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On the Stability of Positive Semigroups

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Abstract

The stability and contraction properties of positive integral semigroups on Polish spaces are investigated. We provide a novel analysis based on an extension of V -norm, Dobrushin-type, contraction techniques on functionally weighted Banach spaces for Markov operators. These are applied to a general class of positive and possibly time-inhomogeneous bounded integral semigroups and their normalised versions. Under mild regularity conditions, the Lipschitz-type contraction analysis presented in this article simplifies and extends several exponential estimates developed in the literature. The spectral-type theorems that we develop can also be seen as an extension of Perron-Frobenius and Krein-Rutman theorems for positive operators to time-varying positive semigroups. Incidentally, in the context of time-homogeneous models, the regularity conditions discussed in the present article appears to be a necessary and sufficient condition for the existence of leading eigenvalues. We review and illustrate in detail the impact of these results in the context of positive semigroups arising in transport theory, physics, mathematical biology and advanced signal processing.

Keywords: Positive semigroups; Boltzmann-Gibbs transformations; Contraction inequalities; Dobrushin's ergodic coefficient; Spectral theorems; Foster-Lyapunov conditions.

Mathematics Subject Classification: Primary: 47D08, 47D06, 47D07, 47H07; Secondary: 47B65, 37A30, 37M25, 60J25.

1 Introduction

Positive semigroups arise in a variety of areas of applied mathematics, including nonlinear filtering, rare event analysis, branching processes, mean field particle methods and sequential Monte Carlo methodologies, physics and molecular chemistry. In this article we will study the possibly time-inhomogeneous linear semigroup $Q_{s,t}$ and its normalisation $\Phi_{s,t}$, where $0 \leq s \leq t$ refers to a discrete or a continuous time parameter, which are formally introduced in (3) and (4) below. Their interpretation depends upon the application model area.

1. In signal processing, the normalised semigroup $\Phi_{s,t}$ depends on a random observation process and describes the solution to the nonlinear filtering equations, the semigroup $Q_{s,t}$ represents the evolution of the unnormalised filters. In this context, the stability of the semigroup $\Phi_{s,t}$ ensures that the optimal filter corrects any wrong initial condition.
2. In the context of killed absorption processes, $\Phi_{s,t}$ represents the evolution of the distribution of a given process conditioned on non-absorption (a.k.a. the Q -process). In this context, the fixed point probability measure $\eta_\infty = \Phi_{0,t}(\eta_\infty)$ of time homogeneous semigroups is sometimes called the Yaglom or quasi-invariant measure.
3. The conditional Markov processes discussed above also arise in risk analysis and rare event simulation. In this context, these semigroups represent the evolution of a conditional Markov process evolving in a rare event regime.
4. In quantum physics and molecular chemistry the top of the spectrum of positive integral semigroups is related to ground state and free energy computations of Schrödinger operators and Feynman-Kac semigroups (see for instance [75, 71] and the relevant literature therein).
5. In dynamic population literature, the semigroups $Q_{s,t}$ represents the evolution of the first moment of a spatial branching process. In this context, these many-to-one formulae are expressed in terms of Feynman-Kac semigroups connecting the free evolution of a single individual with the killing and the branching rates potential functions.

The details of these application areas are considered in [63, 67, 80] and the relevant references therein. The spectral objects discussed above are naturally related to the analysis of quasi-compact operators and Fredholm integral equations, see for instance [9, 107, 119, 93, 151], as well as in large deviations principles associated with the occupation measures and related additive functional of Markov processes [82, 89, 95, 182].

1.1 Literature review

In order to guide the reader through the vast array of stability analysis results developed in these different disciplines, we give a brief overview of the literature in these fields and provide some precise reference points aide with the navigation between applications.

A unifying point of interest in the above applications is the study of the stability of the afore-mentioned semigroups. In the context of dynamic populations, the long time behavior of branching processes certainly goes back to the end of the 1940s with the pioneering work of Yaglom [183] on Galton-Watson processes. The stability analysis of time homogenous birth-and-death-type processes with absorption on finite or countable spaces dates back to the 1950s-1960s with the pioneering works [128, 165] and the later developments [36, 59, 60, 105, 172]. Sufficient conditions ensuring the existence of a quasi-invariant measure are also developed in [97, 138]. Powerful spectral and h -process type techniques are also developed in [34, 104, 159], as well as in [35, 56, 71, 104, 117]. Non asymptotic spectral techniques that apply to possibly transient and unstable mutation-type linear diffusions that do not necessarily have a gradient form and with a quadratic absorption rate are discussed in the more recent work [71].

One of the first well-founded results on the long time behavior of the nonlinear filtering equation is the seminal article by Ocone and Pardoux in the mid-1990s [154]. In this work the authors show that the optimal filter forgets the initial condition without giving a non-asymptotic rate. This latter analysis is critical in the study of, for instance, numerical algorithms such as the particle filter. The stability of the nonlinear filtering equation is also related to the Lyapunov spectrum and the asymptotic properties of products of random positive matrices [20, 21, 113, 118, 127, 101, 165]. In contrast with positive semigroups arising in physics and biology, the stability analysis of the nonlinear filtering equation involves the study of sophisticated stochastic semigroups that depend on the partial and noisy observations.

The systematic, non-asymptotic stability analysis of non-homogeneous sub-Markovian and Feynman-Kac type semigroups on general state spaces has also been considered at the end of the 1990s, mainly in nonlinear filtering theory and rare event analysis. Several techniques have been adopted. Hilbert metric and robustification techniques, based on the seminal article by Birkhoff [15] was used in [8, 24, 58, 131, 155, 139, 140, 144, 156, 148]. On the other hand, Dobrushin's ergodic coefficients, based on the pioneering articles by Dobrushin [87], were used in [63, 67, 69, 70, 141, 142, 144, 148]. We also mention bounded Lipschitz distance techniques [141, 142] and relative entropy-like criteria [51, 63, 79, 143]. Local Doeblin-type minorisation conditions applying to non-compact spaces, including Foster-Lyapunov approaches and coupling, that apply to ergodic signal-observation filtering problems, including stable Gaussian-linear filtering models, are also discussed in the series of articles [90, 91, 103, 124, 125, 178, 180]. We also refer the reader to [174] for related Lipschitz norm techniques as well as the more recent articles [12, 41, 100, 112] in

the context of positivity preserving operators arising in particle absorption models and quasi-invariant measure literature. Functional inequalities, including Poincaré inequalities and Bakry-Emery criteria approaches are discussed in [153].

Spectral techniques, drift conditions and Wasserstein norm approaches for time-homogenous models are also discussed respectively in [71, 73, 75, 132] as well as in [179]. Further, two-sided minorisation conditions are discussed in [48, 49], see also [63, 64, 67], and truncation-type techniques are presented in [57, 100, 116, 178].

General asymptotic stability results are also provided in [175]. Stabilising changes of Feynman-Kac measures and related importance-sampling-type and h -process techniques that apply to possibly unstable killed processes on unbounded domains are also discussed in [12, 100, 163], see also [71] and well as section B.1 in [16] and in section 7.1 in [72] in the context of discrete time models. The stability of the nonlinear filtering equations with deterministic hyperbolic signals is also developed in the recent article [155].

Despite the numerous references given in the introduction, a complete literature review is not possible. For a more detailed discussion on this subject, we refer the reader to [25, 50] for a review on the asymptotic stability of nonlinear filters, to the bibliographies and reviews [55, 147, 158, 173] on the theory of quasi-stationary distributions, the articles [67, 100] as well as the bibliography in [71] in quantum physics, and the books [63, 64] for a detailed discussion on the long time stability of Feynman-Kac semigroups.

The stability analysis of positive semigroups is also crucial in the convergence analysis of numerical approximations of Feynman-Kac semigroups, including the computation of the principal eigen-function and the leading eigenvalues in the context of time homogeneous models (cf. for instance [75] as well as Theorem 2.11 and Theorem 3.27 in [67] in the context of time varying semigroups). Despite their importance, the numerical implications are not covered in the present article. However, to guide the reader, we end this section with some references to the literature on mean-field-type particle methodologies currently used in this context. Mean field and genetic-type particle methodologies are discussed in [52, 63, 64, 65, 66, 68, 69], as well as [7, 27, 28, 126, 163, 181], and [3, 4, 5, 53, 54, 45, 97, 176].

We also mention the pioneering articles from the mid-1950s by Rosenbluth and Rosenbluth [162] on sampling self-avoiding walks and another from the mid-1980s by Hetherington [120] on reconfiguration Monte-Carlo methods. These interacting Monte Carlo methodologies were further extended by Caffarel and his co-authors in the series of articles [29, 30, 31]. See also Buonauro-Sorella [26], as well as the pedagogical introduction to quantum Monte Carlo by Caffarel-Assaraf [32]. Similar heuristic-type bootstrapping methodologies were also used in the mid-1990s in nonlinear filtering [106, 129, 130] and to simulate long chain molecules [108, 109]. See also [1, 110].

In the context of discrete generation positive semigroups arising in nonlinear filtering and genetic-type algorithms, we refer the reader to the first well-founded article [62]. To the best of our knowledge the first articles discussing uniform propagation of chaos estimates with respect to the time parameter seem to be the articles [69, 70],

followed by [67, 74, 75] and the book [63]. While all the Monte Carlo methods discussed above belong to the same class of genetic-type mutation-selection methodology to estimate Feynman-Kac semigroups, they are known under a variety of different names in the applied literature such as Sequential Monte Carlo methods, Particle Filters, Cloning and Pruning as well as Bootstrapping techniques, Diffusion Monte Carlo, Population-Monte Carlo, Reconfiguration Monte Carlo, Moran and Fleming-Viot particle models, to name a few. Related reinforcement and self-interacting Markov chain methodologies and related stochastic approximation-type techniques are presented in [77, 78] and more recently in [14, 17, 145].

1.2 Contribution and structure

The main objective of this article is to further develop the stability analysis of positive semigroups for a general and abstract class of time inhomogeneous models, which, in general, are much more difficult to handle than their time homogeneous counterparts; this is because the operators may drastically change during the semigroup evolution. We will also tackle the problem of non-compact state spaces.

In the present article, we present a new unifying methodology that combines Dobrushin ergodic coefficient techniques developed in [63, 64, 67, 69, 76] with the powerful Foster-Lyapunov methodologies developed in the articles [12, 91, 100, 137, 178] in the context of Feynman-Kac semigroups and nonlinear filtering. Our approach relies on the contraction analysis of a class of triangular arrays of Markov semigroups introduced in [67, 69, 76] in terms of the V -Dobrushin contraction-type ergodic theory for Markov operators presented in [114] and further developed in a systematic way in the book [80]. A brief review on this subject is provided in section 3.2. For a more thorough discussion on these V -contraction coefficient principles, we refer to section 8.2.5 in [80]. The triangular-array-type Markov semigroups discussed above are described in section 1.5.

The article is structured as follows. In section 1.3 we introduce the notation that will be used throughout and state our main results. Section 1.4 and section 1.5 are dedicated to the detailed description of the semigroups considered in the article. The regularity conditions used in our analysis are presented in section 1.6. The main results of the article are described in section 2. In particular, in section 2.2, we present the main results for time-inhomogeneous models, with a more refined analysis of time homogeneous models being given in section 2.3. In section 3, we give a review on integral operators, which connects to some of the major topics that have been considered in the modern literature. Section 4 is dedicated to the proofs of the main theorems presented in this article. The appendix houses most of our technical proofs.

1.3 Some basic notation

Let $\mathcal{B}(E)$ be the space of measurable functions on a Polish metric space (E, d) equipped with its Borel σ -field \mathcal{E} . With a slight abuse of notation, we denote by

0 and 1 the null and unit scalars as well as the null and unit function on E .

We denote by $\mathcal{B}^+(E) \subset \mathcal{B}(E)$ the subspace of functions f such that $\inf_K f > 0$ for any compact set $K \subset E$, by $\mathcal{B}_b(E) \subset \mathcal{B}(E)$ the subspace of bounded measurable functions, and by $\mathcal{B}_0(E) \subset \mathcal{B}_b(E)$ the subspace of functions f that vanish at infinity; that is, for any $\epsilon > 0$ the set $\{f \geq \epsilon\}$ is compact.

We also let $\mathcal{C}(E) \subset \mathcal{B}(E)$ be the subset of continuous functions, $\mathcal{C}_b(E) \subset \mathcal{C}(E)$ the subset of bounded continuous functions and $\mathcal{C}_0(E) \subset \mathcal{C}_b(E)$ the subspace of functions that vanish at infinity.

