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Probabilistic representations of fragmentation equations

Madalina Deaconu* Antoine Lejay†

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In this survey article, we present various probabilistic representations of the fragmentation equation, and show how they are related. We focus on the stochastic process which represents the evolution of the mass of a typical particle subjected to a fragmentation process. These probabilistic representations range from Markov chains to Stochastic Differential Equations with jumps, and we consider how they are inter-related. In particular, we show how these representations leads to easy numerical simulations.

1. Introduction

Fragmentation plays a fundamental role in numerous applications arising both in natural science and in industrial processes. Most often, in these phenomena one needs to describe how the clusters present in the experience evolve in time and more precisely how do they break into smaller clusters. This is an important topic in physics (engine reactions), astrophysics (formation of asteroids), in geophysics (landslide, avalanches), in copper industry, etc.

For answering this question several issues can be concerned. We study in this paper the fragmentation equation which describes the time evolution of a particles system which are characterised by their masses. We will consider here binary fragmentation

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procedure, that means that at some times a particle can split into two particles of smaller masses. We suppose in our model that the described phenomenon preserves the global mass of the system.

Denote by $c(t, x)$ the concentration of particles of mass x in the system at time t . The fragmentation equation is the following deterministic integro-differential equation, for all $x \in \mathbb{R}_+$ and $t \in \mathbb{R}_+$:

$$\begin{cases} \frac{\partial c(t, x)}{\partial t} &= -\frac{1}{2} \int_0^x c(t, x) F(x - y, y) dy + \int_0^{+\infty} c(t, x + y) F(y, x) dy \\ c(0, x) &= c_0(x), \end{cases} \quad (1)$$

where F is the fragmentation kernel which is a non-negative and symmetric function defined on $(0, \infty)^2$. The form of F plays a fundamental role in the problem.

In the first line of equation (1), the first term in the right hand side accounts for the disappearance of particles of mass x after breakage into two particles of masses y and $x - y$, where $0 < y < x$, while the second term accounts for the formation of particles of mass x after the splitting of a particle of mass $x + y$ into two particles of masses x and y , for $y > 0$.

A huge literature deals with this equation. Both deterministic and stochastic interpretations have been considered. In some of these papers the coagulation-fragmentation phenomenon is also treated. Existence results are obtained in [26, 45, 46, 53, 54]. The analytic treatment of fragmentation equations is usually dealt using the theory of semi-groups, which is sometimes delicate due to the variety of behaviors of fragmentation kernels. An extensive account may be found in the book [8].

The main objective of this paper is to discuss and describe the various stochastic approaches of these equations and to highlight possible links between the stochastic tools that can be used. The stochastic approach is namely based on the conservation of mass observed in the physical system which induces naturally the construction of a random process with a distribution of the form $xc(x, t) dx$. This representation leads to a natural interpretation for both discrete and continuous masses. Such analysis was started by A.N. Kolmogorov [42].

In particular, our probabilistic representations are derived one from the other, starting from the simplest one using Continuous Times Markov Chain as in [29] to end up with Stochastic Differential Equations with Jumps as in [31] and passing through Random Point Processes. As all these representations are equivalent, one may choose the most appropriate framework according to the problem to solve, as each representation comes with different tools and viewpoints.

We focus on generic results in the sense we do not consider particular shapes of the kernels and their related properties. This is a wide subject. To keep things simple

and to focus on applied results, we left aside on probabilistic representations in terms of clouds of particles or in branching processes where the states space encodes all the particles' mass. Such representations require higher levels of abstraction. The book of J. Bertoin [14] (see also [11]) presents a details account on continuous time Markov chain. A recent paper gives an interpretation of the fragmentation in terms of branching processes [15]. It is then applied to particular classes of kernels connected to the avalanches to give new results on this context [17] and to construct a numerical algorithm [16].

Coagulation phenomenon is usually studied with fragmentation. Yet, it leads to more intricate results as the coagulation induces some non-linear behaviors. Results on the probabilistic approach for coagulation-fragmentation processes are obtained in [31], where the interpretation is given by the solution of a stochastic differential equation with jumps. Jeon [39] studied the coagulation-fragmentation equation in the discrete setting and approximated it by a sequence of finite Markov chains, and formulated a criterion on the finite chain which detect the gelation phenomena. In [40], a non-linear stochastic process is described for the discrete coagulation-fragmentation equation namely for small initial data or in the case where coagulation is dominating the fragmentation.

Outline. After this introductory part, we develop in Section 2 the fragmentation equation and give two formulations for it. Section 3 contains the Kolmogorov forward and backward expressions of the problem. In Sections 4 and 5 the solution of the fragmentation equation is described via a stochastic process from a Markov chain. More precisely we depicted the embedding of the Markov chain into a jump process, a description via Marked Point Processes and the connection with martingale problems. In Section 6, we describe the process in terms of stochastic differential equations. A brief discussion on the mass conservation is given in Section 7. Section 8 is devoted to a simulation algorithm and numerical developments for particular fragmentation kernels. In Section 9 we state some conclusions. Two short appendices end the paper.

2. The fragmentation equation

Let us consider the evolution of the mass of a fragment of snow or rock which is subject to a fragmentation process. In this article, we constantly refer to the mass, yet the fragmentation may be function of other physical characteristics (see for example [50] for application to bubbles).

We define from now on the particles as x -mers (or x -clusters) as fragments of mass x . A fragmentation equation describes the evolution of the mass of fragments

in time. Such an equation is ruled by a *fragmentation kernel* $F : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ which specifies how a $(x + y)$ -mer breaks up into a x -mer and a y -mer. We assume that $(x, y) \mapsto F(x, y)$ is measurable from $\mathbb{R}_+^2 \rightarrow \mathbb{R}_+$.

For the first results, we are working under a symmetry hypothesis. Additional assumptions will be made later while in a first time, we only perform formal derivations.

Hypothesis 1. The kernel F is symmetric: $F(x, y) = F(y, x)$, for any $x, y \geq 0$.

2.1. The fragmentation equation from the balance condition

We consider first the case of a binary, symmetric fragmentation, in which the fragments breaks into two part while the total mass is conserved.

The quantity of interest is $c(t, x)$, the concentration of x -mers at time t . The balance condition is the following: x -mers break into y -mers at rate $F(x - y, y)$ (and are removed from $c(t, x)$), while $(x + y)$ -mers break into x -mers at rate $F(y, x)$. In a short time Δt ,

$$c(t + \Delta t, x) \approx c(t, x) - \frac{\Delta t}{2} c(t, x) \int_0^x F(x - y, y) dy + \Delta t \int_0^{+\infty} c(t, x + y) F(y, x) dy. \quad (2)$$

The $1/2$ term is to avoid to count twice the fragmentation of a x -mer as $(y, x - y)$ and as $(y - x, y)$.

Passing to the limit as $\Delta t \rightarrow 0$, we recover the fragmentation equation (1):

$$\frac{\partial c(t, x)}{\partial t} = -\frac{1}{2} \int_0^x c(t, x) F(x - y, y) dy + \int_0^{+\infty} c(t, x + y) F(y, x) dy. \quad (3)$$

2.2. The mean rate at which x -mers break

Let us define

$$F(x) := \int_0^x F(x - y, y) \frac{x - y}{x} dy \text{ for } x \geq 0. \quad (4)$$

Lemma 1. *Under Hypothesis 1 (symmetry of the kernel),*

$$F(x) = \frac{1}{2} \int_0^x F(x - y, y) dy \text{ for any } x \geq 0. \quad (5)$$

Proof. Let $x \geq 0$ be fixed. We evaluate

$$\int_0^x F(y, x-y) \frac{x-y}{x} dy = \int_0^x F(y, x-y) dy - \int_0^x F(y, x-y) \frac{y}{x} dy.$$

With the change of variable $z = x - y$, since F is symmetric in its arguments,

$$\begin{aligned} \int_0^x F(y, x-y) \frac{x-y}{x} dy &= \int_0^x F(y, x-y) dy - \int_0^x F(y, x-y) \frac{y}{x} dy \\ &= \int_0^x F(y, x-y) dy - \int_0^x F(z, x-z) \frac{x-z}{x} dz. \end{aligned}$$

This leads to (5). □

With Lemma 1, (3) is rewritten

$$\frac{\partial c(t, x)}{\partial t} = -c(t, x)F(x) + \int_0^{+\infty} c(t, x+y)F(y, x) dy. \quad (6)$$

From (6), during a small time Δt , a fraction $F(x)\Delta t$ is subject to a breakage. The term $F(x)$ may be interpreted as the mean rate at which an x -mer breaks.

Using a change of variable $y = y - x$, we rewrite (6) (which is equivalent to (3)) as

$$\frac{\partial c(t, x)}{\partial t} = -F(x)c(t, x) + \int_x^{+\infty} c(t, y)F(y)B(x|y) dy \quad (7)$$

with

$$B(x|y) := \frac{F(y-x, x)}{F(y)}. \quad (8)$$

We note that from the very definition of $F(y)$,

$$\int_0^y B(x|y) \frac{y}{x} dx = 1.$$

Conversely, given $y \mapsto B(x|y)$ and a rate function $x \mapsto F(x)$, we may recover the kernel $(x, y) \mapsto F(x, y)$ by setting

$$F(x, y) := F(x+y)B(y|x+y).$$

The symmetry assumption Hypothesis 1 is not used here. In the physical literature, the starting point is usually (7) with a given kernel B and a rate F . Under Hypothesis 1, the symmetry relation expresses that $B(y|x+y) = B(x|x+y)$ for any (x, y) . We refer to [59] for considerations on binary fragmentation and symmetric kernels.

2.3. Two formulations of the fragmentation equation

We presented previously a formal derivation of the fragmentation equation (3). However, we do not know whether the concentration exists. We now present two alternative formulations of the fragmentation equation.

Notation 1 (Space of measures). We denote by \mathbf{M} the space of measures on \mathbb{R}_+ . There are several ways to equip \mathbf{M} with a metric so that it becomes a complete, separable metric space with respect to the vague (resp. weak) topology, *i.e.* a family $\{\mu_n\}_n$ converges vaguely to μ if and only if $\int f d\mu_n$ converges to $\int f d\mu$ for any non-negative function f with compact support (resp. any non-negative function f uniformly bounded) [32, 33, 57].

Let $t \mapsto p(t, dx)$ be a \mathcal{M} -valued solution to

$$\frac{\partial}{\partial t} \int_0^{+\infty} p(t, dx) \phi(x) = \int_0^{+\infty} \int_0^x p(t, dx) F(x-y, y) \frac{x-y}{x} (\phi(x-y) - \phi(x)) dy \quad (9)$$

for any continuous function ϕ with compact support on \mathbb{R}_+ . We do not discuss now the existence of such a solution.

Notation 2. From the kernel F , we set for any measurable function ϕ ,

$$\mathcal{F}\phi(x) := \int_0^x F(x-y, y) \frac{x-y}{x} (\phi(x-y) - \phi(x)) dy. \quad (10)$$

Remark 1. At that point and without further assumption on F , nothing ensures that $\mathcal{F}\phi$ is well defined. However, if $F(x) < +\infty$ at some $x < 0$ and ϕ is bounded then $\mathcal{F}\phi(x)$ is finite. If for some bounded interval $[a, b] \subset \mathbb{R}_+$, F is bounded on $[a, b]$ and $x \mapsto F(x, y)$ is continuous for any $y \leq x$, then $x \mapsto \mathcal{F}\phi(x)$ is continuous on $[a, b]$ for any ϕ which is bounded and continuous on $[0, b]$.

