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# Hybrid high-order methods for elliptic PDEs on curved and complicated domains

Zhaonan Dong and Zuodong Wang

**Abstract** We introduce a hybrid high-order method employing Nitsche’s boundary penalty techniques for the Poisson problem on the curved and complicated domain. There are two key ideas in this work: Firstly, the methods employ the Nitsche-type boundary penalty technique to weakly enforce the boundary conditions, which avoids the computation of the parameterized mapping for the curved boundary. Secondly, an optimal  $L^2$  error estimate for the Poisson problem with mixed Dirichlet and Neumann boundary conditions is derived. Moreover, the stability and optimal error estimate for the proposed HHO methods are independent of the number and measure of faces on the domain boundary. Finally, a numerical experiment is presented in this chapter to confirm the theoretical results.

## 1 Introduction

Hybrid high-order (HHO) methods were introduced for solving the linear diffusion in [9] and for locking-free linear elasticity in [8]. In the HHO method, the discrete unknowns are attached to the mesh cells, and the mesh faces. The two key ingredients for HHO methods are local reconstruction operators and local stabilization operators in each mesh cell. HHO methods offer various attractive features, such as the support of polytopal meshes, optimal error estimates, local conservation properties, and computational efficiency due to compact stencils and local elimination of the cell unknowns by static condensation. We refer to two monographs [5, 6] for a comprehensive review.

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The present work aims to analyze and test a variant of the HHO method employing Nitsche's techniques for solving the Poisson problem with mixed boundary conditions on curved and complicated domains. Using Nitsche's techniques for weakly enforcing the boundary conditions or the interface conditions for HHO methods has been applied for second-order, and fourth-order elliptic PDEs on polygonal and polyhedral meshes cells [2, 1, 4, 10, 11]. The HHO methods mentioned above can deal with the curved interface or curved domain boundary. However, to the best of the authors' knowledge, no HHO method in the literature can solve the Poisson problem on a complicated domain containing a lot of small faces on Neumann boundary without being influenced by the number of faces.

There are two main novelties of the HHO methods in this work: Firstly, the proposed HHO methods do not contain any face unknowns on the Dirichlet and Neumann boundary faces. There is no need to compute the parameterized mapping to satisfy the boundary condition on the curved domain. Moreover, the size of the linear system is independent of the number of faces on the domain boundary, which is essential for solving PDEs on a complicated domain with a lot of tiny faces. Secondly, a new  $L^2$ -norm error estimate for the Poisson problem with mixed boundary conditions is proven to be independent of the number and measure of faces on the domain boundary. Finally, we present one numerical example to test the proposed method.

## 2 Model and discrete setting

We follow the standard notations in Sobolev space,  $\|\cdot\|$  means standard  $L^2$  norm and  $|\cdot|_{H^{t+1}}$  means standard  $H^{t+1}$  seminorm, and we simplify the notation of  $L^2$  inner product as  $(\cdot, \cdot)$ , if there is no special statement.

### 2.1 Model problem

Let  $\Omega$  be an open Lipschitz domain in  $\mathbb{R}^d$ ,  $d \in \{2, 3\}$ ,  $\Gamma = \Gamma_D \cup \Gamma_N$ , where  $\Gamma_D$  is the closed Dirichlet boundary satisfying  $|\Gamma_D| > 0$  and  $\Gamma_N$  is the Neumann boundary. We consider the solution  $u \in H^1(\Omega)$  of the following elliptic boundary problem:

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= g_D \quad \text{on } \Gamma_D, \quad \nabla u \cdot \mathbf{n}_\Omega = g_N \quad \text{on } \Gamma_N \end{aligned} \tag{1}$$

with  $f \in L^2(\Omega)$ ,  $g_D \in H^{\frac{1}{2}}(\Gamma_D)$ ,  $g_N \in L^2(\Gamma_N)$  and  $\mathbf{n}_\Omega$  to be the unit outward normal vector of domain  $\Omega$ .

The weak form of the above problem is defined as: Find  $u \in H^1(\Omega)$ , satisfying  $u|_{\Gamma_D} = g_D$ ,

$$a(u, v) = \ell(v), \quad \forall v \in H_{\Gamma_D}^1(\Omega),$$

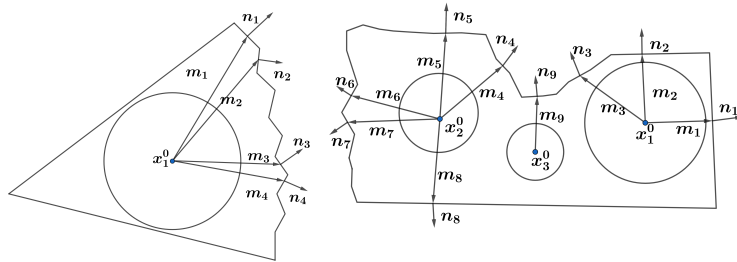
with  $a(u, v) := (\nabla u, \nabla v)_\Omega$  and  $\ell(v) := (f, v)_\Omega + (g_N, v)_{\Gamma_N}$ , where  $H_{\Gamma_D}^1(\Omega) = \{v \in H^1(\Omega) | v = 0 \text{ on } \Gamma_D\}$ .