In the further development of the article $V \geq 1$ stands for some locally bounded measurable function. We let $\mathcal{B}_V(E) \subset \mathcal{B}(E)$ be the Banach space of functions f with $\|f\|_V := \|f/V\| < \infty$, and we set $\mathcal{B}_V^+(E) := \mathcal{B}^+(E) \cap \mathcal{B}_V(E)$. We also consider the sets

$$\begin{aligned} \mathcal{B}_{0,V}(E) &:= \{f \in \mathcal{B}(E) : f/V \in \mathcal{B}_0(E)\} & \mathcal{B}_{0,V}^+(E) &:= \mathcal{B}_{0,V}(E) \cap \mathcal{B}^+(E) \\ \mathcal{C}_V(E) &:= \mathcal{C}(E) \cap \mathcal{B}_V(E) & \mathcal{C}_{0,V}(E) &:= \mathcal{C}(E) \cap \mathcal{B}_{0,V}(E). \end{aligned}$$

We also denote by $\mathcal{M}(E)$ the set of signed Radon measures μ on E and by $\mathcal{M}_V(E) \subset \mathcal{P}(E)$ the subset of measures $\mu \in \mathcal{M}(E)$ such that $|\mu|(V) < \infty$, where $|\mu| := \mu_+ + \mu_-$ stands for with the total variation measure associated with a Hahn-Jordan decomposition $\mu = \mu_+ - \mu_-$ of the measure. We also let $\mathcal{P}_V(E) \subset \mathcal{M}_V(E)$ the set of probability measures $\mu \in \mathcal{P}(E)$ such that $\mu(V) < \infty$.

We define the duality map, as well as the right and dual left action of a bounded integral operator Q using the classical measure theoretic notation, as follows:

$$\begin{aligned} (\mu, f) \in (\mathcal{M}_V(E) \times \mathcal{B}_V(E)) &\mapsto \mu(f) \in \mathbb{R} \quad \text{with} \quad \mu(f) := \int f(x)\mu(dx) \\ f \in \mathcal{B}_V(E) &\mapsto Q(f) \in \mathcal{B}_V(E) \quad \text{with} \quad Q(f)(x) := \int Q(x, dy) f(y) \\ \mu \in \mathcal{M}_V(E) &\mapsto \mu Q \in \mathcal{M}_V(E) \quad \text{with} \quad \mu Q(dy) := \int \mu(dx) Q(x, dy). \end{aligned} \tag{1}$$

Given a pair of integral operators (Q_1, Q_2) , we denote by $Q_1 Q_2$ the integral composition operator defined by

$$(Q_1 Q_2)(x, dz) = \int Q_1(x, dy) Q_2(y, dz).$$

We denote by I the identity integral operator. We say that Q is positive if $Q(f) \geq 0$ whenever $f \geq 0$. Whenever $Q(1) \leq 1$ we say that Q is sub-Markovian, and Q is said to be Markovian when $Q(1) = 1$. We associate with a function $H > 0$ the Boltzmann-Gibbs transformation

$$\Psi_H : \mu \in \mathcal{P}_V(E) \mapsto \Psi_H(\mu) \in \mathcal{P}_{V/H}(E) \tag{2}$$

defined by

$$\Psi_H(\mu)(dx) := \frac{H(x) \mu(dx)}{\mu(H)}.$$

We let $\mathcal{T} = \mathbb{R}_+ := [0, \infty[$ or $\mathcal{T} = \mathbb{N}$ be the continuous or discrete time space, respectively. For a given time horizon $t \in \mathcal{T}$, we set $\mathcal{T}_t := \mathcal{T} \cap [0, t]$. Given some $\tau \in \mathcal{T}$ with $\tau > 0$, we consider the time mesh $\mathcal{T}(\tau) := \{n\tau : n \in \mathbb{N}\}$ and we set $\mathcal{T}_t(\tau) = \mathcal{T}(\tau) \cap [0, t]$. We also consider for any $s \leq t$ the τ -intervals

$$[s, t]_\tau = \mathcal{T}(\tau) \cap [s, t] \quad \text{and} \quad [s, t[_\tau = \mathcal{T}(\tau) \cap [s, t[.$$

When $f = 1_A$ is the indicator function of some set $A \in \mathcal{E}$, we will sometimes slightly abuse notation and write $\mu(A)$ instead of $\mu(1_A)$. We also set $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$, for $a, b \in \mathbb{R}$ and we use the conventions

$$\left(\sum_{\emptyset}, \prod_{\emptyset} \right) = (0, 1) \quad \text{and} \quad \left(\sup_{\emptyset}, \inf_{\emptyset} \right) = (-\infty, +\infty).$$

1.4 Positive integral semigroups

Let $Q_{s,t}$ be a collection of positive integral operators from $\mathcal{B}_V(E)$ to $\mathcal{B}_V(E)$, indexed by parameters $s, t \in \mathcal{T}$ with $s \leq t$, and satisfying for any $s, u, t \in \mathcal{T}$, $s \leq u \leq t$, the semigroup property

$$Q_{s,u}Q_{u,t} = Q_{s,t} \quad \text{with} \quad Q_{s,s} = I. \quad (3)$$

Assume that $Q_{t,t+\tau}(1) > 0$ for some $\tau > 0$ and any $t \in \mathcal{T}$. This irreducibility condition ensures that $Q_{t,t+\tau}(f) > 0$ for any $f > 0$. In the discrete time setting, this condition can be relaxed up to some modification of the state space; see for instance the construction in [63, section 4.4], as well as the one dimensional neutron transport model discussed in [63, Example 4.4.3] and the soft and hard obstacle models discussed in [65, 73].

For sub-Markovian time homogeneous semigroups $Q_{t,t+\tau} = Q_{0,\tau}$ the Borel set $\{Q_{0,\tau}(1) = 0\}$ can be taken to be part of an absorbing set. In this context, to analyse the behavior of a non absorbed particle there is no loss of generality in assuming that $Q_{0,\tau}(1) > 0$.

In discrete time settings, up to a time-rescaling, there is no loss of generality in assuming that $\tau = 1$. In addition, continuous time models satisfying $Q_{t,t+\tau}(1) > 0$ for a particular $\tau > 0$ can be encapsulated by the discrete time setting via the time discretisation $\mathcal{T}(\tau)$.

The continuous time models considered in the present article are assumed to satisfy $Q_{t,t+s}(1) > 0$ for any $s > 0$. For both continuous and discrete time models, unless otherwise is stated, we shall denote by $\tau > 0$ a given and fixed parameter belonging to the continuous or the discrete time index set \mathcal{T} such that $Q_{t,t+\tau}(1) > 0$. From our discussion above, this parameter can be chosen $\tau = 1$ in discrete time settings, and by any parameter $\tau > 0$ in the continuous time framework.

We emphasise that the analysis of continuous time models is sometimes based on a discrete time approach based on regularity conditions that depend on the parameter

τ chosen by the user. In this context, the analysis is performed at the level of the τ -discretised model and the estimation constants presented in our results may depend on the parameter τ .

For any $s, t \in \mathcal{T}$ with $s \leq t$ and for any measure $\eta_s \in \mathcal{P}_V(E)$ we let $\Phi_{s,t}(\eta_s) \in \mathcal{P}_V(E)$ be the normalised distribution defined for any $f \in \mathcal{B}_V(E)$ by the formula

$$\Phi_{s,t}(\eta_s)(f) := \eta_s Q_{s,t}(f) / \eta_s Q_{s,t}(1). \quad (4)$$

The mapping $\Phi_{s,t}$ is a well defined nonlinear map from $\mathcal{P}_V(E)$ into $\mathcal{P}_V(E)$ satisfying for any $s, u, t \in \mathcal{T}$ with $s \leq u \leq t$ the semigroup property

$$\Phi_{s,t} = \Phi_{u,t} \circ \Phi_{s,u} \quad \text{with} \quad \Phi_{s,s}(\mu) = \mu.$$

Unless otherwise stated, all the semigroups discussed in this article are indexed by conformal indices $s \leq t$ in the set \mathcal{T} . To avoid repetition, we often write $Q_{s,t}$ and $\Phi_{s,t}$ without specifying the order $s \leq t$ of the indices $s, t \in \mathcal{T}$.

For a given a function $W \geq 0$ and some probability measure η on E we shall denote by $\kappa_W^-(\eta), \lambda^-(\eta) \in [0, +\infty[$ and $\kappa_W(\eta), \lambda(\eta) \in [0, +\infty]$ the parameters

$$\begin{aligned} \kappa_W^-(\eta) &:= \inf \Phi_{s,t}(\eta)(W) && \leq \kappa_W(\eta) &:= \sup \Phi_{s,t}(\eta)(W) \\ \lambda^-(\eta) &:= \inf \Phi_{s,t}(\eta)(Q_{t,t+\tau}(1)) && \leq \lambda(\eta) &:= \sup \Phi_{s,t}(\eta)(Q_{t,t+\tau}(1)). \end{aligned}$$

In the above display, the infimum and the supremum are taken over all indices (s, t) such that $t \in \mathcal{T}(\tau)$ and $s \in \mathcal{T}_t(\tau)$.

The unnormalised and the normalised semigroups $Q_{s,t}$ and $\Phi_{s,t}$ are connected for any $\mu \in \mathcal{M}_V(E)$, $f \in \mathcal{B}_V(E)$ and $s, t \in \mathcal{T}$ by the formula

$$\mu Q_{s,t}(f) = \Phi_{s,t}(\mu)(f) \prod_{u \in [s,t]_\tau} \Phi_{s,u}(\mu)(Q_{u,u+\tau}(1)). \quad (5)$$

The above formula coincides with the product formula relating the unnormalised operators $Q_{s,t}$ with the normalised semigroup $\Phi_{s,t}$ discussed in [67, section 1.3.2], see also [63, proposition 2.3.1] and [64, section 12.2.1]. For time homogeneous models we use the notation

$$(\Phi_t, Q_t) := (\Phi_{0,t}, Q_{0,t}).$$

In this setting, the product formula (5) reduces to

$$\mu(Q_t(f)) = \Phi_t(\mu)(f) \prod_{s \in [0,t]_\tau} \Phi_s(\mu)(Q_\tau(1)). \quad (6)$$

The general abstract class of time varying semigroups discussed above encapsulates most of the regular discrete or continuous time-varying stochastic models encountered in physics, biology and signal processing, including path-space and genealogical tree based models [3, 63, 64, 74].

In contrast with conventional Hilbert space approaches to the stability of reversible Markov semigroups (cf. for instance [22, 84, 85, 136]), the analysis of time varying models does not rely on a particular reversible measure. The framework of weighted spaces and V -norms considered in the article is a natural but non-unique framework to analyse time-varying positive semigroups.

1.5 A class of R -semigroups

Fix a measure $\bar{\eta}_0 \in \mathcal{P}_V(E)$ and a positive function $H \in \mathcal{B}_V^+(E)$, for some locally bounded function $V \geq 1$. We associate with these objects the functions $H_{s,t}$ and the normalised semigroups $\bar{Q}_{s,t}$ defined for any $s, t \in \mathcal{T}$ by the formulae

$$H_{s,t} := \bar{Q}_{s,t}(H) \quad \text{with} \quad \bar{Q}_{s,t}(f) := Q_{s,t}(f)/\bar{\eta}_s Q_{s,t}(1) \quad \text{and} \quad \bar{\eta}_s = \Phi_{0,s}(\bar{\eta}_0). \quad (7)$$

From the definitions, it follows that

$$\bar{\eta}_s \bar{Q}_{s,t} = \bar{\eta}_t \quad \text{and} \quad \bar{Q}_{s,u} \bar{Q}_{u,t} = \bar{Q}_{s,t} \quad \text{as well as} \quad \bar{\eta}_s(H_{s,t}) = \bar{\eta}_t(H) \quad \text{and} \quad H_{t,t} = H, \quad (8)$$

and

$$Q_{s,v}(H_{v,t}) = \lambda_{s,v} H_{s,t} \quad \text{with} \quad \lambda_{s,v} := \bar{\eta}_s Q_{s,v}(1) = \lambda_{s,u} \lambda_{u,v}.$$

We also set

$$\lambda^-(\bar{\eta}_0) \leq \lambda^- := \inf_{t \in \mathcal{T}(\tau)} \bar{\eta}_t(Q_{t,t+\tau}(1)) \leq \lambda := \sup_{t \in \mathcal{T}(\tau)} \bar{\eta}_t(Q_{t,t+\tau}(1)) \leq \lambda(\bar{\eta}_0). \quad (9)$$

Definition 1.1. We define the triangular array of Markov operators $R_{u,v}^{(t)}$, indexed by the parameter $0 \leq u \leq v \leq t$ and $t \in \mathcal{T}$, for any $f \in \mathcal{B}_b(E)$ by

$$R_{u,v}^{(t)}(f) := \frac{Q_{u,v}(f Q_{v,t}(H))}{Q_{u,v}(Q_{v,t}(H))} = \frac{1}{\lambda_{u,v} H_{u,t}} Q_{u,v}(H_{v,t}f). \quad (10)$$

For any given time horizon $t \in \mathcal{T}$ and any $s \leq u \leq v \leq t$, we have the semigroup property

$$R_{s,v}^{(t)} = R_{s,u}^{(t)} R_{u,v}^{(t)} \quad \text{and} \quad R_{s,s}^{(t)} = I.$$

This shows that $R_{u,v}^{(t)}(x, dy)$ can be interpreted as the Markov transition of a stochastic flow $X_{u,v}^{(t)}(x)$ on the time interval \mathcal{T}_t . In the further development of the article we use the terminology *R-semigroup* and *R-process* to refer to these objects.

