_j(t) & \text{for } \tau \in [s, t], \\ 0 & \text{for } \tau > t, \end{cases} \quad t, \tau \in [s, s + \tau_N]. \quad (\text{A.9})$$

Note that $k(t, \tau) = 0$ when $t - \tau < \tau_1$, and $d_\tau k(t, \tau) = d_\tau \mu(t, \tau - t)$ on $[s, s + \tau_N]$ for fixed t . Hence, (A.8) becomes

$$y(t) = \int_{s^-}^{t^-} dk(t, \tau) y(\tau) + f(t), \quad s \leq t \leq s + \tau_N. \quad (\text{A.10})$$

Now, (A.10) is the Stieltjes-Volterra equation we shall work with.

It suffices to prove (4.4) for $t \in [s, s + \tau_N]$ under the additional assumption that ϕ and the D_j , which are continuous by hypothesis, also have bounded and locally bounded variation, on $[-\tau_N, 0]$ and \mathbb{R} respectively. Indeed, functions of bounded variation are dense in C_s (resp. $C^0(\mathbb{T}, \mathbb{R}^{d \times d})$), for instance because C^1 -functions are, and if ϕ_k converges uniformly to ϕ in C_s while $D_{j,k}$ converges uniformly to D_j in $[s, s + \tau_N]$ as $k \rightarrow \infty$, then the solution to (2.1) with initial condition ϕ_k and coefficients $D_{j,k}$ converges uniformly on $[s, s + \tau_N]$ to the solution with initial condition ϕ and coefficients D_j , as is obvious by inspection. Hence, we shall assume without loss of generality that ϕ has bounded variation and the D_j have locally bounded variation. Then, since it follows from (A.7) and (A.8) that

$$f(t) = \sum_{\tau_\ell \in (t-s, \tau_N]} D_\ell(t) \phi(t - s - \tau_\ell), \quad (\text{A.11})$$

it is clear from (A.11) and (3.2) that $f \in BV_l[s, s + \tau_N] \cap BV_r[s, s + \tau_N]$, since it is continuous.

A.2. Volterra kernels of type B^∞

Volterra equations for functions of a single variable have been studied extensively, see *e.g.* [31, 32]. However, the specific assumption that the kernel has bounded variation seems treated somewhat tangentially. On the one hand, it is subsumed in the measure-valued case presented in [32, Ch. 10], but no convenient criterion is given there for the existence of a resolvent kernel. On the other hand, [1, Ch. 9, Sec. 1] sketches the main arguments needed to handle kernels of bounded variation, but the exposition has issues⁴ which is why we provide a proof in this section.

We define a Stieltjes-Volterra kernel of type B^∞ on $[a, b] \times [a, b]$ as a measurable function $\kappa : [a, b] \times [a, b] \rightarrow \mathbb{R}^{d \times d}$, with $\kappa(t, \tau) = 0$ for $\tau \geq t$, such that the partial maps $\kappa(t, \cdot)$ lie in $BV_l[a, b]$ and $\|\kappa(t, \cdot)\|_{BV([a, b])}$ is uniformly bounded with respect to $t \in [a, b]$. In addition, we require that $\lim_{\tau \rightarrow t^-} W_{[\tau, t]}(\kappa(t, \cdot)) = 0$ uniformly with respect to t ; *i.e.*, to every $\varepsilon > 0$, there exists $\eta > 0$ such that $W_{[\tau, t]}(\kappa(t, \cdot)) < \varepsilon$ as soon as $0 < t - \tau < \eta$. Note that $W_{[\tau, t]}(\kappa(t, \cdot)) \rightarrow 0$ for fixed t as $\tau \rightarrow t^-$ whenever $\kappa(t, \cdot)$ has bounded variation on $[a, b]$, by the very definition (3.1); so, the assumption really is that the convergence is uniform with respect to t . Hereafter, we drop the qualifier “of type B^∞ ” and simply speak of Stieltjes-Volterra kernels on $[a, b]$. We endow the space $\mathcal{K}_{[a, b]}$ of such kernels with the norm $\|\kappa\|_{[a, b]} := \sup_{t \in [a, b]} \|\kappa(t, \cdot)\|_{BV[a, b]}$. If κ_k is a Cauchy sequence in $\mathcal{K}_{[a, b]}$, then κ_k converges uniformly on $[a, b] \times [a, b]$ to a $\mathbb{R}^{d \times d}$ -valued function κ because

$$\|\kappa_k(t, \tau) - \kappa_l(t, \tau)\| = \|(\kappa_k(t, \tau) - \kappa_l(t, \tau)) - (\kappa_k(t, t) - \kappa_l(t, t))\| \leq \|\kappa_k(t, \cdot) - \kappa_l(t, \cdot)\|_{BV[a, b]}.$$

