HAL open science

On the Stability of Positive Semigroups

Pierre del Moral, Emma Horton, Ajay Jasra

To cite this version:

Pierre del Moral, Emma Horton, Ajay Jasra. On the Stability of Positive Semigroups. 2022. hal-03468416v3

HAL Id: hal-03468416

https://inria.hal.science/hal-03468416v3

Preprint submitted on 10 Jul 2022 (v3), last revised 17 Feb 2023 (v6)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Submitted to the Annals of Applied Probability

ON THE STABILITY OF POSITIVE SEMIGROUPS

BY PIERRE DEL MORAL[1], EMMA HORTON[1] AND AJAY JASRA [2]

[1]*Centre de Recherche Inria, Bordeaux Sud-Ouest Talence, 33405, FR, pierre.del-moral@inria.fr, emma.horton@inria.fr*

[2]*Applied Mathematics and Computational Science Program, Computer, Electrical and Mathematical Sciences and Engineering Division, King Abdullah University of Science & Technology, Building 1, Thuwal, 23955-6900, KSA, ajay.jasra@kaust.edu.sa*

The stability and contraction properties of positive integral semigroups on Polish spaces are investigated. Our novel analysis is based on the extension of V-norm contraction methods, associated to functionally weighted Banach spaces for Markov semigroups, to positive semigroups. This methodology is applied to a general class of positive and possibly time-inhomogeneous bounded integral semigroups and their normalised versions. The spectral theorems that we develop are an extension of Perron-Frobenius and Krein-Rutman theorems for positive operators to a class of time-varying positive semigroups. In the context of time-homogeneous models, the regularity conditions discussed in the present article appear to be necessary and sufficient condition for the existence of leading eigenvalues. We review and illustrate the impact of these results in the context of positive semigroups arising in transport theory, physics, mathematical biology and signal processing.

1. Introduction. Positive semigroups arise in a variety of areas of applied mathematics, including nonlinear filtering, rare event analysis, branching processes, physics and molecular chemistry. In this article we will study the possibly time-inhomogeneous linear semigroup $Q_{s,t}$ and its normalisation $\Phi_{s,t}$, where $0 \leq s \leq t$ refers to a discrete or a continuous time parameter, which are formally introduced in section 1.1. Their interpretation depends upon the application model area as we now describe.

1. In signal processing, the normalised semigroup $\Phi_{s,t}$ depends on a random observation process and describes the solution to the nonlinear filtering equations, the semigroup $Q_{s,t}$ represents the evolution of the unnormalised filters. In this context, the stability of the semigroup $\Phi_{s,t}$ ensures that the optimal filter forgets its initial condition.
2. In the context of killed absorption processes, $\Phi_{s,t}$ represents the evolution of the distribution of a given process conditioned on non-absorption (a.k.a. the Q-process). In this context, the fixed point probability measure $\eta_\infty = \Phi_t(\eta_\infty)$ of time homogeneous semigroups $\Phi_t := \Phi_{s,s+t}$ is sometimes called the Yaglom or quasi-invariant measure.
3. The conditional Markov processes discussed above also arise in risk analysis and rare event simulation. In this context, these semigroups represent the evolution of a conditional Markov process evolving in a rare event regime.
4. In quantum physics and molecular chemistry the top of the spectrum of positive integral semigroups is related to ground state and free energy computations of Schrödinger operators and Feynman-Kac semigroups (see for instance [84, 80]).

*Corresponding author.

MSC2020 subject classifications: Primary 47D08, 47D06, 47D07, 47H07; secondary 47B65, 37A30, 37M25, 60J25.

Keywords and phrases: Positive semigroups, Boltzmann-Gibbs transformations, Contraction inequalities, Dobrushin's ergodic coefficient, Spectral theorems, Foster-Lyapunov conditions.

5. In the dynamic population literature, the semigroups $Q_{s,t}$ represent the evolution of the first moment of a spatial branching process. In this context, theses many-to-one formulae are expressed in terms of Feynman-Kac semigroups connecting the free evolution of a single individual with the killing and the branching rates potential functions.

The details of these application areas are considered in [72, 76, 89] and the relevant references therein. The spectral objects discussed above are naturally related to the analysis of quasi-compact operators and Fredholm integral equations, see for instance [12, 102, 118, 130, 166], as well as in large deviations principles associated with the occupation measures and related additive functional of Markov processes [91, 98, 104, 200].

1.1. *Description of the models.* Let $\mathcal{B}(E)$ be the algebra of locally bounded measurable functions on a Polish space E (that is separable completely metrizable topological space). We denote by $\mathcal{B}_b(E) \subset \mathcal{B}(E)$ the sub-algebra of bounded measurable functions. With a slight abuse of notation, we denote by 0 and 1 the null and unit scalars as well as the null and unit function on E. We denote by $\mathcal{M}_b(E)$ the set of bounded signed measures on E. Also let $\mathcal{C}_b(E) \subset \mathcal{B}_b(E)$ be the sub-algebra of continuous and bounded functions, and by $\mathcal{P}(E) \subset \mathcal{M}_b(E)$ be the subset of probability measures on E.

Let $\mathcal{T} = \mathbb{R}_+ := [0, \infty[$ or $\mathcal{T} = \mathbb{N}$ be the continuous or discrete time space, respectively. Consider a collection of positive integral operators $Q_{s,t} : f \mapsto Q_{s,t}(f)$ from $\mathcal{B}_b(E)$ into $\mathcal{B}_b(E)$, indexed by parameters $s, t \in \mathcal{T}$ with $s \leq t$, and satisfying for any $s, u, t \in \mathcal{T}$, $s \leq u \leq t$, the semigroup property

$$Q_{s,u} Q_{u,t} = Q_{s,t} \quad \text{with} \quad Q_{s,s} = I. \tag{1.1}$$

The right action $Q_{s,t} : f \mapsto Q_{s,t}(f)$ and dual left action $\mu \mathcal{M}_b(E) \mapsto \mu Q_{s,t} \in \mathcal{M}_b(E)$ of and integral operator $Q_{s,t}$ are defined in classical measure theoretic notation in (2.1). Assume that $Q_{s,t}(1) > 0$ for any $s \leq t$.

For any measure $\eta_s \in \mathcal{P}(E)$ we let $\Phi_{s,t}(\eta_s) \in \mathcal{P}(E)$ be the normalised distribution defined for any $f \in \mathcal{B}_b(E)$ by the formula

$$\Phi_{s,t}(\eta_s)(f) := \eta_s Q_{s,t}(f) / \eta_s Q_{s,t}(1). \tag{1.2}$$

The mapping $\Phi_{s,t}$ is a well defined nonlinear map from $\mathcal{P}(E)$ into $\mathcal{P}(E)$ satisfying for any $s, u, t \in \mathcal{T}$ with $s \leq u \leq t$ the semigroup property

$$\Phi_{s,t} = \Phi_{u,t} \circ \Phi_{s,u} \quad \text{with} \quad \Phi_{s,s}(\mu) = \mu.$$

Unless otherwise stated, all the semigroups discussed in this article are indexed by conformal indices $s \leq t$ in the set $\mathcal{T}$. To avoid repetition, we often write $Q_{s,t}$ and $\Phi_{s,t}$ without specifying the order $s \leq t$ of the indices $s, t \in \mathcal{T}$. For time homogeneous models we use the notation

$$(\Phi_t, Q_t) := (\Phi_{0,t}, Q_{0,t}).$$

In contrast with conventional Hilbert space approaches to the stability of reversible Markov semigroups (cf. for instance [28, 93, 94, 149]), the analysis of time varying models of the form (1.1) and (1.2) does not rely on a particular reversible measure. The framework of weighted spaces and V-norms considered in the article is a natural but non-unique framework to analyse time-varying positive semigroups.

1.2. *Literature review.* In order to guide the reader through the vast array of stability analysis results developed in these different disciplines, we give a brief overview of the literature in these fields. We also provide some precise reference points to aide with the navigation between applications.

A unifying point of interest in the above applications is the study of the stability of the afore-mentioned semigroups. In the context of dynamic populations, the long time behavior of branching processes certainly goes back to the end of the 1940s with the pioneering work of Yaglom [202] on Galton-Watson processes. The stability analysis of time homogenous birth-and-death processes with absorption on finite or countable spaces dates back to the 1950s-1960s with the pioneering works [139, 180] and the later developments [44, 67, 68, 116, 189]. Sufficient conditions ensuring the existence of a quasi-invariant measure are also developed in [106, 151]. Powerful spectral and h-process techniques are also developed in [42, 43, 64, 80, 115, 128, 174]. Non-asymptotic spectral techniques that apply to possibly transient and unstable mutation linear diffusions that do not necessarily have a gradient form and with a quadratic absorption rate are discussed in the more recent work [80].

One of the first well-founded results on the long time behavior of the nonlinear filtering equation is the seminal article by Ocone and Pardoux in the mid-1990s [169]. In this work the authors show that the optimal filter forgets the initial condition without giving a non-asymptotic rate. This latter analysis is critical in the study of, for instance, numerical algorithms such as the particle filter. The stability of the nonlinear filtering equation is also related to the Lyapunov spectrum and the asymptotic properties of products of random positive matrices [26, 27, 111, 124, 129, 138, 180]. In contrast with positive semigroups arising in physics and biology, the stability analysis of the nonlinear filtering equation involves the study of sophisticated stochastic semigroups that depend on partial and noisy observations.

The systematic, non-asymptotic stability analysis of non-homogeneous sub-Markovian and Feynman-Kac semigroups on general state spaces has also been considered at the end of the 1990s, mainly in nonlinear filtering theory and rare event analysis. Several techniques have been adopted. Hilbert metric and robustification techniques, based on the seminal article by Birkhoff [20] was used in [11, 30, 66, 143, 152, 153, 157, 162, 170, 171]. On the other hand, Dobrushin's ergodic coefficients, based on the pioneering articles by Dobrushin [96], were used in [72, 76, 78, 79, 154, 155, 157, 162]. We also mention bounded Lipschitz distance techniques [154, 155] and relative entropy-like criteria [59, 72, 88, 156]. Local Doeblin minorisation conditions applying to non-compact spaces, including Foster-Lyapunov approaches and coupling, that apply to ergodic signal-observation filtering problems, including stable Gaussian-linear filtering models, are also discussed in the series of articles [99, 100, 113, 135, 136, 196, 198]. We also refer the reader to [191] for related Lipschitz norm techniques as well as the more recent articles [15, 18, 49, 109, 123] in the context of positivity preserving operators arising in particle absorption models and quasi-invariant measure literature. Functional inequalities, including Poincaré inequalities and Bakry-Emery criteria approaches are discussed in [168], quasi-compactness Lyapunov criteria are also discussed in the recent article [18]. Spectral techniques, drift conditions and Wasserstein norm approaches for time-homogenous models are also discussed respectively in [80, 82, 84, 144, 197]. Further, two-sided minorisation conditions are discussed in [56, 57, 72, 73, 76], and truncation techniques are presented in [65, 109, 127, 196].

General asymptotic stability results are also provided in [192]. Stabilising changes of Feynman-Kac measures and related importance-sampling and h-process techniques that apply to possibly unstable killed processes on unbounded domains are also discussed in [15, 109, 178], see also [80] and well as section B.1 in [21] and in section 7.1 in [81] in the context of discrete time models. The stability of the nonlinear filtering equations with deterministic hyperbolic signals is also developed in the recent article [170].

Despite the numerous references given in the introduction, a complete literature review is not possible. For a more detailed discussion on this subject, we refer the reader to [31, 58] for a review on the asymptotic stability of nonlinear filters, to the bibliographies and reviews [63,

161, 173, 190] on the theory of quasi-stationary distributions, the articles [76, 109] as well as the bibliography in [80] in quantum physics, and the books [72, 73] for a detailed discussion on the long time stability of Feynman-Kac semigroups.

The stability analysis of positive semigroups is also crucial in the convergence analysis of numerical approximations of Feynman-Kac semigroups, including the computation of the principal eigen-function and the leading eigenvalues in the context of time homogeneous models (cf. for instance [84] as well as Theorem 2.11 and Theorem 3.27 in [76] in the context of time varying semigroups). Despite their importance, the numerical implications are not covered in the present article. However, to guide the reader, we end this section with some references to the literature on mean-field particle methodologies currently used in this context. Mean field and genetic particle methodologies are discussed in the series of articles on Feynman-Kac semigroups arising in physics and nonlinear filtering [5, 10, 60, 72, 73, 74, 75, 77, 78, 178, 199], as well as in [33, 34] in the context of Dirichlet Laplacian and in [7, 8, 10, 53, 61, 62, 106, 137, 194] in the context of quasistationary measures.

Several pioneering articles from the mid-1950s by Rosenbluth and Rosenbluth [177] on sampling self-avoiding walks and another from the mid-1980s by Hetherington [131] on reconfiguration Monte-Carlo methods are relevant to our work. These interacting Monte Carlo methodologies were further extended by Caffarel and his co-authors in the series of articles [35, 36, 37]. See also Buonaura-Sorella [32], as well as the pedagogical introduction to quantum Monte Carlo by Caffarel-Assaraf [38]. Similar heuristic boostrapping methodologies were also used in the mid-1990s in nonlinear filtering [117, 141, 142] and to simulate long chain molecules [119, 120]. See also the go-with-the-winner methodology discussed in [2, 121] and the Fleming-Viot techniques presented in [34].

In the context of discrete generation positive semigroups arising in nonlinear filtering and genetic algorithms, we refer the reader to the first well-founded article [71]. To the best of our knowledge the first articles discussing time-uniform propagation of chaos estimates seem to be the articles [78, 79], followed by [76, 83, 84] and the book [72]. From a purely mathematical perspective, all of the genetic Monte Carlo methods discussed above can be seen as mean field particle interpretations of Feynman-Kac semigroups. In path space settings, the genealogical tree associated with these branching Monte Carlo methods allows one to compute Feynman-Kac path integrals and provide an unbiased estimate of unnormalised semigroups.

Whilst all the Monte Carlo methods discussed above belong to the same class of genetic mutation-selection methodology to estimate Feynman-Kac semigroups, they are known under a variety of different names in the applied literature such as Sequential Monte Carlo methods, Feynman-Kac particle interpretations, Particle Filters, Cloning and Pruning as well as Bootstrapping techniques, Diffusion Monte Carlo, Population-Monte Carlo, Reconfiguration Monte Carlo, Moran and Fleming-Viot particle models, to name a few. Related reinforcement and self-interacting Markov chain methodologies and stochastic approximation techniques are presented in [86, 87] and more recently in [17, 23, 159].

Most of the terminology encountered in the literature arises from the application domains as well as with the branching/genetic evolution of the Monte Carlo methodology. As underlined in [77], from a mathematical viewpoint, only the terminologies “Fleming-Viot” and ”Moran” are misleading. The reasons are two-fold: firstly, the Moran particle model is a finite population model that converges as the number of particles tends to infinity towards a stochastic Fleming Viot superprocess [69, 110] and secondly, the genetic noise arising in the limit requires a Moran finite population process with symmetric-selection jump rates. In our context, the selection/killing-jump rates are far from being symmetric, the empirical measures of the finite population model are biased and the limiting Feynman-Kac semigroup, as the number of particles tends to infinity, is purely deterministic.

1.3. *Statement of some main results.* The main objective of this article is to review and further develop the stability analysis of positive semigroups for a general and abstract class of time inhomogeneous models, which, in general, are much more difficult to handle than their time homogeneous counterparts; this is because the operators may drastically change during the semigroup evolution. We will also tackle the problem of non-compact state spaces.

We begin with an exposition of discrete time and time homogeneous models. Consider a positive integral operator Q and let $Q_{n+1} = Q_n Q = Q Q_n$ be the associated discrete time semigroup indexed by $n \in \mathbb{N}$. In this context we have

$$\mu(Q^n(1)) = \prod_{0 \leq k < n} \Phi_k(\mu)(g) \quad \text{with} \quad g := Q(1). \tag{1.3}$$

When E is compact, the Schauder fixed-point theorem ensures the existence of an invariant measure

$$\eta_\infty = \Phi_n(\eta_\infty) \in \mathcal{P}(E) \quad \text{and} \quad \eta_\infty Q_n = e^{n\rho}\, \eta_\infty \quad \text{with} \quad \rho := \log \eta_\infty(Q(1)).$$

Note that the right-hand side assertion in the above display is a direct consequence of the fixed point equation and the product formula (1.3). For not necessarily compact spaces, we also quote the following abstract Krylov-Bogolioubov theorem for discrete time Feller semigroups, which is of interest in its own right.

THEOREM 1.1. *Consider a positive integral operator Q such that $Q(\mathcal{C}_b(E)) \subset \mathcal{C}_b(E)$ and let Q_n be the associated discrete time Feller semigroup indexed by $n \in \mathbb{N}$. The normalised semigroup Φ_n has at least one invariant probability measure $\eta_\infty = \Phi_n(\eta_\infty)$ if and only if there exists some probability measure η such that the sequence of probability measures $\Phi_n(\eta)$ indexed by $n \in \mathbb{N}$ is tight and we have*

$$\beta_n(\eta) := \Phi_n(\eta)(g) \in]0,1] \longrightarrow_{n\to\infty} \beta_\infty(\eta) > 0, \tag{1.4}$$

with the function g defined in (1.3). In addition, whenever these conditions are met we have $\beta_\infty(\eta) = \eta_\infty(g)$.

The proof of Theorem 1.1 is provided in section 6.1. Related equivalent conditions for the existence of invariant measures on locally compact spaces E are discussed in [151], for models preserving continuous functions that tend to 0 at ∞, see also [106]. Under the assumptions of Theorem 1.1, using (1.3) we check the product series formulae

$$\begin{aligned} h_{n+1}(x) &:= e^{-(n+1)\rho}\, Q^{n+1}(1)(x) \\ &= e^{-\rho}\, Q(h_n)(x) = \prod_{0 \leq k \leq n} \left(1 + (\Phi_k(\delta_x)(\overline{g}) - \Phi_k(\eta_\infty)(\overline{g}))\right) \end{aligned} \tag{1.5}$$

with the normalisation $\overline{g} := e^{-\rho} g = g/\eta_\infty(g)$ of the function g defined in (1.3). The above rather elementary formulae connect the convergence properties of the functions h_n as $n \to \infty$ with the stability properties of the normalized measures $\Phi_n(\mu)$.

Recall that (see for instance chapter 7 in [146]) the product in the above display is absolutely convergent if and only if we have

$$\sum_{k \geq 0} |\Phi_k(\delta_x)(\overline{g}) - \Phi_k(\eta_\infty)(\overline{g})| < \infty. \tag{1.6}$$

In this situation, the collection of functions $h_n(x)$ converges pointwise as $n \to \infty$ to the measurable function defined by the product series

$$h(x) := \prod_{n \geq 0} \left(1 + (\Phi_n(\delta_x)(\overline{g}) - \Phi_n(\eta_\infty)(\overline{g}))\right). \tag{1.7}$$

At this level of generality we cannot ensure that h is a right eigenvalue of Q. To apply the dominated convergence theorem, we need to ensure that the sequence of functions h_n is uniformly bounded. This property requires to calibrate with some precision the stability properties of the normalized semigroup Φ_n.

This brief discussion already motivates the importance of the stability analysis of the normalised semigroups. Several classes of semigroups with an increasing level of regularity are presented in section 3. All of these different models are all expressed in terms of the triangular array of semigroups defined in section 3.1. The first class of semigroups, termed R-semigroups, is discussed in section 3.2 and it relies on the contraction properties these semigroups with respect to the total variation distance (cf. for instance condition (3.6)). Section 4.1 provides several uniform exponential contraction theorems for this class of models with respect to the total variation norm. The second class of semigroups, termed stable V-positive semigroups, is discussed in section 3.4. These models rely on an regularity property with respect to some Lyapunov function V discussed in section 3.3.

Section 4.2 provides several uniform exponential contraction theorems for this class of models with respect to V-norms. Section 4.3 illustrates the impact of these results in the context of time homogeneous semigroups. We shall see that Q_t is a stable V-positive semigroup if and only if there exists some leading eigenvalues and the semigroup of the process evolving in the ground state is a stable positive semigroup.

These different classes of positive semigroups are not based on any type of absolutely continuity condition, but on different types of regularity properties of integral operator regularity and local contraction properties of the pivotal triangular array of semigroups defined in section 3.1.

In this section, to avoid the details of abstract regularity conditions we have chosen to review and state some of our main results in the context absolute continuity with an increasing level of regularity. We also illustrate these regularity conditions with some elementary examples. We recall that a semigroup of positive integral operators (1.1) is said to be absolutely continuous as soon as for some $\tau > 0$ and any $t \in \mathcal{T}$ we have

$$Q_{t,t+\tau}(x, dy) = q_{t,t+\tau}(x, y)\ \nu_\tau(dy) \tag{1.8}$$

for some density function $q_{t,t+\tau}$ on E^2 with respect to a Radon positive measure ν_τ on E.

1.3.1. *Total variation stability theorems.* Consider the following condition:

($\mathcal{A}$) *There exists some $\tau > 0$ such that density $q_{t,t+\tau}$ is uniformly positive for some parameter $\tau > 0$; that is, we have that*

$$0 < \iota^-(\tau) := \inf q_{t,t+\tau}(x, y) \leq \iota^-(\tau) := \sup q_{t,t+\tau}(x, y) < \infty \tag{1.9}$$

where the infimum and the supremum are taken over all space-time indices $((x, y), t) \in (E^2 \times \mathcal{T})$. For continuous time models, we assume that for any $\mu \in \mathcal{P}(E)$ we have

$$\inf \mu(Q_{t,t+\varepsilon}(1)) > 0 \quad \textit{and} \quad \pi_\tau := \sup \|Q_{t,t+\varepsilon}(1)\| < \infty \tag{1.10}$$

where the infimum is taken over all continuous time indices $t \geq 0$ and any $\varepsilon \in [0, \tau[$.

In the context of discrete time semigroups, there is no loss of generality to assume that $\tau = 1$ (cf. section 2.2). For continuous time sub-Markovian semigroups, the right-hand side condition in (1.10) is automatically met with $\pi_r \leq 1$. Whenever $Q_{s,t}(1) \geq Q_{s,u}(1)$ for any $u \geq t$, the left-hand side condition in (1.9) ensures that $\inf \mu(Q_{t,t+\varepsilon}(1)) \geq \iota^-(\tau) > 0$; and for time homogenous models the right-hand side condition in (1.9) is met as soon as q_τ is bounded.

For continuous time models, condition (1.9) is automatically satisfied for time homogeneous jump elliptic diffusions on compact manifolds S with a bounded jump rate, see for

instance the pioneering work of Aronson [9], Nash [164] and Varopoulos [193] on Gaussian estimates for heat kernels on manifolds. It is also met for uniformly elliptic diffusions on the compact closure $E = \overline{D}$ of some bounded open domain D in $\mathbb{R}^n$ with an oblique reflection on some smooth boundary ∂D, see for instance [41]. Condition (1.9) is also preserved by conditioning *any* of the stochastic processes discussed above by the non-absorption event, as soon as the killing rate is bounded on the state space E, see for instance [85] for a more thorough discussion on this class of sub-Markovian semigroups on compact manifolds. Further references on uniformly positive discrete time semigroups can be found in [78, 82, 85] and the books [72, 73, 76].

The next theorem is a direct consequence of the uniform estimates (3.9) and a rather well known uniform contraction theorem, Theorem 4.1, which is valid for not necessarily absolutely continuous semigroups under weaker conditions on the triangular array of semigroups discussed above [76, 78, 79].

THEOREM 1.2. *Let $Q_{s,t}$ be an absolutely semigroup (1.8) satisfying condition $(\mathcal{A})$ for some parameter $\tau > 0$. In this situation, There exist constants $a < \infty$ and $b > 0$ such that for any $s \leq t$ and any $\mu_1, \mu_2 \in \mathcal{P}(E)$ we have the uniform stability estimate*

$$\|\Phi_{s,t}(\mu_1) - \Phi_{s,t}(\mu_2)\|_{tv} \leq a\, e^{-b(t-s)}, \tag{1.11}$$

with the total variation norm $\|.\|_{tv}$ on $\mathcal{M}_b(E)$ defined in (2.5). In addition, there exists a constant $c(\mu_1)$ and $c(\mu_2) < \infty$ such that for any $s \leq t$ we have the local Lipschitz estimate

$$\|\Phi_{s,t}(\mu_1) - \Phi_{s,t}(\mu_2)\|_{tv} \leq (c(\mu_1) \wedge c(\mu_2))\, e^{-b(t-s)}\, \|\mu_1 - \mu_2\|_{tv}.$$

Returning to the discrete and time homogeneous semigroup $Q = Q_{t,t+1}$ discussed in (1.5) the uniform estimate (1.11) ensures that

$$\sum_{n \geq 0} \sup_{x \in E} |\Phi_n(\delta_x)(\overline{g}) - \Phi_n(\eta_\infty)(\overline{g})| < \infty.$$

In this scenario, the collection of functions h_n defined in (1.5) is uniformly bounded and h_n converges pointwise as $n \to \infty$ to the function $h \in \mathcal{B}_b(E)$ defined in (1.7). Applying the dominated convergence theorem, for any $n \in \mathbb{N}$ we conclude that

$$Q_n(h) = e^{\rho n} h.$$

Following the above comments in the context of discrete or continuous time homogeneous semigroups, we present in Section 4.1 a variety of results that follow almost immediately from the estimates obtained in Theorem 1.2.

These results include the existence of an unique leading eigen-triple $(\rho, \eta_\infty, h) \in (\mathbb{R} \times \mathcal{P}(E) \times \mathcal{B}_b(E))$ of the positive semigroup; that is, for any $t \geq 0$ we have

$$Q_t(h) = e^{\rho t}\, h \quad \text{and} \quad \eta_\infty Q_t = e^{\rho t}\, \eta_\infty \quad \text{with} \quad \eta_\infty(h) = 1. \tag{1.12}$$

The leading eigen-function h is sometimes called the ground state of the semigroup. Defining the finite rank (and hence compact) operator

$$T \,:\, f \in \mathcal{B}_b(E) \mapsto T(f) := \frac{h}{\eta_\infty(h)}\, \eta_\infty(f) \in \mathcal{B}_b(E), \tag{1.13}$$

we also have the following extended and refined version of the Krein-Rutman theorem.

COROLLARY 1. *For any $t \in \mathcal{T}$, we have the operator norm exponential decays*

$$\left|\left|\left| e^{-\rho t} Q_t - T \right|\right|\right| \leq 2\, a\, e^{-b(t-s)} \left(\|h\| / \eta_\infty(h) + c(\eta_\infty)^2 \right), \tag{1.14}$$

with the same parameters $(a, b, c(\eta_\infty))$ as in (4.2).

The details, including a proof, of the above results can be found in section 4.1. As shown at the end of section 4.1, the estimate (1.14) ensures the uniqueness of the eigenfunction h (up to some constant) and that the essential spectral radius of Q_t is strictly smaller than its spectral radius $r(Q) = e^{\rho}$. Thus, the operator Q_t is quasi-compact and by a variant of the Krein-Rutman theorem (cf. for instance Theorem 1.1 in [102, 166]), we recover the fact the existence of non-null eigenfunction h. A brief review of quasi-compact operators is provided in section 2.4. The quasi-compactness property of Q_t is often based on compact localisation arguments and applying Arzela-Ascoli theorem when the semigroups have a continuous density. The above corollary does not rely on these arguments and it also offers an exponential convergence rate that only depends on the stability properties of the semigroup Φ_t.

Unfortunately, (1.9) is rarely satisfied for non-compact state spaces. Some cases where it does hold are for reflected diffusions on smooth boundaries as well as for some particular classes of operators on non-compact spaces (including for ad-hoc truncated drift Gaussian transitions or Laplace transitions on non-compact spaces - see [76, section 5] and [72, Exercise 3.5.2]).

1.3.2. *V-norm stability theorems.* Our starting point is to localise (1.9) using a some locally bounded $V \geq 1$ with compact level sets $K_r = \{V \leq r\} \subset E$. In this notation, the V-localised version of (1.9) is defined as follows:

$(\mathcal{A})_V$: *There exists some parameters $0 < \tau \in \mathcal{T}$ and $r_1 > 1$ such that for any $r \geq r_1$ we have $\nu_\tau(K_r) > 0$ as well as*

$$0 < \inf_{t\in\mathcal{T}} \inf_{K_r^2} q_{t,t+\tau} \leq \sup_{t\in\mathcal{T}} \sup_{K_r^2} q_{t,t+\tau} < \infty. \tag{1.15}$$

For continuous time semigroups, we assume that for any $r \geq r_1$ we have

$$\pi_\tau^-(K_r) := \inf_{x\in K_r} \inf Q_{t,t+\varepsilon}(1_{K_r})(x) > 0 \quad \textit{and} \quad \pi_\tau := \sup \|Q_{t,t+\varepsilon}(1)\| < \infty \tag{1.16}$$

where the infimum is taken over all continuous time indices $t \geq 0$ and any $\varepsilon \in [0, \tau]$.

The condition (1.15) is rather flexible as we will now explain. For time homogeneous models on some open connected domain $E \subset \mathbb{R}^d$ (with respect to the trace $\nu_\tau(dy)$ of the Lebesgue measure dy on E) condition (1.15) is clearly met as soon as q_τ is a bounded continuous positive function on E^2. To check condition $\nu_\tau(\{V \leq r\}) > 0$, simply notice that any closed ball in E (which has clearly positive Lebesgue measure) is included in some r_1-sub-level set of V, thus also included in all upper r-sub-level sets, with $r \geq r_1$.

Absolutely continuous integral operators arise in a natural way in discrete time settings [72, 76, 100, 196] and in the analysis of continuous time elliptic diffusion absorption models [9, 107, 108, 185]. In connection to this, two-sided estimates for stable-like processes are provided in [25, 147, 183, 195]. Two sided Gaussian estimates can also be obtained for some classes of degenerate diffusion processes of rank 2, that is when the Poisson brackets of the first order span the whole space [148]. This class of diffusions includes frictionless Hamiltonian kinetic models. Diffusion density estimates can be extended to sub-Markovian semigroups using the multiplicative functional methodology developed in [85].

Whenever the trajectories of these diffusion flows, say $t \mapsto X_t(x)$, where $x \in E$ is the initial position, are absorbed on the smooth boundary ∂E of a open connected domain E, for any $\tau > 0$ the densities $q_\tau(x, y)$ of the sub-Markovian semigroup Q_τ (with respect to the trace of the Lebesgue measure on E) associated with the non absorption event are null at the boundary. Nevertheless, whenever these densities are positive and continuous on the open set E^2 for some $\tau > 0$, they are uniformly positive and bounded on any compact subsets of E; thus condition (1.15) is satisfied. In addition, whenever $T(x)$ stands for first exit time from

E and $T_r(x)$ the first exit time from the compact level set $K_r \subset E$ starting from $x \in K_r$, for any $\varepsilon \in [0,\tau]$ we have the estimate

$$Q_\varepsilon(1_{K_r})(x) := \mathbb{E}\left(1_{X_\varepsilon(x)\in K_r}\ 1_{T(x)>\varepsilon}\right) \geq \mathbb{P}\left(T_r(x)>\varepsilon\right) \geq \mathbb{P}\left(T_r(x)>\tau\right).$$

In this context, condition (1.16) is met as soon as $\inf_{x\in K_r}\mathbb{P}\left(T_r(x)>\tau\right)>0$, for any $r\geq r_1$.

To state our next main result, we need to introduce some additional terminology. Let $\mathcal{C}(E)\subset\mathcal{B}(E)$ the sub-algebra of continuous functions. Let $\mathcal{B}_V(E)\subset\mathcal{B}(E)$ be the Banach space of measurable functions f on E with $\|f/V\|<\infty$; and $\mathcal{C}_V(E)\subset\mathcal{B}_V(E)$ be the subspace of continuous functions.

We also let $\mathcal{P}_V(E)$ be the convex set of probability measures $\mu_i\in\mathcal{P}(E)$ such that $\mu_i(V)<\infty$ with $i=1,2$ equipped with the operator V-norm

$$|||\mu_1-\mu_2|||_V := \sup\{|(\mu_1-\mu_2)(f)|\ :\ \|f\|_V\leq 1\}.$$

More generally, we denote by $|||\cdot|||_V$ the operator norm defined in (2.4). We also denote by $\mathcal{B}_0(E)\subset\mathcal{B}_b(E)$ the sub-algebra of locally lower bounded that vanish at infinity and by $\mathcal{C}_0(E):=\mathcal{B}_0(E)\cap\mathcal{C}(E)$ the sub-algebra of continuous functions. We also let $\mathcal{B}_\infty(E)\subset\mathcal{B}(E)$ be the subalgebra of locally bounded and uniformly positive functions V that grow at infinity. We shall consider the subspaces

$$\mathcal{B}_{0,V}(E):=\{f\ \in\mathcal{B}(E)\ :\ |f|/V\in\mathcal{B}_0(E)\}\quad\text{and}\quad\mathcal{C}_{0,V}(E):=\mathcal{B}_{0,V}(E)\cap\mathcal{C}(E).$$

We refer to section 2.1.2 for more precise definitions.

The next theorem is a direct consequence of a theorem, Theorem 4.3 which is valid for not necessarily absolutely continuous semigroups under weaker conditions.

THEOREM 1.3. *Assume condition $(\mathcal{A})_V$ is met for some $\tau>0$ and some $V\in\mathcal{B}_\infty(E)$. In addition, there exists some function $\Theta_\tau\in\mathcal{B}_0(E)$ such that for any $s<t$ and any positive function $f\in\mathcal{B}_V(E)$ we have*

$$Q_{s,t}(f)\in\mathcal{B}_{0,V}(E)\quad\text{and}\quad Q_{s,s+\tau}(V)/V\leq\Theta_\tau. \tag{1.17}$$

For continuous time semigroups, we also assume that

$$\pi_\tau(V):=\sup\|Q_{t,t+\varepsilon}(V)/V\|<\infty \tag{1.18}$$

where the supremum is taken over all continuous time indices $t\geq 0$ and any $\varepsilon\in[0,\tau]$. In this situation, for any $\mu_1,\mu_2\in\mathcal{P}_V(E)$ there exists a parameter $b>0$ and some finite constant $c(\mu_1,\mu_2)<\infty$ such that for any $s\leq t$ we have the local Lipschitz estimate

$$|||\Phi_{s,t}(\mu_1)-\Phi_{s,t}(\mu_2)|||_V\leq c(\mu_1,\mu_2)\ e^{-b(t-s)}\ |||\mu_1-\mu_2|||_V. \tag{1.19}$$

A more precise description of the parameters $(b,c(\mu_1,\mu_2))$ is provided in Theorem 4.3. To the best of our knowledge, Theorem 1.3 and its extended version, Theorem 4.3, have not been established elsewhere in the literature.

It should be noted that the left-hand side condition in (1.17) ensures that $Q_{s,t}(V)/V\in\mathcal{B}_0(E)$, for any $s<t$. Thus, the condition on the right-hand side in (1.17) provides uniform control of the functions $Q_{s,s+\tau}(V)/V$ with respect to the time parameter $s\geq 0$. We also mention that left-hand side condition in (1.17) is met as soon as $Q_{s,t}(V)/V\in\mathcal{B}_0(E)$ and $Q_{s,t}$ is a strong Feller semigroup, in the sense that for any $s<t$ we have $Q_{s,t}(\mathcal{B}_V(E))\subset\mathcal{C}_V(E)$. In this situation, for any positive function $f\in\mathcal{B}_V(E)$ and $s<t$ the function $Q_{s,t}(f)$ is positive and continuous; and thus locally lower bounded. In this situation, whenever $\|f\|_V\leq 1$, for any $s<t$ we have the comparison property

$$Q_{s,t}(f)/V\leq Q_{s,t}(V)/V\in\mathcal{B}_0(E)\Longrightarrow Q_{s,t}(f)/V\in\mathcal{B}_0(E)\Longleftrightarrow Q_{s,t}(f)\in\mathcal{C}_{0,V}(E).$$

For time homogeneous models $Q_t := Q_{s,s+t}$, the right-hand side condition in (1.17) becomes the Lyapunov condition

$$Q_\tau(V)/V \leq \Theta_\tau \in \mathcal{B}_0(E). \tag{1.20}$$

In the context of time homogeneous models, the strong Feller condition also ensure that the ground state eigen-function is continuous (cf. (4.28)).

