

Computing the 4-Edge-Connected Components of a Graph in Linear Time*

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Abstract

We present the first linear-time algorithm that computes the 4-edge-connected components of an undirected graph. Hence, we also obtain the first linear-time algorithm for testing 4-edge connectivity. Our results are based on a linear-time algorithm that computes the 3-edge cuts of a 3-edge-connected graph G , and a linear-time procedure that, given the collection of all 3-edge cuts, partitions the vertices of G into the 4-edge-connected components.

1 Introduction

Let $G = (V, E)$ be a connected undirected graph with m edges and n vertices. An (*edge*) *cut* of G is a set of edges $S \subseteq E$ such that $G \setminus S$ is not connected. We say that S is a k -*cut* if its cardinality is $|S| = k$. Also, we refer to the 1-cuts as the *bridges* of G . A cut S is *minimal* if no proper subset of S is a cut of G . The *edge connectivity* of G , denoted by $\lambda(G)$, is the minimum cardinality of an edge cut of G . A graph is k -*edge-connected* if $\lambda(G) \geq k$.

A cut S separates two vertices u and v , if u and v lie in different connected components of $G \setminus S$. Vertices u and v are k -edge-connected, denoted by $u \stackrel{G}{\equiv}_k v$, if there is no $(k - 1)$ -cut that separates them. By Menger’s theorem [15], u and v are k -edge-connected if and only if there are k -edge-disjoint paths between u and v . A k -*edge-connected component* of G is a maximal set $C \subseteq V$ such that there is no $(k - 1)$ -edge cut in G that disconnects any two vertices $u, v \in C$ (i.e., u and v are in the same connected component of $G \setminus S$ for any $(k - 1)$ -edge cut S). We can define, analogously, the *vertex cuts* and the k -*vertex-connected components* of G .

Computing and testing the edge connectivity of a graph, as well as its k -edge-connected components, is a classical subject in graph theory, as it is an important notion in several application areas (see, e.g., [17]), that has been extensively studied since the 1970’s. It is known how to compute the $(k - 1)$ -edge cuts, $(k - 1)$ -vertex cuts, k -edge-connected components and k -vertex-connected components of a graph in linear time for $k \in \{2, 3\}$ [5, 9, 16, 19, 22]. The case $k = 4$

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has also received significant attention [2, 3, 10, 11]. Unfortunately, none of the previous algorithms achieved linear running time. In particular, Kanevsky and Ramachandran [10] showed how to test whether a graph is 4-vertex-connected in $O(n^2)$ time. Furthermore, Kanevsky et al. [11] gave an $O(m + n\alpha(m, n))$ -time algorithm to compute the 4-vertex-connected components of a 3-vertex-connected graph, where α is a functional inverse of Ackermann’s function [21]. Using the reduction of Galil and Italiano [5] from edge connectivity to vertex connectivity, the same bounds can be obtained for 4-edge connectivity. Specifically, one can test whether a graph is 4-edge-connected in $O(n^2)$ time, and one can compute the 4-edge-connected components of a 3-edge-connected graph in $O(m + n\alpha(m, n))$ time. Dinitz and Westbrook [3] presented an $O(m + n \log n)$ -time algorithm to compute the 4-edge-connected components of a general graph G (i.e., when G is not necessarily 3-edge-connected). Nagamochi and Watanabe [18] gave an $O(m + k^2 n^2)$ -time algorithm to compute the k -edge-connected components of a graph G , for any integer k . We also note that the edge connectivity of a simple undirected graph can be computed in $O(m \text{polylog} n)$ time, randomized [7, 12] or deterministic [8, 14]. The best current bound is $O(m \log^2 n \log \log^2 n)$, achieved by Henzinger et al. [8] which provided an improved version of the algorithm of Kawarabayashi and Thorup [14].

Our results and techniques In this paper we present the first linear-time algorithm that computes the 4-edge-connected components of a general graph G , thus resolving a problem that remained open for more than 20 years. Hence, this also implies the first linear-time algorithm for testing 4-edge connectivity. We base our results on the following ideas. First, we extend the framework of Georgiadis and Kosinas [6] for computing 2-edge cuts (as well as mixed cuts consisting of a single vertex and a single edge) of G . Similar to known linear-time algorithms for computing 3-vertex-connected and 3-edge-connected components [9, 22], Georgiadis and Kosinas [6] define various concepts with respect to a depth-first search (DFS) spanning tree of G . We extend this framework by introducing new key parameters that can be computed efficiently and provide characterizations of the various types of 3-edge cuts that may appear in a 3-edge-connected graph. We deal with the general case by dividing G into auxiliary graphs H_1, \dots, H_ℓ , such that each H_i is 3-edge-connected and corresponds to a different 3-edge-connected component of G . Also, for any two vertices x and y , we have $x \stackrel{G}{\equiv}_4 y$ if and only if x and y are both in the same auxiliary graph H_i and $x \stackrel{H_i}{\equiv}_4 y$. Furthermore, this reduction allows us to compute in linear time the number of *minimal 3-edge cuts* in a general graph G . Next, in order to compute the 4-edge-connected components in each auxiliary graph H_i , we utilize the fact that a minimum cut of a graph G separates G into two connected components. Hence, we can define the set V_C of the vertices in the connected component of $G \setminus C$ that does not contain a specified root vertex r . We refer to the number of vertices in V_C as the *r-size* of the cut C . Then, we apply a recursive algorithm that successively splits H_i into smaller graphs according to its 3-cuts. When no more splits are possible, the connected components of the final split graph correspond to the 4-edge-connected components of G . We show that we can implement this procedure in linear time by processing the cuts in non-decreasing order with respect to their *r-size*.

2 Concepts defined on a DFS-tree structure

Let $G = (V, E)$ be a connected undirected graph, which may have multiple edges. For a set of vertices $S \subseteq V$, the induced subgraph of S , denoted by $G[S]$, is the subgraph of G with vertex set S and edge set $\{e \in E \mid \text{both ends of } e \text{ lie in } S\}$. Let T be the spanning tree of G provided by a depth-first search (DFS) of G [19], with start vertex r . The edges in T are called *tree-edges*; the edges in $E \setminus T$ are called *back-edges*, as their endpoints have ancestor-descendant relation in T . A

vertex u is an ancestor of a vertex v (v is a descendant of u) if the tree path from r to v contains u . Thus, we consider a vertex to be an ancestor (and, consequently, a descendant) of itself. We let $p(v)$ denote the parent of a vertex v in T . If u is a descendant of v in T , we denote the set of vertices of the simple tree path from u to v as $T[u, v]$. The expressions $T[u, v)$ and $T(u, v]$ have the obvious meaning (i.e., the vertex on the side of the parenthesis is excluded). From now on, we identify vertices with their preorder number (assigned during the DFS). Thus, v being an ancestor of u in T implies that $v \leq u$. Let $T(v)$ denote the set of descendants of v , and let $ND(v)$ denote the number of descendants of v (i.e. $ND(v) = |T(v)|$). With all $ND(v)$ computed, we can check in constant time whether a vertex u is a descendant of v , since $u \in T(v)$ if and only if $v \leq u$ and $u < v + ND(v)$ [20].

Whenever (x, y) denotes a back-edge, we shall assume that x is a descendant of y . We let $B(v)$ denote the set of back-edges (x, y) , where x is a descendant of v and y is a proper ancestor of v . Thus, if we remove the tree-edge $(v, p(v))$, $T(v)$ remains connected to the rest of the graph through the back-edges in $B(v)$. This implies that G is 2-edge-connected if and only if $|B(v)| > 0$, for every $v \neq r$. Furthermore, G is 3-edge-connected only if $|B(v)| > 1$, for every $v \neq r$. We let $b_count(v)$ denote the number of elements of $B(v)$ (i.e. $b_count(v) = |B(v)|$). $low(v)$ denotes the lowest y such that there exists a back-edge $(x, y) \in B(v)$. Similarly, $high(v)$ is the highest y such that there exists a back-edge $(x, y) \in B(v)$.

We let $M(v)$ denote the nearest common ancestor of all x for which there exists a back-edge $(x, y) \in B(v)$. Note that $M(v)$ is a descendant of v . Let m be a vertex and v_1, \dots, v_k be all the vertices with $M(v_1) = \dots = M(v_k) = m$, sorted in decreasing order. (Observe that v_{i+1} is an ancestor of v_i , for every $i \in \{1, \dots, k-1\}$, since m is a common descendant of all v_1, \dots, v_k .) Then we have $M^{-1}(m) = \{v_1, \dots, v_k\}$, and we define $nextM(v_i) := v_{i+1}$, for every $i \in \{1, \dots, k-1\}$, and $lastM(v_i) := v_k$, for every $i \in \{1, \dots, k\}$. Thus, for every vertex v , $nextM(v)$ is the successor of v in the decreasingly sorted list $M^{-1}(M(v))$, and $lastM(v)$ is the lowest element in $M^{-1}(M(v))$.

The following two simple facts have been proved in [6].

Fact 2.1. *All $ND(v)$, $b_count(v)$, $M(v)$, $low(v)$ and $high(v)$ can be computed in total linear-time, for all vertices v .*

Fact 2.2. *$B(u) = B(v) \Leftrightarrow M(u) = M(v)$, and $high(u) = high(v) \Leftrightarrow M(u) = M(v)$ and $b_count(u) = b_count(v)$.*

Furthermore, [6] implies the following characterization of a 3-edge-connected graph.

Fact 2.3. *G is 3-edge-connected if and only if $|B(v)| > 1$, for every $v \neq r$, and $B(v) \neq B(u)$, for every pair of vertices u and v , $u \neq v$.*

Lemma 2.4. *Let v be an ancestor of u and $M(v)$ a descendant of u . Then, $M(v)$ is a descendant of $M(u)$.*

Proof. Let $(x, y) \in B(v)$. Then x is a descendant of $M(v)$, and therefore a descendant of u . Furthermore, y is a proper ancestor of v , and therefore a proper ancestor of u . This shows that $(x, y) \in B(u)$, and thus we have $B(v) \subseteq B(u)$. This shows that $M(v)$ is a descendant of $M(u)$. \square

The following lemma will be implicitly evoked several times in the following sections.

Lemma 2.5. *Let u be a proper descendant of v such that $M(u) = M(v)$. Then, $B(v) \subseteq B(u)$. Furthermore, if the graph is 3-edge-connected, $B(v) \subset B(u)$.*

Proof. Let $(x, y) \in B(v)$. Then x is a descendant of $M(v)$, and therefore a descendant of $M(u)$, and therefore a descendant of u . Furthermore, y is a proper ancestor of v , and therefore a proper ancestor of u . This shows that $(x, y) \in B(u)$, and thus $B(v) \subseteq B(u)$ is established. If the graph is 3-edge-connected, $B(v) \subset B(u)$ is an immediate consequence of fact 2.3. \square

Now let us provide some extensions of those concepts that will be needed for our purposes. Assume that G is 3-edge-connected, and let $v \neq r$ be a vertex of G . By fact 2.3, $b_count(v) > 1$, and therefore there are at least two back-edges in $B(v)$. Of course, there is at least one back-edge $(x, y) \in B(v)$ such that $y = low(v)$. We let $low1(v)$ denote y , and $low1D(v)$ denote x . That is, $low1(v)$ is the *low* point of v , and $low1D(v)$ is a descendant of v which is connected with a back-edge to its *low* point. (Of course, $low1D(v)$ is not uniquely determined, but we need to have at least one such descendant stored in a variable.) Similarly, we let $highD(v)$ denote a descendant of v which is connected with a back-edge to the *high* point of v . (Again, $highD(v)$ is not uniquely determined.) Then, there may exist another back-edge $(x', y') \in B(v)$ with $x' \neq x$ and $y' = y$. In this case, we let $low2(v)$ denote y' (that is, $low2(v)$ is, again, the *low* point of v) and $low2D(v)$ denote x' . If there is no back-edge $(x', y') \in B(v)$ with $x' \neq x$ and $y' = y$, let $(x', y') \in B(v)$ denote a back-edge with $y' = \min(\{w \mid \exists(z, w) \in B(v)\} \setminus \{y\})$. Then we let $low2(v)$ denote y' and $low2D(v)$ denote x' . Thus, if $v \neq r$, we know that $(low1D(v), low(v))$ and $(low2D(v), low2(v))$ are two distinct back-edges in $B(v)$. We have defined $low1$, $low1D$, $low2$ and $low2D$ because we need to have stored, for every vertex $v \neq r$, two back-edges from $B(v)$ (see section 3.1). Any other pair of back-edges from $B(v)$ could do as well. It is easy to compute all $low1(v)$, $low1D(v)$, $low2(v)$ and $low2D(v)$ during the DFS.

We let $l(v)$ denote the lowest y for which there exists a back-edge (v, y) , or v if no such back-edge exists. Thus, $low(v) \leq l(v)$. Now let c_1, \dots, c_k be the children of v sorted in non-decreasing order w.r.t. their *low* point. Then we call c_1 the *low1* child of v , and c_2 the *low2* child of v . (Of course, the *low1* and *low2* children of v are not uniquely determined after a DFS on G , since we may have $low(c_1) = low(c_2)$.) We let $\tilde{M}(v)$ denote the nearest common ancestor of all x for which there exists a back-edge $(x, y) \in B(v)$ with x a proper descendant of $M(v)$. Formally, $\tilde{M}(v) := nca\{x \mid \exists(x, y) \in B(v) \text{ and } x \neq M(v)\}$. If the set $\{x \mid \exists(x, y) \in B(v) \text{ and } x \neq M(v)\}$ is empty, we leave $\tilde{M}(v)$ undefined. We also define $M_{low1}(v)$ as the nearest common ancestor of all x for which there exists a back-edge $(x, y) \in B(v)$ with x being a descendant of the *low1* child of $M(v)$, and $M_{low2}(v)$ as the nearest common ancestor of all x for which there exists a back-edge $(x, y) \in B(v)$ with x a descendant of the *low2* child of $M(v)$. Formally, $M_{low1}(v) := nca\{x \mid \exists(x, y) \in B(v) \text{ and } x \text{ is a descendant of the } low1 \text{ child of } M(v)\}$ and $M_{low2}(v) := nca\{x \mid \exists(x, y) \in B(v) \text{ and } x \text{ is a descendant of the } low2 \text{ child of } M(v)\}$. If the set in the formal definition of $M_{low1}(v)$ (resp. $M_{low2}(v)$) is empty, we leave $M_{low1}(v)$ (resp. $M_{low2}(v)$) undefined.

2.1 Computing the DFS parameters in linear time

Algorithm 1 shows how we can easily compute $highD(v)$ during the computation of all *high* points. The algorithm uses the static tree disjoint-set-union data structure of Gabow and Tarjan [4] to achieve linear running time.

Algorithm 2 shows how we can compute all $M(v)$ and $nextM(v)$, algorithm 3 shows how we can compute all $\tilde{M}(v)$, and algorithm 4 shows how we can compute all $M_{low1}(v)$ and $M_{low2}(v)$, for all vertices $v \neq r$, in total linear time. These algorithms process the vertices in a bottom-up fashion, and they work recursively on the descendants of a vertex. To perform these computations in linear time, we have to avoid descending to the same vertices an excessive amount of times during the recursion. To achieve this, we use a variable $currentM[w]$, that has the property that, during the

Algorithm 1: Compute all $high(v)$ and $highD(v)$, for all vertices $v \neq r$

```

1 initialize a DSU structure on the vertices of  $G$ , where the link operations are
  predetermined by the edges of  $T$ 
2 for  $v = n$  to  $v = 1$  do
3   foreach  $u$  adjacent to  $v$  do
4     if  $u$  is a descendant of  $v$  then
5        $x \leftarrow find(u)$ 
6       while  $x > v$  do
7          $high[x] \leftarrow v$ 
8          $highD[x] \leftarrow u$ 
9          $next \leftarrow find(p(x))$ 
10         $link(x, p(x))$ 
11         $x \leftarrow next$ 
12      end
13    end
14  end
15 end

```

course of the algorithm, when we process a vertex v , all back-edges that start from a descendant of w and end in a proper ancestor of v have their higher end in $T(currentM[w])$ (this means, of course, that $currentM[w]$ is a descendant of w). And so, if we want e.g. to compute $M_{low1}(v)$, we may descend immediately to $currentM[c_1]$, where c_1 is the *low1* child of $M(v)$. In Lemma 2.7, we give a formal proof of the correctness and linear complexity of Algorithms 3 and 4.

Algorithm 2: Compute all $M(v)$ and $nextM(v)$, for all vertices $v \neq r$

```

// Compute all  $M(v)$  and  $nextM(v)$ 
1 for  $v = n$  to  $v = 2$  do
2    $nextM[v] \leftarrow \emptyset$ 
3    $c \leftarrow v, m \leftarrow v$ 
4   while  $M(v) = \emptyset$  do
5     if  $l(m) < v$  then  $M(v) \leftarrow m$ , break
6      $c_1 \leftarrow low1$  child of  $m$ 
7      $c_2 \leftarrow low2$  child of  $m$ 
8     if  $low(c_2) < v$  then  $M(v) \leftarrow m$ , break
9      $c \leftarrow c_1, m \leftarrow M(c)$ 
10  end
11  if  $c \neq v$  then  $nextM(c) \leftarrow v$ 
12 end

```

Lemma 2.6. Let v and v' be two vertices such that v' is an ancestor of v with $M(v') = M(v)$. Then, $\tilde{M}(v')$ (resp. $M_{low1}(v')$, resp. $M_{low2}(v')$), if it is defined, is a descendant of $M(v)$ (resp. $M_{low1}(v)$, resp. $M_{low2}(v)$).

Proof. Let v' be an ancestor of v such that $M(v') = M(v)$.

Assume, first, that $\tilde{M}(v')$ is defined. Then, there exists a back-edge $(x, y) \in B(v')$ where x is a proper descendant of $M(v')$. Since $M(v') = M(v)$, x is a proper descendant of $M(v)$. Furthermore,

Algorithm 3: Compute all $\tilde{M}(v)$, for all vertices $v \neq r$

```

1 initialize an array currentM with  $n$  entries
  // Compute all  $\tilde{M}(v)$ 
2 foreach vertex  $v$  do currentM[ $v$ ]  $\leftarrow v$ 
3 for  $v = n$  to  $v = 2$  do
4    $m \leftarrow M(v)$ 
5    $c \leftarrow \text{low1 child of } m$ 
6   if  $\text{low}(c) \geq v$  then  $\tilde{M}(v) \leftarrow \emptyset$ , continue
7    $c' \leftarrow \text{low2 child of } m$ 
8   if  $\text{low}(c') < v$  then  $\tilde{M}(v) \leftarrow m$ , continue
9    $m \leftarrow \text{currentM}[c]$ 
10  while  $\tilde{M}(v) = \emptyset$  do
11    if  $l(m) < v$  then  $\tilde{M}(v) \leftarrow m$ , break
12     $c_1 \leftarrow \text{low1 child of } m$ 
13     $c_2 \leftarrow \text{low2 child of } m$ 
14    if  $\text{low}(c_2) < v$  then  $\tilde{M}(v) \leftarrow m$ , break
15     $m \leftarrow \text{currentM}[c_1]$ 
16  end
17  currentM[ $c$ ]  $\leftarrow m$ 
18 end

```

since y is a proper ancestor of v' , it is also a proper ancestor of v . This shows that $(x, y) \in B(v)$, and $\tilde{M}(v)$ is an ancestor of x . Due to the generality of (x, y) , we conclude that $\tilde{M}(v)$ is an ancestor of $\tilde{M}(v')$.

Now assume that $M_{\text{low1}}(v')$ is defined. Then, there exists a back-edge $(x, y) \in B(v')$ where x is a descendant of the *low1* child of $M(v')$. Since $M(v') = M(v)$, x is a descendant of the *low1* child of $M(v)$. Furthermore, since y is a proper ancestor of v' , it is also a proper ancestor of v . This shows that $(x, y) \in B(v)$, and $M_{\text{low1}}(v)$ is an ancestor of x . Due to the generality of (x, y) , we conclude that $M_{\text{low1}}(v)$ is an ancestor of $M_{\text{low1}}(v')$.

