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# Circular Pattern Matching with $k$ Mismatches

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## Abstract

The  $k$ -mismatch problem consists in computing the Hamming distance between a pattern  $P$  of length  $m$  and every length- $m$  substring of a text  $T$  of length  $n$ , if this distance is no more than  $k$ . In many real-world applications, any cyclic rotation of  $P$  is a relevant pattern, and thus one is interested in computing the minimal distance of every length- $m$  substring of  $T$  and any cyclic rotation of  $P$ . This is the circular pattern matching with  $k$  mismatches ( $k$ -CPM) problem. A multitude of papers have been devoted to solving this problem but, to the best of our knowledge, only average-case upper bounds are known. In this paper, we present the first non-trivial worst-case upper bounds for the  $k$ -CPM problem. Specifically, we show an  $\mathcal{O}(nk)$ -time algorithm and an  $\mathcal{O}(n + \frac{n}{m} k^4)$ -time algorithm. The latter algorithm applies in an extended way a technique that was very recently developed for the  $k$ -mismatch problem [Bringmann et al., SODA 2019].

A preliminary version of this work appeared at FCT 2019. In this version we improve the time complexity of the main algorithm from  $\mathcal{O}(n + \frac{n}{m} k^5)$  to  $\mathcal{O}(n + \frac{n}{m} k^4)$ .

# 1 Introduction

Pattern matching is a fundamental problem in computer science [1]. It consists in finding all substrings of a text  $T$  of length  $n$  that match a pattern  $P$  of length  $m$ . In many real-world applications, a measure of similarity is usually introduced allowing for *approximate* matches between the given pattern and substrings of the text. The most widely-used similarity measure is the Hamming distance between the pattern and all length- $m$  substrings of the text.

Computing the Hamming distance between  $P$  and all length- $m$  substrings of  $T$  has been investigated for the past 30 years. The first efficient solution requiring  $\mathcal{O}(n\sqrt{m\log m})$  time was independently developed by Abrahamson [2] and Kosaraju [3] in 1987. The  $k$ -mismatch version of the problem asks for finding only the substrings of  $T$  that are close to  $P$ , specifically, at Hamming distance at most  $k$ . The first efficient solution to this problem running in  $\mathcal{O}(nk)$  time was developed in 1986 by Landau and Vishkin [4]. It took almost 15 years for a breakthrough result by Amir et al. improving this to  $\mathcal{O}(n\sqrt{k\log k})$  [5]. More recently, there has been a resurgence of interest in the  $k$ -mismatch problem. Clifford et al. gave an  $\mathcal{O}((n/m)(k^2\log k) + npolylogn)$ -time algorithm [6], which was subsequently improved further by Gawrychowski and Uznański to  $\mathcal{O}((n/m)(m + k\sqrt{m})polylogn)$  [7]. In [7], the authors have also provided evidence that any further progress in this problem is rather unlikely.

The  $k$ -mismatch problem has also been considered on compressed representations of the text [8, 9, 10, 11], in the parallel model [12], and in the streaming model [13, 6, 14]. Furthermore, it has been considered in non-standard stringology models, such as the parameterized model [15] and the order-preserving model [16].

In many real-world applications, such as in bioinformatics [17, 18, 19, 20] or in image processing [21, 22, 23, 24], any cyclic shift (rotation) of  $P$  is a relevant pattern, and thus one is interested in computing the minimal distance of every length- $m$  substring of  $T$  and any cyclic rotation of  $P$ , if this distance is no more than  $k$ . This is the circular pattern matching with  $k$  mismatches ( $k$ -CPM) problem. A multitude of papers [25, 26, 27, 28, 29, 30] have thus been devoted to solving the  $k$ -CPM problem but, to the best of our knowledge, only average-case upper bounds are known; i.e. in these works the assumption is that text  $T$  is uniformly random. The main result states that, after preprocessing pattern  $P$ , the average-case optimal search time of  $\mathcal{O}(n\frac{k+\log m}{m})$  [31] can be achieved for certain values of the error ratio  $k/m$  (see [29, 25] for more details on the preprocessing costs). Note that the exact (no mismatches allowed) version of the CPM problem can be solved as fast as exact pattern matching; namely, in  $\mathcal{O}(n)$  time [32].

In this paper, we draw our motivation from **(i)** the importance of the  $k$ -CPM problem in real-world applications and **(ii)** the fact that no (non-trivial) worst-case upper bounds are known. Trivial here refers to running the fastest-known algorithm for the  $k$ -mismatch problem [7] separately for each of the  $m$  rotations of  $P$ . This yields an  $\mathcal{O}(n(m + k\sqrt{m})polylogn)$ -time algorithm for the  $k$ -CPM problem. This is clearly unsatisfactory: it is a simple exercise to design an  $\mathcal{O}(nm)$ -time or an  $\mathcal{O}(nk^2)$ -time algorithm. In an effort to tackle this unpleasant situation, we present two much more efficient algorithms: a simple  $\mathcal{O}(nk)$ -time algorithm and an  $\mathcal{O}(n + \frac{n}{m}k^4)$ -time algorithm. Our second algorithm applies in an extended way a technique that was developed very recently for  $k$ -mismatch pattern matching in grammar compressed strings by Bringmann et al. [9]. We also show that both of our algorithms can be implemented in  $\mathcal{O}(m)$  space.

A preliminary version of this work was published as [33].

**Our approach** We first consider a simple version of the problem (called ANCHOR-MATCH) in which we are given a position in  $T$  (an *anchor*) which belongs to potential  $k$ -mismatch circular occurrences of  $P$ . A simple  $\mathcal{O}(k)$ -time algorithm is given (after linear-time preprocessing) to compute all relevant occurrences. By considering separately each position in  $T$  as an anchor we obtain an  $\mathcal{O}(nk)$ -time algorithm. The concept of an anchor is extended to the so-called *matching pairs*: when we know a pair of positions, one in  $P$  and the other in  $T$ , that are aligned. Then comes the idea of a *sample*  $\mathcal{S}$ , which is a fragment of  $P$  of length  $\Theta(m/k)$  which supposedly exactly matches a corresponding fragment in  $T$ . We choose  $\mathcal{O}(k)$  samples and work for each of them and for windows of  $T$  of size  $2m$ . As it is typical in many versions of pattern matching, our solution is split into periodic and non-periodic cases. If  $\mathcal{S}$  is non-periodic the sample occurs only  $\mathcal{O}(k)$  times in a window and each occurrence gives a matching pair (and consequently two possible anchors). Then we perform ANCHOR-MATCH for each such anchor. The hard part is the case when  $\mathcal{S}$  is periodic. Here we compute all exact occurrences of  $\mathcal{S}$  and obtain  $\mathcal{O}(k)$  groups of occurrences, each one being an arithmetic progression. Now each group is processed using the approach “few matches or almost periodicity” of Bringmann et al. [9]. In the latter case periodicity is approximate allowing up to  $k$  mismatches. Finally, we are able to decrease the exponent of  $k$  by one in the complexity using a marking trick.

## 2 Preliminaries

Let  $S = S[0]S[1] \cdots S[n-1]$  be a *string* of length  $|S| = n$  over an integer alphabet  $\Sigma$ . The elements of  $\Sigma$  are called *letters*. For two positions  $i$  and  $j$  on  $S$ , we denote by  $S[i..j] = S[i] \cdots S[j]$  the *fragment* of  $S$  that starts at position  $i$  and ends at position  $j$  (it equals the empty string  $\varepsilon$  if  $j < i$ ). A *prefix* of  $S$  is a fragment that starts at position 0, i.e. of the form  $S[0..j]$ , and a *suffix* is a fragment that ends at position  $n-1$ , i.e. of the form  $S[i..n-1]$ . For an integer  $p$ , we define the  $p$ th *power* of  $S$ , denoted by  $S^p$ , as the string obtained from concatenating  $p$  copies of  $S$ .  $S^\infty$  denotes the string obtained by concatenating infinitely many copies. If  $S$  and  $S'$  are two strings of the same length, then by  $S =_k S'$  we denote the fact that  $S$  and  $S'$  have at most  $k$  mismatches, that is, that the Hamming distance between  $S$  and  $S'$  does not exceed  $k$ .