⁴For example, the equation satisfied by $\tilde{\rho}(t, s)$ at top of page 258 is not right.

Clearly, κ is measurable and left continuous for fixed t with $\kappa(t, \tau) = 0$ for $\tau \geq t$. Also, if m is so large that $\|\kappa_k - \kappa_l\|_{[a,b]} < \varepsilon$ for $k, l \geq m$ and $\eta > 0$ so small that $W_{[\tau,t]}(\kappa_m) < \varepsilon$ when $t - \tau < \eta$, we get that $W_{[\tau,t]}(\kappa_l) \leq W_{[\tau,t]}(\kappa_m) + W_{[\tau,t]}(\kappa_m - \kappa_l) < 2\varepsilon$, and letting $l \rightarrow \infty$ we get from [19, thm. 1.3.5] that $W_{[\tau,t]}(\kappa) \leq 2\varepsilon$. Furthermore, the same reference implies that $\|\kappa\|_{[a,b]} \leq \sup_k \|\kappa_k\|_{[a,b]}$ so that $\kappa \in \mathcal{K}_{[a,b]}$. Finally, writing that $W_{[a,b]}(\kappa_k(t, \cdot) - \kappa_l(t, \cdot)) < \varepsilon$ and passing to the limit as $l \rightarrow \infty$, we see that $\lim_k \|\kappa_k - \kappa\|_{[a,b]} = 0$, whence $\mathcal{K}_{[a,b]}$ is a Banach space. Note that a Stieltjes-Volterra kernel κ is necessarily bounded with $\sup_{[a,b] \times [a,b]} \|\kappa(t, \tau)\| \leq \|\kappa\|_{[a,b]}$. As \mathfrak{H} is left continuous and the D_j are bounded, on $[s, s + \tau_N]$, it is easy to check that $k(t, \tau)$ defined in (A.9) is a Stieltjes-Volterra kernel on $[s, s + \tau_N] \times [s, s + \tau_N]$.

A *resolvent* for the Stieltjes-Volterra kernel κ on $[a, b] \times [a, b]$ is a Stieltjes-Volterra kernel ρ on $[a, b] \times [a, b]$ satisfying

$$\rho(t, \beta) = -\kappa(t, \beta) + \int_{\beta^-}^{t^-} d\kappa(t, \tau) \rho(\tau, \beta), \quad a \leq t, \beta \leq b. \quad (\text{A.12})$$

Lemma 14. *If κ is a Stieltjes-Volterra kernel on $[a, b] \times [a, b]$, a resolvent for κ uniquely exists.*

PROOF. Pick $r > 0$ to be adjusted later, and for $\Psi \in \mathcal{K}_{[a,b]}$ let us define

$$F_r(\Psi)(t, \beta) := \int_{\beta^-}^{t^-} e^{-r(t-\tau)} d\kappa(t, \tau) \Psi(\tau, \beta), \quad a \leq \beta, t \leq b.$$

Then, $F_r(\Psi)(t, \beta) = 0$ for $\beta \geq t$, and for $a \leq \beta_1 < \beta_2 < t$ we have that

$$F_r(\Psi)(t, \beta_2) - F_r(\Psi)(t, \beta_1) = \int_{\beta_2^-}^{t^-} e^{-r(t-\tau)} d\kappa(t, \tau) (\Psi(\tau, \beta_2) - \Psi(\tau, \beta_1)) - \int_{\beta_1^-}^{\beta_2^-} e^{-r(t-\tau)} d\kappa(t, \tau) \Psi(\tau, \beta_1),$$