Finally, assume that $M_{\text{low2}}(v')$ is defined. Then, there exists a back-edge $(x, y) \in B(v')$ where x is a descendant of the *low2* child of $M(v')$. Since $M(v') = M(v)$, x is a descendant of the *low2* child of $M(v)$. Furthermore, since y is a proper ancestor of v' , it is also a proper ancestor of v . This shows that $(x, y) \in B(v)$, and $M_{\text{low2}}(v)$ is an ancestor of x . Due to the generality of (x, y) , we conclude that $M_{\text{low2}}(v)$ is an ancestor of $M_{\text{low2}}(v')$. \square

Lemma 2.7. Algorithms 3 and 4 compute all $\tilde{M}(v)$, $M_{\text{low1}}(v)$ and $M_{\text{low2}}(v)$, for all vertices $v \neq r$, in total linear time.

Proof. Let us show e.g. that Algorithm 4 correctly computes all $M_{\text{low1}}(v)$, for all $v \neq r$, in total linear time. The proofs for the other cases are similar. So let v be a vertex $\neq r$. Since we are interested in the back-edges $(x, y) \in B(v)$ with x a descendant of the *low1* child c of $M(v)$, we first have to check whether $\text{low}(c) < v$. If $\text{low}(c) \geq v$, then there is no such back-edge, and therefore we set $M_{\text{low1}}(v) \leftarrow \emptyset$ (in line 6). If $\text{low}(c) < v$, then $M_{\text{low1}}(v)$ is defined, and in line 7 we assign m the value *currentM*[c]. We claim that, at that moment, *currentM*[c] is an ancestor of $M_{\text{low1}}(v)$, and every *currentM*[c_1] that we will access in the **while** loop in line 13 is also an ancestor of $M_{\text{low1}}(v)$; furthermore, when we reach line 15, *currentM*[c] is assigned $M_{\text{low1}}(v)$. It is not difficult to see this

Algorithm 4: Compute all $M_{low1}(v)$ and $M_{low2}(v)$, for all vertices $v \neq r$

```

1 initialize an array currentM with  $n$  entries
  // Compute all  $M_{low1}(v)$ 
2 foreach vertex  $v$  do currentM[ $v$ ]  $\leftarrow v$ 
3 for  $v = n$  to  $v = 2$  do
4    $m \leftarrow M(v)$ 
5    $c \leftarrow$  low1 child of  $m$ 
6   if  $low(c) \geq v$  then  $M_{low1}(v) \leftarrow \emptyset$ , continue
7    $m \leftarrow$  currentM[ $c$ ]
8   while  $M_{low1}(v) = \emptyset$  do
9     if  $l(m) < v$  then  $M_{low1}(v) \leftarrow m$ , break
10     $c_1 \leftarrow$  low1 child of  $m$ 
11     $c_2 \leftarrow$  low2 child of  $m$ 
12    if  $low(c_2) < v$  then  $M_{low1}(v) \leftarrow m$ , break
13     $m \leftarrow$  currentM[ $c_1$ ]
14  end
15  currentM[ $c$ ]  $\leftarrow m$ 
16 end
  // Compute all  $M_{low2}(v)$ 
17 foreach vertex  $v$  do currentM[ $v$ ]  $\leftarrow v$ 
18 for  $v = n$  to  $v = 2$  do
19    $m \leftarrow M(v)$ 
20    $c \leftarrow$  low2 child of  $m$ 
21   if  $low(c) \geq v$  then  $M_{low2}(v) \leftarrow \emptyset$ , continue
22    $m \leftarrow$  currentM[ $c$ ]
23   while  $M_{low2}(v) = \emptyset$  do
24     if  $l(m) < v$  then  $M_{low2}(v) \leftarrow m$ , break
25      $c_1 \leftarrow$  low1 child of  $m$ 
26      $c_2 \leftarrow$  low2 child of  $m$ 
27     if  $low(c_2) < v$  then  $M_{low2}(v) \leftarrow m$ , break
28      $m \leftarrow$  currentM[ $c_1$ ]
29   end
30   currentM[ $c$ ]  $\leftarrow m$ 
31 end

```

inductively. Suppose, then, that this was the case for every vertex $v' > v$, and let us see what happens when we process v . Let c be the *low1* child of $M(v)$. Initially, *currentM*[c] was set to be c . Now, if *currentM*[c] is still c , $M_{low1}(v)$ is a descendant of c (by definition). Otherwise, due to the inductive hypothesis, *currentM*[c] had been assigned $M_{low1}(v')$ during the processing of a vertex $v' > v$ with $M(v') = M(v)$. This implies that v' is a descendant of v , and by Lemma 2.6 we have that $M_{low1}(v')$ is an ancestor of $M_{low1}(v)$. In any case, then, we have that $m =$ *currentM*[c] in an ancestor of $M_{low1}(v)$. Now we enter the **while** loop in line 8. If either $l(m) < v$ or $low(c_2) < v$, where c_2 is the *low2* child of m , we have that $M_{low1}(v)$ is an ancestor of m . Since m is also an ancestor of $M_{low1}(v)$, we correctly set $M_{low1}(v) \leftarrow m$ (in lines 9 or 12). Otherwise, we have that $M_{low1}(v)$ is a descendant of the *low1* child c_1 of m . Now, due to the inductive hypothesis, *currentM*[c_1] is either c_1 or $M_{low1}(v')$ for a vertex $v' > v$ with $M(v') = m$. In the first case we

obviously have that $currentM[c_1]$ is an ancestor of $M_{low1}(v)$. Now assume that the second case is true, and let (x, y) be a back-edge with x a descendant of c_1 and y a proper ancestor of v . Then, since $v' > v$ and v, v' have m as a common descendant, we have that v is ancestor of v' , and therefore y is a proper ancestor of v' . This shows that x is a descendant of $M_{low1}(v')$. Thus, due to the generality of (x, y) , we have that $M_{low1}(v)$ is a descendant of $M_{low1}(v')$. In any case, then, we have that $currentM[c_1]$ is an ancestor of $M_{low1}(v)$. Thus we set $m \leftarrow currentM[c_1]$ and we continue the **while** loop, until we have that $m = M_{low1}(v)$, in which case we will set $currentM[c] \leftarrow m$ in line 15. Thus we have proved that Algorithm 4 correctly computes $M_{low1}(v)$, for every vertex $v \neq r$, and that, during the processing of a vertex v , every $currentM[c]$ that we access is an ancestor of $M_{low1}(v)$ (until, in line 15, we assign $currentM[c]$ to $M_{low1}(v)$).

Now, to prove linearity, let $S(v) = \{m_1, \dots, m_k\}$, ordered increasingly, denote the (possible empty) set of all vertices that we had to descend to before leaving the **while** loop in lines 8-14. (Thus, if $k \geq 1$, $m_k = M_{low1}(v)$.) In other words, $S(v)$ contains all vertices that were assigned to m in line 13. We will show that Algorithm 4 runs in linear time, by showing that, for every two vertices v and v' , $v \neq v'$ implies that $S(v) \cap S(v') \subseteq \{M_{low1}(v)\}$, where we have $S(v) \cap S(v') = \{M_{low1}(v)\}$ only if $M_{low1}(v) = M_{low1}(v')$. Of course, it is definitely the case that $S(v) \cap S(v') = \emptyset$ if v and v' are not related as ancestor and descendant, since the **while** loop descends to descendants of the vertex under processing. So let v' be a proper ancestor of v . If $M_{low1}(v')$ is not a descendant of the *low1* child c of $M(v)$, then we obviously have $S(v) \cap S(v') = \emptyset$ (since $S(v)$ consists of descendants of c , but the **while** loop during the computation of $M_{low1}(v')$ will not descend to the subtree of c). Thus we may assume that $M_{low1}(v')$ is a descendant of c . Now, let $S(v') = \{m_1, \dots, m_k\}$ and $m_0 = currentM[c']$, where c' is the *low1* child of $M(v')$. We will show that every m_i , for every $i \in \{1, \dots, k\}$, is either an ancestor of $M(v)$ or a descendant of $M_{low1}(v)$. (This obviously implies that $S(v') \cap S(v) \subseteq \{M_{low1}(v)\}$.) First observe that $M(v')$ is either an ancestor of $M(v)$ or a descendant of $M_{low1}(v)$. To see this, suppose that $M(v')$ is not an ancestor of $M(v)$. Since $M_{low1}(v')$ is a descendant of c , there is at least one back-edge (x, y) in $B(v')$ with x a descendant of c . Then, since y is a proper ancestor of v' and v' is a proper ancestor of v , we have that (x, y) is in $B(v)$, and therefore x is a descendant of $M_{low1}(v)$. Now let (x', y') be a back-edge in $B(v')$. If x' is a descendant of a vertex in $T[c, v']$, but not a descendant of c , then the nearest common ancestor of x and x' is in $T[M(v), v']$, and therefore $M(v')$ is an ancestor of $M(v)$, contradicting our supposition. Thus, x' is a descendant of c . Furthermore, y' is a proper ancestor of v , and therefore $(x', y') \in B(v)$. Thus, x' is a descendant of $M_{low1}(v)$. Due to the generality of $(x', y') \in B(v')$, we conclude that $M(v')$ is a descendant of $M_{low1}(v)$. Thus we have shown that $M(v')$ is either an ancestor of $M(v)$ or a descendant of $M_{low1}(v)$.

Now, if $M(v')$ is a descendant of $M_{low1}(v)$, we obviously have $S(v) \cap S(v') = \emptyset$. Let's assume, then, that $M(v')$ is an ancestor of $M(v)$. If $M(v')$ coincides with $M(v)$, then $c' = c$, and so m_0 coincides with $currentM[c]$, which is a descendant of $M_{low1}(v)$ (since $M_{low1}(v)$ has already been calculated), and therefore every m_i , for every $i \in \{1, \dots, k\}$, is a proper descendant of $M_{low1}(v)$ (since m_1 , if it exists, is a proper descendant of m_0), and so we have $S(v') \cap S(v) = \emptyset$. So let's assume that $M(v')$ is a proper ancestor of $M(v)$. Then, c' is an ancestor of $M(v)$. Suppose that m_0 is not an ancestor of $M(v)$. This means that $currentM[c'] \neq c'$, and therefore there is a vertex $\tilde{v} > v'$ with $M(\tilde{v}) = M(v')$ and $M_{low1}(\tilde{v}) = currentM[c']$. Furthermore, since m_0 is not an ancestor of $M(v)$, it must be a descendant of c . Now, since v' is an ancestor of v and $M(v')$ is a proper ancestor of $M(v)$, Lemma 2.4 implies that $M(v')$ is a proper ancestor of v . Since $M(v') = M(\tilde{v})$, this implies that $M(\tilde{v})$ is a proper ancestor of v , and therefore \tilde{v} is a proper ancestor of v . Now let (x, y) be a back-edge in $B(\tilde{v})$ such that x is a descendant of $M_{low1}(\tilde{v}) = currentM[c'] = m_0$. Then, since m_0 is a descendant of c , x is also descendant of c . Furthermore, since \tilde{v} is an ancestor of v , y is a proper ancestor of v . This shows that x is a descendant of $M_{low1}(v)$. Due to the generality

of (x, y) , we conclude that $M_{low1}(\tilde{v})$ is a descendant of $M_{low1}(v)$. Thus we have shown that m_0 is either an ancestor of $M(v)$ or a descendant of $M_{low1}(v)$.

Now let's assume that m_i is either an ancestor of $M(v)$ or a descendant of $M_{low1}(v)$, for some $i \in \{0, \dots, k-1\}$. We will prove that the same is true for m_{i+1} . If m_i is a descendant of $M_{low1}(v)$, then the same is true for m_{i+1} . Let's assume, then, that m_i is an ancestor of $M(v)$. Now we have that $m_{i+1} = \text{current}M[c_1]$, where c_1 is the *low1* child of m_i . If $m_i = M(v)$, then we have $c_1 = c$, and therefore $\text{current}M[c_1] = \text{current}M[c]$ is a descendant of $M_{low1}(v)$ (since $M_{low1}(v)$ has already been computed). Suppose, then, that m_i is a proper ancestor of $M(v)$. Then, c_1 is an ancestor of $M(v)$. If $\text{current}M[c_1] = c_1$, we obviously have that $\text{current}M[c_1]$ is an ancestor of $M(v)$. Otherwise, if $\text{current}M[c_1] \neq c_1$, there is a vertex \tilde{v} such that $M(\tilde{v}) = m_i$ and $\text{current}M[c_1] = M_{low1}(\tilde{v})$. Assume, first, that \tilde{v} is an ancestor of v . Suppose that $M_{low1}(\tilde{v})$ is not an ancestor of $M(v)$. Then it must be a descendant of c . Let (x, y) be a back-edge in $B(\tilde{v})$ with x a descendant of $M_{low1}(\tilde{v})$. Then x is a descendant of c . Furthermore, y is a proper ancestor of \tilde{v} , and therefore a proper ancestor of v . This shows that x is a descendant of $M_{low1}(v)$. Due to the generality of (x, y) , we conclude that $M_{low1}(\tilde{v})$ is a descendant of $M_{low1}(v)$. Thus, if \tilde{v} is an ancestor of v , $M_{low1}(\tilde{v})$ is either an ancestor of $M(v)$ or a descendant of $M_{low1}(v)$. Suppose, now, that \tilde{v} is a descendant of v . Let (x, y) be a back-edge in $B(v)$. Then, x is a descendant of $M(v)$, and therefore a descendant of c_1 . Furthermore, y is a proper ancestor of v , and therefore a proper ancestor of \tilde{v} . This shows that x is a descendant of $M_{low1}(\tilde{v})$. Due to the generality of (x, y) , we conclude that $M(v)$ is a descendant of $M_{low1}(\tilde{v})$. In any case, then, m_{i+1} is either an ancestor of $M(v)$ or a descendant of $M_{low1}(v)$. Thus, $S(v) \cap S(v') \subseteq \{M_{low1}(v)\}$ is established. \square

3 Computing the 3-cuts of a 3-edge-connected graph

In this section we present a linear-time algorithm that computes all the 3-edge-cuts of a 3-edge-connected graph $G = (V, E)$. It is well-known that the number of the 3-edge-cuts of G is $O(n)$ [17] (e.g., it follows from the definition of the cactus graph [1, 13]), but we provide an independent proof of this fact. Then, in Section 4.1, we show how to extend this algorithm so that it can also count the number of minimal 3-edge-cuts of a general graph. Note that there can be $O(n^3)$ such cuts [2].

Our method is to classify the 3-cuts on the DFS-tree T in a way that allows us to compute them efficiently. If $\{e_1, e_2, e_3\}$ is a 3-cut, we can initially distinguish three cases: either e_1 is a tree-edge and both e_2 and e_3 are back-edges (section 3.1), or e_1 and e_2 are two tree-edges and e_3 is a back-edge (section 3.2), or e_1, e_2 and e_3 is a triplet of tree-edges (section 3.3). Then, we divide those cases in subcases based on the concepts we have introduced in the previous section. Figure 1 gives a general overview of the cases we will handle in detail in the following sections.

3.1 One tree-edge and two back-edges

Lemma 3.1. *Let $\{(u, p(u)), e, e'\}$ be a 3-cut such that e and e' are back-edges. Then $B(u) = \{e, e'\}$. Conversely, if for a vertex $u \neq r$ we have $B(u) = \{e, e'\}$ where e and e' are back-edges, then $\{(u, p(u)), e, e'\}$ is a 3-cut.*

Proof. After removing the tree-edge $(u, p(u))$, the edges that connect $T(u)$ with the rest of the graph are precisely those contained in $B(u)$. Let e and e' be two back-edges in $B(u)$. Then it is obvious that $\{(u, p(u)), e, e'\}$ is a 3-cut if and only if $B(u)$ consists precisely of these two back-edges. \square

Thus, to find all 3-cuts of the form $\{(u, p(u)), e, e'\}$, where e and e' are back-edges, we only have to store, for every vertex u , two back-edges $e, e' \in B(u)$. Since $(low1D(u), low1(u))$ and

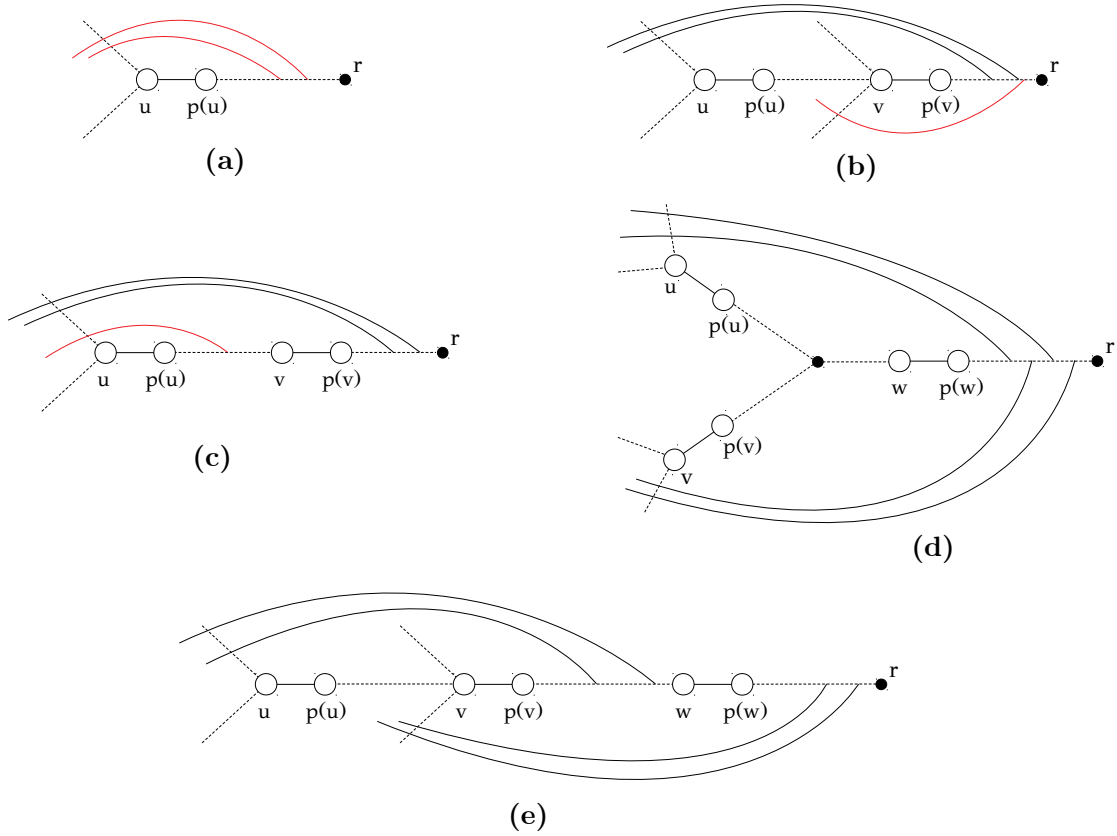


Figure 1: The types of 3-cuts with respect to a DFS-tree. **(a)** One tree-edge $(u, p(u))$ and two back-edges (section 3.1). **(b)** Two tree-edges $(u, p(u))$ and $(v, p(v))$, where u is a descendant of v , and one-back edge in $B(v) \setminus B(u)$ (section 3.2.1). **(c)** Two tree-edges $(u, p(u))$ and $(v, p(v))$, where u is a descendant of v , and one-back edge in $B(u) \setminus B(v)$ (section 3.2.2). **(d)** Three tree-edges $(u, p(u))$, $(v, p(v))$ and $(w, p(w))$, where w is an ancestor of u and v , but u and v are not related as ancestor and descendant (section 3.3.1). **(e)** Three tree-edges $(u, p(u))$, $(v, p(v))$ and $(w, p(w))$, where u is a descendant of v and v is a descendant of w (section 3.3.2).