Using Notation 2, it is convenient to rewrite (9) as

$$\frac{\partial}{\partial t} \int_0^{+\infty} p(t, dx) \phi(x) = \int_0^{+\infty} p(t, dx) \mathcal{F}\phi(x). \quad (11)$$

We also define the equation

$$\frac{\partial p(t, dx)}{\partial t} = dx \int_x^{+\infty} p(t, dy) \frac{x}{y} F(y-x, x) - p(t, dx) F(x). \quad (12)$$

Proposition 1. *The equations (9) and (12) are equivalent.*

Proof. Let ϕ be a continuous function with compact support on \mathbb{R}_+ . After two changes of variables, by inverting first x and y and then changing y to $x - y$,

$$\begin{aligned} \int_0^{+\infty} \phi(x) dx \int_x^{+\infty} p(t, dy) \frac{x}{y} F(y - x, x) \\ &= \int_0^{+\infty} \int_0^{+\infty} p(t, dy) dx \phi(x) \frac{x}{y} F(y - x, x) \mathbb{1}_{x \leq y} \\ &= \int_0^{+\infty} p(t, dx) \int_0^x dy \phi(y) \frac{y}{x} F(x - y, y) \\ &= \int_0^{+\infty} p(t, dx) \int_0^x dy \phi(x - y) \frac{x - y}{x} F(x - y, y). \end{aligned}$$

On the other hand, from the very definition of F in (4), we have

$$\int_0^{+\infty} p(t, dx) F(x) \phi(x) = \int_0^{+\infty} \int_0^x p(t, dx) F(y - x, x) \frac{y - x}{x} \phi(x) dy.$$

This proves the result. \square

Notation 3. We define for a measure μ and a bounded function ϕ with compact support,

$$\langle \mu, \phi \rangle := \int_0^{+\infty} \mu(dx) \phi(x).$$

With $p_t := p(t, \cdot)$, (11) is rewritten in a compact way as the integro-differential equation

$$\frac{\partial}{\partial t} \langle p_t, \phi \rangle = \langle p_t, \mathcal{F} \phi \rangle. \quad (13)$$

We also define for a bounded, finitely additive signed measure μ ,

$$\mathcal{F}^* \mu(dx) := \left(\int_x^{+\infty} \mu(dy) \frac{x}{y} F(y - x, x) \right) dx - F(x) \mu(dx).$$

As in the proof of Proposition 1,

$$\langle \mathcal{F}^* \mu, \phi \rangle = \langle \mu, \mathcal{F} \phi \rangle$$

so that (12) is

$$\frac{\partial}{\partial t} \langle p_t, \phi \rangle = \langle \mathcal{F}^* p_t, \phi \rangle. \quad (14)$$

Hence, the operators \mathcal{F} and \mathcal{F}^* are in duality.

2.4. Link to the fragmentation equation

We relate our new equation to the fragmentation equation. Actually, $p(t, dx)$ relates to the mass distribution, while $c(t, x)$ is the concentration of the masses.

Proposition 2. *Assume Hypothesis 1, that is F is symmetric. Let us assume that $p(t, dx)$ has a density for any $t > 0$, which we write $p(t, dx) = x \cdot c(t, x) dx$ for a function c . Then c solves (3).*

Proof. Using $\psi(x) := \phi(x)/x$ as a test function in (9),

$$\begin{aligned} \frac{\partial}{\partial t} \langle c_t, \phi \rangle &= \frac{\partial}{\partial t} \langle p_t, \psi \rangle \\ &= \int_0^{+\infty} \int_0^x c(t, x) F(x - y, y) \phi(x - y) dy - \int_0^{+\infty} c(t, x) F(x) \phi(x) dx. \end{aligned}$$

With a change of variable ($z = x - y, y = y$), we deduce

$$\int_0^{+\infty} \int_0^x c(t, x) F(y, x - y) \phi(x - y) dy = \int_0^{+\infty} \int_0^{+\infty} c(t, z + y) F(y, z) dy \phi(z) dz.$$

With Lemma 1,

$$\int_0^{+\infty} c(t, x) F(x) \phi(x) dx = \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} c(t, x) F(y, x - y) dy \phi(x) dx.$$

This proves that c solves (3). □

3. The Kolmogorov forward and backward equations

We intend in this part to come closer to a stochastic description. This also allows one to consider the well-posedness and uniqueness of the fragmentation equation under mild hypotheses on the kernel F . Other conditions using functional analysis may be found for example in [5, 52, 56] and many other papers. We observe that despite its apparent simplicity, the fragmentation equation is not that simple as various pathologies may appear, see *e.g.* [6].

3.1. The kernels

From the fragmentation kernel, we deduce another kernel with natural probabilistic representation.

Notation 4. We define the kernels

$$G(x, dy) := \mathbb{1}_{[0,x]}(y)F(x-y, y)\frac{y}{x} dy \text{ and } \mathbf{G}(x, dy) := \frac{G(x, dy)}{F(x)}. \quad (15)$$

Remark 2. We are concerned here with kernels $\mathbf{G}(x, \cdot)$ which have a density. Yet most of the results presented here may be extended to the case where F is not necessarily symmetric or when $\mathbf{G}(x, \cdot)$ presents some atoms. For example, for the binary fragmentation, one may take $\mathbf{G}(x, \cdot) = \delta_{x/2}$ so that when a breakage occurs, the x -mer is broken in two parts of equal masses.

Notation 5. We denote by $\text{Bor}(\mathbb{R}_+)$ the set of Borel sets of \mathbb{R}_+ .

With (4) and the change of variable ($z = x - y$), $\mathbf{G}(x, [0, x]) = \mathbf{G}(x, \mathbb{R}_+) = 1$ and $G(x, \Lambda) \geq 0$ for any $\Lambda \in \text{Bor}(\mathbb{R}_+)$. Besides, $\mathbf{G}(x, \{x\}) = 0$ for any $x \geq 0$.

This means that for each x , $G(x, \cdot)$ may be interpreted as the probability density function of a random variable Y taking its values in $[0, x]$, with $\mathbb{P}[Y \in [a, b]] = \int_a^b G(x, dy)$.

It is also convenient (for reasons explained later, LM means ‘‘loss of mass’’) to introduce

$$G^{\text{LM}}(x, dy) := F(x-y, y)\frac{x-y}{x}\mathbb{1}_{0 \leq y \leq x} dy \text{ and } \mathbf{G}^{\text{LM}}(x, dy) := \frac{G^{\text{LM}}(x, dy)}{F(x)}. \quad (16)$$

Since $\int_0^x G^{\text{LM}}(x, dy) = F(x)$, it holds that $\int_0^x \mathbf{G}^{\text{LM}}(x, dy) = 1$.

Let Y be a random variables with distribution $\mathbf{G}(x, \cdot)$ for some $x > 0$. Set $Z = x - Y$. Using the change of variable $y' = x - y$ and the symmetry of F ,

$$\begin{aligned} \mathbb{P}[Z \leq z] &= 1 - \mathbb{P}[Y \leq x - z] = 1 - \int_0^x \mathbb{1}_{y \leq x-z} F(x-y, y)\frac{y}{xF(x)} dy \\ &= \int_0^x \mathbb{1}_{y \geq x-z} F(x-y, y)\frac{y}{xF(x)} dy = \int_0^x \mathbb{1}_{y \leq z} F(x-y, y)\frac{x-y}{xF(x)} dy, \end{aligned}$$

so that the probability density function of Z is $\mathbf{G}^{\text{LM}}(x, \cdot)$.

Remark 3. With Notation 2, the operator \mathcal{F} defined by (10) is also equal to

$$\mathcal{F}\phi(x) = \int_0^{+\infty} (\phi(x-y) - \phi(x))G^{\text{LM}}(x, dy).$$

3.2. The Kolmogorov forward equations

Let $\Lambda \in \text{Bor}(\mathbb{R}_+)$. If we apply formally the operator \mathcal{F} defined in Notation 2 to the indicator function $\mathbb{1}_\Lambda$, we obtain that

$$\mathcal{F}\mathbb{1}_\Lambda(x) = \int_0^x F(x-y, y) \frac{x-y}{x} \mathbb{1}_\Lambda(x-y) dy - \int_0^x F(x-y, y) \frac{x-y}{x} \mathbb{1}_\Lambda(x) dy.$$

Using a change of variable $z = y - x$ and the definition of $F(x)$ in (4),

$$\mathcal{F}\mathbb{1}_\Lambda(x) = \int_0^x F(x-y, y) \frac{y}{x} \mathbb{1}_\Lambda(y) - F(x) \mathbb{1}_\Lambda(x) = G(x, \Lambda) - F(x) \mathbb{1}_\Lambda(x).$$

Formally, when using $\mathbb{1}_\Lambda$ as a test function in (13), since $\int_{\mathbb{R}_+} p(t, dy) \mathbb{1}_\Lambda(y) = p(t, \Lambda)$ for any $t > 0$, we are led to consider

$$\frac{\partial p(t, \Lambda)}{\partial t} = \int_{\mathbb{R}_+} p(t, dy) G(y, \Lambda) - \int_{\mathbb{R}_+} p(t, dy) F(y) \mathbb{1}_\Lambda(y). \quad (17)$$

We now consider solving equations of type (17). For this, we consider the solutions when the initial conditions is concentrated at a given point. This means that we consider solving

$$\frac{\partial P(t, x, \Lambda)}{\partial t} = \int_{\mathbb{R}_+} P(t, x, dy) G(y, \Lambda) - \int_{\mathbb{R}_+} P(t, x, dy) F(y) \mathbb{1}_\Lambda(y) \quad (18)$$

$$\text{with } P(0, x, \Lambda) = \mathbb{1}_\Lambda(x). \quad (19)$$

The system (18)-(19) is stronger than (9) as we are concerned by solving a family of equations.

Notation 6 (Transition probability). A *transition probability* is a family $P = \{P(t, x, \Lambda)\}_{t \geq 0, x \in \mathbb{R}_+, \Lambda \in \text{Bor}(\mathbb{R}_+)}$ such that

- (i) $t \mapsto P(t, x, \Lambda)$ is continuous for any $x \in \mathbb{R}_+, \Lambda \in \text{Bor}(\mathbb{R}_+)$;
- (ii) $x \mapsto P(t, x, \Lambda)$ is $\text{Bor}(\mathbb{R}_+)$ -measurable for any $t \geq 0$ and $\Lambda \in \text{Bor}(\mathbb{R}_+)$;
- (iii) $\Lambda \mapsto P(t, x, \Lambda)$ is σ -additive with $0 \leq P(t, x, \Lambda) \leq 1$ for any $t \geq 0, x \in \mathbb{R}_+$ and $\Lambda \in \text{Bor}(\mathbb{R}_+)$;
- (iv) For any $x \in \mathbb{R}_+, \Lambda \in \text{Bor}(\mathbb{R}_+)$,

$$P(t, x, \Lambda) \xrightarrow[t \rightarrow 0+]{} \mathbb{1}_\Lambda(x).$$

Hypothesis 2. For any $x < +\infty, F(x) < +\infty$.

Definition 1 (Kolmogorov forward equation). We say that the transition P solves the Kolmogorov forward equation whenever for any $x \in \mathbb{R}_+$, any bounded set $\Lambda \in \text{Bor}(\mathbb{R}_+)$, (19) is satisfied and $\partial_t P(t, x, dy)$ exists for every time $t > 0$ and (18) is satisfied.

Remark 4. The Kolmogorov forward equation is also called the *Fokker-Planck equation* or the *master equation*.

We define the operator \mathcal{A}^+ which acts on transitions as

$$\mathcal{A}^+ P(t, x, \Lambda) := \int_0^t ds \int_{\mathbb{R}_+} P(s, x, dy) e^{-(t-s)F(y)} G(y, \Lambda). \quad (20)$$

We then define recursively

$$P^{(0)}(t, x, \Lambda) := \mathbb{1}_\Lambda(x) \exp(-tF(x)), \quad (21)$$

$$P^{(n)}(t, x, \Lambda) := \mathcal{A}^+ P^{(n-1)} \text{ for } n \geq 1. \quad (22)$$

Finally, we set

$$P(t, x, \Lambda) := \sum_{n=0}^{+\infty} P^{(n)}(t, x, \Lambda) = \sum_{n=0}^{+\infty} (\mathcal{A}^+)^n \mathbb{1}_\Lambda(x) \exp(-tF(x)). \quad (23)$$

Actually, it is easily checked that for $n \geq 1$,

$$\frac{\partial P^{(n)}(t, x, \Lambda)}{\partial t} = \int_{\mathbb{R}_+} P^{(n-1)}(t, x, dy) G(y, \Lambda) - \int_{\mathbb{R}_+} P^{(n)}(t, x, dy) F(y) \mathbb{1}_\Lambda(y).$$

Proposition 3 ([29, Theorem 1], [28, Theorem 4.1]). *Under Hypothesis 2, P given by (23) is a transition probability and solves the Kolmogorov forward equation (18)-(19).*

Remark 5. It follows from [29, Eq. (23)] that $\partial_t P(t, x, \Lambda)$ exists for any time.