where we used that $\kappa(t, \cdot)$ is left continuous to assign the lower (resp. upper) bound β_2^- to the first (resp. second) integral in the above right hand side. Now, the first integral goes to 0 as $\beta_1 \rightarrow \beta_2$ by dominated convergence, since $\Psi(t, \cdot)$ is left-continuous; the second integral also goes to 0, because $|\nu_{\kappa(t, \cdot)}|([\beta_1, \beta_2)) \rightarrow 0$ when $\beta_1 \rightarrow \beta_2$, by standard properties of finite measures. Altogether, we see

that $F_r(\Psi)(t, \cdot)$ is left-continuous. Moreover, for $[c, d] \subset [a, t]$ and $c = \beta_0 < \beta_1 < \dots < \beta_N = d$,

$$\begin{aligned}
& \sum_{i=1}^N \|F_r(\Psi)(t, \beta_i) - F_r(\Psi)(t, \beta_{i-1})\| \\
& \leq \sum_{i=1}^N \left\| \int_{\beta_{i-1}^-}^{t^-} e^{-r(t-\tau)} d\kappa(t, \tau) (\Psi(\tau, \beta_i) - \Psi(\tau, \beta_{i-1})) \right\| + \sum_{i=1}^N \left\| \int_{\beta_{i-1}^-}^{\beta_i^-} e^{-r(t-\tau)} d\kappa(t, \tau) \Psi(\tau, \beta_{i-1}) \right\| \\
& \leq \sum_{i=1}^N \int_{\beta_{i-1}^-}^{t^-} e^{-r(t-\tau)} d|\nu_{\kappa(t, \cdot)}| \|(\Psi(\tau, \beta_i) - \Psi(\tau, \beta_{i-1}))\| + \sum_{i=1}^N \int_{\beta_{i-1}^-}^{\beta_i^-} e^{-r(t-\tau)} d|\nu_{\kappa(t, \cdot)}| \|\Psi(\tau, \beta_{i-1})\| \\
& \leq \int_{d^-}^{t^-} d|\nu_{\kappa(t, \cdot)}| \sum_{i=1}^N \|(\Psi(\tau, \beta_i) - \Psi(\tau, \beta_{i-1}))\| \\
& \quad + e^{-r(t-d)} \int_{c^-}^{d^-} d|\nu_{\kappa(t, \cdot)}| \sum_{i=1}^N \|(\Psi(\tau, \beta_i) - \Psi(\tau, \beta_{i-1}))\| + \sup_{[a, t] \times [a, t]} \|\Psi\| \int_{c^-}^{d^-} e^{-r(t-\tau)} d|\nu_{\kappa(t, \cdot)}| \\
& \leq 2W_{[d, t]}(\kappa(t, \cdot)) \sup_{\tau \in [d, t]} W_{[c, d]}(\Psi(\tau, \cdot)) \\
& \quad + 2e^{-r(t-d)} W_{[c, d]}(\kappa(t, \cdot)) \sup_{\tau \in [c, d]} W_{[c, d]}(\Psi(\tau, \cdot)) + 2e^{-r(t-d)} \sup_{[a, t] \times [a, t]} \|\Psi\| W_{[c, d]}(\kappa(t, \cdot)).
\end{aligned}$$

When $d = t$, the same inequality holds but then $W_{[d, t]}(\kappa(t, \cdot))$ is zero. Setting $c = a$ and $d = t$, we get from the above majorization that $W_{[a, t]}(F_r(\Psi)(t, \cdot)) \leq 4\|\kappa\|_{[a, b]} \|\Psi\|_{[a, b]}$, and since $F_r(\Psi)(t, \tau) = 0$ for $\tau \geq t$ we deduce that $W_{[a, b]}(F_r(\Psi)(t, \cdot)) = W_{[a, t]}(F_r(\Psi)(t, \cdot))$ is bounded, uniformly with respect to t . Next, if we fix $\varepsilon > 0$ and pick $\eta > 0$ so small that $W_{[\tau, t]}(\kappa(t, \cdot)) \leq \varepsilon$ as soon as $t - \tau \leq \eta$ (this is possible because $\kappa \in \mathcal{K}_{[a, b]}$), the same estimate yields

$$W_{[c, t]}(F_r(\Psi)(t, \cdot)) \leq 4W_{[c, t]}(\kappa(t, \cdot)) \|\Psi\|_{[a, b]} \leq 4\varepsilon \|\Psi\|_{[a, b]}, \quad t - c \leq \eta. \quad (\text{A.13})$$