Several remarks are of interest here:

1. The stability properties of these stochastic models plays a crucial role in the analysis of positive operators. An illustration in the context of coupled harmonic oscillators arising in physics is discussed in (65).
2. The use of these triangular arrays of Markov semigroups in the stability analysis of the time varying positive semigroups has been considered in [67, 69, 70, 76] and more recently in [44] in the context of Feynman-Kac semigroups and when $H = 1$. We also refer the reader to chapter 4 in [63], and chapter 12 in [64] for a systematic analysis of the contraction properties of these semigroups with respect to the total variation norm.

3. The same class of R -semigroups associated with some positive function H are also discussed in the article [12] in the context of time-homogeneous sub-Markovian models, under different set of regularity conditions. We shall discuss these conditions in section 2.4.
4. The interpretation of the R -semigroups depends on the application. For time homogeneous semigroups, the R -semigroups associated with the positive eigenstate $H = h$ of the semigroup Q_t coincides with the semigroup of the so-called h -process, see for instance (47) and (50). In the context of nonlinear filtering with $H = 1$, these semigroups represent the forward evolution of the optimal smoother on some observation time interval [67, 69, 70]. In branching processes theory, these semigroups reflect the evolution of an auxiliary process often used to describe the ancestral lineage of an individual [13, 146]. The link between these seemingly disconnected subjects is the so-called many-to-one formula connecting the first moment of the occupation measure of a spatial branching process with a Feynman-Kac semigroup (see for instance [13, 146], as well as section 1.4.4 in [63] and section 28.4 in [80]).

We now provide some important Markov transport formulae relating the semigroups introduced so far. The first formula connecting the R -semigroup with the flow of measures $\eta_u = \Phi_{s,u}(\eta_s)$ is given by

$$\Psi_{H_{s,t}}(\eta_s) R_{s,u}^{(t)}(f) = \frac{\eta_s(Q_{s,u}(H_{u,t}f))}{\eta_s(Q_{s,u}(H_{u,t}))} = \frac{\eta_u(H_{u,t}f)}{\eta_u(H_{u,t})}$$

This yields the formula

$$\Psi_{H_{s,t}}(\eta_s) R_{s,u}^{(t)} = \Psi_{H_{u,t}}(\eta_u).$$

Choosing $u = t$ and $\eta_s = \eta$ so that $\eta_t = \Phi_{s,t}(\eta)$ in the above display, we obtain the following lemma.

Lemma 1.2. *For any $t \in \mathcal{T}$ and $s \in \mathcal{T}_t$ and any $\eta \in \mathcal{P}_V(E)$ we have*

$$\Psi_H(\Phi_{s,t}(\eta)) = \Psi_{H_{s,t}}(\eta) R_{s,t}^{(t)} \quad \text{and} \quad \eta_s(H_{s,t}) \Psi_{H_{s,t}}(\eta_s) (1/H_{s,t}) = 1, \quad (11)$$

with the flow of probability measures

$$\forall 0 \leq s \leq u \leq t \quad \eta_u := \Phi_{0,u}(\eta_0) = \Phi_{s,u}(\eta_s) \quad \text{and} \quad \Psi_{H_{s,t}}(\eta_s) R_{s,u}^{(t)} = \Psi_{H_{u,t}}(\eta_u).$$

1.6 Regularity conditions

Let us begin with our first regularity condition.

(\mathcal{H}_0) *There exists a non-increasing function $\theta : [1, \infty[\rightarrow \mathbb{R}_+$, a constant $r_0 \geq 0$ and a function $\alpha : r \in [r_0, \infty[\mapsto \alpha(r) \in]0, 1]$ such that for any $t \in \mathcal{T}$, $s + \tau \in \mathcal{T}_t$ and $r \geq r_0$ we have*

$$\sup_{V_\theta(x) \vee V_\theta(y) \leq r} \left\| \delta_x R_{s,s+\tau}^{(t)} - \delta_y R_{s,s+\tau}^{(t)} \right\|_{tv} \leq 1 - \alpha(r) \quad \text{with} \quad V_\theta := 1/\theta(V). \quad (12)$$

The condition (12) is satisfied for some parameter $\alpha(r) \in]0, 1]$ *if and only if* the following minorisation condition holds

$$\forall t \in \mathcal{T} \quad \forall s + \tau \in \mathcal{T}_t \quad \forall r \geq r_0 \quad \forall (x, y) \in E^2 : V_\theta(x) \vee V_\theta(y) \leq r$$

$$\exists \nu \in \mathcal{P}(E) \quad \text{such that} \quad \delta_x R_{s, s+\tau}^{(t)} \geq \alpha(r) \nu \quad \text{and} \quad \delta_y R_{s, s+\tau}^{(t)} \geq \alpha(r) \nu.$$

Note that the measure ν in the above display may depend upon the parameters (x, y) as well as (r, s, t, τ) . The strength of the above condition depends on the choice of the functions V . For instance, we remark that (12) is satisfied for the unit function $V = 1 = r_0$ and the function $\theta(v) = 1/v$ *if and only if* the following *uniform* minorisation condition holds

$$\forall t \in \mathcal{T} \quad \forall s + \tau \in \mathcal{T}_t \quad \forall (x, y) \in E^2 \tag{13}$$

$$\exists \nu \in \mathcal{P}(E) \quad \text{such that} \quad \delta_x R_{s, s+\tau}^{(t)} \geq \alpha(1) \nu \quad \text{and} \quad \delta_y R_{s, s+\tau}^{(t)} \geq \alpha(1) \nu.$$

This latter condition provides a practical and natural way to check condition (\mathcal{H}_0) . For Markovian semigroups, this condition reduces to the well-known Dobrushin's condition [87]. In addition, when the measure ν in (13) does not depend on the state variables (x_1, x_2) , condition (13) coincides with Doeblin's condition [86]. To the best of our knowledge, the use of the triangular arrays $R_{s,u}^{(t)}$ and the extension of Dobrushin's contraction stability theory to time-varying and possibly random positive semigroups go back to the end of the 1990s with the article [70], see also [67, 69, 76] as well as the books [63, 64] and references therein.

Dobrushin's and Doeblin's conditions are popular conditions in probability theory. They are not always easily checked but are met for a large class of discrete or continuous time irreducible stochastic processes, mainly on compact domains. For instance, in the context of continuous time sub-Markovian semigroups this condition is satisfied for elliptic diffusions on compact manifolds killed at a bounded rate, as well as elliptic diffusions killed at the boundary of a bounded domain, cf. [76, Proposition 3.1] and [61, Lemma 3]. Combining the proof of [76, Proposition 3.1] with the two-sided estimates presented in [11, Lemma 3.3], one can check that (13) is also met for reflected one-dimensional diffusions killed at a bounded rate on some compact interval.

Condition (13) is also met for absorbed time homogenous neutron transport processes for sufficiently smooth domains with an absorbing boundary, as well as for time varying multidimensional birth and death processes with mass extinction, see [38, sections 4.1 & 4.2]. Further examples of sub-Markovian semigroups satisfying (13) are discussed in [63, 64, 67]. See also [65, 73, 43, 44]. Sufficient conditions and further examples are discussed in section 2.4.

Apart from the fact that the uniform minorisation condition (13) is met for some particular classes of operators on non-compact spaces (including for ad-hoc truncated drift Gaussian transitions or Laplace-type transitions on non-compact spaces - see [67,

section 5] and [63, exercise 3.5.2]) it is rarely satisfied for non-compact state spaces and thus our starting point is to localise the uniform minorisation condition (13).

Next, we consider a locally bounded function V with compact level sets. Note that in this case the function $\theta(V)$ vanishes at infinity and V_θ also has compact level sets defined by

$$\begin{aligned} \forall 0 < \epsilon < \epsilon_0 := \theta(\inf_E V) \\ \mathcal{V}_\epsilon := \{V \leq r_\epsilon\} = \{\theta(V) \geq \epsilon\} = \{V_\theta \leq 1/\epsilon\} \neq \emptyset \quad \text{with} \quad \inf_E V < r_\epsilon := \theta^{-1}(\epsilon). \end{aligned} \quad (14)$$

Note that the bounded compactness level set condition ensures that V is necessarily lower-semicontinuous and its infimum on every compact set is attained.

In this notation, condition (12) is satisfied for some $r_0 \geq \theta^{-1}(\epsilon_0)$ if and only if the following *local* minorisation condition holds

$$\begin{aligned} \forall 0 < \epsilon \leq \theta(r_0) \quad \forall t \in \mathcal{T} \quad \forall s + \tau \in \mathcal{T}_t \quad \forall (x, y) \in \mathcal{V}_\epsilon^2 \\ \exists \nu \in \mathcal{P}(E) \quad \text{such that} \quad \delta_x R_{s, s+\tau}^{(t)} \geq \alpha(r_\epsilon) \nu \quad \text{and} \quad \delta_y R_{s, s+\tau}^{(t)} \geq \alpha(r_\epsilon) \nu. \end{aligned} \quad (15)$$

Note that the measure ν in the above display may depend upon the parameters (x, y) and (r_0, s, t, τ) as well as on the level set parameter $0 < \epsilon \leq \theta(r_0)$.

The local Doebelin-type minorisation condition discussed above is rather standard in the stability analysis of Markov semigroups (see for instance [150, Theorem 6.15], [171, Proposition 2], and [149, Theorem 16.2.3]).

We strengthen this local minorisation property with the following condition.

(\mathcal{H}_1) Condition (\mathcal{H}_0) is met for some (α, θ) and some locally bounded function $V \geq 1$ with compact level sets. For any $t \in \mathcal{T}$ and $\mu \in \mathcal{P}_V(E)$ we have

$$Q_{t, t+\tau}(V)/V \leq \theta(V) \quad \text{as well as} \quad \kappa_V(\mu) < \infty \quad \text{and} \quad \lambda^- > 0. \quad (16)$$

In addition, there exists some $\epsilon_H > 0$ such that for any $0 < \epsilon \leq \epsilon_H$ we have

$$\varsigma_\epsilon(H) := \inf_{t \in \mathcal{T}(\tau)} \inf_{s \in \mathcal{T}_t(\tau)} \inf_{\mathcal{V}_\epsilon} H_{s,t} > 0 \quad \text{and} \quad \|H\|_V := \sup_{t \in \mathcal{T}(\tau)} \sup_{s \in \mathcal{T}_t(\tau)} \|H_{s,t}\|_V < \infty. \quad (17)$$

Whenever (\mathcal{H}_1) is satisfied, by replacing H with $H/\|H\|_V$, there is no loss of generality if we assume that $\|H_{s,t}\|_V \leq 1$, so that

$$H_{s,t} \leq V \quad \text{for any } s, t \in \mathcal{T}(\tau) \quad \text{and} \quad \|H\|_V \leq 1.$$

Note that condition (\mathcal{H}_1) depends on the choice of the functions V and $H_{s,t}$, with the latter, defined in (7), further depending on the choice of the function H and the initial probability measure $\bar{\eta}_0$.

Also note that the growth/Lyapunov condition stated in the left-hand side of (16) is satisfied with $\theta(v) = v^{-\delta}$ as soon as

$$Q_\tau(V)/V^{1-\delta} \in \mathcal{B}_b(E) \quad \text{for some} \quad 0 < \delta < 1. \quad (18)$$

Several ways to design functions V satisfying the above condition are discussed in section 3.3.

In the context of sub-Markovian semigroups arising in the literature, we emphasise that these growth conditions don't impose that the underlying free evolution is stable. For a more thorough discussion on this subject, we refer to the analysis of the harmonic oscillator (63) and the illustrations provided in the end of section 3.3.

Checking the estimates (17) may involve delicate calculations. For time homogeneous models, condition (\mathcal{H}_1) is equivalent to the existence of a leading eigen-pair of the positive semigroup (cf. for instance Theorem 2.16). For a more thorough discussion, we refer to section 2.3 dedicated to time homogenous models.

Next we present another stronger but simpler and more tractable condition that applies to absolutely continuous semigroups.

(\mathcal{H}_2) : For any $t \in \mathcal{T}$, the positive operator $Q_{t,t+\tau}$ has the form

$$Q_{t,t+\tau}(x, dy) = q_{t,t+\tau}(x, y) \nu_\tau(dy) \quad \text{and} \quad Q_{t,t+\tau}(V)/V \leq \theta(V) \quad (19)$$

for some non-increasing function $\theta : [1, \infty[\rightarrow \mathbb{R}_+$ and some locally bounded function $V \geq 1$ with compact level sets. In the above display, $q_{t,t+\tau}$ is a density with respect to some Radon positive measure ν_τ on E , such that for some $\epsilon_1 > 0$ and any $0 < \epsilon \leq \epsilon_1$ we have

$$0 < \iota_\epsilon^- := \inf_{t \in \mathcal{T}} \inf_{\mathcal{V}_\epsilon^2} q_{t,t+\tau} \leq \iota_\epsilon := \sup_{t \in \mathcal{T}} \sup_{\mathcal{V}_\epsilon^2} q_{t,t+\tau} < \infty \quad \text{and} \quad 0 < \nu_\tau(\mathcal{V}_\epsilon) < \infty. \quad (20)$$

For time homogeneous models $q_{t,t+\tau} = q_\tau$ condition (20) is clearly met as soon as q_τ is a positive continuous density on the compact sets \mathcal{V}_ϵ^2 . These absolutely continuous integral operators arise in a natural way in discrete time settings [63, 67] and continuous time elliptic diffusion-type absorption models [6, 98, 99, 168]. In connection to this, two-sided estimates for stable-like processes are provided in [167, 177]. See also [19, 134]. Two sided Gaussian-type estimates can also be obtained for some classes of degenerate diffusion processes of rank 2, that is when the Poisson brackets of the first order span the whole space [135]. This class of diffusions includes frictionless Hamiltonian kinetic models. Diffusion density estimates can be extended to sub-Markovian semigroups using the multiplicative functional methodology developed in [76]. Jump type models can be handled directly from (\mathcal{H}_1) or using the methodology and minorisation conditions discussed in sections 2.1 and 2.3.