We say that a string  $S$  has period  $q$  if  $S[i] = S[i+q]$  for all  $i = 0, \dots, |S| - q - 1$ . String  $S$  is periodic if it has a period  $q$  such that  $2q \leq |S|$ . We denote the smallest period of  $S$  by  $\text{per}(S)$ . Fine and Wilf’s periodicity lemma [34] asserts that if a string of length  $n$  has periods  $p$  and  $q$  and  $n \geq p + q - 1$ , then the string has a period  $\text{gcd}(p, q)$ .

For a string  $S$ , by  $\text{rot}_x(S)$  for  $0 \leq x < |S|$ , we denote the string that is obtained from  $S$  by moving the prefix of  $S$  of length  $x$  to its suffix. We call the string  $\text{rot}_x(S)$  (or its representation  $x$ ) a *rotation* of  $S$ . More formally, we have

$$\text{rot}_x(S) = VU, \text{ where } S = UV \text{ and } |U| = x.$$

### 2.1 Anatomy of Circular Occurrences

In what follows, we denote by  $m$  the length of the pattern  $P$  and by  $n$  the length of the text  $T$ . We say that  $P$  has a  $k$ -mismatch circular occurrence (in short  $k$ -occurrence) in  $T$  at position  $p$  if  $T[p..p+m-1] =_k \text{rot}_x(P)$  for some rotation  $x$ . In this case, the position  $x$  in the pattern is called the *split point* of the pattern and  $p + (m - x) \bmod m$ <sup>1</sup> is called the *anchor* in the text. In other

<sup>1</sup>The modulo operation is used to handle the trivial rotation with  $x = 0$ .

words, if  $P = UV$  and its rotation  $VU$  occurs in  $T$ , then the first position of  $V$  in  $P$  is the split point of this occurrence, and the first position of  $U$  in  $T$  is the anchor of this occurrence (see Fig. 1).

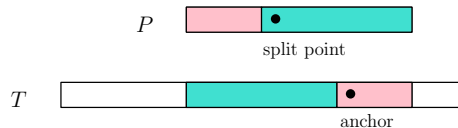


Figure 1: The split point and the anchor for a  $k$ -occurrence of  $P$  in  $T$ .

The main problem in scope can now be stated as follows.

**$k$ -CPM PROBLEM**

**Input:** Text  $T$  of length  $n$ , pattern  $P$  of length  $m$ , and positive integer  $k$ .

**Output:** All positions of  $T$  that contain a  $k$ -occurrence of  $P$ .

For an integer  $z$ , let us denote  $\mathbf{W}_z = [z \dots z + m - 1]$  (*window of size  $m$* ). Intuitively, this window corresponds to a length- $m$  fragment of the text  $T$ . For a  $k$ -occurrence at position  $p$  of  $T$  with rotation  $x$ , we introduce a set of pairs of positions in the fragment of the text and the corresponding positions from the original (unrotated) pattern  $P$ :

$$M(p, x) = \{(i, (i - p + x) \bmod m) : i \in \mathbf{W}_p\}.$$

The pairs  $(i, j) \in M(p, x)$  are called *matching pairs* of an occurrence  $p$  with rotation  $x$ . In particular,  $(p + ((m - x) \bmod m), 0) \in M(p, x)$ . An example is provided in Fig. 2.

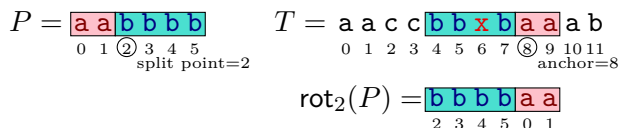


Figure 2: A 1-occurrence of  $P = \text{aabbbb}$  in text  $T = \text{aaccbbxbaaab}$  at position  $p = 4$  with rotation  $x = 2$ ;  $M(4, 2) = \{(4, 2), (5, 3), (6, 4), (7, 5), (8, 0), (9, 1)\}$ .

### 3 An $\mathcal{O}(nk)$ -time Algorithm

We first introduce an auxiliary problem in which one wants to compute all  $k$ -occurrences of  $P$  in  $T$  with a given anchor  $\mathbf{a}$ . This problem describes the core computational task in our first solution.

**ANCHOR-MATCH PROBLEM**

**Input:** Text  $T$  of length  $n$ , pattern  $P$  of length  $m$ , positive integer  $k$ , and position  $\mathbf{a}$ .

**Output:** All  $k$ -occurrences  $p$  of  $P$  in  $T$  with anchor  $\mathbf{a}$ , represented as a collection of  $\mathcal{O}(k)$  intervals.

For a string  $X$  let us denote by  $X_{(i)}$  and  $X^{(i)}$  its fragments of length  $m$  starting at position  $i$  and ending at position  $i$ , respectively. Moreover, for a binary string  $X$ , by  $\|X\|$  we denote the arithmetic sum of characters in  $X$ . We define the following auxiliary problem.

LIGHT-FRAGMENTS PROBLEM

**Input:** Positive integers  $m, k$  and a string  $V$  of length  $n$  over alphabet  $\{0, 1\}$  containing  $\mathcal{O}(k)$  non-zero characters. The string  $V$  is specified by its positions with non-zero characters (sorted increasingly).

**Output:** The set  $A = \{i : \|V_{(i)}\| \leq k\}$  represented as a collection of  $\mathcal{O}(k)$  intervals.

**Lemma 1.** *The LIGHT-FRAGMENTS problem can be solved in  $\mathcal{O}(k)$  time.*

*Proof.* Let  $I$  be the set of positions of  $V$  with non-zero characters;  $|I| = \mathcal{O}(k)$  by definition. We define a piecewise constant function  $f$  on integers such that  $f(x) = |I \cap [0, x]|$ . Let  $g(x) = f(x + m - 1) - f(x - 1)$ . Then  $g(x) = |I \cap [x, x + m - 1]|$ . Function  $f$  has  $\mathcal{O}(k)$  pieces, so  $g$  has  $\mathcal{O}(k)$  pieces as well and both can be computed in  $\mathcal{O}(k)$  time since  $I$  is sorted. In the end we report the pieces where  $g$  has value at most  $k$ .  $\square$

Let us recall a standard algorithmic tool for preprocessing text  $T$ . We denote the length of the longest common prefix (resp. suffix) of two strings  $U$  and  $V$  by  $\text{lcp}(U, V)$  (resp.  $\text{lcs}(U, V)$ ). There is an  $\mathcal{O}(n)$ -sized data structure answering such queries over suffixes (resp. prefixes) of  $T$  in  $\mathcal{O}(1)$  time after  $\mathcal{O}(n)$ -time preprocessing. It consists of the suffix array of  $T$  and a data structure for answering range minimum queries; see [1]. Using the kangaroo method [4, 12], these queries can handle mismatches; after an  $\mathcal{O}(n)$ -time preprocessing of  $T$ , longest common prefix (resp. suffix) queries with up to  $k$  mismatches can be answered in  $\mathcal{O}(k)$  time.

**Lemma 2.** *After  $\mathcal{O}(n)$ -time preprocessing, the answer to ANCHOR-MATCH problem, represented as a union of  $\mathcal{O}(k)$  intervals, can be computed in  $\mathcal{O}(k)$  time.*

*Proof.* In the preprocessing we prepare a data structure for  $\text{lcp}$  queries in  $P\#T$ , for a special character  $\#$  that occurs neither in  $P$  nor in  $T$ .

Consider now a query for an anchor  $\mathbf{a}$  over  $T$ . Let  $L = T^{(\mathbf{a}^{-1})}$  and  $R = T_{(\mathbf{a})}$ . We define a binary string  $L'$  such that  $L'[i] = 1$  if and only if  $L[i] \neq P[i]$ . We define  $R'$  analogously. Let  $L''$  be the longest suffix of  $L'$  such that  $\|L''\| \leq k$ . Let  $R''$  be the longest prefix of  $R'$  such that  $\|R''\| \leq k$ .