3.3. The Kolmogorov backward equation

Similarly to \mathcal{A}^+ given by (20), we define

$$\mathcal{A}^- P(t, x, \Lambda) := \int_0^t ds \int_{\mathbb{R}_+} e^{-sF(x)} G(y, dz) P(t-s, z, \Lambda), \quad (24)$$

and we define recursively

$$Q^{(0)}(t, x, \Lambda) := \exp(-tF(x)) \mathbb{1}_\Lambda(x) = P^{(0)}(t, x, \Lambda)$$

and $Q^{(n+1)}(t, x, \Lambda) := \mathcal{A}^- Q^{(n)}(t, x, \Lambda).$

The function $Q^{(n+1)}$ solves

$$\frac{\partial Q^{(n+1)}(t, x, \Lambda)}{\partial t} = F(x)Q^{(n+1)}(t, x, \Lambda) - \int_{\mathbb{R}_+} G(x, dy)Q^{(n)}(t, y, \Lambda).$$

Finally, we set

$$Q(t, x, \Lambda) := \sum_{n=0}^{+\infty} Q^{(n)}(t, x, \Lambda).$$

Lemma 2. For any $t \geq 0$, $x \in \mathbb{R}_+$ and $\Lambda \in \text{Bor}(\mathbb{R}_+)$, $P(t, x, \Lambda) = Q(t, x, \Lambda)$.

Proof. Since $P^{(0)}(t, x, dy) = \delta_y(x) \exp(-tF(x))$, it is easily checked that $P^{(1)}(t, x, dy) = Q^{(1)}(t, x, dx)$. In addition, using the Fubini-Tonelli theorem, the operators \mathcal{A}^+ and \mathcal{A}^- commute so that if $P^{(m)} = Q^{(m)}$ for any $1 \leq m \leq n$,

$$P^{(n+1)} := \mathcal{A}^+ P^{(n)} = \mathcal{A}^+ \mathcal{A}^- Q^{(n-1)} = \mathcal{A}^- \mathcal{A}^+ P^{(n-1)} = \mathcal{A}^- P^{(n)} = \mathcal{A}^- Q^{(n)} = Q^{(n+1)}.$$

This proves the result. \square

Definition 2 (Kolmogorov backward equation). A transition function P solves the Kolmogorov backward equation¹ if $\partial_t P(t, x, \Lambda)$ exists for any $t \geq 0$, $x \in \mathbb{R}_+$, $\Lambda \in \text{Bor}(\mathbb{R}_+)$ and

$$\frac{\partial P(t, x, \Lambda)}{\partial t} = \int_{\mathbb{R}_+} G(x, dy)P(t, y, \Lambda) - F(x)P(t, x, \Lambda), \quad (25)$$

$$P(0, x, \Lambda) = \mathbb{1}_\Lambda(x). \quad (26)$$

We introduce a new property of the transition probability.

Definition 3 (Chapman-Kolmogorov equation). For a transition probability, the Chapman-Kolmogorov equation is that

$$P(t + s, x, \Lambda) = \int_{\mathbb{R}_+} P(t, x, dy)P(s, y, \Lambda) \quad (27)$$

for any $s, t \geq 0$, $x \in \mathbb{R}_+$, $\Lambda \in \text{Bor}(\mathbb{R}_+)$.

Proposition 4 ([29, Theorems 2-5], [28, theorem 3.1]). Under Hypothesis 2, the function P given by (23) solves the Kolmogorov backward equation (25) and is the unique transition probability to do so among all the $P(t, x, \Lambda)$ that solves (25)-(26) for any t . Besides, it satisfies the Kolmogorov-Chapman equation (27).

¹Here, backward refers to the fact that the operator acts on the value x which represents the initial point.

3.4. Comments on the Kolmogorov equations

For more detail we rewrite the Kolmogorov forward and backward equations using operators.

Let us introduce the two operators \mathcal{M} (multiplication) and \mathcal{G} (kernel) which we consider as acting on the left to transform measures μ to measures and on the right to transform measurable functions ϕ to measurable functions as

$$\mu\mathcal{M}(dx) := \mu(dx)F(x) \text{ and } \mathcal{M}\phi(x) := F(x)\phi(x)$$

and

$$\begin{aligned} \mu\mathcal{G}(dx) &:= \int \mu(dx)G(x, dy) = \int \mu(dx)F(x)\mathbf{G}(x, dy) = \int \mu\mathcal{M}(dx)\mathbf{G}(x, dy) \\ \text{and } \mathcal{G}\phi(x) &:= \int G(x, dy)\phi(y) = F(x) \int \mathbf{G}(x, dy)\phi(y) = \mathcal{M} \int \mathbf{G}(x, dy)\phi(y). \end{aligned}$$

The Kolmogorov forward equation (18) is written

$$\partial_t P(t, x, dy) = P(t, x, \cdot)(\mathcal{G} - \mathcal{M})(dy), \quad (28)$$

while the Kolmogorov backward equation (25) is written

$$\partial_t P(t, x, dy) = (\mathcal{G} - \mathcal{M})P(t, \cdot, dy)(x). \quad (29)$$

The Kolmogorov forward and backward equations express that the transition, seen as acting on the right on functions and on the left on measures, commutes with $\mathcal{G} - \mathcal{M}$.

The relationship between (28) and (29) may also be understood from the Chapman-Kolmogorov equation (27): Since

$$\frac{P(t + \varepsilon, x, \Lambda) - P(t, x, \Lambda)}{\varepsilon} = \frac{1}{\varepsilon} \int_{\mathbb{R}_+} P(t, x, dz)(P(\varepsilon, z, \Lambda) - \mathbb{1}_\Lambda(x)),$$

it follows formally that letting $\varepsilon \rightarrow 0$,

$$\begin{aligned} \partial_t P(t, x, \Lambda) &= \int_{\mathbb{R}_+} P(t, x, dz)\partial_t P(0, z, \Lambda) \\ &= \int_{\mathbb{R}_+} P(t, x, dz)((\mathcal{G} - \mathcal{M})P(0, z, \lambda)) = \int_{\mathbb{R}_+} P(t, x, dz)((\mathcal{G} - \mathcal{M})\mathbb{1}_\Lambda(z)). \end{aligned}$$

On the other hand, since formally $P(0, x, dz) = \delta_x(dz)$ (the Dirac mass at x),

$$\begin{aligned} \frac{P(t + \varepsilon, x, \Lambda) - P(t, x, \Lambda)}{\varepsilon} &= \frac{1}{\varepsilon} \int_{\mathbb{R}_+} (P(\varepsilon, x, dz) - \delta_x(dz)) P(t, z, \Lambda) \\ &= \frac{1}{\varepsilon} \int_{\mathbb{R}_+} \int_0^\varepsilon P(\eta, x, dz) (\mathcal{G} - \mathcal{M}) P(t, z, \Lambda) d\eta dz. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$, leads to the Kolmogorov backward equation.

Notation 7 (Semi-group). We set for a continuous, bounded function ϕ , $t \geq 0$ and $x \in \mathbb{R}_+$,

$$P_t \phi(x) := \int_{\mathbb{R}_+} P(t, x, dz) \phi(z).$$

Such family $\{P_t\}_{t \geq 0}$ is called a *semi-group*.

As any continuous, bounded function may be approximated by step functions, the Kolmogorov forward equation (18)-(19) implies that

$$\partial_t P_t \phi(x) = \int_{\mathbb{R}_+} P(t, x, dz) \mathcal{G} \phi(z) - \int_{\mathbb{R}_+} P(t, x, dz) \mathcal{M} \phi(z) = P_t (\mathcal{G} - \mathcal{M}) \phi(x). \quad (30)$$

With Remark 3, we note that $\mathcal{G} - \mathcal{M} = \mathcal{F}$. Therefore, for any finite measure μ on \mathbb{R}_+ , setting $p_t(dz) = \int_{\mathbb{R}_+} \mu(dx) P(t, x, dz)$, we get that $\partial_t \langle p_t, \phi \rangle = \langle p_t, \mathcal{F} \phi \rangle$, which is nothing more than (13).

3.5. Probabilistic representation

Our main point of interest is to use the transition P as the transition probability for a stochastic process ξ defined on a probability space $(\Omega, \mathcal{H}, \mathbb{P})$ with a filtration $(\mathcal{H}_t)_{t \geq 0}$. The filtration is a family of σ -algebras which contains the information up to time t . Here, for any $\omega \in \Omega$, $t \mapsto \xi_t(\omega)$ is a path which describes the evolution of a typical particle. Averaging over many of such paths gives the average behavior of the evolution of the mass and allows one to recover the concentration.

More precisely, the relation between ξ and P that solves (18)-(19) or (25)-(26) is the following

$$\mathbb{P}[\xi(t+s) \in \Lambda \mid \mathcal{H}_t] = P(s, \xi(t), \Lambda) \text{ for any } s, t \geq 0 \text{ and } \Lambda \in \text{Bor}(\mathbb{R}_+).$$

Knowing the stochastic process contains much more information than just knowing its marginal distributions $\xi(t)$ for any time t , as it encodes a dynamical behavior.

4. Evolution of the mass as a stochastic process from a Markov chain

We now present several structures that specify the dynamical evolution of the mass of a typical fragment as a stochastic process, that is, as a random evolution in time.

We first describe the evolution of the mass as a discrete event indexed by the number of breakages, with

- Y_k , the mass after k -th breakage,
- σ_k , the time span between two breakages,
- τ_k the time at which the k -th breakage occurs.

4.1. A description by a Markov chain

Let us consider first a Markov chain $\{(\sigma_k, Y_k)\}_{k \geq 0}$ living in $\mathbb{R}_+ \times \mathbb{R}_+$ defined by $Y_0 = x$, $\sigma_0 = 0$ and the Y_k and the σ_k are drawn independently with the distributions

$$\sigma_{k+1} \sim \mathcal{E}(F(Y_k)), \quad Y_{k+1} \sim \mathbf{G}(Y_k, \cdot), \quad (31)$$

where $\mathcal{E}(\lambda)$ is the exponential distribution of parameter λ . The distribution of (σ_{k+1}, Y_{k+1}) given $Y_k = x$ has density

$$\mathbf{R}_x(ds, dy) = F(x) \exp(-sF(x)) \mathbf{G}(x, dy) ds. \quad (32)$$

We note that $\sigma_k > 0$ a.s. for any $k \geq 0$. The time of the events are defined as $\tau_0 = 0$ and

$$\tau_k = \sigma_0 + \sigma_1 + \cdots + \sigma_k \text{ for } k \geq 0. \quad (33)$$

The sequence $\{\tau_k\}_{k \geq 0}$ is increasing. Since $G(x, \Lambda) = G(x, \Lambda \cap [0, x])$ for any $x \in \mathbb{R}_+$ and $\Lambda \in \text{Bor}(\mathbb{R}_+)$, the sequence $\{Y_k\}_{k \geq 0}$ is decreasing.

The sequence $\{(\tau_k, Y_k)\}_{k \geq 0}$ is also a Markov chain: it represents the typical evolution of a fragment. At time τ_k , the Y_{k-1} -mer breaks up into two new fragments, one being a Y_k -mer. The σ_k 's represent the time span between two breakages.