Altogether, we just showed that $F_r(\Psi) \in \mathcal{K}_{[a, b]}$. Moreover, if we take r so large that $e^{-r\eta} < \varepsilon$, then either $t - a \leq \eta$ and then (A.13) with $c = a$ gives us $W_{[a, t]}(F_r(\Psi)(t, \cdot)) \leq 4\varepsilon \|\Psi\|_{[a, b]}$, or else $t - \eta > a$ in which case (A.13) with $c = t - \eta$, together with our initial estimate when $c = a$ and $d = t - \eta$, team up to produce:

$$\begin{aligned}
W_{[a, t]}(F_r(\Psi))(t, \cdot) &= W_{[a, t-\eta]}(F_r(\Psi)(t, \cdot)) + W_{[t-\eta, t]}(F_r(\Psi)(t, \cdot)) \\
&\leq 2\varepsilon \sup_{\tau \in [t-\eta, t]} W_{[a, t-\eta]}(\Psi(\tau, \cdot)) + 2\varepsilon W_{[a, t-\eta]}(\kappa(t, \cdot)) \sup_{\tau \in [a, t-\eta]} W_{[a, t-\eta]}(\Psi(\tau, \cdot)) \\
&\quad + 2\varepsilon \sup_{[a, t] \times [a, t]} \|\Psi\| W_{[a, t-\eta]}(\kappa(t, \cdot)) + 4\varepsilon \|\Psi\|_{[a, b]} \\
&\leq 2\varepsilon \|\Psi\|_{[a, b]} (3 + 2\|\kappa\|_{[a, b]}). \quad (\text{A.14})
\end{aligned}$$

Consequently, as $W_{[a, t]}(F_r(\Psi)(t, \cdot)) = W_{[a, t]}(F_r(\Psi)(t, \cdot))$ by the left continuity of $F_r(\Psi)(t, \cdot)$, we can ensure upon choosing r sufficiently large that the operator $F_r : \mathcal{K}_{[a, b]} \rightarrow \mathcal{K}_{[a, b]}$ has arbitrary small norm. Hereafter, we fix r so that $\|F_r\| < \lambda < 1$.

Now, let $\tilde{\rho}_0 = 0$ and define inductively:

$$\tilde{\rho}_{k+1}(t, \beta) = -e^{-rt} \kappa(t, \beta) + F_r(\tilde{\rho}_k)(t, \beta).$$

Clearly $(t, \beta) \mapsto e^{-rt} \kappa(t, \beta)$ lies in $\mathcal{K}_{[a, b]}$, so that $\tilde{\rho}_k \in \mathcal{K}_{[a, b]}$ for all k . Moreover, we get from what precedes that $\|\tilde{\rho}_{k+1} - \tilde{\rho}_k\|_{[a, b]} \leq \lambda \|\tilde{\rho}_k - \tilde{\rho}_{k-1}\|_{[a, b]}$. Thus, by the shrinking lemma, $\tilde{\rho}_k$ converges in

$\mathcal{K}_{[a,b]}$ to the unique $\tilde{\rho} \in \mathcal{K}_{[a,b]}$ such that

$$\begin{aligned}\tilde{\rho}(t, \beta) &= -e^{-rt}\kappa(t, \beta) + F_r(\tilde{\rho})(t, \beta) \\ &= -e^{-rt}\kappa(t, \beta) + \int_{\beta^-}^{t^-} e^{-r(t-\tau)} d\kappa(t, \tau) \tilde{\rho}(\tau, \beta), \quad a \leq t, \beta \leq b.\end{aligned}\tag{A.15}$$

Putting $\rho(t, \beta) := e^{rt}\tilde{\rho}(t, \beta)$, one can see that ρ lies in $\mathcal{K}_{[a,b]}$ if and only if $\tilde{\rho}$ does, and that (A.15) is equivalent to (A.12). This achieves the proof. \square