When $Q_{s,t} = P_{s,t}$ is a Markovian semigroup $P_{s,t}$ on $\mathcal{B}_V(E)$, the semigroup $\Phi_{s,t}(\mu) = \mu P_{s,t}$ is linear, and the right-hand side Lyapunov condition in (1.17) ensures that $\mu P_{s,t}(V)$ is uniformly bounded with respect to the time parameters $s \leq t$ (cf. Lemma 3.2 or the Lyapunov condition (A.4) applied to the unit function $H = 1$). In this context, the constant $c = c(\mu_1, \mu_2)$ in (1.19) does not depend on the pair of measures (μ_1, μ_2), and the estimate (1.19) reduces to the V-norm contraction of time varying Markov semigroups

$$|||\mu_1 P_{s,t} - \mu_2 P_{s,t}|||_V \leq c\, e^{-b(t-s)}\, |||\mu_1 - \mu_2|||_V.$$

For a direct proof of the above estimate based on V-norm contraction properties of Markov semigroups we refer to section 2.3.

For some measure $\mu \in \mathcal{P}_V(E)$ and some positive function $H \in \mathcal{B}_{0,V}(E)$, we define the finite rank (and hence compact) operator

$$f \in \mathcal{B}_V(E) \mapsto T^{\mu,H}_{s,t}(f) := \frac{Q_{s,t}(H)}{\mu_s Q_{s,t}(1)}\, \mu_t(f) \in \mathcal{B}_V(E) \tag{1.21}$$

with the flow of measures $\mu_t = \Phi_{s,t}(\mu_s)$ starting at some $\mu_0 = \mu$. In this notation, we have the following time inhomogenous version of the Krein-Rutman theorem.

COROLLARY 2. *For any $s \leq t$ we have the operator norm exponential decay*

$$\left|\left|\left| \frac{Q_{s,t}}{\mu_s Q_{s,t}(1)} - T^{\mu,H}_{s,t} \right|\right|\right|_V \leq c_H(\mu)\, e^{-b(t-s)}$$

for some finite constant $c_H(\mu)$ that depends on (μ, H) and with the same constant $b > 0$ as in (1.19).

A precise description of the parameters $(b, c_H(\mu))$ is provided in Corollary 7 (applied to $\overline{\eta}_0 = \mu$, see also Lemma 3.2). To the best of our knowledge, the above corollary is the first result of this type for time varying positive semigroups.

We emphasize that for locally compact Polish spaces, our methodology also applies to Feller V-positive semigroups, that is $Q_{s,t}(\mathcal{C}_V(E)) \subset \mathcal{C}_V(E)$ and $Q_{s,t}(V)/V \in \mathcal{C}_0(E)$ for any $s < t$, as soon as the Lyapunov function V can be continuous (also choosing a function $H \in \mathcal{C}_{0,V}(E)$ in the definition of the class of non necessarily absolutely continuous V-positive semigroups introduced in Definition 3.4). As shown in (4.29), for time homogeneous semigroups, in this context the continuity of the ground function is granted by compact convergence arguments.

Now, choosing $V \in \mathcal{C}_\infty(E) := \mathcal{B}_\infty(E) \cap \mathcal{C}(E)$, Theorem 1.3 is also valid when we replace the left-hand side condition in (1.17) by the Feller property $Q^V_{s,t}(\mathcal{C}_b(E)) \subset \mathcal{C}_b(E)$ with $s < t$ of the conjugate semigroup $Q^V_{s,t}(f) := Q_{s,t}(fV)/V$ and the V-positiveness property by the condition $Q^V_{s,t}(f) \in \mathcal{C}_0(E)$ for any positive function $f \in \mathcal{C}_b(E)$. In this context, the right-hand side condition in (1.17) and (1.18) become

$$Q^V_{s,s+\tau}(1) \leq \Theta_\tau \in \mathcal{B}_0(E) \quad \text{and} \quad \pi_\tau(V) := \sup_{\varepsilon[0,\tau[} \|Q^V_{t,t+\varepsilon}(1)\| < \infty.$$

For normed finite spaces $(E, \|\cdot\|)$, the right-hand side Lyapunov condition in (1.17) only relies on the design of a function $x \in E \mapsto \Theta_\tau(x)$ that tends to 0 as $\|x\| \to \infty$. For sub-Markovian semigroups with hard obstacles, the right-hand side Lyapunov condition in (1.17) allows one to consider diffusion processes conditional to non absorption on the boundaries of some open connected domains. For instance, for a time homogeneous semigroup $Q_t(x, dy) = q_t(x, y)\, dy$ with a bounded density q_t on some bounded domain (open connected) $E \subset \mathbb{R}^n$ with Lipschitz boundary ∂E, for any $0 < \delta < 1$ we have

$$V(x) := 1/d(x, \partial E)^{1-\delta} \quad \text{with} \quad d(x, \partial E) := \inf\{\|x - y\| \,:\, y \in \partial E\}$$

$$\Longrightarrow Q_\tau(V)/V \leq \Theta_\tau := c_\tau/V \in \mathcal{B}_0(E) \quad \text{with } c_\tau := \|Q_\tau(V)\| < \infty.$$

For a more concrete example, we also mention that Brownian motion conditioned to non absorption on $E :=]0, 1[$ has a Dirichlet heat kernel $q_t(x, y) > 0$ on the open cell $E^2 =]0, 1[^2$ that satisfies the conditions of Theorem 1.3 with the Lyapunov function $V(x) = 1/\sqrt{x} + 1/\sqrt{1-x}$. For a more detailed discussion on the design of Lyapunov functions V, we refer to the articles [6, 100, 109, 196]

For time homogeneous models, the r.h.s. condition in (1.17) takes the form $Q_\tau(V)/V \leq \Theta_\tau \in \mathcal{B}_0(E)$. In terms of the compact sets $\mathcal{K}_\varepsilon := \{\Theta_\tau \geq \varepsilon\}$ this condition ensures that for any $\varepsilon > 0$ we have the Foster-Lyapunov inequality

$$Q_\tau(V)(x) \leq \varepsilon\, V(x) + 1_{\mathcal{K}_\varepsilon}(x)\, c_\varepsilon \quad \text{with} \quad c_\varepsilon := \sup_{\mathcal{K}_\varepsilon}(V\Theta_\tau) < \infty. \tag{1.22}$$

We can also slightly relax the above by assuming that for any $n \geq 1$ we have

$$Q_\tau(V)(x) \leq \varepsilon_n\, V(x) + 1_{\mathcal{K}_{\varepsilon_n}}(x)\, c_{\varepsilon_n} \tag{1.23}$$

where $\mathcal{K}_{\varepsilon_n} \subset E$ stands for some increasing sequence of compacts sets and c_{ε_n} some finite constants, indexed by a decreasing sequence of parameters $\varepsilon_n \in [0, 1]$ such that $\varepsilon_n \longrightarrow_{n\to\infty} 0$. Assuming that $Q_\tau(V)/V$ is locally lower bounded and lower semicontinuous, condition (1.23) ensures that $Q_\tau(V)/V \in \mathcal{B}_0(E)$. Indeed, for any $\delta > 0$, there exists some $n \geq 1$ such that $\varepsilon_n < \delta$ and we have

$$\{Q_\tau(V)/V \geq \delta\} \subset \{Q_\tau(V)/V > \varepsilon_n\} \subset \mathcal{K}_{\varepsilon_n}$$

Since $\{Q_\tau(V)/V \geq \delta\}$ is a closed subset of a compact set it is also compact. Whenever the sequence of compact sets K_{ε_n} is exhausting, in the sense that for any compact subset $K \subset E$ there exists some $n \geq 1$ such that $K \subset K_{\varepsilon_n}$, we have

$$\inf_K Q_\tau(V)/V \geq \inf_{K_{\varepsilon_n}} Q_\tau(V)/V \geq \varepsilon_n$$

This condition ensures that the function $Q_\tau(V)/V$ is locally lower bounded. In this situation, we have $Q_\tau(V)/V \in \mathcal{B}_0(E)$ as soon as $Q_\tau(V)/V$ is lower semicontinuous.

For time varying semigroups, working with conditions of the form (1.23) we also need to ensure that the same sequence of compacts can be used at every time horizon. Last, but not least, sub-Markov semigroup associated with hard-obstacles may be defined on some domain with a complex topological structure. In this context, the design of the Lyapunov function and the compact subsequence (1.23) depends on the stability properties of the free evolution as well as on the topological structure of the domain.

We remark that our methodology also applies to non necessarily absolutely continuous semigroups. These models, termed stable V-positive semigroups, are defined in section 3.4. They relies on the contraction properties of the triangular array of Markov operators discussed above on the compact level sets of the function V (cf. for instance condition (3.15)). In lemma 3.2 (mainly due to N. Whiteley in [196, Proposition 1 & Lemma 10]) we shall

see that the semigroups considered in Theorem 1.3 are particular classes of stable V-positive semigroups.

In section 4 we present a new unifying methodology that combines Dobrushin ergodic coefficient techniques developed in [72, 73, 76, 78, 85] with the powerful Foster-Lyapunov methodologies developed in the articles [15, 100, 109, 150, 196] in the context of Feynman-Kac semigroups and nonlinear filtering. Our approach relies on the contraction analysis of a class of triangular arrays of Markov semigroups introduced in [76, 78, 85] in terms of the V-norm contraction ergodic theory for Markov operators presented in [125] and further developed in a systematic way in the book [89]. A brief review on this subject is provided in section 2.3. For a more thorough discussion on these V-contraction coefficient principles, we refer to in [89, Section 8.2.5].

1.3.3. *Time homogeneous semigroups.* As one might expect, for time homogeneous semigroups, a variety of results that follow almost immediately from the estimates obtained in Theorem 1.3. These results are described in section 4.3 and they include the existence of an unique leading eigen-triple

$$(\rho, \eta_\infty, h) \in (\mathbb{R} \times \mathcal{P}_V(E) \times \mathcal{B}_{0,V}(E)) \quad \text{satisfying (1.12).} \tag{1.24}$$

Choosing $(\mu, H) = (\eta_\infty, h)$ in (1.21), the operator $T_{s,t}^{\eta_\infty,h}$ simplifies to the operator T introduced in (1.13); that is, for any $f \in \mathcal{B}_V(E)$ we have

$$T_{s,t}^{\eta_\infty,h}(f) = \frac{h}{\eta_\infty(h)}\, \eta_\infty(f) = T(f) \in \mathcal{B}_{0,V}(E).$$

In addition to the fact that the ground state function h discussed above is generally unknown, whenever its existence is assured, the stability of a non necessarily absolutely continuous positive semigroup Q_t is reduced to the one of a more conventional Markov semigroup defined by the h-transform of Q_t given by

$$f \in \mathcal{B}_{V^h}(E) \mapsto P_t^h(f) := Q_t(fh)/Q_t(h) \in \mathcal{B}_{V^h}(E) \quad \text{with} \quad V^h := V/h. \tag{1.25}$$

The condition $Q_\tau(V)/V \in \mathcal{B}_0(E)$ ensures that $V^h \in \mathcal{B}_\infty(E)$ and there exists some $0 < \varepsilon < 1$ and some constant $c > 0$ such that

$$P_\tau^h(V^h) \leq \varepsilon\, V^h + c. \tag{1.26}$$

A proof of the above assertion is provided in Lemma 6.2. We now introduce another condition.

$(\mathcal{H}^h)$ *There exists some* $r_0 > 0$ *and some* $\alpha : r \in [r_0, \infty[\mapsto \alpha(r) \in]0,1]$ *such that for any* $r \geq r_0$ *we have*

$$\sup_{V^h(x) \vee V^h(y) \leq r} \left\| \delta_x P_\tau^h - \delta_y P_\tau^h \right\|_{tv} \leq 1 - \alpha(r). \tag{1.27}$$

Using the Lyapunov inequality (1.26) and the local contraction estimate (1.27) the stability properties of P_t^h follows the conventional V-norm contraction methodology for Markov semigroups developed in Section 2.3.

For the rest of the section, we assume that Q_t is a (non necessarily absolutely continuous) time homogeneous V-positive semigroup in the sense that $Q_t(\mathcal{B}_V(E)) \subset \mathcal{B}_{0,V}(E)$, for any $t > 0$. In addition, there exists and leading eigen-triple (1.24) with $h(x) > 0$ for any $x \in E$, and the Doob's h-transform satisfies condition $(\mathcal{H}^h)$. In this situation, the finite rank operator T defined in (1.13) maps $\mathcal{B}_V(E)$ into $\mathcal{B}_{0,V}(E)$. As shown in Section 4.3 these conditions are met under our regularity conditions (cf. (4.30), (4.31) and Corollary 9).

The next theorem is a synthesis of Theorem 4.5 and its corollaries, Corollary 11 and Corollary 12, adapting the arguments of (4.15) to V-norms. To the best of our knowledge, the next theorem does not exist in the literature.

THEOREM 1.4. *The Markov semigroup P_t^h has a single invariant measure $\eta_\infty^h \in \mathcal{P}_{V^h}(E)$ and Q_t is a quasi-compact operator on $\mathcal{B}_V(E)$. In addition, there exists some $a < \infty$ and $b > 0$, such that for any $\mu, \eta \in \mathcal{P}_{V^h}(E)$ and $t \in \mathcal{T}$ we have the contraction estimate*

$$\|\mu P_t^h - \eta P_t^h\|_{V^h} \leq a\, e^{-bt}\, \|\mu - \eta\|_{V^h} \qquad \text{and} \qquad \left|\!\left|\!\left| e^{-\rho t}\, Q_t - T \right|\!\right|\!\right|_V \leq a\, e^{-bt}. \tag{1.28}$$

The conjugate measure $\eta_\infty(dx) := 1/h(x)\ \eta_\infty^h(dx)/\eta_\infty^h(1/h) \in \mathcal{P}_V(E)$ is the unique invariant measure of the semigroup Φ_t. For any $\mu_1, \mu_2 \in \mathcal{P}_V(E)$ there also exists some finite constant $c(\mu_1, \mu_2) < \infty$ such that for any $t \in \mathcal{T}$ we have

$$|||\Phi_t(\mu_1) - \Phi_t(\mu_2)|||_V \leq\ c(\mu_1, \mu_2)\, e^{-bt}\ |||\mu_1 - \mu_2|||_V.$$

1.4. *On the design of Lyapunov functions.* Condition (1.23) is often presented in the literature as an initial Lyapunov condition to analyze the stability property of time homogenous sub-Markov semigroups (see for instance [109, 123], as well as section 17.5 in [89] in the context of Markov semigroups and the references therein). In this context, the Lyapunov condition in the right-hand side of (1.20) provides a practical way to design Lyapunov functions satisfying (1.23). This section provides some elementary principles to design these Lyapunov functions.

Assume that $Q_{s,s+\tau}$ is dominated by some auxiliary positive integral operators $\mathcal{Q}_{s,s+\tau}$ in the sense that

$$Q_{s,s+\tau}(x, dy) \leq c_\tau\ \mathcal{Q}_{s,s+\tau}(x, dy) \quad \text{for some finite constant } c_\tau < \infty. \tag{1.29}$$

Then, we readily check that

$$\mathcal{Q}_{s,s+\tau}(V)/V \leq \Theta_\tau \in \mathcal{B}_0(E) \Longrightarrow Q_{s,s+\tau}(V)/V \leq c_\tau \Theta_\tau \in \mathcal{B}_0(E).$$

Given a time homogeneous semigroup $\mathcal{Q}_t$, and some pair of functions $V \in \mathcal{B}_\infty(E)$, and $H \in \mathcal{B}_V(E)$ such that $V^H := V/H \in \mathcal{B}_\infty(E)$ and $\mathcal{Q}_\tau(V)/V \in \mathcal{B}_0(E)$ we have the conjugate principle

$$\begin{aligned} & Q_{s,s+\tau}(x, dy) \leq c_\tau\ \mathcal{Q}_\tau(x, dy) H(y)/H(x) \quad \text{for some } c_\tau < \infty \\ & \Longrightarrow Q_{s,s+\tau}(V^H)/V^H \leq \Theta_\tau := c_\tau\ \mathcal{Q}_\tau(V)/V \in \mathcal{B}_0(E). \end{aligned} \tag{1.30}$$

For instance, a sub-Markovian semigroup Q_t associated with a linear hypoelliptic diffusion on $E := \mathbb{R}^n$ evolving in an absorbing potential that grows at least quadratically is dominated by the sub-Markovian semigroup $\mathcal{Q}_t$ of a coupled harmonic oscillator. The latter of which has an explicit solution with an exponential decay total mass $\mathcal{Q}_\tau(1)(x) \longrightarrow_{\|x\| \to \infty} 0$, with also well known leading triplet and Lyapunov functions, such as the functions $V(x) = 1 + \|x\|^n$, for any given $n \geq 1$. In this context the ground state function $h \in \mathcal{C}_0(E)$ is the centered Gaussian density with a covariance matrix satisfying an algebraic Riccati equation, and the corresponding h-process is an Ornstein-Ulhenbeck diffusion. In this scenario, we recall that $\mathcal{Q}_t$ and the semigroup $\mathcal{P}_t^h$ of the h-process diffusion flow are connected by the formula $\mathcal{Q}_t(f) = e^{\rho t} h\, \mathcal{P}_t^h(f/h)$, for some $\rho < 0$. We refer to [80] for a detailed discussion on coupled harmonic oscillators.

Whenever the domination property (1.29) is met for some time homogeneous semigroup $\mathcal{Q}_\tau := \mathcal{Q}_{s,s+\tau}$, such that $\mathcal{Q}_\tau(1) \in \mathcal{B}_0(E)$, for any $V \in \mathcal{B}_\infty(E)$ have

$$c_\tau := \|\mathcal{P}_\tau(V)/V\| < \infty \quad \text{with the Markov operator} \quad \mathcal{P}_\tau(f) := \mathcal{Q}_\tau(f)/\mathcal{Q}_\tau(1)$$

$$\Longrightarrow Q_{s,s+\tau}(V)/V \leq \Theta_\tau := c_\tau\ \mathcal{Q}_\tau(1) \in \mathcal{B}_0(E).$$

If $c'_\tau := \|\mathcal{P}_\tau(V)\| < \infty$ and $\mathcal{Q}_\tau(1)/V \in \mathcal{B}_0(E)$ we also have

$$Q_{s,s+\tau}(V)/V \leq \Theta_\tau := c'_\tau\ \mathcal{Q}_\tau(1)/V \in \mathcal{B}_0(E).$$

For instance, when $\mathcal{Q}_\tau$ is the semigroup associated with the half-harmonic oscillator on $E =]0,\infty[$, choosing the function $V(x) = x^n + 1/x$ for any given $n \geq 1$, we have $\|\mathcal{Q}_\tau(V)\| < \infty$, for any $\tau > 0$.

The semigroup Q_t of a non absorbed Ornstein-Ulenbeck diffusion flow on $E =]0,\infty[$ killed at the origin can be seen as the h-transform of a solvable sub-Markov semigroup associated with a one dimensional linear diffusion evolving in a quadratic potential well and killed at the origin. In this context, the ground state function h is a centered Gaussian density and we can choose the Lyapunov function $V^h(x) = (x^n + 1/x)/h(x)$, for any given $n \geq 1$. The stability properties of this class of non absorbed one dimensional Ornstein-Ulenbeck diffusions are also discussed in [158] and more recently in [168] in terms of the tangent of the h-process.

Next, we illustrate (1.30) with a one-dimensional Langevin diffusion killed at the origin. Let $X_t(x)$ be the stochastic flow with generator L defined by

$$L(f) = \frac{1}{2}\, e^{2W} \partial \left(e^{-2W} \partial f\right)$$

for some non negative function W. We denote by Q_t the sub-Markovian semigroup associated with the flow $X_t(x)$ starting at $x \in E =]0,\infty[$ killed at the boundary $\partial E = \{0\}$; that is, we have that

$$Q_t(f)(x) := \mathbb{E}(f(X_t(x))\, 1_{T^X_{\partial E}(x) > t}) \quad \text{with} \quad T^X_{\partial E}(x) := \inf\{t \geq 0 \,:\, X_t(x) \in \partial E\}.$$

Consider the sub-Markovian semigroup $\mathcal{Q}_t$ associated with a Brownian flow $B_t(x)$ on $E =]0,\infty[$ killed at the origin and at rate

$$U := \frac{1}{2}\, \left((\partial W)^2 - \partial^2 W\right) = H^{-1} \frac{1}{2}\, \partial^2 H \quad \text{with} \quad H := e^{-W}.$$

When $W(x) = \varsigma\, x^2/2$, we have $U := \varsigma\left(\varsigma\, x^2 - 1\right)/2$ and the semigroup $\mathcal{Q}_t$ coincides with the semigroup of an half-harmonic oscillator for which we know that

$$V(x) := x + 1/x \Longrightarrow c_\tau := \|\mathcal{Q}_\tau(V)\| < \infty.$$

This implies that $V^H \in \mathcal{B}_\infty(E)$ and by a change of probability measure we have

$$Q_\tau(V^H)/V^H = \mathcal{Q}_\tau(V)/V \leq c_\tau/V \in \mathcal{B}_0(E).$$

More generally, assume that W is chosen so that

$$U(x) \geq \varsigma_0 + \varsigma_1\, x^2/2 \quad \text{for some parameters } \varsigma_0 \in \mathbb{R} \text{ and } \varsigma_1 > 0.$$

In this situation $\mathcal{Q}_t$ is now dominated by the semigroup of an half-harmonic oscillator. The special case $\partial W(x) = 1/(2x) - a\, x + b\, x^3$ with $a, b > 0$ yields the logistic diffusion on the half-line discussed in [42] using spectral arguments.

When the dominating operator $\mathcal{Q}_t = \mathcal{P}_t$ is a Markov integral operator, the literature is also abounds with the design of Lyapunov functions for Markov operators $\mathcal{P}_\tau$ which can be used without further work to check the right-hand side condition in (1.17) as soon as $Q_{s,s+\tau}$ is dominated by a Markov operator $\mathcal{P}_\tau$. For instance, when $\mathcal{Q}_t = \mathcal{P}_t$ is the Markov semigroup associated with a Riccati matrix-valued diffusion $X_t(x)$ evolving in the space E of positive definite matrices with real entries, under appropriate controllability and observability conditions, for any $t > 0$ we have $\|\mathcal{P}_t(V)\| < \infty$ with $V(x) := \mathrm{Tr}(x) + 1/\mathrm{Tr}(x)$, where $\mathrm{Tr}(x)$ stands for the trace of a positive definite matrix $x \in E$. A proof of the above assertion is provided in [22]. The same Riccati-type analysis applies to the logistic birth and death processes and the competitive and multivariate Lotka-Volterra birth and death process on $E := \mathbb{N}^n - \{0\}$ discussed in Theorem 1.1 in [51], with the Lyapunov function defined for any $x = (x_i)_{1 \leq i \leq n}$ by $V(x) = \sum_{1 \leq i \leq n} x_i$.

For more discussion on the design of Lyapunov functions V on not necessarily bounded domains, we refer to the articles [6, 100, 109, 196] for more illustrations, as well as to section 2.1.2.

1.5. *Organisation of the article.* The article is structured as follows: In section 2 we introduce the notation that will be used throughout and state our main results. Section 3 is dedicated to the detailed description of the different classes of semigroups considered in the article. The main stability and contraction theorems of the article are described in section 4. In particular, in section 4.2, we present the main results for time-inhomogeneous models, with a more refined analysis of time homogeneous models being given in section 4.3. In section 5, we illustrate the impact of our results with some selected illustrations on nonlinear conditional processes, sub-Markov models and related Feynman-Kac measures on path spaces. Some comparisons between our regularity conditions and the ones used in existing literature are discussed in section 5.4. Section 6 is dedicated to the proofs of the main theorems presented in this article. The appendix houses most of our technical proofs.

2. Preliminary results.

2.1. *Some basic notation.*

2.1.1. *Measure theoretic notation.* We equip the set $\mathcal{M}_b(E)$ of bounded signed measures μ on E with the total variation norm $\|\mu\|_{tv} := |\mu|(E)/2$, where $|\mu| := \mu_+ + \mu_-$ stands for the total variation measure associated with a Hahn-Jordan decomposition $\mu = \mu_+ - \mu_-$ of the measure.

We define the duality map, as well as the right and dual left action of a bounded integral operator Q using the classical measure theoretic notation, as follows:

$$
\begin{aligned}
&(\mu, f) \in (\mathcal{M}_b(E) \times \mathcal{B}_b(E)) \mapsto \mu(f) \in \mathbb{R} \quad \text{with} \quad \mu(f) := \int f(x)\mu(dx) \\
&f \in \mathcal{B}_b(E) \mapsto Q(f) \in \mathcal{B}_b(E) \quad \text{with} \quad Q(f)(x) := \int Q(x, dy)\ f(y) \\
&\mu \in \mathcal{M}_b(E) \mapsto \mu Q \in \mathcal{M}_b(E) \quad \text{with} \quad \mu Q(dy) := \int \mu(dx)\ Q(x, dy).
\end{aligned} \tag{2.1}
$$

Given a pair of integral operators (Q_1, Q_2), we denote by Q_1Q_2 the integral composition operator defined by

$$(Q_1Q_2)(x, dz) = \int Q_1(x, dy)\ Q_2(y, dz).$$

For any $n \in \mathbb{N}$ we also write $Q^n = Q^{n-1}Q$ with the convention $Q^0 = I$ the identity operator. We denote by I the identity integral operator. We say that Q is positive if $Q(f) \geq 0$ whenever $f \geq 0$. Whenever $Q(1) \leq 1$ we say that Q is sub-Markovian, and Q is said to be Markovian when $Q(1) = 1$. The Boltzmann-Gibbs transformation Ψ_h associated with some bounded positive function $h > 0$ and defined by

$$\Psi_h \ :\ \mu \in \mathcal{P}(E) \mapsto \Psi_h(\mu) \in \mathcal{P}(E) \quad \text{with} \quad \Psi_h(\mu)(dx) := \frac{h(x)\ \mu(dx)}{\mu(h)}. \tag{2.2}$$

We recall the Lipschitz estimate

$$\|\Psi_h(\mu_1) - \Psi_h(\mu_2)\|_{tv} \leq \frac{\|h\|}{\mu_1(h) \vee \mu_2(h)}\ \|\mu_1 - \mu_2\|_{tv}.$$

For a detailed proof of the above assertion we refer to [134, lemma 9.5], or [92, appendix B], see also [78] as well as [156, proposition 3.1] and [73, proposition 12.1.7].

When $f = 1_A$ is the indicator function of some measurable subset $A \subset E$, we will sometimes slightly abuse notation and write $\mu(A)$ instead of $\mu(1_A)$. We also set $a \wedge b = \min(a,b)$ and $a \vee b = \max(a,b)$, for $a, b \in \mathbb{R}$ and we use the conventions

$$\left(\sum_{\emptyset}, \prod_{\emptyset}\right) = (0,1) \quad \text{and} \quad \left(\sup_{\emptyset}, \inf_{\emptyset}\right) = (-\infty, +\infty).$$

2.1.2. *Functional analysis notation.* We let $\mathcal{B}_\infty(E) \subset \mathcal{B}(E)$ the subalgebra of locally bounded and uniformly positive functions V that grow at infinity; that is, $\sup_K V < \infty$ for any compact set $K \subset E$, and for any $r \geq V_\star := \inf_E V > 0$ the r-sub-level set $\{V \leq r\} \subset E$ is a non empty compact subset. For instance the function $V(x) := x + 1/x$ when $E =]0, \infty[$ belongs to $\mathcal{B}_\infty(E)$. Note that the compactness level set condition ensures that V is necessarily lower-semicontinuous (abbreviated l.s.c.) and its infimum on every compact set is attained.

We check that $\mathcal{B}_\infty(E)$ is an algebra by recalling that the product $V = V_1 V_2$ of non negative l.s.c functions $V_1, V_2 \in \mathcal{B}_\infty(E)$ is also l.s.c. so that the r-sub-level set $\{V \leq r\}$ is closed. In addition, it is included in the union of the compact $\sqrt{r}$-sub-level sets of the functions V_1 and V_2; that is

$$\{V \leq r\} \subset \{V_1 \leq \sqrt{r}\} \cup \{V_2 \leq \sqrt{r}\}.$$

Observe that for any locally bounded l.s.c. function V_1 we have

$$V_1 \geq V_2 \in \mathcal{B}_\infty(E) \Longrightarrow V_1 \in \mathcal{B}_\infty(E).$$

Since V is locally bounded, any compact set $K \subset E$ is included in some sub-level set of V. Indeed, choosing $r_K := \sup_K V$ we have

$$K \subset \{V \leq r_K\}. \tag{2.3}$$

Thus, for any pair of functions $V_1, V_2 \in \mathcal{B}_\infty(E)$ and for any $r_1 \geq V_{1,\star}$ there exists some parameters $r_2 \geq V_{2,\star}$ and $r_3 \geq V_{1,\star}$ such that

$$\{V_1 \leq r_1\} \subset \{V_2 \leq r_2\} \subset \{V_1 \leq r_3\}.$$

Let $\mathcal{B}_0(E) \subset \mathcal{B}_b(E)$ the subalgebra of bounded positive functions h locally lower bounded that vanish at infinity; that is, $\inf_K h > 0$ for any compact set $K \subset E$ and for any $0 < \varepsilon \leq \|h\| < \infty$ the ε-super-level set $\{h \geq \varepsilon\} \subset E$ is a non empty compact subset. Observe that

$$V \in \mathcal{B}_\infty(E) \quad \Longleftrightarrow \quad 1/V \in \mathcal{B}_0(E).$$

In addition, for any locally lower bounded u.s.c. function h_1 we have

$$h_1 \leq h_2 \in \mathcal{B}_0(E) \Longrightarrow h_1 \in \mathcal{B}_0(E).$$

Also notice that ε-super-level set of a non necessarily u.s.c. $h_1 \leq h_2 \in \mathcal{B}_0(E)$ is included in the compact set $\{h_2 \geq \varepsilon\}$.

Finally note that the sub-algebras $\mathcal{B}_\infty(E)$ and $\mathcal{B}_0(E)$ have no unit unless E is compact and the null function $0 \notin \mathcal{B}_0(E)$, nevertheless the unit function $1 \in \mathcal{C}_{0,V}(E)$ for any $V \in \mathcal{B}_\infty(E)$.

REMARK 2.1. *When E is a locally compact space, its topology coincides with the weak topology induced by $\mathcal{C}_0(E) := \mathcal{B}_0(E) \cap \mathcal{C}_b(E)$, and inversely (cf. Proposition 2.1 in [3]). In this context a continuous function h vanishes at infinity if and only if its extension to the one point compactification (a.k.a. Alexandroff compactification) $E_\infty := E \cup \{\infty\}$ (obtained by setting $h(\infty) = 0$) is continuous. For locally compact spaces, we recall that the one point extension E_∞ is compact.*

REMARK 2.2. *When $(E,\|\cdot\|)$ is a finite dimensional normed spaces, by (2.3) for any positive locally lower bounded u.s.c. function h on E, and any $\varepsilon>0$ there exists some $r>0$ such that*

$$\overline{\mathbb{B}}(r):=\{x\in E\ :\ \|x\|\leq r\}\subset\{h\geq\varepsilon\}\quad\text{so that}\quad h\in\mathcal{B}_0(E)\Longleftrightarrow\lim_{\|x\|\to\infty}h(x)=0.$$

In this context, we have

$$0\leq h_1\leq h_2\in\mathcal{B}_0(E)\Longrightarrow\forall\varepsilon>0\quad\exists r>0\quad\text{s.t.}\quad\forall x\notin\overline{\mathbb{B}}(r)\qquad h_1(x)\leq\varepsilon.$$

We let $\mathcal{C}(E)\subset\mathcal{B}(E)$ the sub-algebra of continuous functions. For a given $V\in\mathcal{B}_\infty(E)$, we let $\mathcal{B}_V(E)\subset\mathcal{B}(E)$ be the Banach space of functions $f\in\mathcal{B}(E)$ with $\|f\|_V:=\|f/V\|<\infty$; and by $\mathcal{C}_V(E)\subset\mathcal{B}_V(E)$ be the subset of continuous functions. We also denote by $|\!|\!|Q|\!|\!|_V$ the operator norm of a bounded linear operator $Q:f\in\mathcal{B}_V(E)\mapsto Q(f)\in\mathcal{B}_V(E)$; that is

$$|\!|\!|Q|\!|\!|_V:=\sup\{\|Q(f)\|_V\ :\ f\in\mathcal{B}_V(E)\quad\text{such that}\quad\|f\|_V\leq1\}.\tag{2.4}$$

We also denote by $\mathcal{M}(E)$ the set of signed Radon measures on E and by $\mathcal{M}_V(E)\subset\mathcal{M}(E)$ the subset of measures $\mu\in\mathcal{M}(E)$ such that $|\mu|(V)<\infty$.

For a given function $V\geq1/2$, the V-oscillation of a function $f\in\mathcal{B}(E)$ is given by

$$\mathrm{osc}_V(f):=\sup_{x,y}\frac{|f(x)-f(y)|}{V(x)+V(y)}\leq\|f\|_V$$

and with a slight abuse of notation, the V-norm on $\mathcal{M}(E)$ is given by

$$|\!|\!|\mu|\!|\!|_V=\sup\{|\mu(f)|\ :\ \|f\|_V\leq1\}=\sup\{|\mu(f)|\ :\ \mathrm{osc}_V(f)\leq1\}=|\mu|(V).$$

For a detailed proof of the equivalent formulations in the latter definition, we refer to Proposition 8.2.16 in [89]. The choice of condition $V\geq1/2$ in the above two definitions is imposed only to recover the conventional total variation dual distance between probability measures when choosing $V=1/2$ in the dual norms.

When $V=1/2$ we recover the conventional total variation norm, that is for any $\mu_1,\mu_2\in\mathcal{P}(E)$ we have

$$|\!|\!|\mu_1-\mu_2|\!|\!|_{1/2}=\|\mu_1-\mu_2\|_{tv}\tag{2.5}$$

we recall that

$$\|\mu_1-\mu_2\|_{tv}\leq1-\varepsilon\Longleftrightarrow(\exists\nu\in\mathcal{P}(E)\ :\ \mu_1\geq\varepsilon\,\nu\quad\text{and}\quad\mu_2\geq\varepsilon\,\nu)\,.\tag{2.6}$$

REMARK 2.3. *Whenever the Polish space E is locally compact metric space the integral map $(\mu,f)\mapsto\mu(f)$ gives the isometry*

$$\mathcal{M}_V(E)\simeq\mathcal{C}_{0,V}(E)'.$$

In this context, the set of Radon measures reduces to the set of locally finite Borel regular measures. The above assertion is a direct consequence of [187, Theorem 3.1] applied to the Nachbin family $\mathcal{V}:=\{V_\alpha=\alpha/V\ :\ \alpha>0\}$. See also [186, Theorem 3.26] as well as [29, Theorem 2.1].

Finally, we present a technical lemma regarding the $V-$norm estimates of the Boltzmann-Gibbs operators; the proof is given in appendix A.3.

LEMMA 2.1. *For any $V \in \mathcal{B}_\infty(E)$ and $0 < h \in \mathcal{B}_{0,V}(E)$ and any $\mu_1, \mu_2 \in \mathcal{P}_{V/h}(E)$ we have the estimate*

$$\left|\left|\left|\Psi_{1/h}(\mu_1) - \Psi_{1/h}(\mu_2)\right|\right|\right|_V \leq \frac{1}{\mu_1(1/h)} \left(1 + \frac{\mu_2(V/h)}{\mu_2(1/h)}\right) \left|\left|\left|\mu_1 - \mu_2\right|\right|\right|_{V/h}. \tag{2.7}$$

In addition, for any $\mu_1, \mu_2 \in \mathcal{P}_V(E)$ we have

$$\left|\left|\left|\Psi_h(\mu_1) - \Psi_h(\mu_2)\right|\right|\right|_{V/h} \leq \frac{1}{\mu_1(h)} \left(1 + \frac{\mu_2(V)}{\mu_2(h)}\right) \left|\left|\left|\mu_1 - \mu_2\right|\right|\right|_V. \tag{2.8}$$

2.2. *Discrete and continuous time models.* For a given $s \in \mathcal{T}$ and $\tau \in \mathcal{T}$ with $\tau > 0$, we consider the time mesh

$$[s,t]_\tau := \{s + n\tau \in [s,t] \,:\, n \in \mathbb{N}\} \quad \text{and} \quad [s,\infty[_\tau := \{s + n\tau \in [s,\infty[\,:\, n \in \mathbb{N}\}.$$

We define $[s,t[_\tau$, $]s,t]_\tau$ and $]s,t[_\tau$ by replacing respectively in the above display $[s,t]$ by $[s,t[$, $]s,t]$ and $]s,t[$. For continuous time indices $\mathcal{T} = \mathbb{R}_+$, we shall denote by $\lfloor t/\tau \rfloor$ the integer part of t/τ and by $\{t/\tau\}$ the fractional part so that $t = \lfloor t/\tau \rfloor \tau + \{t/\tau\}\tau$. For discrete time indices $\mathcal{T} = \mathbb{N}$, choosing $\tau = 1$ we have $[0,\infty[_\tau = \mathbb{N}$.