$(low2D(u), low2(u))$ are two such back-edges, we mark the triplet $\{(u, p(u)), (low1D(u), low1(u)), (low2D(u), low2(u))\}$, for every u that has $b_count(u) = 2$.

3.2 Two tree-edges and one back-edge

Lemma 3.2. *Let $\{(u, p(u)), (v, p(v)), e\}$ be a 3-cut such that e is a back-edge. Then u and v are related as ancestor and descendant.*

Proof. Suppose that u and v are not related as ancestor or descendant. Since the graph is 3-edge-connected, $b_count(u) > 1$, and therefore there is least one back-edge $(x, y) \in B(u) \setminus \{e\}$. Since v is not a descendant of u , $v \notin T[x, u]$; and since v is not an ancestor of u , $v \notin T[p(u), y]$. Thus, by removing the edges $(u, p(u))$, $(v, p(v))$, and e , from the graph, u remains connected with $p(u)$, through the path $T[u, x], (x, y), T[p(u), y]$. This contradicts that fact that $\{(u, p(u)), (v, p(v)), e\}$ is a 3-cut. \square

Proposition 3.3. *Let $\{(u, p(u)), (v, p(v)), e\}$ be a 3-cut, where e is a back-edge. Then, either (1) $B(v) = B(u) \sqcup \{e\}$ or (2) $B(u) = B(v) \sqcup \{e\}$. Conversely, if there exists a back-edge e such that (1) or (2) is true, then $\{(u, p(u)), (v, p(v)), e\}$ is a 3-cut.*

Proof. (\Rightarrow) By Lemma 3.2, we may assume, without loss of generality, that v is an ancestor of u . Now, suppose that (1) does not hold; we will prove that (2) does. Since (1) is not true, there must exist a back-edge e' such that $e' \in B(v)$ and $e' \notin B(u) \cup \{e\}$, or $e' \notin B(v)$ and $e' \in B(u) \cup \{e\}$. Suppose the first is true: that is, there exists a back-edge (x, y) such that $(x, y) \in B(v)$ and $(x, y) \notin B(u) \cup \{e\}$. Then y is an ancestor of v , and therefore an ancestor of u . But, since $(x, y) \notin B(u)$, x cannot be a descendant of u , and thus it belongs to $T(v) \setminus T(u)$. Now, by removing the edges $(u, p(u))$, $(v, p(v))$ and e from the graph, we can see that v remains connected with $p(v)$ through the path $T[v, x], (x, y), T[y, p(v)]$. This contradicts the fact that $\{(u, p(u)), (v, p(v)), e\}$ is a 3-cut. Thus we have shown that there exists a back-edge e' such that $e' \notin B(v)$ and $e' \in B(u) \cup \{e\}$, and also that $B(v) \subseteq B(u) \cup \{e\}$. Now, suppose that there exists a back-edge $(x, y) \neq e$ such that $(x, y) \notin B(v)$ and $(x, y) \in B(u)$. Then x is a descendant of u , and therefore a descendant of v . But, since $(x, y) \notin B(v)$, y is not a proper ancestor of v , and thus belongs to $T[p(u), v]$. Now, by removing the edges $(u, p(u))$, $(v, p(v))$ and e from the graph, we can see that u remains connected with $p(u)$ through the path $T[u, x], (x, y), T[y, p(u)]$. This contradicts the fact that $\{(u, p(u)), (v, p(v)), e\}$ is a 3-cut. Thus we have shown that e is the unique back-edge such that $e \notin B(v)$ and $e \in B(u)$, and also that $B(u) \subseteq B(v) \cup \{e\}$. In conjunction with $B(v) \subseteq B(u) \cup \{e\}$, this implies that $B(u) = B(v) \sqcup \{e\}$.

(\Leftarrow) First, observe that both (1) and (2) imply that u and v are related as ancestor and descendant: Since the graph is 2-edge-connected, we have $b_count(x) > 0$, for every vertex $x \neq r$; and whenever we have $B(u) \cap B(v) \neq \emptyset$, for two vertices u and v , (and such is the case if either (1) or (2) is true), we can infer that u and v are related as ancestor and descendant. Now, due to the symmetry of the relations (1) and (2), we may assume, without loss of generality, that v is an ancestor of u . Let's assume first that (1) is true, and let $e = (x, y)$. Since $(x, y) \in B(v)$, y is a proper ancestor of v , and therefore a proper ancestor of u . But, since $(x, y) \notin B(u)$, x cannot be a descendant of u , and thus it belongs to $T(v) \setminus T(u)$. Furthermore, this is the only back-edge that starts from $T(v) \setminus T(u)$ and ends in a proper ancestor of v , since $B(v) \setminus \{e\} = B(u)$. Thus we can see that, by removing the edges $(u, p(u))$, $(v, p(v))$ and e from the graph, the graph becomes disconnected. (For the subgraph $T(v) \setminus T(u)$ becomes disconnected from $T(u) \cup (T(r) \setminus T(v))$.) Now assume that (2) is true, and let $e = (x, y)$. Since $(x, y) \in B(u)$, x is a descendant of u , and therefore a descendant of v . But, since $(x, y) \notin B(v)$, y is not a proper ancestor of v , and thus it belongs to $T[p(u), v]$. Furthermore, it is

the only back-edge that starts from $T(u)$ and ends in $T[p(u), v]$, since $B(u) \setminus \{e\} = B(v)$. Thus we can see that, by removing the edges $(u, p(u))$, $(v, p(v))$ and e from the graph, the graph becomes disconnected. (For the subgraph $T(v) \setminus T(u)$ becomes disconnected from $T(u) \cup (T(r) \setminus T(v))$.) \square

Here we distinguish two cases, depending on whether $B(v) = B(u) \sqcup \{e\}$ or $B(u) = B(v) \sqcup \{e\}$.

3.2.1 v is an ancestor of u and $B(v) = B(u) \sqcup \{e\}$.

Throughout this section let $V(u)$ denote the set of vertices v that are ancestors of u and such that $B(v) = B(u) \sqcup \{e\}$, for a back-edge e . By proposition 3.3, this means that $\{(u, p(u)), (v, p(v)), e\}$ is a 3-cut. The following lemma shows that, for every vertex v , there is at most one vertex u such that $v \in V(u)$.

Lemma 3.4. *Let u, u' be two distinct vertices. Then $V(u) \cap V(u') = \emptyset$.*

Proof. Suppose that there exists a $v \in V(u) \cap V(u')$. Then there are back-edges e, e' such that $B(v) = B(u) \sqcup \{e\}$ and $B(v) = B(u') \sqcup \{e'\}$, and so we have $B(u) \sqcup \{e\} = B(u') \sqcup \{e'\}$. Since $b_count(u) > 1$ and $b_count(u') > 1$ (for the graph is 3-edge-connected), we infer that $B(u) \cap B(u') \neq \emptyset$, and thus u and u' are related as ancestor and descendant. Thus we can assume, without loss of generality, that u' is an ancestor of u . Now let $(x, y) \in B(u)$. Then x is a descendant of u , and therefore a descendant of u' . Furthermore, since $B(v) = B(u) \sqcup \{e\}$, we have $(x, y) \in B(v)$, and so y is a proper ancestor of v , and therefore a proper ancestor of u' . This shows that $(x, y) \in B(u')$, and thus we have $B(u) \subseteq B(u')$. In conjunction with $B(u) \sqcup \{e\} = B(u') \sqcup \{e'\}$ (which implies that $|B(u)| = |B(u')|$), we infer that $B(u) = B(u')$ (and $e = e'$). This contradicts the fact that the graph is 3-edge-connected. \square

Thus, the total number of 3-cuts of the form $\{(u, p(u)), (v, p(v)), e\}$, where u is a descendant of v and e is a back-edge such that $B(v) = B(u) \sqcup \{e\}$, is $O(n)$. Now we will show how to compute, for every vertex v , the vertex u such that $v \in V(u)$ (if such a vertex u exists), together with the back-edge e such that $\{(u, p(u)), (v, p(v)), e\}$ is a 3-cut, in total linear time.

Let u, v, e be such that $v \in V(u)$ and $B(v) = B(u) \sqcup \{e\}$, and let $e = (x, y)$. Then y is a proper ancestor of v , and therefore a proper ancestor of u , so x cannot be a descendant of v (since $e \notin B(u)$). Thus, x is either on the tree-path $T(u, v]$, or it is a proper descendant of a vertex in $T(u, v]$, but not a descendant of u . In the first case we have $\tilde{M}(v) = M(u)$ (and $x = M(v)$); in the second case either $M_{low1}(v) = M(u)$ (and $x = M_{low2}(v)$) or $M_{low2}(v) = M(u)$ (and $x = M_{low1}(v)$). (For an illustration, see figure 2.) The following lemma shows how we can determine u from v .

Lemma 3.5. *Let v be an ancestor of u such that $\tilde{M}(v) = M(u)$ or $M_{low1}(v) = M(u)$ or $M_{low2}(v) = M(u)$, and let $m = \tilde{M}(v)$ or $M_{low1}(v)$ or $M_{low2}(v)$, depending on whether $\tilde{M}(v) = M(u)$ or $M_{low1}(v) = M(u)$ or $M_{low2}(v) = M(u)$. Then, $v \in V(u)$ if and only if u is the lowest element in $M^{-1}(m)$ which is greater than v and such that $high(u) < v$ and $b_count(v) = b_count(u) + 1$.*

Proof. (\Rightarrow) $v \in V(u)$ means that there exists a back-edge e such that $B(v) = B(u) \sqcup \{e\}$. Thus we get immediately $b_count(v) = b_count(u) + 1$ as a consequence. Furthermore, since $B(u) \subset B(v)$, we also get $high(u) < v$ (since for every $(x, y) \in B(u)$ it must be the case that y is a proper ancestor of v , and therefore $high(u)$ is a proper ancestor of v). Now, suppose that there exists a $u' \in M^{-1}(m)$ which is lower than u and greater than v . Then, since $B(u) = B(u')$ (and, in particular, $B(u') \subset B(u)$), there is a back-edge $(x, y) \in B(u)$ with $x \in T(u)$ and $y \in T[p(u), u']$. But this contradicts the fact that $high(u) < v$.

(\Leftarrow) Let $(x, y) \in B(u)$. Then x is a descendant of u , and therefore a descendant of v . Furthermore,

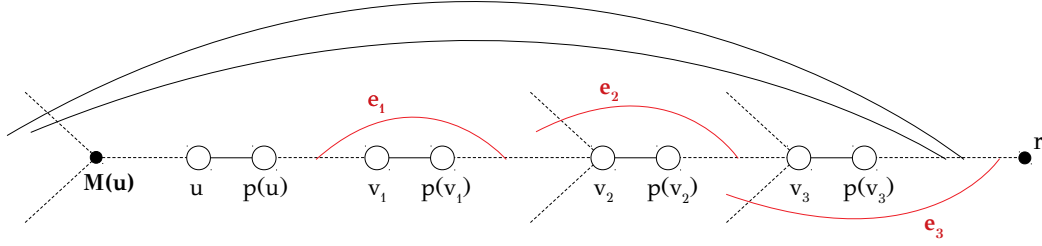


Figure 2: In this example we have $V(u) = \{v_1, v_2, v_3\}$, and every back-edge e_i satisfies $B(v_i) = B(u) \sqcup \{e_i\}$. It should be clear that every $M(v_i)$ is an ancestor of $M(u)$, and $\tilde{M}(v_1) = M(u)$, $M_{low1}(v_2) = M(u)$ and $M_{low2}(v_3) = M(u)$. It is perhaps worth noting that, for every vertex u , we may have many vertices $v \in V(u)$ with $\tilde{M}(v) = M(u)$ or $M_{low1}(v) = M(u)$, but only the lowest v in $V(u)$ may have $M_{low2}(v) = M(u)$.

$high(u) < v$ implies that y is a proper ancestor of v . This shows that $(x, y) \in B(v)$, and thus we have $B(u) \subseteq B(v)$. Then, $b_count(v) = b_count(u) + 1$ implies the existence of a back-edge $e \in B(v) \setminus B(u)$ such that $B(v) = B(u) \sqcup \{e\}$. \square

Thus, for every vertex v , we have to check whether the lowest element u of $M^{-1}(m)$ which is greater than v satisfies $b_count(v) = b_count(u) + 1$, for all $m \in \{\tilde{M}(v), M_{low1}(v), M_{low2}(v)\}$. To do this efficiently, we process the vertices in a bottom-up fashion, and we keep in a variable *currentVertex*[m] the lowest element of $M^{-1}(m)$ currently under consideration, so that we do not have to traverse the list $M^{-1}(m)$ from the beginning each time we process a vertex. Algorithm 5 is an implementation of this procedure.

3.2.2 v is an ancestor of u and $B(u) = B(v) \sqcup \{e\}$.

Throughout this section let $U(v)$ denote the set of vertices u that are descendants of v and such that $B(u) = B(v) \sqcup \{e\}$, for a back-edge e . By proposition 3.3, this means that $\{(u, p(u)), (v, p(v)), e\}$ is a 3-cut. The following lemma shows that, for every vertex u , there is at most one vertex v such that $u \in U(v)$.

Lemma 3.6. *Let v, v' be two distinct vertices. Then $U(v) \cap U(v') = \emptyset$.*

Proof. Suppose that there exists a $u \in U(v) \cap U(v')$. Then v and v' are related as ancestor and descendant, since they have a common descendant. Thus we may assume, without loss of generality, that v' is an ancestor of v . Let (x, y) be a back-edge in $B(v')$. Then, y is a proper ancestor of v' , and therefore a proper ancestor of v . Furthermore, $u \in U(v')$ implies that $B(v') \subseteq B(u)$, and therefore $(x, y) \in B(u)$. Thus, x is a descendant of u , and therefore a descendant of v . This shows that $(x, y) \in B(v)$, and thus we have $B(v') \subseteq B(v)$. Now, $u \in U(v) \cap U(v')$ means that there exist two back-edges e, e' such that $B(u) = B(v) \sqcup \{e\}$ and $B(u) = B(v') \sqcup \{e'\}$, and thus we have $B(v) \sqcup \{e\} = B(v') \sqcup \{e'\}$. Therefore, $|B(v)| = |B(v')|$. In conjunction with $B(v') \subseteq B(v)$,

Algorithm 5: Find all 3-cuts $\{(u, p(u)), (v, p(v)), e\}$, where u is a descendant of v and $B(v) = B(u) \sqcup \{e\}$, for a back-edge e .

```

1 initialize an array currentVertex with  $n$  entries
  //  $m = \tilde{M}(v)$ 
2 foreach vertex  $x$  do currentVertex[ $x$ ]  $\leftarrow x$ 
3 for  $v \leftarrow n$  to  $v = 1$  do
4    $m \leftarrow \tilde{M}(v)$ 
5   if  $m = \emptyset$  then continue
6   // find the lowest  $u \in M^{-1}(m)$  which is greater than  $v$ 
7    $u \leftarrow \text{currentVertex}[m]$ 
8   while  $\text{nextM}(u) \neq \emptyset$  and  $\text{nextM}(u) > v$  do  $u \leftarrow \text{nextM}(u)$ 
9   currentVertex[ $m$ ]  $\leftarrow u$ 
10  // check the condition in lemma 3.5
11  if  $\text{high}(u) < v$  and  $\text{b\_count}(v) = \text{b\_count}(u) + 1$  then
12  | mark the triplet  $\{(u, p(u)), (v, p(v)), (M(v), l(M(v)))\}$ 
13  end
14 end
15 //  $m = M_{low1}(v)$ 
16 foreach vertex  $x$  do currentVertex[ $x$ ]  $\leftarrow x$ 
17 for  $v \leftarrow n$  to  $v = 1$  do
18    $m \leftarrow M_{low1}(v)$ 
19   if  $m = \emptyset$  then continue
20   // find the lowest  $u \in M^{-1}(m)$  which is greater than  $v$ 
21    $u \leftarrow \text{currentVertex}[m]$ 
22   while  $\text{nextM}(u) \neq \emptyset$  and  $\text{nextM}(u) > v$  do  $u \leftarrow \text{nextM}(u)$ 
23   currentVertex[ $m$ ]  $\leftarrow u$ 
24   // check the condition in lemma 3.5
25   if  $\text{high}(u) < v$  and  $\text{b\_count}(v) = \text{b\_count}(u) + 1$  then
26   | mark the triplet  $\{(u, p(u)), (v, p(v)), (M_{low2}(v), l(M_{low2}(v)))\}$ 
27   end
28 end
29 //  $m = M_{low2}(v)$ 
30 foreach vertex  $x$  do currentVertex[ $x$ ]  $\leftarrow x$ 
31 for  $v \leftarrow n$  to  $v = 1$  do
32    $m \leftarrow M_{low2}(v)$ 
33   if  $m = \emptyset$  then continue
34   // find the lowest  $u \in M^{-1}(m)$  which is greater than  $v$ 
35    $u \leftarrow \text{currentVertex}[m]$ 
36   while  $\text{nextM}(u) \neq \emptyset$  and  $\text{nextM}(u) > v$  do  $u \leftarrow \text{nextM}(u)$ 
37   currentVertex[ $m$ ]  $\leftarrow u$ 
38   currentVertex[ $m$ ]  $\leftarrow \text{prev}$ 
39   // check the condition in lemma 3.5
40   if  $\text{high}(u) < v$  and  $\text{b\_count}(v) = \text{b\_count}(u) + 1$  then
41   | mark the triplet  $\{(u, p(u)), (v, p(v)), (M_{low1}(v), l(M_{low1}(v)))\}$ 
42   end
43 end

```

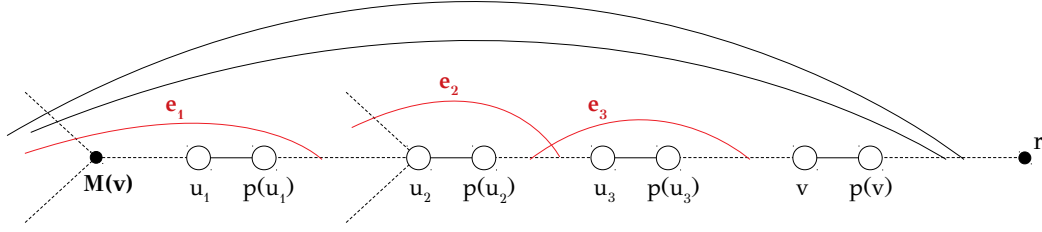


Figure 3: In this example we have $U(v) = \{u_1, u_2, u_3\}$, and every back-edge e_i satisfies $B(u_i) = B(v) \sqcup \{e_i\}$. It should be clear that every $M(u_i)$ is an ancestor of $M(v)$, and $M(u_1) = M(v)$, $M_{low1}(u_2) = M(v)$ and $\tilde{M}(u_3) = M(v)$. It is perhaps worth noting that, for every vertex v , only one $u \in U(v)$ may have $M(u) = M(v)$ (that is, the one satisfying $nextM(u) = v$), but we may have many vertices $u \in V(v)$ with $\tilde{M}(u) = M(v)$ or $M_{low1}(u) = M(v)$.

this implies that $B(v) = B(v')$ (and $e = e'$), contradicting the fact that the graph is 3-edge-connected. \square

Thus, the total number of 3-cuts of the form $\{(u, p(u)), (v, p(v)), e\}$, where u is a descendant of v and e is a back-edge such that $B(u) = B(v) \sqcup \{e\}$, is $O(n)$. We will now show how to compute, for every vertex u , the vertex v such that $u \in U(v)$ (if such a vertex v exists), together with the back-edge e such that $\{(u, p(u)), (v, p(v)), e\}$ is a 3-cut, in total linear time.