Using the kangaroo method [4, 12], the strings  $L'', R''$  can be constructed in  $\mathcal{O}(k)$  time. The ANCHOR-MATCH problem now reduces to the LIGHT-FRAGMENTS problem for string  $V = L''R''$ .  $\square$

For an illustration of Lemma 2 inspect Fig. 3.

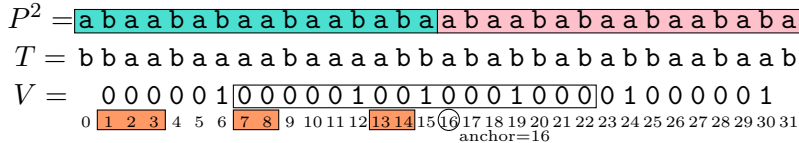


Figure 3: An illustration of the setting in Lemma 2 with  $P = (\text{abaababa})^2$ , the text as in the figure, anchor  $\mathbf{a} = 16$  and  $k = 3$ . The string  $V$ , used in the proof of the lemma, shows the positions of the at most  $k$  mismatches to the left and to the right of the anchor. The output consists of the three intervals  $[1, 3]$ ,  $[7, 8]$  and  $[13, 14]$  shown in orange. For example, 7 is an occurrence since the fragment of  $V$  of length  $|P|$  starting at this position (represented by a rectangle) contains at most  $k$  ones.

**Proposition 1.**  *$k$ -CPM can be solved in  $\mathcal{O}(nk)$  time and  $\mathcal{O}(n)$  space.*

*Proof.* We invoke the algorithm of Lemma 2 for all  $\mathbf{a} \in [0..n-1]$  and obtain  $\mathcal{O}(nk)$  intervals of  $k$ -occurrences of  $P$  in  $T$ . Instead of storing all the intervals, we count how many intervals start and end at each position of  $T$ . We can then compute the union of the intervals by processing these counts from left to right.  $\square$

## 4 Algorithmic tools

In this section we introduce further algorithmic tools to get to our second solution.

### 4.1 Internal Queries in a Text

Let  $T$  be a string of length  $n$  called text. An Internal Pattern Matching (IPM) query, for two given fragments  $F$  and  $G$  of the text, such that  $|G| \leq 2|F|$ , computes the set of all occurrences of  $F$  in  $G$ . If there are more than two occurrences, they form an arithmetic sequence with difference  $\text{per}(F)$ . A data structure for IPM queries in  $T$  can be constructed in  $\mathcal{O}(n)$  time and answers queries in  $\mathcal{O}(1)$  time (see [35] and [36, Theorem 1.1.4]). It can be used to compute all occurrences of a given fragment  $F$  of length  $p$  in  $T$ , expressed as a union of  $\mathcal{O}(n/p)$  pairwise disjoint arithmetic sequences with difference  $\text{per}(F)$ , in  $\mathcal{O}(n/p)$  time.

### 4.2 Simple Geometry of Arithmetic Sequences of Intervals

We next present algorithms that will be used in subsequent proofs for handling regular sets of intervals.

For an interval  $I$  and an integer  $r$ , let  $I \oplus r = \{i + r : i \in I\}$ . We define

$$\text{Chain}_q(I, a) = I \cup (I \oplus q) \cup (I \oplus 2q) \cup \dots \cup (I \oplus aq).$$

This set is further called an *interval chain* (with difference  $q$ ). Note that it can be represented in  $\mathcal{O}(1)$  space using four integers:  $a$ ,  $q$ , and the endpoints of  $I$ . An illustration of an interval chain representing the output for a problem defined in the next subsection can be found in Fig. 4.

For a given value of  $q$ , let us fit the integers from  $[1..n]$  into the cells of a grid of width  $q$  so that the first row consists of numbers 1 through  $q$ , the second of numbers  $q+1$  to  $2q$ , etc. Let us call this grid  $\mathcal{G}_q$ . A chain  $\text{Chain}_q$  can be conveniently represented in the grid  $\mathcal{G}_q$  using the following lemma from [37].

**Lemma 3** ([37]). *The set  $\text{Chain}_q(I, a)$  is a union of  $\mathcal{O}(1)$  orthogonal rectangles in  $\mathcal{G}_q$ . The coordinates of the rectangles can be computed in  $\mathcal{O}(1)$  time.*

Lemma 4 can be used to compute a union of interval chains.

**Lemma 4.** *Given  $c$  interval chains, all of which have difference  $q$  and are subsets of  $[0..n]$ , the union of these chains, expressed as a subset of  $[0..n]$ , can be computed in  $\mathcal{O}(n+c)$  time.*

*Proof.* By Lemma 3, the problem reduces to computing the union of  $\mathcal{O}(c)$  rectangles on a grid of total size  $n$ . Let  $t$  be a 2D array of the same shape as  $\mathcal{G}_q$ , initially set to zeroes. For a rectangle with opposite corners  $(x_1, y_1)$  and  $(x_2, y_2)$ , with  $x_1 \leq x_2$  and  $y_1 \leq y_2$ , we increment  $t[x_1, y_1]$ , decrement

$t[x_2 + 1, y_1]$  and  $t[x_1, y_2 + 1]$ , and increment  $t[x_2 + 1, y_2 + 1]$  (provided that the respective cells are within the array). This takes  $\mathcal{O}(c)$  time. We then compute prefix sums of  $t$ , which are defined as

$$t'[x, y] = \sum_{i=1}^x \sum_{j=1}^y t[i, j].$$

Such values can be computed in time proportional to the size of the grid, i.e., in  $\mathcal{O}(n)$  time. Finally, we note that  $(x, y)$  is contained in  $t'[x, y]$  rectangles, concluding the proof.  $\square$

**Remark 1.** The proof of Lemma 4 is essentially based on an idea that was used, for example, for reducing the decision version of range stabbing queries in 2D to weighted range counting queries in 2D (cf. [38]).

We will also use the following auxiliary lemma.

**Lemma 5.** *Let  $X$  and  $Z$  be intervals and  $q$  be a positive integer. The set*

$$Z' := \{z \in Z : \exists_{x \in X} z \equiv x \pmod{q}\},$$

*represented as a disjoint sum of at most three interval chains, each with difference  $q$ , can be computed in  $\mathcal{O}(1)$  time.*

*Proof.* If  $|X| \geq q$ , then  $Z' = Z$  is an interval and thus an interval chain. If  $|X| < q$ , then  $Z'$  can be divided into disjoint intervals of length smaller than or equal to  $|X|$ . The intervals from the second until the penultimate one (if any such exist), have length  $|X|$ . Hence, they can be represented as a single chain, as the first element of each such interval is equal mod  $q$  to the first element of  $X$ . The two remaining intervals can be treated as chains as well.  $\square$

### 4.3 The Aligned-Light-Sum Problem

We define the following abstract problem that resembles the LIGHT-FRAGMENTS problem from Section 3.

ALIGNED-LIGHT-SUM PROBLEM

**Input:** Positive integers  $m, k, q$  and strings  $U, V$  over alphabet  $\{0, 1\}$ , each containing  $\mathcal{O}(k)$  non-zero characters. The strings are specified by their positions with non-zero characters.

**Output:** The set  $A = \{i : (\exists j) \|U_{(i)}\| + \|V_{(j)}\| \leq k \wedge j \equiv i \pmod{q}\}$ .

$$\begin{array}{l}
 U = 0100000000000000 \\
 V = 000000000000000100000000000000 \\
 \quad \boxed{0\ 1}\ 2\ 3\ \boxed{4\ 5}\ 6\ 7\ \boxed{8\ 9}\ 10\ 11\ \boxed{12\ 13}\ 14\ 15\ 16\ 17\ 18\ 19\ 20\ 21\ 22\ 23\ 24\ 25\ 26\ 27\ 28
 \end{array}$$

Figure 4: An instance of the ALIGNED-LIGHT-SUM problem with  $m = 15$ ,  $k = 2$ ,  $q = 4$ ,  $U = 010^{14}$  and  $V = 0^{14}10^{14}$ . The output is the interval chain  $\text{Chain}_4([0, 1], 3)$  shown in orange.