The algorithmic construction of realizations of such a chain is easy. We detail this in Algorithm 1 in Section 8.2 below.

Notation 8. For convenience, we define

$$\tau_\infty := \lim_{k \rightarrow \infty} \tau_k \leq +\infty.$$

4.2. An alternative description using a Poisson Point Process

We introduce now another construction of the above chain. With this new construction, the chain depends on a single stochastic process, called a Poisson point process, which serves as a “random generator” for all the other random variables. Later, this construction will be helpful to understand the formulation through stochastic differential equations.

We start by recalling standard definitions on Poisson Point process (PPP). Some properties of PPP are recalled in Section B. More information are given in [20, 21, 32, 33, 41].

Definition 4 (Counting measure). Fix $d \geq 1$. A *counting measure* μ on \mathbb{R}^d is a locally finite measure such that $\nu(\Lambda) \in \mathbb{N}$ for any relatively compact set $\Lambda \in \text{Bor}(\mathbb{R}^d)$.

We denote by \mathcal{M} the set of counting measure. We write $\text{Bor}(\mathcal{M})$ the smallest σ -algebra such that $\mu \mapsto \mu(\Lambda)$ is measurable for any relatively compact Borel set Λ of \mathbb{R}^d .

Definition 5 (Point process). Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space. A *point process* N is a measurable mapping from $(\Omega, \mathcal{G}, \mathbb{P})$ to $(\mathcal{M}, \text{Bor}(\mathcal{M}))$, meaning that N is a random counting measure.

Definition 6 (Intensity measure). The *intensity measure* n of a point process N is $n(B) = \mathbb{E}[N(B)]$ for any Borel set B of \mathbb{R}^d (note that possibly, $n(B) = +\infty$).

Definition 7 (Poisson Point process (PPP)). A point process N on \mathbb{R}^d whose intensity n is locally finite is called a *Poisson Point Process* (PPP) if $N(B_1), \dots, N(B_k)$ are independent Poisson random variables with parameters $n(B_1), \dots, n(B_k)$ for any family of pairwise disjoint Borel sets (B_1, \dots, B_k) of \mathbb{R}^d and any $k \geq 1$.

We consider a PPP N on \mathbb{R}_+^3 with intensity $n(ds, du, dv) = ds du dv$ on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$.

Notation 9. For a point $x \in \mathbb{R}_+$, we denote by

$$g(x, y) := F(x - y, y) \frac{y}{x} \mathbb{1}_{0 \leq y \leq x}$$

the density of the measure $G(x, \cdot)$ and we set

$$\Gamma(x) := \{(u, v) \in \mathbb{R}_+^2 \mid v \leq g(x, u)\}.$$

Proposition 5. *Let us construct iteratively a family $\{(\zeta_k, U_k, V_k)\}_{k \geq 0}$ with values in \mathbb{R}_+^3 by setting $\theta_0 := 0$, $U_0 := 0$ and defining $(\theta_{k+1}, U_{k+1}, V_{k+1})$ as the leftmost point of the PPP N in the subset $[\theta_0 + \dots + \theta_k, +\infty) \times \Gamma(U_k)$, for any $k \geq 0$. Then $\{(\theta_k, U_k)\}_{k \geq 0}$ is a Markov chain with the same distribution as $\{(\sigma_k, Y_k)\}_{k \geq 0}$ defined in Section 4.1.*

Proof. We set $\nu(du, dv) := du dv$ so that N has intensity measure $ds \nu(du, dv)$. Hence,

$$\nu(\Gamma(x)) = \int_0^{+\infty} \int_0^{+\infty} \mathbb{1}_{v \leq g(x,u)} du dv = \int_0^{+\infty} g(x, u) du = G(x, \mathbb{R}_+) = F(x).$$

With Lemma 8 and Remark 9 in Appendix B, the leftmost point (ζ, U, V) of N in $[t, +\infty) \times \Gamma(x)$ satisfies:

- (U, V) is independent from ζ and uniform in $\Gamma(x)$,
- $\zeta \sim \mathcal{G}(F(x))$.

With Lemma 9, U has for density $F(x)^{-1}g(x, y) dy = G(x, dy)$. With the rejection principle (Lemma 10), this proves the result. \square

With this construction, the probability space $(\Omega, \mathcal{G}, \mathbb{P})$ of the chain is the one supporting the PPP process. There exists a measurable function Φ from Ω to $\mathbb{R}_+^\infty \times \mathbb{R}_+^\infty$ such that $\{(\sigma_k(\omega), Y_k(\omega))\}_{k \geq 0} = \Phi(\omega)$. In Section 4.1, the probability space has to be constructed inductively as a limit of the finite chains $\{(\sigma_k, Y_k)\}_{0 \leq k \leq n}$ thanks to the Ionescu Tulcea theorem [36].

5. Evolution of the mass as a stochastic process from the Markov chain

5.1. Embedding the Markov chain into a jump process

We have given two ways to construct a Markov chain. We now denote by $\Xi = \{(\tau_k, Y_k)\}_{k \geq 0}$, encoding the times $\{\tau_k\}_{k \geq 0}$ at which the breakages occurs, and $\{Y_k\}_{k \geq 0}$, the mass of the fragments.

Notation 10 (State space). To distinguish the space and the time, we denote the state-space, *i.e.*, the space in which the masses take their values by \mathbb{X} instead of \mathbb{R}_+ .

Notation 11 (Cemetery point and point at infinity). We adjoint to \mathbb{X} a cemetery point \dagger . We write $\overline{\mathbb{X}} := \mathbb{X} \cup \{\dagger\}$ and $\overline{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{\infty\}$. Actually, as we consider only a decreasing mass, the cemetery point could also be identified with 0.

Our goal is now to construct a $\overline{\mathbb{X}}$ -valued continuous process $\xi = \{\xi(t)\}_{t \geq 0}$ indexed by the time and to identify its distribution.

Notation 12 (Embedding the Markov chain). From a $(\mathbb{R}_+ \times \mathbb{X})$ -valued Markov chain $\Xi = \{(\tau_k, Y_k)\}_{k \geq 0}$, we define

$$\xi(t) := \begin{cases} Y_k & \text{if } \tau_k \leq t < \tau_{k+1}, \\ \dagger & \text{if } t \geq \tau_\infty := \lim_n \tau_n \text{ or } Y_0 = \dagger. \end{cases} \quad (34)$$

The process ξ defined above is left-continuous with right-limits, and is piecewise constant on $[\tau_k, \tau_{k+1})$. It has jumps only when breaking occur.

Notation 13 (Counting process). We define the *counting process* $\llbracket \xi \rrbracket$ in the following way: for $J \in \text{Bor}(\mathbb{R}_+)$, $\llbracket \xi \rrbracket J := \#\{\tau_k \mid \tau_k \in J\}$, i.e. $\llbracket \xi \rrbracket J$ is the number of events occurring in J . We also set $\llbracket \xi \rrbracket(\tau) := \llbracket \xi \rrbracket(0, \tau]$.

We now define properly the underlying probabilistic structure as in [28, Sect. 2]:

- **Probability space.** We denote by Ω the set of all sequences $\{(t_k, x_k)\}_{k \geq 0}$ in $\overline{\mathbb{R}}_+ \times \overline{\mathbb{X}}$ such that (i) $x_0 \in \mathbb{X}$ and $t_0 = 0$; (ii) if $t_k < +\infty$, $t_k < t_{k+1}$ and $x_k \in \mathbb{X}$; (iii) if $t_k = +\infty$, $t_k = t_{k+1}$ and $x_k = \dagger$. We also set $t_\infty := \lim_k t_k$ and $x_\infty = \dagger$. This space Ω is a measurable subspace of $\overline{\mathbb{R}}_+ \times \overline{\mathbb{X}}$.

- **σ -algebra.** We then denote by \mathcal{G} the σ -field of measurable subsets of Ω , that is the set of elements of type $\Lambda \cap \Omega$ when Λ is measurable in $\overline{\mathbb{R}}_+ \times \overline{\mathbb{X}}$, where the σ -algebra on $\overline{\mathbb{R}}_+ \times \overline{\mathbb{X}}$ is the product σ -algebra.

- **Filtration.** For any $t \geq 0$, we define $\mathcal{H}_t := \sigma(\text{Bor}(\mathbb{X}), \mathcal{G}_t)$ with

$$\mathcal{G}_t := \sigma(\mathbb{1}_{x_k \in \Lambda} \mathbb{1}_{t_k \leq s} \mid 0 \leq s \leq t, \Lambda \in \text{Bor}(\mathbb{X}), k \geq 1).$$

This filtration $\{\mathcal{H}_t\}_{t \geq 0}$ is an *a.s. jumping filtration* on \mathbb{R}_+ [38, Definition 1, p. 15], i.e. \mathcal{H}_t and \mathcal{H}_{t_k} coincide on $\{t_k \leq t < t_{k+1}\} \in \mathcal{H}_t$.

- **Stopped σ -algebra.** A *stopping time* is a random time τ such that $\{\tau \leq t\} \in \mathcal{H}_t$. For a stopping time τ , we define the *stopped σ -algebra* as

$$\mathcal{H}_\tau := \{A \in \mathcal{G} \mid A \cap \{\tau \leq t\} \in \mathcal{H}_t \text{ for all } t \in \mathbb{R}_+\}.$$

With the above filtration, for any $\Lambda \in \mathcal{G}$, $\Lambda \cap \{t_k \leq t\} \in \mathcal{H}_t$ so that each t_k is a stopping time.

- **Process.** This is the process ξ defined by (34). Its distribution is denoted by \mathbb{P} . Besides, we write \mathbb{P}_x for the distribution of ξ given $\xi(0) = x$, $x \in \mathbb{X}$.

Notation 14. For $u \geq 0$, we denote respectively by $K_u(ds, dy)$ and $H_u(ds)$ the regular conditional distributions of $(\tau_{\llbracket \xi \rrbracket(u)+1}, \xi(\llbracket \xi \rrbracket(u) + 1))$ given \mathcal{H}_u and of $\tau_{\llbracket \xi \rrbracket(u)+1}$ given \mathcal{H}_u . This means that we consider by K_u (resp. H_u) the distribution of the mass and time (resp. time only) of the first breakage occurring after the time u .

We skip the technical details, especially regarding the measurability, of the next lemma. However, thanks to the memoryless property of the exponential distribution (See Lemma 6), its meaning is clear.

Lemma 3 ([28, Lemma 2.1]). *For all $t \geq u \geq 0$,*

$$\begin{aligned} H_u(dt) &= F(\xi(u)) \exp(-(t-u)F(\xi(u))) dt \text{ on } \{\llbracket \xi \rrbracket(u) < +\infty\}, \\ K_u(dt, dy) &= \exp(-(t-u)F(\xi(u))) G(\xi(u), dy) dt \text{ on } \{\llbracket \xi \rrbracket(u) < +\infty\}. \end{aligned}$$

Now, let us consider computing of $\mathbb{P}[\xi(t) \in \Lambda \mid \mathcal{H}_u]$ for $0 \leq u \leq t$ and $\Lambda \in \text{Bor}(\mathbb{X})$.

Lemma 4. *For any $u < t$, $\Lambda \in \text{Bor}(\mathbb{X})$ and $n \geq 0$,*

$$P^{(n)}(t-u, \xi(u), \Lambda) = \mathbb{P}[\xi(t) \in \Lambda, \llbracket \xi \rrbracket(u, t) = n \mid \mathcal{H}_u], \quad (35)$$

where $P^{(n)}$ is defined in (22).

Sketch of the proof. We give only the main argument of the proof. We left the details on measurability and cite the steps in [28].