## 2.2 Polytopal and curved meshes

Let  $\{\mathcal{T}_h\}_{h>0}$  be a mesh family such that each mesh  $\mathcal{T}_h$  covers  $\Omega$  exactly. A generic mesh cell is denoted by  $T \in \mathcal{T}_h$ , its diameter by  $h_T$ , and its unit outward normal vector by  $\mathbf{n}_T$ . We partition the boundary  $\partial T$  of any mesh cell  $T \in \mathcal{T}_h$  by means of the two subsets  $\partial T^i := \overline{\partial T} \cap \Omega$ ,  $\partial T^b := \partial T \cap \partial\Omega$ . Similarly, we partition the mesh as  $\mathcal{T}_h = \mathcal{T}_h^i \cup \mathcal{T}_h^b$ , where  $\mathcal{T}_h^b$  is the collection of all the mesh cells  $T$  such that  $\partial T^b$  has positive measure. Moreover, we further split  $\partial T^b$  into two nonoverlapping subsets  $\partial T^D := \partial T^b \cap \Gamma_D$  and  $\partial T^N := \partial T^b \cap \Gamma_N$ . The mesh faces are collected in the set  $\mathcal{F}_h$ , which is split as  $\mathcal{F}_h = \mathcal{F}_h^i \cup \mathcal{F}_h^b$ , where  $\mathcal{F}_h^i$  is the collection of the interior faces (shared by two distinct mesh cells) and  $\mathcal{F}_h^b$  the collection of the boundary faces. For all  $F \in \mathcal{F}_h$ , we orient  $F$  by means of the fixed unit normal vector  $\mathbf{n}_F$  whose direction is arbitrary for all  $F \in \mathcal{F}_h^i$  and  $\mathbf{n}_F := \mathbf{n}_\Omega$  for all  $F \in \mathcal{F}_h^b$ . For any mesh cell  $T \in \mathcal{T}_h$ , the mesh faces composing its boundary  $\partial T$  are collected in the set  $\mathcal{F}_{\partial T}$ , which is partitioned as  $\mathcal{F}_{\partial T} = \mathcal{F}_{\partial T^i} \cup \mathcal{F}_{\partial T^D} \cup \mathcal{F}_{\partial T^N}$  with obvious notation.

**Assumption** (1). Any interior mesh  $T \in \mathcal{T}_h^i$  is a polytope with planar faces, and the sequence of interior meshes  $\mathcal{T}_h^i$  is shape-regular in the sense of [8].

(2). For any boundary mesh cell  $T \in \mathcal{T}_h^b$ , all the faces in  $\mathcal{F}_{\partial T^i}$  are planar with diameter uniformly equivalent to  $h_T$ . Moreover, for each  $\partial T^b$ ,  $T$  can be decomposed into a finite union of nonoverlapping subsets,  $\{T_{\partial T^b, m}\}_{m \in \{1, \dots, n_{T, \partial T^b}\}}$ , such that  $T_{\partial T^b, m}$  is star-shaped with respect to an interior ball of radius  $r_{T, m}$  satisfying  $\rho h_T \leq r_{T, m}$ , with the mesh-regularity parameter  $\rho > 0$ . We refer to Figure 1 for an illustration with  $n_{T, \partial T^i} = 1$  (left) and  $n_{T, \partial T^i} = 3$  (right).  $\square$



**Fig. 1** Examples for the mesh assumptions

*Remark 1 (Mesh assumptions)* The above mesh assumptions on the mesh sequence are pretty general. The assumption (1) ensures that the interior faces of the mesh are planar is essential for the HHO methods, which employ the discrete unknowns attached to the interior faces. On the other hand, using a Nitsche-like penalty method avoids introducing discrete unknowns on the boundary faces. Therefore, the face on the domain boundary can be curved or complicated. The star-shapedness assumption (2) on the mesh boundary cells is introduced to invoke a Poincaré-type inequality, discrete trace inequality and trace inequality to hold in such cells.

### 2.3 Analysis tools

Let us briefly review the main analysis tools used in this report. We only state the results. In what follows, we always consider a shape-regular mesh sequence satisfying Assumptions. Moreover, in various bounds, we use  $a \lesssim b$  to denote  $a \leq Cb$  with  $C$  to be any generic constant (its value can change at each occurrence) that is independent of mesh size  $h > 0$ , but may depend on the shape-regularity of the mesh sequence and the polynomial degree for all  $T \in \mathcal{T}_h$ . Let  $k \in \mathbb{N}$  be the polynomial degree and  $\mathbb{P}_{k,d}(T)$  to be the polynomial space of order at most  $k$  in any cell  $T \in \mathcal{T}_h$ .