The following lemma will help us to establish the link between the hypotheses (\mathcal{H}_0) , (\mathcal{H}_1) and (\mathcal{H}_2) . It is a slight modification of [178, Proposition 1 & Lemma 10], based on technical approaches from [91] in the context of nonlinear filtering stability.

Lemma 1.3. *Assume that (19) and (20) are satisfied. In this situation, (17) is satisfied with $\epsilon_H := \theta(\bar{\eta}(V) + 1)$. In addition, for any $\eta \in \mathcal{P}_V(E)$ we have*

$$0 < \kappa_H^-(\eta) \leq \kappa_V(\eta) < \infty \quad \text{and} \quad 0 < \lambda^-(\eta) \leq \lambda(\eta) < \infty. \quad (21)$$

For the convenience of the reader a detailed proof in our context is provided in the appendix, see section A.1. The above lemma also ensures for any bounded $f \geq 0$, $x \in \mathcal{V}_\epsilon$ and $0 < \epsilon \leq \epsilon_H := \theta(\bar{\eta}(V) + 1)$ we have

$$R_{s,s+\tau}^{(t)}(f)(x) \geq \frac{\iota_\epsilon^-}{\lambda \|H\|_V} \nu_\tau(1_{\mathcal{V}_\epsilon} H_{s+\tau,t} f) \geq \frac{\iota_\epsilon^- \varsigma_\epsilon(H) \nu_\tau(\mathcal{V}_\epsilon)}{\lambda \|H\|_V} \frac{\nu_\tau(1_{\mathcal{V}_\epsilon} H_{s+\tau,t} f)}{\nu_\tau(1_{\mathcal{V}_\epsilon} H_{s+\tau,t})},$$

which implies (12). Thus, the above lemma ensures that

$$(\mathcal{H}_2) \implies ((\mathcal{H}_1) \quad \text{with} \quad \epsilon_H = \theta(\bar{\eta}(V) + 1)) \implies (\mathcal{H}_0).$$

2 Statement of the main results

2.1 A uniform stability theorem

This short section is concerned with a brief review on the stability properties of the non-linear semigroup $\Phi_{s,t}$ under the uniform minorisation condition (13). The main objective of this article is to extend these results to more general classes of models satisfying the local minorisation condition stated in (15). We recall the rather well known strong stability theorem which is valid on abstract measurable state spaces.

Theorem 2.1 ([67, 69, 70]). *Let (E, \mathcal{E}) be a measurable state space. Assume that (13) is satisfied and $H = 1$. Then for any $s, t \in \mathcal{T}$ and any probability measures $\mu, \eta \in \mathcal{P}(E)$ we have the uniform stability estimate*

$$\|\Phi_{s,t}(\mu) - \Phi_{s,t}(\eta)\|_{tv} \leq (1 - \alpha(1))^{(t-s)/\tau}. \quad (22)$$

In addition, for any $\mu_1, \mu_2 \in \mathcal{P}(E)$ we have the local Lipschitz estimate

$$\|\Phi_{s,t}(\mu_1) - \Phi_{s,t}(\mu_2)\|_{tv} \leq \frac{\|Q_{s,t}(1)\|}{\mu_1(Q_{s,t}(1)) \vee \mu_2(Q_{s,t}(1))} (1 - \alpha(1))^{(t-s)/\tau} \|\mu_1 - \mu_2\|_{tv}. \quad (23)$$

A detailed and remarkably simple proof of the above theorem based on the non-linear transport formula (11) is provided in [67] (cf. for instance Theorem 2.3), see also [69, Lemma 2.1 & Lemma 2.3], [76, Lemma 2.1 & Proposition 2.3], [64, section 12.2], [67, section 2.1.2 & section 3.1.3] and [63, section 4]. The key semigroup oscillation formula [67, Theorem 2.3], [76, Proposition 2.3] connecting the *uniform* exponential decay (22) with Dobrushin's ergodic coefficient of the R -semigroup when $H = 1$ defined in section 3.2 is given by

$$\sup_{(\mu, \eta) \in \mathcal{P}(E)^2} \|\Phi_{s,t}(\mu) - \Phi_{s,t}(\eta)\|_{tv} = \sup_{(x, y) \in E^2} \|\delta_x R_{s,t}^{(t)} - \delta_y R_{s,t}^{(t)}\|_{tv}.$$

The above formula shows that the *uniform* exponential decays (22) are dictated by the contraction properties of the triangular arrays $R_{s,u}^{(t)}$. An extended version of the above theorem for general relative entropy criteria is provided in [63, section 4.3.1].

For the rest of this section, we place ourselves in the time homogeneous setting and E stands for an abstract measurable state space. In this case, a variety of results follows almost immediately from the uniform estimates obtained in this theorem. Before stating them, we first discuss some relevant properties of time homogeneous models.

The uniform estimate (22) implies that $\Phi_t(\mu)$ is a Cauchy sequence in the complete set of probability measures $\mathcal{P}(E)$ equipped with the total variation distance. Thus, for any μ , the flow $\Phi_t(\mu)$ converges, as $t \rightarrow \infty$, exponentially fast to a single probability measure $\eta_\infty \in \mathcal{P}(E)$. Using the fixed point equation, for any $s, t \in \mathcal{T}$ we readily check that

$$\eta_\infty(Q_{s+t}(1)) = \eta_\infty(Q_s(1)) \eta_\infty(Q_t(1)).$$

Thus for any $t \in \mathcal{T}$ we have the exponential formula

$$\eta_\infty(Q_t(1)) = e^{\rho t} \quad \text{for some } \rho \in \mathbb{R}. \quad (24)$$

Choosing $\bar{\eta}_0 = \eta_\infty$ in (7) and using (6) for any $t \in \mathcal{T}(\tau)$ we check that

$$\mu \bar{Q}_t(1) = \prod_{s \in [0, t]_{\mathcal{T}}} \{1 + [\Phi_s(\mu)(\bar{Q}_\tau(1)) - \Phi_s(\eta_\infty)(\bar{Q}_\tau(1))]\}. \quad (25)$$

The exponential version of the above formula in the context of sub-Markovian semi-groups with soft killing is discussed in (60).

Conversely, by (22) for any $n \geq 1$ we have

$$\sum_{s \in \mathcal{T}(\tau)} |\Phi_s(\mu)(\bar{Q}_\tau(1)) - \Phi_s(\eta_\infty)(\bar{Q}_\tau(1))|^n < \infty$$

as well as

$$\|\bar{Q}_t(1)\| \leq \prod_{s \in \mathcal{T}(\tau)} (1 + (1 - \alpha(1))^{s/\tau} \|Q_\tau(1)\| / \eta_\infty Q_\tau(1)) < \infty.$$

The two estimates discussed above ensure that $\mu \bar{Q}_t(1)$ converges, as $\mathcal{T}(\tau) \ni t \rightarrow \infty$, to a non zero number and we have

$$0 < \inf_{t \in \mathcal{T}(\tau)} \mu(\bar{Q}_t(1)) \leq \sup_{t \in \mathcal{T}(\tau)} \|\bar{Q}_t(1)\| < \infty. \quad (26)$$

Note that the convergence of the series ensures that $\Phi_t(\mu)(\bar{Q}_\tau(1)) \rightarrow_{t \rightarrow \infty} 1$. Also observe that (26) implies that

$$\forall \mu \in \mathcal{P}(E) \quad q(\mu) := \sup_{t \in \mathcal{T}(\tau)} \|Q_t(1)\| / \mu(Q_t(1)) < \infty.$$

We are now in a position to state our first corollary of Theorem 2.1.

Corollary 2.2. *Under the assumptions of Theorem 2.1, for any $\mu, \eta \in \mathcal{P}(E)$ and $t \in \mathcal{T}$ we have the local contraction estimate*

$$\|\Phi_t(\mu) - \Phi_t(\eta)\|_{tv} \leq (q(\mu) \wedge q(\eta)) (1 - \alpha(1))^{t/\tau} \|\mu - \eta\|_{tv}. \quad (27)$$

Choosing $\mu = \delta_x$ in (25), we also readily check that there exists $h \in \mathcal{B}_b(E)$ such that $\eta_\infty(h) = 1$ and

$$\lim_{t \rightarrow \infty} \overline{Q}_t(1)(x) = \lim_{t \rightarrow \infty} Q_t(1)(x) = h(x) > 0.$$

By the dominated convergence theorem, for any $s \in \mathcal{T}$ this implies that

$$\overline{Q}_{s+t}(1)(x) = \overline{Q}_s(\overline{Q}_t(1))(x) \xrightarrow{t \rightarrow \infty} h(x) = \overline{Q}_s(h)(x). \quad (28)$$

Choosing $H = h$ in (10), formula (11) reads

$$\Psi_h(\Phi_t(\eta)) = \Psi_h(\eta) P_t^h \quad \text{with} \quad P_s^h(f) := Q_s(hf)/Q_s(h) = R_{0,s}^{(t)}(f). \quad (29)$$

Note that $h \in \mathcal{C}_b(E)$ as soon as $Q_\tau(\mathcal{B}_b(E)) \subset \mathcal{C}_b(E)$. Also observe that

$$\delta_x P_t^h = \Psi_h(\Phi_t(\delta_x)).$$

The second corollary is now a direct consequence of Theorem 2.1.

Corollary 2.3. *Under the assumptions of Theorem 2.1, there exists a positive function $h \in \mathcal{B}_b(E)$ and a constant $\rho \in \mathbb{R}$ such that for any $t \in \mathcal{T}$ we have*

$$Q_t(h) = e^{\rho t} h \quad \text{and} \quad \eta_\infty^h P_t^h = \eta_\infty^h \quad \text{with} \quad \eta_\infty^h := \Psi_h(\eta_\infty). \quad (30)$$

In addition, we have the total variation exponential decays

$$\|\delta_x P_t^h - \eta_\infty^h\|_{tv} \leq \frac{\|h\|}{\eta_\infty(h)} (1 - \alpha(1))^{t/\tau}. \quad (31)$$

Using (26) and applying the uniform estimate (22) to $\mu = \delta_x$ and $\eta = \eta_\infty$ for any $s, t \in \mathcal{T}$ and $x \in E$ we have

$$|\overline{Q}_{t+s}(1)(x)/\overline{Q}_t(1)(x) - 1| = |\Phi_t(\delta_x)(\overline{Q}_s(1)) - 1| \leq 2 q(\eta_\infty) (1 - \alpha(1))^{t/\tau}.$$

This implies that

$$\|\overline{Q}_t(1) - h/\eta_\infty(h)\| = \lim_{s \rightarrow \infty} \|\overline{Q}_t(1) - \overline{Q}_{t+s}(1)\| \leq 2 q(\eta_\infty)^2 (1 - \alpha(1))^{t/\tau}.$$

On the other hand, for any $f \in \mathcal{B}_b(E)$ and $x \in E$ we have

$$\begin{aligned} |\overline{Q}_t(f)(x) - \frac{h(x)}{\eta_\infty(h)} \eta_\infty(f)| &\leq \|f\| |\overline{Q}_t(1)(x) - h(x)/\eta_\infty(h)| \\ &\quad + \frac{\|h\|}{\eta_\infty(h)} \|\Phi_t(\delta_x)(f) - \eta_\infty(f)\|, \end{aligned}$$

which leads us to our final corollary of the section.

Corollary 2.4. *Under the assumptions of Corollary 2.3, we have the operator norm exponential decays*

$$\sup_{\|f\| \leq 1} \left\| \overline{Q}_t(f) - \frac{h}{\eta_\infty(h)} \eta_\infty(f) \right\| \leq 2 (1 - \alpha(1))^{t/\tau} (\|h\|/\eta_\infty(h) + q(\eta_\infty)^2), \quad (32)$$

as well as for any $s, t \in \mathcal{T}$ and $\eta \in \mathcal{P}(E)$ the uniform total variation estimates

$$\begin{aligned} & \|\Psi_{Q_t(1)}(\Phi_s(\eta)) - \Psi_h(\eta_\infty)\|_{tv} \\ & \leq q(\eta_\infty) (1 - \alpha(1))^{s/\tau} + 2 (1 - \alpha(1))^{t/\tau} (\|h\|/\eta_\infty(h) + q(\eta_\infty)^2). \end{aligned} \quad (33)$$

The last assertion comes from the decomposition

$$\begin{aligned} \overline{\eta}_0 = \eta_\infty & \implies \Psi_{Q_t(1)}(\mu)(f) - \Psi_h(\eta_\infty)(f) \\ & = [\Psi_{\overline{Q}_t(1)}(\mu) - \Psi_{\overline{Q}_t(1)}(\eta_\infty)](f) + \eta_\infty(f(\overline{Q}_t(1) - h/\eta_\infty(h))). \end{aligned}$$

The estimate (33) is often used in the analysis of the ergodic properties of particle absorption models, see for instance [40, 115] as well as equation (62) in the present article.