Let u, v, e be such that $u \in U(v)$ and $B(u) = B(v) \sqcup \{e\}$, and let $e = (x, y)$. Then, x is a descendant of u , and therefore a descendant of v . But since $e \notin B(v)$, y is not an ancestor of v , and therefore $y \in T[p(u), v]$. Thus, $y = high(u)$ (and $x = highD(u)$), since every other back-edge $(x', y') \in B(u)$ is also in $B(v)$ and thus has $y' < v \leq y$. This shows how we can determine the back-edge e from a pair of vertices u, v that satisfy $u \in U(v)$. Furthermore, $B(u) = B(v) \sqcup \{e\}$ implies that $M(u)$ is an ancestor of $M(v)$. Thus, either $M(u) = M(v)$, or $M(u)$ is a proper ancestor of $M(v)$. In the second case, we have that either $\tilde{M}(u) = M(v)$ or $M_{low1}(u) = M(v)$ (since the *low* point of u is given by a back-edge in $B(v)$). (For an illustration, see figure 3.) Now the following lemma shows how we can determine v from u .

Lemma 3.7. *Let u be a descendant of v such that $M(u) = M(v)$ or $\tilde{M}(u) = M(v)$ or $M_{low1}(u) = M(v)$, and let $m = M(u)$ or $\tilde{M}(u)$ or $M_{low1}(u)$, depending on whether $M(u) = M(v)$ or $\tilde{M}(u) = M(v)$ or $M_{low1}(u) = M(v)$. Then $u \in U(v)$ if and only if v is the greatest element in $M^{-1}(m)$ which is lower than u and such that $b_count(u) = b_count(v) + 1$.*

Proof. (\Rightarrow) $u \in U(v)$ means that there exists a back-edge e such that $B(u) = B(v) \sqcup \{e\}$. Thus we get immediately that $b_count(u) = b_count(v) + 1$. Now suppose, for the sake of contradiction, that there exists a $v' \in M^{-1}(m)$ which is greater than v and lower than u . Let $(x, y) \in B(v')$. Then y is a proper ancestor of v' , and therefore a proper ancestor of u . Furthermore, x is a descendant of $M(v')$ ($= M(v)$), and so every one of the relations $M(u) = M(v)$, $\tilde{M}(u) = M(v)$

or $M_{low1}(u) = M(v)$ implies that x is a descendant of u . This shows that $(x, y) \in B(u)$, and thus we have $B(v') \subseteq B(u)$. Now, since $M(v) = M(v')$ and v' is a proper ancestor of v , we have $B(v) \subset B(v')$. Since $b_count(u) = b_count(v) + 1$, $B(v) \subset B(v') \subseteq B(u)$ implies that $B(u) = B(v')$, contradicting the fact that the graph is 3-edge-connected.

(\Leftarrow) Let $(x, y) \in B(v)$. Then y is a proper ancestor of v , and therefore a proper ancestor of u . Furthermore, x is a descendant of $M(v)$, and every one of the relations $M(u) = M(v)$, $\tilde{M}(u) = M(v)$ or $M_{low1}(u) = M(v)$ implies that x is a descendant of $M(u)$. This shows that $(x, y) \in B(u)$. Thus we have $B(v) \subseteq B(u)$, and so $b_count(u) = b_count(v) + 1$ implies that there exists a back-edge e such $B(u) = B(v) \sqcup \{e\}$. \square

Thus, for every vertex u , we have to check whether the greatest element v in $M^{-1}(m)$ which is lower than u satisfies $b_count(u) = b_count(v) + 1$, for all $m \in \{M(u), \tilde{M}(u), M_{low1}(u)\}$. To do this efficiently, we process the vertices in a bottom-up fashion, and we keep in a variable `currentVertex[m]` the lowest element of $M^{-1}(m)$ currently under consideration, so that we do not have to traverse the list $M^{-1}(m)$ from the beginning each time we process a vertex. Algorithm 6 is an implementation of this procedure.

3.3 Three tree-edges

Lemma 3.8. *Let $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ be a 3-cut, and assume, without loss of generality, that $w < \min\{v, u\}$. Then w is an ancestor of both u and v .*

Proof. Suppose that w is neither an ancestor of u nor an ancestor of v . Let $(x, y) \in B(w)$. Then x is a descendant of w , and therefore it is not a descendant of either u or v . In other words, $u, v \notin T[x, w]$. Furthermore, y is a proper ancestor of w . Since neither u nor v is an ancestor of w (since $w < \min\{v, u\}$), we have that $u, v \notin T[w, r]$, and therefore $u, v \notin T[w, y]$. Thus, by removing the tree-edges $(u, p(u))$, $(v, p(v))$ and $(w, p(w))$, w remains connected with $p(w)$ through the path $T[w, x], (x, y), T[y, p(w)]$, contradicting the fact that $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ is a 3-cut. This shows that w is an ancestor of either u or v (or both). Suppose, for the sake of contradiction, that w is not an ancestor of u . Then w is an ancestor of v . This implies that u is not a descendant of v (for otherwise it would be a descendant of w). If u is an ancestor of v , it must necessarily be an ancestor of w (because $v \in T(w)$ and $u \notin T(w)$), but $w < u$ forbids this case. Thus, u is not a descendant of v . So far, then, we have that u is not related as ancestor and descendant with either w or v . Thus we may follow the same reasoning as above, to conclude that, by removing the tree-edges $(u, p(u))$, $(v, p(v))$ and $(w, p(w))$, u remains connected with $p(u)$, again contradicting the fact that $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ is a 3-cut. This shows that w is an ancestor of u . Using the same argument we can also prove that w is an ancestor of v . \square

At this point we distinguish two cases, depending on whether u and v are related as ancestor and descendant.

3.3.1 u and v are not related as ancestor and descendant

In what follows we will provide some characterizations of the 3-cuts of the form $\{(u, p(u)), (v, p(v)), (w, p(w))\}$, where w is an ancestor of u and v , and u, v are not related as ancestor and descendant. It will be useful to keep in mind the situation depicted in Figure 4.

Proposition 3.9. *Let u and v be two vertices which are not related as ancestor and descendant, and let w be an ancestor of both u and v . Then, $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ is a 3-cut if and only if $B(w) = B(u) \sqcup B(v)$.*

Algorithm 6: Find all 3-cuts $\{(u, p(u)), (v, p(v)), e\}$, where u is a descendant of v and $B(u) = B(v) \sqcup \{e\}$, for a back-edge e .

```

1 initialize an array currentVertex with  $n$  entries
  //  $m = M(v)$ ; just check whether the condition of Lemma 3.7 is satisfied for
  //  $nextM(u)$ 
2 if  $b\_count(u) = b\_count(nextM(u)) + 1$  then
3   | mark the triplet  $\{(u, p(u)), (nextM(u), p(nextM(u))), (highD(u), high(u))\}$ 
4 end
  //  $m = \tilde{M}(u)$ 
5 foreach vertex  $x$  do currentVertex $[x] \leftarrow x$ 
6 for  $u \leftarrow n$  to  $u = 1$  do
7   |  $m \leftarrow \tilde{M}(u)$ 
8   | if  $m = \emptyset$  then continue
  // find the greatest  $v \in M^{-1}(m)$  which is lower than  $u$ 
9   |  $v \leftarrow currentVertex[m]$ 
10  | while  $v \neq \emptyset$  and  $v \geq u$  do  $v \leftarrow nextM(v)$ 
11  | currentVertex $[m] \leftarrow v$ 
  // check the condition in Lemma 3.7
12  | if  $b\_count(u) = b\_count(v) + 1$  then
13  |   | mark the triplet  $\{(u, p(u)), (v, p(v)), (highD(u), high(u))\}$ 
14  |   end
15 end
  //  $m = M_{low1}(u)$ 
16 foreach vertex  $x$  do currentVertex $[x] \leftarrow x$ 
17 for  $u \leftarrow n$  to  $u = 1$  do
18  |  $m \leftarrow M_{low1}(u)$ 
19  | if  $m = \emptyset$  then continue
  // find the greatest  $v \in M^{-1}(m)$  which is lower than  $u$ 
20  |  $v \leftarrow currentVertex[m]$ 
21  | while  $v \neq \emptyset$  and  $v \geq u$  do  $v \leftarrow nextM(v)$ 
22  | currentVertex $[m] \leftarrow v$ 
  // check the condition in Lemma 3.7
23  | if  $b\_count(u) = b\_count(v) + 1$  then
24  |   | mark the triplet  $\{(u, p(u)), (v, p(v)), (highD(u), high(u))\}$ 
25  |   end
26 end

```

Proof. (\Rightarrow) Let $(x, y) \in B(w)$, and let's assume that $(x, y) \notin B(u)$. Since y is a proper ancestor of w , and therefore a proper ancestor of u , from $(x, y) \notin B(u)$ we infer that x is not a descendant of u . Suppose for the sake of contradiction that x is not a descendant of v , either. This means that neither u nor v is in $T[x, w]$, and so, by removing the edges $(u, p(u))$, $(v, p(v))$ and $(w, p(w))$, w remains connected with $p(w)$ through the path $T[w, x], (x, y), T[y, p(w)]$. This contradicts that fact that $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ is a 3-cut. Thus we have established that x is a descendant of v . Since y is also a proper ancestor of v , we have $(x, y) \in B(v)$. Thus we have shown that $B(w) \subseteq B(u) \cup B(v)$. Conversely, let $(x, y) \in B(u) \cup B(v)$, and assume, without loss of generality, that $(x, y) \in B(u)$. Then, x is a descendant of u , and therefore a descendant of w . Now suppose, for the sake of contradiction, that y is not a proper ancestor of w . Then we have $w \notin T[p(u), y]$, and since w is not a descendant of u , we also have $w \notin T[x, u]$. Furthermore, since u and v are not related as ancestor and descendant, v is not contained neither in $T[p(u), y]$ nor in $T[x, u]$. Thus, by removing the edges $(u, p(u))$, $(v, p(v))$ and $(w, p(w))$, u remains connected with $p(u)$ through the path $T[u, x], (x, y), T[y, p(u)]$. This contradicts that fact that $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ is a 3-cut. Thus we have shown that y is a proper ancestor of w , and so we have that $(x, y) \in B(w)$. Thus we have established that $B(u) \cup B(v) \subseteq B(w)$, and so we have $B(w) = B(u) \cup B(v)$. Since u and v are not related as ancestor and descendant, we have $B(u) \cap B(v) = \emptyset$. We conclude that $B(w) = B(u) \sqcup B(v)$.

(\Leftarrow) Consider the sets of vertices $T(u)$, $T(v)$, $A = T(w) \setminus (T(u) \cup T(v))$ and $B = T(w) \setminus T(u)$. Since u and v are not related as ancestor and descendant, and w is an ancestor of both u and v , these sets are mutually disjoint. Now, since $B(u) \subset B(w)$, all back-edges that start from $T(u)$ end either in $T(u)$ or in B . Similarly, since $B(v) \subset B(w)$, all back-edges that start from $T(v)$ end either in $T(v)$ or in B . Furthermore, a back-edge that starts from A cannot reach B and must necessarily end in A , since it starts from a descendant of w , but not from a descendant of either u or v (while we have $B(w) = B(u) \sqcup B(v)$). Thus, by removing from the graph the tree-edges $(u, p(u))$, $(v, p(v))$ and $(w, p(w))$, the graph becomes separated into two parts: $T(u) \cup T(v) \cup B$ and A . \square

Lemma 3.10. *Let u and v be two vertices which are not related as ancestor and descendant, and let w be an ancestor of both u and v . Then $B(w) = B(u) \sqcup B(v)$ if and only if: $M_{low1}(w) = M(u)$ and $M_{low2}(w) = M(v)$ (or $M_{low1}(w) = M(v)$ and $M_{low2}(w) = M(u)$), and $high(u) < w$, $high(v) < w$, and $b_count(w) = b_count(u) + b_count(v)$.*

Proof. (\Rightarrow) $b_count(w) = b_count(u) + b_count(v)$ is an immediate consequence of $B(w) = B(u) \sqcup B(v)$. Furthermore, since every $(x, y) \in B(u)$ is also in $B(w)$, it has $y < w$, and so $high(u) < w$. With the same reasoning, we also get $high(v) < w$. Now, since $B(w) = B(u) \sqcup B(v)$, we have that $M(w)$ is an ancestor of both $M(u)$ and $M(v)$. Since u and v are not related as ancestor and descendant, $M(u)$ and $M(v)$ are not related as ancestor or descendant, either. This implies that they are both proper descendants of $M(w)$. Now, suppose, for the sake of contradiction, that $M(u)$ and $M(v)$ are descendants of the same child c of $M(w)$. Then there must exist a back-edge $(x, y) \in B(w)$ such that $x = M(w)$ or x is a descendant of a child of $M(w)$ different from c . (Otherwise, $M(w)$ would be a descendant of c , which is absurd.) But this contradicts the fact that $B(w) = B(u) \sqcup B(v)$, since (x, y) does not belong neither in $B(u)$ nor in $B(v)$. Thus, $M(u)$ and $M(v)$ are descendants of different children of $M(w)$. Furthermore, since every back-edge $(x, y) \in B(w)$ has x in $T(u)$ or $T(v)$, there are no other children of $M(w)$ from whose subtrees begin back-edges that end in a proper ancestor of w . Thus, one of $M(u)$ and $M(v)$ is a descendant of the *low1* child of $M(w)$, and the other is a descendant of the *low2* child of $M(w)$. We may assume, without loss of generality, that $M(u)$ is a descendant of the *low1* child of $M(w)$, and $M(v)$ is a descendant of the *low2* child of $M(w)$. Since $B(u) \subset B(w)$, we have that $M(u)$ is a descendant of

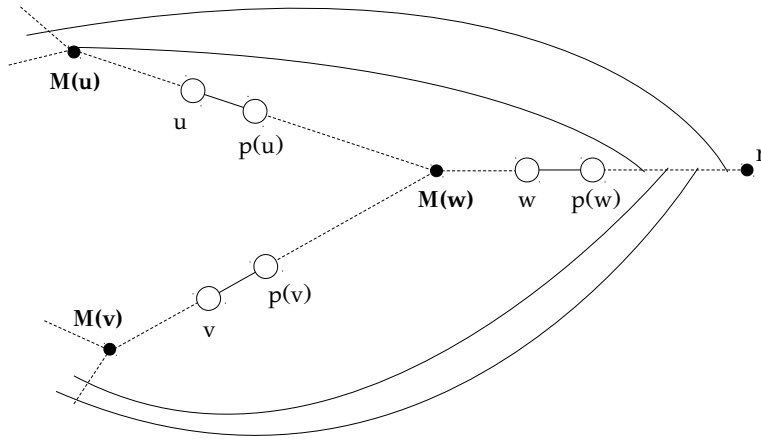


Figure 4: In this example we have $B(w) = B(u) \sqcup B(v)$. Observe that $M_{low1}(w) = M(u)$ and $M_{low2}(w) = M(v)$. Furthermore, $high(u) < w$ and $high(v) < w$. Also, if there is another vertex u' with $M(u') = M(u)$, it must either be a descendant of u or an ancestor of w . Thus, u is the lowest vertex in $M^{-1}(M_{low1}(w))$ which is greater than w . Similarly, v is the lowest vertex in $M^{-1}(M_{low2}(w))$ which is greater than w . By Lemmata 3.10 and 3.11, these properties (together with $b_count(w) = b_count(u) + b_count(v)$) are sufficient to establish $B(w) = B(u) \sqcup B(v)$. Notice also that, if we remove the tree-edges $(u, p(u))$, $(v, p(v))$ and $(w, p(w))$, the graph becomes disconnected into two components: $T(u) \cup T(v) \cup (T(r) \setminus T(w))$ and $T(w) \setminus (T(u) \cup T(v))$. (See also the “ \Leftarrow ” part of the proof of proposition 3.9.)

$M_{low1}(w)$. Furthermore, since $B(w) = B(u) \sqcup B(v)$ and $M(v)$ is not a descendant of the *low1* child of $M(w)$, there are no back-edges (x, y) with x a descendant of the *low1* child of $M(w)$ and y a proper ancestor of w apart from those contained in $B(u)$. Thus, $M(u)$ is an ancestor of $M_{low1}(w)$, and $M_{low1}(w) = M(u)$ is established. With the same reasoning, we also get $M_{low2}(w) = M(v)$.
 (\Leftarrow) Let $(x, y) \in B(u)$. Then x is a descendant of u , and therefore a descendant of w . Furthermore, since $high(u) < w$, we have $y < w$, and therefore y is a proper ancestor of w . This shows that $(x, y) \in B(w)$, and thus $B(u) \subseteq B(w)$. With the same reasoning, we also get $B(v) \subseteq B(w)$. Thus we have $B(u) \cup B(v) \subseteq B(w)$. Since u and v are not related as ancestor and descendant, we have $B(u) \cap B(v) = \emptyset$. From $B(u) \cup B(v) \subseteq B(w)$, $B(u) \cap B(v) = \emptyset$, and $b_count(w) = b_count(u) + b_count(v)$, we conclude that $B(w) = B(u) \sqcup B(v)$. \square

The following lemma shows, that, for every vertex w , there is at most one pair u, v of descendants of w which are not related as ancestor and descendant and are such that $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ is a 3-cut. Thus, the number of 3-cuts of this type is $O(n)$. Furthermore, it allows us to compute u and v (if such a pair of u and v exists).

Lemma 3.11. *Let $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ be a 3-cut such that u and v are not related as ancestor and descendant and let w is an ancestor of both u and v . Assume w.l.o.g. that $M_{low1}(w) = M(u)$ and $M_{low2}(w) = M(v)$, and let $m_1 = M_{low1}(w)$ and $m_2 = M_{low2}(w)$. Then u is the lowest vertex in $M^{-1}(m_1)$ which is greater than w , and v is the lowest vertex in $M^{-1}(m_2)$ which is greater than w .*

Proof. By Proposition 3.9, we have that $B(w) = B(u) \sqcup B(v)$. Now, suppose that there exists a $u' \in M^{-1}(m_1)$ which is lower than u and greater than w . Then, $M(u') = M(u)$ implies that $B(u') \subset B(u)$, and so there is a back-edge $(x, y) \in B(u) \setminus B(u')$. This means that y is not a proper ancestor of u' , and therefore not a proper ancestor of w , either. But this implies that $(x, y) \notin B(w)$, contradicting the fact that $B(u) \subset B(w)$. A similar argument shows that there does not exist a $v' \in M^{-1}(m_2)$ which is lower than v and greater than w . \square

Thus we only have to find, for every vertex w , the lowest element u of $M^{-1}(M_{low1}(w))$ which is greater than w , and the lowest element v of $M^{-1}(M_{low2}(w))$ which is greater than w , and check the condition in Lemma 3.10 - i.e., whether $high(u) < w$, $high(v) < w$, and $b_count(w) = b_count(u) + b_count(v)$. To do this efficiently, we process the vertices in a bottom-up fashion, and we keep in a variable *currentVertex*[x] the lowest element of $M^{-1}(x)$ currently under consideration. Thus, we do not need to traverse the list $M^{-1}(x)$ from the beginning each time we process a vertex. Algorithm 7 is an implementation of this procedure.

3.3.2 u and v are related as ancestor and descendant

Throughout this section it will be useful to keep in mind the situation depicted in Figure 5.