**Lemma 6.** *The ALIGNED-LIGHT-SUM problem can be solved in  $\mathcal{O}(k^2)$  time with the output represented as a collection of  $\mathcal{O}(k^2)$  interval chains, each with difference  $q$ .*



*Proof.* Let  $I$  and  $I'$  be the positions with non-zero characters in  $U'$  and  $V'$ , respectively. We partition the set  $\{0, \dots, |U'| - m\}$  into intervals such that for all  $j$  in an interval, the set  $\mathbf{W}_j \cap I$  is the same. For this, we use a sliding window approach. We generate events corresponding to  $x$  and  $x - m + 1$  for all  $x \in I$  and sort them. When  $j$  crosses an event, the set  $\mathbf{W}_j \cap I$  changes. Thus we obtain a partition of  $\{0, \dots, |U'| - m\}$  into intervals  $Z_1, \dots, Z_{n_1}$ . We obtain a similar partition of  $\{0, \dots, |V'| - m\}$  into intervals  $Z'_1, \dots, Z'_{n_2}$ . We have  $n_1, n_2 = \mathcal{O}(k)$ .

Let us now fix  $Z_j$  and  $Z'_{j'}$ . First we check if the condition on the total number of non-zero characters is satisfied for any  $z \in Z_j$  and  $z' \in Z'_{j'}$ . If so, we compute the set  $Z'_{j'} \bmod q = \{z' \bmod q : z' \in Z'_{j'}\}$ . It is a single circular interval and can be computed in constant time. The required result is

$$\{z \in Z_j : z \bmod q \in X\}.$$

By Lemma 5, this set can be represented as a union of three chains, each with difference  $q$  and, as such, can be computed in  $\mathcal{O}(1)$  time. The conclusion follows.  $\square$

## 5 An $\mathcal{O}(n + k^5)$ -time Algorithm for Short Texts

In this section we proceed by assuming that  $m \leq n \leq 2m$  and aim at an  $\mathcal{O}(n + k^5)$ -time algorithm. In the next sections we remove this assumption and reduce the exponent of  $k$  to 4.

A (*deterministic*) *sample* is a short fragment  $\mathcal{S}$  of the pattern  $P$ . An occurrence in the text without any mismatch is called *exact*. We introduce a problem of SAMPLE-MATCH that consists in finding all  $k$ -occurrences of  $P$  in  $T$  such that  $\mathcal{S}$  matches exactly a fragment of length  $|\mathcal{S}|$  in  $T$ .

We split the pattern  $P$  into  $2k+3$  fragments of length  $\lfloor \frac{m}{2k+3} \rfloor$  or  $\lceil \frac{m}{2k+3} \rceil$  each. In any  $k$ -occurrence of  $P$  in  $T$  at least  $k+2$  of those fragments will occur exactly in  $T$  (up to  $k$  fragments may occur with one mismatch and at most one fragment will contain the split point).

**Remark 2.** We force at least  $k+2$  fragments (instead of just one) to match exactly for two reasons: (1) in order to have more than a half of them match exactly, which will guarantee that the interval chains that are obtained from applications of the ALIGNED-LIGHT-MATCH problem have the same difference and thus can be unioned using Lemma 4 (see the proof of Proposition 2); and (2) for the marking trick in the next section.

Let us henceforth fix a sample  $\mathcal{S}$  as one of these fragments, let  $p_{\mathcal{S}}$  be its starting position in  $P$ , and let  $m_{\mathcal{S}} = |\mathcal{S}|$ . We assume that the split point  $x$  in  $P$  is to the right of  $\mathcal{S}$ , i.e., that  $x \geq p_{\mathcal{S}} + m_{\mathcal{S}}$ . The opposite case—that  $x < p_{\mathcal{S}}$ —can be handled analogously.

### 5.1 Matching Non-Periodic Samples

Let us assume that  $\mathcal{S}$  is non-periodic. Further let  $j$  denote the starting position of  $\mathcal{S}$  in  $P$  and  $i$  denote a starting position of an occurrence of  $\mathcal{S}$  in  $T$ . We introduce a problem in which, intuitively, we compute all  $k$ -occurrences of  $P$  in  $T$  which align  $T[i]$  with  $P[j]$ .

#### PAIR-MATCH PROBLEM

**Input:** Text  $T$  of length  $n$ , pattern  $P$  of length  $m$ , positive integer  $k$ , and two integers  $i \in [0..n-1]$  and  $j \in [0..m-1]$ .

**Output:** The set  $A(i, j)$  of all positions in  $T$  where we have a  $k$ -mismatch occurrence of  $\text{rot}_x(P)$  for some  $x$  such that  $(i, j)$  is a matching pair.

We show how to solve the SAMPLE-MATCH problem for a non-periodic sample  $\mathcal{S}$  in  $\mathcal{O}(k^2)$  time in two steps. First, we show how to solve the PAIR-MATCH problem for a given  $(i, j)$  in  $\mathcal{O}(k)$  time. We then apply this solution for each occurrence  $i$  of  $\mathcal{S}$  in  $T$  by noticing that we can have  $\mathcal{O}(k)$  such occurrences in total.

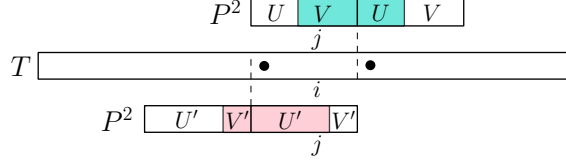


Figure 5: The two possible anchors for the matching pair of positions  $(i, j)$  are shown as bullet points. A possible  $k$ -occurrence of  $P$  in  $T$  corresponding to the left (resp. right) anchor is shown below  $T$  (above  $T$ , resp.). Note that  $P^2 = PP$ .

**Lemma 7.** *After  $\mathcal{O}(n)$ -time preprocessing, the PAIR-MATCH problem can be solved in  $\mathcal{O}(k)$  time, where the output is represented as a union of  $\mathcal{O}(k)$  intervals.*

*Proof.* Recall that the ANCHOR-MATCH problem returns all  $k$ -occurrences of  $P$  in  $T$  with a given anchor. The PAIR-MATCH problem can be essentially reduced to the ANCHOR-MATCH problem, since for a given matching pair of characters in  $P$  and  $T$ , there are at most two ways of choosing the anchor depending on the relation between  $j$  and a split point: these are  $i - j$  and  $i + |P| - j$  (see Fig. 5). Clearly, we choose  $i - j$  as an anchor only if  $i - j \geq 0$  and  $i + |P| - j$  only if  $i + |P| - j < |T|$ . We then have to take the intersection of the answer with  $[i - m + 1 . i]$  to ensure that the  $k$ -occurrence contains position  $i$ .  $\square$

**Lemma 8.** *After  $\mathcal{O}(n)$ -time preprocessing, the SAMPLE-MATCH problem for a non-periodic sample  $\mathcal{S}$  can be solved in  $\mathcal{O}(k^2)$  time and outputs a union of  $\mathcal{O}(k^2)$  intervals of occurrences.*

*Proof.* Since  $\mathcal{S}$  is non-periodic, it has  $\mathcal{O}(k)$  occurrences in  $T$ , which can be computed in  $\mathcal{O}(k)$  time after an  $\mathcal{O}(n)$ -time preprocessing using IPM queries [35, 36] in  $P\#T$ . Let  $j$  be the starting position of  $\mathcal{S}$  in  $P$  and  $i$  be a starting position of an occurrence of  $\mathcal{S}$  in  $T$ . For each of the  $\mathcal{O}(k)$  such pairs  $(i, j)$ , the computation reduces to the PAIR-MATCH problem for  $i$  and  $j$ . The statement follows by Lemma 7.  $\square$

## 5.2 Matching Periodic Samples

Let us assume that  $\mathcal{S}$  is periodic, i.e., it has a period  $q$  with  $2q \leq |\mathcal{S}|$ . A fragment of a string  $S$  containing an inclusion-maximal arithmetic sequence of occurrences of  $\mathcal{S}$  in  $S$  with difference  $q$  is called here an  $\mathcal{S}$ -run. If  $\mathcal{S}$  matches a fragment in the text, then the match belongs to an  $\mathcal{S}$ -run. For example, the underlined fragment of  $S = \mathbf{bbabababaa}$  is an  $\mathcal{S}$ -run for  $\mathcal{S} = \mathbf{abab}$ .