With (21),

$$\begin{aligned} \mathbb{P}[\xi(t) \in \Lambda, \llbracket \xi \rrbracket(u, t) = 0 \mid \mathcal{H}_u] &= \mathbb{P}[\xi(t) \in \Lambda, \tau_{\llbracket \xi \rrbracket(u)+1} > t \mid \mathcal{H}_u] \\ &= \mathbb{P}[\xi(u) \in \Lambda, \tau_{\llbracket \xi \rrbracket(u)+1} > t \mid \mathcal{H}_u] = \mathbb{1}_\Lambda(\xi(u)) H_u([t, +\infty)) = P^{(0)}(t-u, \xi(u), \Lambda). \end{aligned}$$

Now, for some $n \geq 0$, our aim is to show by induction that (35) holds at level n . Then

$$\begin{aligned} &\mathbb{P}[\xi(t) \in \Lambda, \llbracket \xi \rrbracket(u, t) = n+1 \mid \mathcal{H}_u] \\ &= \int_u^t \int_{\mathbb{X}} \mathbb{P}[\xi(t) \in \Lambda, \llbracket \xi \rrbracket(\tau_{\llbracket \xi \rrbracket(u)+1}, t) = n \mid \mathcal{H}_u, \tau_{\llbracket \xi \rrbracket(u)+1} = s, Y_{\llbracket \xi \rrbracket(u)+1} = y] K_u(ds, dy) \\ &= \int_u^t \int_{\mathbb{X}} \mathbb{P}[\xi(t) \in \Lambda, \llbracket \xi \rrbracket(s, t) = n \mid \mathcal{H}_{\tau_{\llbracket \xi \rrbracket(u)+1}}, \tau_{\llbracket \xi \rrbracket(u)+1} = s, Y_{\llbracket \xi \rrbracket(u)+1} = y] K_u(ds, dy). \end{aligned}$$

Using the induction hypothesis and the definition of \mathcal{A}^+ in (20) as well as the expression of K_u in Lemma 3, we obtain

$$\begin{aligned} \mathbb{P}[\xi(t) \in \Lambda, \llbracket \xi \rrbracket(u, t) = n+1 \mid \mathcal{H}_u] &= \int_u^t \int_{\mathbb{X}} P^{(n)}(t-s, y, \Lambda) K_u(ds, dy) \\ &= \mathcal{A}^+ P^{(n)}(t-u, \xi(u), \Lambda) = P^{(n+1)}(t-u, \xi(u), \Lambda). \quad (36) \end{aligned}$$

Hence we get the result. \square

Proposition 6. *The process ξ defined by (34) is a Markov process whose transition P satisfies the Kolmogorov forward and backward equations.*

Proof. From Lemma 4, by decomposition over the possible values in $[[\xi]](u, t]$,

$$\begin{aligned} \mathbb{P}[\xi(t) \in \Lambda \mid \mathcal{H}_u] &= \sum_{n \geq 0} \mathbb{P}[\xi(t) \in \Lambda, [[\xi]](u, t] = n \mid \mathcal{H}_u] \\ &= \sum_{n \geq 0} P^{(n)}(t - u, \xi(u), \Lambda) = P(t - u, \xi, \Lambda). \end{aligned}$$

The result follows from Propositions 3 and 4. \square

This process is not only Markov but also strongly Markov, *i.e.* the independence with respect to the σ -algebra \mathcal{H}_τ for any suitable stopping time τ can be assessed also when a fixed time is replaced by a stopping time.

Proposition 7 ([38]). *The process ξ defined by (34) is a strong Markov process.*

Such a process ξ is a *Pure Jump Markov Process*. This class of stochastic processes was introduced by W. Feller in [29] under the name *Purely Discontinuous Markov Process*. This class of processes is included in the class of *Piecewise Deterministic Markov Process* (PDMP) [22], itself included in the class of *Jump Markov Process* [38]. Heuristically, this process is Markov as a consequence of the memoryless property of the exponential distribution.

5.2. A description using a Marked Point Process

Using the kernel G , we have constructed a Markov chain $\{(\tau_k, Y_k)\}_{k \geq 0}$ which we have embedded into a continuous time stochastic process $\{\xi(t)\}_{t \geq 0}$ with jumps at times τ_k such that $\xi(\tau_k) = Y_k$. Here, Y_k represents the mass after a breakage that occurs at time τ_k .

Notation 15 (Loss of mass at breakage). We set $\Delta_k \xi := -(\xi(\tau_{k+1}) - \xi(\tau_k)) \geq 0$.

The sequence $\{(\tau_k, \Delta_k \xi)\}_{k \geq 0}$ is not Markov as it depends on the remaining mass.

We consider now another representation using random measures on $(0, +\infty) \times \mathbb{X}$. For this aim, we define the random measure

$$\mu(\omega; dt, dx) := \sum_{k \geq 1} \delta_{\tau_k(\omega), \Delta_k \xi(\omega)}(dt, dx) \mathbb{1}_{\tau_k(\omega) < +\infty}, \quad (37)$$

where $\delta_{(s,x)}$ is the Dirac mass at (s, x) . This measure is identified with points on $\mathbb{R}_+ \times \mathbb{X}$, where the coordinates of the points represent the time and the loss of mass when a breakage occurs.

Such a measure can be integrated against any measurable, bounded function f on $\overline{\mathbb{R}}_+ \times \mathbb{X}$ with $f(+\infty, x) = 0$ by (omitting the ω),

$$\int_0^{+\infty} \int_{\mathbb{X}} f(s, x) \mu(dt, dx) = \sum_{k \geq 1} f(\tau_k, \Delta_k \xi). \quad (38)$$

The set $\{(\tau_k, \Delta_k \xi)\}_{\geq 1}$ is called a *multivariate point process*, or a *marked point process*.

From the knowledge of μ and (38), we recover many information on ξ . For example, with $f : (t; s, x) \mapsto \mathbb{1}_{(0,t]}(s)$,

$$\int_0^{+\infty} \int_{\mathbb{X}} f(t; s, x) \mu(dt, dx) = \sum_{\substack{k \geq 1 \\ \tau_k \leq t}} 1 = \llbracket \xi \rrbracket(0, t].$$

Using the function $f : (t; s, x) \mapsto -\mathbb{1}_{(0,t]}(s)x$,

$$\int_0^{+\infty} \int_{\mathbb{X}} f(t; s, x) \mu(dt, dx) = - \sum_{\substack{k \geq 1 \\ \tau_k \leq t}} \Delta_k \xi = \xi(t) - \xi(0). \quad (39)$$

The random measure is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. As it contains temporal evolution, we consider also a filtration $(\mathcal{H}_t)_{t \geq 0}$ such that $\mathcal{H}_t \subset \mathcal{F}$ and $\mu((0, t], \cdot)$ is \mathcal{H}_t -measurable for any $t > 0$. Given $\Lambda \in \text{Bor}(\mathbb{X})$ and $t \geq 0$, $h > 0$, one may ask what is the mean number of points of μ falling in $[t, t+h) \times \Lambda$ given \mathcal{H}_{t-} , where \mathcal{H}_{t-} represents the information seen up to time $t-$. For h small,

$$\mathbb{P}[\llbracket \xi \rrbracket[t, t+h) = \ell \mid \mathcal{H}_{t-}] = \begin{cases} \exp(-hF(\xi(\tau_{\llbracket \xi \rrbracket(t-)}))) & \text{if } \ell = 0, \\ hF(\xi(\tau_{\llbracket \xi \rrbracket(t-)})) & \text{if } \ell = 1, \\ o(h) & \text{if } \ell \geq 2. \end{cases}$$

Most likely, there will be no points or one point. Given there is one point, the probability it falls in Λ is $\int \mathbf{G}(\xi(\tau_{\llbracket \xi \rrbracket(t-)}), dx) \mathbb{1}_{\Lambda}(x)$.

Thus, the mean number of points in the small volume $[t, t+dt) \times [y, y+dy)$ is approximatively

$$F(\xi(\tau_{\llbracket \xi \rrbracket(t-)})) \mathbf{G}^{\text{LM}}(\xi(\tau_{\llbracket \xi \rrbracket(t-)}), dx) dt = G^{\text{LM}}(\xi(\tau_{\llbracket \xi \rrbracket(t-)}), dx) dt.$$

This suggests to define a new random measure

$$\begin{aligned}\nu(dt, dx) &= \sum_{k \geq 0} \mathbb{1}_{\tau_k < t \leq \tau_{k+1}} F(\xi(\tau_k)) G^{\text{LM}}(\xi(\tau_k), dx) dt \\ &= \sum_{k \geq 0} \mathbb{1}_{\tau_k < t \leq \tau_{k+1}} G^{\text{LM}}(\xi(\tau_k), dx) dt = G^{\text{LM}}(\xi(\tau_{\llbracket \xi \rrbracket(t-)}), dx) dt.\end{aligned}\quad (40)$$

This random measure ν has two main features: it is $(\mathcal{H}_{t-})_{t \geq 0}$ -measurable, hence depends only on the past. Besides, it is such that $\mathbb{E}[\mu([t, t+h), \Lambda)] = \mathbb{E}[\nu([t, t+h), \Lambda)]$.

We now formalize this construction.

Hypothesis 3. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting $\{(\tau_k, \Delta_k \xi)\}_{k \geq 0}$, we assume that there exists a filtration $\{\mathcal{H}_t\}_{t \geq 0}$ which is right-continuous, such that $\mathcal{H}_t \subset \mathcal{F}$, τ_k is a stopping time with respect to $(\mathcal{H}_t)_{t \geq 0}$ and $\Delta_k \xi$ is \mathcal{H}_{τ_k} -measurable. Furthermore, we assume that

$$\mathcal{H}_t = \mathcal{F}_0 \vee \sigma(\mu((0, s) \times B)); \quad s \leq t, \quad B \in \text{Bor}(\mathbb{X}) \quad \text{for any } t \geq 0. \quad (41)$$

Such a filtration may be called the *intrinsic history*² [21, p. 358].

In (41), $\mu((0, s) \times B)$ is the number of points of $\{(\tau_k, \Delta_k \xi)\}_{k \geq 0}$ that falls in $(0, s) \times B$.

Definition 8 (Predictable process). A process $X : \Omega \times \mathbb{R}_+$ is *predictable* if $(\omega, t) \mapsto X_t(\omega)$ is measurable with respect to the σ -algebra \mathcal{P} generated by left-continuous process which are $\{\mathcal{H}_t\}_{t \geq 0}$ -adapted.

Definition 9. A random measure ν is called *predictable* if for each non-negative process $X : \Omega \times \mathbb{R}_+ \times \mathbb{X}$ which is measurable with respect to $\mathcal{P} \otimes \text{Bor}(\mathbb{X})$, the process $(\nu X) : \Omega \times \mathbb{R}_+$ defined by

$$(\nu X)_t(\omega) := \int_{\mathbb{X}} \int_0^t X(\omega, s, x) \nu(\omega; ds, dx)$$

is predictable.

Our measure ν given by (40) is clearly predictable.

Theorem 1 (Theorem (2.1), (2.4), [37]). *There exists a unique predictable random measure ν (up to a \mathbb{P} -null set) such that*

$$\mathbb{E} \left[\int_{\mathbb{X}} X(t, x) \mu(dt, dx) \right] = \mathbb{E} \left[\int_{\mathbb{X}} X(t, x) \nu(dt, dx) \right] \quad (42)$$

²Using more general filtrations hinders the explicit computation of the compensator, see [21, p. 367].

for any non-negative process $X : \Omega \times \mathbb{R}_+ \times \mathbb{X}$ which is measurable with respect to $\mathcal{P} \otimes \text{Bor}(\mathbb{X})$. Moreover, it may be chosen so that $\nu(\{t\} \times \mathbb{X}) \leq 1$ and $\nu([\tau_\infty, +\infty[\times \mathbb{X}) = 0$. This measure ν is called the dual predictable projection of μ . This measure ν is also characterized by the fact that for any $\Lambda \in \text{Bor}(\mathbb{X})$, $\{\nu((0, t] \times \Lambda)\}_{t \geq 0}$ is predictable and $\{\mu((0, t \wedge \tau_k] \times \Lambda) - \nu((0, t \wedge \tau_k] \times \Lambda)\}_{t \geq 0}$ is a uniformly integrable martingale.