Proposition 3.12. *Let u, v, w be three vertices such that u is a descendant of v and v is a descendant of w . Then $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ is a 3-cut if and only if $B(v) = B(u) \sqcup B(w)$.*

Proof. (\Rightarrow) Let $(x, y) \in B(v)$, and assume that $(x, y) \notin B(u)$. $(x, y) \in B(v)$ implies that y is a proper ancestor of v , and therefore a proper ancestor of u . Thus, $(x, y) \notin B(u)$ implies that x is not a descendant of u . Furthermore, $(x, y) \in B(v)$ implies that x is a descendant of v , and therefore a descendant of w . Now suppose, for the sake of contradiction, that y is not a proper ancestor of w . Then, $w \notin T[p(v), y]$. Now we see that, by removing the edges $(u, p(u))$, $(v, p(v))$ and $(w, p(w))$ from the graph, v remains connected with $p(v)$ through the path $T[v, x], (x, y), T[y, p(v)]$ (since

Algorithm 7: Find all 3-cuts $\{(u, p(u)), (v, p(v)), (w, p(w))\}$, where w is an ancestor of u and v , and u, v are not related as ancestor and descendant

```

1 initialize an array currentVertex with  $n$  entries
2 foreach vertex  $x$  do currentVertex $[x] \leftarrow x$ 
3 for  $w \leftarrow n$  to  $w = 1$  do
4    $m_1 \leftarrow M_{low1}(w), m_2 \leftarrow M_{low2}(w)$ 
5   if  $m_1 = \emptyset$  or  $m_2 = \emptyset$  then continue
6   // find the lowest  $u$  in  $M^{-1}(m_1)$  which is greater than  $w$ 
7    $u \leftarrow \text{currentVertex}[m_1]$ 
8   while  $\text{nextM}(u) \neq \emptyset$  and  $\text{nextM}(u) > w$  do  $u \leftarrow \text{nextM}(u)$ 
9   currentVertex $[m_1] \leftarrow u$ 
10  // find the lowest  $v$  in  $M^{-1}(m_2)$  which is greater than  $w$ 
11   $v \leftarrow \text{currentVertex}[m_2]$ 
12  while  $\text{nextM}(v) \neq \emptyset$  and  $\text{nextM}(v) > w$  do  $v \leftarrow \text{nextM}(v)$ 
13  currentVertex $[m_2] \leftarrow v$ 
14  // check the condition in Lemma 3.10
15  if  $b\_count(w) = b\_count(u) + b\_count(v)$  and  $high(u) < w$  and  $high(v) < w$  then
16    | mark the triplet  $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ 
17  end
18 end

```

$u, w \notin T[v, x] \cup T[p(v), y]$). This contradicts the fact that $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ is a 3-cut. Therefore, y is a proper ancestor of w , and thus $(x, y) \in B(w)$. Thus far we have established that $B(v) \subseteq B(u) \cup B(w)$. Now let $(x, y) \in B(u)$. Then x is a descendant of u , and therefore a descendant of v . Suppose, for the sake of contradiction, that y is not a proper ancestor of v . Then, $v \notin T[p(u), y]$. Now we see that, by removing the edges $(u, p(u)), (v, p(v))$ and $(w, p(w))$ from the graph, u remains connected with $p(u)$ through the path $T[u, x], (x, y), T[y, p(u)]$. This contradicts the fact that $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ is a 3-cut. Therefore, y is a proper ancestor of v , and thus $(x, y) \in B(v)$. This shows that $B(u) \subseteq B(v)$. Now let $(x, y) \in B(w)$. Then y is a proper ancestor of w , and therefore a proper ancestor of v . Suppose, for the sake of contradiction, that x is not a descendant of v . Then x is not a descendant of u , either, and so $u, v \notin T[x, w]$. Thus we see that, by removing the edges $(u, p(u)), (v, p(v))$ and $(w, p(w))$ from the graph, w remains connected with $p(w)$ through the path $T[w, x], (x, y), T[y, p(w)]$. This contradicts the fact that $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ is a 3-cut. Therefore, x is a descendant of v , and thus $(x, y) \in B(v)$. This shows that $B(w) \subseteq B(v)$. Thus we have established that $B(u) \cup B(w) \subseteq B(v)$, and so we have $B(v) = B(u) \cup B(w)$.

Now suppose, for the sake of contradiction, that there is a back-edge $(x, y) \in B(u) \cap B(w)$. Since $B(u) \neq B(w)$ (for otherwise $u = w$), there must exist a back-edge (x', y') in $B(u) \setminus B(w)$ or in $B(w) \setminus B(u)$. Take the first case, first. Then, since $B(u) \subseteq B(v)$, y' is a proper ancestor of v . But since $(x', y') \notin B(w)$, y' cannot be a proper ancestor of w . Let P be a path connecting x' with x in $T(u)$. Then, by removing the tree-edges $(u, p(u)), (v, p(v))$ and $(w, p(w))$, w remains connected with $p(w)$ through the path $T[w, y'], (x', y'), P, (x, y), T[y, p(w)]$, which contradicts the assumption that $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ is a 3-cut. Now take the case $\exists (x', y') \in B(w) \setminus B(u)$. Then, since $B(w) \subseteq B(v)$, x' is a descendant of v . But since $(x', y') \notin B(u)$, x' cannot be a descendant of u . Let P be a path connecting y with y' in $T(v) \setminus T(u)$, and Q be a path connecting x' with $p(u)$ in $T(u) \setminus T(v)$. Then, by removing the tree-edges $(u, p(u)), (v, p(v))$ and $(w, p(w))$, u remains

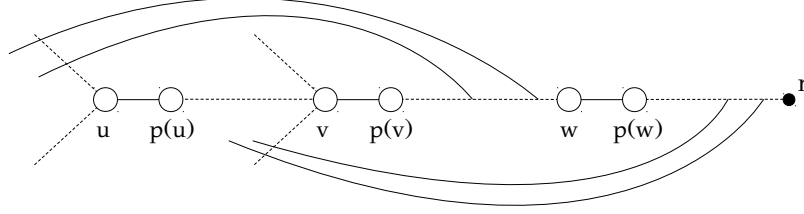


Figure 5: In this example we have $B(v) = B(u) \sqcup B(w)$. By removing the tree-edges $(u, p(u))$, $(v, p(v))$ and $(w, p(w))$, the graph becomes disconnected into two components: $T(u) \cup (T(w) \setminus T(v))$ and $(T(v) \setminus T(u)) \cup (T(r) \setminus T(w))$. (See also the “ \Leftarrow ” part of the proof of proposition 3.12.)

connected with $p(u)$ through the path $T[u, x], (x, y), P, (y', x'), Q$, which contradicts the assumption that $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ is a 3-cut. This shows that $B(u) \cap B(w) = \emptyset$. We conclude that $B(v) = B(u) \sqcup B(w)$.

(\Leftarrow) Consider the sets of vertices $A = T(u)$, $B = T(v) \setminus T(u)$, $C = T(w) \setminus T(v)$ and $D = T(r) \setminus T(w)$. Since u is a descendant of v and v is a descendant of w , these sets are mutually disjoint. Now, since $B(u) \subset B(v)$ and $B(u) \cap B(w) = \emptyset$, every back-edge that starts from A ends either in A or in $T(v, w]$, and thus in C . Furthermore, every back-edge that starts from B and does not end in B , is a back-edge that starts from $T(v)$, but not from $T(u)$, and ends in a proper ancestor of v ; thus, since $B(v) = B(u) \sqcup B(w)$, it ends in $T(w, r]$, and thus in D . Finally, every back-edge that starts from C must end in C , since $B(w) \subset B(v)$. Thus we see, that, by removing from the graph the tree-edges $(u, p(u))$, $(v, p(v))$ and $(w, p(w))$, the graph becomes separated into two parts: $A \cup C$ and $B \cup D$. \square

Corollary 3.13. *If $(u, p(u))$, $(v, p(v))$ are two tree-edges, there is at most one w such that $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ is a 3-cut.*

Proof. This is a consequence of propositions 3.9 and 3.12. \square

Here we distinguish two cases, depending on whether $M(v) = M(w)$ or $M(v) \neq M(w)$.

$M(v) \neq M(w)$

Lemma 3.14. *Let u be a descendant of v and v a descendant of w , and $M(v) \neq M(w)$. Then, $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ is a 3-cut if and only if: $M(w) = M_{low1}(v)$ and w is the greatest vertex with $M(w) = M_{low1}(v)$ which is lower than v , $M(u) = M_{low2}(v)$ and u is the lowest vertex with $M(u) = M_{low2}(v)$, $high(u) < v$ and $b_count(v) = b_count(u) + b_count(w)$. (See Figure 6.)*

Proof. (\Rightarrow) By proposition 3.12, we have $B(v) = B(u) \sqcup B(w)$. This immediately establishes both $high(u) < v$ and $b_count(v) = b_count(u) + b_count(w)$. Now, since $B(v) = B(u) \sqcup B(w)$, both $M(u)$ and $M(w)$ are descendants of $M(v)$. We will show that $M(u)$ and $M(w)$ are not related as ancestor and descendant. First, suppose that $M(u)$ is an ancestor of $M(w)$. Now let $(x, y) \in B(w)$.

Then x is a descendant of $M(w)$, and therefore a descendant of $M(u)$. Furthermore, y is a proper ancestor of w , and therefore a proper ancestor of u . This shows that $(x, y) \in B(u)$, contradicting the fact that $B(u) \cap B(w) = \emptyset$. Now suppose that $M(w)$ is an ancestor of $M(u)$. Let $(x, y) \in B(v)$. Since $B(v) = B(u) \sqcup B(w)$, x is a descendant of either $M(u)$ or $M(w)$. In either case, x is a descendant of w . Due to the generality of (x, y) , this shows that $M(v)$ is a descendant of $M(w)$. Since $M(w)$ is also a descendant of $M(v)$, we get $M(w) = M(v)$, contradicting $M(w) \neq M(v)$. Thus we have established that $M(u)$ and $M(w)$ are not related as ancestor and descendant. Since $M(u)$ and $M(v)$ are descendants of $M(v)$, they must be proper descendants of $M(v)$. Now we will show that $M(u)$ and $M(w)$ are descendants of different children of $M(v)$. Suppose, for the sake of contradiction, that $M(u)$ and $M(w)$ are descendants of the same child c of $M(v)$. Then, there must exist a back-edge $(x, y) \in B(v)$ such that $x = M(v)$ or x is a descendant of a child of $M(v)$ different from c . (Otherwise, we would have that $M(v)$ is a descendant of c , which is absurd.) But this means that (x, y) is neither in $B(u)$ nor in $B(w)$, contradicting the fact that $B(v) = B(u) \sqcup B(w)$. Thus, one of $M(u)$ and $M(w)$ is a descendant of the *low1* child of $M(v)$, and the other is a descendant of the *low2* child of $M(v)$. Observe that there does not exist a back-edge $(x, y) \in B(u)$ such that $y = \text{low}(v)$, for this would imply that $(x, y) \in B(w)$ (since u is a descendant of w), and $B(u)$ does not meet $B(w)$. Thus, since $B(v) = B(u) \sqcup B(w)$, v gets its *low* point from $B(w)$. This shows that $M(w)$ is a descendant of the *low1* child of $M(v)$ and $M(u)$ is a descendant of the *low2* child of $M(v)$. Since $B(w) \subset B(v)$, we have that $M(w)$ is a descendant of $M_{\text{low1}}(v)$. Furthermore, since $B(v) = B(u) \sqcup B(w)$ and $M(u)$ is not a descendant of the *low1* child of $M(v)$, there are no back-edges (x, y) with x a descendant of the *low1* child of $M(v)$ and y a proper ancestor of v apart from those contained in $B(w)$. Thus, $M(w)$ is an ancestor of $M_{\text{low1}}(v)$, and $M_{\text{low1}}(v) = M(w)$ is established. With the same reasoning, we also get $M_{\text{low2}}(v) = M(u)$.

Now suppose, for the sake of contradiction, that there exists a vertex w' with $M(w') = M(w)$ and $v > w' > w$. This implies that $B(w) \subset B(w')$, and thus there is a back-edge $(x, y) \in B(w') \setminus B(w)$. Then x is a descendant of $M(w')$, and therefore a descendant of $M_{\text{low1}}(v)$. Furthermore, y is a proper ancestor of w' , and therefore a proper ancestor of v . This shows that $(x, y) \in B(v)$, and therefore, since $B(v) = B(u) \sqcup B(w)$ and $(x, y) \notin B(w)$, we have $(x, y) \in B(u)$. But x is not a descendant of $M(u)$, since it is a descendant of $M(w)$ which is not related as ancestor or descendant with $M(u)$. That's a contradiction. Thus we have established that w is the greatest vertex with $M(w) = M_{\text{low1}}(v)$ which is lower than v . Finally, suppose for the sake of contradiction that there exists a vertex u' with $M(u') = M(u)$ and $u' < u$. This implies that $B(u') \subset B(u)$, and therefore there exists a back-edge $(x, y) \in B(u) \setminus B(u')$. Then, y is a proper ancestor of u and a descendant of u' . Since $\text{high}(u) < v$, we have $y < v$, and therefore u' is an ancestor of v . Now suppose that u' is an ancestor of w . Let $(x', y') \in B(u')$. Then x' is a descendant of $M(u')$, and therefore a descendant of $M(u)$, and therefore a descendant of u , and therefore a descendant of w . Furthermore, y' is a proper ancestor of u' , and therefore a proper ancestor of w . This shows that $(x', y') \in B(w)$. But this cannot be the case, since $(x', y') \in B(u') \subset B(u)$ and $B(u) \cap B(w) = \emptyset$. Thus, u' is a descendant of w . Since u' is an ancestor of v , it is also an ancestor of $M_{\text{low1}}(v) = M(w)$. Thus, Lemma 2.4 implies that $M(u')$ is an ancestor of $M(w)$. But, since $M(u') = M(u)$, this contradicts the fact that $M(u)$ and $M(w)$ are not related as ancestor and descendant. Thus we have established that u is the lowest vertex with $M(u) = M_{\text{low2}}(v)$.

(\Leftarrow) By proposition 3.12, it is sufficient to prove that $B(v) = B(u) \sqcup B(w)$. First, let $(x, y) \in B(u)$. Then x is a descendant of u , and therefore a descendant of v . Furthermore, $y \leq \text{high}(u) < v$ implies that y is a proper ancestor of v . This shows that $B(u) \subseteq B(v)$. Now let $(x, y) \in B(w)$. Then y is a proper ancestor of w , and therefore a proper ancestor of v . Since $M(w) = M_{\text{low1}}(v)$, we have that x is a descendant of v . This shows that $B(w) \subseteq B(v)$. Thus we have $B(u) \cup B(w) \subseteq B(v)$. Since $M(u)$ and $M(w)$ are not related as ancestor and descendant (for they are descendants of different children

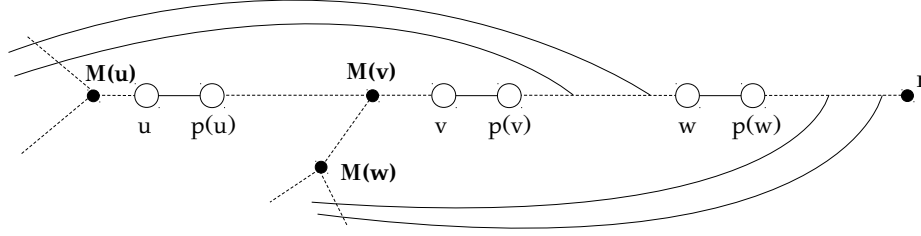


Figure 6: In this example we have $B(v) = B(u) \sqcup B(w)$. Observe that $M_{low1}(v) = M(w)$ and $M_{low2}(v) = M(u)$. u is the last vertex in $M^{-1}(M(u))$, and w is the greatest vertex in $M^{-1}(M(w))$ which is lower than v .

of $M(v)$), we have that $B(u) \cap B(w) = \emptyset$. In conjunction with $b_count(v) = b_count(u) + b_count(w)$, from $B(u) \cup B(w) \subseteq B(v)$ and $B(u) \cap B(w) = \emptyset$ we conclude that $B(u) \sqcup B(w) = B(v)$. \square

This lemma shows that, for every vertex v , there is at most one pair of vertices u, w , where u is a descendant of v , w is an ancestor of v , $M(v) \neq M(w)$, and $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ is a 3-cut. In particular, we have that w is the greatest vertex with $M(w) = M_{low1}(v)$ which is lower than v , u is the last vertex in $M^{-1}(M_{low2}(v))$, $high(u) < v$ and $b_count(v) = b_count(u) + b_count(w)$. Thus, Algorithm 8 shows how we can compute all 3-cuts of this type. We only have to make sure that we can compute w without having to traverse the list $M^{-1}(M_{low1}(v))$ from the beginning, each time we process a vertex v . To achieve this, we process the vertices in a bottom-up fashion, and we keep in an array $currentM[x]$ the current element of $M^{-1}(x)$ under consideration, so that we do not need to traverse the list $M^{-1}(x)$ from the beginning each time we process a vertex.

$M(v) = M(w)$ Let w be a proper ancestor of v such that $M(v) = M(w)$. By corollary 3.13, there is at most one descendant u of v such that $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ is a 3-cut. In order to find this u (if it exists), we distinguish two cases, depending on whether $w = nextM(v)$ or $w \neq nextM(v)$. In any case, we will need the following lemma, which gives a necessary condition for the existence of u .

Lemma 3.15. *Let u, v, w be three vertices such that u is a descendant of v , v is a descendant of w , and $M(v) = M(w)$. Then, $B(v) = B(u) \sqcup B(w)$ only if $high(u) = high(v)$ and $nextM(u) = \emptyset$.*

Proof. Let $(x, y) \in B(u)$ be such that $y = high(u)$. Then, since $B(v) = B(u) \sqcup B(w)$, we have $(x, y) \in B(v)$, and so $y \leq high(v)$. Suppose for the sake of contradiction that $y \neq high(v)$. Then, since $B(v) = B(u) \sqcup B(w)$, there exists a $(x', y') \in B(w)$ such that $y' = high(v)$. Furthermore, since $y \neq high(v)$ and $(x, y) \in B(v)$, we have $y' > y$, which means that y is a proper ancestor of w . But then, since x is a descendant of u , it is also a descendant of w , and thus $(x, y) \in B(w)$, contradicting the fact that $B(u) \cap B(w) = \emptyset$. Thus we have shown that $high(u) = high(v)$.

Now suppose, for the sake of contradiction, that there exists a u' which is a proper ancestor of u with $M(u') = M(u)$. Then we have $B(u') \subset B(u)$. Now suppose, for the sake of contradiction, that u' is an ancestor of v . Suppose that u' is an ancestor of w . Let $(x, y) \in B(u')$. Then x

Algorithm 8: Find all 3-cuts $\{(u, p(u)), (v, p(v)), (w, p(w))\}$, where u is a descendant of v , v is a descendant of w , and $M(v) \neq M(w)$.

```

1 foreach vertex  $v$  do  $currentVertex[v] \leftarrow v$ 
2 for  $v \leftarrow n$  to  $v = 1$  do
3    $m_1 \leftarrow M_{low1}(v), m_2 \leftarrow M_{low2}(v)$ 
4   if  $m_1 = \emptyset$  or  $m_2 = \emptyset$  then continue
5   // find the greatest  $w$  in  $M^{-1}(m_1)$  which is lower than  $v$ 
6    $w \leftarrow currentVertex(m_1)$ 
7   while  $w \neq \emptyset$  and  $w \geq v$  do  $w \leftarrow nextM(w)$ 
8    $currentVertex[m_1] \leftarrow w$ 
9   //  $u$  is the last element of  $M^{-1}(m_2)$ 
10   $u \leftarrow lastM(m_2)$ 
11  // check the condition in Lemma 3.14
12  if  $w \neq \emptyset$  and  $high(u) < v$  and  $b\_count(v) = b\_count(u) + b\_count(w)$  then
13  |   mark the triplet  $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ 
14  |   end
15 end

```

is a descendant of $M(u')$, and therefore a descendant of $M(u)$, and therefore a descendant of u , and therefore a descendant of w . Furthermore, y is a proper ancestor of u' , and therefore a proper ancestor of w . This means that $(x, y) \in B(w)$, and thus we have $B(u') \subseteq B(w)$. But this contradicts $B(u) \cap B(w) = \emptyset$, since $B(u') \subset B(u)$. Thus, we have that u' is a descendant of w . Then, since u' is an ancestor of v , it is also an ancestor of $M(v) = M(w)$, and thus, by Lemma 2.4, $M(u') = M(u)$ is an ancestor of $M(v)$. Since $B(v) = B(u) \sqcup B(w)$, we have that $M(v)$ is an ancestor of $M(u)$, and thus $M(u) = M(v)$. In conjunction with $high(u) = high(v)$, this implies that $B(v) = B(u)$, contradicting the fact that the graph is 3-edge-connected. Thus, we have that u' is not an ancestor of v . Since v and u' have u as a common descendant, we infer that u' is a descendant of v . Now, since $B(u') \subset B(u)$, we have that there exists a back-edge $(x, y) \in B(u) \setminus B(u')$. Then, y is descendant of u' , and therefore a descendant of v . But this means that $(x, y) \notin B(v)$, contradicting the fact that $B(u) \subset B(v)$. We conclude that there is not $u' \in M^{-1}(M(u))$ which is a proper ancestor of u . \square

Case $w = nextM(v)$. Now we will show how to find, for every vertex v , the unique u (if it exists) which is a descendant of v and such that $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ is a 3-cut, where $w = nextM(v)$. Obviously, the number of 3-cuts of this type is $O(n)$. According to Lemma 3.15, $high(u) = high(v)$, and therefore it is sufficient to seek this u in $high^{-1}(high(v))$.