**Lemma 9.** *If  $\mathcal{S}$  is periodic, the number of  $\mathcal{S}$ -runs in the text is  $\mathcal{O}(k)$  and they can all be computed in  $\mathcal{O}(k)$  time after  $\mathcal{O}(n)$ -time preprocessing.*

*Proof.* We construct the data structure for IPM queries on  $P\#T$ . This allows us to compute the set of all occurrences of  $\mathcal{S}$  in  $T$  as a collection of  $\mathcal{O}(k)$  arithmetic sequences with difference  $\text{per}(\mathcal{S})$ . We then check for every two consecutive sequences if they can be joined together. This takes  $\mathcal{O}(k)$  time and results in  $\mathcal{O}(k)$   $\mathcal{S}$ -runs.  $\square$

For two equal-length strings  $S$  and  $S'$ , we denote the set of their *mismatches* by

$$\text{Mis}(S, S') = \{i = 0, \dots, |S| - 1 : S[i] \neq S'[i]\}.$$

We say that position  $a$  in  $S$  is a *misperiod* with respect to the fragment  $S[i..j]$  if  $S[a] \neq S[b]$  where  $b$  is the unique position such that  $b \in [i..j]$  and  $(j - i + 1) \mid (b - a)$ .

We define the set  $\text{LeftMisper}_k(S, i, j)$  as the set of  $k$  maximal misperiods that are smaller than  $i$  and  $\text{RightMisper}_k(S, i, j)$  as the set of  $k$  minimal misperiods that are greater than  $j$ . Each of the sets can have less than  $k$  elements if the corresponding misperiods do not exist. We further define

$$\text{Misper}_k(S, i, j) = \text{LeftMisper}_k(S, i, j) \cup \text{RightMisper}_k(S, i, j)$$

and  $\text{Misper}(S, i, j) = \bigcup_{k=0}^{\infty} \text{Misper}_k(S, i, j)$ .

The following lemma captures a combinatorial property behind the new technique of Bringmann et al. [9]. The intuition is shown in Fig. 6.

**Lemma 10.** *Assume that  $S =_k S'$  and that  $S[i..j] = S'[i..j]$ . Let*

$$I = \text{Misper}_{k+1}(S, i, j) \text{ and } I' = \text{Misper}_{k+1}(S', i, j).$$

*If  $I \cap I' = \emptyset$ , then  $\text{Mis}(S, S') = I \cup I'$ ,  $I = \text{Misper}(S, i, j)$ , and  $I' = \text{Misper}(S', i, j)$ .*

*Proof.* Let  $J = \text{Misper}(S, i, j)$  and  $J' = \text{Misper}(S', i, j)$ . We first observe that  $I \cup I' \subseteq \text{Mis}(S, S')$  since  $I \cap I' = \emptyset$ . Then,  $S =_k S'$  implies that  $|\text{Mis}(S, S')| \leq k$  and hence  $|I| \leq k$  and  $|I'| \leq k$ , which in turn implies that  $I = J$  and  $I' = J'$ . The observation that  $\text{Mis}(S, S') \subseteq J \cup J'$  concludes the proof.  $\square$

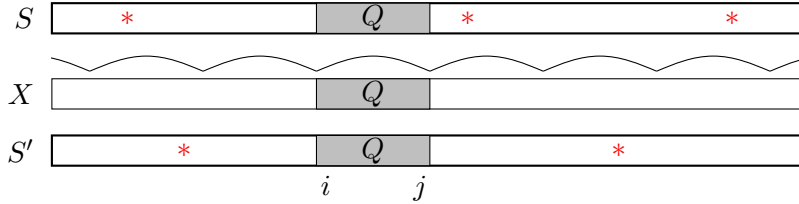


Figure 6: Let  $S$ ,  $S'$ , and  $X$  be equal-length strings such that  $X$  is a factor of  $Q^\infty$  and  $S[i..j] = S'[i..j] = X[i..j] = Q$ . The asterisks in  $S$  denote the positions in  $\text{Mis}(S, X)$ , or equivalently, the misperiods with respect to  $S[i..j]$ . Similarly for  $S'$ . One can observe that  $\text{Mis}(S, X) \cap \text{Mis}(S', X) = \emptyset$  and that  $\text{Mis}(S, X) \cup \text{Mis}(S', X) = \text{Mis}(S, S')$ .

A string  $S$  is *k-periodic w.r.t. an occurrence  $i$  of  $Q$*  if  $|\text{Misper}(S, i, i + |Q| - 1)| \leq k$ . In this case,  $|Q|$  is called the *k-period*. In particular, in the conclusion of the above lemma  $S$  and  $S'$  are  $|I|$ -periodic and  $|I'|$ -periodic, respectively, w.r.t.  $Q = S[i..j] = S'[i..j]$ . This notion forms the basis of the following auxiliary problem in which we search for  $k$ -occurrences in which the rotation of the pattern and the fragment of the text are  $k$ -periodic for the same period  $Q$ .

Let  $U$  and  $V$  be two strings and  $J$  and  $J'$  be sets containing positions in  $U$  and  $V$ , respectively. We say that length- $m$  fragments  $U[p..p + m - 1]$  and  $V[x..x + m - 1]$  are  $(J, J')$ -disjoint if the sets  $(\mathbf{W}_p \cap J) \ominus p$  and  $(\mathbf{W}_x \cap J') \ominus x$  are disjoint. For example, if  $J = \{2, 4, 11, 15, 16, 17\}$ ,  $J' = \{5, 6, 15, 18, 19\}$ , and  $m = 12$ , then  $U[3..14]$  and  $V[6..17]$  are  $(J, J')$ -disjoint for:

$$\begin{array}{r}
U = \quad ab \bullet \boxed{a \bullet b \ abc \ ab \bullet \ abc} \bullet \bullet \bullet \\
V = \ abc \ ab \bullet \boxed{\bullet bc \ abc \ abc \ \bullet bc} \bullet \bullet c
\end{array}$$

Let us introduce an auxiliary problem that is obtained in the case that is shown in the conclusion of the above lemma (i.e., misperiods in the rotation of  $P$  and the corresponding fragment of  $T$  are not aligned); see also Fig. 7.

**PERIODIC-PERIODIC-MATCH PROBLEM**

**Input:** A string  $U$  which is  $2k$ -periodic w.r.t. an exact occurrence  $i$  of a length- $q$  string  $Q$  and a string  $V$  which is  $2k$ -periodic w.r.t. an exact occurrence  $i'$  of the same string  $Q$  such that  $m \leq |U|, |V| \leq 2m$  and

$$J = \text{Misper}(U, i, i + q - 1), \quad J' = \text{Misper}(V, i', i' + q - 1).$$

(The strings  $U$  and  $V$  are not stored explicitly.)