We have assumed that $Q_{t,t+\tau}(1) > 0$ for any $\tau > 0$ and any $t \in \mathcal{T}$. This irreducibility condition ensures that $Q_{t,t+\tau}(f) > 0$ for any $f > 0$. For sub-Markovian time homogeneous semigroups $Q_{t,t+\tau} = Q_{0,\tau}$ this condition can be relaxed by considering the Borel set $\{Q_{0,\tau}(1) = 0\}$ as a part of an absorbing set. In this context, to analyse the behavior of a non absorbed particle there is no loss of generality in assuming that $Q_{0,\tau}(1) > 0$.

In the discrete time setting, this condition can also be relaxed up to a time-rescaling or up to some modification of the state space; see for instance the construction in [72, section 4.4], as well as the one dimensional neutron transport model discussed in [72, Example 4.4.3] and the soft and hard obstacle models discussed in [74, 82].

Nevertheless, we emphasise that the analysis of continuous time models is sometimes based on a discrete time approach based on regularity conditions that depend on some parameter τ chosen by the user. In this context, the analysis is performed at the level of the τ-discretised model and the estimation constants presented in our results may depend on the parameter τ. These estimation constants are defined and discussed in some details below.

DEFINITION 2.1. *For a given a function $W \geq 0$ and some probability measure η on E and some parameter $\tau > 0$ we shall denote by $\kappa^-_{\tau,W}(\eta), \lambda^-_\tau(\eta) \in [0,+\infty[$ and $\kappa_{\tau,W}(\eta), \lambda_\tau(\eta) \in [0,+\infty]$ the parameters*

$$\begin{aligned} \kappa^-_{\tau,W}(\eta) &:= \inf \Phi_{s,t}(\eta)(W) &&\leq \quad \kappa_{\tau,W}(\eta) := \sup \Phi_{s,t}(\eta)(W) \\ \lambda^-_\tau(\eta) &:= \inf \Phi_{s,t}(\eta)(Q_{t,t+\varepsilon}(1)) &&\leq \quad \lambda_\tau(\eta) \quad := \sup \Phi_{s,t}(\eta)(Q_{t,t+\varepsilon}(1)). \end{aligned} \tag{2.9}$$

In the above display, the infimum and the supremum are taken over all time indices $s \in \mathcal{T}$ and $t \in [s,\infty[_\tau$ and $\varepsilon = \tau$. With a slight abuse of notation, we also denote by $\kappa^-_W(\eta), \lambda^-(\eta), \kappa_W(\eta), \lambda(\eta)$, the parameters defined as above by taking the infimum and the supremum are taken over all time indices $s \in \mathcal{T}$ and $s \leq t \in \mathcal{T}$ and $\varepsilon \in \mathcal{T} \cap [0,\tau]$.

For discrete time models, we have assumed that $\tau = 1$ so that $(\kappa^-_{\tau,W}, \kappa_{\tau,W}) = (\kappa^-_W, \kappa_W)$ and $(\lambda^-_\tau(\eta), \lambda_\tau(\eta)) = (\lambda^-(\eta), \lambda(\eta))$. For continuous time models, the next lemma provides conditions under which the infimum and the supremum are taken over all continuous time time indices $s \geq 0$ and $t \geq s$.

LEMMA 2.2. *For continuous time models, assume $\kappa^-_{\tau,H}(\mu) > 0$ and $\kappa_{\tau,V}(\mu) < \infty$, and condition (1.16) are met for some $\tau > 0$, $\mu \in \mathcal{P}_V(E)$, $V \in \mathcal{B}_\infty(E)$ and some locally lower bounded positive function H on E. In this situation, we have $\kappa^-_H(\mu) > 0$ and $\kappa_V(\mu) < \infty$ as well as*

$$0 < \lambda^-(\mu) \leq \lambda(\mu) < \infty \quad \text{and} \quad \sup \frac{\|Q_{t,t+\varepsilon}(1)\|}{\Phi_{s,t}(\mu)Q_{t,t+\varepsilon}(1)} < \infty \tag{2.10}$$

where the infimum and the supremum are taken over all continuous time indices $s \in \mathcal{T}$ and $t \geq s$ and $\varepsilon \in \mathcal{T} \cap [0, \tau]$.

The proof of the above technical lemma is provided in the appendix on page 50. We end this section with a brief discussion on unnormalised and the normalised semigroups $Q_{s,t}$ and $\Phi_{s,t}$. These linear and nonlinear semigroups are connected for any $\mu \in \mathcal{P}(E)$, $f \in \mathcal{B}_b(E)$ and $s \in \mathcal{T}$ and $t \in [s, \infty[_\tau$ by the formula

$$\mu Q_{s,t}(f) = \Phi_{s,t}(\mu)(f) \prod_{u \in [s,t[_\tau} \Phi_{s,u}(\mu)(Q_{u,u+\tau}(1)). \tag{2.11}$$

The above formula coincides with the product formula relating the unnormalised operators $Q_{s,t}$ with the normalised semigroup $\Phi_{s,t}$ discussed in [76, section 1.3.2], see also [72, proposition 2.3.1] and [73, section 12.2.1]. For time homogeneous models, the product formula (2.11) applied to $f = 1$ reduces for any $t \in \mathcal{T}$ to

$$\mu(Q_t(1)) = \Phi_{\lfloor t/\tau \rfloor \tau}(\mu)\left(Q_{\{t/\tau\}\tau}(1)\right) \prod_{0 \leq n < \lfloor t/\tau \rfloor} \Phi_{n\tau}(\mu)(Q_\tau(1)). \tag{2.12}$$

Rewritten in a slightly different form, we have

$$\mu(Q_t(1)) = \frac{1}{\Phi_t(\mu)\left(Q_{(1-\{t/\tau\})\tau}(1)\right)} \prod_{0 \leq n \leq \lfloor t/\tau \rfloor} \Phi_{n\tau}(\mu)(Q_\tau(1)).$$

2.3. *V-Dobrushin coefficient.* Let us fix $0 \leq s \leq t$ and let V_s, V_t denote a couple of measurable functions such that $V_s, V_t \geq 1$. The V-Dobrushin coefficient of a Markov integral operator $P_{s,t}$ (non necessary a semigroup) from $\mathcal{B}_{V_t}(E)$ into $\mathcal{B}_{V_s}(E)$ is the norm operator defined by

$$\beta_{V_s,V_t}(P_{s,t}) = \sup_{\mu,\eta \in \mathcal{P}_{V_s}(E)} |||(\mu - \eta)P_{s,t}|||_{V_t} / |||\mu - \eta|||_{V_s}. \tag{2.13}$$

We also have the equivalent formulation

$$\begin{aligned} \beta_{V_s,V_t}(P_{s,t}) &= \sup\left\{\mathrm{osc}_{V_s}(P_{s,t}(f)) \,:\, \mathrm{osc}_{V_t}(f) \leq 1\right\} \\ &= \sup_{(x,y) \in E^2} |||\delta_x P_{s,t} - \delta_y P_{s,t}|||_{V_t} / (V_s(x) + V_s(y)). \end{aligned} \tag{2.14}$$

When $V_s = V_t = V$, we write $\beta_V(P_{s,t})$ instead of $\beta_{V,V}(P_{s,t})$. For a more thorough discussion on these contraction coefficients, we refer the reader to sections 8.2.5 - 8.2.7 in [89]. If in addition $V = 1/2$ we write $\beta(P_{s,t})$ instead of $\beta_{1/2}(P_{s,t})$, to denote the conventional Dobrushin ergodic coefficient with respect to the total variation distance

$$\beta(P_{s,t}) = \sup_{(x,y) \in E^2} \|\delta_x P_{s,t} - \delta_y P_{s,t}\|_{tv}.$$

By (2.6) the contraction condition $\beta(P_{s,t}) \leq (1 - \alpha)$ is satisfied for some parameter $\alpha \in]0, 1]$ if and only if the following minorisation condition holds

$$\forall (x,y) \in E^2 \quad \exists \nu \in \mathcal{P}(E) \,:\, \delta_x P_{s,t} \geq \alpha\, \nu \quad \text{and} \quad \delta_y P_{s,t} \geq \alpha\, \nu.$$

The next lemma provides some contraction conditions in terms of a Foster-Lyapunov inequality and a local minorisation condition on compact level sets. The proof is in appendix A.4.

LEMMA 2.3. *Assume that there exist locally bounded functions $V_s, V_t \geq 1$ with compact level sets, $\varepsilon \in [0,1[$ as well as a function $\alpha : r \in [r_0, \infty[\mapsto \alpha(r) \in]0,1]$, for some $r_0 \geq 1$ such that for any $r \geq r_0$ we have*

$$P_{s,t}(V_t) \leq \varepsilon\, V_s + 1 \quad \text{and} \quad \sup_{V_s(x) \vee V_s(y) \leq r} \|\delta_x P_{s,t} - \delta_y P_{s,t}\|_{tv} \leq 1 - \alpha(r). \tag{2.15}$$

Then, for any $0 < \delta \leq 1$ we have the contraction coefficient estimate

$$\beta_{V_s^{\varepsilon,\delta}, V_t^{\varepsilon,\delta}}(P_{s,t}) \leq 1 - \alpha_\delta(\varepsilon) \quad \text{with} \quad \alpha_\delta(\varepsilon) := \alpha(r_\varepsilon(\delta))(1 - \delta/2) \in]0,1[. \tag{2.16}$$

In the above display, $V_u^{\varepsilon,\delta}$ with $u \in \{s,t\}$ stands for the functions defined by

$$V_u^{\varepsilon,\delta} := 1/2 + \rho_\varepsilon(\delta)\, V_u \quad \text{and} \quad \rho_\varepsilon(\delta) := \frac{\delta}{1+\varepsilon} \frac{\alpha(r_\varepsilon(\delta))}{2 r_\varepsilon(\delta)} \tag{2.17}$$

with the level set parameter $r_\varepsilon(\delta)$ defined by

$$r_\varepsilon(\delta) := r_0 \vee \frac{2}{1-\varepsilon}\left(1 + \frac{1}{1-\varepsilon}\left(3 + 6\left(\frac{1}{\delta} - 1\right)\right)\right).$$

Under the assumptions of lemma 2.3, using (2.13) for any $\mu, \eta \in \mathcal{P}_{V_s}(E)$ we readily check the contraction estimate

$$|||\mu P_{s,t} - \eta P_{s,t}|||_{V_t^{\varepsilon,\delta}} \leq (1 - \alpha_\delta(\varepsilon))\ |||\mu - \eta|||_{V_s^{\varepsilon,\delta}}. \tag{2.18}$$

with the parameter $\alpha_\delta(\varepsilon)$ defined in (2.16).

REMARK 2.4. *Observe that*

$$r_0(\delta) := r_0 \vee 4\left(2 + 3\left(\frac{1}{\delta} - 1\right)\right) > 1 \text{ and } \rho_0(\delta) := \frac{\delta}{2 r_0(\delta)}\ \alpha(r_0(\delta)) \longrightarrow_{\delta \to 0} 0.$$

Thus, whenever the estimates (2.15) are met with the unit function $V_s = V_t = r_0 = 1$, choosing $\varepsilon = 0$ in (2.15) we have, for any $r > 1$,

$$\lim_{\delta \to 0} \beta_{V_s^{0,\delta}, V_t^{0,\delta}}(P_{s,t}) = \beta(P_{s,t}) \leq 1 - \alpha(r).$$

Whenever $P_{s,t}$ is a Markov semigroup the Foster-Lyapunov inequality, on the left-hand side of (2.15), applied to some time homogeneous function $V_t = V$ ensures that for any initial probability measure $\mu \in \mathcal{P}_V(E)$, the distribution flow $\mu_t := \mu P_{0,t}$ indexed by $t \in [0, \infty[_\tau$ for some $\tau > 0$ is tight. To check this claim, notice that

$$(2.15) \Longrightarrow \mu_t(V) \leq \varepsilon^{t/\tau} \mu(V) + (1-\varepsilon)^{-1} \leq \mu(V) + (1-\varepsilon)^{-1}. \tag{2.19}$$

By the Markov inequality, for any $\overline{\varepsilon} \in]0,1[$ this implies that

$$K_{\overline{\varepsilon}} := \{x \in E : \overline{\varepsilon} V(x) \leq \mu(V) + (1-\varepsilon)^{-1}\} \Longrightarrow \sup_{t \in [0,\infty[_\tau} \mu_t\left(E - K_{\overline{\varepsilon}}\right) \leq \overline{\varepsilon}. \tag{2.20}$$

For continuous time models, for any $\tau > 0$ we assume that

$$\pi_\tau(V) := \sup_{s \geq 0} \sup_{\delta \in [0,\tau[} \|P_{s,s+\delta}(V)/V\| < \infty \quad \Longrightarrow \quad \sup_{t \geq 0} \mu P_{0,t}(V) < \infty. \tag{2.21}$$

We check the right-hand side assertion in the above display using for any $t = [0, \tau[$ the estimate

$$\mu P_{0,n\tau+t}(V) = \mu P_{0,n\tau}(P_{n\tau,n\tau+t}V) \leq \pi_\tau(V)\ \mu P_{0,n\tau}(V).$$

We further assume $P_{s,t}$ is a continuous time semigroup and (2.15) is met for any $s \geq 0$ and $t = s + \tau$, for some parameter $\varepsilon \in [0, 1[$, some function $\alpha : r \in [r_0, \infty[\mapsto \alpha(r) \in]0, 1]$, and some function $V = V_s = V_t$ that may depends on some given parameter $\tau > 0$. In this situation, iterating (2.18) for any $n \geq 0$ and any $\mu, \eta \in \mathcal{P}_V(E)$ we have

$$|||\mu P_{s,s+n\tau} - \eta P_{s,s+n\tau}|||_{V^{\varepsilon,\delta}} \leq (1 - \alpha_\delta(\varepsilon)))^n\ |||\mu - \eta|||_{V^{\varepsilon,\delta}} \tag{2.22}$$

with the function

$$V^{\varepsilon,\delta} := 1/2 + \rho_\varepsilon(\delta)\ V.$$

Observe that for any $s \geq 0$ and $t = [0, \tau[$ we have

$$|||\mu P_{s,s+t} - \eta P_{s,s+t}|||_{V^{\varepsilon,\delta}} \leq (1 \vee \pi_\tau(V))\ |\mu - \eta|(V^{\varepsilon,\delta}) = (1 \vee \pi_\tau(V))\ |||\mu - \eta|||_{V^{\varepsilon,\delta}}$$

with the parameter $\pi_\tau(V)$ defined in (2.21). This yields the estimate

$$|||\mu P_{s,s+t} - \eta P_{s,s+t}|||_{V^{\varepsilon,\delta}} \leq (1 \vee \pi_\tau(V))\ (1 - \alpha_\delta(\varepsilon))^{\lfloor t/\tau \rfloor}\ |||\mu - \eta|||_{V^{\varepsilon,\delta}}.$$

For time homogeneous semigroups $P_t := P_{s,s+t}$ the above estimate ensures the existence of a single invariant probability measure $\mu_\infty = \mu_\infty P_t \in \mathcal{P}_V(E)$. The analysis discussed above combines the ergodic theory for Markov operators presented in [125] with the V-Dobrushin contraction methodology developed in [89]. Alternative contraction approaches mainly based on [125] are discussed in [15, 50, 160, 163].

2.4. *Quasi-compact operators.* Next, we recall some standard definitions and compactness principles from time homogeneous positive semigroup theory. Let $(Q_t)_{t\geq 0}$ be a time homogenous positive semigroup from $\mathcal{B}_V(E)$ into $\mathcal{B}_V(E)$. Then Q_t is said to be irreducible (a.k.a. ideal irreducible) if there exists no closed Q_t-invariant ideals distinct from $\{0\}$ and $\mathcal{B}_V(E)$ (cf. [172, Definition 4.2.1]). It is also well known (e.g. [101, Proposition 2.1] or [112, Proposition 4.1]) that this condition is met if and only if for any non zero function $f \geq 0$ on $\mathcal{B}_V(E)$ and any non zero positive measure μ on E, there exists some $t \in \mathcal{T}$ such that $\mu Q_t(f) > 0$.

The spectral radius of $(Q_t)_{t\geq 0}$ is defined as

$$r_V(Q) := \lim_{t\to\infty} |||Q_t|||_V^{1/t} = \lim_{t\to\infty} \|Q_t(V)\|_V^{1/t} = \inf_{t\geq 0} \|Q_t(V)\|_V^{1/t}. \tag{2.23}$$

The semigroup Q_t is said to be quasi-compact if its essential spectral radius

$$\overline{r}_V(Q) := \lim_{t\to\infty} (\inf\{|||Q_t - T|||_V\ :\ T \quad \text{compact}\})^{1/t}$$

satisfies $\overline{r}_V(Q) < r_V(Q)$. Recalling that the product of positive operators $Q_1 Q_2$ is compact as soon as the Q_1 is compact (cf. [175, Theorem VI.12]), the quasi-compactness property of the operator Q_t (for sufficiently large t) is clearly met as soon Q_τ is a compact operator for some $\tau \in \mathcal{T}$. Such one-parameter semigroups are sometimes called eventually compact semigroups (see [103, Section 3]).

Now assume there exists some $\tau \in \mathcal{T}$ such that the discrete generation semigroup Q_t indexed by $t \in [0, \infty[_\tau$ is irreducible and quasi-compact on $\mathcal{B}_V(E)$. By a variant of the Krein-Rutman theorem (cf. for instance Theorem 1.1 in [102, 166]), $r_V(Q) > 0$ is an eigenvalue

corresponding to a non null eigenfunction $h \in \mathcal{B}_V(E)$. More precisely, there exists some a non null function $h \in \mathcal{B}_V(E)$ such that

$$\forall t \in [0,\infty[_\tau \qquad Q_t(h) = e^{\rho t}\, h \quad \text{with} \quad \rho := \log r_V(Q).$$

Note that since Q_τ is irreducible, for any $x \in E$ there exists some $t \in [0,\infty[_\tau$ such that

$$Q_t(h)(x) = e^{\rho t}\, h(x) > 0.$$

Whenever $Q_\tau(\mathcal{B}_V(E)) \subset \mathcal{C}_{0,V}(E)$, the function h belongs to $\mathcal{C}_{0,V}(E)$.

Sufficient conditions in terms of the Lyapunov functions $V \geq 1$ ensuring the compactness of Q_t for some $t \in \mathcal{T}$ are discussed in [109, 176]. For instance, we have the following lemma, the proof of which is housed in the appendix on page 58..

LEMMA 2.4. *Assume that $Q_\tau(V)/V \in \mathcal{B}_0(E)$ for some $\tau > 0$. In addition, assume that for any compact set $K \subset E$ the operator $Q_\tau^K(f) := 1_K Q_\tau(1_K f)$ is compact on $\mathcal{B}_V(E)$. In this situation, for any $t \in \mathcal{T}$ with $t \geq \tau$, the operator Q_t is a compact operator from $\mathcal{B}_V(E)$ into itself.*

REMARK 2.5. *The condition $Q_\tau(V)/V \in \mathcal{B}_0(E)$ allows one to localise the operators on compact sets. The compactness condition of the semigroup Q_τ^K is readily checked for absolutely continuous operators of the form*

$$Q_\tau(x,dy) = q_\tau(x,y)\, \nu_\tau(dy) \tag{2.24}$$

where $q_\tau(x,y)$ is a continuous density with respect to some Radon measure ν_τ on E. The proof of the above assertion is rather standard. For completeness, it is provided in the appendix on page 58.

The above models encapsulate Markov transitions restricted to a compact set $K \subset \mathbb{R}^r$, for some $r \geq 1$, defined by

$$Q_\tau^K(x,dy) := 1_K(x)\, p_\tau(x,y)\, 1_K(y)\, dy$$

where $p_\tau(x,y)$ is a continuous probability transition density with respect to the Lebesgue measure dy on $\mathbb{R}^r$. For a more thorough discussion on this class of compact integral operators we refer to [118] and the more recent article [40].

REMARK 2.6. *Now assume that E is a countable space. In this situation, whenever $V = 1$ we can also use the following equivalence principle (see for instance Theorem 2.1 in [204])*

$$Q_\tau \text{ is compact on } \mathcal{B}(E) \Longleftrightarrow \forall \varepsilon > 0\ \exists K_\varepsilon \text{ finite s.t. } \sup_{x\in E} Q_\tau(1_{E-K_\varepsilon})(x) \leq \varepsilon.$$

More generally, assume that $Q_\tau(V)/V \in \mathcal{B}_0(E)$, In this situation we have

$$Q_t \quad \text{compact on } \mathcal{B}_V(E) \Longleftrightarrow \forall \varepsilon > 0 \quad \exists K_\varepsilon \quad \text{finite} \quad \text{s.t.} \quad \|Q_t(1_{K_\varepsilon^c} V)\|_V \leq \varepsilon. \tag{2.25}$$

The proof of the above assertion is provided in the appendix on page 59.

Whenever E is compact, by a theorem of de Pagter (cf. [172, Theorem 4.2.2]), a compact and irreducible positive operator Q_t on $\mathcal{C}(E)$ has a positive spectral radius $r(Q_t) > 0$, while its essential spectral radius is null. Applying the Krein-Rutman theorem (cf. [102, 166, Theorem 1.1]), there exists some non-negative and non-zero measure $\nu_\infty \in \mathcal{M}(E) = \mathcal{C}(E)'$ and a non-negative and non-zero function $h \in \mathcal{C}(E)$ such that

$$\nu_\infty Q_t = e^{\rho t}\, \nu_\infty \quad \text{and} \quad Q_t(h) = e^{\rho t}\, h \quad \text{with} \quad \rho = \log r(Q_1). \tag{2.26}$$

This yields the fixed point equation

$$\eta_\infty := \nu_\infty/\nu_\infty(1) = \Phi_t(\eta_\infty).$$

Several sufficient conditions in terms of the Lyapunov functions $V \geq 1$ ensuring the quasi-compactness properties of Q_τ are discussed in [109, 130, 176], see also Lemma 2.4 and Remark 2.5 in the present article. For a more detailed discussion on quasi-compact and compact operators we refer to [103, 166, 172, 175] and references therein.

3. Some classes of positive semigroups.

3.1. *Triangular array semigroups.* Fix a measure $\overline{\eta}_0 \in \mathcal{P}(E)$ and some locally bounded positive functions $H > 0$. We associate with these objects the functions $H_{s,t}$ and the normalised semigroups $\overline{Q}_{s,t}$ defined for any $s \leq t$ by the formulae

$$H_{s,t} := \overline{Q}_{s,t}(H) \quad \text{with} \quad \overline{Q}_{s,t}(f) := Q_{s,t}(f)/\overline{\eta}_s Q_{s,t}(1) \qquad \text{and} \quad \overline{\eta}_s = \Phi_{0,s}(\overline{\eta}_0). \tag{3.1}$$

From the definitions, for any $s \leq u \leq t$ it also follows that

$$\overline{\eta}_s \overline{Q}_{s,t} = \overline{\eta}_t \quad \text{and} \quad \overline{Q}_{s,u}\overline{Q}_{u,t} = \overline{Q}_{s,t}, \tag{3.2}$$

as well as

$$\overline{\eta}_s(H_{s,t}) = \overline{\eta}_t(H) \ \text{ and } \ H_{t,t} = H.$$

For any $s \leq u \leq v \leq t$, we also have

$$Q_{s,v}(H_{v,t}) = \lambda_{s,v}\ H_{s,t} \quad \text{with} \quad \lambda_{s,v} := \overline{\eta}_s Q_{s,v}(1) = \lambda_{s,u}\ \lambda_{u,v}.$$

Observe that

$$\lambda^-(\overline{\eta}_0) \leq \lambda^- := \inf_{t\geq 0} \overline{\eta}_t(Q_{t,t+\tau}(1)) \leq \lambda := \sup_{t\geq 0} \overline{\eta}_t(Q_{t,t+\tau}(1)) \leq \lambda(\overline{\eta}_0). \tag{3.3}$$

with the parameters $\lambda^-(\overline{\eta}_0)$ and $\lambda(\overline{\eta}_0)$ introduced in Definition 2.1.

DEFINITION 3.1. *We define the triangular array of Markov operators $R^{(t)}_{u,v}$, indexed by the parameter $t \geq 0$, for any $0 \leq u \leq v \leq t$ and $f \in \mathcal{B}_b(E)$ by*

$$R^{(t)}_{u,v}(f) := \frac{Q_{u,v}(f\ Q_{v,t}(H))}{Q_{u,v}(Q_{v,t}(H))} = \frac{1}{\lambda_{u,v}\ H_{u,t}}\ Q_{u,v}(H_{v,t}f). \tag{3.4}$$

For any given time horizon $t \geq 0$ and any $s \leq u \leq v \leq t$, we have the semigroup property

$$R^{(t)}_{s,v} = R^{(t)}_{s,u} R^{(t)}_{u,v} \quad \text{and} \quad R^{(t)}_{s,s} = I.$$

Several remarks are of interest here:

1. The stability properties of these stochastic models play a crucial role in the analysis of positive operators. The use of these triangular arrays of Markov semigroups in the stability analysis of the time varying positive semigroups has been considered in [76, 78, 79, 85] and more recently in [52] in the context of Feynman-Kac semigroups and when $H = 1$. We also refer the reader to chapter 4 in [72], and chapter 12 in [73] for a systematic analysis of the contraction properties of these semigroups with respect to the total variation norm.
2. The same class of triangular array semigroups associated with some positive function H are also discussed in the article [15] in the context of time-homogeneous sub-Markovian models, under different set of regularity conditions. We shall discuss these conditions in section 5.4.

3. The interpretation of the triangular array semigroups discussed above depends on the application. For time homogeneous semigroups, the semigroups associated with the positive eigenstate $H = h$ of the semigroup Q_t coincides with the semigroup of the so-called h-process, see for instance (4.32) and (4.35). In the context of nonlinear filtering with $H = 1$, these semigroups represent the forward evolution of the optimal smoother on some observation time interval [76, 78, 79]. In branching processes theory, these semigroups reflect the evolution of an auxiliary process often used to describe the ancestral lineage of an individual [16, 160]. The link between these seemingly disconnected subjects is the so-called many-to-one formula connecting the first moment of the occupation measure of a spatial branching process with a Feynman-Kac semigroup (see for instance [16, 19, 160], as well as [72, section 1.4.4] and [89, section 28.4]).

We now provide some important Markov transport formulae relating the semigroups introduced so far. The first formula connecting the triangular array semigroup discussed above with the flow of measures $\eta_u = \Phi_{s,u}(\eta_s)$ is given for any $s \leq u \leq t$ by

$$\Psi_{H_{s,t}}(\eta_s)\, R^{(t)}_{s,u}(f) = \frac{\eta_s(Q_{s,u}(H_{u,t}f))}{\eta_s(Q_{s,u}(H_{u,t}))} = \frac{\eta_u(H_{u,t}f)}{\eta_u(H_{u,t})}.$$

This yields the formula

$$\Psi_{H_{s,t}}(\eta_s)\, R^{(t)}_{s,u} = \Psi_{H_{u,t}}(\eta_u)\,.$$

Choosing $(u, \eta_s) = (t, \eta)$ so that $\eta_t = \Phi_{s,t}(\eta)$ in the above display, we obtain the next lemma.

LEMMA 3.1. *For any $s \leq t$ and any $\eta \in \mathcal{P}(E)$ we have*

$$\Psi_H(\Phi_{s,t}(\eta)) = \Psi_{H_{s,t}}(\eta) R^{(t)}_{s,t} \quad \text{and} \quad \eta_s(H_{s,t})\, \Psi_{H_{s,t}}(\eta_s)\,(1/H_{s,t}) = 1, \tag{3.5}$$

with the flow of probability measures given for any $0 \leq s \leq u \leq t$ by the formulae

$$\eta_u := \Phi_{0,u}(\eta_0) = \Phi_{s,u}(\eta_s) \quad \text{and} \quad \Psi_{H_{s,t}}(\eta_s)\, R^{(t)}_{s,u} = \Psi_{H_{u,t}}(\eta_u)\,.$$

3.2. *A class of R-semigroups.* In the further development of this section, $R^{(t)}_{u,v}$ is the triangular array of Markov operators defined in (3.4) with $H = 1$.

DEFINITION 3.2. *We say that $Q_{s,t}$ is an R-semigroup as soon as there exists some parameters $\tau > 0$ and $\varepsilon_\tau \in]0, 1[$ s.t. for any $s + \tau \leq t$ we have*

$$\beta\left(R^{(t)}_{s,s+\tau}\right) := \sup_{(x,y)\in E^2} \left\| \delta_x R^{(t)}_{s,s+\tau} - \delta_y R^{(t)}_{s,s+\tau} \right\|_{tv} \leq 1 - \varepsilon_\tau. \tag{3.6}$$

The parameter $\beta\left(R^{(t)}_{s,s+\tau}\right)$ defined above is called the Dobrushin ergodic coefficient of the Markov transition $R^{(t)}_{s,s+\tau}$. The above condition is satisfied *if and only if* the following condition holds

$$\begin{aligned} &\forall s \geq 0 \quad \forall t \geq s + \tau \quad \forall (x, y) \in E^2 \\ &\exists \nu \in \mathcal{P}(E) \quad \text{such that} \quad \delta_x R^{(t)}_{s,s+\tau} \geq \varepsilon_\tau\, \nu \quad \text{and} \quad \delta_y R^{(t)}_{s,s+\tau} \geq \varepsilon_\tau\, \nu. \end{aligned} \tag{3.7}$$

Note that the measure ν in the above display may depend upon the parameters (x, y) as well as (s, t, τ).

For Markovian semigroups, this condition reduces to the well-known Dobrushin's condition [96]. In addition, when the measure ν in (3.7) does not depend on the state variables (x, y), condition (3.7) coincides with Doeblin's condition [95].

REMARK 3.1. *The condition* $(\mathcal{A})$ *discussed in section 1.3 ensures that for any* $x_1, x_2 \in E$ *and* $t \in \mathcal{T}$ *we have*

$$\iota(\tau)\, Q_{t,t+\tau}(x_1, dy) \leq Q_{t,t+\tau}(x_2, dy) \quad \text{with} \quad \iota(\tau) := \iota^-(\tau)/\iota^+(\tau). \tag{3.8}$$

Thus, (3.6) is met with $\varepsilon_\tau := \iota(\tau)^2$. *In addition, for any* $t \in \mathcal{T}$ *and* $s \in]0, \infty[_\tau$ *and* $\varepsilon \in [0, \tau]$ *we also have*

$$0 < \iota(\tau) \leq \frac{Q_{t,u+\varepsilon}(1)}{\mu Q_{t,u+\varepsilon}(1)} = \frac{Q_{t,u}(Q_{u,u+\varepsilon}(1))}{\mu Q_{t,u}(Q_{u,u+\varepsilon}(1))} \leq \frac{1}{\iota(\tau)} \quad \textit{with} \quad u := t + s.$$

Therefore, for any $\mu \in \mathcal{P}(E)$ *we have the estimates*

$$0 < \iota(\tau) \leq q_\tau^-(\mu) := \inf \frac{Q_{t,t+s}(1)(x)}{\mu Q_{t,t+s}(1)} \leq q_\tau(\mu) := \sup \frac{Q_{t,t+s}(1)(x)}{\mu Q_{t,t+s}(1)} \leq 1/\iota(\tau) \tag{3.9}$$

where the infimum and the supremum are taken over all $x \in E$ *and all pair of indices* $t \in \mathcal{T}$ *and* $s \in [0, \infty[_\tau$. *For continuous time semigroups, the supremum in (3.9) can be also be taken over all continuous time indices* $s, t \geq 0$, *as soon as for any* $\mu \in \mathcal{P}(E)$ *we have the uniform local estimate*

$$\sup_{t \geq 0} \sup_{\varepsilon \in [0,\tau]} \frac{\|Q_{t,t+\varepsilon}(1)\|}{\mu Q_{t,t+\varepsilon}(1)} < \infty.$$

The above condition is clearly met as soon as (1.10) is satisfied.

To the best of our knowledge, the use of the triangular arrays $R_{s,u}^{(t)}$ and the extension of Dobrushin's contraction stability theory to time-varying and possibly random positive semigroups goes back to the 1990s with the article [79], see also [72, 73, 76, 78, 85].

Dobrushin's and Doeblin's conditions are popular conditions in probability theory. They are not always easily checked but are met for a large class of discrete or continuous time irreducible stochastic processes, mainly on compact domains. For instance, in the context of continuous time sub-Markovian semigroups this condition is satisfied for elliptic diffusions on compact manifolds killed at a bounded rate, as well as elliptic diffusions killed at the boundary of a bounded domain, cf. [85, Proposition 3.1] and [70, Lemma 3]. Combining the proof of [85, Proposition 3.1] with the two-sided estimates presented in [14, Lemma 3.3], one can check that (3.7) is also met for sub-Markovian semigroups associated with reflected diffusions on bounded connected domains.

Condition (3.7) is also met for absorbed time homogenous neutron transport processes for sufficiently smooth domains with an absorbing boundary, as well as for time varying multidimensional birth and death processes with mass extinction, see [46, sections 4.1 & 4.2]. Further examples of sub-Markovian semigroups satisfying (3.7) are discussed in [51, 52, 72, 73, 74, 76, 82]. Sufficient conditions and further examples are discussed in section 5.4.

3.3. *A class of* V*-positive semigroups.* In the further development of this section $V \in \mathcal{B}_\infty(E)$ stands for some function such that $V_\star \geq 1$.

DEFINITION 3.3. *A semigroup of positive integral operators* $Q_{s,t}$ *on* $\mathcal{B}_b(E)$ *is called a* V*-positive semigroup as soon as there exists some* $\tau > 0$ *and some function* $\Theta_\tau \in \mathcal{B}_0(E)$ *such that for any positive function* $f \in \mathcal{B}_V(E)$ *and any* $0 \leq s < t$ *we have*

$$Q_{s,t}(f) \in \mathcal{B}_{0,V}(E) \quad \textit{and} \quad Q_{s,s+\tau}(V)/V \leq \Theta_\tau. \tag{3.10}$$

For continuous time semigroups, we assume that (1.16) and (1.18) are satisfied.

The function Θ_τ in condition (3.10) is used to control, uniformly in time, the compact super-level sets of the time varying u.s.c. function $Q_{t,t+\tau}(V)/V$ in terms of a time homogenous function $\Theta_\tau \in \mathcal{B}_0(E)$ that often depends on V. For instance, we can choose in (3.10) a function Θ_τ of the form $\theta_\tau(V) \in \mathcal{B}_0(E)$ for some decreasing function $\theta_\tau : [1,\infty[\to \mathbb{R}_+$. Indeed, for any $\Theta_\tau \in \mathcal{B}_0(E)$ and $\varepsilon > 0$ there exists some $\varepsilon_1, \varepsilon_2 > 0$ such that

$$\mathcal{V}_\varepsilon := \{\Theta_\tau \geq \varepsilon\} \subset \{V \leq \theta^{-1}(\varepsilon_1)\} = \{\theta_\tau(V) \geq \varepsilon_1\} \subset \mathcal{V}_{\varepsilon_2}. \tag{3.11}$$

Similarly, for any $r > 1$ there exists some $\varepsilon > 0$ and $r_1 > 1$ such that

$$\{V \leq r\} \subset \mathcal{V}_\varepsilon \subset \{V \leq r_1\}. \tag{3.12}$$

The right-hand side condition in (3.10) ensures that for any $t \geq \tau$ and $s \geq 0$ we have

$$Q_{s,s+t}(V)/V \leq c_{s,t}(\tau)\, \Theta_\tau \in \mathcal{B}_0(E) \quad \text{with} \quad c_{s,t}(\tau) := \|Q_{s+\tau,s+t}(V)/V\|.$$

Condition (3.10) ensures that the normalized semigroup $\Phi_{s,t}$ maps $\mathcal{P}_V(E)$ into itself, and the right action operator $f \mapsto Q_{s,t}(f)$ maps $\mathcal{B}_V(E)$ into itself.