Proposition 3.16. *Let $h = high(v)$ and $w = nextM(v)$, and suppose that the list $high^{-1}(h)$ is sorted in decreasing order. Then, u is a descendant of v such that $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ is a 3-cut if and only if u is a predecessor of v in $high^{-1}(h)$, $nextM(u) = \emptyset$, $low(u) \geq w$, $b_count(u) = b_count(v) - b_count(w)$, and all elements of $high^{-1}(h)$ between u and v are ancestors of u .*

Proof. (\Rightarrow) By proposition 3.12, we have $B(v) = B(u) \sqcup B(w)$. This shows that $b_count(u) = b_count(v) - b_count(w)$ and $low(u) \geq w$ (for if we had $low(u) < w$, then $B(u)$ would intersect $B(w)$). Lemma 3.15 shows that $high(u) = high(v)$ and $nextM(u) = \emptyset$. Since u is a descendant of v , it is greater than v , and thus it is a predecessor of v in $high^{-1}(x)$. Now suppose that there exists a $u' \in high^{-1}(x)$ which is lower than u and greater than v , but it is not an ancestor of u .

Since u is a descendant of v , $v < u' < u$ implies that u' is also a descendant of v . Let (x, h) be a back-edge with x a descendant of u' . Then x is also a descendant of v , and thus $(x, h) \in B(v)$. But since u' is not a descendant of u , x cannot be a descendant of u either, and so $(x, h) \in B(v)$ and $B(v) = B(u) \sqcup B(w)$ both imply that $(x, h) \in B(w)$. However, $h = \text{high}(u) \geq \text{low}(u) \geq w$. A contradiction.

(\Leftarrow) By proposition 3.12, it is sufficient to show that $B(v) = B(u) \sqcup B(w)$. Let $(x, y) \in B(u)$. Then x is a descendant of u , and therefore a descendant of v . Furthermore, since $\text{high}(u) = \text{high}(v)$, we have that y is a proper ancestor of v . This shows that $(x, y) \in B(v)$, and thus we have $B(u) \subseteq B(v)$. Now, since $M(v) = M(w)$ and $w = \text{nextM}(v) < v$, we have that $B(w) \subset B(v)$. Thus we have established that $B(u) \cup B(w) \subseteq B(v)$. Now observe that $B(u) \cap B(w) = \emptyset$: for if $(x, y) \in B(u)$, then $y \geq \text{low}(u)$, and we have assumed that $\text{low}(u) \geq w$; thus, $(x, y) \notin B(w)$. Now, since $b_count(u) = b_count(v) - b_count(w)$ and $B(u) \cup B(w) \subseteq B(v)$ and $B(u) \cap B(w) = \emptyset$, we conclude that $B(v) = B(u) \sqcup B(w)$. \square

Now let h be a vertex. Based on proposition 3.16, we will show how to find, for every v in the decreasingly sorted list $\text{high}^{-1}(h)$, the unique vertex $u \in \text{high}^{-1}(h)$ (if it exists) such that $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ is a 3-cut, where $w = \text{nextM}(v)$. To do this, we need an array A of size m (the number of edges of the graph), and a stack S . We begin by traversing the list $\text{high}^{-1}(h)$ from its first element, and every u we meet that satisfies $\text{nextM}(u) = \emptyset$ and is an ancestor of its predecessor (or the first element of the list) we push it in S and also store it in $A[b_count(u)]$. If u is not an ancestor of its predecessor, we set $A[z] = \emptyset$, for every $z \in S$, while we pop out all elements from S ; then we push u in S and also store it in $A[b_count(u)]$. Now, if we meet a vertex v that satisfies $\text{nextM}(v) \neq \emptyset$ and is ancestor of its predecessor, we check whether the entry $u = A[b_count(v) - b_count(\text{nextM}(v))]$ is not \emptyset , and if $\text{low}(u) \geq \text{nextM}(v)$ we mark the triplet $\{(u, p(u)), (v, p(v)), (\text{nextM}(v), p(\text{nextM}(v)))\}$ (observe that u satisfies all conditions of proposition 3.16). If v is not an ancestor of the top element of S , we set $A[u] = \emptyset$, for every $u \in S$, while we pop out all elements from S . In any case, we keep traversing the list, following the same procedure, until we reach its end. This process is implemented in Algorithm 9.

Case $w \neq \text{nextM}(v)$. Now we will show how to find, for every vertex v , the set of all u which are descendants of v with the property that there exists a w with $M(w) = M(v)$ and $w < \text{nextM}(v)$, such that $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ is a 3-cut. Let $U(v)$ denote this set. (An illustration is given in Figure 7.) According to Lemma 3.15, for every $u \in U(v)$ we have $\text{high}(u) = \text{high}(v)$, and therefore it is sufficient to seek those u in $\text{high}^{-1}(\text{high}(v))$.

To do this, we use a stack $\text{stackU}[v]$, for every vertex v , in which we store vertices u from $\text{high}^{-1}(\text{high}(v))$. By the time we have filled all stacks $\text{stackU}[v]$, the following three properties will be satisfied: (1) for every vertex v , $U(v) \subseteq \text{stackU}[v]$, (2) if $v \neq v'$, then $\text{stackU}[v] \cap \text{stackU}[v'] = \emptyset$, and (3) every u in $\text{stackU}[v]$ is a descendant of its successors in $\text{stackU}[v]$. The contents of $\text{stackU}[v]$ will be all those u satisfying the necessary condition described in the following lemma.

Lemma 3.17. *Let $h = \text{high}(v)$, and assume that the list $\text{high}^{-1}(h)$ is sorted in decreasing order. Then, $u \in U(v)$ only if u is a predecessor of v in $\text{high}^{-1}(h)$ such that $\text{nextM}(u) = \emptyset$, $\text{low}(u) < \text{nextM}(v)$, $\text{low}(u) \geq \text{lastM}(v)$, and all elements of $\text{high}^{-1}(h)$ between u and v are ancestors of u .*

Proof. $u \in U(v)$ means that u is a descendant of v and there is an ancestor w of v such that $M(v) = M(w)$, $w \neq \text{nextM}(v)$, and $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ is a 3-cut. By proposition 3.12, we have $B(v) = B(u) \sqcup B(w)$. From this we infer that $\text{low}(u) \geq w$ (for otherwise, since u is a descendant of w , we would have that $B(u)$ meets $B(w)$). This shows that $\text{low}(u) \geq \text{lastM}(v)$. Lemma 3.15 implies that $\text{high}(u) = \text{high}(v)$ and $\text{nextM}(u) = \emptyset$. Furthermore, since u is a descendant

Algorithm 9: Find all 3-cuts $\{(u, p(u)), (v, p(v)), (w, p(w))\}$, where u is a descendant of v and $w = nextM(v)$.

```

1 initialize an array  $A$  with  $m$  entries (where  $m$  is the number of edges of the graph)
2 initialize a stack  $S$ 
3 sort the elements of every list  $high^{-1}(h)$ , for every vertex  $h$ , in decreasing order
4 foreach vertex  $h$  do
5    $u \leftarrow$  first element of  $high^{-1}(h)$ 
6   while  $u \neq \emptyset$  do
7      $z \leftarrow$  next element of  $high^{-1}(h)$ 
8     if  $z = \emptyset$  then break
9     if  $z$  is not an ancestor of  $u$  then
10      while  $S$  is not empty do
11         $u' \leftarrow S.pop()$ 
12         $A[b\_count(u')] \leftarrow \emptyset$ 
13      end
14    end
15    if  $nextM(z) = \emptyset$  then
16       $S.push(z)$ 
17       $A[b\_count(z)] \leftarrow z$ 
18    end
19    else if  $nextM(z) \neq \emptyset$  then
20       $v \leftarrow z, w \leftarrow nextM(v)$ 
21      if  $A[b\_count(v) - b\_count(w)] \neq \emptyset$  then
22         $u \leftarrow A[b\_count(v) - b\_count(w)]$ 
23        if  $low(u) \geq w$  then
24          mark the triplet  $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ 
25        end
26      end
27    end
28     $u \leftarrow z$ 
29  end
30 end

```

of v , it is greater than v , and thus it is a predecessor of v in $high^{-1}(h)$. Now suppose, for the sake of contradiction, that $low(u) \geq nextM(v)$. Since there is a $w < nextM(v)$ such that $M(w) = M(v)$, there must exist a back-edge $(x, y) \in B(v)$ with $y \in T(nextM(v), w]$. Since $low(u) \geq nextM(v)$, it cannot be the case that $(x, y) \in B(u)$, and therefore $B(v) = B(u) \sqcup B(w)$ implies that $(x, y) \in B(w)$, which is absurd, since $y \geq w$. Thus, $low(u) < nextM(v)$. Finally, suppose, for the sake of contradiction, that there exists a $u' \in high^{-1}(h)$ which is lower than u and greater than v , but it is not an ancestor of u . Since u is a descendant of v , $v < u' < u$ implies that u' is also a descendant of v . Let (x, h) be a back-edge with x a descendant of u' . Then x is also a descendant of v , and thus $(x, h) \in B(v)$. But since u' and u are not related as ancestor or descendant, x cannot be a descendant of u . Thus, $(x, h) \notin B(u)$. Since $(x, h) \in B(v)$ and $B(v) = B(u) \sqcup B(w)$, this implies that $(x, h) \in B(w)$. However, $h = high(u) \geq low(u) \geq w$. A contradiction. \square

Thus, $stackU[v]$ contains all u that are predecessors of v in $high^{-1}(high(v))$ and satisfy $nextM(u) =$

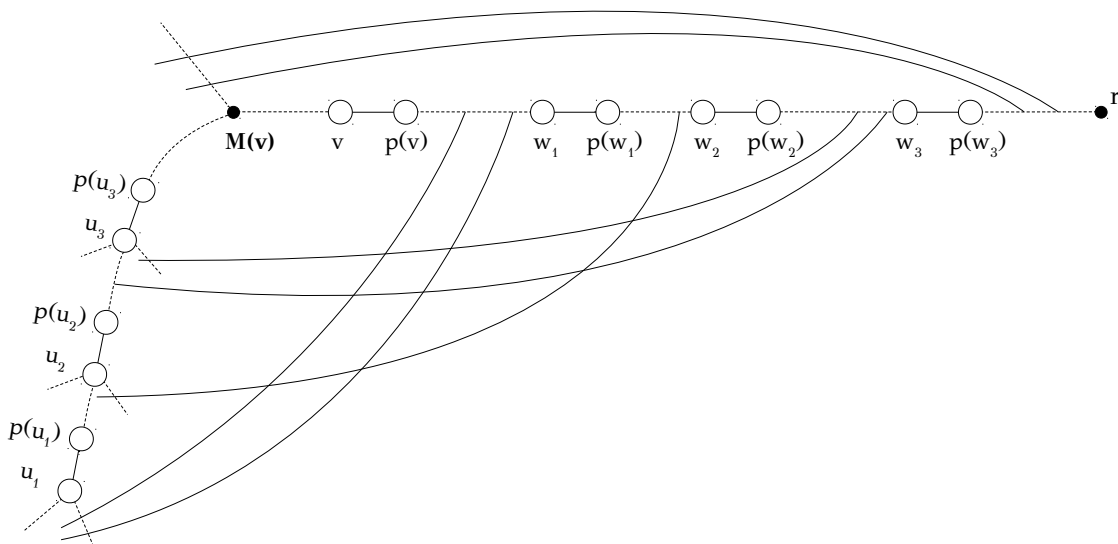


Figure 7: In this example we have $M(v) = M(w_1) = M(w_2) = M(w_3)$, $U(v) = \{u_1, u_2, u_3\}$, and the triplets $\{(u_i, p(u_i)), (v, p(v)), (w_i, p(w_i))\}$, for $i \in \{1, 2, 3\}$, are 3-cuts. Observe that all $\{u_1, u_2, u_3\}$ are related as ancestor and descendant. This property is proved in Lemma 3.17. Furthermore, all $u \in U(v)$ have $high(u) = high(v)$.

\emptyset , $low(u) < nextM(v)$, $low(u) \geq lastM(v)$ and all elements of $high^{-1}(high(v))$ between u and v are ancestors of u . By Lemma 3.17, property (1) of the stacks $stackU[v]$ is satisfied. The following lemma shows that property (2) is also satisfied.

Lemma 3.18. *Let v, v' be two vertices such that v' is a proper ancestor of v with $high(v') = high(v)$, and let $u \in stackU[v]$. Then $u \notin stackU[v']$.*

Proof. First observe that the stacks $stackU[v]$ and $stackU[v']$ are non-empty only if $nextM(v) \neq \emptyset$ and $nextM(v') \neq \emptyset$. Now, since $high(v') = high(v)$, by Lemma 3.20, we have that $nextM(v) < lastM(v')$. Since $u \in stackU[v]$, it has $low(u) < nextM(v)$. But then $low(u) < lastM(v')$, and so $u \notin stackU[v']$. \square

This implies that the total number of elements in all stacks $stackU[v]$ (by the time we have filled them) is $O(n)$. Now let h be a vertex, and let us show how to fill the stacks $stackU[v]$, for all v in the decreasingly sorted list $high^{-1}(h)$. To do this, we will need a stack S . We begin traversing the list $high^{-1}(h)$ from its first element, and when we process a vertex u such that $nextM(u) = \emptyset$ we push it in S if it is an ancestor of its predecessor (or the first elements of the list). Otherwise, we drop all elements from S , push u in S , and keep traversing the list. When we meet a vertex v that satisfies $nextM(v) \neq \emptyset$ and is also an ancestor of its predecessor, we check whether the top element u of S satisfies $low(u) < lastM(v)$, in which case we start popping elements out of S , until the top element u of S (if S is not left empty) satisfies $low(u) \geq lastM(v)$. Then, as long as the top element u of S satisfies $low(u) < nextM(v)$, we repeatedly pop out the top element from S and push it in $stackU[v]$. If v is not an ancestor of its predecessor, we drop all elements from S . In any case, we keep traversing the list, following the same procedure, until we reach its end. This process is implemented in Algorithm 10. Property (3) of the stacks $stackU$ is satisfied due to the way we fill them with this algorithm. To prove the correctness of Algorithm 10 - i.e., that by the time we reach the end of $high^{-1}(h)$, every stack $stackU[v]$, for every $v \in high^{-1}(h)$, contains all elements u satisfying the necessary condition in Lemma 3.17 -, we need the following two lemmata.

Lemma 3.19. *If u' is an ancestor of u with $high(u) = high(u')$, then $low(u') \leq low(u)$.*

Proof. Let $(x, y) \in B(u)$. Then x is a descendant of u , and therefore a descendant of u' . Furthermore, $y \leq high(u) = high(u')$, and therefore y is a proper ancestor of u' . This shows that $(x, y) \in B(u')$, and thus we have $B(u) \subseteq B(u')$. $low(u') \leq low(u)$ is an immediate consequence of this fact. \square

Lemma 3.20. *Let v, v' be two vertices such that v' is a proper ancestor of v , $nextM(v) \neq \emptyset$, $nextM(v') \neq \emptyset$, and $high(v') = high(v)$. Then, $nextM(v) < lastM(v')$.*

Proof. Let $(x, y) \in B(v)$. Then x is a descendant of v , and therefore a descendant of v' . Furthermore, since $y \leq high(v)$ and $high(v) = high(v')$ and $high(v') < v'$, we have that y is a proper ancestor of v' . This shows that $(x, y) \in B(v')$, and thus $B(v) \subseteq B(v')$. From this we infer that $M(v)$ is a descendant of $M(v')$. Now, since $M(nextM(v)) = M(v)$ and $nextM(v) < v$, we have that $B(nextM(v)) \subset B(v)$. This means that there exists a back-edge (x, y) such that x is a descendant of $M(v)$ and y is a proper ancestor of v but not a proper ancestor of $nextM(v)$. Then, since $(x, y) \in B(v)$, we have $y \leq high(v)$, and so $high(v)$ is not a proper ancestor of $nextM(v)$, and thus $nextM(v)$ is an ancestor of $high(v)$. Since $high(v) = high(v')$ and $high(v')$ is a proper ancestor of v' , we infer that $nextM(v)$ is a proper ancestor of v' . Now suppose, for the sake of contradiction, that $lastM(v')$ is an ancestor of $nextM(v)$. Let $(x, y) \in B(lastM(v'))$. Then, x is a descendant of $M(lastM(v'))$, and thus a descendant of $M(v')$, and thus a descendant of v' , and thus a descendant

of $nextM(v)$. Furthermore, y is a proper ancestor of $lastM(v')$, and therefore a proper ancestor of $nextM(v)$. This shows that $(x, y) \in B(nextM(v))$, and thus we have $B(lastM(v')) \subseteq B(nextM(v))$. From this we infer that $M(lastM(v'))$ is a descendant of $M(nextM(v))$. But $M(lastM(v')) = M(v')$ and $M(nextM(v)) = M(v)$. Thus, $M(v')$ is a descendant of $M(v)$. Since $M(v)$ is a descendant of $M(v')$, we conclude that $M(v') = M(v)$. But this implies, in conjunction with $high(v') = high(v)$, that $B(v) = B(v')$, contradicting the fact that the graph is 3-edge-connected. This shows that $nextM(v)$ is a proper ancestor of $lastM(v')$. \square

Now, to prove the correctness of Algorithm 10, we have to show that the elements we push into $stackU[v]$ satisfy the necessary condition in Lemma 3.17, and the elements we pop out from S do not satisfy this condition either for v or for any successor of v in the list $high^{-1}(h)$. So, let v be a vertex in $high^{-1}(h)$ such that $nextM(v) \neq \emptyset$, and let v' be a successor of v in $high^{-1}(h)$ such that $nextM(v') \neq \emptyset$. Now, when we meet v as we traverse $high^{-1}(x)$, we pop out the top elements u from S that have $low(u) < lastM(v)$. By the definition of $stackU[v]$, these are not included in $stackU[v]$. Now, by Lemma 3.20, we have $nextM(v) < lastM(v')$. Since $low(u) < lastM(v) \leq nextM(v)$, we have $low(u) < lastM(v')$, and thus u is not in $stackU[v']$ either, so it does not matter that we pop those u out of S . Then, once we reach a \tilde{u} in S that satisfies $low(\tilde{u}) \geq lastM(v)$, we pop out the top elements u of S that have $low(u) < nextM(v)$, and push them into $stackU[v]$. This is according to the definition of $stackU[v]$. Since $nextM(v) < lastM(v')$ and $low(u) < nextM(v)$, we have $low(u) < lastM(v')$, and so, again, these u are not included in $stackU[v']$, and thus it does not matter that we pop them out of S . Now, when we reach a u in S that has $low(u) \geq nextM(v)$, we can be certain, by Lemma 3.19, that no u' in S has $low(u') < nextM(v)$, since all elements of S are descendants of u (by the way we fill the stack S), and thus they have $low(u') \geq low(u) \geq nextM(v)$. Then it is proper to move on to the next element of $high^{-1}(h)$.