**Output:** The set of positions  $p$  in  $U$  for which there exists a  $(J, J')$ -disjoint  $k$ -occurrence  $U[p..p + m - 1]$  of  $V[x..x + m - 1]$  for  $x$  such that

$$i - p \equiv i' - x \pmod{q}.$$

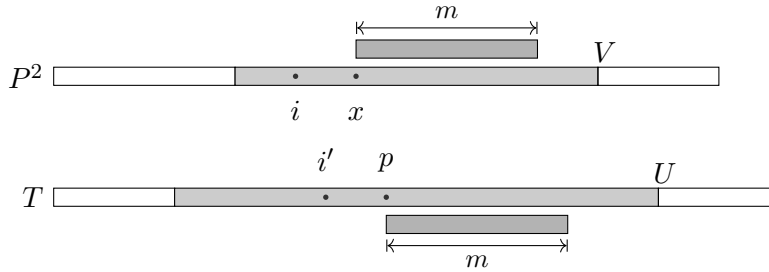


Figure 7: In the case of a periodic sample, we have two  $2k$ -periodic fragments  $U$  and  $V$ . We find all  $p$  in  $U$  such that for some position  $x$  in  $V$ , the fragments of length  $m$  starting at positions  $p$  and  $x$  are at Hamming distance at most  $k$ . If  $q$  is a period, then  $i - p \equiv i' - x \pmod{q}$  due to synchronization of periodicities.

Intuitively, the modulo condition on the output of the PERIODIC-PERIODIC-MATCH problem corresponds to the fact that the approximate periodicity is aligned.

In the PERIODIC-PERIODIC-MATCH problem we search for  $k$ -occurrences in which none of the misperiods in  $J$  and  $J'$  are aligned. In this case each of the misperiods accounts for one mismatch in the  $k$ -occurrence. In the lemma below we reduce the PERIODIC-PERIODIC-MATCH problem to the ALIGNED-LIGHT-SUM problem, in which we only require that the total number of misperiods in an occurrence is at most  $k$ . This way all the  $(J, J')$ -disjoint  $k$ -occurrences can be found. Also additional occurrences where two misperiods are aligned can be reported, but they are still valid  $k$ -occurrences (actually,  $k'$ -occurrences for some  $k' < k$ ).

**Lemma 11.** *We can compute in  $\mathcal{O}(k^2)$  time a set of  $k$ -occurrences of  $P$  in  $T$  represented as  $\mathcal{O}(k^2)$  interval chains, each with difference  $q$ , that is a superset of the solution to the PERIODIC-PERIODIC-MATCH problem.*

*Proof.* In the PERIODIC-PERIODIC-MATCH problem the modulo condition forces the exact occurrences of the approximate period to match. Hence, it guarantees that all the positions except the

misperiod positions match. Now the ALIGNED-LIGHT-SUM problem highlights these positions inside the string.

**Claim.** PERIODIC-PERIODIC-MATCH can be reduced in  $\mathcal{O}(k)$  time to the ALIGNED-LIGHT-SUM problem so that we obtain a superset of the desired result. The potential extra positions do not satisfy only the  $(J, J')$ -disjointness condition.

*Proof.* Let the parameters  $m, k$  and  $q$  remain unchanged. We create strings  $U'$  and  $V'$  of length  $|U|$  and  $|V|$ , respectively, with positions with non-zero characters in the sets  $I$  and  $I' = J'$ , respectively. Then we prepend  $U'$  with  $z = (i' - i) \bmod q$  zeros. Let  $A$  be the solution to the ALIGNED-LIGHT-SUM problem for  $U'$  and  $V'$ . Then  $(A \ominus z) \cap \mathbb{Z}_{\geq 0}$  is a superset of the solution to PERIODIC-PERIODIC-MATCH; the elements of the set that correspond to matches where non-zero elements of the strings  $U', V'$  were aligned do not satisfy the disjointness condition.  $\square$

Now the thesis follows from Lemma 6.  $\square$

Let us further define

$$\text{PAIRS-MATCH}(T, I, P, J) = \bigcup_{i \in I, j \in J} \text{PAIR-MATCH}(T, i, P, j).$$

Let  $A$  be a set of positions in a string  $S$  and  $m$  be a positive integer. We then denote  $A \bmod m = \{a \bmod m : a \in A\}$  and by  $\text{frag}_A(S)$  we denote the fragment  $S[\min(A) \dots \max(A)]$ . We provide pseudocode for an algorithm that computes all  $k$ -occurrences of  $P$  such that  $\mathcal{S}$  matches a fragment of a given  $\mathcal{S}$ -run (see Algorithm 1); inspect also Fig. 8.

**Data:** A periodic fragment  $\mathcal{S}$  of pattern  $P$ , an  $\mathcal{S}$ -run  $R$  in text  $T$ ,  $q = \text{per}(\mathcal{S})$ , and  $k$ .  
**Result:** A compact representation of  $k$ -occurrences of  $P$  in  $T$  including all  $k$ -occurrences where  $\mathcal{S}$  in  $P$  matches a fragment of  $R$  in  $T$ .  
Let  $R = T[s \dots s + |R| - 1]$ ;  
 $J := \text{Misper}_{k+1}(T, s, s + q - 1)$ ;  $\{ \mathcal{O}(k) \text{ time} \}$   
 $J' := \text{Misper}_{k+1}(P^2, m + p_{\mathcal{S}}, m + p_{\mathcal{S}} + q - 1)$ ;  $\{ \mathcal{O}(k) \text{ time} \}$   
 $U := \text{frag}_J(T)$ ;  $V := \text{frag}_{J'}(P^2)$ ;  
 $Y := \text{PERIODIC-PERIODIC-MATCH}(U, V)$ ;  $\{ \mathcal{O}(k^2) \text{ time} \}$   
 $Y := Y \oplus \min(J)$ ;  
 $J' := J' \bmod m$ ;  
 $X := \text{PAIRS-MATCH}(T, J, P, J')$ ;  $\{ \mathcal{O}(k^3) \text{ time} \}$   
**return**  $X \cup Y$ ;

**Algorithm 1:** Run-Sample-Matching

**Lemma 12.** After  $\mathcal{O}(n)$ -time preprocessing, algorithm Run-Sample-Matching works in  $\mathcal{O}(k^3)$  time and returns a compact representation that consists of  $\mathcal{O}(k^3)$  intervals and  $\mathcal{O}(k^2)$  interval chains, each with difference  $q$ . Moreover, if there is at least one interval chain, then some rotation of the pattern  $P$  is  $k$ -periodic with a  $k$ -period  $\text{per}(\mathcal{S})$ .

*Proof.* See Algorithm 1. The sets  $J$  and  $J'$  can be computed in  $\mathcal{O}(k)$  time:

**Claim.** If  $S$  is a string of length  $n$ , then the sets  $\text{RightMisper}_k(S, i, j)$  and  $\text{LeftMisper}_k(S, i, j)$  can be computed in  $\mathcal{O}(k)$  time after  $\mathcal{O}(n)$ -time preprocessing.

*Proof.* For  $\text{RightMisper}_k(S, i, j)$ , we use the kangaroo method [4, 12] to compute the longest common prefix with at most  $k$  mismatches of  $S[j+1..n-1]$  and  $U^\infty$  for  $U = S[i..j]$ . The value  $\text{lcp}(X^\infty, Y)$  for a fragment  $X$  and a suffix  $Y$  of a string  $S$ , occurring at positions  $a$  and  $b$ , respectively, can be computed in constant time as follows. If  $\text{lcp}(S[a..n-1], S[b..n-1]) < |X|$  then we are done. Otherwise the answer is given by  $|X| + \text{lcp}(S[b..n-1], S[b+|X|..n-1])$ . The computations for  $\text{LeftMisper}_k(S, i, j)$  are symmetric.  $\square$

The  $\mathcal{O}(k^3)$  and  $\mathcal{O}(k^2)$  time complexities of computing  $X$  and  $Y$  follow from Lemmas 7 and 11, respectively (after  $\mathcal{O}(n)$ -time preprocessing). The sets  $X$  and  $Y$  consist of  $\mathcal{O}(k^3)$  intervals and  $\mathcal{O}(k^2)$  interval chains, each with difference  $q$ .