Proposition 8 ([37, Proposition (3.1)]). *The dual predictable projection ν of μ is given by (40).*

Remark 6. In [37, Theorem (3.6)], it is possible to construct μ from a suitable predictable random measure. We note that however, to apply such a construction, we need to construct ξ as in our setting, ν depends on ξ . The issue of simultaneously constructing μ and ξ is solved when considering the approach *via* stochastic differential equations.

Corollary 1. *The counting process $[[\xi]]$ is a non-homogeneous Poisson process with random intensity $t \mapsto F(\xi(t-))$.*

5.3. Marked point processes and the martingale problem

As ξ is piecewise constant, we easily obtain the following change of variable formula using a telescopic sum argument.

Lemma 5 (Change of variable formula). *Let ϕ be a bounded, continuous function from \mathbb{X} to \mathbb{R} . Then for any $t \geq r \geq 0$,*

$$\begin{aligned} \phi(\xi(t)) &= \phi(\xi(r)) + \sum_{\substack{k \geq 0 \\ r < \tau_k \leq t}} (\phi(\xi(\tau_k)) - \phi(\xi(\tau_k-))) \\ &= \phi(\xi(0)) + \sum_{\substack{k \geq 0 \\ \tau_k \leq t}} (\phi(\xi(\tau_k-)) - \Delta_k \xi - \phi(\xi(\tau_k-))). \end{aligned} \quad (43)$$

Let ϕ be a bounded, continuous function, and $t \geq r \geq 0$. Eq. (43) suggests to introduce the process $X_{r,t,\phi}$ from $\Omega \times \mathbb{R}_+ \times \mathbb{X}$ to \mathbb{R} by

$$X_{r,t,\phi}(\omega; s, z) := -\mathbb{1}_{(r,t]}(s)(\phi(\xi(\omega; s-)) - z - \phi(\xi(\omega; s-))).$$

Since $X_{t,\phi}(\omega; s, z)$ involves $s \mapsto \xi(s-)$, $X_{r,t,\phi}$ is $\{\mathcal{H}_t\}_{t \geq 0}$ -adapted and is $\mathcal{P} \otimes \text{Bor}(\mathbb{X})$ -measurable.

With (43) and the definition of μ in (37), we have

$$\phi(\xi(\omega; t)) = \phi(\xi(r; 0)) + \int_0^{+\infty} \int_{\mathbb{X}} X_{r,t,\phi}(\omega; s, z) \mu(\omega; ds, dz).$$

With (40), we obtain

$$\begin{aligned} & \int_0^{+\infty} \int_{\mathbb{X}} X_{r,t,\phi}(\omega; r, z) \nu(\omega; v ds, dz) \\ &= \sum_{\substack{k \geq 0 \\ r < \tau_k \leq t}} \int_{\tau_{k-1}}^{\tau_k} (\phi(\xi(s-) - z) - \phi(\xi(s-))) G^{\text{LM}}(\xi(s-), dz) ds \\ & \qquad \qquad \qquad \int_0^t (\phi(\xi(s) - z) - \phi(\xi(s))) G^{\text{LM}}(\xi(s), dz) ds. \end{aligned}$$

By using the Remark (3) on the operator \mathcal{F} , we get

$$\int_0^{+\infty} \int_{\mathbb{X}} X_{r,t,\phi}(\omega; r, z) \nu(\omega; v ds, dz) = \int_r^t \mathcal{F}\phi(\xi(s)) ds.$$

Proposition 9 ([37, Proposition 3.5]). *Let us assume that $\int_0^t \int_{\mathbb{X}} |X_{r,t,\phi}(\omega; r, z)| \nu(\omega; v ds, dz)$ is finite on $\{t < \tau_\infty\}$. Then*

$$\mathbb{E} \left[\int_0^{+\infty} \int_{\mathbb{X}} X_{r,t,\phi}(\omega; r, z) \mu(\omega; v ds, dz) \right] = \mathbb{E} \left[\int_r^t \mathcal{F}\phi(\xi(s)) ds \right].$$

Moreover, there exists a sequence $\{\theta_k\}_k$ of stopping times, increasing almost surely toward τ_∞ such that $\{Z_{t \wedge \theta_k}\}_k$ is a uniformly integrable martingale for each $k \geq 0$, where Z is a right-continuous $\{\mathcal{H}_t\}_{t \geq 0}$ -adapted process such that

$$Z_t := \xi(0) + \int_0^t \int_{\mathbb{X}} X_{r,t,\phi}(s, x) (\mu(ds, dz) - \nu(ds, dz)) \text{ a.s. on } \{t < \tau_\infty\}.$$

Definition 10 ((Local) Martingale problem). Fix ν a probability distribution on \mathbb{R}_+ . A process $Y \in \mathcal{D}(\mathbb{R}_+, \mathbb{R})$ on a filtered probability space $(\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t \geq 0}, \mathbb{P}')$ is said to solve the (local) *martingale problem* if for any $\phi \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R})$, the process M^ϕ defined by

$$M_t^\phi := \phi(Y_t) - \phi(Y_0) - \int_0^t \mathcal{F}\phi(Y_s) ds \text{ for } t \geq 0,$$

is a (local) martingale with respect to $(\mathcal{F}'_t)_{t \geq 0}$, where Y_0 has the distribution ν .

Corollary 2. *Under suitable assumptions, the distribution of the process ξ solves the local martingale problem.*

5.4. Uniqueness in the martingale problem

Definition 11 (Well-posedness of the martingale problem). The martingale problem given in Definition 10 is *well-posed* whenever for any $y \in \mathbb{R}_+$, there exists one and only one solution to the martingale problem for any initial distribution δ_y .

The next proposition is an extension found in [44] to local martingale problems of [27, Theorem 4.4.2].

Proposition 10. *Assume that $\mathcal{F}\phi$ is bounded on compact subsets of \mathbb{R}_+ for any ϕ which is continuous and bounded. Assume that for any initial distribution ν , any two solutions \mathbb{P} and \mathbb{P}' to the local martingale problem have the same one-dimensional distributions, that is $\mathbb{P}[\xi(t) \in \Lambda] = \mathbb{P}'[\xi(t) \in \Lambda]$ for any $\Lambda \in \text{Bor}(\mathbb{R}_+)$. Then there is at most one solution to the martingale problem, which is moreover strong Markov.*

If moreover, for any $y \in \mathbb{R}_+$, there exists a unique solution \mathbb{P}_x to the martingale problem with initial measure δ_x such that $x \mapsto \mathbb{P}_x[\Lambda]$ for any $\Lambda \in \text{Bor}(\mathbb{R}_+)$, then for any initial distribution μ , there exists a unique solution \mathbb{P}_μ to the martingale problem.

Hypothesis 4. Assume that $x \mapsto g^{\text{LM}}(x, \cdot)$ from \mathbb{R}_+ to \mathbf{M}^+ (space of positive measures) is uniformly bounded and is continuous with respect to the weak topology of \mathcal{M}^+ .

Proposition 11 (Proposition 2.2, [43]). *Under Hypothesis 4, there exists a unique solution Y to the martingale problem, which is also a Markov process.*

Remark 7. Proposition 2.2 contains also a statement on existence and uniqueness of the martingale problem for non bounded kernel g^{LM} .

6. Stochastic differential equations

6.1. Motivation

The martingale problem (Definition 10) was defined in terms of distribution of the process, through the notion of test functions. Through the construction of a marked point process, we have seen that the process ξ constructed by embedding a Markov chain solves the local martingale problem, to that

$$\phi(\xi(t)) = \phi(\xi(0)) + M_t^\phi + \int_0^t \mathcal{F}\phi(\xi(s-)) ds$$

for a continuous function ϕ , for a local martingale M^ϕ . However, can we describe M_t^ϕ ?

6.2. Stochastic Differential Equations driven by a Poisson Point Process

Definition 12 (Compensated Poisson process). Let N be a Poisson Point Process on \mathbb{R}_+^3 with intensity $ds du dv$. The *compensated Poisson process* of N is

$$\tilde{N}(ds, du, dv) := N(ds, du, dv) - ds du dv. \quad (44)$$

We set

$$H(y, u, v) := -u \cdot \mathbb{1}_{u \leq y} \mathbb{1}_{v \leq g^{\text{LM}}(y, u)} \quad (45)$$

$$\text{and } \ell(y) := \int_{\mathbb{X}} \int_{\mathbb{X}} H(y, u, v) du dv = \int_0^y z \cdot G^{\text{LM}}(y, dz). \quad (46)$$

We consider finding a process $X \in \mathfrak{D}(\mathbb{R}_+, \mathbb{R})$ and a PPP N such that

$$X_t = X_0 + \int_0^t \int_{\mathbb{X}} \int_{\mathbb{X}} H(X_{s-}, u, v) N(ds, du, dv) \text{ for } t \geq 0 \quad (47)$$

or equivalently using the compensated Poisson process,

$$X_t = X_0 + \int_0^t \int_{\mathbb{X}} \int_{\mathbb{X}} H(X_{s-}, u, v) \tilde{N}(ds, du, dv) + \int_0^t \ell(X_s) ds \text{ for } t \geq 0. \quad (48)$$

Eq. (47) is called a *stochastic differential equation (SDE) driven by a random measure*. For general accounts on this theory, see for example [9, 35]

Remark 8. As X has a countable number of discontinuities, we could write either $\int_0^t \ell(X_{s-}) ds$ or $\int_0^t \ell(X_s) ds$, as these two quantities are almost surely equal.

6.3. Weak solutions

We consider only existence of weak solutions. The problem of existence of a strong solution remains open, although this seems plausible thanks to the results of Section 4.2.

Definition 13 (Weak solution). Given a distribution ν on \mathbb{R}_+ , a *weak solution* to (47) is any 3-uple (X, N, ζ) defined on a probability space $(\Omega', \mathfrak{G}, (\mathfrak{G}_t)_{t \geq 0}, \mathbb{P}')$ such that the distribution of ζ is ν , the distribution of N is that of a PPP with intensity $ds du dv$ and X solves (47), while ζ is \mathfrak{G}_0 -measurable, and X and N are both $(\mathfrak{G}_t)_{t \geq 0}$ -adapted.

Weak solutions are also solutions to the martingale problem.

Proposition 12 (Itô's formula). *Let us consider X to be a weak solution to the SDE (47). Let ϕ be a continuous, bounded function. Then for any $t \geq 0$,*

$$\begin{aligned} \phi(X_t) = \phi(\zeta) + \int_0^t \int_{\mathbb{R}_+^2} (\phi(X_{s-} + H(X_{s-}, u, v)) - \phi(X_{s-})) \tilde{N}(ds, du, dv) \\ + \int_0^t \mathcal{F} \phi(X_s) ds, \end{aligned} \quad (49)$$

for the compensated Poisson Process \tilde{N} of N . Moreover, X solves the local martingale problem with respect to \mathcal{F} .