Lemma 3.21. *Let v be a vertex and u, u' two elements in $stackU[v]$, where u is a predecessor of u' in $stackU[v]$. Then, $low(u') \leq low(u)$.*

Proof. Since $u, u' \in stackU[v]$, we have $high(u) = high(v) = high(u')$. Since u is a predecessor of u' in $stackU[v]$, by property (3) of $stackU[v]$ we have that u is a descendant of u' . Thus, by Lemma 3.19, we get $low(u') \leq low(u)$. \square

The next lemma is the basis to find all 3-cuts of the form $\{(u, p(u)), (v, p(v)), (w, p(w))\}$, where u is a descendant of v , $M(v) = M(w)$, and $w \neq nextM(v)$.

Lemma 3.22. *Let u be a vertex in $stackU[v]$ and w a proper ancestor of v such that $M(w) = M(v)$. Then, if $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ is a 3-cut, we have that $b_count(v) = b_count(u) + b_count(w)$ and w is the greatest element of $M^{-1}(M(v))$ such that $w \leq low(u)$. Conversely, if $b_count(v) = b_count(u) + b_count(w)$ and $w \leq low(u)$, then $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ is a 3-cut.*

Proof. (\Rightarrow) By proposition 3.12, we have $B(v) = B(u) \sqcup B(w)$. This explains both $b_count(v) = b_count(u) + b_count(w)$ and $w \leq low(u)$. (For if we had $low(u) < w$, then, since u is a descendant of w , $B(u)$ would meet $B(w)$.) Now suppose, for the sake of contradiction, that there is a vertex w' such that $M(w') = M(v)$ and $w < w' \leq low(u)$. Since $B(v) = B(u) \sqcup B(w)$, we have that $low(u) < v$, and therefore $w' < v$. Since $M(w') = M(v)$, this means that $B(w') \subset B(v)$. Furthermore, since $M(w) = M(w')$ and $w < w'$, we infer that $B(w) \subset B(w')$, and therefore there exists a back-edge $(x, y) \in B(w') \setminus B(w)$. Then, by $B(w') \subset B(v)$, we have that $(x, y) \in B(v)$, and $B(v) = B(u) \sqcup B(w)$ implies that $(x, y) \in B(u)$ or $(x, y) \in B(w)$. Since $(x, y) \notin B(w)$, $(x, y) \in B(u)$ is the only option left. But y is a proper ancestor of w' , and therefore a proper ancestor of $low(u)$ (since $w' \leq low(u)$).

Algorithm 10: Fill all stacks $stackU[v]$, for all vertices v

```

1 initialize a stack  $S$ 
2 foreach vertex  $v$  do initialize a stack  $stackU[v]$ 
3 foreach vertex  $h$  do
4    $u \leftarrow$  first element of  $high^{-1}(h)$ 
5   while  $u \neq \emptyset$  do
6      $z \leftarrow$  next element of  $high^{-1}(h)$ 
7     if  $z = \emptyset$  then break
8     if  $z$  is not an ancestor of  $u$  then
9       | pop out all elements from  $S$ 
10    end
11    if  $nextM(z) = \emptyset$  then
12      |  $S.push(z)$ 
13    end
14    else if  $nextM(z) \neq \emptyset$  then
15      | while  $low(S.top()) < lastM(v)$  do  $S.pop()$ 
16      | while  $low(S.top()) < nextM(v)$  do
17        |  $u \leftarrow S.pop()$ 
18        |  $stackU[v].push(u)$ 
19      | end
20    end
21     $u \leftarrow z$ 
22  end
23 end

```

This implies that $(x, y) \notin B(u)$, which is absurd. We conclude that w is the greatest element of $M^{-1}(M(v))$ such that $w \leq low(u)$.

(\Leftarrow) By proposition 3.12, it is sufficient to show that $B(v) = B(u) \sqcup B(w)$. $u \in stackU[v]$ implies that u is a descendant of v such that $high(u) = high(v)$. Now let $(x, y) \in B(u)$. Then x is a descendant of u , and therefore a descendant of v . Furthermore, $y \leq high(u) = high(v)$, and therefore y is a proper ancestor of v . This shows that $(x, y) \in B(v)$, and thus we have $B(u) \subseteq B(v)$. Since $M(w) = M(v)$ and $w < v$, we have $B(w) \subset B(v)$. Thus we have established that $B(u) \cup B(w) \subseteq B(v)$. Notice that no $(x, y) \in B(u)$ is contained in $B(w)$, since $y \geq low(u) \geq w$, and thus y is not a proper ancestor of w . Thus we have $B(u) \cap B(w) = \emptyset$. Now $B(v) = B(u) \sqcup B(w)$ follows from $B(u) \cup B(w) \subseteq B(v)$, $B(u) \cap B(w) = \emptyset$ and $b_count(v) = b_count(u) + b_count(w)$. \square

Now our goal is to find, for every $u \in stackU[v]$, for every vertex v , the vertex w (if it exists) which has $M(w) = M(v)$ and $w < nextM(v)$, and is such that $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ is a 3-cut. By Lemma 3.22, w has the property that it is the greatest vertex in $M^{-1}(M(v))$ which has $w \leq low(u)$. Let us describe a simple method to find the w with this property, which will give us the intuition to provide a linear-time algorithm for our problem. So let v be a vertex, $m = M(v)$, and u be a vertex in $stackU[v]$. A simple idea is to start from v and keep traversing the list $M^{-1}(m)$, through the pointers $nextM$, until we reach a $w \in M^{-1}(m)$ such that $w \leq low(u)$. The problem here is that we may have to pass from the same elements of $M^{-1}(m)$ an excessive amount of times (depending on the number of elements in $stackU[v]$). We can remedy this by keeping in a variable $lowestW$ the w that we reached the last time we processed a $u \in stackU[v]$. Then, when we process

the successor of u in $stackU[v]$, we begin the search in $M^{-1}(m)$ from $lowestW$. This will work, since the every $u \in stackU[v]$ is a descendant of its successor u' in $stackU[v]$ (due to the way we have filled the stacks $stackU$ with Algorithm 10), and we have $high(u) = high(u')$, and therefore, by Lemma 3.19, $low(u') \leq low(u)$. However, this is, again, not a linear-time procedure, since, for every vertex v , when we start processing the first vertex in $stackU[v]$, we begin traversing the list $M^{-1}(M(v))$ from v , and therefore, every time we process a vertex v' with $M(v') = M(v)$, we may have to pass again from the same vertices that we passed from during the processing of v , exceeding the time bound in total. Now, to achieve linear time, we process the vertices from the lowest to the highest, and, for every v that we process, we keep in a variable $lowestW[v]$ the w that we reached the last time we processed a $u \in stackU[v]$. Then, when we have to process a $u \in stackU[v]$, we traverse the list $M^{-1}(M(v))$ through the pointers $lowestW$, starting from $lowestW[v]$. (Initially, we set every $lowestW[v]$ to $nextM(v)$.) Thus we perform a kind of path-compression method, which is shown Algorithm 11. The next three lemmata will be used in proving the correctness and linear complexity of Algorithm 11.

Lemma 3.23. *Let v be a vertex and $u \in stackU[v]$. When we reach line 8 during the processing of u , we have that w is a vertex in $M^{-1}(M(v))$ such that $w \leq low(u)$ and $w \leq \min\{low(u') \mid \exists v' \text{ with } M(v') = M(v), w < v' < v \text{ and } u' \in stackU[v']\}$.*

Proof. First observe that, during the processing of a vertex v , the variables w and $lowestW[v]$ are members of $M^{-1}(M(v))$, and w is an ancestor of v while $lowestW[v]$ is a proper ancestor of v . (It is easy to see this inductively. For if this holds for all vertices $v' < v$, then it is also true for v , since the **while** loop in line 7 assigns w to $lowestW[w]$, and w is assumed to be an ancestor of v with $M(w) = M(v)$, and thus $lowestW[w]$ is also an ancestor of v with $M(lowestW[w]) = M(v)$, due to the inductive hypothesis.) Then it is obvious that, when we reach line 8 during the processing of $u \in stackU[v]$, we have that $M(w) = M(v)$ and $w \leq low(u)$, since the **while** loop in line 7 terminates precisely when such a w is found. Now we will show that, when we process a $u \in stackU[v]$, every time w is assigned $lowestW[w]$ during the execution of the **while** loop in line 7, we have $w \leq low(u')$, for every $u' \in stackU[v']$, for every v' with $M(v') = M(v)$ and $w < v' < v$. It is easy to see this inductively. Suppose, then, that this was the case for every vertex that we processed before v , for every predecessor of u in $stackU[v]$ that we already processed, and for every step of the **while** loop in line 7 in the processing of u so far. Thus, now w has the property that $w \leq low(u')$, for every $u' \in stackU[v']$, for every v' with $M(v') = M(v)$ and $w < v' < v$. So let us perform $w \leftarrow lowestW[w]$ once more (which means that we still have $w > low(u)$), and let \tilde{w} be the current value of w , to distinguish it from the previous one which we will denote simply as w . Now, due to the inductive hypothesis, we have that $\tilde{w} \leq low(u')$ for every $u' \in stackU[v']$, for every v' with $M(v') = M(v)$ and $\tilde{w} < v' < w$. We also have (again, due to the inductive hypothesis) that $w \leq low(u')$ for every $u' \in stackU[v']$, for every v' with $M(v') = M(v)$ and $w < v' < v$. Since $\tilde{w} < w$, we thus have $\tilde{w} \leq low(u')$, for every $u' \in stackU[v']$, for every v' with $M(v') = M(v)$ and $\tilde{w} < v' < w$ or $w < v' < v$. Thus we only have to consider the case $v' = w$, and prove that every $u' \in stackU[w]$ satisfies $\tilde{w} \leq low(u')$. Observe that $lowestW[w]$ was updated for the last time in line 8 when we were processing the last element \tilde{u} of $stackU[w]$. Then, since $\tilde{w} = lowestW[w]$, due to the inductive hypothesis we have that $\tilde{w} \leq low(\tilde{u})$. Since every $u' \in stackU[w]$ has $high(u') = high(\tilde{u})$ and \tilde{u} is an ancestor of its predecessors in $stackU[w]$ (due to the way we have filled the stacks $stackU$ with Algorithm 10), by Lemma 3.19 we have that $low(\tilde{u}) \leq low(u')$, and therefore $\tilde{w} \leq low(u')$. Thus we have shown that $\tilde{w} \leq low(u')$, for every $u' \in stackU[v']$, for every v' with $M(v') = M(v)$ and $\tilde{w} < v' < v$. \square

Lemma 3.24. *Let v be a vertex and $u \in \text{stack}U[v]$. When we reach line 8 during the processing of u , we have that w is the greatest vertex in $M^{-1}(M(v))$ such that $w \leq \text{low}(u)$ and $w \leq \min\{\text{low}(u') \mid \exists v' \text{ with } M(v') = M(v), w < v' < v \text{ and } u' \in \text{stack}U[v']\}$.*

Proof. We will prove this lemma by induction. Let's assume, then, that, for every vertex $v' \leq v$, and every $u' \in \text{stack}U[v']$ that we processed so far, whenever we reached line 8 w was the greatest vertex with $M(w) = M(v')$ such that $w \leq \text{low}(u)$ and $w \leq \min\{\text{low}(u'') \mid \exists v'' \text{ with } M(v'') = M(v'), w < v'' < v' \text{ and } u'' \in \text{stack}U[v'']\}$. Now let u be the next element of $\text{stack}U[v]$ that we process. Let \tilde{w} be the greatest vertex with $M(\tilde{w}) = M(v)$ such that $\tilde{w} \leq \text{low}(u)$ and $\tilde{w} \leq \min\{\text{low}(u') \mid \exists v' \text{ with } M(v') = M(v), \tilde{w} < v' < v \text{ and } u' \in \text{stack}U[v']\}$. (The existence of such a \tilde{w} is guaranteed by Lemma 3.23.) Let w be the last vertex during the execution of the **while** loop in line 7 that had $w > \text{low}(u)$, and let $w' = \text{lowest}W[w]$. Then we have that $w' = \text{lowest}W[w] \leq \text{low}(u)$, and the **while** loop terminates here. We will show that $w' = \tilde{w}$. We distinguish two cases, depending on whether $w' = \text{next}M(w)$ or $w' \neq \text{next}M(w)$. In the first case, we have that $w > \text{low}(u)$, but $\text{next}M(w) \leq \text{low}(u)$. Thus, $w' = \text{next}M(w)$ is the greatest vertex with $M(w') = M(v)$ such that $w' \leq \text{low}(u)$, and so we have $w' = \tilde{w}$ (since w' satisfies also $w' \leq \min\{\text{low}(u') \mid \exists v' \text{ with } M(v') = M(v), w < v' < v \text{ and } u' \in \text{stack}U[v']\}$), by Lemma 3.23). Now, if $w' \neq \text{next}M(w)$, this means, due to the inductive hypothesis (and since $w' = \text{lowest}W[w]$), that w' is the greatest vertex with $M(w') = M(w)$ such that $w' \leq \text{low}(\tilde{u})$ and $w' \leq \min\{\text{low}(u') \mid \exists v' \text{ with } M(v') = M(w), w' < v' < w \text{ and } u' \in \text{stack}U[v']\}$, where \tilde{u} is the last element in $\text{stack}U[w]$. Now, since \tilde{w} satisfies $\tilde{w} \leq \min\{\text{low}(u') \mid \exists v' \text{ with } M(v') = M(v), \tilde{w} < v' < v \text{ and } u' \in \text{stack}U[v']\}$ and $\tilde{w} < w < v$, we have $\tilde{w} \leq \text{low}(\tilde{u})$ and $\tilde{w} \leq \min\{\text{low}(u') \mid \exists v' \text{ with } M(v') = M(w), \tilde{w} < v' < w \text{ and } u' \in \text{stack}U[v']\}$. Thus, \tilde{w} cannot be greater than w' , and so we have $w' \geq \tilde{w}$. Since $w' \leq \text{low}(u)$, and, as a consequence of Lemma 3.23, $w' \leq \min\{\text{low}(u') \mid \exists v' \text{ with } M(v') = M(v), w' < v' < v \text{ and } u' \in \text{stack}U[v']\}$, it must be the case that $w' = \tilde{w}$. \square

Lemma 3.25. *Let $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ be a 3-cut where u is a descendant of v , v is a descendant of w with $M(v) = M(w)$, and $w \neq \text{next}M(v)$. Then, w is the greatest vertex in $M^{-1}(M(v))$ such that $w \leq \text{low}(u)$ and $w \leq \min\{\text{low}(u') \mid \exists v' \text{ with } M(v') = M(v), w < v' < v \text{ and } u' \in \text{stack}U[v']\}$.*

Proof. Suppose, for the sake of contradiction, that there exists a vertex v' with $M(v') = M(v)$ and $w < v' < v$, such that there exists a $u' \in \text{stack}U[v']$ with $\text{low}(u') < w$. Since $u' \in \text{stack}U[v']$, we have that u' is a proper descendant of v' with $\text{high}(u') = \text{high}(v')$. Let $(x, y) \in B(u')$ (of course, $B(u')$ is not empty, since the graph is 3-edge-connected). Then x is a descendant of u' , and therefore a descendant of v' . Furthermore, $y \leq \text{high}(u') = \text{high}(v')$, and therefore y is a proper ancestor of v' . This shows that $(x, y) \in B(v')$. Thus we have $B(u') \subset B(v')$. Since $M(v') = M(v)$ and $v' < v$, we have $B(v') \subset B(v)$. Thus, $B(u') \subset B(v)$. Now we will prove that u' is not related as ancestor or descendant with u . First, since $\text{low}(u') < w \leq \text{low}(u)$, it cannot be the case that u' is a descendant of u (for a back-edge $(x, \text{low}(u')) \in B(u')$ would also be a back-edge in $B(u)$, and thus we would have $\text{low}(u) \leq \text{low}(u')$, which is a absurd). Suppose, then, that u' is an ancestor of u . Since v' is a proper ancestor of v with $M(v') = M(v)$, we must have $\text{high}(v') < \text{high}(v)$; and since $\text{high}(u') = \text{high}(v')$, we therefore have $\text{high}(u') < \text{high}(v)$. This means that u' (which is related as ancestor or descendant with v , since we supposed it is an ancestor of u) is a proper ancestor of v , and therefore a proper ancestor of $M(v)$. Since, then, u' is a descendant v' and $M(v') = M(v)$, by Lemma 2.4 we have that $M(u')$ is an ancestor of $M(v)$. But $B(u') \subset B(v)$ implies that $M(u')$ is a descendant of $M(v)$, and therefore $M(u') = M(v)$. Since $M(v) = M(v')$ and $\text{high}(v') = \text{high}(u')$, we get that $B(u') = B(v')$, which implies that $v' = u'$ - a contradiction. Thus we have shown that u' is not related as ancestor or descendant with u .

Now let (x, y) , with $y = \text{high}(u')$, be a back-edge in $B(u')$. Then we have $(x, y) \in B(v)$. By proposition 3.12, we have $B(v) = B(u) \sqcup B(w)$, and therefore $(x, y) \in B(u)$ or $(x, y) \in B(w)$. Since u' is not related as ancestor of descendant with u , it cannot be the case that x (which is a descendant of u') is a descendant of u , and therefore $(x, y) \in B(u)$ is rejected. Now, since $B(u') \subset B(v')$, we have $(x, y) \in B(v')$. Since $M(v') = M(w)$ and $w < v'$, we have that $B(w) \subset B(v')$, and thus there exists a back-edge $(x', y') \in B(v')$ such that $y' \in T(v', w]$. But since $y = \text{high}(u') = \text{high}(v')$, we must have $y' \leq y$. Thus, y is not a proper ancestor of w , and so $(x, y) \notin B(w)$, either. We have arrived at a contradiction, as a consequence of our initial supposition. This shows that there is no vertex v' with $M(v') = M(v)$ and $w < v' < v$, such that there exists a $u' \in \text{stack}U[v']$ with $\text{low}(u') < w$. Thus, $w \leq \min\{\text{low}(u') \mid \exists v' \text{ with } M(v') = M(v), w < v' < v \text{ and } u' \in \text{stack}U[v']\}$. Now, by Lemma 3.22, w is the greatest vertex in $M^{-1}(M(v))$ with $w \leq \text{low}(u)$. Thus, w must be the greatest vertex in $M^{-1}(M(v))$ that satisfies both $w \leq \text{low}(u)$ and $w \leq \min\{\text{low}(u') \mid \exists v' \text{ with } M(v') = M(v), w < v' < v \text{ and } u' \in \text{stack}U[v']\}$. \square

Algorithm 11: Find all 3-cuts $\{(u, p(u)), (v, p(v)), (w, p(w))\}$, where u is a descendant of v , v is a descendant of w with $M(v) = M(w)$, and $w \neq \text{next}M(v)$.

```

1 initialize an array lowestW with  $n$  entries
2 foreach vertex  $v$  do  $\text{lowest}W[v] \leftarrow \text{next}M(v)$ 
3 for  $v \leftarrow 1$  to  $v \leftarrow n$  do
4   while  $\text{stack}U[v].\text{top}() \neq \emptyset$  do
5      $u \leftarrow \text{stack}U[v].\text{pop}()$ 
6      $w \leftarrow \text{lowest}W[v]$ 
7     while  $w > \text{low}(u)$  do  $w \leftarrow \text{lowest}W[w]$ 
8      $\text{lowest}W[v] \leftarrow w$ 
9     if  $b\_count(v) = b\_count(u) + b\_count(w)$  then
10    | mark the triplet  $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ 
11    end
12  end
13 end

```

Proposition 3.26. *Algorithm 11 identifies all 3-cuts $\{(u, p(u)), (v, p(v)), (w, p(w))\}$, where u is a descendant of v , v is a descendant of w with $M(v) = M(w)$, and $w \neq \text{next}M(v)$. Furthermore, it runs in linear time.*

Proof. Let $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ be a 3-cut, where u is a descendant of v , v is a descendant of w with $M(v) = M(w)$, and $w \neq \text{next}M(v)$. By Lemma 3.25, w is the greatest vertex in $M^{-1}(M(v))$ such that $w \leq \text{low}(u)$ and $w \leq \min\{\text{low}(u') \mid \exists v' \text{ with } M(v') = M(v), w < v' < v \text{ and } u' \in \text{stack}U[v']\}$. By Lemma 3.24, Algorithm 11 will identify w during the processing of u in line 8. As a consequence of proposition 3.12, we have $b_count(v) = b_count(u) + b_count(w)$, and thus the triplet $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ will be marked in line 10. Conversely, let $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ be a triplet that gets marked by Algorithm 11 in line 10. Then, we have $u \in \text{stack}U[v]$. Furthermore, Lemma 3.24 implies that w has $M(w) = M(v)$ and $w \leq \text{low}(u)$. Then, since $u \in \text{stack}U[v]$, we have $\text{low}(u) < \text{next}M(v)$, and therefore w is a proper ancestor of v . Now, since $b_count(v) = b_count(u) + b_count(w)$, Lemma 3.22 implies that $\{(u, p(u)), (v, p(v)), (w, p(w))\}$ is a 3-cut. Thus, the correctness of Algorithm 11 is established.