3.4. *A class of stable V-positive semigroups.*

DEFINITION 3.4. *A given V-positive semigroup $Q_{s,t}$ with respect to the Lyapunov function V and some parameter $\tau > 0$ is said to be stable when the following conditions are satisfied:*

- *There exists some $\overline{\eta}_0 \in \mathcal{P}_V(E)$ such that*

  $$\lambda^-(\overline{\eta}_0) > 0 \quad \textit{and for any } \mu \in \mathcal{P}_V(E) \textit{ we have} \quad \kappa_V(\mu) < \infty. \tag{3.13}$$

  with the parameters $\lambda^-(\overline{\eta}_0)$ and $\kappa_V(\mu)$ introduced in Definition 2.1.
- *There exists some positive function $H \in \mathcal{B}_{0,V}(E)$ as well as some $r_H > 0$ such that for any $r \geq r_H$ we have*

  $$\varsigma_r(H) := \inf \inf_{V(x) \leq r} H_{s,t}(x) > 0 \quad \textit{and} \quad |||H|||_V := \sup \|H_{s,t}\|_V < \infty \tag{3.14}$$

  where the infimum is taken over all indices $s \in \mathcal{T}$ and $s \leq t \in \mathcal{T}$, and $H_{s,t}$ stands for the normalized function depending on the measure $\overline{\eta}_0$ introduced in (3.13) and defined in (3.1). For continuous time semigroups, we also assume that $\pi_\tau(H) < \infty$, with $\pi_\tau(H)$ defined as (1.18) by replacing V by H.
- *There exists $r_0 \geq 1$ and some function $\alpha : r \in [r_0, \infty[\mapsto \alpha(r) \in]0,1]$ such that for any $s \in \mathcal{T}$ and $r \geq r_0$ we have*

  $$\sup_{V(x) \vee V(y) \leq r} \left\| \delta_x R^{(t)}_{s,s+\tau} - \delta_y R^{(t)}_{s,s+\tau} \right\|_{tv} \leq 1 - \alpha(r) \tag{3.15}$$

  where $R^{(t)}_{u,v}$ stands for the triangular array of Markov operators defined in (3.4) with the function H introduced in (3.14).

The local Doeblin minorisation condition (3.15) is rather standard in the stability analysis of Markov semigroups (see for instance [165, Theorem 6.15], [188, Proposition 2], and [163, Theorem 16.2.3]). For Markov semigroups $Q_{s,t}(1) = 1$, choosing $H = 1$ the left-hand side condition in (3.13) and (3.14) are clearly met. In this context, we recall that the Lyapunov condition in (3.10) ensures that $\Phi_{s,t}(\mu) = \mu Q_{s,t}(V)$ is uniformly bounded with respect to the time parameters $s \leq t$ (cf. for instance Lemma 3.2 or the Lyapunov condition (A.4) applied to the function $H = 1$).

In summary, a Markov semigroup $P_{s,t}$ is a stable V-positive semigroup when $P_{s,s+\tau}$ satisfies the Lyapunov condition in (3.10) as well as the local Doeblin minorisation condition (3.15) (applied to $H = 1$ and $Q_{s,t} = P_{s,t}$ so that $R^{(t)}_{s,s+\tau} = P_{s,s+\tau}$).

REMARK 3.2. *For continuous time models, by lemma 2.2 , it suffices to check conditions (3.13) with the parameters $\lambda_\tau^-(\overline{\eta}_0)$ and $\kappa_{\tau,V}(\mu)$ introduced in definition 2.1. In the same vein, it suffices to check (3.14) by taking the infimum and the supremum over time indices $s \in \mathcal{T}$ and $t \in [s,\infty[_\tau$. To check this claim, observe that for any $s \geq 0$, $n \in \mathbb{N}$ and $t \in [0,\tau[$ we have*

$$s_n = s + n\tau \quad \text{and} \quad u := s + t$$

$$\Longrightarrow u_n := u + n\tau = s_n + t \quad \text{and} \quad H_{s,s_n+t} = \frac{Q_{s,u}Q_{u,u_n}(H)}{\overline{\eta}_s Q_{s,u}Q_{u,u_n}(1)}.$$

Using (1.16) for any $x \in K_r := \{V \leq r\}$ we check that

$$H_{s,s_n+t}(x) = \frac{Q_{s,u}(H_{u,u_n})(x)}{\overline{\eta}_s Q_{s,u}(1)} \geq (\varsigma_r(H)/\pi_\tau)\, Q_{s,u}(1_{K_r})(x) \geq \varsigma_r(H)\, \pi_\tau^-(K_r)/\pi_\tau.$$

Correspondingly, recalling that $\overline{\eta}_s$ is a tight sequence, for any $\delta \in]0,1[$ there exists some K such that $\overline{\eta}_s(K) \geq (1-\delta)$. In this notation, by (1.16)we have

$$H_{s,s_n+t}(x)/V(x) \leq \frac{Q_{s,u}(V(H_{u,u_n}/V))(x)/V(x)}{\overline{\eta}_s Q_{s,u}(1)} \leq \frac{\pi_\tau(V)\, \|H_{u,u_n}\|_V}{(1-\delta)\pi_\tau^-(K)}.$$

We conclude that $|||H_{s,t}|||_V$ is uniformly bounded with respect to the continuous time parameters $s \geq 0$ and $t \geq s$.

Also note that for any compact set $K \subset E$ we have

$$(V \geq 1 \quad \text{and} \quad H/V \in \mathcal{B}_0(E)) \Longrightarrow 1/\inf_K H \leq \sup_K (V/H) < \infty \Longrightarrow \inf_K H > 0.$$

Whenever (3.14) is met, replacing H with $H/|||H|||_V$, there is no loss of generality if we assume the uniform estimate $\|H_{s,t}\|_V \leq 1$. Also notice that for any $s < t$ and

$$H \in \mathcal{B}_{0,V}(E) \Longrightarrow H_{s,t} = Q_{s,t}(H)/\overline{\eta}_s Q_{s,t}(1) \in \mathcal{B}_{0,V}(E).$$

Checking the estimates (3.13) and (3.14) may involve delicate calculations. For time homogeneous models, conditions (3.13) and (3.14) are equivalent to the existence of a leading eigen-pair of the positive semigroup (cf. for instance Theorem 4.4 and Corollary 9). For a more thorough discussion, we refer to section 4.3 dedicated to time homogenous models. Next we present another stronger but simpler and more tractable condition that applies to absolutely continuous semigroups.

The following lemma is a slight modification of [196, Proposition 1 & Lemma 10], based on technical approaches from [100] in the context of stability for nonlinear filtering

LEMMA 3.2. *Under the assumptions of theorem 1.3, for any locally bounded positive functions $H \in \mathcal{B}_{0,V}(E)$ and any $\overline{\eta}_0, \mu \in \mathcal{P}_V(E)$, we have (3.14) as well as*

$$0 < \kappa_H^-(\mu) \leq \kappa_V(\mu) < \infty \quad \text{and} \quad 0 < \lambda^-(\mu) \leq \lambda(\mu) < \infty. \tag{3.16}$$

For the convenience of the reader a detailed proof in our context is provided in the appendix, see section A.2. In the context of absolutely continuity, the above lemma also ensures for any bounded $f \geq 0$, and any $V(x) \leq r$ with $r \geq r_1$ we have

$$Q_{s,s+\tau}(H_{s+\tau,t}f)(x) \geq c_1(r)\, \nu_\tau(1_{V\leq r}\, H_{s+\tau,t}f) \quad \text{and} \quad \lambda_{s,s+\tau}\, H_{s,t} \leq \lambda\, |||H|||_V$$

for some constant $c_1(r) > 0$. By (3.4) and (3.14), this yields

$$R^{(t)}_{s,s+\tau}(f)(x) \geq \frac{c_1(r)}{\lambda\, |||H|||_V}\, \nu_\tau(1_{V\leq r}\, H_{s+\tau,t}f) \geq c_2(r)\, \frac{\nu_\tau(1_{V\leq r}\, H_{s+\tau,t}f)}{\nu_\tau(1_{V\leq r}\, H_{s+\tau,t})},$$

for some constant $c_2(r) > 0$, which implies (3.15). Thus, the above lemma ensures that absolutely continuous semigroups satisfying condition $(\mathcal{A})_V$ are stable V-positive semigroups.

4. Stability and contraction theorems.

4.1. *Contraction of R-semigroups.* In the further development of this section, $R^{(t)}_{u,v}$ is the triangular array of Markov operators defined in (3.4) with $H=1$. Also assume that $Q_{s,t}$ is an R-semigroup (that is (3.7) is satisfied with $H=1$).

4.1.1. *An uniform stability theorem.* This short section is concerned with a brief review on the stability properties of the non-linear semigroup $\Phi_{s,t}$ under the uniform minorisation condition (3.7). We recall the rather well known strong stability theorem which is valid when E is a measurable space.

THEOREM 4.1 ([76, 78, 79]). *Then for any $s,t\in\mathcal{T}$ and any $\mu_1,\mu_2\in\mathcal{P}(E)$ we have the uniform stability estimate*

$$\|\Phi_{s,t}(\mu_1)-\Phi_{s,t}(\mu_2)\|_{tv}\leq(1-\varepsilon_\tau)^{\lfloor(t-s)/\tau\rfloor}. \tag{4.1}$$

In addition we have the local Lipschitz estimate

$$\|\Phi_{s,t}(\mu_1)-\Phi_{s,t}(\mu_2)\|_{tv}\leq\frac{\|Q_{s,t}(1)\|}{\mu_1(Q_{s,t}(1))\vee\mu_2(Q_{s,t}(1))}(1-\varepsilon_\tau)^{\lfloor(t-s)/\tau\rfloor}\|\mu_1-\mu_2\|_{tv}. \tag{4.2}$$

A detailed and remarkably simple proof of Theorem 4.1 based on the nonlinear transport formula (3.5) is provided in [76, Theorem 3], see also [78, Lemma 2.1 & Lemma 2.3], [85, Lemma 2.1 & Proposition 2.3], [73, section 12.2], [76, section 2.1.2 & section 3.1.3] and [72, section 4]. The key semigroup oscillation formula [76, Theorem 2.3], [85, Proposition 2.3] connecting the *uniform* exponential decay (4.1) with Dobrushin's ergodic coefficient of the R-semigroup defined in section 2.3 is given by

$$\beta\left(R^{(t)}_{s,t}\right)=\sup_{(\mu,\eta)\in\mathcal{P}(E)^2}\|\Phi_{s,t}(\mu)-\Phi_{s,t}(\eta)\|_{tv}.$$

The above formula shows that the *uniform* exponential decays (4.1) are dictated by the contraction properties of the triangular arrays $R^{(t)}_{s,u}$. An extended version of Theorem 4.1 for general relative entropy criteria is provided in [72, section 4.3.1].

4.1.2. *Quasi-invariant measures.* For the rest of this section, we place ourselves in the time homogeneous setting. In this case, a variety of results follows almost immediately from the uniform estimates obtained in this theorem. Before stating them, we first discuss some relevant properties of time homogeneous models.

The uniform estimate (4.1) implies that $\Phi_t(\mu)$ is a Cauchy sequence in the complete set of probability measures $\mathcal{P}(E)$ equipped with the total variation distance. Thus, for any $\mu\in\mathcal{P}(E)$, the flow $\Phi_s(\mu)$ converges, as $s\to\infty$, exponentially fast to a single probability measure $\eta_\infty\in\mathcal{P}(E)$ that doesn't depends on μ. Choosing $\mu=\eta_\infty$ for any $s,t\in\mathcal{T}$ and $f\in\mathcal{B}_b(E)$ we have

$$\Phi_t(\Phi_s(\eta_\infty))(f)=\frac{\Phi_s(\eta_\infty)Q_t(f)}{\Phi_s(\eta_\infty)Q_t(1)}=\Phi_{s+t}(\eta_\infty)(f)\longrightarrow_{s\to\infty}\Phi_t(\eta_\infty)(f)=\eta_\infty(f).$$

In continuous time settings, note that the fixed point $\eta^{[\tau]}_\infty=\Phi_\tau(\eta^{[\tau]}_\infty)$ doesn't depend on the time step $\tau>0$. To check this claim, note that for any $\mu\in\mathcal{P}(E)$ and $t\geq 0$ we have the decomposition

$$\eta^{[\tau_1]}_\infty-\eta^{[\tau_2]}_\infty=\left(\eta^{[\tau_1]}_\infty-\Phi_{\lfloor t/\tau_1\rfloor\tau_1}\left(\Phi_{\{t/\tau_1\}\tau_1}(\mu)\right)\right)+\left(\Phi_{\lfloor t/\tau_2\rfloor\tau_2}\left(\Phi_{\{t/\tau_2\}\tau_2}(\mu)\right)-\eta^{[\tau_2]}_\infty\right).$$

The uniform estimate (4.1) yields the estimate

$$\|\eta_\infty^{[\tau_1]} - \eta_\infty^{[\tau_2]}\|_{tv} \leq (1-\varepsilon_{\tau_1})^{\lfloor t/\tau_1 \rfloor} + (1-\varepsilon_{\tau_2})^{\lfloor t/\tau_2 \rfloor} \longrightarrow_{t\to\infty} 0.$$

The invariant measure η_∞ is sometimes called the quasi-invariant measure of the semigroup Q_t.

Using the fixed point equation $\Phi_t(\eta_\infty) = \eta_\infty$, for any $s,t \in \mathcal{T}$ we readily check that

$$\eta_\infty(Q_{s+t}(1)) = \eta_\infty(Q_s(1))\ \eta_\infty(Q_t(1)).$$

Thus for any $t \in \mathcal{T}$ we have the exponential formula

$$\eta_\infty(Q_t(1)) = e^{\rho t} \quad \text{for some} \quad \rho \in \mathbb{R} \quad \text{and} \quad \overline{Q}_t(1) = e^{-\rho t}\ Q_t(1). \tag{4.3}$$

Choosing $\overline{\eta}_0 = \eta_\infty$ in (3.1) and using (2.12), for any $\mu \in \mathcal{P}(E)$ we check that

$$\frac{1}{\|\overline{Q}_{(1-\{t/\tau\})\tau}(1)\|}\ \mu\overline{Q}_{(\lfloor t/\tau \rfloor+1)\tau}(1) \leq \mu\overline{Q}_t(1) \leq \|\overline{Q}_{\{t/\tau\}\tau}(1)\|\ \mu\overline{Q}_{\lfloor t/\tau \rfloor\tau}(1). \tag{4.4}$$

Notice that

$$\sup_{\varepsilon\in[0,1]} \|\overline{Q}_{\varepsilon\tau}(1)\| = e^{-\rho\varepsilon\tau} \sup_{\varepsilon\in[0,1]} \|Q_{\varepsilon\tau}(1)\| < \infty.$$

On the other hand, for any $t \in [0,\infty[_\tau$ we have

$$\mu\overline{Q}_t(1) = \prod_{s\in[0,t[_\tau} \left\{1 + \left[\Phi_s(\mu)(\overline{Q}_\tau(1)) - \Phi_s(\eta_\infty)(\overline{Q}_\tau(1))\right]\right\}. \tag{4.5}$$

The exponential version of the above formula in the context of sub-Markovian semigroups with soft killing is discussed in (5.3).

Conversely, by (4.1) for any $n \geq 1$ we have

$$\sum_{s\in[0,\infty[_\tau} \left|\Phi_s(\mu)(\overline{Q}_\tau(1)) - \Phi_s(\eta_\infty)(\overline{Q}_\tau(1))\right|^n < \infty$$

as well as

$$\|\overline{Q}_t(1)\| \leq \prod_{s\in[0,\infty[_\tau} \left(1 + (1-\varepsilon_\tau)^{\lfloor s/\tau \rfloor}\|\overline{Q}_\tau(1)\|\right) < \infty.$$

The two estimates discussed above ensure that $\mu\overline{Q}_t(1)$ converges, as $[0,\infty[_\tau \ni t \to \infty$, to a non-zero number and by (4.4) we have

$$0 < \inf_{t\in\mathcal{T}} \mu(\overline{Q}_t(1)) \leq \sup_{t\in\mathcal{T}} \|\overline{Q}_t(1)\| < \infty. \tag{4.6}$$

Also observe that (4.6) implies that for any $\mu \in \mathcal{P}(E)$ we have

$$q(\mu) := \sup_{t\in\mathcal{T}} \|Q_t(1)\|/\mu(Q_t(1)) < \infty.$$

We are now in a position to state our first corollary of Theorem 4.1.

COROLLARY 3. *Under the assumptions of Theorem 4.1, for any $\mu,\eta \in \mathcal{P}(E)$ and $t \in \mathcal{T}$ we have the local contraction estimate*

$$\|\Phi_t(\mu) - \Phi_t(\eta)\|_{tv} \leq (q(\mu) \wedge q(\eta))\ (1-\varepsilon_\tau)^{\lfloor t/\tau \rfloor}\ \|\mu - \eta\|_{tv}. \tag{4.7}$$

4.1.3. *Ground state functions.* Choosing $\mu = \delta_x$ in (4.5), we also readily check that there exists $h \in \mathcal{B}_b(E)$ such that $\eta_\infty(h) = 1$ and for any $x \in E$ we have the pointwise convergence

$$\lim_{n\to\infty} \overline{Q}_{n\tau}(1)(x) = h(x) > 0.$$

By (4.6) and the dominated convergence theorem, for any $s \in [0,\infty[_\tau$ this implies that

$$\overline{Q}_{s+n\tau}(1)(x) = \overline{Q}_s(\overline{Q}_{n\tau}(1))(x) \longrightarrow_{n\to\infty} h(x) = \overline{Q}_s(h)(x) = e^{-\rho s}\, Q_s(h)(x). \tag{4.8}$$

In continuous time settings, note that the ground state function $h^{[\tau]} = e^{\rho\tau}\, Q_\tau(h^{[\tau]})$ doesn't depend on the time step $\tau > 0$. To check this claim, observe that

$$Q_t(h^{[\tau]}) = e^{\rho\tau}\, Q_\tau(Q_t(h^{[\tau]}))$$

is also an eigenfunction of Q_τ with the same eigenvalue $e^{\rho\tau}$. By uniqueness we conclude that $Q_t(h^{[\tau]}) = e^{\rho t} h^{[\tau]}$. Alternatively, applying the uniform estimate (4.1) to $\mu = \delta_x$ and $\eta = \eta_\infty$ for any $s, t \in \mathcal{T}$ and $x \in E$ we have

$$|\overline{Q}_{t+s}(1)(x)/\overline{Q}_t(1)(x) - 1| = |\Phi_t(\delta_x)(\overline{Q}_s(1)) - 1| \leq 2\, q(\eta_\infty)\, (1-\varepsilon_\tau)^{\lfloor t/\tau \rfloor}$$

and therefore

$$\|\overline{Q}_{t+s}(1) - \overline{Q}_t(1)\| \leq 2\, q(\eta_\infty)^2\, (1-\varepsilon_\tau)^{\lfloor t/\tau \rfloor}. \tag{4.9}$$

This shows that $\overline{Q}_t(1)$ is an uniformly bounded Cauchy sequence in $\mathcal{B}_b(E)$ that converges to $h \in \mathcal{B}_b(E)$ as $t \to \infty$. If in addition E is a Polish space and Q_t is Feller, that is $Q_t(\mathcal{C}_b(E)) \subset \mathcal{C}_b(E)$ arguing as above we also check that $h \in \mathcal{C}_b(E)$. The eigenfunction h is sometimes called the ground state of the semigroup Q_t.

Choosing $H = h$ in (3.4), formula (3.5) reads

$$\Psi_h(\Phi_t(\eta)) = \Psi_h(\eta) P_t^h \quad \text{with} \quad P_s^h(f) := Q_s(hf)/Q_s(h) = R_{0,s}^{(t)}(f). \tag{4.10}$$

Also observe that

$$\delta_x P_t^h = \Psi_h\left(\Phi_t(\delta_x)\right).$$

The second corollary is a direct consequence of Theorem 4.1.

COROLLARY 4. *There exists a positive function $h \in \mathcal{B}_b(E)$ and a constant $\rho \in \mathbb{R}$ such that for any $t \in \mathcal{T}$ we have*

$$Q_t(h) = e^{\rho t}\, h \quad \textit{and} \quad \eta_\infty^h P_t^h = \eta_\infty^h \quad \textit{with} \quad \eta_\infty^h := \Psi_h(\eta_\infty). \tag{4.11}$$

In addition, we have the total variation exponential decays

$$\|\delta_x P_t^h - \eta_\infty^h\|_{tv} \leq \frac{\|h\|}{\eta_\infty(h)}\, (1-\varepsilon_\tau)^{\lfloor t/\tau \rfloor}. \tag{4.12}$$

By (4.9) we have

$$\|\overline{Q}_t(1) - h/\eta_\infty(h)\| = \lim_{s\to\infty} \|\overline{Q}_t(1) - \overline{Q}_{t+s}(1)\| \leq 2\, q(\eta_\infty)^2\, (1-\varepsilon_\tau)^{\lfloor t/\tau \rfloor}.$$

On the other hand, for any $f \in \mathcal{B}_b(E)$ with $\|f\| \leq 1$ and any $x \in E$ we have

$$|\overline{Q}_t(f)(x) - \frac{h(x)}{\eta_\infty(h)}\, \eta_\infty(f)| \leq |\overline{Q}_t(1)(x) - h(x)/\eta_\infty(h)| + \frac{\|h\|}{\eta_\infty(h)}\, \|\Phi_t(\delta_x)(f) - \eta_\infty(f)|.$$

This leads us to our final corollary of the section.

COROLLARY 5. *For any $t \in \mathcal{T}$, we have the operator norm exponential decays (1.14) with $c(\eta_\infty) = q(\eta_\infty)$, as well as for any $s, t \in \mathcal{T}$ and $\eta \in \mathcal{P}(E)$ the uniform total variation estimates*

$$\begin{aligned} &\|\Psi_{Q_t(1)}(\Phi_s(\eta)) - \Psi_h(\eta_\infty)\|_{tv} \\ &\qquad \leq q(\eta_\infty)\,(1-\varepsilon_\tau)^{\lfloor s/\tau \rfloor} + 2\,(1-\varepsilon_\tau)^{\lfloor t/\tau \rfloor}\,\left(\|h\|/\eta_\infty(h) + \, q(\eta_\infty)^2\right). \end{aligned} \tag{4.13}$$

The last assertion comes from the decomposition

$$\begin{aligned} \overline{\eta}_0 = \eta_\infty \Longrightarrow \Psi_{Q_t(1)}(\mu)(f) - \Psi_h(\eta_\infty)(f) \\ = [\Psi_{\overline{Q}_t(1)}(\mu) - \Psi_{\overline{Q}_t(1)}(\eta_\infty)](f) + \eta_\infty(f(\overline{Q}_t(1) - h/\eta_\infty(h))). \end{aligned}$$

The estimate (4.13) is often used in the analysis of the ergodic properties of particle absorption models, see for instance [48, 126] as well as equation (5.6) in the present article.

The above corollary can be interpreted as an extended version of the Krein-Rutman theorem to positive semigroups satisfying the uniform minorisation condition (3.7). In this connection, note that Corollary 5 readily yields the uniqueness of the eigenfunction h (up to some constant). Indeed, by (1.14) we have

$$\left(\forall t \in \mathcal{T} \qquad \overline{Q}_t(g) = g \in \mathcal{B}_b(E)\right) \Longrightarrow g = \frac{\eta_\infty(g)}{\eta_\infty(h)}\, h. \tag{4.14}$$

Letting $T_t := e^{\rho t} T$, we also have

$$\begin{aligned} \lim_{t\to\infty} |||Q_t - T_t|||^{1/t} &\leq e^{\rho}\,(1-\varepsilon_\tau)^{1/\tau} \\ &< e^{\rho} = \eta_\infty(Q_t(1))^{1/t} \leq \|Q_t(1)\|^{1/t} \longrightarrow_{t\to\infty} \lim_{t\to\infty} |||Q_t|||^{1/t} \end{aligned} \tag{4.15}$$

where $|||\cdot|||$ denotes the operator norm. The estimates stated in (4.15) ensure that the essential spectral radius of Q_t is strictly smaller than its spectral radius. For a more thorough discussion on these spectral quantities, we refer to section 2.4.

A more refined analysis under weaker conditions, including exponential stability theorems and contraction properties of time homogeneous semigroups, is provided in section 4.3.

4.2. *Contraction of stable V-positive semigroups.* In the further development of this section $V \in \mathcal{B}_\infty(E)$ stands for some function such that $V_\star \geq 1$. We consider a stable V-positive semigroup $Q_{s,t}$ satisfying the Lyapunov condition (3.10) for some $\tau > 0$ as well as (3.13) and (3.14) for some measure $\overline{\eta}_0 \in \mathcal{P}_V(E)$ and some positive function $H \in \mathcal{B}_{0,V}(E)$. In addition, $H_{s,t}$ and $R^{(t)}_{u,v}$ stands for the corresponding normalized function defined in (3.1) and the triangular array of Markov operators defined in (3.4).

For continuous time semigroups, we recall that $\pi_\tau(H) < \infty$ and (1.16) and (1.18) are satisfied. By Remark 3.2, these conditions ensures that all infimum and supremum in (3.14) as well as in the definition of the parameters $\lambda^-(\overline{\eta}_0)$ and $\kappa_V(\mu)$ defined in (2.9) can be taken over continuous time indices. The proof of the following theorem and its corollary can be found in section 6.2.

THEOREM 4.2. *There exist constants $a < \infty$ and $b > 0$ such that for any $s \in \mathcal{T}$, $u \in [s,t]_\tau$, $t \in [s,\infty[_\tau$ and and $\mu, \eta \in \mathcal{P}_{V/H_{s,t}}$, we have the uniform contraction estimate*

$$\|\mu R^{(t)}_{s,u} - \eta R^{(t)}_{s,u}\|_{V/H_{u,t}} \leq a\, e^{-b(u-s)}\, \|\mu - \eta\|_{V/H_{s,t}}. \tag{4.16}$$

As we shall see in Lemma 6.1 our regularity conditions ensure the existence of some $0<\varepsilon<1$ and some constant $c>0$ such that for any time horizon $s\in\mathcal{T}$ and $t\in[s,\infty[_\tau$ we have the Lyapunov estimate

$$R^{(t)}_{s,s+\tau}(V/H_{s+\tau,t})\leq \varepsilon\, V/H_{s,t}+c. \tag{4.17}$$

In this direction, we also emphasise that the main ingredient of the proof of Theorem 4.2 is the V-contraction for Markov operators discussed in Lemma 2.3.

COROLLARY 6. *For continuous time stable V-positive semigroups, there exist constants $a<\infty$ and $b>0$ such that for any $s\leq u\leq t$ and $\mu,\eta\in\mathcal{P}_{V/H_{s,t}}$, we have the uniform contraction estimate*

$$\|\mu R^{(t)}_{s,u}-\eta R^{(t)}_{s,u}\|_{V/H_{u,t}}\leq a\,(\pi_\tau(V)/\lambda^-(\overline{\eta}_0))\,e^{-b(u-s)}\,\|\mu-\eta\|_{V/H_{s,t}}. \tag{4.18}$$

The estimate on the left-hand side of (3.14) allows one to control, uniformly with respect to the time parameter, the quantities $\mu(H_{s,t})$ as a function of $\mu(V)$, for any $\mu\in\mathcal{P}_V(E)$. Since these uniform estimates will be used several times in the sequel, we present them here in a general form. Applying the Markov inequality, for any $\mu\in\mathcal{P}_V(E)$ the left-hand side condition in (3.14) ensures the existence of some $n\geq 1$ such that

$$r_n:=\mu(V)+n\Longrightarrow \mu(H_{s,t})\geq \varsigma_{r_n}(H)\mu\,(V\leq r_n)\geq \varsigma_{r_n}(H)/(1+\mu(V)/n)>0. \tag{4.19}$$

For any $\mu\in\mathcal{P}_V(E)$, we conclude that

$$0<\omega_H(\mu):=\inf_{s\geq 0}\inf_{t\geq s}\mu(H_{s,t})\leq \mu(V). \tag{4.20}$$

Similarly, we check that the condition $\kappa_V(\mu)<\infty$ ensures the tightness of sequence of measures $\Phi_{s,t}(\mu)$ indexed by $s\geq 0$ and $t\geq s$, for any $\mu\in\mathcal{P}_V(E)$. In the same vein, we check that the flow of measures $\overline{\eta}_t$ is tight. Thus, choosing

$$r_n=\overline{\eta}(V)+n\geq r_H \quad \text{with} \quad \overline{\eta}(V):=\sup_{t\geq 0}\overline{\eta}_t(V)$$

we also check that

$$\overline{\eta}_-(H):=\inf_{t\geq 0}\overline{\eta}_t(H)\geq \varsigma_{r_n}(H)/(1+\overline{\eta}(V)/n)>0.$$

We are now in position to discuss some direct consequences of Theorem 4.2. Defining the finite rank (and hence compact) operator

$$f\in\mathcal{B}_V(E)\mapsto T_{s,t}(f):=\frac{H_{s,t}}{\overline{\eta}_s(H_{s,t})}\,\overline{\eta}_t(f)\in\mathcal{B}_{0,V}(E),$$

the first corollary and its time homogeneous version discussed in Corollary 12 can be interpreted as an extended version of the Krein-Rutman theorem to time varying positive semigroups.

COROLLARY 7. *For any $s\in\mathcal{T}$ and $t\in[s,\infty[_\tau$ we have the exponential decay*

$$\left|\left|\left|\overline{Q}_{s,t}-T_{s,t}\right|\right|\right|_V\leq\ a\,e^{-b(t-s)}\,\left(1+|||H|||_V\,\overline{\eta}(V)/\overline{\eta}_-(H)\right),$$

where (a,b) were defined in (4.16). In addition, we have the uniform norm estimate

$$\left|\left|\left|\overline{Q}_{s,t}\right|\right|\right|_V\leq(1+a)\,\left(1+|||H|||_V\,\overline{\eta}(V)/\overline{\eta}_-(H)\right). \tag{4.21}$$

For continuous time semigroups, the above estimates remain valid for any continuous time indices $s\leq t$ with the parameter a replaced by the parameter

$$a(\overline{\eta}_0):=a\,(\pi_\tau(V)/\lambda^-(\overline{\eta}_0)). \tag{4.22}$$

PROOF. Using (3.2) and (3.5) we check that

$$\Psi_{H_{s,t}}(\overline{\eta}_s)\, R^{(t)}_{s,t}(f/H) = \Phi_{s,t}(\overline{\eta}_s)(f)/\Phi_{s,t}(\overline{\eta}_s)(H) = \overline{\eta}_t(f)/\overline{\eta}_t(H).$$

This yields the decomposition

$$\overline{Q}_{s,t}(f) - \frac{H_{s,t}}{\overline{\eta}_s(H_{s,t})}\,\overline{\eta}_t(f) = H_{s,t}\,\left(R^{(t)}_{s,t}(f/H) - \Psi_{H_{s,t}}(\overline{\eta}_s)\, R^{(t)}_{s,t}(f/H)\right).$$

Applying (4.16) to $u = t$, $\mu = \delta_x$ and $\eta = \Psi_{H_{s,t}}(\overline{\eta}_s)$, for any $\|f\|_V = \|f/H\|_{V/H} \leq 1$ we check the estimate

$$\left|\overline{Q}_{s,t}(f) - \frac{H_{s,t}}{\overline{\eta}_s(H_{s,t})}\,\overline{\eta}_t(f)\right| \leq a\, e^{-b(t-s)}\,\left(V + \frac{H_{s,t}}{\overline{\eta}_s(H_{s,t})}\,\overline{\eta}_s(V)\right) \tag{4.23}$$

where (a, b) were defined in (4.16). This concludes the proof. □

When $H = 1$, the extended version of the above corollary in the context of random semigroups arising in filtering is provided in [196]. The proof in [196] relies on rather sophisticated coupling and decomposition techniques given in [145], which were further developed in [100].

REMARK 4.1. *In Theorem 4.2 and Corollary 7, the tightness condition $\kappa_V(\mu) < \infty$ for any $\mu \in \mathcal{P}_V(E)$ in the right-hand side of (3.13) can be replaced by the condition $\overline{\eta}(V) < \infty$. In this situation, choosing $f = 1$ in (4.23) for any $\mu \in \mathcal{P}_V(E)$ we readily check that*

$$\left|\mu\overline{Q}_{s,t}(1) - \mu(H_{s,t})/\overline{\eta}_t(H)\right| \leq a\,\mu(V)\, e^{-b(t-s)}\,\left(1 + |||H|||_V\,\overline{\eta}(V)/\overline{\eta}_-(H)\right), \tag{4.24}$$

where (a, b) were defined in (4.16).

Our next result transfers the stability of the R-semigroup to that of the normalised semigroup Φ.

THEOREM 4.3. *For any $s \in \mathcal{T}$ and $t \in [s, \infty[_\tau$, and any $\mu, \eta \in \mathcal{P}_V(E)$ we have the local contraction estimate*

$$|||\Phi_{s,t}(\mu) - \Phi_{s,t}(\eta)|||_V \leq a\,\kappa(\eta, \mu)\, e^{-b(t-s)}\;\; |||\mu - \eta|||_V \tag{4.25}$$

with (a, b) as in (4.16) and $\kappa(\eta, \mu)$ given by

$$\kappa(\eta, \mu) := \kappa_H(\mu)\,(1 + \kappa_V(\eta))\;\;(1 + \eta(V)/\omega_H(\eta))\,/\omega_H(\mu).$$

For continuous time semigroups, the above estimates remain valid for any continuous time indices $s \leq t$ with the parameter a replaced by the paremeter $a(\overline{\eta}_0)$ defined in (4.22).

This result is a direct consequence of the V-contraction estimates (4.16) stated in Theorem 4.2 and the rather elementary Boltzmann-Gibbs estimates (2.7) and (2.8). The full proof is provided in section 6.3.

As in Remark 4.1, the condition $\kappa_V(\mu) < \infty$ for any $\mu \in \mathcal{P}_V(E)$ in Theorem 4.3 can be replaced by condition

$$\overline{\eta}(V) < \infty \quad \text{and} \quad \kappa_H(\mu) < \infty \quad \text{for any } \mu \in \mathcal{P}_V(E).$$

The right-hand side condition in the above display is clearly met for any bounded function H. In this situation, following word-for-word the proof of Theorem 4.3, we check that

$$|||\Phi_{s,t}(\mu) - \overline{\eta}_t|||_V \leq a\,\kappa(\overline{\eta}, \mu)\, e^{-b(t-s)}\;\; |||\mu - \overline{\eta}_s|||_V$$

with

$$\kappa(\overline{\eta},\mu) := \kappa_H(\mu)\,(1+\overline{\eta}(V))\,\big(1+\overline{\eta}(V)/\overline{\eta}_-(H)\big)\,/\omega_H(\mu).$$

Using the above estimate we readily check that $\kappa_V(\mu) < \infty$ for any $\mu \in \mathcal{P}_V(E)$.

The V-norm stability of the semigroup $\Phi_{s,t}$ is also discussed in [196] (for instance [196, Corollary 1]). The proof in [196] is based on Corollary 7 and it does not provide local Lipschitz contraction estimates.

REMARK 4.2. *The time varying Lyapunov function $V/H_{u,t}$ associated with the triangular array of Markov operators $R^{(t)}_{s,u}$ discussed in (4.17) depends on the terminal time horizon t. This property allows one to control the exponential decays (4.16) of the corresponding $(V/H_{u,t})$-norms uniformly in t. These estimates are crucial in the proof of Theorem 4.3.*

We note that more conventional approaches, based on time homogeneous Lyapunov functions V, are discussed in [160]. This approach also ensures that the Markov semigroup $R^{(t)}_{s,u}$ forgets its initial state with respect to a common time homogenous V-norm. However, it seems difficult to deduce any local Lipschitz estimates of the form (4.25) from these uniform estimates.

Choosing $H = 1$ in (3.5) we have

$$[\Phi_{s,t}(\mu) - \Phi_{s,t}(\eta)]\,(f) = \frac{1}{\mu(H_{s,t})}\,(\mu-\eta)\,(D_\eta\Phi_{s,t}(f)) \tag{4.26}$$

with the first order linear operator $D_\eta\Phi_{s,t}$ defined by the formula

$$D_\eta\Phi_{s,t}(f)(x) := H_{s,t}(x)\;\left(\Phi_{s,t}(\delta_x) - \Phi_{s,t}(\eta)\right)(f).$$

Taylor expansions of higher order are also discussed in [5]. A weak version of the total variation estimate (4.25) is now easily obtained from the above perturbation formula.