#### Lemma 1 (Discrete trace inequalities)

For any  $T \in \mathcal{T}_h$  satisfying the mesh assumptions, the following relation holds for all  $v \in \mathbb{P}_{k,d}(T)$ :

$$\|v\|_{\partial T} \lesssim h_T^{-\frac{1}{2}} \|v\|_T. \quad (2)$$

#### Lemma 2 (Multiplicative trace inequality)

Let element  $T \in \mathcal{T}_h$ . Then, the following statement holds true for all  $v \in H^1(\Omega)$ :

$$\|v\|_{\partial T} \lesssim (h_T^{-\frac{1}{2}} \|v\|_T + \|v\|_T^{\frac{1}{2}} \|\nabla v\|_T^{\frac{1}{2}}). \quad (3)$$

#### Lemma 3 (Polynomial approximation)

Let  $k \geq 0$  be the polynomial degree, and there is a real number  $t \in [0, k + 1]$ , all  $m \in \{0, \dots, \lfloor t \rfloor\}$ , and  $v \in H^{t+1}(T)$ ,

$$|v - \Pi_T^k(v)|_{H^m(T)} \lesssim h_T^{t-m} |v|_{H^t(T)}, \quad (4)$$

where  $\Pi_T^k$  denoted the  $L^2$ -orthogonal projection onto  $\mathbb{P}_{k,d}(T)$ .

The following approximation result holds by using (4) combined with the multiplicative trace inequality (3): Let  $k \geq 0$  be the polynomial degree, the following estimates holds for all  $v \in H^{t+1}(T)$ ,

$$\begin{aligned} & \|v - \Pi_T^{k+1}(v)\|_T + h_T^{\frac{1}{2}} \|v - \Pi_T^{k+1}(v)\|_{\partial T} + h_T \|\nabla(v - \Pi_T^{k+1}(v))\|_T \\ & + h_T^{\frac{3}{2}} \|\nabla(v - \Pi_T^{k+1}(v))\|_{\partial T} \lesssim h_T^{t+1} |v|_{H^{t+1}(T)} \end{aligned} \quad (5)$$

*Remark 2* The proofs of the above lemmas on mesh cells having flat faces can be found in [7, Sec. 1.4.3] and [14, 12]. More general results on general shaped polygonal and curved meshes have been developed recently; we refer to [2, 3].

### 3 HHO discretization

In this section, we will introduce the mixed-order HHO method employing Nitsche's boundary penalty techniques. We first start to introduce the HHO space. Let  $k \geq 0$  be the polynomial degree. We consider a pair  $\hat{v}_T^k := (v_T^{k+1}, v_{\partial T^i}^k)$ , where  $v_T^{k+1}$  is defined on  $T$  and  $v_{\partial T^i}^k$  is defined on the inner faces (facewise)  $F \in \mathcal{F}_{\partial T^i}$  composing the boundary  $\partial T^i$  of  $T$ . We define the local HHO space

$$\hat{V}_T^k := \mathbb{P}_{k+1,d}(T) \times \mathbb{P}_{k,d-1}(\mathcal{F}_{\partial T^i}),$$

noticing the above local HHO space does not contain face unknowns on the domain boundary.

Next, the construction of the HHO methods are as follows: We start with the reconstruction of the gradient operator and then add the stabilization operator. The reconstruction operator is defined as:  $R_T^i : \hat{V}_T^k \rightarrow \mathbb{P}_{k+1,d}(T)$  s.t. for every pair  $\hat{v}_T^k = (v_T^{k+1}, v_{\partial T^i}^k) \in \hat{V}_T^k$ , the function  $R_T^i(\hat{v}_T^k)$  is s.t. for all  $q \in \mathbb{P}_{k+1,d}(T)$ ,

$$\begin{aligned} (\nabla R_T^i(\hat{v}_T^k), \nabla q)_T &:= (\nabla v_T^{k+1}, \nabla q)_T - (v_T^{k+1} - v_{\partial T^i}^k, \partial_n q)_{\partial T^i} - (v_T^{k+1}, \partial_n q)_{\partial T^D} \\ &= -(v_T^{k+1}, \Delta q)_T + (v_{\partial T^i}^k, \partial_n q)_{\partial T^i} + (v_T^{k+1}, \partial_n q)_{\partial T^N} \end{aligned} \quad (6)$$

and together with  $(R_T^i(\hat{v}_T^k) - v_T^{k+1}, 1)_T := 0$ . To alleviate the notation, let  $\partial_n$  denote the outward normal derivative on  $\partial T$ . In addition, we define the lifting operator  $L_T^{k+1} : L^2(\partial T^D) \rightarrow \mathbb{P}_{k+1,d}(T)$  for all  $T \in \mathcal{T}_h$  such that, for all  $g_D \in L^2(\partial T^D)$  and all  $q \in \mathbb{P}_{k+1,d}(T)$ ,

$$(\nabla L_T^{k+1}(g_D), \nabla q)_T := (g_D, \partial_n q)_{\partial T^D},$$

together with the condition  $(L_T^{k+1}(g_D), 1)_T := 0$ . Noticing that  $L_T^{k+1}(g_D) := 0$  for all interior cell  $T \in \mathcal{T}_h^i$ .

Next, we consider the stabilization operators. For the interior faces  $F \in \mathcal{F}_{\partial T^i}$ , the LS stabilization operators are defined as

$$S_T^i(\hat{v}_T^k) := \Pi_{\partial T^i}^k(v_T^{k+1} - v_{\partial T^i}^k).$$

where  $\Pi_{\partial T^i}^k$  denotes the  $L^2$ -orthogonal projection onto the broken polynomial space  $\mathbb{P}_{k,d-1}(\mathcal{F}_{\partial T^i})$ .