To prove that Algorithm 11 runs in linear time, we will count the number of times that we access the array *lowestW* during the **while** loop in line 7. Specifically, we will show that, by the time

the algorithm is terminated, the v entry of $lowestW$, for every vertex v , will have been accessed at most once in line 7. We will prove this inductively, using the inductive proposition: $\Pi(v) \equiv$ after processing v , we have that $\forall v' < v$ $lowestW[v']$ has been accessed at most once in line 7 during the course of the algorithm so far **and** $\forall v' \leq v$ we have that every $w \in T(v', lowestW[v'])$ has $lowestW[w] \geq lowestW[v']$. Thus, (the first part of) $\Pi(n)$ implies the linearity of Algorithm 11. Now, suppose that $\Pi(v-1)$ is true for a $v \in \{1, \dots, n\}$ (observe that $\Pi(0)$ is trivially true). We will prove that $\Pi(v)$ is also true. Thus we have to show that: after we have processed every $u \in stackU[v]$, we have that $\forall v' < v$ $lowestW[v']$ has been accessed at most once in line 7 during the course of the algorithm so far **and** $\forall v' \leq v$ we have that every $w \in T(v', lowestW[v'])$ has $lowestW[w] \geq lowestW[v']$ (1). Now, suppose that this was a case for a specific $\tilde{u} \in stackU[v]$. We will show that it is still true for the successor u of \tilde{u} in $stackU[v]$. (Of course, due to the inductive hypothesis, (1) is definitely true before we have begun processing the elements of $stackU[v]$, and therefore we may also have that u is the first element of $stackU[v]$ in what follows.) Let \tilde{w} be the value of $lowestW[v]$ after the assignment in line 8, during the processing of u . Thus, all vertices that we traversed during the execution of the **while** loop, during the processing of u , are contained in $T[v, \tilde{w}]$. Now let $v' < v$ be a vertex with the property that $lowestW[v']$ has been accessed once in line 7 during the course of the algorithm before the processing of u , and let \tilde{v} be the vertex during whose processing we had to access $lowestW[v']$ in the **while** loop. We will show that $lowestW[v']$ will not be accessed in line 7 during the processing of u . Of course, we may assume that v' is in $T[v, \tilde{w}]$, for otherwise it is clear that the v' entry of $lowestW$ will not be accessed during the execution of the **while** loop (since the traversal in **while** loop will not reach vertices lower than \tilde{w} , and when it reaches \tilde{w} it will terminate). We note that, since the v' entry of $lowestW$ was accessed during the execution of the **while** loop during the processing of \tilde{v} , we have that $lowestW[\tilde{v}]$ is an ancestor of $lowestW[v']$, and therefore a proper ancestor of v' . Now, if $\tilde{v} = v$, then $lowestW[v]$ was assigned $lowestW[\tilde{v}]$, in line 8, during the processing of a predecessor of u in $stackU[v]$. Thus, when we begin processing u , w is assigned a proper ancestor of v' in line 6, before entering the **while** loop, and so the v' entry of $lowestW$ will not be accessed during the execution of the **while** loop. So let's assume that $\tilde{v} < v$. Initially, the variable w is assigned $lowestW[v]$ in line 6. We claim that $lowestW[v]$ is either a descendant of \tilde{v} or a proper ancestor of v' . To see this, suppose, for the sake of contradiction, that $lowestW[v]$ is in $T(\tilde{v}, v')$. Then, we have $\tilde{v} \in T(v, lowestW[v])$, and therefore, since (1) is true for \tilde{u} (the predecessor of u in $stackU[v]$), we have that $lowestW[\tilde{v}] \geq lowestW[v]$. Since $lowestW[\tilde{v}]$ is a proper ancestor of v' , this implies that $v' > lowestW[v]$, contradicting the supposition $lowestW[v] \leq v'$. Thus, before executing the **while** loop, we have that w is either a descendant of \tilde{v} or a proper ancestor of v' . Now suppose that the **while** loop has been executed 0 or more times, and w is assigned a descendant of \tilde{v} or a proper ancestor of v' . We will show that if we execute the **while** loop once more, w will either be assigned a descendant of \tilde{v} or a proper ancestor of v' . Of course, if w is a proper ancestor of v' , the same is true for $lowestW[w]$. Moreover, if $w = \tilde{v}$, then, as noted above, we have that $lowestW[w]$ is a proper ancestor of v' . So let's assume that w is a proper descendant of \tilde{v} , and suppose, for the sake of contradiction, that $lowestW[w]$ is in $T(\tilde{v}, v')$. Then, since $\tilde{v} \in T(w, lowestW[w])$, due to the inductive hypothesis we have that $lowestW[\tilde{v}] \geq lowestW[w]$. Since we also have $v' > lowestW[\tilde{v}]$, this contradicts the supposition $lowestW[w] \geq v'$. Thus, if w is a proper descendant of \tilde{v} , $lowestW[w]$ is either a descendant of \tilde{v} or a proper ancestor of v' . In any case, then, during the execution of the **while** loop, w will be assigned either a descendant of \tilde{v} or a proper ancestor of v' , and thus the v' entry of $lowestW$ will not be accessed.

It remains to show that, after the processing of u , for every $w \in T(v, \tilde{w})$ we have $lowestW[w] \geq \tilde{w}$. Due to the inductive hypothesis, this is definitely true for every $w \in T(v, lowestW[v])$ (where $lowestW[v]$ here has the value after the processing of \tilde{u} and before the processing of u), since

$lowestW[v] \geq \tilde{w}$, and every such w has $lowestW[w] \geq lowestW[v]$. Now let's assume that $w \in T[lowestW[v], \tilde{w})$, and suppose, for the sake of contradiction, that $lowestW[w] < \tilde{w}$. Then it cannot be that case that $w = lowestW[v]$, since $\tilde{w} \leq lowestW[lowestW[v]]$ (for the existence of a $w \in T[lowestW[v], \tilde{w})$ implies that $\tilde{w} \neq lowestW[v]$). Now, since $lowestW[v] > w > \tilde{w}$, there must exist a w' such that $w' \in T[lowestW[v], w]$, $lowestW[w'] < w$ and $lowestW[w'] \geq \tilde{w}$. Since $lowestW[w] < \tilde{w}$, we cannot $w' = w$. Then, $w \in T(w', lowestW[w'])$, and thus, due to the inductive hypothesis, we have $lowestW[w] \geq lowestW[w']$. Since $lowestW[w'] \geq \tilde{w}$, this implies that $lowestW[w] \geq \tilde{w}$, contradicting the supposition $lowestW[w] < \tilde{w}$. Thus, every $w \in T(v, \tilde{w})$ has $lowestW[w] \geq \tilde{w}$. The proof that (1) is true for u is complete. Due to the generality of $u \in stackU[v]$, this implies that $\Pi(v)$ is true. This shows, by induction, that $\Pi(n)$ is true, and the linearity of Algorithm 11 is thus established. \square

4 Computing the 4-edge-connected components in linear time

Now we consider how to compute the 4-edge-connected components of an undirected graph G in linear time. First, we reduce this problem to the computation of the 4-edge-connected components of a collection of auxiliary 3-edge-connected graphs.

4.1 Reduction to the 3-edge-connected case

Given a (general) undirected graph G , we execute the following steps:

- Compute the connected components of G .
- For each connected component, we compute the 2-edge-connected components which are subgraphs of G .
- For each 2-edge-connected component, we compute its 3-edge-connected components C_1, \dots, C_ℓ .
- For each 3-edge-connected component C_i , we compute a 3-edge-connected auxiliary graph H_i , such that for any two vertices x and y , we have $x \stackrel{G}{\equiv}_4 y$ if and only if x and y are both in the same auxiliary graph H_i and $x \stackrel{H_i}{\equiv}_4 y$.
- Finally, we compute the 4-edge-connected components of each H_i .

Steps 1–3 take overall linear time [19, 22]. We describe step 5 in the next section, so it remains to give the details of step 4. Let H be a 2-edge-connected component (subgraph) of G . We can construct a compact representation of the 2-cuts of H , which allows us to compute its 3-edge-connected components C_1, \dots, C_ℓ in linear time [6, 22]. Now, since the collection $\{C_1, \dots, C_\ell\}$ constitutes a partition of the vertex set of H , we can form the quotient graph Q of H by shrinking each C_i into a single node. Graph Q has the structure of a tree of cycles [2]; in other words, Q is connected and every edge of Q belongs to a unique cycle. Let (C_i, C_j) and (C_i, C_k) be two edges of Q which belong to the same cycle. Then (C_i, C_j) and (C_i, C_k) correspond to two edges (x, y) and (x', y') of G , with $x, x' \in C_i$. If $x \neq x'$, we add a virtual edge (x, x') to $G[C_i]$. (The idea is to attach (x, x') to $G[C_i]$ as a substitute for the cycle of Q which contains (C_i, C_j) and (C_i, C_k) .) Now let \bar{C}_i be the graph $G[C_i]$ plus all those virtual edges. Then \bar{C}_i is 3-edge-connected and its 4-edge-connected components are precisely those of G that are contained in C_i [2]. Thus we can compute the 4-edge-connected components of G by computing the 4-edge-connected components of the graphs $\bar{C}_1, \dots, \bar{C}_\ell$ (which can easily be constructed in total linear time). Since every \bar{C}_i is 3-edge-connected, we can apply Algorithm 12 of the following section to compute its 4-edge-connected

components in linear time. Finally, we define the multiplicity $m(e)$ of an edge $e \in \bar{C}_i$ as follows: if e is virtual, $m(e)$ is the number of edges of the cycle of Q which corresponds to e ; otherwise, $m(e)$ is 1. Then, the number of minimal 3-cuts of H is given by the sum of all $m(e_1) \cdot m(e_2) \cdot m(e_3)$, for every 3-cut $\{e_1, e_2, e_3\}$ of \bar{C}_i , for every $i \in \{1, \dots, l\}$ [2]. Since the 3-cuts of every \bar{C}_i can be computed in linear time, the minimal 3-cuts of H can also be computed within the same time bound.

4.2 Computing the 4-edge-connected components of a 3-edge-connected graph

Now we describe how to compute the 4-edge-connected components of a 3-edge-connected graph G in linear time. Let r be a distinguished vertex of G , and let C be a minimum cut of G . By removing C from G , G becomes disconnected into two connected components. We let V_C denote the connected component of $G \setminus C$ that does not contain r , and we refer to the number of vertices of V_C as the r -size of the cut C . (Of course, these notions are relative to r .)

Let $G = (V, E)$ be a 3-edge-connected graph, and let \mathcal{C} be the collection of the 3-cuts of G . If the collection \mathcal{C} is empty, then G is 4-edge-connected, and V is the only 4-edge-connected component of G . Otherwise, let $C \in \mathcal{C}$ be a 3-cut of G . By removing C from G , G is separated into two connected components, and every 4-edge-connected component of G lies entirely within a connected component of $G \setminus C$. This observation suggests a recursive algorithm for computing the 4-edge-connected components of G , by successively splitting G into smaller graphs according to its 3-cuts. Thus, we start with a 3-cut C of G , and we perform the splitting operation shown in Figure 8. Then we take another 3-cut C' of G and we perform the same splitting operation on the part which contains (the corresponding 3-cut of) C' . We repeat this process until we have considered every 3-cut of G . When no more splits are possible, the connected components of the final split graph correspond (by ignoring the newly introduced vertices) to the 4-edge-connected components of G .

To implement this procedure in linear time, we must take care of two things. First, whenever we consider a 3-cut C of G , we have to be able to know which ends of the edges of C belong to the same connected component of $G \setminus C$. And second, since an edge e of a 3-cut of the original graph may correspond to two virtual edges of the split graph, we have to be able to know which is the virtual edge that corresponds to e . We tackle both these problems by locating the 3-cuts of G on a DFS-tree T of G rooted at r , and by processing them in increasing order with respect to their r -size. By locating a 3-cut $C \in \mathcal{C}$ on T we can answer in $O(1)$ time which ends of the edges of C belong to the same connected component of $G \setminus C$. And then, by processing the 3-cuts of G in increasing order with respect to their size, we ensure that (the 3-cut that corresponds to) a 3-cut $C \in \mathcal{C}$ that we process lies in the split part of G that contains r .

Now, due to the analysis of the preceding sections, we can distinguish the following types of 3-cuts on a DFS-tree T (see also Figure 1):

- (I) $\{(v, p(v)), (x_1, y_1), (x_2, y_2)\}$, where (x_1, y_1) and (x_2, y_2) are back-edges.
- (IIa) $\{(u, p(u)), (v, p(v)), (x, y)\}$, where u is a descendant of v and $(x, y) \in B(v)$.
- (IIb) $\{(u, p(u)), (v, p(v)), (x, y)\}$, where u is a descendant of v and $(x, y) \in B(u)$.
- (III) $\{(u, p(u)), (v, p(v)), (w, p(w))\}$, where w is an ancestor of both u and v , but u, v are not related as ancestor and descendant.
- (IV) $\{(u, p(u)), (v, p(v)), (w, p(w))\}$, where u is a descendant of v and v is a descendant of w .

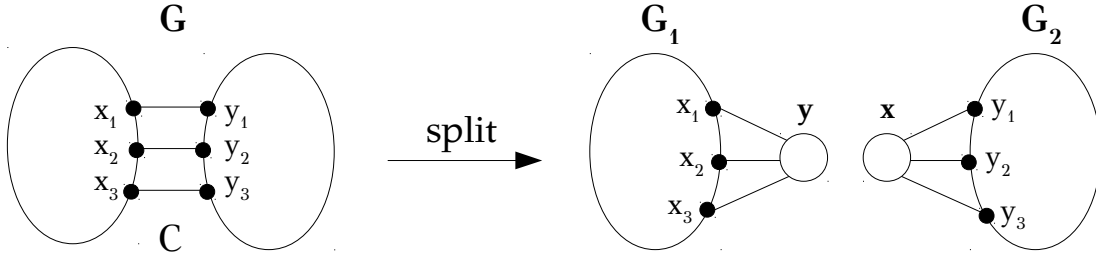


Figure 8: $C = \{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$ is a 3-cut of G , with $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ lying in different connected components of $G \setminus C$. The split operation of G at C consists of the removal the edges of C from G , and the introduction of two new nodes x, y , and six virtual edges $(x_1, y), (x_2, y), (x_3, y), (x, y_1), (x, y_2), (x, y_3)$. Now, the split graph is made of two connected components, G_1 and G_2 . Every 3-cut $C' \neq C$ of G (or more precisely: a 3-cut that corresponds to C') lies entirely within G_1 or G_2 . Conversely, every 3-cut of either G_1 or G_2 corresponds to a 3-cut of G . Thus, every 4-edge-connected component of G lies entirely within G_1 or G_2 .

Let r be the root of T . Then, for every 3-cut $C \in \mathcal{C}$, V_C is either $T(v)$, or $T(v) \setminus T(u)$, or $T(w) \setminus (T(u) \cup T(v))$, or $T(u) \cup (T(w) \setminus T(v))$, depending on whether C is of type (I), (II), (III), or (IV), respectively. Thus we can immediately calculate the size of C and the ends of its edges that lie in V_C . In particular, the size of C is either $ND(v)$, or $ND(v) - ND(u)$, or $ND(w) - ND(u) - ND(v)$, or $ND(u) + ND(w) - ND(v)$, depending on whether it is of type (I), (II), (III), or (IV), respectively; V_C contains either $\{v, x_1, x_2\}$, or $\{p(u), v, x\}$, or $\{p(u), v, y\}$, or $\{p(u), p(v), w\}$, or $\{u, p(v), w\}$, depending on whether C is of type (I), (IIa), (IIb), (III), or (IV), respectively.

Algorithm 12 shows how we can compute the 4-edge-connected components of G in linear time, by repeatedly splitting G into smaller graphs according to its 3-cuts. When we process a 3-cut C of G , we have to find the edges of the split graph that correspond to those of C , in order to delete them and replace them with (new) virtual edges. That is why we use the symbol v' , for a vertex $v \in V$, to denote a vertex that corresponds to v in the split graph. (Initially, we set $v' \leftarrow v$.) Now, if (x, y) is an edge of C with $x \in V_C$, the edge of the split graph corresponding to (x, y) is (x', y') . Then we add two new vertices v_C and \tilde{v}_C to G , and the virtual edges (x', \tilde{v}_C) and (v_C, y') . Finally, we let x correspond to v_C , and so we set $x' \leftarrow v_C$. This is sufficient, since we process the 3-cuts of G in increasing order with respect to their size, and so the next time we meet the edge (x, y) in a 3-cut, we can be certain that it corresponds to (v_C, y') .

Algorithm 12: Compute the 4-edge-connected components of a 3-edge-connected graph $G = (V, E)$

- 1 Find the collection \mathcal{C} of the 3-cuts of G
 - 2 Locate and classify the 3-cuts of G on a DFS-tree of G rooted at r
 - 3 For every $C \in \mathcal{C}$, calculate $size(C)$ (relative to r)
 - 4 Sort \mathcal{C} in increasing order w.r.t. the $size$ of its elements
 - 5 **foreach** $v \in V$ **do** Set $v' \leftarrow v$
 - 6 **foreach** $C = \{(x_1, y_1), (x_2, y_2), (x_3, y_3)\} \in \mathcal{C}$ **do**
 - 7 Find the ends of the edges of C that lie in V_C // *Let those ends be x_1, x_2 and x_3*
 - 8 Remove the edges $(x'_1, y'_1), (x'_2, y'_2), (x'_3, y'_3)$ from G
 - 9 Introduce two new vertices v_C and \tilde{v}_C to G
 - 10 Add the edges $(x'_1, \tilde{v}_C), (x'_2, \tilde{v}_C), (x'_3, \tilde{v}_C), (v_C, y'_1), (v_C, y'_2), (v_C, y'_3)$ to G
 - 11 Set $x'_1 \leftarrow v_C, x'_2 \leftarrow v_C, x'_3 \leftarrow v_C$
 - 12 **end**
 - 13 Output the connected components of G , ignoring the newly introduced vertices
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