COROLLARY 8. *Consider the triangular array semigroup (3.5) associated to the unit function $H = 1$. In this case, for any $s \in \mathcal{T}$, $t \in [s,\infty[_\tau$ and any $\eta \in \mathcal{P}_V(E)$ we have*

$$\sup_{\|f\|_{V/H_{s,t}}\leq 1} \|D_\eta\Phi_{s,t}(f)\|_V \leq a\,e^{-b(t-s)}\;(1+\eta(V)/\omega_H(\eta))\,. \tag{4.27}$$

PROOF. We have

$$D_\eta\Phi_{s,t}(f)(x) := H_{s,t}(x)\int\;\eta(dy)\;\frac{H_{s,t}(y)}{\eta(H_{s,t})}\;(\delta_x R^{(t)}_{s,t} - \delta_y R^{(t)}_{s,t})(f).$$

Using (4.16) we check that

$$\|\delta_x R^{(t)}_{s,t} - \delta_y R^{(t)}_{s,t}\|_{V/H_{s,t}} \leq a\,e^{-b(t-s)}\;((V/H_{s,t})(x) + (V/H_{s,t})(y)).$$

This implies that

$$\sup_{\|f\|_{V/H_{s,t}}\leq 1} \|D_\eta\Phi_{s,t}(f)\|_V \leq a\,e^{-b(t-s)}\;(1+\eta(V)\;\|H_{s,t}\|_V/\eta(H_{s,t}))$$

and we can now conclude. □

REMARK 4.3. *These weak form estimates are particularly useful in the convergence analysis of the mean field particle models associated with sub-Markovian integral operators. Taylor expansions at any order are discussed in section 3.1.3 and chapter 10 in [73], see also section 2.3 in the more recent article [5].*

4.3. *Time homogenous models.*

4.3.1. *Leading eigen-triple.* Consider a time homogenous version Q_t of the stable V-positive semigroup discussed in section 4.2. This section is concerned with the existence of an unique leading eigen-triple $(\rho, \eta_\infty, h) \in (\mathbb{R} \times \mathcal{P}_V(E) \times \mathcal{B}_{0,V}(E))$ satisfying (1.12). We follow word-for-word the arguments developed in the end of section 4.1 in terms of V-normed spaces. In this context, for any $s \in \mathcal{T}$ we have

$$\kappa_H(\Phi_s(\overline{\eta}_0)) \leq \kappa_H(\overline{\eta}_0) < \infty \quad \text{and} \quad 0 < \omega_H(\overline{\eta}_0) \leq \omega_H(\Phi_s(\overline{\eta}_0)).$$

Thus, by (4.25) we readily check that

$$|||\Phi_t(\overline{\eta}_0) - \Phi_{s+t}(\overline{\eta}_0)|||_V \leq a\ \kappa(\overline{\eta}_0)\ e^{-bt} \quad |||\overline{\eta}_0 - \Phi_s(\overline{\eta}_0)|||_V \leq 2\ a\ \kappa(\overline{\eta}_0)\kappa_V(\overline{\eta}_0)\ e^{-bt}$$

with $\kappa(\overline{\eta}_0) := \kappa(\overline{\eta}_0, \overline{\eta}_0)$. The above Lipschitz exponential decay estimate ensures that $\overline{\eta}_t$ is a Cauchy sequence in the complete set $\mathcal{P}_V(E)$ equipped with the V-distance. Thus it converges exponentially fast to a single probability $\eta_\infty = \Phi_t(\eta_\infty) \in \mathcal{P}_V(E)$. The fixed point equation yields the exponential formula (4.3). Observe that

$$\frac{Q_t(f)}{\eta_\infty Q_t(1)} = \frac{\overline{Q}_t(f)}{\eta_\infty(\overline{Q}_t(1))}$$

with the semigroup $\overline{Q}_t$ defined in (3.1) in terms of the measure $\overline{\eta}_0 \in \mathcal{P}_V(E)$ satisfying (3.13) and (3.14). Combining (4.21) with (4.24) we check that the function $t \mapsto \|e^{-\rho t} Q_t(1)\|_V$ is uniformly bounded. Now choosing $\overline{\eta}_0 = \eta_\infty$ in (3.1) the normalized semigroup takes the following form

$$\overline{Q}_t(1) = Q_t(1)/\eta_\infty Q_t(1) = e^{-\rho t} Q_t(1) \quad \text{and we have} \quad \sup_{t \geq 0} \|\overline{Q}_t(1)\|_V < \infty.$$

As in (1.6), now applying Theorem 4.25 we have the pointwise convergence of product series expansion (4.5); that is, for any $x \in E$ we have the product series formula

$$h(x) := \lim_{n \to \infty} \overline{Q}_{n\tau}(1)(x) = \prod_{n \geq 0} \left\{1 + \left[\Phi_{n\tau}(\delta_x)(\overline{Q}_\tau(1)) - \Phi_{n\tau}(\eta_\infty)(\overline{Q}_\tau(1))\right]\right\} > 0.$$

Applying the dominated convergence theorem as in (4.8), we also check that $\eta_\infty(h) = 1$ as well as for any $s \in [0, \infty[_\tau$ the formulae

$$Q_s(h) = e^{\rho s}\ h \in \mathcal{B}_{0,V}(E). \tag{4.28}$$

For continuous time models, the ground state doesn't depend on the time step so that the above formula is satisfied for any $s \geq 0$. Choosing $\eta = \eta_\infty$ and H such that $\eta_\infty(H) = 1$ and $\|H\|_V \leq 1$ in (4.25) we check that for any $V(x) \leq r$ and $s, t \in \mathcal{T}$ we have the exponential estimate

$$|\Phi_t(\delta_x)(\overline{Q}_s(H)) - 1| \leq \|\overline{Q}_s(H)\|_V |||\Phi_t(\delta_x) - \eta_\infty|||_V \leq c(r)\ e^{-bt}$$

with some finite constant $c(r) < \infty$ and the parameter b as in (4.16). Since $\|\overline{Q}_t(H)\|_V$ is uniformly bounded with respect to the time parameter, this implies that for any compact subset $K \subset E$ we have

$$\sup_K |\overline{Q}_{t+s}(H) - \overline{Q}_s(H)| \leq c_K\ e^{-bt} \quad \text{with some finite constant } c_K < \infty. \tag{4.29}$$

This shows that $\overline{Q}_t(H)$ is a uniformly Cauchy sequence on compact sets and we have the pointwise convergence $\overline{Q}_t(H)(x) \longrightarrow_{t \to \infty} h(x)$. Applying the dominated convergence theorem, for any $s > 0$ we have that

$$\overline{Q}_s(\overline{Q}_t(H)) = \overline{Q}_{t+s}(H) \longrightarrow_{t \to \infty} h = \overline{Q}_s(h) = e^{\rho s}\ h \in \mathcal{B}_{0,V}(E).$$

For strong Feller semigroups (in the sense that $Q_{s,t}(\mathcal{B}_V(E)) \subset \mathcal{C}_V(E)$, for any $s < t$), $\overline{Q}_t(H)$ compactly converges as $t \to \infty$ to some continuous function $h = \overline{Q}_s(h) \in \mathcal{C}_{0,V}(E)$ as soon as the Polish space E is locally compact. The same result applies when $H \in \mathcal{C}_V(E)$ and $Q_t(\mathcal{C}_V(E)) \subset \mathcal{C}_V(E)$. We recall that Polish spaces are separable metric spaces so they are second countable (and thus locally compact Polish spaces are σ-compact). In this context, $\mathcal{C}(E)$ equipped with the compact uniform topology is a complete metric space. This ensures that $h \in \mathcal{C}(E)$. We also check that $\|h\|_V < \infty$ by recalling that $\|Q_t(H)\|_V$ is uniformly bounded with respect to the time horizon.

The next theorem connects the stability of the semigroup Q_t with the one of the Doob's h-transform P_t^h defined in (1.25). The proof is provided in section 6.4.

THEOREM 4.4. *The semigroup Q_t is V-positive and stable if and only if there exists an eigen-triple (ρ, η_∞, h) satisfying (4.28) and P_t^h is stable V^h-positive semigroup, with the function $V^h = V/h$.*

4.3.2. *Sub-integral semigroups.* In Section 3.4 we have seen that absolutely continuous semigroups satisfying condition $(\mathcal{A})_V$ are stable V-positive semigroups. Our next objective is to relax this absolutely continuity condition. Consider the following sub-integral condition

$$Q_\tau(x_1, dx_2) \geq q_\tau(x_1, x_2)\ \chi_\tau(dx_2), \tag{4.30}$$

for some $\tau > 0$, some density function $q_\tau(x_1, x_2) > 0$ and some positive Radon measure χ_τ. Also assume that for any compact set K there exists some positive measurable function $q_\tau^K(x_2)$ such that

$$\inf_{x_1 \in K} q_\tau(x_1, x_2) \geq q_\tau^K(x_2) > 0 \quad \text{and} \quad \chi_\tau(q_\tau^K) > 0.$$

For instance, the left-hand side condition is satisfied for lower semi-continuous function $q_\tau(x_1, x_2)$ with respect to the first variable, and upper-semicontinuous with respect to the second. In this situation, for any compact set $K \subset E$ we have

$$\forall x \in K \qquad Q_\tau(h)(x) = e^{\rho\tau}\ h(x) \geq\ \chi_\tau(h q_\tau^K) > 0 \quad \text{and thus} \quad \inf_K h > 0.$$

Whenever (4.30) is satisfied, for any compact set $K \subset E$ we have

$$\forall x \in K \qquad Q_\tau(x, dz) \geq \iota_K\ \nu_K(dz) \tag{4.31}$$

for some Radon probability measure ν_K and some $\iota_K > 0$ whose values may depends on the parameter τ. This minorisation condition ensures that (6.2) is satisfied. To check this claim, observe that for any $x \in K_r := \{V \leq r\}$ we have

$$\begin{aligned} P_\tau^h(x, dz) &\geq \frac{1}{e^{\rho\tau} \sup_{K_r} h}\ Q_\tau(x, dz)\ h(z) \\ &\geq \alpha(r)\ \nu_{K_r}^h(dz) \ \text{ with } \ \alpha(r) = \iota_{K_r}\ e^{-\rho\tau} \frac{\nu_{K_r}(h)}{\sup_{K_r} h} \ \text{ and } \ \nu_{K_r}^h := \Psi_h(\nu_{K_r}). \end{aligned}$$

This clearly implies (6.2).

As we shall see in Lemma 6.2, the condition $Q_\tau(V)/V \in \mathcal{B}_0(E)$ implies the Lyapunov inequality (1.26). This property also ensures that for any $\eta \in \mathcal{P}_V(E)$ we have $\kappa_V(\eta) < \infty$. For a more thorough discussion on the consequences of the Foster-Lyapunov inequality we refer the reader to Proposition 6.1.

This yields the following corollary of Theorem 4.4.

COROLLARY 9. *Consider a V-positive semigroup Q_t satisfying the local minorisation condition (4.30). In this situation, Q_t is stable if and only if there exists a leading eigen-triple $(\rho, \eta_\infty, h) \in (\mathbb{R} \times \mathcal{P}_V(E) \times \mathcal{B}_{0,V}(E))$.*

Note that the minorisation condition (4.30) is less stringent than the absolutely continuous condition (1.8) imposed by condition $(\mathcal{A})_V$. In the context of particle absorption models, it applies to jump processes including regular piecewise deterministic processes as well as Metropolis-Hastings transitions.

4.3.3. *Doob's h-tranform semigroup.* This section presents a more refined analysis of a time homogenous V-positive semigroups Q_t satisfying the following weaker condition:

$$Q_t(h) = e^{\rho t}\, h > 0 \quad \text{for some } \tau > 0,\ \rho \in \mathbb{R}, \ \text{ and } h \in \mathcal{B}_{0,V}(E). \tag{4.32}$$

Recall that for any compact set $K \subset E$ we have

$$V \geq 1 \quad \text{and} \quad h/V \in \mathcal{B}_0(E) \Longrightarrow \inf_K h \geq \inf_K (h/V) > 0.$$

Arguing as above condition $(\mathcal{H}^h)$ introduced in (1.27) is satisfied as soon as the local minorisation condition (4.31) is satisfied. The main drawback of condition $(\mathcal{H}^h)$ is that it requires some knowledge of the function h which is often unknown.

Several illustrations and some sufficient conditions ensuring the existence of the leading eigen-pair (ρ, h) satisfying (4.32) are discussed in section 2.4, which is dedicated to the study of quasi-compact positive operators. See also Corollary 4. For instance, in remark 2.5 we shall see that the existence of a leading eigen-pair (ρ, h) satisfying (4.32) is granted for absolutely continuous semigroups of the form (1.8) equipped with a continuous density. Note that in this case condition $(\mathcal{A})_V$ is satisfied.

For a more detailed discussion on the design of functions V satisfying condition $Q_\tau(V)/V \in \mathcal{B}_0(E)$, we refer to [6, 100, 109, 196]. The article [15] also provides different conditions ensuring the existence of a leading eigen-pair (ρ, h) for semigroups that are not necessarily absolutely continuous. We shall discuss these conditions in section 5.4, dedicated to comparisons with the existing literature on this subject. We also refer the reader to [15] for some additional illustrations of these conditions in the context of the growth-fragmentation equations.

We are now in position to state the main result of this section, the proof of which can be found in section 6.5.

THEOREM 4.5. *Assume that $(\mathcal{H}^h)$ is satisfied. Then the Markov semigroup P_t^h has a single invariant measure $\eta_\infty^h \in \mathcal{P}_{V/h}(E)$. In addition, there also exists some finite constant $a_h < \infty$ and some parameter $b_h > 0$, such that for any $\mu, \eta \in \mathcal{P}_{V/h}(E)$ and $t \in \mathcal{T}$ we have the contraction estimate*

$$\|\mu P_t^h - \eta P_t^h\|_{V/h} \leq a_h\, e^{-b_h t}\, \|\mu - \eta\|_{V/h}. \tag{4.33}$$

In addition, for any $\eta \in \mathcal{P}_V(E)$ we have the estimates

$$0 < \kappa_h^-(\eta) \leq \kappa_V(\eta) < \infty. \tag{4.34}$$

.

As with Theorem 4.2, the main ingredient of the proof is the V-contraction for Markov operators discussed in Lemma 2.3. Also note that for continuous time semigroups, we have

$$\pi_\tau^h(V^h) := \sup_{s \geq 0} \sup_{\delta \in [0,\tau[} \|P_\delta^h(V^h)/V^h\| \leq e^{|\rho|\delta} \pi_\tau(V) < \infty.$$

REMARK 4.4. *Theorem 4.5 ensures the uniqueness of the invariant measure* $\eta^h_\infty = \eta^h_\infty P^h_t \in \mathcal{P}_{V/h}(E)$ *and exponential decay to equilibrium of the* h*-process.*

Choosing $\overline{\eta}_0 = \eta_\infty := \Psi_{1/h}(\eta^h_\infty)$ *we have* $\overline{\eta}_0 = \Phi_\tau(\overline{\eta}_0)$*. In this scenario, we readily check (3.14) choosing* $H = h$ *and using the fact that*

$$H_{s,t} = h \quad \text{and} \quad \overline{\eta}_0\, Q_{t,t+\tau}(1) = e^{\rho\tau}. \tag{4.35}$$

We illustrate the impact of Theorem 4.5 with some direct corollaries.

COROLLARY 10. *Under condition* $(\mathcal{H}^h)$*, for any* $\eta \in \mathcal{P}_V(E)$ *and* $s, t \geq 0$ *we have*

$$\eta^h_t := \Psi_h(\eta) P^h_t \Longrightarrow \eta^h_s \left| \frac{P^h_t(1/h)}{\eta^h_s P^h_t(1/h)} - 1 \right| \leq c(\eta)\; a_h\; e^{-b_h t}, \tag{4.36}$$

where (a_h, b_h) *were introduced in (4.33) and*

$$c(\eta) := 2\; \kappa_V(\eta) \kappa_h(\eta) / \kappa^-_h(\eta),$$

with $\kappa_V(\eta)$ *and* $\kappa^-_V(\eta)$ *defined in* (2.9).

PROOF. Recall that $\eta_t := \Phi_t(\eta) = \Psi_{1/h}(\eta^h_t)$ so that

$$\eta^h_t(1/h)\; \eta_t(h) = 1 \quad \text{and} \quad \eta^h_t(V/h) = \eta_t(V)/\eta_t(h).$$

The estimate (4.36) is now easily checked applying (4.33) to $(\mu, \eta) = (\delta_x, \delta_y)$ and using the inequality

$$\eta^h_s \left| \frac{P^h_t(1/h)}{\eta^h_s P^h_t(1/h)} - 1 \right| \leq \kappa_h(\eta) \int \eta^h_s(dx) \eta^h_s(dy)\; \left| P^h_t(1/h)(x) - P^h_t(1/h)(y) \right|.$$

This ends the proof of the Corollary. □

COROLLARY 11. *Under condition* $(\mathcal{H}^h)$*, the measure* $\eta_\infty := \Psi_{1/h}(\eta^h_\infty) \in \mathcal{P}_V(E)$ *is the unique invariant measure of the semigroup* Φ_t*.*

In addition, for any $\mu, \eta \in \mathcal{P}_V(E)$ *and* $t \in \mathcal{T}$ *we have* $\kappa_V(\mu) < \infty$ *and*

$$|||\Phi_t(\mu) - \Phi_t(\eta)|||_V \leq\; a_h\; \kappa(\mu, \eta)\; e^{-b_h t}\; |||\mu - \eta|||_V, \tag{4.37}$$

with (a_h, b_h) *as in (4.33) and the parameters* $\kappa(\mu, \eta)$ *defined by*

$$\kappa(\mu, \eta) := \kappa_h(\mu)\, (1 + \kappa_V(\eta))\, (1 + \eta(V)/\eta(h))\, / \mu(h).$$

PROOF. Combining (4.10) with the Boltzman-Gibbs estimate (2.7) we readily check the estimate

$$|||\Phi_t(\mu) - \Phi_t(\eta)|||_V \leq\; \kappa_h(\mu)\, (1 + \kappa_V(\eta))\; \left|\left|\left| (\Psi_h(\mu) - \Psi_h(\eta)) P^h_t \right|\right|\right|_{V/h}.$$

The contraction estimate (4.33) now implies that

$$\left|\left|\left| (\Psi_h(\mu) - \Psi_h(\eta)) P^h_t \right|\right|\right|_{V/h} \leq a_h\; \kappa_h(\mu)\, (1 + \kappa_V(\eta))\; e^{-b_h t}\; |||\Psi_h(\mu) - \Psi_h(\eta)|||_{V/h}.$$

Using the Boltzman-Gibbs estimate (2.8) we also have

$$|||\Psi_h(\mu) - \Psi_h(\eta)|||_{V/h} \leq \frac{1}{\mu(h)} \left(1 + \frac{\eta(V)}{\eta(h)} \right)\; |||\mu - \eta|||_V.$$

This ends the proof of the corollary. □

Corollary 11 is closely related to the exponential decay estimates stated in Theorem 1 of [109] under different regularity conditions. In contrast with [109], our approach is based on the V-contraction properties of the h-semigroups and it allows one to derive local contraction estimates.

Defining the finite rank (and hence compact) operator

$$f \in \mathcal{B}_V(E) \mapsto T(f) := \frac{h}{\eta_\infty(h)}\ \eta_\infty(f) \in \mathcal{B}_{0,V}(E),$$

the next corollary follows word-for-word the same arguments as the proof of Corollary 7, thus it is skipped.

COROLLARY 12. *Under condition $(\mathcal{H}^h)$, for any $t \in \mathcal{T}$ we have the operator norm exponential decay*

$$\left|\left|\left|\overline{Q}_t - T\right|\right|\right|_V \leq a_h\ e^{-b_h t}\ (1 + \eta_\infty(V)/\eta_\infty(h)) \tag{4.38}$$

with $\overline{Q}_t := e^{-t\rho}\ Q_t$ and the same parameters (a_h, b_h) as in (4.33).

5. Some illustrations.

5.1. *Nonlinear conditional processes.* Whenever $Q_{s,t}$ is sub-Markovian, we have the nonlinear transport equation

$$\Phi_{s,t}(\mu) = \mu M^\mu_{s,t}$$

with the collection of Markov transition $M^\mu_{s,t}$ indexed by $\mu \in \mathcal{P}(E)$ given by the formula

$$M^\mu_{s,t}(f)(x) = Q_{s,t}(1)(x)\ \frac{Q_{s,t}(f)(x)}{Q_{s,t}(1)(x)} + (1 - Q_{s,t}(1)(x))\ \Phi_{s,t}(\mu)(f).$$

We also recall that for any $s \leq u \leq t$ we have the nonlinear semigroup equation

$$M^\mu_{s,t} = M^\mu_{s,u} M^{\Phi_{s,u}(\mu)}_{u,t}.$$

This shows that the normalised semigroup $\Phi_{s,t}$ is the semigroup of a nonlinear Markov process sometimes called process conditioned to non-absorption at every time step. For time homogeneous models, unless μ coincides with the quasi-invariant measure $\Phi_t(\eta_\infty) = \eta_\infty$, the process is a nonlinear interacting jump process. In this interpretation, the distribution on path-space of the nonlinear process coincides with the McKean-measure associated with a jump process whose jumps intensity depends on the distribution of the random states.

For a more thorough discussion on the nonlinear interacting jump processes associated with these nonlinear Markov semigroups we refer to section 12.3 in [73] and the articles [5, 76, 77]. Next proposition is a direct consequence of Theorem 4.3.

PROPOSITION 5.1. *Under the assumptions of Theorem 4.2, for any $s \leq t$ and any $\mu, \eta \in \mathcal{P}_V(E)$ we have the local Lipschitz operator norm estimate*

$$\left|\left|\left|\delta_x M^\mu_{s,t} - \delta_x M^\eta_{s,t}\right|\right|\right|_V \leq a\ \kappa(\eta, \mu)\ e^{-b(t-s)}\ \ |||\mu - \eta|||_V$$

with $(a, b, \kappa(\eta, \mu))$ as in (4.25).

5.2. *Sub-Markovian semigroups.* Sub-Markovian operators are naturally associated with killed or absorbed stochastic processes. Consider a stochastic flow $t \in [s,\infty[\mapsto X^c_{s,t}(x)$ starting at $X^c_{s,s}(x) = x \in E$ when $t = s$ and absorbed in a cemetery state c at some random time $T^c_s(x)$. For instance, suppose we are given an auxiliary stochastic flow $X_{s,t}(x)$ evolving on E, which is sent to the cemetery at some uniformly bounded rate $U_t(y) \geq 0$ when at $y \in E$. In this situation, we have the so-called Feynman-Kac propagator formulae

$$\begin{aligned} Q_{s,t}(f)(x) &= \mathbb{E}\left(f(X^c_{s,t}(x))\, 1_{T^c_s(x)>t}\right) \\ &= \mathbb{E}\left(f(X_{s,t}(x))\, \exp\left(-\int_s^t U_u(X_{s,u}(x))\, du\right)\right). \end{aligned} \tag{5.1}$$

In this context, it is readily checked that the normalised and unnormalised semigroups are connected by the formula

$$\mu Q_{s,t}(f) = \Phi_{s,t}(\mu)(f)\, \exp\left(-\int_s^t \Phi_{s,u}(\mu)(U_u)\, du\right). \tag{5.2}$$

In terms of the absorption time, the above formula reads

$$\int \mu(dx)\, \mathbb{P}(X^c_{s,t}(x) \in dy,\ T^c_s(x) > t) = \Phi_{s,t}(\mu)(dy)\, \exp\left(-\int_s^t \Phi_{s,u}(\mu)(U_u)\, du\right).$$

This shows that the killing time of the process starting from μ at time s is a Poisson process with a time varying rate function $\Phi_{s,t}(\mu)(U_t)$ that depends on μ. The discrete time version of the above formula coincides with the product formula (2.11). For a more thorough discussion on this subject we refer to [76, section 1.3.2], [72, proposition 2.3.1] or [73, section 12.2.1]. As noted in [63, 161], in the context of time homogeneous models, we readily check that the killing time is exponentially distributed as soon as $\mu = \eta_\infty = \Phi_t(\eta_\infty)$.

Applying the above to $f = 1$ for any $\mu_1, \mu_2 \in \mathcal{P}(E)$ and $s \leq t$ we check that

$$\mu_1 Q_{s,t}(1)/\mu_2 Q_{s,t}(1) = \exp\left(\int_s^t (\Phi_{s,u}(\mu_2)(U_u) - \Phi_{s,u}(\mu_1)(U_u))\, du\right). \tag{5.3}$$

The discrete time version of the above formula coincides with (2.11). Under the assumptions of Theorem 4.1 the norm of the first order operator introduced in (4.26) decays exponentially. That is for any f such that $\mathrm{osc}(f) := \sup_{(x,y)\in E^2} |f(x) - f(y)| = 1$, we have

$$\|D_{\mu_2}\Phi_{s,t}(f)\| \leq q\, (1-\varepsilon_\tau)^{(t-s)/\tau} \quad \text{with} \quad q := \sup \frac{\mu_1 Q_{s,t}(1)}{\mu_2 Q_{s,t}(1)} < \infty. \tag{5.4}$$

where the supremum is taken over all indices (s,t) such that $s \in \mathcal{T}$, $t \in [s,\infty[_\tau$ and $\mu_1, \mu_2 \in \mathcal{P}(E)$.

In the context of time homogeneous models $X_{s,s+t}(x) = X_t(x) := X_{0,t}(x)$ and $U_t = U$, using formula (5.2) we readily check that the ground state h discussed in (4.8) takes the following form

$$h(x) = \lim_{t\to\infty} \frac{\delta_x Q_t(1)}{\eta_\infty Q_t(1)} = \exp\left(\int_0^\infty (\Phi_s(\eta_\infty)(U) - \Phi_s(\delta_x)(U))\, ds\right).$$

5.3. *Path space Feynman-Kac measures.* Let $\Omega = D([0,\infty[, E)$ be the space of càdlàg paths $\omega : s \in \mathbb{R}_+ := [0,\infty[\mapsto \omega_s \in E$. Consider a canonical Markov process $(\Omega, (X_s)_{s\geq 0}, (\mathcal{F}_s)_{s\geq 0}, \mathbb{P}_\mu)$ with generator L and initial distribution $\mu \in \mathcal{P}_V(E)$. In this notation, the Feynman-Kac measure on path-space associated with the time homogeneous version of (5.1) is defined for any $t \geq 0$ and $\omega \in \Omega$ by the formula

$$\mathbb{Q}_{\mu,t}(d\omega) := \frac{1}{\mathbb{Z}_{\mu,t}}\, \exp\left(-\int_0^t U(\omega_s)\, ds\right) \mathbb{P}_\mu(d\omega), \tag{5.5}$$

where $\mathbb{Z}_{\mu,t}$ is a normalising constant. In continuous time, the leading eigen-pair (ρ, h), discussed in (4.32), is obtained by solving the equation

$$L(h) - Uh = \rho h \Longleftrightarrow -U = \rho - L(h)/h.$$

Let $\mathbb{P}^h_{\Psi_h(\mu)}$ be the distribution of the h-process starting with the initial probability measure $\Psi_h(\mu) \in \mathcal{P}_{V/h}(E)$. In this notation, an exponential change of probability measure yields, for any $t \geq s \geq 0$ and any bounded $\mathcal{F}_s$-measurable function F_s,

$$\mathbb{Q}_{\mu,t}(F_s) - \mathbb{P}^h_{\Psi_h(\mu)}(F_s) = \int_\Omega \mathbb{P}^h_{\Psi_h(\mu)}(d\omega)\ F_s(\omega)\ \left(\frac{P^h_{t-s}(1/h)(\omega_s)}{\eta^h_s P^h_{t-s}(1/h)} - 1\right).$$

Using (4.36) we check the following estimate.

COROLLARY 13. *Under the assumptions of Corollary 10 for any $s \leq t$ and $\mu \in \mathcal{P}_V(E)$ we have*

$$|\mathbb{Q}_{\mu,t}(F_s) - \mathbb{P}^h_{\Psi_h(\mu)}(F_s)| \leq c(\mu)\ a_h\ e^{-b_h(t-s)}\ \|F_s\|,$$

where the parameters $(a_h, b_h, c(\mu))$ were defined in Corollary 10.

The above exponential estimate improves the asymptotic result presented in Proposition 6.1 in [201], and simplifies the analysis in [46, 48]. We also mention that the Feynman-Kac measures on path space $\mathbb{Q}_{\mu,t}$ and thus the path-distribution of the h-process can be approximated using genealogical tree based Monte Carlo methods, see for instance [5, 72, 73, 83] and references therein. In Quantum physics and more particularly in statistical mechanics, the measure $\mathbb{Q}_{\mu,t}$ is sometimes called the grand-ensemble associated with the interaction potential U [114, 201]. In particle absorption literature, the distribution $\mathbb{P}^h_{\Psi_h(\mu)}$ of the h-process is sometimes called the distribution of the Q-process, that is the process conditioned to never be extinct [46, 48].

We now state some results pertaining to the limiting behaviour of the occupation measure of the time homogeneous stochastic flow. To simplify the presentation, we shall only work under the strong regularity conditions stated in Theorem 4.1. In this context, these results are direct consequences of our semigroup analysis that build on the analysis developed in [48, 55, 126].

Observe that for any $s \leq u \leq t$ we have

$$\mathbb{E}\left(f(X^c_{s,u}(x)) \mid T^c_s(x) > t\right) = \Psi_{Q_{u,t}(1)}\left(\Phi_{s,u}(\delta_x)\right)(f).$$

Setting $X^c_u(x) := X^c_{0,u}(x)$ and $T^c(x) := T^c_0(x)$, due to (4.13) we obtain the following corollary.

COROLLARY 14. *Under the assumptions of Corollary 3, we have*

$$\begin{aligned}\sup_{\|f\|\leq 1}\sup_{x\in E}&\left|\mathbb{E}\left(\frac{1}{t}\int_0^t f(X^c_u(x))\,du \mid T^c(x) > t\right) - \Psi_h(\eta_\infty)(f)\right| \\ &\leq \frac{\tau}{t}\,\frac{1}{|\log(1-\varepsilon_\tau)|}\left(q(\eta_\infty) + 2\left(\|h\|/\eta_\infty(h) + q(\eta_\infty)^2\right)\right).\end{aligned} \tag{5.6}$$

In the above display, h stands for the eigenfunction defined in Corollary 5. The measure $\Psi_h(\eta_\infty)$ which coincides with the invariant measure of the h-process is sometimes called the quasi-ergodic measure of the non-absorbed process. Similarly, we also obtain a uniform bound on the L^2 distance.

PROPOSITION 5.2. *Under the assumptions of Corollary 3, for any* $f \in \mathcal{B}(E)$ *with* $\mathrm{osc}(f) \leq 1$ *and any* $t \in \mathcal{T}$ *we have the uniform estimate*

$$\left|\mathbb{E}\left(\left(\frac{1}{t}\int_0^t f(X_u^c(x))\,du - \Psi_h(\eta_\infty)(f)\right)^2 \mid T^c(x) > t\right)\right|$$

$$\leq \frac{8\tau}{t}\,\frac{\left(\|h\|/\eta_\infty(h) + (1 \vee q(\eta_\infty))^2\right)}{|\log(1-\varepsilon_\tau)|}.$$

PROOF. First observe that for any $s \leq u \leq v \leq t$ we have

$$\mathbb{E}\left(f(X_{s,u}^c(x))\,f(X_{s,v}^c(x)) \mid T_s^c(x) > t\right)$$

$$= \frac{\delta_x Q_{s,u}\left(f\,Q_{u,t}(1)\,Q_{u,v}(f\,Q_{v,t}(1))/Q_{u,v}(Q_{v,t}(1))\right)}{\delta_x Q_{s,u}(Q_{u,t}(1))}$$

$$= \int \Psi_{Q_{u,t}(1)}\left(\Phi_{s,u}(\delta_x)\right)(dy)\,f(y)\int \Psi_{Q_{v,t}(1)}\left(\Phi_{u,v}(\delta_y)\right)(dz)\,f(z).$$

Replacing f by $(f - \Psi_h(\eta_\infty)(f))$ there is no loss of generality to assume that $\Psi_h(\eta_\infty)(f) = 0$. In this situation, due to (4.13), we have

$$\left|\mathbb{E}\left(\left(\frac{1}{t}\int_0^t f(X_u^c(x))\,du\right)^2 \mid T^c(x) > t\right)\right|$$

$$\leq \frac{4}{t^2}\,\left(\|h\|/\eta_\infty(h) + (1 \vee q(\eta_\infty))^2\right)\int_{0\leq u\leq v\leq t}\left(e^{-a(v-u)} + e^{-a(t-v)}\right)dv\,du$$

with

$$a := |\log(1-\varepsilon_\tau)|/\tau.$$

This implies that

$$\left|\mathbb{E}\left(\left(\frac{1}{t}\int_0^t f(X_u^c(x))\,du\right)^2 \mid T^c(x) > t\right)\right| \leq \frac{8}{at}\,\left(\|h\|/\eta_\infty(h) + (1 \vee q(\eta_\infty))^2\right).$$

□

We end our discussion of sub-Markov operators with bounded potentials by noting that we may also extend Proposition 5.2 to also obtain bounds in $\mathbb{L}^p$ of the order $t^{-1/2}$. The idea behind the proof is to write the averages in terms of the h-process (in particular, the trajectorial version) and then apply $\mathbb{L}^p$ bounds for occupation measures of a Markov process, using the Poisson equation associated with the h-semigroup (see for instance Lemma 8.4.11 in [89] in discrete time settings).

5.4. *Comparisons of our conditions with the literature.* In this section, we highlight some of the comparisons between the models and the regularity conditions discussed in the present article and conditions often used in the literature for positive integral operators.

We begin by remarking that the class of positive semigroups discussed in this article encapsulates discrete generation Feynman-Kac semigroups defined for any $t \in [0,\infty[_\tau$ by the formula

$$Q_{t,t+\tau}(x,dy) = G_{t,t+\tau}(x)\,P_{t,t+\tau}(x,dy)$$

with the potential function G_t and the Markov transition $P_{t,t+\tau}$ defined by

$$G_{t,t+\tau} := Q_{t,t+\tau}(1) \quad \text{and} \quad P_{t,t+\tau}(f) := Q_{t,t+\tau}(f)/Q_{t,t+\tau}(1). \tag{5.7}$$

This class of probabilistic models arises in a variety of disciplines including statistical physics, biology, signal processing, rare event analysis, and many others; see [72, 73, 76, 89] and the relevant references therein.

In this context, the uniform minorization condition (3.7) is a well known strong condition ensuring the stability of the semigroups $\Phi_{s,t}$ and the existence of fixed point invariant measures for time homogeneous semigroups; see for instance [76, Theorem 2.3], [78, Lemma 2.1 & Lemma 2.3], [85, Lemma 2.1], [73, section 12.2], as well as [76, section 2.1.2 & section 3.1.3] and [72, section 4].

The next condition is taken from [46]. In terms of the Markov transition $P_{s,s+\tau}$ discussed in (5.7) it takes the following form:

$(\mathcal{P})$: *For any $s \in \mathcal{T}$ and $(x_1, x_2) \in E^2$ there exists some $\nu \in \mathcal{P}(E)$ such that for $i = 1, 2$ and any $t \geq s + \tau$ we have the estimates*

$$\delta_{x_i} P_{s,s+\tau} \geq \varepsilon_1\, \nu \quad \textit{and} \quad \varepsilon_2\, \|Q_{s+\tau,t}(1)\| \leq \nu(Q_{s+\tau,t}(1)) \tag{5.8}$$

for some parameters $0 < \varepsilon_i \leq 1$ whose values do not depend on x_i, nor on $s \in \mathcal{T}$.

We also refer the reader to conditions (A1) and (A2) in [47], as well as [45, 48, 52] for further work and discussion on this condition. The time-homogeneous version of the above condition is appears in a variety of contexts in the literature. For example, as shown in [46, Theorem 2.1], condition $(\mathcal{P})$ is a sufficient and necessary condition for the *uniform* exponential decay (4.1), which was also applied in [132, Theorem 10] and [133, Theorem 7.1] to neutron transport models. In addition, we refer the reader to [14, section 2.2] for the use of condition $(\mathcal{P})$ in the design of admissible coupling constants (a.k.a. generalised Doeblin's conditions) and to [54] on birth-and-death processes where it is the main ingredient of the proof of Theorem 3.1, and to [126, section 2]. Moreover, for time-homogeneous models satisfying (3.7), the right-hand side estimate in (5.8) is a direct consequence of the right-hand side estimate in (4.6).

When $(\mathcal{P})$ is met, for any $f \geq 0$ and $i = 1, 2$ choosing $H = 1$ we have the lower bound estimate

$$R^{(t)}_{s,s+\tau}(f)(x_i) \geq \varepsilon_1\, \frac{\nu\,(Q_{s+\tau,t}(1)\; f)}{P_{s,s+\tau}\,(Q_{s+\tau,t}(1))\,(x_i)} \geq \varepsilon_1 \varepsilon_2\; \Psi_{Q_{s+\tau,t}(1)}(\nu_s)(f).$$

This implies that condition $(\mathcal{P})$ is stronger than the Dobrushin's condition discussed in (3.7). More precisely, we have

$$(\mathcal{P}) \Longrightarrow (3.7) \quad \text{with} \quad H = 1 \qquad \varepsilon_\tau := \varepsilon_1 \varepsilon_2 \quad \text{and} \quad \nu = \Psi_{Q_{s+\tau,t}(1)}(\nu_s).$$

As previously mentioned, the uniform minorisation condition (3.7) as well as $(\mathcal{P})$ are difficult to check in practice; several sufficient conditions are discussed in [72, 73, 76, 78, 84, 85].

The next condition is a slight extension of [199, condition (68)] and [13, condition (3.24)].