Proof. Let ϕ be a continuous, bounded function. Since X is piecewise constant,

$$\phi(X_t) - \phi(\zeta) = \sum_{t \text{ s.t. } \Delta X_t \neq 0} (\phi(X_t + \Delta_t X) - \phi(X_{t-})) \text{ with } \Delta_t X = X_t - X_{t-}. \quad (50)$$

Using (48), we obtain

$$\phi(X_t) = \phi(\zeta) + \int_0^t \int_{\mathbb{R}_+^2} (\phi(X_{s-} + H(X_{s-}, u, v)) - \phi(Y_{s-})) N(ds, du, dv). \quad (51)$$

For any event A and any $0 \leq u \leq y$,

$$\phi(y - u \mathbb{1}_A) - \phi(y) = (\phi(y - u) - \phi(y)) \mathbb{1}_A.$$

Hence,

$$\begin{aligned} \int_0^t \int_{\mathbb{R}_+^2} (\phi(X_{s-} + H(X_{s-}, u, v)) - \phi(X_{s-})) du dv ds \\ = \int_0^t \int_0^{X_{s-}} (\phi(X_{s-} - u) - \phi(X_{s-})) g^{\text{LM}}(X_{s-}, u) du ds \\ = \int_0^t \mathcal{F} \phi(X_{s-}) ds = \int_0^t \mathcal{F} \phi(X_s) ds. \end{aligned}$$

Using the compensated Poisson process, we rewrite (50) and (51) as (49). As ϕ is bounded,

$$M_t^\phi := \int_0^t \int_{\mathbb{R}_+^2} (\phi(X_{s-} + H(X_{s-}, u, v)) - \phi(Y_{s-})) \tilde{N}(ds, du, dv), \quad t \geq 0$$

is a martingale [35, Lemma 3.1]. □

Corollary 3. *If the fragmentation kernel is such that the martingale problem is well posed, then the associated process is a Markov process. Besides, any weak solution to (47) has a transition kernel which satisfies the Kolmogorov forward equation.*

Proof. Let X be a weak solution to (47), which is also a Markov process. Using (49),

$$\mathbb{E}[\phi(X_t)] = \mathbb{E}[\phi(\zeta)] + \int_0^t \mathbb{E}[\mathcal{F}\phi(X_s)] ds, \quad \forall t \geq 0.$$

As X is a Markov process, it is characterized by its probability transition function $P(t, x, \Lambda)$, $t \geq 0$, $x \in \mathbb{X}$, $\Lambda \in \text{Bor}(\mathbb{X})$. Therefore, if μ is the distribution of the initial condition ζ ,

$$\begin{aligned} \int_{\mathbb{X}} \int_{\mathbb{X}} \mu(dx) P(t, x, dy) \phi(y) &= \mathbb{E}[\phi(X_t)] \\ &= \int_{\mathbb{X}} \mu(dx) \phi(x) + \int_0^t \int_{\mathbb{X}} \int_{\mathbb{X}} \mu(dx) P(t, x, dy) \mathcal{F}\phi(y). \end{aligned} \quad (52)$$

Using Notation 7,

$$\langle \mu, \partial_t P_t \phi \rangle = \langle \mu, P_t \mathcal{F}\phi \rangle,$$

which is nothing more than the Kolmogorov forward equation (13). \square

6.4. A weak existence condition

We now give a condition for existence of weak solution of SDE.

Hypothesis 5. The kernel F is symmetric (Hypothesis 1). The map ℓ defined by (46) is continuous with

$$\ell(x) \leq C(1 + x^p) \quad (53)$$

for some constants $C \geq 0$ and $p \geq 1$. Moreover,

$$\lim_{n \rightarrow \infty} \sup_{\varepsilon \leq x \leq \varepsilon^{-1}} \ell_n(x) = 0 \text{ for any } \varepsilon \in (0, 1)$$

with

$$\ell_n(x) = \int_0^x \frac{y(x-y)}{x} F(x-y, y) \mathbb{1}_{F(x-y, y) \geq n} dy.$$

Proposition 13 ([31, Proposition 3.1]). *Under Hypothesis 5, when ζ is finite moment of order $p+1$ (where p appears in (53)), then there exists a weak solution to the SDE (47).*

7. Mass conservation

The *mass conservation* means that the density satisfies $P(t, x, \mathbb{R}_+) = 1$ for any time $t > 0$ and any starting point $x > 0$. In the framework of Markov chains, it means that Y_k never reaches the cemetery point \dagger in finite time, *i.e.* $\lim_{k \rightarrow \infty} \tau_k = +\infty$. In our case, as the mass is non-increasing with respect to the time, one can use 0 as the cemetery point \dagger .

If $P(t, x, \mathbb{R}_+) = 1$ for any $t > 0$, then the associated semi-group is said to be *stochastic*. Otherwise, it is *substochastic*.

Several sufficient conditions on the kernel can be given to ensure that the semi-group is stochastic [52, 56, 58, 61]. We just give one such condition. Discussions about cases where the mass is not conserved due to the creation of an infinite number of particles of zero mass (dust) may be found in [6, 51, 58, 59, 60] among others. Such a phenomenon is called *shattering* and was pointed out first by A.F. Filippov [30].

Proposition 14 ([56, Corollary 6.1]). *If F is bounded on bounded subsets of $(0, +\infty)$, then the mass is conserved.*

8. A simulation algorithm

8.1. Computation for some kernels

In Table 1, we consider a variety of standard kernels and compute the quantities associated to some usual fragmentation kernels.

In Figures 1 and Figures 2, we present the quantiles (10 %, 20 %, 50 %, 75 %, 90 %) and the means at different masses for the evolution of a set of particles over the time, with a constant initial mass, and with an initial mass distributed being log-normally distributed. Although almost all the kernels in Table 1 have the same breaking dynamics $G(x, \cdot)$, the rate strongly influence the behavior of the dynamics of the cloud of particles.

We summarize the relation between the different kernels and functions for a

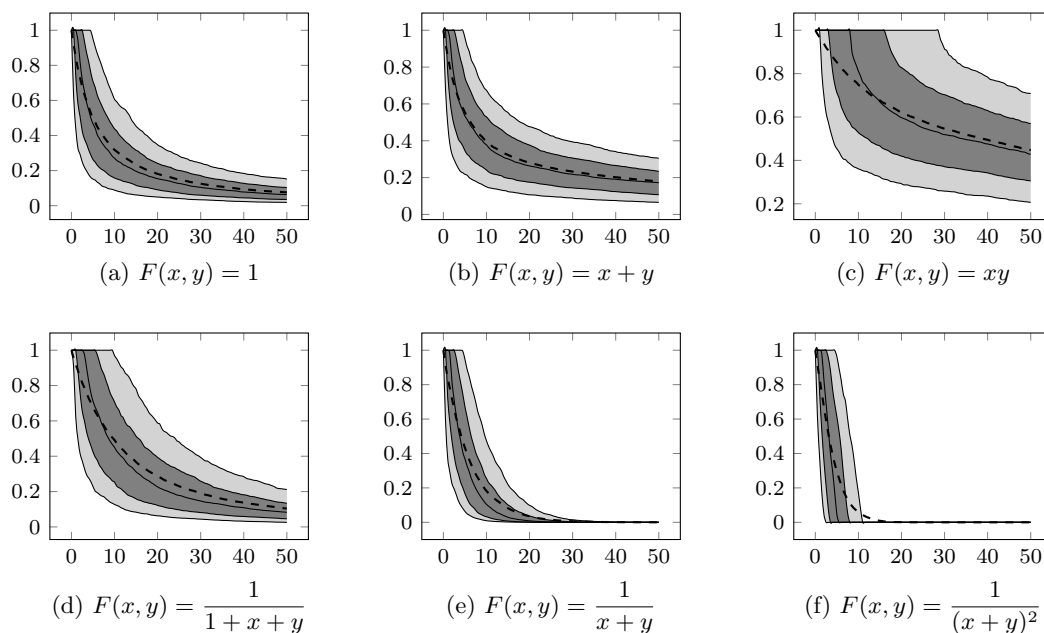


Figure 1: Statistics of the evolution of a cloud of particles with the time. Each line represents a quantile (10 %, 20 %, 50 %, 75 %, 90 %, the lines are ordered from bottom to top), and the dashed line represents the mean. The initial mass is 1. The cloud has 2000 particles.

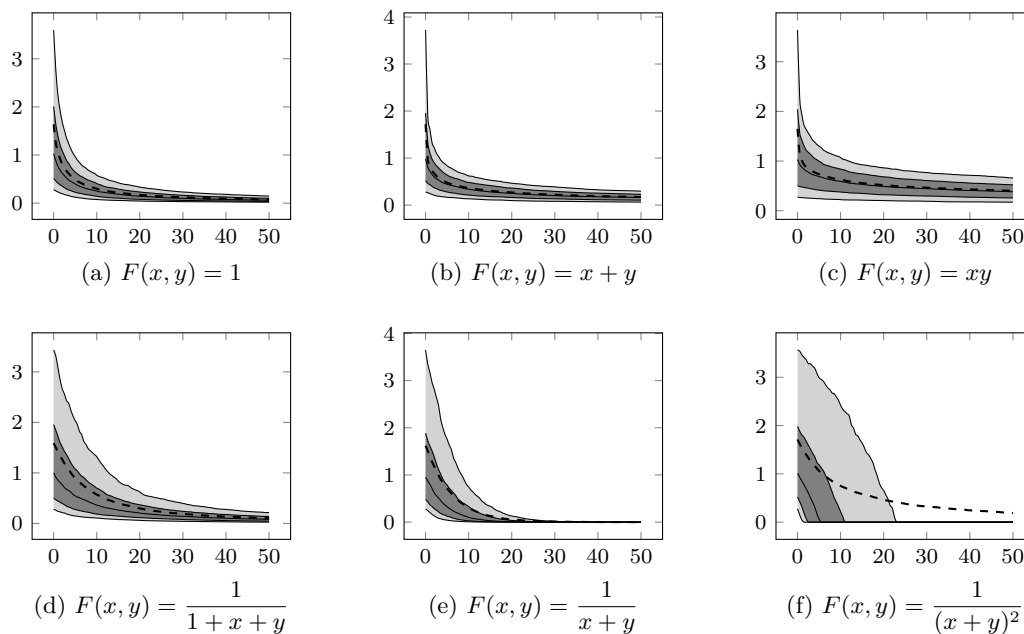


Figure 2: Statistics of the evolution of a cloud of particles with the time. Each line represents a quantile (10 %, 20 %, 50 %, 75 %, 90 %, the lines are ordered from bottom to top), and the dashed line represents the mean. The initial mass follows the log-normal distribution of log-mean 0 and log-variance 1.0. The cloud has 2000 particles.

symmetric kernel $F(x, y)$:

$$F(x) = \int_0^x F(x-y, y) \frac{x-y}{x} dy = \int_0^x F(x-y, y) \frac{y}{x} dy \quad \text{at (4),}$$

$$G(x, dy) = \mathbb{1}_{[0, x]}(y) F(x-y, y) \frac{y}{x} dy \quad \text{at (15),}$$

$$\mathbf{G}(x, dy) = \frac{G(x, dy)}{F(x)} \quad \text{at (15),}$$

$$G^{\text{LM}}(x, dy) = F(x-y, y) \frac{x-y}{x} \mathbb{1}_{0 \leq y \leq x} dy \quad \text{at (16),}$$

$$\mathbf{G}^{\text{LM}}(x, dy) = \frac{G^{\text{LM}}(x, dy)}{F(x)} \quad \text{at (16),}$$

$$B(y|x) dy = \frac{F(x-y, y)}{F(x)} dy = \mathbf{G}(x, dy) \frac{x}{y} \quad \text{at (8).}$$

The distribution function of the mass of a fragment after a breakage of a x -mer is then $\mathbf{G}(x, [0, y]) = \int_0^y \mathbf{G}(x, dz)$.

Self-similar fragmentation. In many situations, there exists a function H such that $\mathbf{G}(x, [0, y]) = H(y/x)$ for $y \leq x$. For $x = 1$, $y \in [0, 1] \mapsto H(y)$ is the distribution function of a random variable we denote by Y . Then $Y(x) = xY$ has for distribution function $H : y \in [0, x] \mapsto H(y/x) = \mathbf{G}(x, [0, y])$. In such situation, this means that the breakage of the particle x to news particles of mass y and $x-y$ depends only on the relative mass y/x . Regarding simulation, one has only to know how to simulate the random variable Y .

If $B(y|x) = b(y/x)/x$ for some function $b : [0, 1] \rightarrow \mathbb{R}_+$ such that $\int_0^1 b(u) du = 1$, then the corresponding kernel has for probability density function $\mathbf{G}(x, dy) = b(y/x) \cdot y/x^2$ and

$$\mathbf{G}(x, [0, y]) = \int_0^y b\left(\frac{z}{x}\right) \frac{z}{x} \frac{dz}{x} = H\left(\frac{y}{x}\right) \text{ where } H(v) := \int_0^v b(u)u du.$$

Examples of such functions b and their characteristics, inference and applications may be found in [24].