For the Dirichlet face  $F \in \mathcal{F}_{\partial T^D}$ , we employ the Nitsche's boundary penalty techniques, i.e.,  $S_T^D(\hat{v}_T^k) := v_T^{k+1}|_{\partial T^D}$ .

Then, the local bilinear form  $\hat{a}_T$  in each cell  $T$  is defined as follow:

$$\begin{aligned} \hat{a}_T(\hat{u}_T^k, \hat{v}_T^k) &:= (\nabla R_T^i(\hat{u}_T^k), \nabla R_T^i(\hat{v}_T^k))_T \\ &\quad + h_T^{-1}(S_T^i(\hat{u}_T^k), S_T^i(\hat{v}_T^k))_{\partial T^i} + h_T^{-1}(u_T^{k+1}, v_T^{k+1})_{\partial T^D}. \end{aligned}$$

Next, we define the global bilinear forms  $\hat{a}_h, s_h$ , and linear form  $\ell_h$  as follows:

$$\begin{aligned} \hat{a}_h(\hat{u}_h^k, \hat{v}_h^k) &:= \sum_{T \in \mathcal{T}_h} \hat{a}_T(\hat{u}_T^k, \hat{v}_T^k), \\ \ell_h(\hat{v}_h^k) &:= \sum_{T \in \mathcal{T}_h} (f, v_T^{k+1})_T + (g_N, v_T^{k+1})_{\partial T^N} - (g_D, \partial_n R_T^i(\hat{v}_T^k) - h_T^{-1} v_T^{k+1})_{\partial T^D} \\ s_h(\hat{u}_h^k, \hat{v}_h^k) &:= \sum_{T \in \mathcal{T}_h} (h_T^{-1}(S_T^i(\hat{u}_T^k), S_T^i(\hat{v}_T^k))_{\partial T^i} + h_T^{-1}(u_T^{k+1}, v_T^{k+1})_{\partial T^D}). \end{aligned} \tag{7}$$

Finally, the discrete problem can be defined as follows:

$$\begin{cases} \text{Find } \hat{u}_h^k \in \hat{V}_h^k \text{ such that} \\ \hat{a}_h(\hat{u}_h^k, \hat{v}_h^k) = \ell_h(\hat{v}_h^k), \forall \hat{v}_h^k \in \hat{V}_h^k, \end{cases} \tag{8}$$

*Remark 3* We mention that there is another version for HHO method with Nitsche's boundary penalty techniques which does not contain the lifting operator, see section 2 in [4]. Without the lifting operator, the stability of the method in [4] is achieved by choosing the penalty parameter large enough. On the contrary, the proposed HHO methods does not suffer from it.

## 4 Main results

### 4.1 Stability and well-posedness

The stability and approximation results presented in this section follows the same technique in [13, Chapter 39]. For simplification of the presentation, the details are not presented in this work. Furthermore, we assume in the this work that that the global regularity is higher or equal to local regularity, i.e.,  $t \geq k + 1$ .

For all  $T \in \mathcal{T}_h$  and all  $\hat{v}_T^k \in \hat{V}_T^k$ , we define the local seminorm and the global norm:

$$|\hat{v}_T^k|_{\hat{V}_T^k}^2 := \|\nabla v_T^{k+1}\|_T^2 + h_T^{-1} \|v_{\partial T^i}^k - v_T^{k+1}\|_{\partial T^i}^2 + h_T^{-1} \|v_T^{k+1}\|_{\partial T^D}^2.$$

#### Lemma 4 (Stability and well-posedness)

Let all  $T \in \mathcal{T}_h$  satisfy the mesh assumptions. Then, the local stability holds

$$|\hat{v}_T^k|_{V_T^k}^2 \lesssim \hat{a}_T(\hat{v}_T^k, \hat{v}_T^k) \lesssim |\hat{v}_T^k|_{V_T^k}^2.$$

We equip the space  $\hat{v}_h^k$  with the norm  $\|\hat{v}_h^k\|_{\hat{V}_h^k}^2 := \sum_{T \in \mathcal{T}_h} |\hat{v}_T^k|_{V_T^k}^2$ . The discrete bilinear form  $\hat{a}_h$  is coercive on  $\hat{V}_h^k$ :

$$\|\hat{v}_h^k\|_{\hat{V}_h^k}^2 \lesssim a_T^k(\hat{v}_h^k, \hat{v}_h^k).$$

Invoking the Lax–Milgram lemma the (8) is well-posed.

## 4.2 Approximation and $H^1$ error estimate

For any  $T \in \mathcal{T}_h$ , we define the local reduction operator  $\hat{I}_T^k : H^1(T) \rightarrow \mathbb{P}_{k+1,d}(T)$ , for all  $u \in H^1(T)$ ,

$$\hat{I}_T^k(v) := (\Pi_T^{k+1}(v), \Pi_{\partial T^i}^k(v)) \in \hat{V}_T^k.$$

We then define the energy projection operator  $\mathcal{E}_T : H^1(T) \rightarrow \mathbb{P}_{k+1,d}(T)$  such that

$$\mathcal{E}_T(v) := R_T^i(\hat{I}_T^k(v)) + L_T^{k+1}(v), \quad \forall v \in H^1(T).$$

Similarly, the global energy projection is defined  $\mathcal{E}(v)|_T := \mathcal{E}_T(v)$ , for all  $T \in \mathcal{T}_h$ .