The above corollary can be interpreted as an extended version of the Krein-Rutman theorem to positive semigroups satisfying the uniform minorisation condition (13). In this connection, note that (32) readily yields the uniqueness of the eigenfunction h (up to some constant). Indeed, by (32) we have

$$(\forall t \in \mathcal{T} \quad \overline{Q}_t(g) = g \in \mathcal{B}_V(E)) \implies g = \frac{\eta_\infty(g)}{\eta_\infty(h)} h.$$

Defining the finite rank (and hence compact) operator

$$T_t(f) := e^{\rho t} \frac{h}{\eta_\infty(h)} \eta_\infty(f),$$

we also have

$$\begin{aligned} \lim_{t \rightarrow \infty} \|\|Q_t - T_t\|\|^{1/t} & \leq e^\rho (1 - \alpha(1))^{1/\tau} \\ & < e^\rho = \eta_\infty(Q_t(1))^{1/t} \leq \|Q_t(1)\|^{1/t} \xrightarrow{t \rightarrow \infty} \lim_{t \rightarrow \infty} \|\|Q_t\|\|^{1/t} \end{aligned} \quad (34)$$

where $\|\|\cdot\|\|$ denotes the operator norm. The estimates stated in (34) ensure that the essential spectral radius of Q_t is strictly smaller than its spectral radius. For a more thorough discussion on these spectral quantities, we refer to section 3.4.

A more refined analysis under weaker conditions, including exponential stability theorems and contraction properties of time homogeneous semigroups, is provided in section 2.3.

2.2 Contraction of R -semigroups

We now present the first main result of the article, the proof of which can be found in section 4.1.

Theorem 2.5. *Assume that (\mathcal{H}_1) is satisfied. Then there exist constants $a < \infty$ and $b > 0$ such that for any $t \in \mathcal{T}$, $s, u \in \mathcal{T}_t$ and $\mu, \eta \in \mathcal{P}_{V/H_{s,t}}$, we have the uniform contraction estimate*

$$\|\mu R_{s,u}^{(t)} - \eta R_{s,u}^{(t)}\|_{V/H_{u,t}} \leq a e^{-b(u-s)} \|\mu - \eta\|_{V/H_{s,t}}. \quad (35)$$

Remark 2.6. *As we shall see in Lemma 4.1, condition (\mathcal{H}_1) also ensures the existence of some $0 < \epsilon < 1$ and some constant $c > 0$ such that for any time horizon $t \in \mathcal{T}$ and any $s + \tau \in \mathcal{T}_t$ we have the Lyapunov estimate*

$$R_{s,s+\tau}^{(t)}(V/H_{s+\tau,t}) \leq \epsilon V/H_{s,t} + c. \quad (36)$$

In this direction, we also emphasise that the main ingredient of the proof of Theorem 2.5 is the V -contraction for Markov operators discussed in Lemma 3.2.

The estimate on the left-hand side of (17) allows one to control, uniformly with respect to the time parameter, the quantities $\mu(H_{s,t})$ in terms of the parameters $\mu(V)$, for any $\mu \in \mathcal{P}_V(E)$. Since these uniform estimates will be used several times in the sequel, we present them here in a general form. Applying the Markov inequality, for any $\mu \in \mathcal{P}_V(E)$ the left-hand side condition in (17) ensures the existence of some $n \geq 1$ and some $\delta_n > 0$ such that

$$\epsilon_n = \theta(\mu(V) + n) \leq \epsilon_H \quad \text{and} \quad \mu(H_{s,t}) \geq \delta_n \mu(\mathcal{V}_{\epsilon_n}) \geq \frac{\delta_n}{1 + \mu(V)/n} > 0. \quad (37)$$

We conclude that

$$(\mathcal{H}_1) \implies \mu \in \mathcal{P}_V(E) \quad 0 < \omega_H(\mu) := \inf_{t \in \mathcal{T}(\tau)} \inf_{s \in \mathcal{T}_t(\tau)} \mu(H_{s,t}) \leq \mu(V). \quad (38)$$

Similarly, we check that the condition $\kappa_V(\mu) < \infty$ ensures the tightness of sequence of measures $\Phi_{s,t}(\mu)$ indexed by $s, t \in \mathcal{T}$, for any $\mu \in \mathcal{P}_V(E)$. In the same vein, by (9), condition (\mathcal{H}_1) ensures that the flow of measures $\bar{\eta}_t$ is tight. Thus, choosing

$$\epsilon_n = \theta(\bar{\eta}(V) + n) \leq \epsilon_H \quad \text{with} \quad \bar{\eta}(V) := \sup_{t \in \mathcal{T}(\tau)} \bar{\eta}_t(V)$$

we also check that

$$\bar{\eta}_-(H) := \inf_{t \in \mathcal{T}(\tau)} \bar{\eta}_t(H) \geq \varsigma_{\epsilon_n}(H)/(1 + \bar{\eta}(V)/n) > 0.$$

Arguing as above we present a simple way to check that $\lambda^- > 0$. Assume that $\inf_{t \in \mathcal{T}} Q_{t,t+\tau}(1) \geq g_\tau > 0$ for some continuous function $g_\tau > 0$. As a result, we have that

$$\bar{\eta}(V) \leq \kappa_V(\bar{\eta}_0) < \infty \implies \lambda^- > 0.$$

The right-hand side assertion follows from the fact that

$$\epsilon_n = \theta(\bar{\eta}(V) + n) \implies \bar{\eta}_t(Q_{t,t+\tau}(1)) \geq \frac{\inf_{V \leq \epsilon_n} g_\tau}{1 + \bar{\eta}(V)/n} > 0.$$

We are now in position to discuss some direct consequences of the above theorem. The first corollary and its time homogeneous version discussed in Corollary 2.20 can be interpreted as an extended version of the Krein-Rutman theorem to time varying positive semigroups.

Corollary 2.7. *Assume that (\mathcal{H}_1) is satisfied. Then there exists some finite constant $c < \infty$ such that for any $t \in \mathcal{T}$ and $s \in \mathcal{T}_t$ we have the operator norm exponential decay*

$$\sup_{\|f\|_V \leq 1} \left\| \bar{Q}_{s,t}(f) - \frac{H_{s,t}}{\bar{\eta}_s(H_{s,t})} \bar{\eta}_t(f) \right\|_V \leq a e^{-b(t-s)} (1 + \|H\|_V \bar{\eta}(V)/\bar{\eta}_-(H)),$$

where (a, b) were defined in (35).

Proof. Using (8) and (11) we check that

$$\Psi_{H_{s,t}}(\bar{\eta}_s) R_{s,t}^{(t)}(f/H) = \Phi_{s,t}(\bar{\eta}_s)(f)/\Phi_{s,t}(\bar{\eta}_s)(H) = \bar{\eta}_t(f)/\bar{\eta}_t(H).$$

This yields the decomposition

$$\bar{Q}_{s,t}(f)(x) - \frac{H_{s,t}}{\bar{\eta}_s(H_{s,t})} \bar{\eta}_t(f) = H_{s,t}(x) \left(R_{s,t}^{(t)}(f/H) - \Psi_{H_{s,t}}(\bar{\eta}_s) R_{s,t}^{(t)}(f/H) \right)$$

Applying (35) to $u = t$, $\mu = \delta_x$ and $\eta = \Psi_{H_{s,t}}(\bar{\eta}_s)$, for any $\|f\|_V = \|f/H\|_{V/H} \leq 1$ we check the estimate

$$\left| \bar{Q}_{s,t}(f) - \frac{H_{s,t}}{\bar{\eta}_s(H_{s,t})} \bar{\eta}_t(f) \right| \leq a e^{-b(t-s)} \left(V + \frac{H_{s,t}}{\bar{\eta}_s(H_{s,t})} \bar{\eta}_s(V) \right) \quad (39)$$

where (a, b) were defined in (35). This concludes the proof. \blacksquare

When $H = 1$, the extended version of the above corollary in the context of random semigroups arising in filtering is provided in [178]. The proof in [178] relies on rather sophisticated coupling and decomposition techniques given in [133], which were further developed in [91].

The time homogeneous version of the above result is provided in Corollary 2.20. In this context, condition (\mathcal{H}_1) is readily checked by choosing $H = h$, where h is the positive eigenstate of the positive semigroup, see for instance (47) and (50).

Remark 2.8. *In Theorem 2.5 and Corollary 2.7, the tightness condition $\kappa_V(\mu) < \infty$ for any $\mu \in \mathcal{P}_V(E)$ imposed by (\mathcal{H}_1) can be replaced by the condition $\bar{\eta}(V) < \infty$. In this situation, choosing $f = 1$ in (39) for any $\mu \in \mathcal{P}_V(E)$ we readily check that*

$$|\mu \bar{Q}_{s,t}(1) - \mu(H_{s,t})/\bar{\eta}_t(H)| \leq a \mu(V) e^{-b(t-s)} (1 + \|H\|_V \bar{\eta}(V)/\bar{\eta}_-(H)), \quad (40)$$

where (a, b) were defined in (35).

Our second main result transfers the stability of the R -semigroup to that of the normalised semigroup Φ .

Theorem 2.9. *Under the assumptions of Theorem 2.5, for any $t \in \mathcal{T}$ and $s \in \mathcal{T}_t$ and any $\mu, \eta \in \mathcal{P}_V(E)$ we have the local contraction estimate*

$$\|\Phi_{s,t}(\mu) - \Phi_{s,t}(\eta)\|_V \leq a \kappa(\eta, \mu) e^{-b(t-s)} \|\mu - \eta\|_V \quad (41)$$

with (a, b) as in (35) and $\kappa(\eta, \mu)$ given by

$$\kappa(\eta, \mu) := \kappa_H(\mu) (1 + \kappa_V(\eta)) (1 + \eta(V)/\omega_H(\eta)) / \omega_H(\mu).$$

This result is a direct consequence of the V -contraction estimates (35) stated in Theorem 2.5 and rather elementary Boltzmann-Gibbs estimates (68) and (69) presented in section 3.1. The full proof is provided in section 4.2.

Remark 2.10. *As in Remark 2.8, the condition $\kappa_V(\mu) < \infty$ for any $\mu \in \mathcal{P}_V(E)$ imposed by (\mathcal{H}_1) in Theorem 2.9 can be replaced by condition*

$$\bar{\eta}(V) < \infty \quad \text{and} \quad \kappa_H(\mu) < \infty \quad \text{for any } \mu \in \mathcal{P}_V(E).$$

The right-hand side condition in the above display is clearly met for any bounded function H . More generally it can be checked using the estimate (40). In this situation, following word-for-word the proof of Theorem 2.9, we check that

$$\|\Phi_{s,t}(\mu) - \bar{\eta}_t\|_V \leq a \kappa(\bar{\eta}, \mu) e^{-b(t-s)} \|\mu - \bar{\eta}_s\|_V$$

with

$$\kappa(\bar{\eta}, \mu) := \kappa_H(\mu) (1 + \bar{\eta}(V)) (1 + \bar{\eta}(V)/\bar{\eta}_-(H)) / \omega_H(\mu).$$

Using the above estimate we readily check that $\kappa_V(\mu) < \infty$ for any $\mu \in \mathcal{P}_V(E)$.

Remark 2.11. *The V -norm stability of the semigroup $\Phi_{s,t}$ is also discussed in [178] (for instance [178, Corollary 1]). The proof in [178] is based on Corollary 2.7 and it does not provide local Lipschitz contraction estimates.*

Remark 2.12. *The time varying Lyapunov function $V/H_{u,t}$ associated with the triangular array of Markov operators $R_{s,u}^{(t)}$ discussed in (36) depends on the terminal time horizon t . This property allows one to control the exponential decays (35) of the corresponding $(V/H_{u,t})$ -norms uniformly in t . These estimates are crucial in the proof of Theorem 2.9.*

We note that more conventional approaches, based on time homogeneous Lyapunov functions V , are discussed in [146]. This approach also ensures that the Markov semigroup $R_{s,u}^{(t)}$ forgets its initial state with respect to a common time homogenous V -norm. However, it seems difficult to deduce any local Lipschitz estimates of the form (41) from these uniform estimates.