Lemma 15. *Let κ be a Stieltjes-Volterra kernel on $[a, b] \times [a, b]$ and ρ its resolvent. For each \mathbb{R}^d -valued function $g \in BV_r([a, b])$, the unique bounded measurable solution to the equation*

$$y(t) = \int_{a^-}^{t^-} d\kappa(t, \tau)y(\tau) + g(t), \quad a \leq t \leq b,\tag{A.16}$$

is given by

$$y(t) = g(t) - \int_{a^-}^{t^-} d\rho(t, \alpha)g(\alpha), \quad a \leq t \leq b.\tag{A.17}$$

PROOF. Define y through (A.17) so that $y(a) = g(a)$, by inspection. Since $g \in BV_r[a, b]$ and $\rho(t, \cdot), \kappa(t, \cdot)$ lie in $BV_l[a, b]$, integrating by parts [19, thm. 7.5.9]⁵ while using (A.12) along with Fubini's theorem and the relations $\kappa(t, \alpha) = \rho(t, \alpha) = 0$ for $\alpha \geq t$, we get that

$$\begin{aligned}\int_{a^-}^{t^-} d\kappa(t, \alpha)y(\alpha) &= (\kappa(t, a^+) - \kappa(t, a))y(a) + \int_{a^+}^{t^-} d\kappa(t, \alpha)y(\alpha) \\ &= (\kappa(t, a^+) - \kappa(t, a))g(a) + \int_{a^+}^{t^-} d\kappa(t, \alpha)g(\alpha) - \int_{a^+}^{t^-} d\kappa(t, \alpha) \int_{a^-}^{\alpha^-} d\rho(\alpha, \beta)g(\beta) \\ &= (\kappa(t, a^+) - \kappa(t, a))g(a) + \int_{a^+}^{t^-} d\kappa(t, \alpha)g(\alpha) - \int_{a^+}^{t^-} d\kappa(t, \alpha) \int_{a^+}^{\alpha^-} d\rho(\alpha, \beta)g(\beta) \\ &\quad - \int_{a^+}^{t^-} d\kappa(t, \alpha)(\rho(\alpha, a^+) - \rho(\alpha, a))g(a) \\ &= (\kappa(t, a^+) - \kappa(t, a))g(a) + \int_{a^+}^{t^-} d\kappa(t, \alpha)g(\alpha) + \int_{a^+}^{t^-} d\kappa(t, \alpha) \int_{a^+}^{\alpha^-} \rho(\alpha, \beta)dg(\beta) \\ &\quad - \int_{a^+}^{t^-} d\kappa(t, \alpha) [\rho(\alpha, \beta)g(\beta)]_{\beta=a^+}^{\beta=\alpha^-} - \int_{a^+}^{t^-} d\kappa(t, \alpha)(\rho(\alpha, a^+) - \rho(\alpha, a))g(a) \\ &= (\kappa(t, a^+) - \kappa(t, a))g(a) + \int_{a^+}^{t^-} d\kappa(t, \alpha)g(\alpha) + \int_{a^+}^{t^-} \left(\int_{\beta^-}^{t^-} d\kappa(t, \alpha)\rho(\alpha, \beta) \right) dg(\beta) \\ &\quad + \int_{a^+}^{t^-} d\kappa(t, \alpha)\rho(\alpha, a^+)g(a) - \int_{a^+}^{t^-} d\kappa(t, \alpha)(\rho(\alpha, a^+) - \rho(\alpha, a))g(a) \\ &= (\kappa(t, a^+) - \kappa(t, a))g(a) + \int_{a^+}^{t^-} d\kappa(t, \alpha)g(\alpha) \\ &\quad + \int_{a^+}^{t^-} (\rho(t, \beta) + \kappa(t, \beta))dg(\beta) + \int_{a^+}^{t^-} d\kappa(t, \alpha)\rho(\alpha, a)g(a) \\ &= (\kappa(t, a^+) - \kappa(t, a))g(a) + [\kappa(t, \alpha)g(\alpha)]_{\alpha=a^+}^{\alpha=t^-} + \int_{a^+}^{t^-} \rho(t, \beta)dg(\beta) + \int_{a^+}^{t^-} d\kappa(t, \alpha)\rho(\alpha, a)g(a) \\ &= -\kappa(t, a)g(a) + [\rho(t, \beta)g(\beta)]_{\beta=a^+}^{\beta=t^-} - \int_{a^+}^{t^-} d\rho(t, \beta)g(\beta) + \int_{a^-}^{t^-} d\kappa(t, \alpha)\rho(\alpha, a)g(a) \\ &= -\kappa(t, a)g(a) - \rho(t, a^+)g(a) - \int_{a^+}^{t^-} d\rho(t, \beta)g(\beta) + (\kappa(t, a) + \rho(t, a))g(a) \\ &= -\int_{a^-}^{t^-} d\rho(t, \beta)g(\beta) = y(t) - g(t).\end{aligned}$$