Time-change by scaling the kernel. Let $\lambda > 0$ be a scalar factor, and we consider replacing a kernel $F(x, y)$ by $F^{[\lambda]}(x, y) := \lambda F(x, y)$. The rate then become $F^{[\lambda]}(x) = \lambda F(x)$. This means that the mean of the exponential distribution drawn at each jump is divided by λ .

$F(x, y)$	$G(x, y)$	$F(x)$	$G(x, [0, y])$
1	$\frac{y}{x}$	$\frac{x}{2}$	$\frac{y^2}{x^2}$
$x + y$	y	$\frac{x^2}{2}$	$\frac{y^2}{x^2}$
xy	$\frac{y^2(x - y)}{x}$	$\frac{x^3}{12}$	$\frac{4y^3}{x^3} - \frac{3y^4}{x^4}$
$\frac{1}{1 + x + y}$	$\frac{y}{x(x + 1)}$	$\frac{x}{2(1 + x)}$	$\frac{y^2}{x^2}$
$\frac{1}{(x + y)^n}, n \geq 1$	$\frac{y}{x^{n+1}}$	$\frac{1}{2x^{n-1}}$	$\frac{y^2}{x^2}$

Table 1: Computations for different fragmentation kernels for $y \in [0, x]$.

Let $p^{[\lambda]}(t, dy)$ be the solution to the corresponding fragmentation equation. Then, a simple computation shows that $p^{[\lambda]}(t, \cdot) = p^{[1]}(\lambda t, \cdot)$ for any $t > 0$. Thus, scaling the kernel corresponds to scale the time dynamic, and to scale accordingly the parameter of the exponential distributions giving the breakages' times.

8.2. A general simulation algorithm

The algorithm follows the considerations of Section 4.1. Unless the kernel has a simple form, we implement a rejection principle as in Section 4.2. The algorithm is really simple to set up. The memoryless property of the exponential (Lemma 6) means that by restarting the algorithm using the position of the particle at time T up to time T' , the particles' distribution is the same as the one at horizon $T + T'$.

8.3. Practical implementation

Algorithm 1 is easy to implement.

The ways to draw a random variate Y with density $G(x, \cdot)$ depends on the expression of its distribution function $y \mapsto G(x, [0, y])$. If $H := y \in [0, x] \mapsto G(x, [0, y]) \in [0, 1]$ is easy to invert, then $H^{-1}(U)$ for a uniform random variable U on $[0, 1]$.

Data: A fragmentation kernel F , a time T and a mass m

Result: The mass of a particle initially of mass m at time T

```

1  $t \leftarrow 0$ ;
2  $\xi \leftarrow m$ ;
3 while  $t < T$  do
4   Draw a random variate  $\zeta$  with law  $\mathcal{G}(F(\xi))$ ;
5   Draw a random variate  $\xi'$  with law  $\mathbf{G}(\xi, \cdot)$ ;
6    $\xi \leftarrow \xi'$ ;
7    $t \leftarrow \min\{T, t + \zeta\}$ ;
8 end
9 return  $\xi$ ;

```

Algorithm 1: Monte Carlo algorithm for the simulation of the mass of a typical particle at time T after a fragmentation process.

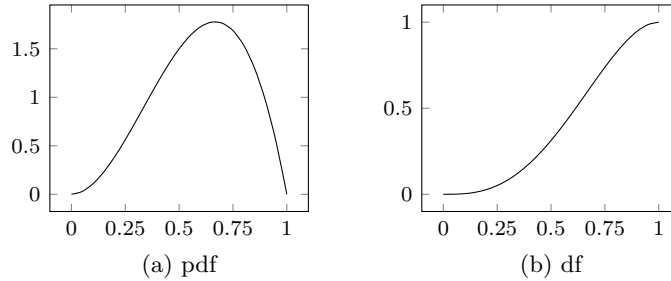


Figure 3: Probability density function (a) and distribution function (b) of $\mathbf{G}(x, \cdot)$ for the multiplicative kernel $F(x, y) = xy$.

For example, when $H(y) = y^2/x^2$, then

$$H^{-1}(u|x) = x\sqrt{u} \text{ for } 0 \leq u \leq 1.$$

This is the case for several kernels in Table 1. In Figure 3, we present the density and distribution functions associated to the multiplicative kernel.

If H^{-1} has no tractable form or is not easily computed numerically, we may rely on the rejection algorithm 2 in Appendix C. Other *ad hoc* simulation algorithms may be found in [23].

9. Conclusion

We have presented a survey on the probabilistic representations of the solution to the fragmentation equation. These representations allow to perform easily some

Monte Carlo simulation. Besides, they can be extended in several directions, among them

- The breakage mechanism may occur from different phenomena: it is easy to deal with several competing kernels.
- The mass can be subject to a continuous abrasion [18] or mass accumulation (growth-fragmentation) [12, 13, 19]. In such case, the times of the next breakage is the first time of a time-inhomogeneous Poisson process [47].
- The evolution of the mass may be coupled with the position and the speed.
- The evolution of mass may be of “mean-field type”, meaning that the fragmentation kernel actually depends on the distribution of the mass at a given time.

The question of the inference of the kernel remains largely opened. Some methods and algorithms may be found in [3, 4, 48] for Monte Carlo approach, [25, 34] for an approach based on PDE, or from the observations of a single path [1, 2].

A. Properties of the exponential distribution

We recall here some fundamental properties of the exponential distribution.

Lemma 6 (Memoryless property). *Let $\theta \sim \mathcal{E}(\lambda)$. Then*

$$\mathbb{P}[\theta > t + u \mid \theta > t] = \mathbb{P}[\theta > u] \text{ for any } u, t \geq 0.$$

Lemma 7. *Let $\xi_1 \sim \mathcal{E}(\lambda_1), \dots, \xi_n \sim \mathcal{E}(\lambda_n)$ for some positive parameters $\lambda_1, \dots, \lambda_n$. Then $\zeta := \min\{\xi_1, \dots, \xi_n\}$ has the distribution $\mathcal{E}(\lambda)$ with $\lambda := \lambda_1 + \dots + \lambda_n$. Besides, for $J := \arg \min_{i=1, \dots, n} \xi_i$ is independent from ζ and*

$$\mathbb{P}[J = i] = \frac{\lambda_i}{\lambda}.$$

B. Random measures and Poisson Point Processes

We present here some facts regarding Poisson Point Processes (PPP) and random measures. More may be found in [32, 33, 41].

Notation 16 (Poisson distribution). We denote by $\mathcal{P}(\lambda)$ the Poisson distribution of parameter λ , that is $X \sim \mathcal{P}(\lambda)$ has its distribution characterized by $\mathbb{P}[X = k] = \lambda^k \exp(-\lambda)/k!$ for $k = 0, 1, 2, \dots$.

Lemma 8. Let N be a PPP on $\mathbb{R}_+ \times \mathbb{R}^d$ with intensity measure n of the form $n(ds, dx) = ds \cdot \nu(dx)$ for a measure ν on \mathbb{R}^d . Let $\Lambda \in \text{Bor}(\mathbb{R}^d)$ such that $\nu(\Lambda) < +\infty$. Denote by (ζ, Y) the coordinate of the leftmost point of $N \cap \mathbb{R}_+ \times \mathbb{R}^d$. Then $\zeta \sim \mathcal{E}(\nu(\Lambda))$ while $\mathbb{P}[Y \in \Lambda'] = \nu(\Lambda')/\nu(\Lambda)$ for any $\Lambda' \in \text{Bor}(\mathbb{R}^d)$, $\Lambda' \subset \Lambda$. At last, ζ and Y are independent.

Remark 9. In the above statement, we may replace the leftmost point of $N \cap \mathbb{R}_+ \times \mathbb{R}^d$ by the leftmost point of $N \cap [t, +\infty) \times \mathbb{R}^d$ for any $t \geq 0$.

Proof. For any $t, h \geq 0$ and $\Lambda' \in \text{Bor}(\mathbb{R}^d)$, set $\Lambda'(t, t+h) = [t, t+h) \times \Lambda'$. Remark that $n(\Lambda(t, t+h)) = h\nu(\Lambda')$.

The key point is that when h is small,

$$\begin{aligned} \mathbb{P}[(\xi, Y) \in \Lambda'(t, t+h)] &= \mathbb{P}[N(\Lambda'(t, t+h)) = 1, N(\Lambda(0, t)) = 0] + o(h^2) \\ &= \mathbb{P}[N(\Lambda'(t, t+h)) = 1] \mathbb{P}[N(\Lambda(0, t)) = 0] + o(h^2) \\ &= h\nu(\Lambda') \exp(-h\nu(\Lambda')) \exp(-t\nu(\Lambda)) + o(h^2). \end{aligned}$$

Thus, set $h = 1/n$ for some integer n and $t_k = a + (b-a)kh$ for $0 \leq a \leq b$. Hence,

$$\begin{aligned} \mathbb{P}[(\xi, Y) \in \Lambda'(a, b)] &= \sum_{k=0}^{n-1} \mathbb{P}[(\xi, Y) \in \Lambda'(t_k, t_{k+1})] \\ &= \sum_{k=0}^{n-1} \frac{b-a}{n} \nu(\Lambda') \exp\left(-\frac{1}{n}\nu(\Lambda')\right) \exp(-t_k\nu(\Lambda)) + o(n^{-1}) \\ &\xrightarrow{n \rightarrow \infty} \frac{\nu(\Lambda')}{\nu(\Lambda)} (\exp(-b\nu(\Lambda)) - \exp(-a\nu(\Lambda))). \end{aligned}$$

This proves the result. □

C. The rejection principle

Introduced by J. Von Neumann, the *rejection principle* is a convenient way to simulate random variables from the knowledge of their densities [23].

Lemma 9. Let p be proportional to the density of a random variable Y with $p(x) = 0$ when $x \notin [0, K]$. Let (U, V) be two random variables that are uniformly distributed under $\{(u, v) \mid 0 \leq v \leq p(u)\}$. Then U is distributed as Y .

Proof. Let (U, V) be as in the statement of the lemma. Then

$$\begin{aligned} \mathbb{P}[U \leq x] &= \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} \mathbb{P} \left[(U, V) \in \left[\frac{kx}{n}, \frac{(k+1)x}{n} \right] \times \left[0, p \left(\frac{(k+1/2)x}{n} \right) \right] \right] \\ &= \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} \frac{\sum \frac{x}{n} p \left(\frac{(k+1/2)x}{n} \right)}{\int_0^K p(y) \, dy} = \frac{\int_0^x p(y) \, dy}{\int_0^K p(y) \, dy}. \end{aligned} \quad (54)$$

Hence the result. \square

From this, we derive the following result, as the distribution of a random variable $Z \in \mathbb{R}^2$ given $Z \in \Lambda$ for a Borel set Λ is uniformly distributed in Λ when Z is uniformly distributed in \mathbb{R}^2 , or any subset of \mathbb{R}^2 containing Λ . Using a rectangular set Λ in Lemma 10, this leads to the simple Monte Carlo algorithm 2.

Lemma 10 (Monte Carlo simulation using the rejection principle). *Let p be proportional to the density of a random variable Y with $0 \leq p(x) \leq M$ for $x \in [0, K]$ and $p(x) = 0$ otherwise. Let (U, V) be two uniform random variables on $[0, K] \times [0, M]$. Then the distribution of U given $V \leq p(U)$ is the one of Y .*

Data: A function $p : [0, K] \rightarrow [0, M]$.

Result: A random variate whose density is proportional to p

1 **repeat**

2 | Draw a random variate $U \sim \mathcal{U}(0, 1)$;

3 | Draw a random variate $V \sim \mathcal{U}(0, 1)$;

4 **until** $V \cdot M \leq p(K \cdot U)$;

5 **return** U ;

Algorithm 2: Monte Carlo rejection algorithm.

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