Choosing $H = 1$ in (11) we have

$$[\Phi_{s,t}(\mu) - \Phi_{s,t}(\eta)](f) = \frac{1}{\mu(H_{s,t})} (\mu - \eta)(D_\eta \Phi_{s,t}(f)) \quad (42)$$

with the first order linear operator $D_{\mu_2} \Phi_{s,t}$ defined by the formula

$$D_\eta \Phi_{s,t}(f)(x) := H_{s,t}(x) (\Phi_{s,t}(\delta_x) - \Phi_{s,t}(\eta))(f).$$

Taylor-type expansions of higher order are also discussed in [3]. A weak version of the total variation estimate (41) is now easily obtained from the above perturbation formula.

Corollary 2.13. *Consider the R -semigroup (11) associated to the unit function $H = 1$. Under the assumptions of Theorem 2.5, for any $t \in \mathcal{T}$, $s \in \mathcal{T}_t$ and $\eta \in \mathcal{P}_V(E)$ we have*

$$\sup_{\|f\|_{V/H_{s,t}} \leq 1} \|D_\eta \Phi_{s,t}(f)\|_V \leq a e^{-b(t-s)} (1 + \eta(V)/\omega_H(\eta)). \quad (43)$$

Proof. We have

$$D_\eta \Phi_{s,t}(f)(x) := H_{s,t}(x) \int \eta(dy) \frac{H_{s,t}(y)}{\eta(H_{s,t})} (\delta_x R_{s,t}^{(t)} - \delta_y R_{s,t}^{(t)})(f).$$

Using (35) we check that

$$\|\delta_x R_{s,t}^{(t)} - \delta_y R_{s,t}^{(t)}\|_{V/H_{s,t}} \leq a e^{-b(t-s)} ((V/H_{s,t})(x) + (V/H_{s,t})(y)).$$

This implies that

$$\sup_{\|f\|_{V/H_{s,t}} \leq 1} \|D_\eta \Phi_{s,t}(f)\|_V \leq a e^{-b(t-s)} (1 + \eta(V) \|H_{s,t}\|_V / \eta(H_{s,t}))$$

and we can now conclude. ■

Remark 2.14. *These weak form estimates are particularly useful in the convergence analysis of the mean field particle models associated with sub-Markovian integral operators. Taylor-type expansions at any order are discussed in section 3.1.3 and chapter 10 in [64], see also section 2.3 in the more recent article [3].*

2.3 Time homogenous models

Time homogenous semigroups can be handle directly from the analysis developed in section 2.2. In this context, condition (\mathcal{H}_1) ensures that for any $s \in \mathcal{T}(\tau)$ we have

$$\kappa_H(\Phi_s(\bar{\eta}_0)) \leq \kappa_H(\bar{\eta}_0) < \infty \quad \text{and} \quad 0 < \omega_H(\bar{\eta}_0) \leq \omega_H(\Phi_s(\bar{\eta}_0))$$

Thus, by (41) we readily check that

$$\|\Phi_t(\bar{\eta}_0) - \Phi_{s+t}(\bar{\eta}_0)\|_V \leq a \kappa(\bar{\eta}_0) e^{-bt} \quad \|\bar{\eta}_0 - \Phi_s(\bar{\eta}_0)\|_V \leq 2 a \kappa(\bar{\eta}_0) \kappa_V(\bar{\eta}_0) e^{-bt}$$

with $\kappa(\bar{\eta}_0) := \kappa(\bar{\eta}_0, \bar{\eta}_0)$. The above Lipschitz exponential decay estimate ensures that $\bar{\eta}_t$ is a Cauchy sequence in the complete set $\mathcal{P}_V(E)$ equipped with the V -distance. Thus it converges exponentially fast to a single probability $\eta_\infty = \Phi_s(\eta_\infty) \in \mathcal{P}_V(E)$. The fixed point equation yields the exponential formula (24). In the same vein, using the product series expansion (25) and extending word-for-word the arguments developed in the end of section 2.1 (cf. (28)), condition (\mathcal{H}_1) ensures the existence of a function $h \in \mathcal{B}_V(E)$ such that

$$\bar{\eta}_0 = \eta_\infty \implies \forall x \in E \quad \bar{Q}_t(1)(x) \xrightarrow{t \rightarrow \infty} h(x) > 0$$

a well as

$$Q_t(h) = e^{\rho t} h \quad \text{and} \quad \eta_\infty Q_t = e^{\rho t} \eta_\infty$$

A triple (ρ, η_∞, h) satisfying the above properties is called a leading eigen-triple of the positive semigroup. Choosing h such that $\|h\|_V = 1$ we have

$$h/V = e^{-\rho\tau} Q_\tau(h)/V \leq e^{-\rho\tau} Q_\tau(V)/V \leq e^{-\rho\tau} \theta(V) \in \mathcal{B}_0(E) \implies h \in \mathcal{B}_{0,V}(E)$$

Since $h \leq V$ and V is locally bounded, the function h is locally bounded, thus V and h are bounded in compact sets. Assume there exists a strictly positive Radon measure χ_τ (i.e. non empty balls have positive measure) such that

$$Q_\tau(x_1, dx_2) \geq q_\tau(x_1, x_2) \chi_\tau(dx_2), \tag{44}$$

for some density function $q_\tau(x_1, x_2) > 0$. Also assume that for any compact set K there exists some positive measurable function $q_\tau^K(x_2)$ such that

$$\inf_{x_1 \in K} q_\tau(x_1, x_2) \geq q_\tau^K(x_2) > 0$$

For instance, this condition is satisfied for lower semi-continuous function $q_\tau(x_1, x_2)$ with respect to the first variable, and upper-semicontinuous with respect to the second (see for instance [80, Proposition 8.2.18]). In this situation, for any compact set $K \subset E$ we have

$$\forall x \in K \quad Q_\tau(h)(x) = e^{\rho\tau} h(x) \geq \chi_\tau(q_\tau^K) > 0 \quad \text{and thus} \quad h \in \mathcal{B}_{0,V}^+(E)$$

In the reverse angle, choosing $(\bar{\eta}_0, H) = (\eta_\infty, h) \in (\mathcal{P}_V(E) \times \mathcal{B}_{0,V}^+(E))$ in (7) and (10), we readily check that

$$\lambda_{t,t+\tau} = e^{\rho\tau} = \lambda^- \quad H_{s,t} = h \quad \text{and} \quad R_{0,s}^{(t)}(f) = Q_s(fh)/Q_s(h) = P_s^h(f)$$

In this context, the Doob h-transform, P_s^h is a semigroup of Markov operators from $\mathcal{B}_{V/h}(E)$ to itself. Note that in this case, whenever (44) is met the quantities $(\varsigma_\epsilon(H), \|\|H\|\|_V)$ defined in (17) resume to

$$\varsigma_\epsilon(H) = \inf_{V_\epsilon} h > 0 \quad \text{and} \quad \|\|H\|\|_V = \|h\|_V = 1$$

In addition, condition (\mathcal{H}_0) is met for $H = h$ as soon as there exists some $r_0 > 0$ and some $\alpha : r \in [r_0, \infty[\mapsto \alpha(r) \in]0, 1]$ such that for any $r \geq r_0$ we have

$$\sup_{V_\theta(x) \vee V_\theta(y) \leq r} \|\delta_x P_\tau^h - \delta_y P_\tau^h\|_{tv} \leq 1 - \alpha(r). \quad (45)$$

In the above display, V_θ stands for the function defined in (12). Whenever the minorisation condition (44) is satisfied, for any compact set $K \subset E$ we have

$$\forall x \in K \quad Q_\tau(x, dz) \geq \iota_K \nu_K(dz) \quad (46)$$

for some Radon probability measure ν_K and some $\iota_K > 0$ whose values may depend on the parameter τ . This minorisation condition ensures that (45) is satisfied. To check this claim, observe that for any $x \in K_r := \{V_\theta \leq r\}$ we have

$$\begin{aligned} P_\tau^h(x, dz) &\geq \frac{1}{e^{\rho\tau} \sup_{K_r} h} Q_\tau(x, dz) h(z) \\ &\geq \alpha(r) \nu_{K_r}^h(dz) \quad \text{with} \quad \alpha(r) = \iota_{K_r} e^{-\rho\tau} \frac{\nu_{K_r}(h)}{\sup_{K_r} h} \quad \text{and} \quad \nu_{K_r}^h := \Psi_h(\nu_{K_r}). \end{aligned}$$

This clearly implies (45).

Remark 2.15. *As we shall see in Lemma 4.2, the condition $Q_\tau(V)/V \in \mathcal{B}_0(E)$ ensures the existence of some $0 < \epsilon < 1$ and some constant $c > 0$ such that*

$$P_\tau^h(V/h) \leq \epsilon V/h + c.$$

This property also ensures that for any $\eta \in \mathcal{P}_V(E)$ we have $\kappa_V(\eta) < \infty$. For a more thorough discussion on the consequences of the Foster-Lyapunov inequality we refer the reader to Proposition 4.3.

We summarize the above discussion with the following theorem.

Theorem 2.16. *Consider a positive semigroup Q_t satisfying the minorisation condition (44) and such that $Q_\tau(V)/V \leq \theta(V)$, for some non-increasing function θ and some locally bounded function $V \geq 1$ with compact level sets.*

In this situation, condition (\mathcal{H}_1) is met for some positive function $(\bar{\eta}_0, H) \in (\mathcal{P}_V(E) \times \mathcal{B}_V^+(E))$ if and only if there exists a leading eigen-triple

$$(\rho, \eta_\infty, h) \in (\mathbb{R} \times \mathcal{P}_V(E) \times \mathcal{B}_V^+(E)).$$

Note that the minorisation condition (44) is less stringent than the absolutely continuous condition (19) imposed by condition (\mathcal{H}_2) . In the context of particle absorption models, it applies to jump-type processes including regular piecewise deterministic processes as well as Metropolis-Hasting type transitions arising in sequential Monte Carlo literature. In the context of time homogeneous models, the growth condition $Q_\tau(V)/V \leq \theta(V)$ in (\mathcal{H}_1) and Theorem 2.16 can be relaxed and replaced by the condition $Q_\tau(V)/V \in \mathcal{B}_0(E)$.

We end of this section with a more refined analysis of time homogenous semigroups satisfying the following weaker condition

$$\begin{cases} Q_\tau(V)/V \in \mathcal{B}_0(E) \\ Q_\tau(h) = e^{\rho\tau} h > 0 \text{ for some } \rho \in \mathbb{R}, \text{ and } h \in \mathcal{B}_V^+(E) \text{ s.t. } \|h\|_V = 1. \end{cases} \quad (47)$$

We recall that the parameter τ in the above display stands for any positive parameter such that $Q_\tau(1) > 0$. Also observe that

$$h/V = e^{-\rho\tau} Q_\tau(h)/V \leq e^{-\rho\tau} Q_\tau(V)/V \in \mathcal{B}_0(E) \implies h \in \mathcal{B}_{0,V}^+(E)$$

This also shows that the function $V^h := V/h \geq 1$ has compact level sets. We now introduce another condition.

(\mathcal{H}^h) *There exists some $r_0 > 0$ and some $\alpha : r \in [r_0, \infty[\mapsto \alpha(r) \in]0, 1]$ such that for any $r \geq r_0$ we have*

$$\sup_{V^h(x) \vee V^h(y) \leq r} \|\delta_x P_\tau^h - \delta_y P_\tau^h\|_{tv} \leq 1 - \alpha(r) \quad (48)$$

Arguing as above the above condition is satisfied as soon as the local minorisation condition (46) is satisfied. Condition (48) is weaker than condition (\mathcal{H}_1) applied to the leading eigen-triple (ρ, η_∞, h) . Indeed, since $V^h \geq e^{\rho\tau} V_\theta$, the above condition is weaker than (44) and (45). The main drawback of condition (\mathcal{H}^h) is that it requires some knowledge of the function h which is often unknown.

Several illustrations and some sufficient conditions ensuring the existence of the leading eigen-pair (ρ, h) satisfying (47) are discussed in section 3.4, which is dedicated to the study of quasi-compact positive operators. See also Corollary 2.3. For instance, in remark 3.5 we shall see that the existence of a leading eigen-pair (ρ, h) satisfying (47) is granted for absolutely continuous semigroups of the form (19) equipped with a continuous density. Note that in this case condition (\mathcal{H}_2) is satisfied.

The article [12] also provides different conditions ensuring the existence of a leading eigen-pair (ρ, h) for semigroups that are not necessarily absolutely continuous. We shall discuss these conditions in section 2.4, dedicated to comparisons with the existing literature on this subject. We also refer the reader to [12] for some additional illustrations of these conditions in the context of the growth-fragmentation equations.

We are now in position to state the main result of this section, the proof of which can be found in section 4.3.

Theorem 2.17. *Assume that (\mathcal{H}^h) is satisfied. Then the Markov semigroup P_t^h has a single invariant measure $\mu_\infty^h \in \mathcal{P}_{V/h}(E)$. In addition, there also exists some finite constant $a_h < \infty$ and some parameter $b_h > 0$, such that for any $\mu, \eta \in \mathcal{P}_{V/h}(E)$ and $t \in \mathcal{T}$ we have the contraction estimate*

$$\|\mu P_t^h - \eta P_t^h\|_{V/h} \leq a_h e^{-b_h t} \|\mu - \eta\|_{V/h}. \quad (49)$$

In addition, we have $\kappa_V(\mu) < \infty$ for any $\mu \in \mathcal{P}_V(E)$.