### Lemma 5 (Energy projection)

Let  $\mathcal{E}_T$  be the energy projection. Then, the following relation holds,

$$(\nabla(\mathcal{E}_T(v) - v), \nabla q)_T = (\Pi_T^{k+1}(v) - v, \partial_n q)_{\partial T^N}, \quad (9)$$

and the condition  $(\mathcal{E}_T(v), 1)_T = (v, 1)_T$  for all  $q \in \mathbb{P}_{k+1,d}(T)$  for all  $v \in H^1(T)$ .

*Remark 4* By noticing the term  $(\Pi_T^{k+1}(v) - v, \partial_n q)_{\partial T^N}$  depends on the Neumann boundary, the  $\mathcal{E}_T(v)$  will be the elliptic projection for the cells do not contains any face on the Neumann boundary.

We start with the approximation properties of operators.

### Lemma 6 (Approximation results)

For all  $v \in H^{t+1}(\Omega)$ ,  $t > \frac{1}{2}$  and  $v|_T \in H^{k+2}(T)$  for all  $T \in \mathcal{T}_h$ , we have

$$\|\nabla(\mathcal{E}_T(v) - v)\|_T + h_T^{\frac{1}{2}} \|\nabla(\mathcal{E}_T(v) - v)\|_{\partial T^i \cup \partial T^D} \lesssim h_T^{k+1} |v|_{H^{k+2}(T)}, \quad (10)$$

$$h_T^{-\frac{1}{2}} \|S_T^i(\hat{I}_T^k(v))\|_{\partial T^i} + h_T^{-\frac{1}{2}} \|S_T^D(\hat{I}_T^k(v)) - v\|_{\partial T^D} \lesssim h_T^{k+1} |v|_{H^{k+2}(T)}. \quad (11)$$

### Lemma 7 (Consistency and boundedness)

For all  $\hat{v}_h^k \in \hat{V}_h^k$ , we define the consistency error as  $\mathcal{D}_h(\hat{v}_h^k) := \sum_{T \in \mathcal{T}_h} (\ell_T(\hat{v}_T^k) - \hat{a}_T^k(\hat{I}_T^k(u), \hat{v}_T^k))$ . Assume that  $u \in H^{t+1}(\Omega)$  with  $t > \frac{1}{2}$  and  $u|_T \in H^{k+2}(T)$  for all



$T \in \mathcal{T}_h$ . Then we have

$$|\mathcal{D}_h(\hat{v}_h^k)| \leq \left( \sum_{T \in \mathcal{T}_h} \|g_T\|_{*T}^2 + \|\xi_T\|_{\#T}^2 \right)^{\frac{1}{2}} \times \|\hat{v}_h^k\|_{\hat{V}_h^k},$$

where

$$\begin{aligned} \|g_T\|_{*T}^2 &:= \|\nabla(u - \mathcal{E}(u))\|_T^2 + h_T \|\nabla(u - \mathcal{E}(u))\|_{\partial T^i \cup \partial T^D}^2, \\ \|\xi_T\|_{\#T}^2 &:= h_T^{-1} \|u - \Pi_T^{k+1}(u)\|_{\partial T^i}^2 + h_T^{-1} \|u - \Pi_T^{k+1}(u)\|_{\partial T^D}^2. \end{aligned}$$

**Lemma 8 (Broken  $H^1$  error estimate)**

For all  $v \in H^{t+1}(\Omega)$ ,  $t > \frac{1}{2}$  and  $v|_T \in H^{k+2}(T)$  for all  $T \in \mathcal{T}_h$ , let  $\hat{u}_h^k \in \hat{V}_h^k$  be the HHO solution and we define the discrete error as  $\hat{e}_h := (e_{\mathcal{T}_h}, e_{\mathcal{F}_h^i}) := \hat{I}_h^k(u) - \hat{u}_h^k \in \hat{V}_h^k$ . Then we have

$$\|\hat{e}_h\|_{\hat{V}_h^k}^2 + \sum_{T \in \mathcal{T}_h} \|\nabla(u - R_T^i(\hat{u}_h^k) - L_T^{k+1}(g_D))\|_T^2 \lesssim \left( \sum_{T \in \mathcal{T}_h} h_T^{2(k+1)} |u|_{H^{k+2}(T)}^2 \right). \quad (12)$$

### 4.3 $L^2$ error estimate

In order to derive the  $L^2$  error estimate, we assume our mesh is quasi-uniform and the domain is convex. Let's define the following dual problem

$$-\Delta \zeta_e = e_{\mathcal{T}_h} \quad \text{in } \Omega, \quad (13)$$

with  $\zeta_e = 0$  on  $\Gamma_D$ ,  $\partial_n \zeta_e = 0$  on  $\Gamma_N$ . We assume the elliptic regularity property holds, such that  $\|\zeta_e\|_{H^2(\Omega)} \lesssim \|e_{\mathcal{T}_h}\|_{\Omega}$ .