Thus, y is a solution to (A.16). Clearly, it is measurable, and it is also bounded since $\|\rho(t, \cdot)\|_{BV([a,b])}$ is bounded independently of t and g is bounded. If \tilde{y} is another solution to (A.16) then $\tilde{y}(a) = y(a) = g(a)$ by inspection, so that $z := y - \tilde{y}$ is a bounded measurable solution to the homogeneous equation:

$$z(t) = \int_{a^+}^{t^-} d\kappa(t, \tau)z(\tau), \quad a \leq t \leq b.$$

⁵This reference deals with open intervals only, and we stick to this case at the cost of a slightly lengthier computation

Pick $r > 0$ to be adjusted momentarily, and set $\tilde{z}(t) := e^{-rt}z(t)$ so that

$$\tilde{z}(t) = \int_{a^+}^{t^-} e^{-r(t-\tau)} d\kappa(t, \tau) \tilde{z}(\tau). \quad (\text{A.18})$$

Let $\eta > 0$ be so small that $W_{[\tau, t]}(\kappa(t, \cdot)) \leq 1/4$ as soon as $t - \tau \leq \eta$; this is possible because $\kappa \in \mathcal{K}_{[a, b]}$. Then, it follows from (A.18) that for $t - \eta > a$:

$$\begin{aligned} |\tilde{z}(t)| &\leq \left| \int_{a^+}^{(t-\eta)^+} e^{-r(t-\tau)^+} d\kappa(t, \tau) \tilde{z}(\tau) \right| + \left| \int_{(t-\eta)^+}^{t^-} e^{-r(t-\tau)^-} d\kappa(t, \tau) \tilde{z}(\tau) \right| \\ &\leq 2e^{-r\eta} W_{(a, t-\eta]}(\kappa(t, \cdot)) \sup_{(a, t-\eta]} |\tilde{z}| + \frac{1}{2} \sup_{(t-\eta, t)} |\tilde{z}| \leq \sup_{(a, t)} |\tilde{z}| (2e^{-r\eta} \|\kappa\|_{[a, b]} + \frac{1}{2}), \end{aligned}$$

while for $t - \eta \leq a$ we simply get $|\tilde{z}(t)| \leq \sup_{(a, t)} |\tilde{z}|/2$. Hence, choosing r large enough, we may assume that $|\tilde{z}(t)| \leq \lambda \sup_{(a, t)} |\tilde{z}|$ for some $\lambda < 1$ and all $t \in [a, b]$. Thus, if we choose $\lambda' \in (\lambda, 1)$ and $t_0 \in (a, b]$, we can find $t_1 \in (a, t_0]$ such that $|\tilde{z}(t_1)| \geq (1/\lambda') |\tilde{z}(t_0)|$, and proceeding inductively we construct a sequence (t_n) in $(a, t_0]$ with $|\tilde{z}(t_n)| \geq (1/\lambda')^n |\tilde{z}(t_0)|$. If we had $|\tilde{z}(t_0)| > 0$, this would contradict the boundedness of \tilde{z} , therefore $\tilde{z} \equiv 0$ on $(a, b]$, whence $z \equiv 0$ so that $y = \tilde{y}$. \square

A.3. Proof of formula (4.4).