As with Theorem 2.5, the main ingredient of the proof is the V -contraction for Markov operators discussed in Lemma 3.2.

Remark 2.18. *The above theorem ensures the uniqueness of the invariant measure $\eta_\infty^h = \eta_\infty^h P_t^h \in \mathcal{P}_{V/h}(E)$ and exponential decay to equilibrium of the h -process. Choosing $\bar{\eta}_0 = \eta_\infty := \Psi_{1/h}(\eta_\infty^h)$ we have $\bar{\eta}_0 = \Phi_\tau(\bar{\eta}_0)$. In this scenario, we readily check (17) using the fact that*

$$\lambda_{t,t+\tau} := \bar{\eta}_0 Q_{t,t+\tau}(1) = e^{\rho\tau} \quad \text{and} \quad H_{s,t} = h \quad (50)$$

We illustrate the impact of the above theorem with some direct corollaries.

Corollary 2.19. *Under condition (\mathcal{H}^h) , the measure $\eta_\infty := \Psi_{1/h}(\eta_\infty^h) \in \mathcal{P}_V(E)$ is the unique invariant measure of the semigroup Φ_t . In addition, for any $\mu, \eta \in \mathcal{P}_V(E)$ and $t \in \mathcal{T}$ we have $\kappa_V(\mu) < \infty$ and*

$$\|\Phi_t(\mu) - \Phi_t(\eta)\|_V \leq a_h \kappa(\mu, \eta) e^{-b_h t} \|\mu - \eta\|_V, \quad (51)$$

with (a_h, b_h) as in (49) and the parameters $\kappa(\mu, \eta)$ defined by

$$\kappa(\mu, \eta) := \kappa_h(\mu) (1 + \kappa_V(\eta)) (1 + \eta(V)/\eta(h)) / \mu(h).$$

Proof. Combining (29) with the Boltzman-Gibbs estimate (68) we readily check the estimate

$$\|\Phi_t(\mu) - \Phi_t(\eta)\|_V \leq \kappa_h(\mu) (1 + \kappa_V(\eta)) \|(\Psi_h(\mu) - \Psi_h(\eta))P_t^h\|_{V/h}.$$

The contraction estimate (49) now implies that

$$\|(\Psi_h(\mu) - \Psi_h(\eta))P_t^h\|_{V/h} \leq a_h \kappa_h(\mu) (1 + \kappa_V(\eta)) e^{-b_h t} \|\Psi_h(\mu) - \Psi_h(\eta)\|_{V/h}.$$

Using the Boltzman-Gibbs estimate (69) we also have

$$\|\Psi_h(\mu) - \Psi_h(\eta)\|_{V/h} \leq \frac{1}{\mu(h)} \left(1 + \frac{\eta(V)}{\eta(h)}\right) \|\mu - \eta\|_V.$$

This ends the proof of the corollary. ■

Corollary 2.19 is closely related to the exponential decay estimates stated in Theorem 1 of [100] under different regularity conditions. In contrast with [100], our approach is based on the V -contraction properties of the h -semigroups and it allows one to derive local contraction estimates. The proof of the next corollary follows word-for-word the same arguments as the proof of Corollary 2.7, thus it is skipped.

Corollary 2.20. *Under condition (\mathcal{H}^h) , for any $t \in \mathcal{T}$ we have the operator norm exponential decay*

$$\sup_{\|f\|_V \leq 1} \left\| e^{-t\rho} Q_t(f) - \frac{h}{\eta_\infty(h)} \eta_\infty(f) \right\|_V \leq a_h e^{-b_h t} (1 + \eta_\infty(V)/\eta_\infty(h)) \quad (52)$$

with the same parameters (a_h, b_h) as in (49).

2.4 Illustrations and comparisons

In this section, we highlight some of the comparisons between the models and the regularity conditions discussed in the present article and conditions often used in the literature for positive integral operators.

Comparison of regularity conditions

We begin by remarking that the class of positive semigroups discussed in this article encapsulates discrete generation Feynman-Kac semigroups defined for any $t \in \mathcal{T}(\tau)$ by the formula

$$Q_{t,t+\tau}(x, dy) = G_t(x) P_{t,t+\tau}(x, dy)$$

with the potential function G_t and the Markov transition $P_{t,t+\tau}$ defined by

$$G_t := Q_{t,t+\tau}(1) \quad \text{and} \quad P_{t,t+\tau}(f) := Q_{t,t+\tau}(f)/Q_{t,t+\tau}(1). \quad (53)$$

This class of probabilistic models arises in a variety of disciplines including statistical physics, biology, signal processing, rare event analysis, and many others; see [63, 64, 67, 80] and the relevant references therein.

In this context, the uniform minorization condition (13) is a well known strong condition ensuring the stability of the semigroups $\Phi_{s,t}$ and the existence of fixed point invariant measures for time homogeneous semigroups; see for instance [67, Theorem 2.3], [69, Lemma 2.1 & Lemma 2.3], [76, Lemma 2.1], [64, section 12.2], as well as [67, section 2.1.2 & section 3.1.3] and [63, section 4].

The next condition is taken from [38]. In terms of the Markov transition $P_{s,s+\tau}$ discussed in (53) it takes the following form:

(\mathcal{P}) : *For any $s \in \mathcal{T}$ and $(x_1, x_2) \in E^2$ there exists some $\nu \in \mathcal{P}(E)$ such that for $i = 1, 2$ and any $t \geq s + \tau$ we have the estimates*

$$\delta_{x_i} P_{s,s+\tau} \geq \epsilon_1 \nu \quad \text{and} \quad \epsilon_2 \|Q_{s+\tau,t}(1)\| \leq \nu(Q_{s+\tau,t}(1)) \quad (54)$$

for some parameters $0 < \epsilon_i \leq 1$ whose values don't depend on x_i , nor on $s \in \mathcal{T}$.

We also refer the reader to conditions (A1) and (A2) in [39], as well as [37, 40, 44] for further work and discussion on this condition.

The time-homogeneous version of the above condition appears in a variety of contexts in the literature. For example, as shown in [38, Theorem 2.1], condition (\mathcal{P}) is a sufficient and necessary condition for the *uniform* exponential decay (22), which was also applied in [121, Theorem 10] and [122, Theorem 7.1] to neutron transport models.

We also refer the reader to [11, section 2.2] for the use of condition (\mathcal{P}) in the design of admissible coupling constants (a.k.a. generalised Doeblin's conditions), to [46] on multi-type birth-and-death processes where it is the main ingredient of the proof of Theorem 3.1, and to [115, section 2]. Moreover, for time-homogeneous models satisfying (13), the right-hand side estimate in (54) is a direct consequence of the right-hand side estimate in (26).

When (\mathcal{P}) is met, for any $f \geq 0$ and $i = 1, 2$ choosing $H = 1$ we have the lower bound estimate

$$R_{s,s+\tau}^{(t)}(f)(x_i) \geq \epsilon_1 \frac{\nu(Q_{s+\tau,t}(1) f)}{P_{s,s+\tau}(Q_{s+\tau,t}(1))(x_i)} \geq \epsilon_1 \epsilon_2 \Psi_{Q_{s+\tau,t}(1)}(\nu_s)(f).$$

This implies that condition (\mathcal{P}) is stronger than the Dobrushin's condition discussed in (13). More precisely, we have

$$(\mathcal{P}) \implies (13) \quad \text{with} \quad H = 1 \quad \alpha(1) := \epsilon_1 \epsilon_2 \quad \text{and} \quad \nu = \Psi_{Q_{s+\tau,t}(1)}(\nu_s).$$

As previously mentioned, the uniform minorisation condition (13) as well as (\mathcal{P}) are difficult to check in practice; several sufficient conditions are discussed in [63, 64, 67, 69, 75, 76].

The next condition is a slight extension of [181, condition (68)] and [10, condition (3.24)].

(\mathcal{Q}) : *There exists a positive measurable function $\varsigma_s(x) > 0$, a constant $\rho > 0$ and probability measures ν_s such that*

$$\varsigma_s(x) \nu_s(dy) \leq \delta_x Q_{s,s+\tau}(dy) \leq \rho \varsigma_s(x) \nu_s(dy). \quad (55)$$

Choosing $H = 1$, this condition implies that for any $f \geq 0$ we have

$$R_{s,u}^{(t)}(f) := \frac{Q_{s,u}(Q_{u,t}(1) f)}{Q_{s,u}(Q_{u,t}(1))} \geq \rho^{-1} \nu_{s,u}^{(t)}(f) \quad \text{with} \quad \nu_s^{(t)}(f) := \frac{\nu_s(Q_{s+\tau,t}(1) f)}{\nu_s(Q_{s+\tau,t}(1))}.$$

This shows that

$$(\mathcal{Q}) \implies (13) \quad \text{with} \quad H = 1 \quad \alpha(1) := \rho^{-1} \quad \text{and} \quad \nu = \nu_s^{(t)}.$$

Condition (\mathcal{Q}) with $\zeta_s(x) = 1$ is also discussed in [63, section 4.3.2], as well as in [139] and [100, section 3.3]. We also refer the reader to [37], where it was shown that condition (\mathcal{Q}) implies the uniform exponential decay (22) in the time-homogeneous case.

Also notice that

$$(\mathcal{Q}) \iff \rho^{-1} \nu_s \leq \delta_x P_{s,s+\tau} \leq \rho \nu_s,$$

where the Markov transition $P_{s,s+\tau}$ was defined in (53). These rather strong two-sided minorisation conditions are well-known: see for instance the uniformly positive condition discussed in [15], condition (19) in [8] in the framework of Hilbert projective metrics, condition (B) in [69] and Theorem 2.3 in [67]. From the above discussion it should be clear that conditions (\mathcal{P}) and (\mathcal{Q}) are stronger than the Dobrushin condition presented in (13).

The class of R -semigroups introduced in (10) are also considered in the article [12] in the context of time homogeneous sub-Markovian models. In our framework, the authors assume the existence of positive functions $H \leq V$, a probability measure ν defined on some compact $K \subset E$, and some finite constant c , such that

$$\sup_K V/H < \infty \quad Q_\tau(V) \leq a V + c 1_K H \quad \text{with} \quad 0 < a < \inf_E (Q_\tau(H)/H). \quad (56)$$

In addition, there exists some $\epsilon \in]0, 1]$ such that for any positive function $f \in \mathcal{B}_{V/H}(E)$ and any $x \in K$ we have

$$Q_\tau(fH)(x)/Q_\tau(H)(x) \geq \epsilon \nu \quad \text{and} \quad \sup_{t \in \mathcal{T}(\tau)} \sup_K \frac{(Q_t(H)/H)}{\nu(Q_t(H)/H)} < \infty. \quad (57)$$

In Lemma 3.1 in [12], using the right-hand side condition in the above display, the authors obtain a Lyapunov equation defined as (36) by replacing $H_{s,t}$ by the function

$$\mathcal{H}_{s,t} := \frac{Q_{t-s}(H)}{\nu(Q_{t-s}(H)/H)} \neq H_{s,t} = \frac{Q_{t-s}(H)}{\bar{\eta}_s Q_{t-s}(1)}.$$

Theorem 2.1 in [12] ensures the existence of a leading triple (ρ, η_∞, h) , as well as exponential estimates similar to the ones discussed in Corollary 2.20. Inversely, the authors show that the existence of a leading triple (ρ, η_∞, h) satisfying these exponential decays imply that the pair (V, h) satisfies condition (56) and (57). Thus, for the class of considered in Theorem 2.16 the conditions (56) and (57) are stronger than condition (\mathcal{H}_1) applied to the leading pair $(\bar{\eta}_0, H) = (\eta_\infty, h)$.

In the context of time homogeneous models, up to a change of Lyapunov function as discussed above, Proposition 3.3 in [12] is closely related to Theorem 2.5 in the present article. In contrast with the V -norm Lipschitz estimates stated in Corollary 2.19 in presented in this article, Corollary 3.7 in [12] doesn't provide any Lipschitz type estimates but also yields some exponential decays of the normalised semigroup Φ_t to equilibrium with respect to the total variation norm.

Absorbed processes with bounded potentials

Whenever the operator $Q_{s,t}$ is sub-Markovian, we have the nonlinear transport equation

$$\Phi_{s,t}(\mu) = \mu M_{s,t}^\mu$$

with the collection of Markov transition $M_{s,t}^\mu$ indexed by $\mu \in \mathcal{P}(E)$ given by the formula

$$M_{s,t}^\mu(f)(x) = Q_{s,t}(1)(x) \frac{Q_{s,t}(f)(x)}{Q_{s,t}(1)(x)} + (1 - Q_{s,t}(1)(x)) \Phi_{s,t}(\mu)(f).$$

We also recall that for any $s \leq u \leq t$ we have the nonlinear semigroup equation

$$M_{s,t}^\mu = M_{s,u}^\mu M_{u,t}^{\Phi_{s,u}(\mu)}.$$

This shows that the normalised semigroup $\Phi_{s,t}$ is the semigroup of a nonlinear Markov process sometimes called process conditioned to non-absorption at every time step. For time homogeneous models, unless μ coincides with the quasi-invariant measure $\Phi_t(\eta_\infty) = \eta_\infty$, the process is a nonlinear interacting jump process. In this interpretation, the distribution on path-space of the nonlinear process coincides with the McKean-measure associated with a jump type process whose jumps intensity depends on the distribution of the random states.