As for the “moreover” statement, by Lemma 10, if any occurrence  $q$  is reported in the PERIODIC-PERIODIC-MATCH problem, then it implies the existence of  $x$  such that  $V[x..x+m-1]$  is  $k$ -periodic with a  $k$ -period  $q$ . However,  $V[x..x+m-1]$  is a rotation of the pattern  $P$ . This concludes the proof.  $\square$

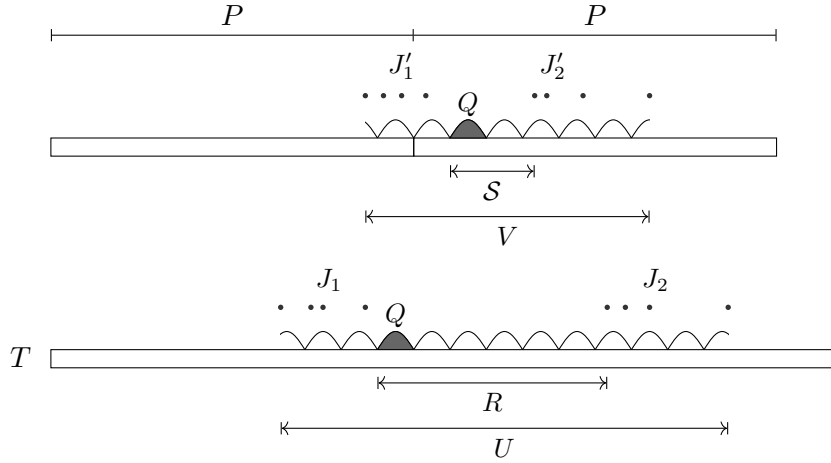


Figure 8: Detailed setting in Algorithm 1;  $J = J_1 \cup J_2$ ,  $J' = J'_1 \cup J'_2$ .

The correctness of the algorithm follows from Lemma 10, as shown in the lemma below.

**Lemma 13.** Assume  $n \leq 2m$ . Let  $\mathcal{S}$  be a periodic sample in  $P$  with smallest period  $q$  and  $R$  be an  $\mathcal{S}$ -run in  $T$ . Let  $X$  and  $Y$  be defined as in the pseudocode of *Run-Sample-Matching*. Then  $X \cup Y$  is a set of  $k$ -occurrences of  $P$  in  $T$  which is a superset of the solution to *SAMPLE-MATCH* for  $\mathcal{S}$  in  $R$ .

*Proof.* Both *PAIR-MATCH* and *PERIODIC-PERIODIC-MATCH* problems return positions of  $k$ -occurrences of  $P$  in  $T$ . Assume that  $\text{rot}_x(P)$ , for  $x \geq ps + m_S$ , has a  $k$ -mismatch occurrence in  $T$  at position  $p$  such that the designated fragment  $\mathcal{S}$  matches a fragment of  $R$  exactly. Hence, it suffices to show that  $p \in X \cup Y$ .

Let  $J = \text{Misper}_{k+1}(T, s, s + q - 1)$  and  $J' = \text{Misper}_{k+1}(P^2, m + p_S, m + p_S + q - 1)$ . We define  $L_1$  and  $L_2$  as the subsets of  $J$  and  $J'$ , respectively, that are relevant for this  $k$ -occurrence, i.e.,

$$L_1 = J \cap \mathbf{W}_p, \quad L_2 = J' \cap \mathbf{W}_x.$$

Further let  $L'_2 = L_2 \bmod m$ . If any  $i \in L_1$  and  $j \in L'_2$  are a matching pair for this  $k$ -occurrence, then it will be found in the PAIRS-MATCH problem, i.e.  $p \in X$ . Let us henceforth consider the opposite case.

Let  $S = T[p..p + m - 1]$ ,  $S' = \text{rot}_x(P)$ , and  $i = m - x + p_S$  be the starting position of  $Q = S[0..q - 1]$  in both strings. Further let  $I = L_1 \ominus p$  and  $I' = L_2 \ominus x$ . We have that  $I \cap I' = \emptyset$  by our assumption that misperiods do not align. We make the following claim.

**Claim.**  $\text{Mis}(S, S') = I \cup I'$ .

*Proof.* Note that  $I = \text{Misper}_{k+1}(S, i, i + q - 1)$  and  $I' = \text{Misper}_{k+1}(S', i, i + q - 1)$ . The latter equality follows from the fact that  $\text{Misper}_{k+1}(T, s, s + q - 1) = \text{Misper}_{k+1}(T, t, t + q - 1)$  for any  $t \in [s, s + |R| - q]$ . We can thus directly apply Lemma 10 to strings  $S$  and  $S'$ .  $\square$

In particular,  $|I| + |I'| \leq k$ . Moreover,  $\min(J) < p$  and  $p + m - 1 < \max(J)$  as well as  $\min(J') < x$  and  $x + m - 1 < \max(J')$ , since otherwise we would have  $|I| \geq k + 1$  or  $|I'| \geq k + 1$ . In conclusion, this  $k$ -occurrence will be found in the PERIODIC-PERIODIC-MATCH problem, i.e.  $p \in Y$ .  $\square$

The following proposition summarizes the results of this section.

**Proposition 2.** *If  $m \leq n \leq 2m$ ,  $k$ -CPM can be solved in  $\mathcal{O}(n + k^5)$  time and  $\mathcal{O}(n)$  space.*

*Proof.* We split the pattern into  $2k + 3$  fragments and choose a sample  $\mathcal{S}$  among them in every possible way.

If the sample  $\mathcal{S}$  is not periodic, we use the algorithm of Lemma 8 for SAMPLE-MATCH in  $\mathcal{O}(k^2)$  time (after  $\mathcal{O}(n)$ -time preprocessing). It returns a representation of  $k$ -occurrences as a union of  $\mathcal{O}(k^2)$  intervals.

If the sample  $\mathcal{S}$  is periodic, we need to find all  $\mathcal{S}$ -runs in  $T$ . By Lemma 9, there are  $\mathcal{O}(k)$  of them and they can all be computed in  $\mathcal{O}(k)$  time (after  $\mathcal{O}(n)$ -time preprocessing). For every such  $\mathcal{S}$ -run  $R$ , we apply the Run-Sample-Matching algorithm. Its correctness follows from Lemma 13. By Lemma 12, it takes  $\mathcal{O}(k^3)$  time and returns  $\mathcal{O}(k^3)$  intervals and  $\mathcal{O}(k^2)$  interval chains, each with difference  $\text{per}(\mathcal{S})$ , of  $k$ -occurrences of  $P$  in  $T$  (after  $\mathcal{O}(n)$ -time preprocessing). Over all  $\mathcal{S}$ -runs, this takes  $\mathcal{O}(k^4)$  time after the preprocessing and returns  $\mathcal{O}(k^4)$  intervals and  $\mathcal{O}(k^3)$  interval chains.

By Lemma 12, if any interval chains are reported in Run-Sample-Matching, then some rotation of the pattern is  $k$ -periodic with a  $k$ -period  $\text{per}(\mathcal{S})$ . Then, at least  $k + 2$  of the  $2k + 3$  pattern fragments do not contain misperiods and hence they must have a period  $q = \text{per}(\mathcal{S})$ . This is actually their smallest period, for if one of these fragments  $\mathcal{S}'$  had a period  $q' < q$ , then  $|\mathcal{S}'| \geq |\mathcal{S}| - 1$  and, by Fine and Wilf's periodicity lemma [34],  $\mathcal{S}'$  would have a period  $q'' = \text{gcd}(q, q') < q$ , which would imply that  $Q$  would also have a period  $q''$  and hence  $\mathcal{S}$  as well. Thus, throughout the course of the algorithm, Run-Sample-Matching can only return interval chains of period  $\text{per}(\mathcal{S})$  by the pigeonhole principle.

In total, SAMPLE-MATCH takes  $\mathcal{O}(k^4)$  time for a given sample (after preprocessing),  $\mathcal{O}(n + k^5)$  time in total, and returns  $\mathcal{O}(k^5)$  intervals and  $\mathcal{O}(k^4)$  interval chains of  $k$ -occurrences, each with the same difference  $q$ . Let us note that an interval is a special case of an interval chain with difference,

say, 1. We then apply Lemma 4 to compute the union of all chains of occurrences and the union of all intervals in  $\mathcal{O}(n + k^5)$  total time. In the end we return the union of the two unions.