Let ρ denote the resolvent of the Stieltjes-Volterra kernel k on $[s, s + \tau_N] \times [s, s + \tau_N]$ defined in (A.9). As f defined in (A.8) lies in $BV_r[s, s + \tau_N]$ (it is even continuous), the solution y to (A.10) is given, in view of Lemma 15, by

$$y(t) = f(t) - \int_{s^-}^{t^-} d\rho(t, \alpha) f(\alpha), \quad s \leq t \leq s + \tau_N. \quad (\text{A.19})$$

Since $\rho(t, \alpha) = 0$ when $\alpha \geq t$, the integral $\int_{s^-}^{t^-}$ can be replaced by $\int_{s^-}^{(s+\tau_N)^+}$ in (A.19). Thus, setting $\tilde{X}(t, \alpha) := I_d \mathcal{H}(t - \alpha) + \rho(t, \alpha)$ with $\mathcal{H}(\tau)$ the “standard” Heaviside function which is 0 for $\tau < 0$ and 1 for $\tau \geq 0$, we deduce from (A.11), since $d_\alpha \mathcal{H}(t - \alpha) = -\delta_t$ on $[s, s + \tau_N]$ for $s \leq t < s + \tau_N$, that

$$y(t) = - \int_{s^-}^{(s+\tau_N)^+} d\tilde{X}(t, \alpha) f(\alpha) = - \int_{s^-}^{(s+\tau_N)^+} d\tilde{X}(t, \alpha) \left(\sum_{\tau_\ell \in (\alpha - s, \tau_N]} D_\ell(\alpha) \phi(\alpha - s - \tau_\ell) \right), \quad s \leq t < s + \tau_N.$$

Rearranging, we get that

$$y(t) = - \sum_{j=1}^N \int_{s^-}^{(s+\tau_j)^-} d\tilde{X}(t, \alpha) D_j(\alpha) \phi(\alpha - s - \tau_j), \quad s \leq t < s + \tau_N,$$

which is what we want (namely: formula (4.4) for $s \leq t < s + \tau_N$) if only we can show that $\tilde{X}(t, \alpha)$ coincides with $X(t, \alpha)$ when $\alpha \in [s, s + \tau_j)$ for each j and every $t \in [s, s + \tau_N]$; here, $X(t, \alpha)$ is defined by (4.1) where we set $s = \alpha$.

For this, we first observe that $X(t, \alpha) = \tilde{X}(t, \alpha) = 0$ when $\alpha > t$ and that $X(t, t) = \tilde{X}(t, t) = Id$. Hence, we need only consider the case $\alpha \in [s, t)$ with $s < t < s + \tau_N$. For $s \leq \alpha < t$, we get that

$$-k(t, \alpha) = k(t, t^-) - k(t, \alpha) = \int_{\alpha^-}^{t^-} dk(t, \tau).$$

Thus, (A.12) (where $\kappa = k$) in concert with the definition of $\tilde{X}(t, \alpha)$ imply that

$$\begin{aligned}\tilde{X}(t, \alpha) &= I_d \mathcal{H}(t - \alpha) - k(t, \alpha) + \int_{\alpha^-}^{t^-} dk(t, \tau) \rho(\tau, \alpha) = I_d + \int_{\alpha^-}^{t^-} dk(t, \tau) (I_d + \rho(\tau, \alpha)) \\ &= I_d + \int_{\alpha^-}^{t^-} dk(t, \tau) \tilde{X}(\tau, \alpha).\end{aligned}$$

Now, on $[\alpha, t)$, we compute from (A.7) and (A.9) that $d_\tau k(t, \tau) = \sum_{t-\tau_j \geq \alpha} D_j(t) \delta_{t-\tau_j}$ and hence, since $\tilde{X}(t - \tau_j, \alpha) = 0$ when $\alpha > t - \tau_j$, the previous equation becomes:

$$\tilde{X}(t, \alpha) = I_d + \sum_{j=1}^N D_j(t) \tilde{X}(t - \tau_j, \alpha) \quad \text{for } s \leq \alpha < t \text{ and } s \leq t < s + \tau_N. \quad (\text{A.20})$$

Comparing (A.20) and (4.1), we see that $\tilde{X}(t, \alpha)$ and $X(t, \alpha)$ coincide on $[s, s + \tau_N) \times [s, s + \tau_N)$, thereby ending the proof.