**Theorem 1 (Discrete  $L^2$  error estimate)**

Let  $u$  be the exact solution and  $\hat{u}_h^k$  be the HHO solution. Let  $u \in H^{k+2}(\Omega)$ ,  $k \geq 0$  and the above elliptic regularity property holds. The following bound holds,

$$\|e_{\mathcal{T}_h}\|_{\Omega} \lesssim h^{k+2} |u|_{H^{k+2}(\Omega)}.$$

**Proof** Using the fact  $\zeta_e$  is the solution of (13) and integration by parts, we have

$$\|e_{\mathcal{T}_h}\|_{\Omega}^2 = \sum_{T \in \mathcal{T}_h} ((\nabla e_T, \nabla \zeta_e)_T + (e_{\partial T^i} - e_T, \partial_n \zeta_e)_{\partial T^i} + (-e_T, \partial_n \zeta_e)_{\partial T^D}).$$

where we have used the fact that  $e_{\partial T^i}$  is single valued on the interior face. Let  $\eta = \zeta_e - \mathcal{E}(\zeta_e)$ , then, using the definition of the energy projection, we have

$$\begin{aligned}
\|e_{\mathcal{T}_h}\|_{\Omega}^2 &= \sum_{T \in \mathcal{T}_h} ((\nabla e_T, \nabla \zeta_e)_T + (e_{\partial T^i} - e_T, \partial_n \zeta_e)_{\partial T^i} + (-e_T, \partial_n \zeta_e)_{\partial T^D}) \\
&= \sum_{T \in \mathcal{T}_h} ((\nabla e_T, \nabla \eta)_T + (\nabla e_T, \nabla R_T^i(\hat{I}_h^k(\zeta_e)))_T + (e_{\partial T^i} - e_T, \partial_n \zeta_e)_{\partial T^i} + (-e_T, \partial_n \zeta_e)_{\partial T^D}) \\
&= \sum_{T \in \mathcal{T}_h} ((\nabla e_T, \nabla \eta)_T + (\nabla R_T^i(\hat{e}_T), \nabla R_T^i(\hat{I}_h^k(\zeta_e)))_T + (e_{\partial T^i} - e_T, \partial_n \eta)_{\partial T^i} \\
&\quad - (e_T, \partial_n \eta)_{\partial T^D}) = \mathcal{I}_1 + \mathcal{I}_2,
\end{aligned}$$

where we used  $\eta = \zeta_e - \mathcal{E}(\zeta_e)$ , the definition of the energy projection and the definition of the reconstruction operator (6) for  $\hat{e}_T$  in the third line, with

$$\begin{aligned}
\mathcal{I}_1 &:= \sum_{T \in \mathcal{T}_h} ((\nabla e_T, \nabla \eta)_T + (e_{\partial T^i} - e_T, \partial_n \eta)_{\partial T^i} - (e_T, \partial_n \eta)_{\partial T^D}), \\
\mathcal{I}_2 &:= \sum_{T \in \mathcal{T}_h} (\nabla R_T^i(\hat{e}_T), \nabla R_T^i(\hat{I}_h^k(\zeta_e)))_T.
\end{aligned}$$

We firstly derive the bound for  $\mathcal{I}_1$ . The Cauchy-schwarz inequality gives

$$\begin{aligned}
\mathcal{I}_1 &\lesssim \sum_{T \in \mathcal{T}_h} (\|\nabla(\Pi_T^{k+1}(u) - u_T)\|_T \|\nabla \eta\|_T \\
&\quad + h_T^{-\frac{1}{2}} \|e_{\partial T^i} - e_T\|_{\partial T^i} \times h_T^{\frac{1}{2}} \|\partial_n \eta\|_{\partial T^i} + h_T^{-\frac{1}{2}} \|e_T\|_{\partial T^D} \times h_T^{\frac{1}{2}} \|\partial_n \eta\|_{\partial T^D})
\end{aligned}$$

Using  $\Pi_T^{k+1}(u) - u_T = (\Pi_T^{k+1}(u) - u) + (u - u_T)$ , the the elliptic regularity for dual problem, approximation result (4) in Lemma 3 and (12) in Lemma 8, the relation  $\|\nabla \eta\|_{\Omega} \leq h|\zeta_e|_{H^2(\Omega)} \lesssim h\|e_{\mathcal{T}_h}\|_{\Omega}$  holds. So we have  $\mathcal{I}_1 \lesssim h^{k+2}|u|_{H^{k+2}(\Omega)}\|e_{\mathcal{T}_h}\|_{\Omega}$ .