$(\mathcal{Q})$: *There exists a positive measurable function $\varsigma_s(x) > 0$, a constant $\rho > 0$ and probability measures ν_s such that*

$$\varsigma_s(x)\; \nu_s(dy) \leq \delta_x Q_{s,s+\tau}(dy) \leq \rho\; \varsigma_s(x)\; \nu_s(dy). \tag{5.9}$$

Choosing $H = 1$ in (3.4), this condition implies that for any $f \geq 0$ we have

$$R^{(t)}_{s,u}(f) := \frac{Q_{s,u}(Q_{u,t}(1)\; f)}{Q_{s,u}(Q_{u,t}(1))} \geq \rho^{-1}\; \nu^{(t)}_{s,u}(f) \quad \text{with} \quad \nu^{(t)}_s(f) := \frac{\nu_s(Q_{s+\tau,t}(1) f)}{\nu_s(Q_{s+\tau,t}(1))}.$$

This shows that

$$(\mathcal{Q}) \Longrightarrow (3.7) \quad \text{with} \quad H = 1 \qquad \varepsilon_\tau := \rho^{-1} \quad \text{and} \quad \nu = \nu_s^{(t)}.$$

Condition $(\mathcal{Q})$ with $\varsigma_s(x) = 1$ is also discussed in [72, section 4.3.2], as well as in [152] and [109, section 3.3]. We also refer the reader to [45], where it was shown that condition $(\mathcal{Q})$ implies the uniform exponential decay (4.1) in the time-homogeneous case.

Also notice that

$$(\mathcal{Q}) \Longleftrightarrow \rho^{-1}\, \nu_s \leq \delta_x P_{s,s+\tau} \leq \rho\, \nu_s,$$

where the Markov transition $P_{s,s+\tau}$ was defined in (5.7). These rather strong two-sided minorisation conditions are well-known: see for instance the uniformly positive condition discussed in [20], [11, condition (19)] in the framework of Hilbert projective metrics, [78, condition (B)] and [76, Theorem 2.3]. From the above discussion it should be clear that conditions $(\mathcal{P})$ and $(\mathcal{Q})$ are stronger than the Dobrushin condition presented in (3.7).

The class of triangular array semigroups introduced in (3.4) are also considered in the article [15] in the context of time homogeneous sub-Markovian models. In our framework, the authors assume the existence of positive functions $H \leq V$, a probability measure ν defined on some compact $K \subset E$, and some finite constant c, such that

$$\sup_K V/H < \infty \qquad Q_\tau(V) \leq a\, V + c\, 1_K H \quad \text{with} \quad 0 < a < \inf_E (Q_\tau(H)/H). \tag{5.10}$$

In addition, there exists some $\varepsilon \in]0,1]$ such that for any positive function $f \in \mathcal{B}_{V/H}(E)$ and any $x \in K$ we have

$$Q_\tau(fH)(x)/Q_\tau(H)(x) \geq \varepsilon\, \nu \quad \text{and} \quad \sup_{t \in [0,\infty[_\tau} \sup_K \frac{Q_t(H)/H}{\nu(Q_t(H)/H)} < \infty. \tag{5.11}$$

In Lemma 3.1 in [15], using the right-hand side condition in the above display, the authors obtain a Lyapunov equation defined as (4.17) by replacing $H_{s,t}$ by the function

$$\mathcal{H}_{s,t} := \frac{Q_{t-s}(H)}{\nu(Q_{t-s}(H)/H)} \neq H_{s,t} = \frac{Q_{t-s}(H)}{\overline{\eta}_s Q_{t-s}(1)}.$$

Theorem 2.1 in [15] ensures the existence of a leading triple (ρ, η_∞, h), as well as exponential estimates similar to the ones discussed in Corollary 12. Thus, for the class of semigroups considered in Corollary 9 in the present article, the conditions (5.10) and (5.11) ensures that the semigroup Q_t is a V-positive semigroup. Conversely, the authors show that the existence of a leading triple (ρ, η_∞, h) satisfying these exponential decays imply that the pair (V, h) satisfies condition (5.10) and (5.11).

In the context of time homogeneous models, up to a change of Lyapunov function as discussed above, Proposition 3.3 in [15] is closely related to Theorem 4.2 in the present article. In contrast with the V-norm Lipschitz estimates stated in Corollary 11 presented in this article, Corollary 3.7 in [15] doesn't provide any Lipschitz estimates but also yields some exponential decays of the normalised semigroup Φ_t to equilibrium with respect to the total variation norm.

6. Proofs of the stability theorems.

6.1. *Proof of Theorem 1.1.*

PROOF. One direction of the proof is obvious. Indeed, if Φ_1 has at least one invariant probability measure η_∞ choosing $\eta = \eta_\infty$ the measure $\Phi_n(\eta_\infty) = \eta_\infty$ is tight and $\beta_n(\eta_\infty) = \eta_\infty(Q(1))$.

Conversely, assume that for some η the sequence of probability measures $\Phi_n(\eta)$ is tight and (1.4) is satisfied. In this situation, for any $\varepsilon > 0$ there exists a compact $K_\varepsilon \subset E$ such that

$$\Phi_n(\eta)(Q(1)) \geq \Phi_n(\eta)(1_{K_\varepsilon} Q(1)) \geq (1-\varepsilon) \inf_{K_\varepsilon} Q(1) \quad \text{and} \quad \beta_\infty(\eta) > 0.$$

We simplify the notation and write β_n and β_∞ instead of $\beta_n(\eta)$ and $\beta_\infty(\eta)$. Consider the probability measures

$$\eta_m := \frac{1}{m} \sum_{0 \leq k < m} \Phi_k(\eta) \Longrightarrow \eta_m(Q(1)) = \frac{1}{m} \sum_{0 \leq k < m} \beta_k \longrightarrow_{m \to \infty} \beta_\infty.$$

There exists at least one probability measure $\eta_\infty := \varpi(\eta)$ and a sub-sequence $m_k \to_{k \to \infty} \infty$ such that η_{m_k} converges weakly to η_∞ as $k \to \infty$. Hence, η_{m_k} and $\Phi_1(\eta_{m_k})$ converge weakly to η_∞ and $\Phi_1(\eta_\infty)$, respectively, as $m \to \infty$. In addition, we have

$$\Phi_n(\eta)(Q(f)) = \Phi_n(\eta)(Q(1))\ \Phi_{n+1}(\eta)(f).$$

This yields the formula

$$\Phi_1(\eta_m) = \sum_{0 \leq k < m} \frac{\beta_k}{\sum_{0 \leq l < m} \beta_l}\ \Phi_{k+1}(\eta)$$

from which we check that

$$\begin{aligned}\Phi_1(\eta_m) - \eta_m = \frac{1}{m} \sum_{0 \leq k < m} \left(\frac{\beta_k}{\frac{1}{m} \sum_{0 \leq l < m} \beta_l} - 1 \right)\ \Phi_{k+1}(\eta) \\ + \frac{1}{m} \sum_{0 \leq k < m} \left(\Phi_{k+1}(\eta) - \Phi_k(\eta) \right).\end{aligned}$$

As a result

$$\begin{aligned}&\Phi_1(\eta_m) - \eta_m \\ &= \frac{1}{m} \sum_{0 \leq k < m} \left(\frac{\beta_k}{\frac{1}{m} \sum_{0 \leq l < m} \beta_l} - 1 \right)\ \Phi_{k+1}(\eta) + \frac{1}{m}\ (\Phi_m(\eta) - \eta).\end{aligned}$$

On the other hand, we have

$$\frac{1}{m} \sum_{0 \leq k < m} \left| \beta_k - \frac{1}{m} \sum_{0 \leq l < m} \beta_l \right| \leq \frac{2}{m} \sum_{0 \leq k < m} |\beta_k - \beta_\infty| \longrightarrow_{m \to \infty} 0.$$

For any $f \in \mathcal{C}_b(E)$ we conclude that

$$|(\Phi_1(\eta_{m_k}) - \eta_{m_k})(f)| \longrightarrow_{k \to \infty} 0 \quad \text{and therefore} \quad \eta_\infty = \Phi_1(\eta_\infty).$$

The last assertion follows from the fact that

$$\eta_{m_k}(Q(1)) = \frac{1}{m_k} \sum_{l=0}^{m_k - 1} \beta_l \longrightarrow_{m \to \infty} \beta_\infty = \eta_\infty(Q(1)).$$

This ends the proof of the theorem. □

6.2. *Proof of Theorem 4.2.* This section is mainly concerned with the proof of Theorem 4.2. For any $\mu \in \mathcal{P}_V(E)$ and $n \geq 1$, applying Markov's inequality we have

$$r_n := \kappa_V(\mu) + n \Longrightarrow \Phi_{s,t}(\mu)(V > r_n) \leq \kappa_V(\mu)/(\kappa_V(\mu) + n) > 0.$$

We conclude that

$$\inf \Phi_{s,t}(\mu)(V \leq r_n) \geq 1/(1 + \kappa_V(\mu)/n) \longrightarrow_{n\to\infty} 1. \tag{6.1}$$

In the above display, the infimum are taken over all $s \in \mathcal{T}$ and $t \geq s$.

LEMMA 6.1. *The estimate (4.17) holds as soon as (3.13) and (3.14) are satisfied.*

PROOF. For any $n \geq 1$ we set $r_n := \overline{\eta}(V) + n$. For any $s \in \mathcal{T}$ and $r > \lambda^- r_1$ we have the estimate

$$Q_{s,s+\tau}(V)/V < (\lambda^-/r)\, 1_{K_r(s)^c} + \|\Theta_\tau\|\, 1_{K_r(s)},$$

with the sets $K_r(s)$ are defined by

$$K_r(s) := \left\{Q_{s,s+\tau}(V)/V \geq \lambda^-/r\right\} \subset \mathcal{K}_r^- := \left\{\Theta_\tau \geq \lambda^-/r\right\}.$$

On the other hand, for any $s \in \mathcal{T}$ such that $s + \tau \leq t$ we have

$$\frac{R^{(t)}_{s,s+\tau}(V/H_{s+\tau,t})}{(V/H_{s,t})} = \frac{1}{\lambda_{s,s+\tau}}\, \frac{Q_{s,s+\tau}(V)}{V}.$$

This yields the estimate

$$\begin{aligned}\frac{R^{(t)}_{s,s+\tau}(V/H_{s+\tau,t})}{(V/H_{s,t})} &\leq \frac{\lambda^-}{\lambda_{s,s+\tau}} \left(\frac{1}{r} 1_{K_r(s)^c} + \frac{\|\Theta_\tau\|}{\lambda^-} 1_{K_r(s)}\right) \\ &\leq \frac{1}{r}\, 1_{K_r(s)^c} + \frac{\|\Theta_\tau\|}{\lambda^-}\, 1_{K_r(s)},\end{aligned}$$

from which we check that

$$R^{(t)}_{s,s+\tau}(V/H_{s+\tau,t}) \leq \frac{1}{r}\, (V/H_{s,t}) + \frac{\|\Theta_\tau\|}{\lambda^-} \sup_{\mathcal{K}_r^-}(V/H_{s,t}).$$

Now, choosing $n \geq 1$ sufficiently large such that

$$r_n \geq r/\lambda^- \geq r_1 \vee r_H,$$

by (2.3) there exists some $\overline{r}_n$ such that

$$\mathcal{K}_r^- = \left\{\Theta_\tau \geq \lambda^-/r\right\} \subset \mathcal{K}_{r_n} := \{\Theta_\tau \geq 1/r_n\} \subset \{V \leq \overline{r}_n\}.$$

By (3.14) this yields the uniform estimate

$$\sup_{\mathcal{K}_r^-}(V/H_{s,t}) \leq \overline{r}_n/\varsigma_{\overline{r}_n}(H) < \infty.$$

This ends the proof of the lemma. □

PROOF OF THEOREM 4.2. Observe that

$$\|H\|_V = 1 \Longrightarrow H \leq V \quad \text{and} \quad H_{s,t} = \overline{Q}_{s,t}(H) \leq \overline{Q}_{s,t}(V).$$

This implies that

$$\lambda^-\, H_{s,t}/V \leq \lambda_{s,s+\tau}\, H_{s,t}/V = Q_{s,s+\tau}(H_{s+\tau,t})/V \leq Q_{s,s+\tau}(V)/V.$$

from which we check that

$$\lambda^- \, V_\theta \leq V/H_{s,t} \quad \text{with} \quad V_\theta := 1/\Theta_\tau.$$

Notice that V_θ has compact level sets and for any $r \geq 1$ there exists some $\varphi(r) \geq 1$ such that for any $\overline{r} \geq \varphi(r)$ we have

$$\{V/H_{s,t} \leq r\} \subset \{V_\theta \leq r/\lambda^-\} \subset \{V \leq \varphi(r)\} \subset \{V \leq \overline{r}\}.$$

By (3.15) for any $\overline{r} \geq \varphi(r) \vee r_0$ we have

$$\sup_{(V/H_{s,t})(x) \vee (V/H_{s,t})(y) \leq r} \left\| \delta_x R^{(t)}_{s,s+\tau} - \delta_y R^{(t)}_{s,s+\tau} \right\|_{tv} \leq 1 - \alpha(\overline{r}) < 1.$$

Now, due to the previous lemma, (4.17) holds which in turn implies that

$$R^{(t)}_{s,s+\tau}(W_{s+\tau,t}) \leq \varepsilon \, W_{s,t} + 1$$

with the collection of functions $W_{s,t} \geq 1$ defined by

$$\frac{\varepsilon}{c} \, \frac{V}{H_{s,t}} \leq W_{s,t} := 1 + \frac{\varepsilon}{c} \, \frac{V}{H_{s,t}} \leq \left(1 + \frac{\varepsilon}{c}\right) \, \frac{V}{H_{s,t}}.$$

Then, for any $\overline{r} \geq \varepsilon(\varphi(r) \vee r_0)/c$ we have

$$\sup_{W_{s,t}(x) \vee W_{s,t}(y) \leq r} \|\delta_x R^{(t)}_{s,s+\tau} - \delta_y R^{(t)}_{s,s+\tau}\|_{tv} \leq 1 - \alpha\left(c\overline{r}/\varepsilon\right) < 1.$$

Applying Lemma 2.3 (see for instance the contraction estimate (2.18)). for any $s \in \mathcal{T}$, $u \in [s,t]_\tau$, $t \in [s,\infty[_\tau$ and $\mu, \eta \in \mathcal{P}_{V/H_{s,t}}$, we have the uniform contraction estimate

$$\|\mu R^{(t)}_{s,u} - \eta R^{(t)}_{s,u}\|_{V/H_{u,t}} \leq a \, e^{-b(u-s)} \, \|\mu - \eta\|_{V/H_{s,t}}.$$

This ends the proof of the theorem. □

PROOF OF COROLLARY 6. For continuous time semigroups, for any $s_n = s + n\tau$ and $u \in [0,\tau[$ we have

$$\begin{aligned} R^{(t)}_{s_n,s_n+u}(V/H_{s_n+u,t}) &= (V/H_{s_n,t}) \, \frac{R^{(t)}_{s_n,s_n+u}(V/H_{s_n+u,t})}{(V/H_{s_n,t})} \\ &= (V/H_{s_n,t}) \, \frac{1}{\lambda_{s_n,s_n+u}} \, \frac{Q_{s_n,s_n+u}(V)}{V} \leq \frac{\pi_\tau(V)}{\lambda^-(\overline{\eta}_0)} \, (V/H_{s_n,t}). \end{aligned}$$

This implies that

$$\begin{aligned} \|\mu R^{(t)}_{s,s_n+t} - \eta R^{(t)}_{s,s_n+t}\|_{V/H_{s_n+u,t}} &\leq |(\mu R^{(t)}_{s,s_n} - \eta R^{(t)}_{s,s_n})| R^{(t)}_{s_n,s_n+u}(V/H_{s_n+u,t}) \\ &\leq \frac{\pi_\tau(V)}{\lambda^-(\overline{\eta}_0)} \, \|\mu R^{(t)}_{s,s_n} - \eta R^{(t)}_{s,s_n}\|_{V/H_{s_n,t}}. \end{aligned}$$

This ends the proof of the theorem. □

6.3. *Proof of Theorem 4.3.*

PROOF. Observe that

$$\lambda_{s,t}\, H_{s,t}\, R^{(t)}_{s,t}(f/H) = \, Q_{s,t}(f),$$

which implies that

$$\Psi_{H_{s,t}}(\mu) R^{(t)}_{s,t}(f/H) = \frac{\mu(Q_{s,t}(f))}{\mu(Q_{s,t}(H))} = \frac{\Phi_{s,t}(\mu)(f)}{\Phi_{s,t}(\mu)(H)}.$$

Combining this with (3.5) and the estimate (2.7) we have

$$\begin{aligned}\|\!|\Phi_{s,t}(\mu) - \Phi_{s,t}(\eta)|\!\|_V &= \left\|\!\left|\Psi_{1/H}(\Psi_{H_{s,t}}(\mu) R^{(t)}_{s,t}) - \Psi_{1/H}(\Psi_{H_{s,t}}(\eta) R^{(t)}_{s,t})\right|\!\right\|_V \\ &\leq \Phi_{s,t}(\mu)(H)\,(1 + \Phi_{s,t}(\eta)(V)) \times \left\|\!\left|\left(\Psi_{H_{s,t}}(\mu) - \Psi_{H_{s,t}}(\eta)\right) R^{(t)}_{s,t}\right|\!\right\|_{V/H}.\end{aligned}$$

Applying (4.16) to $u = t$ we find that

$$\|\!|\Phi_{s,t}(\mu) - \Phi_{s,t}(\eta)|\!\|_V \leq \kappa_H(\mu)\,(1 + \kappa_V(\eta))\;\, a\, e^{-b(t-s)} \|\!|\Psi_{H_{s,t}}(\mu) - \Psi_{H_{s,t}}(\eta)|\!\|_{V/H_{s,t}}.$$

On the other hand, applying (2.8) we check that

$$\|\!|\Psi_{H_{s,t}}(\mu) - \Psi_{H_{s,t}}(\eta)|\!\|_{V/H_{s,t}} \leq \frac{1}{\mu(H_{s,t})}\left(1 + \frac{\eta(V)}{\eta(H_{s,t})}\right)\, \|\!|\mu - \eta|\!\|_V.$$

This ends the proof of the theorem. □

6.4. *Proof of Theorem 4.4.*

PROOF. Assume that Q_t is a stable V-positive semigroup. In this context, there exists an eigen-triple $(\rho, \eta_\infty, h) \in (\mathbb{R} \times \mathcal{P}_V(E) \times \mathcal{B}_{0,V}(E))$ satisfying (4.28). Choosing $(\overline{\eta}_0, H)$ in (3.1) and (3.4), we readily check that

$$R^{(t)}_{s,s+u}(f) = P^h_u(f) \quad \text{and} \quad Q_t(V)/V = e^{\rho t}\, P^h_t(V^h)/V^h \quad \text{with} \quad V^h := V/h \in \mathcal{B}_\infty(E).$$

In this situation, the Doob h-transform, P^h_t is a V^h-positive semigroup of Markov operators from $\mathcal{B}_{V^h}(E)$ to itself; P^h_t maps $\mathcal{B}_{V^h}(E)$ into $\mathcal{B}_{0,V^h}(E)$ for any $t > 0$ and we have

$$P^h_\tau(V^h)/V^h \leq e^{-\rho\tau}\Theta_\tau$$

In addition, condition (3.15) applied to $H = h$ takes the form

$$\sup_{V(x)\vee V(y)\leq r} \left\|\delta_x P^h_\tau - \delta_y P^h_\tau\right\|_{tv} \leq 1 - \alpha(r). \tag{6.2}$$

Observe that

$$V^h(x) \leq r_h \Longrightarrow V(x) \leq r = r_h \sup_{V\leq r_h} h$$

This implies that

$$\sup_{V^h(x)\vee V^h(y)\leq r} \left\|\delta_x P^h_\tau - \delta_y P^h_\tau\right\|_{tv} \leq 1 - \alpha_h(r) \quad \text{with} \quad \alpha_h(r) := \alpha(r \sup_{V\leq r} h). \tag{6.3}$$

We conclude that P^h_t is a stable V^h-positive semigroup.

In the reverse angle, choosing $(\overline{\eta}_0, H) = (\eta_\infty, h) \in (\mathcal{P}_V(E) \times \mathcal{B}_{0,V}(E))$ in (3.1) and (3.4), we readily check that

$$\overline{\eta}_0 Q_{t,t+\tau} = e^{\rho\tau} = \lambda^- = \lambda^-(\eta_\infty) \quad \text{and} \quad H_{s,t} = h$$

Note that in this case, the quantities $(\varsigma_r(H), |\!|\!| H |\!|\!|_V)$ defined in (3.14) become

$$\varsigma_r(H) = \inf_{V \leq r} h > 0 \quad \text{and} \quad |\!|\!| H |\!|\!|_V = \|h\|_V < \infty.$$

Now assume that P_t^h satisfies (6.3) for some function $\alpha_h(r)$. Using the fact that

$$V(x) \leq r \Longrightarrow V^h(x) \leq r / \inf_{V^h \leq r} h$$

we check condition (3.15) applied to $H = h$. We conclude that Q_t is a stable V-positive semigroup as soon as P_t^h is a stable V^h-positive semigroup. This ends the proof of the theorem. □

6.5. *Proof of Theorem 4.5.* This section is mainly concerned with the proof of Theorem 4.5 on the stability of the time homogeneous models discussed in (4.32). In what follow, we set $V^h := V/h$.

LEMMA 6.2. *For any $\varepsilon > 0$ we have the Foster-Lyapunov inequality*

$$P_\tau^h \left(V^h\right) < \varepsilon\, V^h\, 1_{K_{\varepsilon,\tau}^c} + c_{\varepsilon,\tau}\, 1_{K_{\varepsilon,\tau}} \tag{6.4}$$

with the parameter $c_{\varepsilon,\tau}$ and the compact set $K_{\tau,\varepsilon}$ given by

$$c_{\varepsilon,\tau} := e^{-\rho\tau}\, \|Q_\tau(V)/V\| \sup_{K_{\varepsilon,\tau}} V^h \quad \textit{and} \quad K_{\tau,\varepsilon} := \{Q_\tau(V)/V \geq \varepsilon\, e^{\rho\tau}\}.$$

PROOF. For any $\varepsilon > 0$ we have

$$P_\tau^h(V^h)/V^h = e^{-\rho\tau} Q_\tau(V)/V < \varepsilon 1_{K_\varepsilon^c} + e^{-\rho\tau} \|Q_\tau(V)/V\| 1_{K_{\varepsilon,\tau}}.$$

This readily yields the estimate (6.4). □

PROPOSITION 6.1. *For any $\mu \in \mathcal{P}_{V/h}(E)$ we have*

$$0 < \kappa_\tau(\mu) := \inf_{t \in [0,\infty[_\tau} \mu P_t^h(1/h) \leq \kappa_{\tau,V^h}^h(\mu) := \sup_{t \in [0,\infty[_\tau} \mu P_t^h(V^h) < \infty.$$

In addition, for any $\eta \in \mathcal{P}_V(E)$ we have the estimates (4.34).

PROOF. Following word-for-word the same arguments as the proof of (2.19) the Foster-Lyapunov estimate (6.4) implies that

$$\kappa_{\tau,V^h}^h(\mu) \leq \mu(V^h) + c_{\varepsilon,\tau}(1-\varepsilon)^{-1} < \infty.$$

Now, we come to the proof of the left-hand side estimate. Since $0 < h \in \mathcal{B}_{0,V}(E)$ and $\|h\|_V = 1$ the function $V^h \geq 1$ has compact level sets and h is bounded on compact sets. Consider the compact sets $\mathcal{K}_{\tau,V^h}^h(\delta)$ indexed by $\delta \in]0,1[$ and defined by

$$\mathcal{K}_{V^h}^h(\mu,\delta) := \{\delta V^h \leq \kappa_{\tau,V/h}^h(\mu)\}.$$

Arguing as in (2.20), we have

$$\inf_{t \in [0,\infty[_\tau} \mu P_t^h \left(\mathcal{K}_{V/h}^h(\mu,\delta)\right) \geq 1-\delta \quad \text{and} \qquad \kappa(\mu) \geq (1-\delta) / \sup_{\mathcal{K}_{V^h}^h(\mu,\delta)} h.$$

Observe that

$$\sup_{t\in[0,\infty[_\tau} \Psi_h(\eta)P_t^h(V^h) \leq \eta(V)/\eta(h) + c_{\varepsilon,\tau}(1-\varepsilon)^{-1}$$

and consider the compact sets

$$\mathcal{K}_V(\eta,\delta) := \{x : \delta V(x) \leq \eta(V)/\eta(h) + c_{\varepsilon,\tau}(1-\varepsilon)^{-1}\} \subset E.$$

Arguing as above, we check that

$$\inf_{t\in[0,\infty[_\tau} \Psi_h(\eta)P_t^h\left(\mathcal{K}_V(\eta,\delta)\right) \geq 1-\delta \;\text{ and }\; \inf_{t\in[0,\infty[_\tau} \Psi_h(\eta)P_t^h(1/h) \geq \frac{(1-\delta)}{\sup_{\mathcal{K}_V(\eta,\delta)} h}.$$

This yields for any $t \in [0,\infty[_\tau$ and $\delta \in]0,1[$ the uniform estimate

$$\Phi_t(\eta)(V) = \frac{\Psi_h(\eta)P_t^h(V/h)}{\Psi_h(\eta)P_t^h(1/h)} \leq \frac{\eta(V)/\eta(h) + c_{\varepsilon,\tau}(1-\varepsilon)^{-1}}{(1-\delta)/\sup_{\mathcal{K}_V(\eta,\delta)} h}.$$

This implies that $\kappa_{\tau,V}(\eta) < \infty$. By lemma 2.2, for continuous time indices we also have $\kappa_V(\eta) < \infty$. This also ensures that the sequence of probability measures $\Phi_{s,t}(\eta)$ indexed by $s \leq t$ is tight. Choosing the compact set

$$K(\delta,\eta) := \{\delta\, V \leq \kappa_V(\eta)\} \tag{6.5}$$

we readily check that

$$\Phi_{s,t}(\eta)(h) \geq \left(\inf_{K(\delta,\eta)} h\right)\, \Phi_{s,t}(\eta)(K(\delta,\eta)) \geq (1-\delta)\, \left(\inf_{K(\delta,\eta)} h\right) > 0.$$

We conclude that $\kappa_h^-(\eta) > 0$. This ends the proof of the proposition. □

PROOF OF THEOREM 4.5. The estimates (4.34) have been checked in Proposition 6.1. Observe that

$$(6.4) \Longrightarrow P_\tau^h(W) \leq \varepsilon\, W + 1$$

with the function $W \geq 1$ defined by

$$\frac{\varepsilon}{c_{\varepsilon,\tau}}\,\frac{V}{h} \leq W := 1 + \frac{\varepsilon}{c_{\varepsilon,\tau}}\,\frac{V}{h} \leq \left(1 + \frac{\varepsilon}{c_{\varepsilon,\tau}}\right)\,\frac{V}{h}, \tag{6.6}$$

which has compact level sets. On the other hand, by (1.27) we have

$$\sup_{W(x)\vee W(y)\leq r} \|\delta_x P_\tau^h - \delta_y P_\tau^h\|_{tv} \leq 1 - \alpha(c_{\varepsilon,\tau} r/\varepsilon).$$

The estimate (4.33) is now a direct consequence of (2.18) and the estimates (6.6). The proof of the theorem is now completed. □

APPENDIX A: PROOFS OF SEVERAL TECHNICAL REAULTS

A.1. Proof of Lemma 2.2. Condition $\kappa_{\tau,V}(\mu) < \infty$ ensures that that the sequence of probability measures $\Phi_{s,t}(\mu)$ indexed by $s \in \mathcal{T}$ and $t \in [s,\infty[_\tau$ is tight. In this situation, for any $\delta \in]0,1[$ there exists some compact set K such that $\pi_\tau^-(K) > 0$ and $\Phi_{s,t}(\mu)(K) \geq (1-\delta)$ for any $s \in \mathcal{T}$ and $t \in [s,\infty[_\tau$. Thus, for any $s_n := s + n\tau$ and $\varepsilon \in [0,\tau[$ we have

$$\Phi_{s,s_n+\varepsilon}(\mu)(V) = \frac{\Phi_{s,s_n}(\mu)Q_{s_n,s_n+\varepsilon}(V)}{\Phi_{s,s_n}(\mu)Q_{s_n,s_n+\varepsilon}(1)} \leq \left(\pi_\tau(V)/\pi_\tau^-(K)\right)\, \kappa_V(\mu)/(1-\delta)$$

with the parameter $\pi_\tau^-(K)$ defined in (1.16). In the same vein, using (1.18) we have

$$\Phi_{s,s_n+\varepsilon}(\mu)(H) = \frac{\Phi_{s,s_n+\varepsilon}(\mu)(H)}{\Phi_{s,s_n}(\mu)Q_{s_n,s_n+\varepsilon}(1)} \geq \inf_K H\ (1-\delta)/\pi_\tau > 0.$$

This shows that $\Phi_{s,t}(\mu)(V)$ is uniformly bounded and $\Phi_{s,t}(\mu)(H)$ is uniformly positive constant with respect to the parameters $s \in \mathcal{T}$ and $t \geq s$. In addition, the tightness of $\Phi_{s,t}(\mu)$ combined with (1.16) ensures that $\Phi_{s,t}(\mu)(Q_{t,t+\varepsilon}(1))$ is uniformly positive with respect to $s \in \mathcal{T}$ and $t \geq s$ and $\varepsilon \in [0,\tau[$.

For any $\delta \in]0,1[$ there exists some compact set K such that $\pi_\tau^-(K) > 0$ and $\Phi_{s,t}(\mu)(K) \geq (1-\delta)$ and

$$(1.16) \Longrightarrow \sup_{s\geq 0}\sup_{t\geq s}\sup_{\varepsilon\in[0,\tau]} \frac{\|Q_{t,t+\varepsilon}(1)\|}{\Phi_{s,t}(\mu)Q_{t,t+\varepsilon}(1)} \leq \frac{\pi_\tau}{(1-\delta)\pi_\tau^-(K)} < \infty.$$

Finally observe that for any $s \leq t$ and $\varepsilon \in [0,\tau]$ we have

$$(1.18) \Longrightarrow \Phi_{s,t}(\mu)(Q_{t,t+\varepsilon}(1)) \leq \Phi_{s,t}(\mu)(V\ Q_{t,t+\varepsilon}(V)/V) \leq \pi_\tau(V)\ \kappa_V(\mu).$$

This shows that $\Phi_{s,t}(\mu)(Q_{t,t+\varepsilon}(1))$ is uniformly bounded with respect to $s \in \mathcal{T}$ and $t \geq s$. This ends the proof of the lemma. ∎

A.2. Proof of Lemma 3.2. Let H be some locally bounded positive functions s.t. $V/H \in \mathcal{B}_\infty(E)$. Recall that for any $u,t \in \mathcal{T}$, we have

$$\lambda_{u,t}\ H_{u,t} = Q_{u,t}(H) =: h_{u,t} \Longrightarrow Q_{s,u}(h_{u,t}) = h_{s,t} > 0. \tag{A.1}$$

By (3.11) and (3.12) there exists some for some $\varepsilon_1 > 0$ and any $0 < \varepsilon \leq \varepsilon_1$ we have

$$0 < \iota_\varepsilon^- := \inf_{t\in\mathcal{T}}\inf_{\mathcal{V}_\varepsilon^2} q_{t,t+\tau} \leq \iota_\varepsilon := \sup_{t\in\mathcal{T}}\sup_{\mathcal{V}_\varepsilon^2} q_{t,t+\tau} < \infty \quad \text{and} \quad 0 < \nu_\tau(\mathcal{V}_\varepsilon) < \infty \tag{A.2}$$

with the compact ε-super-level sets $\mathcal{V}_\varepsilon$ of the function Θ_τ defined in (3.11). To simplify the notation, we write ν instead of ν_τ. In this notation, we have

$$\iota_\varepsilon^-\ 1_{\mathcal{V}_\varepsilon}(x)\ \nu(dy)\ 1_{\mathcal{V}_\varepsilon}(y) \leq 1_{\mathcal{V}_\varepsilon}(x)\ Q_{t,t+\tau}(x,dy)\ 1_{\mathcal{V}_\varepsilon}(y) \leq\ \iota_\varepsilon\ 1_{\mathcal{V}_\varepsilon}(x)\ \nu(dy)\ 1_{\mathcal{V}_\varepsilon}(y). \tag{A.3}$$

The next lemma is a slight modification of [196, Proposition 1].

LEMMA A.1. *There exists $\lambda_0 > 0$ and a finite constant $c_0 < \infty$ such that for any $s \in \mathcal{T}$ and $t \in [s,\infty[_\tau$ and $\eta \in \mathcal{P}_V(E)$ we have*

$$R_{s,t}^{(t)}\left(\frac{V}{H}\right) \leq e^{-\lambda_0(t-s)}\ \frac{V}{h_{s,t}} + c_0 \quad \textit{and} \quad \limsup_{t\to\infty} \frac{e^{-\lambda_0(t-s)}}{\eta(h_{s,t})} = 0. \tag{A.4}$$

In addition, we have the uniform estimates stated in (3.16).