In order to bound the space required by our algorithm by  $\mathcal{O}(n)$ , we do not store each interval chain explicitly throughout the execution of the algorithm. Instead, for each interval chain we increment/decrement a constant number of cells in a ( $\mathcal{G}_q$ -shaped for interval chains or  $\mathcal{G}_1$ -shaped for intervals) 2D array of size  $\mathcal{O}(n)$  as in the proof of Lemma 4, and compute the union of all such interval chains in the end.  $\square$

## 6 An $\mathcal{O}(n + \frac{n}{m} k^4)$ -time Algorithm

Let us observe that for each non-periodic fragment  $\mathcal{S}$  we have to solve  $\mathcal{O}(k)$  instances of PAIR-MATCH, while for each periodic fragment  $\mathcal{S}$  and each  $\mathcal{S}$ -run, we obtain two sets  $J$  and  $J'$ , each of cardinality  $\mathcal{O}(k)$ , where each pair of elements in  $J \times J'$  requires us to solve an instance of PAIR-MATCH. Further recall that each instance of PAIR-MATCH reduces to two calls to our  $\mathcal{O}(k)$ -time algorithm for ANCHOR-MATCH. We thus consider  $\mathcal{O}(k^4)$  ANCHOR-MATCH instances in total, yielding a total time complexity of  $\mathcal{O}(k^5)$ . As can be seen in the proof of Proposition 2 and the pseudocode, this is the bottleneck of our algorithm, with everything else requiring  $\mathcal{O}(n + k^4)$  time. We will decrease the number of calls to PAIR-MATCH by using a marking trick.

We first present a simple application of the marking trick. Suppose that we are in the standard  $k$ -mismatch problem, where we are to find all  $k$ -occurrences (not circular ones) of a pattern  $P$  of length  $m$  in a text  $T$  of length  $n$ , and  $n \leq 2m$ . Further suppose that  $P$  is square-free, or, in other words, that it is nowhere periodic. Let us consider the following algorithm. We split the pattern into  $k + 1$  fragments of length roughly  $m/k$  each. Then, at each  $k$ -occurrence of  $P$  in  $T$ , at least one of the  $k + 1$  fragments must match exactly. We then find the  $\mathcal{O}(k)$  such exact matches of each fragment in  $T$  and each of them nominates a position for a possible  $k$ -occurrence of  $P$ . We thus have  $\mathcal{O}(k^2)$  candidate positions in total to verify.

Now consider the following refinement of this algorithm. We split the pattern into  $2k$  fragments (instead of  $k + 1$ ), each of length roughly  $m/(2k)$ . Then, at each  $k$ -occurrence of  $P$  in  $T$ , at least  $k$  of the fragments must match exactly; we exploit this fact as follows. Each exact occurrence of a fragment in  $T$  gives a mark to the corresponding position for a  $k$ -occurrence of  $P$ . There are thus  $\mathcal{O}(k^2)$  marks given in total. However, we only need to verify positions with at least  $k$  marks and these are now  $\mathcal{O}(k)$  in total. An illustration is provided in Fig. 9.

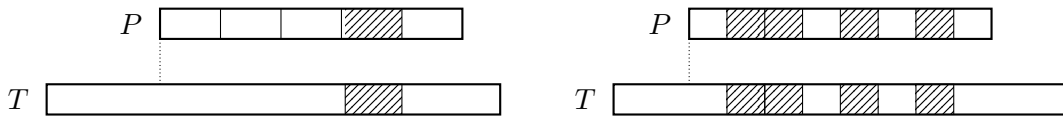


Figure 9: We consider  $k = 4$ . To the left: a candidate starting position for  $P$  given by an exact match of one of the 5 fragments. To the right: a candidate starting position for  $P$  given by exact matches of 4 of the 8 fragments.

Let us get back to the  $k$ -CPM problem. Recall that each  $k$ -occurrence implies that at least  $k + 2$  fragments match exactly. We run the algorithm yielding Proposition 2 with a single difference. Instead of processing each instance of PAIR-MATCH separately, we apply the marking trick in order to decrease the exponent of  $k$  by one. This is achieved by a reduction in the number of calls to the algorithm that solves ANCHOR-MATCH. For each of the  $\mathcal{O}(k^4)$  instances of PAIR-MATCH we mark



the two possible anchors for a  $k$ -occurrence and note that only anchors with at least  $k + 2$  marks need to be verified; these are  $\mathcal{O}(k^3)$  in total. Finally, for each such anchor we apply our solution to the ANCHOR-MATCH problem, which requires  $\mathcal{O}(k)$  time, hence obtaining an  $\mathcal{O}(n + k^4)$ -time algorithm.

The correctness of this approach follows from the following fact. (Note that PERIODIC-PERIODIC-MATCH is not affected by this modification.) If there is a  $k$ -occurrence at position  $i$  of  $T$  that has not been returned by any of the calls to PERIODIC-PERIODIC-MATCH, then, the algorithm would return  $i$  through at least  $k + 2$  calls to ANCHOR-MATCH. We arrive at the following result.

**Proposition 3.** *If  $m \leq n \leq 2m$ ,  $k$ -CPM can be solved in  $\mathcal{O}(n + k^4)$  time and  $\mathcal{O}(n)$  space.*

Both Propositions 1 and 3 use  $\mathcal{O}(n)$  space. Moreover, Proposition 3 assumes that  $n \leq 2m$ . In order to solve the general version of the  $k$ -CPM problem, where  $n$  is arbitrarily large, efficiently and using  $\mathcal{O}(m)$  space, we use the so-called standard trick: we split the text into  $\mathcal{O}(n/m)$  fragments, each of length  $2m$  (perhaps apart from the last one), starting at positions equal to  $0 \pmod m$ .

We need, however, to ensure that the data structures for answering lcp, lcs, and other internal queries over each such fragment of the text can be constructed in  $\mathcal{O}(m)$  time when the input alphabet  $\Sigma$  is large. As a preprocessing step we hash the letters of the pattern using perfect hashing. For each key, we assign a unique identifier from  $\{1, \dots, m\}$ . This takes  $\mathcal{O}(m)$  (expected) time and space [39]. When reading a fragment  $F$  of length (at most)  $2m$  of the text we look up its letters using the hash table. If a letter is in the hash table we replace it in  $F$  by its rank value; otherwise we replace it by rank  $m + 1$ . We can now construct the data structures in  $\mathcal{O}(m)$  time and thus our algorithms can be implemented in  $\mathcal{O}(m)$  space.

If  $\Sigma = \{1, \dots, n^{\mathcal{O}(1)}\}$ , the same bounds can be achieved deterministically. Specifically, we consider two cases. If  $m > \sqrt{n}$  we sort the letters of every text fragment and of the pattern in  $\mathcal{O}(m)$  time per fragment because  $n$  is polynomial in  $m$  and  $|\Sigma|$  is polynomial in  $n$ . Then we can merge the two sorted lists and replace the letters in the pattern and the text fragments by their ranks. Otherwise ( $m \leq \sqrt{n}$ ), we construct a deterministic dictionary for the letters of the pattern in  $\mathcal{O}(m \log^2 \log m)$  time [40]. The dictionary uses  $\mathcal{O}(m)$  space and answers queries in constant time; we use it instead of perfect hashing in the previous solution.

We combine Propositions 1 and 3 with the above discussion to get our final result.

**Theorem 4.** *Circular Pattern Matching with  $k$  Mismatches can be solved in  $\mathcal{O}(\min(nk, n + \frac{n}{m} k^4))$  time and  $\mathcal{O}(m)$  space.*

Our algorithms output all positions in the text where some rotation of the pattern occurs with  $k$  mismatches. It is not difficult to extend the algorithms to output, for each of these positions, a corresponding rotation of the pattern.

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