Let us now turn to estimate  $\mathcal{I}_2$ . Using the definition of bilinear form in (7), the definition of  $\hat{e}_h$  and the HHO solution  $\hat{u}_h$ , we have

$$\mathcal{I}_2 = -s_h(\hat{e}_h, \hat{I}_h^k(\zeta_e)) - \ell_h(\hat{I}_h^k(\zeta_e)) + \hat{a}_h(\hat{I}_h^k(u), \hat{I}_h^k(\zeta_e)).$$

Then, invoking the weak form of the problem  $(f, \zeta_e)_{\Omega} = (\nabla u, \nabla \zeta_e)_{\Omega} - (g_N, \zeta_e)_{\Gamma_N}$  and  $(\nabla u, \nabla \zeta_e)_T = (\nabla u, \nabla(\zeta_e - \Pi_T^{k+1}(\zeta_e)))_T + (\nabla u, \nabla \Pi_T^{k+1}(\zeta_e))_T$ , we have

$$\begin{aligned}
\mathcal{I}_2 &= -s_h(\hat{e}_h, \hat{I}_h^k(\zeta_e)) - \sum_{T \in \mathcal{T}_h} ((f, \Pi_T^{k+1}(\zeta_e) - \zeta_e)_T + (g_N, \Pi_T^{k+1}(\zeta_e) - \zeta_e)_{\partial T^N} \\
&\quad + (h_T^{-1} \Pi_T^{k+1}(\zeta_e) - \partial_n R_T^i(\hat{I}_h^k(\zeta_e)), g_D)_{\partial T^D} + (\nabla u, \nabla(\zeta_e - \Pi_T^{k+1}(\zeta_e)))_T \\
&\quad + (\nabla u, \nabla \Pi_T^{k+1}(\zeta_e))_T) + a_h^i(\hat{I}_h^k(u), \hat{I}_h^k(\zeta_e)).
\end{aligned}$$

Since  $u$  solves the model problem, we have

$$\begin{aligned}
(\nabla u, \nabla(\zeta_e - \Pi_T^{k+1}(\zeta_e)))_T &= (-\Delta u, \zeta_e - \Pi_T^{k+1}(\zeta_e))_T + (\partial_n u, \zeta_e - \Pi_T^{k+1}(\zeta_e))_{\partial T} \\
&= (f, \zeta_e - \Pi_T^{k+1}(\zeta_e))_T + (\partial_n u, \zeta_e - \Pi_T^{k+1}(\zeta_e))_{\partial T^i \cup \partial T^D} + (g_N, \zeta_e - \Pi_T^{k+1}(\zeta_e))_{\partial T^N}.
\end{aligned}$$

Next, we use the definition of the reconstruction operator  $R_T^i$ , adding and subtracting  $\sum_{T \in \mathcal{T}_h} (\nabla L_T^{k+1}(g_D), \nabla R_T^i(\hat{I}_h^k(\zeta_e)))_T$  in  $\mathcal{I}_2$ , using once more the definition of the reconstruction operator  $R_T^i$  for  $-\sum_{T \in \mathcal{T}_h} (\nabla L_T^{k+1}(g_D), \nabla R_T^i(\hat{I}_h^k(\zeta_e)))_T$  and noticing that for any  $q \in \mathbb{P}_{k,d}(T)$ , we have  $(q, \Pi_{\partial T^i}^k \zeta_e)_{\partial T^i} = (q, \zeta_e)_{\partial T^i}$ . With the above results, the term  $\mathcal{I}_2$  can be split into 4 parts,

$$\begin{aligned} \mathcal{I}_3 &:= -s_h(\hat{e}_h, \hat{I}_h^k(\zeta_e)), \\ \mathcal{I}_4 &:= -\sum_{T \in \mathcal{T}_h} ((g_D, h_T^{-1} \Pi_T^{k+1}(\zeta_e))_{\partial T^D}) + s_h(\hat{I}_h^k(u), \hat{I}_h^k(\zeta_e)), \\ \mathcal{I}_5 &:= -\sum_{T \in \mathcal{T}_h} ((\partial_n u, \zeta_e - \Pi_T^{k+1}(\zeta_e))_{\partial T^D \cup \partial T^i} \\ &\quad - (\partial_n(R_T^i(\hat{I}_T^k(u)) + L_T^{k+1}(g_D)), \zeta_e - \Pi_T^{k+1}(\zeta_e))_{\partial T^i} \\ &\quad + (\partial_n(R_T^i(\hat{I}_T^k(u)) + L_T^{k+1}(g_D)), \Pi_T^{k+1}(\zeta_e))_{\partial T^D}), \\ \mathcal{I}_6 &:= -\sum_{T \in \mathcal{T}_h} (\nabla(u - R_T^i(\hat{I}_T^k(u)) - L_T^{k+1}(g_D)), \nabla \Pi_T^{k+1}(\zeta_e))_T. \end{aligned}$$

Using (12) of Lemma 8, (11) of Lemma 6 and the elliptic regularity property of the dual problem,  $\mathcal{I}_3$  and  $\mathcal{I}_4$  can be bounded by  $h^{k+2}|u|_{H^{k+2}(\Omega)} \|e_{\mathcal{T}_h}\|_{\Omega}$ .