PROOF. We set

$$h_{s,t}^V := h_{s,t}/\|h_{s,t}\|_V \quad \text{and} \quad \beta_{s,u}^{(t)} = \|h_{s,t}\|_V/\|h_{u,t}\|_V.$$

We have

$$R_{s,s+\tau}^{(t)}\left(\frac{V}{h_{s+\tau,t}^V}\right) := \frac{1}{\beta_{s,s+\tau}^{(t)}}\ \frac{V}{h_{s,t}^V}\ \frac{Q_{s,s+\tau}(V)}{V} \ \text{ and } \ Q_{s,s+\tau}(h_{s+\tau,t}^V) = \beta_{s,s+\tau}^{(t)}\ h_{s,t}^V.$$

Using (3.11) and (A.3) and recalling that $Q_{t,t+\tau}(V)/V \leq \Theta_\tau$, for any $0 \leq \varepsilon < \varepsilon_1$ we check that

$$R^{(t)}_{s,s+\tau}\left(\frac{V}{h^V_{s+\tau,t}}\right) \leq \frac{\varepsilon}{\beta^{(t)}_{s,s+\tau}}\,\frac{V}{h^V_{s,t}} + \frac{a_\varepsilon}{\nu(1_{\mathcal{V}_\varepsilon} h^V_{s+\tau})} \quad \text{with} \quad a_\varepsilon := \sup_{\mathcal{V}_\varepsilon}(V\Theta_\tau)/\iota^-_\varepsilon.$$

This implies that

$$R^{(t)}_{s,s+n\tau}\left(\frac{V}{h^V_{s+n\tau,t}}\right) \leq \frac{\varepsilon}{\beta^{(t)}_{s+(n-1)\tau,s+n\tau}} R^{(t)}_{s,s+(n-1)\tau}\left(\frac{V}{h^V_{s+(n-1)\tau,t}}\right) + \frac{a_\varepsilon}{\nu(1_{\mathcal{V}_\varepsilon} h^V_{s+n\tau,t})}.$$

Applying the above to $s+n\tau = t$ we obtain the formula

$$R^{(t)}_{s,t}\left(\frac{V}{H}\right)$$

$$\leq \frac{\varepsilon^n}{\beta^{(t)}_{s,t}}\frac{V}{h^V_{s,t}} + \sum_{1\leq k\leq n}\frac{\varepsilon^{n-k}}{\beta^{(t)}_{s+k\tau,t}}\frac{a_\varepsilon}{\nu(1_{\mathcal{V}_\varepsilon} h^V_{s+k\tau,t})} = \frac{\varepsilon^n}{h_{s,t}}\,V + \sum_{1\leq k\leq n}\varepsilon^{n-k}\frac{a_\varepsilon}{\nu(1_{\mathcal{V}_\varepsilon} h_{s+k\tau,t})}.$$

Observe that for any $0 < \varepsilon \leq \varepsilon_2 < \varepsilon_1$ we have $\mathcal{V}_{\varepsilon_2} \subset \mathcal{V}_\varepsilon$. Thus, for any $k \leq n$ and $t = s+n\tau$ we have the lower bound estimate

$$\begin{aligned} 1_{\mathcal{V}_\varepsilon}(x) h_{s+k\tau,s+n\tau}(x) &\geq 1_{\mathcal{V}_{\varepsilon_2}}(x) Q_{s+k\tau,s+(k+1)\tau} 1_{\mathcal{V}_{\varepsilon_2}} \ldots Q_{s+(n-1)\tau,s+n\tau}(1_{\mathcal{V}_{\varepsilon_2}} H)(x) \\ &\geq 1_{\mathcal{V}_{\varepsilon_2}}(x)\left(\inf_{\mathcal{V}_{\varepsilon_2}} H\right)(\iota^-_{\varepsilon_2}\nu(\mathcal{V}_{\varepsilon_2}))^{n-k}. \end{aligned}$$

Choosing $0 < \varepsilon < \varepsilon_3 := \varepsilon_2 \wedge (\iota^-_{\varepsilon_2}\nu(\mathcal{V}_{\varepsilon_2})/2)$ we conclude that

$$R^{(t)}_{s,t}\left(\frac{V}{H}\right) \leq \frac{\varepsilon^n}{h_{s,t}}\,V + \frac{2a_\varepsilon}{\nu(\mathcal{V}_{\varepsilon_2})}\,\frac{1}{\inf_{\mathcal{V}_{\varepsilon_2}} H}.$$

In the same vein, for any $\eta \in \mathcal{P}_V(E)$ we have

$$\eta(h_{s,s+n\tau}) \geq \eta(\mathcal{V}_{\varepsilon_2})\,\left(\inf_{\mathcal{V}_{\varepsilon_2}} H\right)\,(\iota^-_{\varepsilon_2}\,\nu(\mathcal{V}_{\varepsilon_2}))^n \geq \eta(\mathcal{V}_{\varepsilon_2})\,\left(\inf_{\mathcal{V}_{\varepsilon_2}} H\right)\,(2\varepsilon)^n.$$

This yields the estimate

$$\frac{\varepsilon^{(t-s)/\tau}}{\eta(h_{s,t})} \leq 2^{-(t-s)/\tau}\,\frac{1}{\eta(\mathcal{V}_{\varepsilon_2})\,\inf_{\mathcal{V}_{\varepsilon_2}} H}.$$

To summarise: there exists some $\varepsilon_3 > 0$ such that for any $s \in \mathcal{T}$ s.t. $t \in [s,\infty[_\tau$ and for any $0 < \varepsilon < \varepsilon_3$ we have

$$R^{(t)}_{s,t}(V/H) \leq \varepsilon^{(t-s)/\tau}\,V/h_{s,t} + c_\varepsilon \quad \text{and} \quad \limsup_{t\to\infty}\frac{\varepsilon^{(t-s)/\tau}}{\eta(h_{s,t})} = 0.$$

This ends the proof of (A.4). The last assertion follows from the fact that

$$\left(\Psi_{h_{s,t}}(\eta)\,R^{(t)}_{s,t}\right)(V/H) \leq \frac{e^{-\lambda_0(t-s)}}{\eta(h_{s,t})}\,\eta(V) + c_0 \leq c_1(\eta)$$

for some finite constant $c_1(\eta) < \infty$. This implies that the sequence of probability measures

$$\Psi_{h_{s,t}}(\eta)\,R^{(t)}_{s,t}$$

indexed by $s \in \mathcal{T}$ s.t. $t \in [s, \infty[_\tau$ is tight. More precisely, choosing the compact set

$$K_{\varepsilon,\eta} := \{x \in E \,:\, \varepsilon\,(V/H)(x) \leq c_1(\eta)\}$$

we have

$$\left(\Psi_{h_{s,t}}(\eta)\, R^{(t)}_{s,t}\right)(K^c_{\varepsilon,\eta}) \leq \varepsilon \quad \text{so that} \quad \left(\Psi_{h_{s,t}}(\eta)\, R^{(t)}_{s,t}\right)(1/H) \geq \frac{1-\varepsilon}{\sup_{K_{\varepsilon,\eta}} H}.$$

On the other hand, we have

$$\Phi_{s,t}(\eta)(V) = \Psi_{1/H}\left(\Psi_{h_{s,t}}(\eta)\, R^{(t)}_{s,t}\right)(V) \leq c_1(\eta)\, \frac{\sup_{K_{\varepsilon,\eta}} H}{1-\varepsilon}.$$

We conclude that $\kappa_{\tau,V}(\eta) < \infty$. This also shows that the sequence of probability measures $\Phi_{s,t}(\eta)$ indexed by $s \in \mathcal{T}$ and $t \in [s, \infty[_\tau$ is tight. Choosing the compact set

$$K_{\varepsilon,\eta} := \{x \in E \,:\, \varepsilon\, V(x) \leq \kappa_V(\eta)\} \tag{A.5}$$

we readily check that

$$\Phi_{s,t}(\eta)(H) \geq \left(\inf_{K_{\varepsilon,\eta}} H\right)\, \Phi_{s,t}(\eta)(K_{\varepsilon,\eta}) \geq (1-\varepsilon)\, \left(\inf_{K_{\varepsilon,\eta}} H\right) > 0.$$

We conclude that $\kappa^-_{\tau,H}(\eta) > 0$. To take the final step, observe that for any $0 < \varepsilon < \varepsilon_0$ we have

$$(A.3) \Longrightarrow \inf_{t\in\mathcal{T}} \inf_{\mathcal{V}_\varepsilon} Q_{t,t+\tau}(1) \geq \nu(\mathcal{V}_\varepsilon) > 0. \tag{A.6}$$

Thus, for any $t \in [s, \infty[_\infty$ and $0 < \varepsilon < \varepsilon_0$ we have

$$\Phi_{s,t}(\eta)(Q_{t,t+\tau}(1)) \geq \Phi_{s,t}(\eta)\,(1_{\mathcal{V}_\varepsilon} Q_{t,t+\tau}(1)) \geq \Phi_{s,t}(\eta)(\mathcal{V}_\varepsilon)\, \nu(\mathcal{V}_\varepsilon).$$

Similarly, we have

$$\Phi_{s,t}(\eta)(Q_{t,t+\tau}(1)) \leq \Phi_{s,t}(\eta)(V\, Q_{t,t+\tau}(V)/V) \leq \|\Theta_\tau\|\, \kappa_V(\eta).$$

By lemma 2.2, this ends the proof of the estimates in (3.16) for continuous or discrete time indices. The proof of the lemma is completed. □

The next result is a variation of [196, lemma 10].

LEMMA A.2. *We have the estimates (3.14). In addition, for any $0 < \delta < 1$ there exists some $0 < \varepsilon \leq \varepsilon_1$ such that the following uniform compact-approximation estimate holds*

$$\sup_{t\in\mathcal{T}} \left|\left|\left|\overline{Q}_{t,t+\tau} - 1_{\mathcal{V}_\varepsilon}\, \overline{Q}_{t,t+\tau}\right|\right|\right|_V < \delta \quad \textit{and} \quad \sup_{s\in\mathcal{T}} \sup_{t\geq s} \left|\left|\left|\overline{Q}_{s,t}\right|\right|\right|_V < \infty. \tag{A.7}$$

PROOF. We start by proving the estimates given in (A.7). We use the same notation as in the proof of Lemma 6.1 and we set $\mathcal{V}_\varepsilon := \{\Theta_\tau \geq \varepsilon\}$. For any $n \geq 1$ such that $\varepsilon^-_n := \varepsilon_n/\lambda^- < 1$

$$\left|\left|\left|1_{\mathcal{V}^c_{\varepsilon_n}}\, \overline{Q}_{s,s+\tau}\right|\right|\right|_V$$

$$= \left|\left|\left|\overline{Q}_{s,s+\tau} - 1_{\mathcal{V}_{\varepsilon_n}}\, \overline{Q}_{s,s+\tau}\right|\right|\right|_V = \frac{1}{\lambda_{s,s+\tau}}\, \|(Q_{s,s+\tau}(V)/V)\, 1_{\mathcal{V}^c_{\varepsilon_n}}\| < \varepsilon^-_n.$$

This yields the left-hand side estimate in (A.7).

For the right-hand side, for any $i \in \{0,1\}^n$ and $1 \leq k \leq n$, we set

$$l(i) := \inf\{1 \leq k < n : (i_k, i_{k+1}) = (1,1)\}$$

and

$$\{0,1\}_k^n = \{i \in \{0,1\}^n : l(i) = k\}$$

with the convention that

$$\{0,1\}_n^n = \{i \in \{0,1\}^n \ : \ \forall 1 \leq k < n \ : \ (i_k, i_{k+1}) \neq (1,1)\}.$$

Further, set

$$\mathcal{V}_\varepsilon(0) = \mathcal{V}_\varepsilon^c \quad \text{and} \quad \mathcal{V}_\varepsilon(1) = \mathcal{V}_\varepsilon.$$

Then we may decompose $\overline{Q}_{s_0,s_n}$ as follows:

$$\begin{aligned}\overline{Q}_{s_0,s_n} &= \sum_{i\in\{0,1\}^n} 1_{\mathcal{V}_\varepsilon(i_1)}\overline{Q}_{s_0,s_1}\ldots 1_{\mathcal{V}_\varepsilon(i_n)}\overline{Q}_{s_0,s_n} \\ &= \sum_{1\leq k\leq n}\sum_{i\in\{0,1\}_k^n} 1_{\mathcal{V}_\varepsilon(i_1)}\overline{Q}_{s_0,s_1}\ldots 1_{\mathcal{V}_\varepsilon(i_n)}\overline{Q}_{s_{n-1},s_n}.\end{aligned}$$

Our aim is to obtain suitable bounds for the summands in the above decomposition. To this end, set for any $0 < \varepsilon \leq \varepsilon_n$

$$\nu_\varepsilon(dx) := \frac{\nu(dx)1_{\mathcal{V}_\varepsilon}(x)}{\nu(\mathcal{V}_\varepsilon)}.$$

Using this and the notation introduced in (A.3), we have

$$\begin{aligned}&1_{\mathcal{V}_\varepsilon}(x)\,\overline{Q}_{s,s+\tau}\left(1_{\mathcal{V}_\varepsilon}\,\overline{Q}_{s+\tau,t}(V)\right)(x) \\ &\leq \iota_\varepsilon\,\frac{\nu\left(1_{\mathcal{V}_\varepsilon}\,Q_{s+\tau,t}(V)\right)}{\overline{\eta}_s\left(Q_{s,s+\tau}Q_{s+\tau,t}(1)\right)} \leq \iota_\varepsilon\,\frac{\nu\left(1_{\mathcal{V}_\varepsilon}\,Q_{s+\tau,t}(V)\right)}{\overline{\eta}_s\left(1_{\mathcal{V}_\varepsilon}\,Q_{s,s+\tau}1_{\mathcal{V}_\varepsilon}\,Q_{s+\tau,t}(1)\right)} \\ &\leq \frac{\iota_\varepsilon}{\overline{\eta}_s(\mathcal{V}_\varepsilon)}\,\frac{\nu\left(1_{\mathcal{V}_\varepsilon}\,Q_{s+\tau,t}(V)\right)}{\nu\left(1_{\mathcal{V}_\varepsilon}\,Q_{s+\tau,t}(1)\right)} = \frac{\iota_\varepsilon/\iota_\varepsilon^-}{\overline{\eta}_s(\mathcal{V}_\varepsilon)}\,\Phi_{s+\tau,t}\left(\nu_\varepsilon\right)(V) \leq \kappa_V(\nu_\varepsilon)\,\frac{\iota_\varepsilon/\iota_\varepsilon^-}{\overline{\eta}_s(\mathcal{V}_\varepsilon)}.\end{aligned}$$

This yields the uniform estimate

$$1_{\mathcal{V}_{\varepsilon_n}}(x)\overline{Q}_{s,s+\tau}\left(1_{\mathcal{V}_{\varepsilon_n}}\overline{Q}_{s+\tau,t}(V)\right)(x) \leq \iota'_{\varepsilon_n} := \left(\frac{\iota_{\varepsilon_n}}{\iota_{\varepsilon_n}^-}\right)\kappa_V(\nu_{\varepsilon_n})\left(1+\frac{\overline{\eta}(V)}{n}\right).$$

Next fix $1 \leq k < n$. Then one has

$$\begin{aligned}\left|\left|\left|1_{\mathcal{V}_\varepsilon(i_1)}\overline{Q}_{s_0,s_1}\ldots 1_{\mathcal{V}_\varepsilon(i_n)}\overline{Q}_{s_{n-1},s_n}\right|\right|\right|_V &\leq \iota'_{\varepsilon_n}\left|\left|\left|1_{\mathcal{V}_\varepsilon(i_1)}\overline{Q}_{s_0,s_1}\ldots 1_{\mathcal{V}_\varepsilon(i_k)}\overline{Q}_{s_{k-1},s_k}\right|\right|\right|_V \\ &\leq \iota'_{\varepsilon_n}\prod_{1\leq m\leq k}\left|\left|\left|1_{\mathcal{V}_\varepsilon(i_m)}\overline{Q}_{s_{m-1},s_m}\right|\right|\right|_V \qquad \text{(A.8)} \\ &\leq \iota'_{\varepsilon_n}\prod_{1\leq m\leq k}\|\Theta_\tau\|^{1_{i_m=1}}\left(\varepsilon_n^-\right)^{1_{i_m=0}} \\ &= \iota'_{\varepsilon_n}\|\Theta_\tau\|^{\sum_{m=1}^k 1_{i_m=1}}\left(\varepsilon_n^-\right)^{\sum_{m=1}^k 1_{i_m=0}}. \qquad \text{(A.9)}\end{aligned}$$

Similarly, for the case $k = n$, we have

$$\left|\left|\left|1_{\mathcal{V}_\varepsilon(i_1)}\overline{Q}_{s_0,s_1}\ldots 1_{\mathcal{V}_\varepsilon(i_n)}\overline{Q}_{s_{n-1},s_n}\right|\right|\right|_V \leq \|\Theta_\tau\|^{\sum_{m=1}^n 1_{i_m=1}}\left(\varepsilon_n^-\right)^{\sum_{m=1}^n 1_{i_m=0}}. \qquad \text{(A.10)}$$

Now note that for any $i \in \{0,1\}^n$ we have

$$\sum_{1\leq k\leq n} 1_{i_k=1} \leq \frac{n+1}{2} + \frac{1}{2}\sum_{1\leq k<n} 1_{(i_k,i_{k+1})=(1,1)}.$$

This follows from the fact that

$$\begin{aligned} n-1 &\geq \sum_{1\leq k<n} \left(1_{(i_k,i_{k+1})=(1,0)} + 1_{(i_k,i_{k+1})=(0,1)}\right) \\ &= \left(2\sum_{1\leq k\leq n} 1_{i_k=1} - (1_{i_n=1} + 1_{i_1=1})\right) - \sum_{1\leq k<n} 1_{(i_k,i_{k+1})=(1,1)} \\ &\geq 2\left(\sum_{1\leq k\leq n} 1_{i_k=1} - 1\right) - \sum_{1\leq k<n} 1_{(i_k,i_{k+1})=(1,1)}. \end{aligned}$$

Equivalently, we have

$$\sum_{1\leq k\leq n} 1_{i_k=0} \geq \frac{n-1}{2} - \frac{1}{2}\sum_{1\leq k<n} 1_{(i_k,i_{k+1})=(1,1)}.$$

Further observe that for any $i \in \{0,1\}_k^n$ with $1 \leq k \leq n$ we have

$$\sum_{m=1}^{k} 1_{(i_m,i_{m+1})=(1,1)} = 0 \Longrightarrow \sum_{m=1}^{k} 1_{i_m=1} \leq \frac{k+1}{2} \text{ and } \sum_{m=1}^{k} 1_{i_m=0} \geq \frac{k-1}{2}.$$

From (A.9) and (A.10), respectively, we conclude that for any $i \in \{0,1\}_k^n$ with $1 \leq k < n$ we have

$$\left|\left|\left|1_{\mathcal{V}_\varepsilon(i_1)}\overline{Q}_{s_0,s_1}\ldots 1_{\mathcal{V}_\varepsilon(i_n)}\overline{Q}_{s_{n-1},s_n}\right|\right|\right|_V \leq \iota'_{\varepsilon_n}\ (1\vee\|\Theta_\tau\|)^{(k+1)/2}\ \left(\varepsilon_n^-\right)^{(k-1)/2}.$$

and for $k = n$, we have

$$\left|\left|\left|1_{\mathcal{V}_\varepsilon(i_1)}\overline{Q}_{s_0,s_1}\ldots 1_{\mathcal{V}_\varepsilon(i_n)}\overline{Q}_{s_{n-1},s_n}\right|\right|\right|_V \leq (1\vee\|\Theta_\tau\|)^{(n+1)/2}\ \left(\varepsilon_n^-\right)^{(n-1)/2}.$$

We end the proof of the right-hand side estimate in (A.7) by choosing $n \geq 1$ such that

$$\varepsilon_n^- < 1 \wedge (1/\|\Theta_\tau\|).$$

Now for the estimates in (3.14), observe that

$$H \leq V \Longrightarrow \sup_{s\in\mathcal{T}}\sup_{t\in[s,\infty[_\tau} \|\overline{Q}_{s,t}(H)\|_V \leq \sup_{s\in\mathcal{T}}\sup_{t\in[s,\infty[_\tau} \left|\left|\left|\overline{Q}_{s,t}\right|\right|\right|_V < \infty.$$

By remark 3.2, this yields right-hand side estimate in (3.14).

The proof of the left-hand side estimate in (3.14) follows the same lines of arguments as those given for (A.7), thus it is only sketched. Using the same notation as above, we have

$$\overline{\eta}_{s_0}Q_{s_0,s_n}(1) = \sum_{1\leq k\leq n}\sum_{i\in\{0,1\}_k^n} \overline{\eta}_{s_0}1_{\mathcal{V}_\varepsilon(i_1)}Q_{s_0,s_1}\ldots 1_{\mathcal{V}_\varepsilon(i_n)}Q_{s_{n-1},s_n}(1).$$

For any $i \in \{0,1\}_k^n$ for some $1 \leq k < n$, we also have

$$\begin{aligned} &\overline{\eta}_{s_0}1_{\mathcal{V}_\varepsilon(i_1)}Q_{s_0,s_1}\ldots 1_{\mathcal{V}_\varepsilon(i_n)}Q_{s_{n-1},s_n}(1) \\ &\leq \iota_\varepsilon\ \overline{\eta}_{s_0}(V1_{\mathcal{V}_\varepsilon(i_1)}Q^V_{s_0,s_1}\ldots 1_{\mathcal{V}_\varepsilon(i_k)}Q^V_{s_{k-1},s_k}(1_{\mathcal{V}_\varepsilon}))\ \nu(1_{\mathcal{V}_\varepsilon}Q_{s_{k+1},s_n}(1)) \\ &\leq \iota_\varepsilon\ \overline{\eta}(V)\ (1\vee\|\Theta_\tau\|)^{(k+1)/2}\ \varepsilon^{(k-1)/2}\ \nu(1_{\mathcal{V}_\varepsilon}Q_{s_{k+1},s_n}(1)), \end{aligned}$$

with $Q^V_{s,t}(f) := Q_{s,t}(fV)/V$. For any $\varepsilon \leq \varepsilon_1$ we have

$$\begin{aligned} &1_{\mathcal{V}_\varepsilon}(x)Q_{s_0,s_n}(H)(x) \\ &\geq 1_{\mathcal{V}_\varepsilon}(x)\ 1_{\mathcal{V}_{\varepsilon_1}}(x)\left(Q_{s_0,s_1}1_{\mathcal{V}_{\varepsilon_1}}Q_{s_1,s_n}(H)\right)(x) \geq \iota_\varepsilon^-\ 1_{\mathcal{V}_\varepsilon}(x)\ \nu\left(1_{\mathcal{V}_{\varepsilon_1}}Q_{s_1,s_n}(H)\right). \end{aligned}$$

In addition, we have

$$\begin{aligned}
&\nu\left(1_{\mathcal{V}_{\varepsilon_1}} Q_{s_1,s_n}(H)\right)\\
&\geq \nu\left(1_{\mathcal{V}_{\varepsilon_1}} Q_{s_1,s_2} 1_{\mathcal{V}_\varepsilon} \ldots Q_{s_{k-1},s_k} 1_{\mathcal{V}_\varepsilon} Q_{s_k,s_{k+1}} 1_{\mathcal{V}_\varepsilon} Q_{s_{k+1},s_n}(H)\right)\\
&\geq \iota_\varepsilon^- \; \nu\left(1_{\mathcal{V}_{\varepsilon_1}} Q_{s_1,s_2} 1_{\mathcal{V}_\varepsilon} \ldots Q_{s_{k-1},s_k} 1_{\mathcal{V}_\varepsilon}\right) \nu(1_{\mathcal{V}_\varepsilon} Q_{s_{k+1},s_n}(H)).
\end{aligned}$$

This yields the estimate

$$1_{\mathcal{V}_\varepsilon}(x)\; Q_{s_0,s_n}(H) \geq 1_{\mathcal{V}_\varepsilon}(x)\; (\iota_\varepsilon^-)^2\; (\iota_{\varepsilon_1}^- \nu\left(\mathcal{V}_{\varepsilon_1}\right))^k\; \nu(1_{\mathcal{V}_\varepsilon} Q_{s_{k+1},s_n}(H)).$$

from which we check that

$$\begin{aligned}
&1_{\mathcal{V}_\varepsilon}(x)\; \left(\overline{\eta}_{s_0} 1_{\mathcal{V}_\varepsilon(i_1)} Q_{s_0,s_1} \ldots 1_{\mathcal{V}_\varepsilon(i_n)} Q_{s_{n-1},s_n}(1)\right) / Q_{s_0,s_n}(H)(x)\\
&\leq (\iota_\varepsilon/\varepsilon)\; \left(\overline{\eta}(V)/\kappa_H^-(\nu_\varepsilon)\right)\; \left(\varepsilon\; (1 \vee \|\Theta_\tau\|)/(\iota_{\varepsilon_1}^- \nu\left(\mathcal{V}_{\varepsilon_1}\right))^2\right)^{k/2}.
\end{aligned}$$

The end of the proof of the left-hand side assertion in (3.14) now follows word-for-word the same lines of arguments as the proof of the right-hand side estimate in (3.14), thus it is skipped. The proof of the lemma is now completed. □

A.3. Proof of Lemma 2.1.

PROOF. Observe that for any f such that $\|f\|_V \leq 1$, we have

$$(\Psi_{1/h}(\mu_1) - \Psi_{1/h}(\mu_2))(f) = \frac{1}{\mu_1(1/h)}\; (\mu_1 - \mu_2)(g)$$

with the function

$$g := (1/h)(f - \Psi_{1/h}(\mu_2)(f)) \in \mathcal{B}_{V/h}(E).$$

Note that

$$\frac{g}{V/h} = \frac{f - \Psi_{1/h}(\mu_2)(f)}{V} = \frac{f}{V} - \frac{\Psi_{1/h}(\mu_2)(Vf/V)}{V} \Longrightarrow \|g\|_{V/h} \leq 1 + \frac{\mu_2(V/h)}{\mu_2(1/h)}.$$

This ends the proof of the first assertion. Now, for any $\|f\|_{V/h} \leq 1$ we have

$$(\Psi_h(\mu_1) - \Psi_h(\mu_2))(f) = \frac{1}{\mu_1(h)}\; (\mu_1 - \mu_2)(g)$$

with the function

$$\begin{aligned}
&g := h(f - \Psi_h(\mu_2)(f))\\
&\Longrightarrow \frac{|g|}{V} \leq 1 + \frac{h}{V}\; \frac{\mu_2(V)}{\mu_2(h)} \leq 1 + \frac{\mu_2(V)}{\mu_2(h)} \Longrightarrow \|g\|_V \leq 1 + \frac{\mu_2(V)}{\mu_2(h)}.
\end{aligned}$$

This ends the proof of the lemma. □

A.4. Proof of Lemma 2.3.

PROOF. We set $(s,t) = (1,2)$ and $P = P_{1,2}$, and $V_i^\rho := 1/2 + \rho V_i$, with $i \in \{1,2\}$ and $\rho \in]0,1[$. We also consider the function Δ_ρ on E_1^2 defined for any $(x,y) \in E_1^2$ by

$$\begin{aligned}
\Delta_\rho(x,y) &:= \frac{\|\delta_x P - \delta_y P\|_{V_2^\rho}}{\|\delta_x - \delta_y\|_{V_1^\rho}}\\
&\leq \frac{\|\delta_x P - \delta_y P\|_{tv}}{1 + \rho(V_1(x) + V_1(y))} + \frac{\rho(P(V_2)(x) + P(V_2)(y))}{1 + \rho(V_1(x) + V_1(y))}.
\end{aligned}$$

Using the left-hand side estimate in (2.15), we have

$$\begin{aligned} P(V_2)(x) + P(V_2)(y) &\leq \varepsilon(V_1(x) + V_1(y)) + 2 \\ &= (V_1(x) + V_1(y)) \left(\varepsilon + \frac{2}{(V_1(x) + V_1(y))} \right). \end{aligned}$$

When $V_1(x) + V_1(y) \geq r$ this yields the estimate

$$\begin{aligned} \Delta_\rho(x,y) &\leq \frac{1}{1+\rho(V_1(x)+V_1(y))} + \frac{\rho(V_1(x)+V_1(y))}{1+\rho(V_1(x)+V_1(y))} \left(\varepsilon + \frac{2}{r} \right) \\ &= 1 - \left(1 - \frac{1}{1+\rho((V_1(x)+V_1(y))} \right) \left(1 - \left(\varepsilon + \frac{2}{r} \right) \right) \leq 1 - d_\rho^1, \end{aligned}$$

with

$$d_\rho^1 := \left(1 - \frac{1}{1+2\rho r} \right) \left(1 - \left(\varepsilon + \frac{2}{r} \right) \right).$$

Recalling that $V_i \geq 1$ we have

$$P(V_2)/V_1 \leq 1 + \varepsilon.$$

This implies that for any (x,y) such that $V_1(x) + V_1(y) \leq r$ we have

$$\frac{\rho V_1(x)(P(V_2)(x)/V_1(x)) + \rho V_1(y)\ (P(V_2)(y)/V_1(y))}{1+\rho(V_1(x)+V_1(y))} \leq (1+\varepsilon)\ \frac{\rho r}{1+\rho},$$

where we have also used the fact that $\rho < 1$. This yields the estimate

$$\Delta_\rho(x,y) \leq 1 - d_\rho^2 := \frac{1-\alpha(r)}{1+2\rho} + \frac{\rho r}{1+2\rho}(1+\varepsilon).$$

Observe that

$$\rho \leq \frac{1}{1+\varepsilon}\ \frac{\alpha(r)}{r} \Longrightarrow 1 - d_\rho^2 \leq \frac{1}{1+2\rho} < 1.$$

Recalling that $\alpha(r) > 0$ for some $r > r_\varepsilon := 2/(1-\varepsilon)$ we have

$$\frac{2}{r} < 1 - \varepsilon \Longrightarrow 0 < d_\rho^1 = \left(1 - \frac{1}{1+2\rho r} \right) \left(1 - \left(\varepsilon + \frac{2}{r} \right) \right) < 1.$$

Choosing

$$r = r_\varepsilon(1+\delta_1) \quad \text{and} \quad \rho = \frac{\delta_2}{1+\varepsilon}\ \frac{\alpha(r)}{2r}$$

for some $0 < \delta_1 \leq 1$ and $0 < \delta_2 \leq 1$ we have

$$0 < \frac{\delta_2}{3}\ \alpha(r)\ (1-\varepsilon)\ \frac{\delta_1}{2} < d_\rho^1 = \frac{\alpha(r)\delta_2}{(1+\varepsilon)+\alpha(r)\delta_2}\ (1-\varepsilon)\ \frac{\delta_1}{1+\delta_1} < 1,$$

which implies that

$$1 - d_\rho^1 \leq 1 - \alpha(r)\ (1-\varepsilon)\ \frac{\delta_1\delta_2}{6}$$

and

$$1 - d_\rho^2 = \frac{1-\alpha(r)+\rho r(1+\varepsilon)}{1+2\rho} = \frac{1-\alpha(r)(1-\delta_2/2)}{1+2\rho} \leq 1 - \alpha(r)(1-\delta_2/2).$$

Then setting

$$\delta_1 = \frac{6}{1-\varepsilon}\frac{1-\delta_2/2}{\delta_2}$$

we can check that

$$(1-d^1_\rho)\vee(1-d^2_\rho)\leq 1-\alpha(r)(1-\delta_2/2).$$

For any $0<\delta\leq 1$ we set

$$r_\varepsilon(\delta) := r_0 \vee r_\varepsilon\left(1+\frac{1}{1-\varepsilon}\left(3+6\left(\frac{1}{\delta}-1\right)\right)\right) \qquad \rho_\varepsilon(\delta) := \frac{\delta}{1+\varepsilon}\,\frac{\alpha(r_\varepsilon(\delta))}{2r_\varepsilon(\delta)}.$$

We conclude that

$$\begin{aligned}\beta_{V_1^{\delta,\varepsilon},V_2^{\delta,\varepsilon}}(P) = \sup_{x,y\in E}\frac{\|\delta_x P-\delta_y P\|_{V_2^{\delta,\varepsilon}}}{V_1^{\delta,\varepsilon}(x)+V_1^{\delta,\varepsilon}(y)} &\leq 1-\left(d^1_{\rho_\varepsilon(\delta)}\wedge d^2_{\rho_\varepsilon(\delta)}\right)\\ &\leq 1-\alpha(r_\varepsilon(\delta))(1-\delta/2).\end{aligned}$$

This ends the proof of the lemma. □

A.4.1. *Proof of lemma 2.4.*

PROOF. For any $t\in\mathcal{T}$, we set $R^K_t(f) := 1_K Q_t(f)$ so that

$$Q_t = R^K_t + R^{K^c}_t \quad \text{and} \quad Q_{t+\tau} - Q^K_\tau Q_t = Q_\tau R^{K^c}_t + R^{K^c}_\tau R^K_t.$$

This decomposition implies that

$$\begin{aligned}\left|\!\left|\!\left|Q_{t+\tau}-Q^K_\tau Q_t\right|\!\right|\!\right|_V &\leq \left(|\!|\!|Q_\tau|\!|\!|_V + \left|\!\left|\!\left|R^K_t\right|\!\right|\!\right|_V\right)\ \left|\!\left|\!\left|R^{K^c}_\tau\right|\!\right|\!\right|_V\\ &\leq \left(|\!|\!|Q_\tau|\!|\!|_V + |\!|\!|Q_t|\!|\!|_V\right)\ \left|\!\left|\!\left|R^{K^c}_\tau\right|\!\right|\!\right|_V.\end{aligned}$$

Also note that

$$\left|\!\left|\!\left|R^{K^c}_\tau\right|\!\right|\!\right|_V = \|1_{K^c}\ Q_\tau(V)/V\|.$$

Thus, choosing a sequence of compact sets K_n such that

$$1_{K^c_n}\ Q_\tau(V)\leq \frac{1}{n}\ V$$

implies that

$$\left|\!\left|\!\left|Q_{t+\tau}-Q^{K_n}_\tau Q_t\right|\!\right|\!\right|_V \leq \frac{1}{n}\ \left(|\!|\!|Q_\tau|\!|\!|_V + |\!|\!|Q_t|\!|\!|_V\right).$$

Since the product operator $Q^{K_n}_\tau Q_t$ is compact, $Q_{t+\tau}$ is the limit in norm of compact operators, hence it is compact. This ends the proof of the lemma. □

A.4.2. *Proof of the compactness of (2.24)* . For any collection of functions $(f_n)_{n\geq 0}\in\mathcal{B}_V(E)^{\mathbb{N}}$ in the unit ball $\{f\in\mathcal{B}_V(E):\|f\|_V\leq 1\}$ we have

$$\sup_{n\geq 0}\|Q^K_\tau(f_n)\|_V \leq \|Q^K_\tau(V)/V\|.$$

In addition, for any $(x,y)\in K^2$ we have

$$|Q^K_\tau(f_n)(x)-Q^K_\tau(f_n)(y)| \leq \int |q_\tau(x,z)-q_\tau(y,z)|\ 1_K(z)\nu_\tau(dz).$$

Since q_τ is continuous on the compact set $(K \times K)$, it is uniformly continuous. Thus for any $\varepsilon > 0$, there is some $\delta > 0$ such that $d(x,y) \leq \delta$ implies that for any $n \geq 0$ we have

$$|Q_\tau^K(f_n)(x) - Q_\tau^K(f_n)(y)| \leq \varepsilon\, \nu_\tau(K).$$

By the Arzela-Ascoli theorem, $Q_\tau^K(f_n)$ converges uniformly to a continuous limit extended by 0 outside the set K. Recalling that $V \geq 1$ is also converges on $\mathcal{B}_V(E)$. Q_τ^K is an irreducible and compact operator on $\mathcal{B}_V(E)$. This ends the proof of compactness of (2.24).

Proof of (2.25). For any $\varepsilon > 0$ the set $K_\varepsilon := \{Q_\tau(V)/V \geq \varepsilon\}$ is finite, and following the proof of the above lemma, we have

$$\left|\left|\left|Q_{t+\tau} - Q_\tau^{K_\varepsilon} Q_t\right|\right|\right|_V \leq \varepsilon\ \left(|||Q_\tau|||_V + |||Q_t|||_V\right).$$

Since for any finite set K, the operator

$$Q_\tau^K(f)(x) = 1_K(x) \sum_{y \in K} Q_\tau(x,y)\ f(y)$$

is bounded and has finite range, it is compact and we thus conclude that Q_t is compact for any $t \geq \tau$ as soon as $Q_\tau(V)/V \in \mathcal{B}_0(E)$.

The set E can be written as the union $E = \cup_{n \geq 0} K_n$ of an increasing sequence of compact sets K_n. Whenever Q_t is compact on $\mathcal{B}_V(E)$, up to a subsequence extraction the functions $Q_t(V 1_{K_n^c}) \in \mathcal{B}_V(E)$ forms a Cauchy sequence. Thus, for any $\varepsilon > 0$ there exists some $n_\varepsilon \geq 1$ such that for any $n_\varepsilon \leq m \leq n$ we have

$$\|Q_t(V 1_{K_n^c}) - Q_t(V 1_{K_m^c})\|_V = \|Q_t(V 1_{K_n - K_m})\|_V \leq \varepsilon$$

and therefore

$$\|Q_t(V 1_{K_m^c})/V\| \leq \varepsilon.$$

Now, assume that for any $\varepsilon > 0$ there exists some $n_\varepsilon \geq 1$ such that

$$\forall n \geq n_\varepsilon \qquad \|Q_t(V 1_{K_n^c})/V\| \leq \varepsilon.$$

For any $\|f\|_V \leq 1$, we have

$$\|Q_t(f) - Q_t(1_{K_n} f)\|_V \leq \|Q_t(V 1_{K_n^c})/V\| \leq \varepsilon.$$

Arguing as above, using the fact that $f \in \mathcal{B}_V(E) \mapsto 1_{K_n} f \in \mathcal{B}_V(E)$ is a finite range compact operator we conclude that Q_t is compact on $\mathcal{B}_V(E)$. This ends the proof of (2.25).

Acknowledgements. Ajay Jasra was supported by KAUST baseline funding and Pierre Del Moral was partially supported by the ANR project QuAMProcs: ANR-19-CE40-0010. We thank two referees and the editors for their comments, which have greatly improved the paper.

REFERENCES

[1] M. Aizenman and B. Simon. Brownian motion and Harnack's inequality for Schrödinger operators. Comm. Pure. Appl. Math, vol. 35, pp. 209–271 (1982).

[2] D. Aldous and U. Vazirani. Go with the winners algorithms; in Proc. 35th IEEE Sympos. on Foundations of Computer Science (1994)

[3] A.R. Aliabad, F. Azarpanah, M. Namdari. Rings of continuous functions vanishing at infinity. Commentationes Mathematicae Universitatis Carolinae, vol. 45, no. 3, pp. 519-533 (2004). .

[4] A. Arlotto, J.M. Steele. A central limit theorem for temporally nonhomogenous markov chains with applications to dynamic programming. Mathematics of Operations Research, vol. 41, no. 4, pp. 1448–1468 (2016).

[5] M. Arnaudon, P. Del Moral. A duality formula and a particle Gibbs sampler for continuous time Feynman-Kac measures on path spaces. Electronic Journal of Probability, vol. 25, pp. 1–54 (2020).
[6] M. Arnaudon, P. Del Moral, E. Horton, E.M. Ouhabaz. Lyapunov functions for positive semigroups (working article).
[7] A. Asselah, P.A. Ferrari, P. Groisman, P. Quasistationary distributions and Fleming-Viot processes in finite spaces". Journal of Applied Probability, vol. 48, no. 2, pp. 322–332 (2011).
[8] A. Asselah, P.A. Ferrari, P. Groisman, M. Jonckheere, Fleming-Viot selects the minimal quasi-stationary distribution: the Galton-Watson case, Ann. Inst. Henri Poincaré, Probab. Stat., vol. 52, no. 2, pp. 647–668 (2016).
[9] D. G. Aronson. Bounds for the fundamental solution of a parabolic equation. Bull. Amer. Math. Soc. 73, pp. 890–896 (1967).
[10] R. Assaraf, M. Caffarel, and A. Khelif. Diffusion Monte Carlo methods with a fixed number of walkers. Physical Review E, vol. 61, no. 4-4566, (2000).
[11] R. Atar, O. Zeitouni. Exponential stability for nonlinear filtering. Annales de l'Institut Henri Poincare (B) Probability and Statistics. Vol. 33. No. 6 (1997).
[12] K. E. Atkinson. A Survey of Numerical Methods for the Solution of Fredholm Integral Equations of the Second Kind. SIAM, Philadelphia (1976).
[13] M. Baudel, A. Guyader, T. Lelièvre. On the Hill relation and the mean reaction time for metastable processes. Arxiv no:2008.09790, August (2020).
[14] V. Bansaye, B. Cloez, P. Gabriel. Ergodic behavior of non-conservative semigroups via generalized Doeblin's conditions. Acta Applicandae Mathematicae, vol. 166, no. 1, pp. 29–72 (2020).
[15] V. Bansaye, B. Cloez, P. Gabriel, A. Marguet. A non-conservative Harris ergodic theorem. arXiv preprint arXiv:1903.03946 (2019).
[16] V. Bansaye. Ancestral lineages and limit theorems for branching Markov chains in varying environment. Journal of Theoretical Probability, vol. 32, no. 1, pp. 249–281 (2019).
[17] M. Benaim, B. Cloez, F. Panloup. Stochastic approximation of quasi-stationary distributions on compact spaces and applications. The Annals of Applied Probability, vol. 28, no. 4, pp. 2370–2416 (2018).
[18] M. Benaim, N. Champagnat, W. Oçafrain, D. Villemonais. Quasi-compactness criterion for strong Feller kernels with an application to quasi-stationary distributions. Arxiv-hal-03640205 (2022)
[19] J. Bertoin, Jean, A. R. Watson. The strong Malthusian behavior of growth-fragmentation processes. Annales Henri Lebesgue, vol. 3, pp. 795–823 (2020).
[20] G. Birkhoff. Extensions of Jentzsch's theorem. Transactions of the American Mathematical Society, vol. 85, no..1, pp. 219–227 (1957).
[21] A.N. Bishop, P. Del Moral, K. Kamatani, B. Remillard. On one-dimensional Riccati diffusions. Annals of Applied Probability. volume 29, Number 2, pp. 1127–1187 (2019).
[22] A.N. Bishop, P. Del Moral. On the stability of matrix-valued Riccati diffusions. Electronic Journal of Probability, vol. 24, pp. 1–40 (2019).
[23] J. Blanchet, P. Glynn, and S. Zheng. Analysis of a stochastic approximation algo- rithm for computing quasi-stationary distributions. Advances in Applied Probability, vol. 48, no. 3, pp. 792–811 (2016).
[24] V.I. Bogachev. Weak convergence of measures. Providence, RI: American Mathematical Society (2018).
[25] K. Bogdan, K. Szczypkowski. Gaussian estimates for Schrödinger perturbations. arXiv preprint arXiv:1301.4627 (2013).
[26] P. Bougerol. Comparaison des exposants de Lyapounov des processus markoviens multiplicatifs. Annales de l'Institut Henri Poincaré, Probabilités et Statistiques, vol. 24, no. 4, pp. 439–489 (1988).
[27] P. Bougerol. Théorèmes limites pour les systèmes linéaires à coefficients markoviens. Probability Theory and Related Fields, vol. 78, no. 2, pp. 193–221, (1988).
[28] S.P. Brooks and G.O. Roberts. Convergence assessment techniques for Markov chain Monte Carlo. Statistics and Computing, vol. 8, pp. 319–335 (1998).
[29] I. Bucur and G. Paltineanu. De Branges Type Lemma and Approximation in Weighted Spaces Mediterr. J. Math., vol. 18, no. 120 (2021)
[30] A. Budhiraja, D. Ocone. Exponential stability of discrete-time filters for bounded observation noise. Systems & Control Letters, vol. 30, no.4, pp. 185–193 (1997).
[31] A. Budhiraja. Asymptotic stability, ergodicity and other asymptotic properties of the nonlinear filter. Annales de l'IHP Probabilités et statistiques. Vol. 39. No. 6. (2003).
[32] M.C. Buonaura, S. Sorella. Numerical study of the two-dimensional Heisenberg model using a Green function Monte Carlo technique with a fixed number of walkers. Physical Review B, vol. 57, no. 18, 11446 (1998).
[33] K. Burdzy, R. Holyst, and P. March. A Fleming-Viot particle representation of the Dirichlet Laplacian. Comm. Math. Phys., vol. 214, no. 3, pp. 679–703 (2000).

[34] K. Burdzy, R. Holyst, D. Ingerman, P. March. Configurational transition in a Fleming-Viot-type model and probabilistic interpretation of laplacian eigenfunctions. J. Phys. A, vol. 29,no. 29, pp. 2633–2642 (1996).

[35] M. Caffarel, P. Claverie. Treatment of the Schrödinger equation through a Monte Carlo method based upon the generalized Feynman-Kac formula. Journal of Statistical Physics, vol. 43, no. 5, pp. 797–801 (1986).

[36] M. Caffarel, P. Claverie. Development of a pure diffusion quantum Monte Carlo method using a full generalized Feynman-Kac formula. I. Formalism & II Applications to simple systems. The Journal of chemical physics, vol. 88, no. 2, pp. 1088–1099 and pp. 1100-1109 (1988).

[37] M. Caffarel. Stochastic methods in quantum mechanics. In Numerical Determination of the Electronic Structure of Atoms, Diatomic and Polyatomic Molecules (pp. 85-105). Springer, Dordrecht (1989).

[38] M. Caffarel, R. Assaraf. A pedagogical introduction to quantum Monte Carlo. In Mathematical models and methods for ab initio Quantum Chemistry in Lecture Notes in Chemistry, eds. M. Defranceschi and C.Le Bris, Springer p.45 (2000).

[39] R. Carmona, W. Ch. Masters and B. Simon. Relativistic Schrödinger operators: Asymptotic behaviour of the eigenfunction. J. Funct. Anal, vol. 91, pp. 117–142 (1990).

[40] M. M. Castro, J.S.W. Lamb, G.O. Mendez, M. Rasmussen. Existence and uniqueness of quasi-stationary and quasi-ergodic measures for absorbing Markov processes. A Banach space approach. Arxiv-2111.13791 November (2021).

[41] P. Cattiaux. Stochastic calculus and degenerate boundary value problems. Ann. Institut Fourier, vol. 42, no. 3, pp. 541–624 (1992).

[42] P. Cattiaux, P. Collet, A. Lambert, S. Martinez, S. Méléard, and J. San Martin. Quasi-stationary distributions and diffusion models in population dynamics. Ann. Probab., vol. 37, no. 5, pp. 1926–1969 (2009).

[43] P. Cattiaux and S. Méléard. Competitive or weak cooperative stochastic Lotka-Volterra systems conditioned to non-extinction. J. Math. Biology, vol. 60, no. 6, pp. 797–829 (2010).

[44] J. A. Cavender. Quasi-stationary distributions of birth-and-death processes. Adv. Appl. Probab., vol. 10, no. 3, pp. 570–586 (1978).

[45] N. Champagnat, K.A. Coulibaly-Pasquier, D. Villemonais. Criteria for exponential convergence to quasi-stationary distributions and applications to multi-dimensional diffusions. In Séminaire de Probabilités XLIX (pp. 165–182). Springer, (2018).

[46] N. Champagnat, D. Villemonais. Exponential convergence to quasi-stationary distribution and Q-process. Probability Theory and Related Fields, Springer Verlag, vol. 164, no. 1, pp. 243–283 (2016).

[47] N. Champagnat, D. Villemonais. Population processes with unbounded extinction rate conditioned to non-extinction. ArXiv:1611.03010 (2016).

[48] N. Champagnat, D. Villemonais. Uniform convergence to the Q-process. Electronic Communications in Probability, vol. 22, pp. 1–7 (2017).

[49] N. Champagnat, D. Villemonais. Practical criteria for R-positive recurrence of unbounded semigroups. Electronic Communications in Probability, vol. 25, pp. 1–11 (2020).

[50] N. Champagnat, D. Villemonais. General criteria for the study of quasi-stationarity. Arxiv:1712.08092 (2018).

[51] N. Champagnat, D. Villemonais. Lyapunov criteria for uniform convergence of conditional distributions of absorbed Markov processes. Stochastic Processes and their Applications, (Arxiv) vol. 135, pp 51-74 (2021).

[52] N. Champagnat, D. Villemonais. Uniform convergence of penalized time-inhomogeneous Markov processes. ESAIM: Probability and Statistics, vol. 22, pp 129–162 (2018).

[53] N. Champagnat and D. Villemonais, Convergence of the Fleming-Viot process toward the minimal quasi-stationary distribution, ALEA, Lat. Am. J. Probab. Math. Stat., vol. 18, no. 1, pp. 1–15 (2021).

[54] J.R. Chazottes, P. Collet, S. Méléard. On time scales and quasi-stationary distributions for multitype birth-and-death processes. Annales de l'Institut Henri Poincaré, Probabilités et Statistiques, vol. 55, no. 4 (2019).

[55] J. Chen, L. Haitao, J. Siqi. Some limit theorems for absorbing Markov processes." Journal of Physics A: Mathematical and Theoretical, vol. 45, no. 34, 345003 (2012).

[56] P. Chigansky, R. Liptser. Stability of nonlinear filters in nonmixing case. The Annals of Applied Probability, vol. 14, no. 4, pp. 2038–2056 (2004).

[57] P. Chigansky, R. Liptser, and R. Van Handel. Intrinsic methods in filter stability. Handbook of Nonlinear Filtering (2009).

[58] P. Chigansky. Stability of nonlinear filters: A survey. Lecture notes, Petropolis, Brazil (2006).

[59] J.M.C. Clark, D. L. Ocone, and C. Coumarbatch. Relative entropy and error bounds for filtering of Markov processes. Mathematics of Control, Signals and Systems , vol. 12, no. 4, pp. 346–360 (1999).

[60] B. Cloez, J. Corujo. Uniform in time propagation of chaos for a Moran model. arXiv preprint arXiv:2107.10794 (2021).
[61] B. Cloez and M.-N. Thai, Quantitative results for the Fleming-Viot particle system and quasi-stationary distributions in discrete space, Stochastic Process. Appl., vol. 126, no. 3, pp. 680–702 (2016).
[62] B. Cloez and M.-N. Thai, Fleming-Viot processes: two explicit examples, ALEA Lat. Am. J. Probab. Math. Stat., vol. 13, no. 1, pp. 337–356 (2016).
[63] P. Collet, S. Martinez, J. San Martin. Quasi-stationary distributions: Markov chains, diffusions and dynamical systems. Springer Science & Business Media (2012).
[64] P. Collet, S. Martinez, S. Méléard, J. San Martin. Quasi-stationary distributions for structured birth and death processes with mutations. Probability Theory and Related Fields, vol. 151, pp. 191–231 (2011).
[65] D. Crisan, K. Heine. Stability of the discrete time filter in terms of the tails of noise distributions. Journal of the London Mathematical Society, vol. 78, no. 2, pp. 441–458 (2008).
[66] G. Da Prato, M. Fuhrman, P. Malliavin. Asymptotic ergodicity of the process of conditional law in some problem of nonlinear filtering. Journal of Functional Analysis, vol. 164, no. 2, pp. 356–377 (1999).
[67] J. N. Darroch, E. Seneta. On quasi-stationary distributions in absorbing discrete-time finite Markov chains. J. Appl. Probability, vol. 2, pp. 88–100, (1965).
[68] J.N.Darroch, E.Seneta. On quasi-stationary distributions in absorbing continuous-time finite Markov chains. J. Appl. Probab., vol.4, pp. 192–196 (1967).
[69] D. Dawson. Measure-valued Markov processes. École d'été de probabilités de Saint-Flour XXI-1991. Springer, Berlin, Heidelberg, pp. 1–260 (1993).
[70] P. Del Moral, D. Villemonais. Exponential mixing properties for time inhomogeneous diffusion processes with killing. 2014. hal-01083297v1. Bernoulli, vol. 24, no. 2, pp. 1010–1032 (2018).
[71] P. Del Moral. Nonlinear Filtering: Interacting Particle Solution. Markov Processes and Related Fields, vol. 2, no. 4, pp. 555–580 (1996).
[72] P. Del Moral. Feynman-Kac formulae. Springer, New York (2004).
[73] P. Del Moral. Mean field simulation for Monte Carlo integration. CRC press (2013).
[74] P. Del Moral, A. Doucet. Particle Motions in Absorbing Medium with Hard and Soft Obstacles. Stochastic Analysis and Applications. vol. 22, no. 5, pp. 1175–1207 (2004).
[75] P. Del Moral, A. Doucet, A. Jasra. Sequential monte carlo samplers. Journal of the Royal Statistical Society: Series B (Statistical Methodology), vol. 68, no. 3, pp. 411–436 (2006).
[76] P. Del Moral and L. Miclo. Branching and interacting particle systems approximations of Feynman-Kac formulae with applications to non-linear filtering.
In *Séminaire de Probabilités, XXXIV*, volume 1729, pp. 1–145 (2000).
[77] P. Del Moral, L. Miclo. A Moran particle system approximation of Feynman-Kac formulae. Stochastic processes and their applications, vol. 86, no. 2, pp. 193–216 (2000).
[78] P. Del Moral, A. Guionnet. On the stability of interacting processes with applications to filtering and genetic algorithms. Publication du Laboratoire de Statistique et Probabilitb 3-98, Université Paul Sabatier, Toulouse (1998). Annales de l'Institut Henri Poincaré Probab. & Statist. vol. 37, No. 2, pp. 155–194 (2001).
[79] P. Del Moral, A. Guionnet. On the stability of measure valued processes with applications to filtering. Comptes Rendus de l'Académie des Sciences-Series I-Mathematics, vol. 329, no. 5, pp. 429–434 (1999).
[80] P. Del Moral, E. Horton. Quantum harmonic oscillators and Feynman-Kac path integrals for linear diffusive particles. arXiv preprint arxiv:2106.14592 (2021).
[81] P. Del Moral, E. Horton. A theoretical analysis of one-dimensional discrete generation ensemble Kalman particle filters. arXiv preprint arxiv:2107.01855 (2021).
[82] P. Del Moral, A. Jasra. A Note on Random Walks with Absorbing barriers and Sequential Monte Carlo Methods. Stochastic Analysis and Applications vol.36, no. 3, pp. 413–442 (2018).
[83] P. Del Moral, L. Miclo. Genealogies and increasing propagation of chaos for Feynman-Kac and genetic models. The Annals of Applied Probability, vol. 11, no. 4, pp. 1166–1198 (2001).
[84] P. Del Moral, L. Miclo. Particle approximations of Lyapunov exponents connected to Schrödinger operators and Feynman-Kac semigroups. ESAIM: Probability and Statistics, no. 7, pp. 171–208 (2003).
[85] P. Del Moral, L. Miclo. On the stability of nonlinear Feynman-Kac semigroups. Annales de la Faculté des sciences de Toulouse: Mathématiques, serie 6, t. 11, no.2, pp. 135–175 (2002).
[86] P. Del Moral,L. Miclo. Self-interacting Markov chains. Stochastic Analysis and Applications, vol. 24, no. 3, pp. 615–660 (2006).
[87] P. Del Moral,L. Miclo. On convergence of chains with occupational self-interactions. Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences, 460(2041), pp. 325–346 (2004).

[88] P. Del Moral, M. Ledoux, L. Miclo. On contraction properties of Markov kernels. Probability theory and related fields, vol. 126, no. 3, pp. 395–420 (2003).
[89] P. Del Moral, S. Penev. Stochastic Processes: From Applications to Theory. CRC Press, (2017).
[90] P. Del Moral, D. Villemonais. Exponential mixing properties for time inhomogeneous diffusion processes with killing. Bernoulli, vol. 24, no. 2, pp. 1010–1032 (2018).
[91] A. Dembo and O. Zeitouni. Large Deviations Techniques and Applications, volume 38 of Stochastic Modelling and Applied Probability. Springer-Verlag, Berlin (2010)
[92] P. Diaconis and D. Freedman. On the consistency of Bayes estimates (with discussion). The Annals of Statistics, vol. 14, no. 1, pp. 1–67, (1986).
[93] P. Diaconis and L. Saloff-Coste. What do we know about the Metropolis algorithm? 27th Annual ACM Symposium on the Theory of Computing. J. Comput. System Sci., vol. 57, no. 1, pp. 20–36 (1998).
[94] P. Diaconis, K. Khare, and L. Saloff-Coste. Gibbs sampling, exponential families and orthogonal polynomials. Statist. Sci., vol. 23, no. 2, pp. 151–200 (2008).
[95] W. Doeblin, Sur les propriétés asymptotiques de mouvements régis par certains types de chaînes simples. Bull. Soc. Math. Roumaines des Sciences, vol. 39, no. 1, pp. 57-115, vol. 39, no. 2, pp. 3–61 (1937).
[96] R.L. Dobrushin, Central limit theorem for nonstationnary Markov chains, I,II, Theory Probab. Appl., vol. 1, no. 1,4, pp. 66–80, pp. 330–385 (1956).
[97] R.L. Dobrushin, Prescribing a system of random variables by conditional distributions, Theor. Prob. Appl. , vol. 15, no. 3 (1970).
[98] M. D. Donsker and S. R. S. Varadhan. On a variational formula for the principal eigenvalue for operators with maximum principle. Proc. Natl. Acad. Sci., vol. 72, no. 3, pp. 780–783 (1975).
[99] R. Douc, E. Moulines, J. Olsson. Long-term stability of sequential Monte Carlo methods under verifiable conditions. The Annals of Applied Probability, vol. 24, no. 5, pp. 1767–1802 (2014).
[100] R. Douc, E. Moulines, Y. Ritov. Forgetting of the initial condition for the filter in general state-space hidden Markov chain: a coupling approach. Electronic Journal of Probability, vol. 14, pp. 27–49 (2009).
[101] P. Drnovsek, M. Kandic. Ideal-triangularizability of semigroups of positive operators. Integral Equations and Operator Theory, vol. 64, no. 4, pp. 539–552 (2009).
[102] Y. Du. Order structure and topological methods in nonlinear partial differential equations: Vol. 1: Maximum principles and applications (Vol. 2). World Scientific (2006).
[103] K.J. Engel, R. Nagel, One-parameter semigroups for linear evolution equations. In Semigroup forum Springer-Verlag, vol. 63, no. 2, pp. 278–280 (2001).
[104] J. Feng and T. G. Kurtz. Large Deviations for Stochastic Processes, volume 131 of Mathematical Surveys and Monographs. American Mathematical Soc. (2006).
[105] P.A. Ferrari, N. Maric. Quasi stationary distributions and Fleming-Viot processes in countable spaces. Electron. J. Prob., vol. 12, pp. 684–702 (2007).
[106] P.A. Ferrari, H. Kesten, S. Martinez, D. Picco. Existence of quasi-stationary distributions: a renewal dynamical approach. Ann. Probab., vol.23, pp. 501–521 (1995).
[107] G. Ferré. Large Deviations Theory in Statistical Physics: Some Theoretical and Numerical Aspects. Diss. Université Marne La Vallée (2019).
[108] G. Ferré, M. Rousset, and G. Stoltz. Large deviations of empirical measures of diffusions in weighted topologies." Electronic Journal of Probability, vol. 25, pp. 1–52 (2020).
[109] G. Ferré, M. Rousset, and G. Stoltz. More on the long time stability of Feynman-Kac semigroups. Stochastics and Partial Differential Equations: Analysis and Computations, vol. 9, no. 3, pp. 630–673 (2021).
[110] W. H. Fleming and M. Viot. Some measure-valued markov processes in population genetics theory. Indiana University Mathematics Journal, vol. 28, no. 5, pp. 817–843 (1979).
[111] H. Furstenberg and H. Kesten. Product of random matrices. The Annals of Mathematical Statistics, vol. 31, no. 2, pp. 457–469 (1960).
[112] N. Gao, V. G. Troitsky Irreducible Semigroups of Positive Operators on Banach Lattices. Linear and Multilinear Algebra, vol. 62, no. 1, pp. 74–95 (2014).
[113] M. Gerber, N. Whiteley. Stability with respect to initial conditions in V-norm for nonlinear filters with ergodic observations. Journal of Applied Probability, vol. 54; no. 1, pp. 118–133 (2017).
[114] J. Glimm and A. Jaffe. Quantum Physics - A Functional Integral Point of View. 2nd ed., Springer-Verlag, New York (1987).
[115] G. L. Gong, M. P. Qian, and Z. X. Zhao. Killed diffusions and their conditioning. Probab. Theory Related Fields, vol. 80, no. 1, pp. 151–167 (1988).
[116] P. Good. The limiting behavior of transient birth and death processes conditioned on survival. J. Austral. Math. Soc., vol. 8, pp. 716–722 (1968).
[117] N. J. Gordon, D. J. Salmond, A. F. M. Smith. Novel approach to nonlinear/non-Gaussian Bayesian state estimation. IEE PROCEEDINGS-, Vol. 140, pp. 107–113 (1993).

[118] I.G. Graham, I. H. Sloan. On the compactness of certain integral operators. Journal of Mathematical Analysis and Applications, vol. 68, no. 2, pp. 580–594 (1979).

[119] P. Grassberger. Pruned-enriched Rosenbluth method: Simulations of polymers of chain length up to 1000000, Physical Review E, Vol. 56, pp. 3682–3693 (1997).

[120] P. Grassberger, H. Fraunenkron, W. Nadler. PERM: A Monte Carlo strategy for simulating polymers and other things, Monte Carlo Approach to Biopolymers and Protein Folding, ed. P. Grassberger, G. T. Barkema and W. Nadler (World Scientific, Singapore, 1998) pp. 301–315 (1998).

[121] P. Grassberger, Go with the winners: A general Monte Carlo strategy. Computer Physics Communications, vol. 147, no.1–2, pp. 64–70 (2002).

[122] D. Guibourg, L. Hervé and J. Ledoux. Quasi-compactness of Markov kernels on weighted-supremum spaces and geometrical ergodicity. arXiv:1110.3240 (2011).

[123] A. Guillin, B. Nectoux, and L. Wu, Quasi-stationary distribution for strongly Feller Markov processes by Lyapunov functions and applications to hypoelliptic Hamiltonian systems, working paper or preprint, December (2020).

[124] Y. Guivarc'h. Exposants caractéristiques des produits de matrices aléatoires en dépendance markovienne. In H. Heyer, editor, Probability Measures on Groups VII, Oberwolfach 1983, pages 161–181. Volume 1064 of Lecture Notes in Mathematics. Springer-Verlag, Berlin (1984).

[125] M. Hairer, J. C. Mattingly. Yet another look at Harris' ergodic theorem for Markov chains. Seminar on Stochastic Analysis, Random Fields and Applications VI. Springer, Basel (2011).

[126] G. He, Hanjun Zhang, and Yixia Zhu. On the quasi-ergodic distribution of absorbing Markov processes. Statistics & Probability Letters, vol. 149, pp. 116–123 (2019).

[127] K. Heine. Stability of the discrete time filter in terms of the tails of noise distributions. ESAIM: Proceedings. Vol. 19. EDP Sciences (2007).

[128] A. Hening, W. Qi, Z. Shen, Y. Yi . Quasi-stationary distributions of multi-dimensional diffusion processes. arXiv preprint arXiv:2102.05785 (2021).

[129] H. Hennion. Limit theorems for products of positive random matrices. The Annals of Proba- bility, vol. 25, no. 4, pp. 1545–1587, (1997).

[130] H. Hennion. Quasi-compactness and absolutely continuous kernels. Probability theory and related fields, vol. 139, no. 3–4, pp. 451–471 (2007).

[131] J. H. Hetherington, Observations on the statistical iteration of matrices. Physical Review A, 30(5), 2713 (1984).

[132] E. Horton. Stochastic Analysis of the neutron transport equation. PhD Uni. Bath, November (2019).

[133] E. Horton, A. Kyprianou, D. Villemonais. Stochastic methods for the neutron transport equation I: Linear semigroup asymptotics. The Annals of Applied Probability vol. 30, no. 6, pp. 2573–2612 (2020).

[134] M. Hürzeler. Statistical Methods for General State-Space Models. Ph.D. Thesis, Department of Mathematics, ETH Zurich, Zürich (1998).

[135] A. Jasra. On the behaviour of the backward interpretation of Feynman-Kac formulae under verifiable conditions. J. Applied Probability. vol. 52, no. 2, pp. 339–359 (2015).

[136] A. Jasra, A. Doucet. Stability of sequential Monte Carlo samplers via the Foster-Lyapunov condition. Statistics & probability letters, vol. 78, no. 17, pp. 3062–3069 (2008).

[137] L. Journel, P. Monmarché. Convergence of a particle approximation for the quasi-stationary distribution of a diffusion process: uniform estimates in a compact soft case. arXiv preprint arXiv:1910.05060 (2019).

[138] T. Kaijser. A limit theorem for partially observed Markov chains. The Annals of Probability, vol. 3, no. 4, pp. 677–696 (1975).

[139] S. Karlin and J. L. McGregor. The differential equations of birth-and- death processes, and the Stieltjes moment problem. Trans. Amer. Math. Soc., vol. 85, pp. 489–546, (1957).

[140] S. Karlin and J. L. McGregor. The classification of birth and death processes. Transactions of the American Mathematical Society, vol. 86, no. 2, pp. 366–400 (1957).

[141] G. Kitagawa. Monte Carlo filter and smoother for non-Gaussian nonlinear state space models. Journal of Computational and Graphical Statistics, Vol. 5, pp. 1–25 (1996).

[142] G. Kitagawa. A self-organizing state-space model. Journal of American Statistical Association, Vol. 93, pp. 1203–1215 (1998).

[143] M.L. Kleptsyna, A.Y. Veretennikov. On robustness of discrete time optimal filters. Mathematical Methods of Statistics, vol. 25, no. 3, pp. 207–218 (2016).

[144] Kleptsyna, M. L., and A. Yu Veretennikov. On continuous time ergodic filters with wrong initial data. Theory of Probability & Its Applications vol. 53, no. 2, pp. 269–300 (2009).

[145] Kleptsyna, M. L., and A. Yu Veretennikov. On discrete time ergodic filters with wrong initial data. Probab. Theory Related Fields, vol. 141, pp. 411–444 (2008).

[146] K. Knopp. Theory and Application of Infinite Series. London: Blackie & Son Ltd (1954)

[147] V. Knopova. On the Feynman-Kac semigroup for some Markov processes." Modern Stochastics: Theory and Applications, vol. 2, n.2 pp. 107–129 (2015).
[148] V. Konakov, S. Menozzi, S. Molchanov. Explicit parametrix and local limit theorems for some degenerate diffusion processes. Annales de l'IHP Probabilités et statistiques. Vol. 46. No. 4., pp. 908–923 (2010).
[149] I. Kontoyiannis and S.P. Meyn. Spectral theory and limit theorems for geometrically ergodic Markov processes. Ann. Appl. Probab., vol. 13, pp. 304–362 (2003).
[150] I. Kontoyiannis, S. P. Meyn. Geometric ergodicity and the spectral gap of non-reversible Markov chains. Probability Theory and Related Fields, vol. 154, no. 1–2, pp. 327–339 (2012).
[151] J.B. Lasserre, C.E.M. Pearce. On the existence of a quasistationary measure for a Markov chain. Technical Report no. 98109, LAAS-CNRS, Toulouse, France (1998). The Annals of Probability, vol. 29, no. 1, pp. 437-446 (2001).
[152] F. Le Gland, N. Oudjane. Stability and uniform approximation of nonlinear filters using the Hilbert metric and application to particle filters. The Annals of Applied Probability, vol. 14, no. 1, pp. 144–187 (2001).
[153] F. Le Gland, N. Oudjane. A robustification approach to stability and to uniform particle approximation of nonlinear filters: the example of pseudo-mixing signals. Stochastic Processes and their Applications, vol. 106, no. 2, pp. 279–316 (2003).
[154] F. Le Gland, L. Mevel. Exponential forgetting and geometric ergodicity in hidden Markov models. Mathematics of Control, Signals and Systems, vol. 13, no. 1, pp. 63–93 (2000).
[155] F. Le Gland, L. Mevel. Exponential forgetting and geometric ergodicity in HMM's. Proceedings of the 36th Conference on Decision and Control, San Diego. vol. 1. (1997).
[156] F. Le Gland, L. Mevel. Stability and Approximation of nonlinear filters: an information theoretic approach. Proceedings of the 38th Conference on Decision and Control, Phoenix Arizona (1999).
[157] F. Le Gland, L. Mevel. Basic properties of the projective product with application to products of column-allowable nonnegative matrices. Mathematics of Control, Signals and Systems, vol. 13, no. 1 pp. 41–62 (2000).
[158] M. Lladser, J. San Martìn. Domain of attraction of the quasi-stationary distributions for the Ornstein-Uhlenbeck process. Journal of applied probability, vol. 37, no. 2, pp. 511–520 (2000).
[159] C. Mailler, D. Villemonais. Stochastic approximation on non-compact measure spaces and application to measure-valued Pólya processes. The Annals of Applied Probability 30, no. 5, pp. 2393-2438 (2020).
[160] A. Marguet. A law of large numbers for branching Markov processes by the ergodicity of ancestral lineages. ESAIM: Probability and Statistics, 23, pp. 638–661 (2019).
[161] S. Méléard, D. Villemonais. Quasi-stationary distributions and population processes. Probab. Surv., vol. 9, pp. 340–410 (2012).
[162] C. McDonald, S. Yüksel. Exponential filter stability via Dobrushin's coefficient. Electronic Communications in Probability, vol. 25, pp. 1–13 (2020).
[163] S.P. Meyn, R.L. Tweedie. Markov Chains and Stochastic Stability. Springer-Verlag, New York (1993).
[164] J. Nash, Continuity of solutions of parabolic and elliptic equations, American J. Math. vol. 80, pp 931–954 (1958).
[165] E. Nummelin. A splitting technique for Harris recurrent Markov chains, Z. Wahsch. verw. Grebeite, vol. 43, pp. 309–318 (1978).
[166] R. D. Nussbaum. Eigenvectors of order-preserving linear operators. J. London Math. Soc., vol. 58, no. 2, pp. 480–496 (1998).
[167] R. D. Nussbaum. The radius of the essential spectral radius. Duke Math. J., vol. 37, pp. 473–478 (1970)
[168] W. Oçafrain. Convergence to quasi-stationarity through Poincaré inequalities and Bakry-Emery criteria. Electronic Journal of Probability, vol. 26, pp. 1–30 (2021).
[169] D.L. Ocone, E. Pardoux. Asymptotic stability of the optimal filter wit respect to its initial condition. SIAM Control and Optimization, vol. 34, no. 1, pp. 226-243 (1996).
[170] L. Oljaca, T. Kuna, J. Bröcker. Exponential stability and asymptotic properties of the optimal filter for signals with deterministic hyperbolic dynamics. arXiv preprint arXiv:2103.01190 (2021).
[171] N. Oudjane, S. Rubenthaler. Stability and uniform particle approximation of nonlinear filters in case of non ergodic signals. Stochastic Analysis and applications, vol. 23, no. 3, pp. 421–448 (2005).
[172] P. Meyer-Nieberg. Banach lattice, Springer-Verlag (1991).
[173] P. Pollett. Quasi-stationary distributions : a bibliography available at http://www.maths.uq.edu.au/ pkp/papers/qsds/qsds.html (2015).
[174] R. G. Pinsky. On the convergence of diffusion processes conditioned to remain in a bounded region for large time to limiting positive recurrent diffusion processes. Ann. Probab., vol.13, no. 2, pp. 363–378 (1985).
[175] M. Reed and B. Simon. Methods of Modern Mathematical Physics I: Functional Analysis. Academic Press, San Diego (1980).

[176] L. Rey-Bellet. Ergodic properties of Markov processes, volume 1881 of Lecture Notes in Mathematics, pages 1–39. Springer (2006).
[177] M. N. Rosenbluth and A. W. Rosenbluth. Monte Carlo calculation of the average extension of molecular chains. J. Chem. Phys., vol. 23, no. 2, pp. 356–359 (1955).
[178] M. Rousset. On the control of an interacting particle estimation of Schrödinger ground states SIAM journal on mathematical analysis, vol. 38, no. 3, pp. 824–844 (2006).
[179] H. H. Schaefer. Topological vector spaces. Springer-Verlag, New York. Third printing corrected (1971).
[180] E. Seneta and D. Vere-Jones. On quasi-stationary distributions in discrete-time Markov chains with a denumerable infinity of states. J. Appl. Probab., vol. 3, pp. 403–434 (1966).
[181] E. Seneta. Non-Negative Matrices and Markov Chains, 2nd edition. Springer Series in Statistics. Springer-Verlag, New York (1981).
[182] A.S. Sznitman. The Feynman-Kac Formula and Semigroups. In: Brownian Motion, Obstacles and Random Media. Springer Monographs in Mathematics. Springer, Berlin, Heidelberg (1998).
[183] R. Song. Two-sided estimates on the density of the Feynman-Kac semigroups of stable-like processes. Electronic Journal of Probability, vol. 11, pp. 146–161 (2006).
[184] F.M. Spieksma, Countable state space Markov processes: Non explosiveness and moment function. Probability in the Engineering and Informational Sciences, vol. 29, pp. 623–637 (2015).
[185] D. W. Stroock. Diffusion semigroups corresponding to uniformly elliptic divergence form operators. Séminaire de Probabilités, XXII, pp. 316– 347. Springer, Berlin (1988).
[186] W.H. Summers. Weighted Locally Convex Spaces of Continuous Functions. LSU Historical Dissertations and Theses, no. 1520 (1968)
[187] W.H. Summers. Dual spaces of weighted spaces. Trans. Americal Soc., vol. 151, pp. 323–333 (1970).
[188] L. Tierney. Markov chains for exploring posterior distributions Ann. of Statist., vol. 22, no. 4, pp. 1701–1762 (1994).
[189] E. A. van Doorn. Quasi-stationary distributions and convergence to quasi-stationarity of birth-death processes. Adv. in Appl. Probab., vol. 23, no. 4, pp. 683–700 (1991).
[190] E.A. van Doorn and P.K. Pollett. Quasi-stationary distributions for discrete-state models. European J. Oper. Res., vol. 230, pp. 1–14 (2013).
[191] R. Van Handel. The stability of conditional Markov processes and Markov chains in random environments. Ann. Probab., vol. 37, pp. 1876–1925 (2009).
[192] R. Van Handel. Discrete time nonlinear filters with informative observations are stable. Electronic Communications in Probability, vol. 13, pp. 562–575 (2008).
[193] N.T. Varopoulos. Small time gaussian estimates of heat diffusion kernels. II. The theory of large deviations. Journal oh Functional Analysis, vol. 93, pp. 1-33 (1990).
[194] D. Villemonais, General approximation method for the distribution of Markov processes conditioned not to be killed, ESAIM Probab. Stat., vol. 18, pp. 441–467 (2014).
[195] C. Wang. On estimates of the density of Feynman-Kac semigroups of α stable-like processes. Journal of mathematical analysis and applications, vol. 348, no.2, pp. 938–970 (2008).
[196] N. Whiteley. Stability properties of some particle filters. The Annals of Applied Probability, vol. 23, no. 6, pp. 2500–2537 (2013).
[197] N. Whiteley. Dimension-free Wasserstein contraction of nonlinear filters. Stochastic Processes and their Applications, vol. 135(C), pp. 31–50 (2021).
[198] N. Whiteley, N. Kantas, A. Jasra. Linear variance bounds for particle approximations of time-homogeneous Feynman-Kac formulae. Stochastic Processes and their Applications, vol. 122, pp. 1840–1865 (2012).
[199] N. Whiteley, N. Kantas,. Calculating principal eigen-functions of non-negative integral kernels: particle approximations and applications. Mathematics of Operations Research, vol. 42, no. 4, pp. 1007–1034 (2017).
[200] L. Wu. Large and moderate deviations and exponential convergence for stochastic damping Hamiltonian systems. Stoch. Process. Appl., vol. 91, no. 2, pp. 205–238 (2001).
[201] L. Wu. Uniformly integrable operators and large deviations for Markov processes. Journal of Functional Analysis, vol. 172, no.2, pp. 300—376 (2000).
[202] A.M. Yaglom. Certain limit theorems of the theory of branching random processes. Dokl. Akad. Nauk SSSR (N.S.), vol. 56, pp. 795–798 (1947).
[203] Y. Zhang. On the non explosion and explosion for non homogeneous Markov pure jump processes. Journal of Theoretical Probability, vol. 31, no. 3, 1322–1355 (2018).
[204] F. Zucca. A note on compact Markov operators. ArXiv preprint math/0402302 (2004).