Noticing that  $\zeta_e = 0$  on  $\Gamma_D$ ,  $\mathcal{I}_5 = -\sum_{T \in \mathcal{T}_h} (\partial_n(u - R_T^i(\hat{I}_T^k(u)) - L_T^{k+1}(g_D)), \zeta_e - \Pi_T^{k+1}(\zeta_e))_{\partial T^D \cup \partial T^i}$ . Then, using the Cauchy-Schwarz inequality, the elliptic regularity property of the dual problem, the discrete trace inequality (2), (10), (11) of Lemma 6, we have

$$\begin{aligned} \mathcal{I}_5 &\lesssim \left( \sum_{T \in \mathcal{T}_h} h_T^{\frac{1}{2}} \|\nabla(u - \mathcal{E}(u))\|_{\partial T^D \cup \partial T^i} \right) \times h_T^{-\frac{1}{2}} \|\zeta_e - \Pi_T^{k+1}(\zeta_e)\|_{\partial T^D \cup \partial T^i} \\ &\lesssim h^{k+1} |u|_{H^{k+2}(\Omega)} \times h \|e_{\mathcal{T}_h}\|_{\Omega}. \end{aligned}$$

Finally, invoke  $\partial_n \zeta_e = 0$  on  $\Gamma_N$  into  $\mathcal{I}_6$  and use the energy projection property (9) of Lemma 5 for  $\zeta_e$  shows  $\mathcal{I}_6 = -\sum_{T \in \mathcal{T}_h} (\Pi_T^{k+1}(u) - u, \partial_n(\Pi_T^{k+1}(\zeta_e) - \zeta_e))_{\partial T^N}$ . Similar to the bound  $\mathcal{I}_5$ , we have

$$\mathcal{I}_6 \lesssim \left( \sum_{T \in \mathcal{T}_h} \|\Pi_T^{k+1}(u) - u\|_{\partial T^N} \right) \|\nabla(\Pi_T^{k+1}(\zeta_e) - \zeta_e)\|_{\partial T^N} \lesssim h^{k+2} |u|_{H^{k+2}(\Omega)} \|e_{\mathcal{T}_h}\|_{\Omega}.$$

Summing up the above bounds, we derive the final error bound.

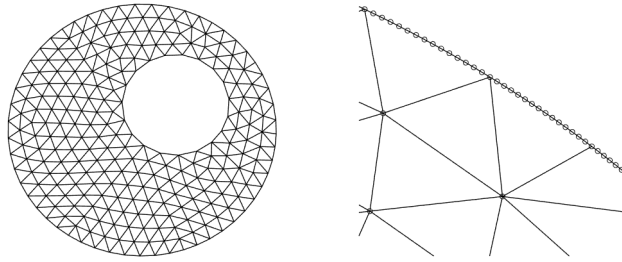
### Corollary 2 ( $L^2$ error estimate for the reconstruction operator)

Let  $u$  be the exact solution of model problem (1) and  $R_T^i(\hat{u}_h^k)$  be the reconstructed HHO solution (8). Let  $u \in H^{k+2}(\Omega)$ ,  $k \geq 0$  and the elliptic regularity property holds. Then, the following bound holds,

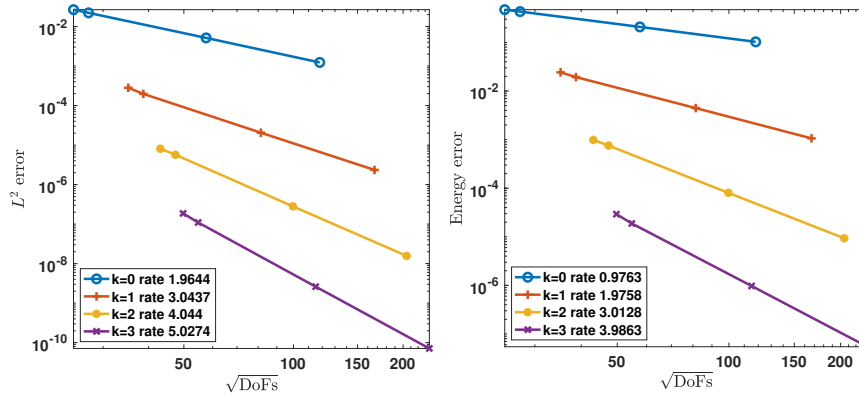
$$\sum_{T \in \mathcal{T}_h} \|u - R_T^i(\hat{u}_h^k) - L_T^{k+1}(g_D)\|_T \lesssim h^{k+2} |u|_{H^{k+2}(\Omega)}.$$

*Proof* The same argument as in [5, chapter 2 ] gives the proof.  $\square$

### 5 Numeral experiment



**Fig. 2** Complicated domain



**Fig. 3** Convergence rates in the  $L^2$ -norm and energy seminorm for  $k = 0, 1, 2, 3$

Let domain  $\Omega$  be a polygon with 5175 small edges inscribed in the unit circle centered at  $(0, 0)$  and remove a circular hole centered at  $(0.25, 0.25)$  with radius 0.4, see Figure 2. The HHO method is tested with the exact solution  $u = \exp(\sin x) \exp(\sin y)$ , with Dirichlet boundary condition imposed on the boundary edges allocated on  $x \leq 0.5$ , and Neumann boundary condition on the remaining boundary. We consider a quasi-uniform sequence of meshes composed of 437, 527, 2266, and 9411

triangular cells. All meshes fit the domain  $\Omega$  exactly, and for every mesh in the sequence, each interior cell has only straight edges, whereas each boundary cell has a lot of small edges that exactly fit the boundary of  $\Omega$ . The convergence result is given in Figure 3. The key observation is that the convergence rates are  $O(h^{k+2})$  to the  $O(h^{k+1})$  for the  $L^2$  error and the energy error, respectively. We emphasize that the total number of global coupled unknowns for the proposed HHO methods is independent of the number of edges on the domain boundary.

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