For a more thorough discussion on the nonlinear interacting jump processes associated with these nonlinear Markov semigroups we refer to section 12.3 in [64] and the articles [3, 67, 68]. Next proposition is a direct consequence of Theorem 2.9.

Proposition 2.21. *Under the assumptions of Theorem 2.5, for any $t \in \mathcal{T}$ and $s \in \mathcal{T}_t$ and any $\mu, \eta \in \mathcal{P}_V(E)$ we have the local Lipschitz operator norm estimate*

$$\|\delta_x M_{s,t}^\mu - \delta_x M_{s,t}^\eta\|_V \leq a \kappa(\eta, \mu) e^{-b(t-s)} \|\mu - \eta\|_V$$

with $(a, b, \kappa(\eta, \mu))$ as in (41).

Sub-Markovian operators are naturally associated with killed or absorbed stochastic processes. Consider a stochastic flow $t \in [s, \infty[\mapsto X_{s,t}^c(x)$ starting at $X_{s,s}^c(x) = x \in E$ when $t = s$ and absorbed in a cemetery state c at some random time $T_s^c(x)$. For instance, suppose we are given an auxiliary stochastic flow $X_{s,t}(x)$ evolving on E , which is sent to the cemetery at some uniformly bounded rate $U_t(y) \geq 0$ when at $y \in E$. In this situation, we have the so-called Feynman-Kac propagator formulae

$$\begin{aligned} Q_{s,t}(f)(x) &= \mathbb{E} (f(X_{s,t}^c(x)) 1_{T_s^c(x) > t}) \\ &= \mathbb{E} \left(f(X_{s,t}(x)) \exp \left(- \int_s^t U_u(X_{s,u}(x)) du \right) \right). \end{aligned} \quad (58)$$

In this context, it is readily checked that the normalised and unnormalised semigroups are connected by the formula

$$\mu Q_{s,t}(f) = \Phi_{s,t}(\mu)(f) \exp \left(- \int_s^t \Phi_{s,u}(\mu)(U_u) du \right). \quad (59)$$

In terms of the absorption time, the above formula reads

$$\int \mu(dx) \mathbb{P}(X_{s,t}^c(x) \in dy, T_s^c(x) > t) = \Phi_{s,t}(\mu)(dy) \exp\left(-\int_s^t \Phi_{s,u}(\mu)(U_u) du\right).$$

This shows that the killing time of the process starting from μ at time s is a Poisson process with a time varying rate function $\Phi_{s,t}(\mu)(U_t)$ that depends on μ . The discrete time version of the above formula coincides with the product formula (5). For a more thorough discussion on this subject we refer to [67, section 1.3.2], [63, proposition 2.3.1] or [64, section 12.2.1]. As noted in [55, 147], in the context of time homogeneous models, we readily check that the killing time is exponentially distributed as soon as $\mu = \eta_\infty = \Phi_t(\eta_\infty)$.

Applying the above to $f = 1$ for any $\mu_1, \mu_2 \in \mathcal{P}(E)$ and $s \leq t$ we check that

$$\mu_1 Q_{s,t}(1) / \mu_2 Q_{s,t}(1) = \exp\left(\int_s^t (\Phi_{s,u}(\mu_2)(U_u) - \Phi_{s,u}(\mu_1)(U_u)) du\right). \quad (60)$$

The discrete time version of the above formula coincides with (5). Under the assumptions of Theorem 2.1 the norm of the first order operator introduced in (42) decays exponentially. That is for any f such that $\text{osc}(f) := \sup_{(x,y) \in E^2} |f(x) - f(y)| = 1$, we have

$$\|D_{\mu_2} \Phi_{s,t}(f)\| \leq q (1 - \alpha(1))^{(t-s)/\tau} \quad \text{with} \quad q := \sup \frac{\mu_1 Q_{s,t}(1)}{\mu_2 Q_{s,t}(1)} < \infty. \quad (61)$$

In the above display, the supremum is taken over all indices (s, t) such that $t \in \mathcal{T}(\tau)$, $s \in \mathcal{T}_t(\tau)$ and $\mu_1, \mu_2 \in \mathcal{P}(E)$.

We now state some results pertaining to the limiting behaviour of the occupation measure of the time homogeneous stochastic flow. These results are direct consequences of our semigroup analysis that build on the analysis developed in [40, 47, 115].

Observe that for any $s \leq u \leq t$ we have

$$\mathbb{E}(f(X_{s,u}^c(x)) \mid T_s^c(x) > t) = \Psi_{Q_{u,t}(1)}(\Phi_{s,u}(\delta_x))(f).$$

Setting $X_u^c(x) := X_{0,u}^c(x)$ and $T^c(x) := T_0^c(x)$, due to (33) we obtain the following corollary.

Corollary 2.22. *Under the assumptions of Corollary 2.2, we have*

$$\begin{aligned} & \sup_{\|f\| \leq 1} \sup_{x \in E} \left| \mathbb{E}\left(\frac{1}{t} \int_0^t f(X_u^c(x)) du \mid T^c(x) > t\right) - \Psi_h(\eta_\infty)(f) \right| \\ & \leq \frac{\tau}{t} \frac{1}{|\log(1 - \alpha(1))|} (q(\eta_\infty) + 2(\|h\|/\eta_\infty(h) + q(\eta_\infty)^2)) \end{aligned} \quad (62)$$

In the above display, h stands for the eigenfunction defined in Corollary 2.4. The measure $\Psi_h(\eta_\infty)$ which coincides with the invariant measure of the h -process is sometimes called the quasi-ergodic measure of the non absorbed process.

In the same vein, we also obtain a uniform bound on the L^2 distance.

Proposition 2.23. *Under the assumptions of Corollary 2.2, for any $f \in \mathcal{B}(E)$ with $\text{osc}(f) \leq 1$ and any $t \in \mathcal{T}$ we have the uniform estimate*

$$\begin{aligned} & \left| \mathbb{E} \left(\left(\frac{1}{t} \int_0^t f(X_u^c(x)) \, du - \Psi_h(\eta_\infty)(f) \right)^2 \mid T^c(x) > t \right) \right| \\ & \leq \frac{8\tau}{t} \frac{(\|h\|/\eta_\infty(h) + (1 \vee q(\eta_\infty))^2)}{|\log(1 - \alpha(1))|}. \end{aligned}$$

Proof. First observe that for any $s \leq u \leq v \leq t$ we have

$$\begin{aligned} & \mathbb{E} \left(f(X_{s,u}^c(x)) f(X_{s,v}^c(x)) \mid T_s^c(x) > t \right) \\ & = \frac{\delta_x Q_{s,u}(f Q_{u,t}(1) Q_{u,v}(f Q_{v,t}(1)) / Q_{u,v}(Q_{v,t}(1)))}{\delta_x Q_{s,u}(Q_{u,t}(1))} \\ & = \int \Psi_{Q_{u,t}(1)}(\Phi_{s,u}(\delta_x))(dy) f(y) \int \Psi_{Q_{v,t}(1)}(\Phi_{u,v}(\delta_y))(dz) f(z). \end{aligned}$$

Replacing f by $(f - \Psi_h(\eta_\infty)(f))$ there is no loss of generality to assume that $\Psi_h(\eta_\infty)(f) = 0$. In this situation, thanks again to (33), we have

$$\begin{aligned} & \left| \mathbb{E} \left(\left(\frac{1}{t} \int_0^t f(X_u^c(x)) \, du \right)^2 \mid T^c(x) > t \right) \right| \\ & \leq \frac{4}{t^2} (\|h\|/\eta_\infty(h) + (1 \vee q(\eta_\infty))^2) \int_{0 \leq u \leq v \leq t} (e^{-a(v-u)} + e^{-a(t-v)}) \, dv \, du \end{aligned}$$

with

$$a := |\log(1 - \alpha(1))|/\tau.$$

This implies that

$$\left| \mathbb{E} \left(\left(\frac{1}{t} \int_0^t f(X_u^c(x)) \, du \right)^2 \mid T^c(x) > t \right) \right| \leq \frac{8}{at} (\|h\|/\eta_\infty(h) + (1 \vee q(\eta_\infty))^2).$$

■

Remark 2.24. *We end our discussion of sub-Markov operators with bounded potentials by noting that we may also extend Proposition 2.23 to also obtain bounds in L^p of the order $t^{-1/2}$. The idea behind the proof is to write the averages in terms of the h -process (in particular, the trajectorial version) and then apply L^p bounds for occupation measures of a Markov process, using the Poisson equation associated with the h -semigroup (see for instance Lemma 8.4.11 in [80] in discrete time settings).*

Unbounded potentials

The analysis of unbounded potential functions $U_t(x)$ requires more care. The prototype solvable model is the \mathbb{R}^{d_0} -valued linear Gaussian diffusion

$$dX_{s,t}(x) = \mathcal{A}_t X_{s,t}(x) dt + \mathcal{R}_t^{1/2} d\mathcal{W}_t \quad \text{with quadratic potential } U_t(x) = \frac{1}{2} x' \mathcal{S}_t x \quad (63)$$

for some multidimensional Brownian motion $\mathcal{W}_t \in \mathbb{R}^{d_1}$ with $d_1 \geq d_0 \geq 1$ and some conformable matrices $(\mathcal{A}_t, \mathcal{R}_t, \mathcal{S}_t)$ such that $\mathcal{R}_t, \mathcal{S}_t > 0$. Solving the Riccati time reversal equation

$$-\partial_s \mathcal{Q}_s := \mathcal{A}'_s \mathcal{Q}_s + \mathcal{Q}_s \mathcal{A}_s - \mathcal{Q}_s \mathcal{R}_s \mathcal{Q}_s + \mathcal{S}_s,$$

for $s \in [0, t]$ with terminal condition $\mathcal{Q}_t = \mathcal{Q}$, we check that

$$\begin{aligned} h_{s,t}(x) &:= \exp\left(-\frac{1}{2} x' \mathcal{Q}_s x\right) \quad \text{and} \quad h_{t,t}(x) = h(x) := \exp\left(-\frac{1}{2} x' \mathcal{Q} x\right) \\ \implies h_{s,t}^{-1}(\partial_s + L_s)(h_{s,t})(x) - U_s(x) &= \lambda_s := -\text{Tr}(\mathcal{R}_s \mathcal{Q}_s). \end{aligned}$$

In the above display, L_s stands for the generator of the stochastic process $X_{0,s}(x)$. This yields the formula

$$Q_{s,t}(h)(x) = \exp\left(-\int_s^t \text{Tr}(\mathcal{R}_u \mathcal{Q}_u) du\right) h_{s,t}(x).$$

On the other hand, for any $s \leq t$ we have the exponential change of measure formula

$$\begin{aligned} Q_{s,t}(f)(x) &= e^{\int_s^t \lambda_u du} \mathbb{E}\left(f(X_{s,t}^{(t)}(x)) h_{s,t}(x)/h(X_{s,t}^{(t)}(x))\right) \\ &= e^{\int_s^t \lambda_u du} h_{s,t}(x) R_{s,t}^{(t)}(f/h)(x), \end{aligned} \quad (64)$$

where $(X_{s,u}^{(t)}(x))_{s \leq u \leq t}$ is the stochastic flow with generator

$$L_u^{(t)}(f) := L_u(f) + \Gamma_{L_u}(h_{u,t}, f)/h_{u,t} \quad \text{and} \quad R_{s,u}^{(t)}(x, dy) := \mathbb{P}(X_{s,u}^{(t)}(x) \in dy).$$

Equivalently, we have

$$R_{s,t}^{(t)}(f) = \frac{Q_{s,t}(h f)}{Q_{s,t}(h)} = \frac{Q_{s,u}(Q_{u,t}(h f))}{Q_{s,u}(Q_{u,t}(h))} = \frac{Q_{s,u}(h_{u,t} P_{u,t}^h(f))}{Q_{s,u}(h_{u,t})} = R_{s,u}^{(t)} R_{u,t}^{(t)}(f).$$

In this situation, the stochastic flow $X_{s,u}^{(t)}(x)$ of time varying R -process is given for any $0 \leq s \leq u \leq t$ by

$$dX_{s,u}^{(t)}(x) = (\mathcal{A}_u - \mathcal{R}_u \mathcal{Q}_u) X_{s,u}^{(t)}(x) du + \mathcal{R}_u^{1/2} dW_u. \quad (65)$$

For time homogenous models $(\mathcal{A}_t, \mathcal{R}_t, \mathcal{S}_t) = (\mathcal{A}, \mathcal{R}, \mathcal{S})$ it is more judicious to choose $\mathcal{Q} = \mathcal{Q}_\infty$ where \mathcal{Q}_∞ stands for the unique positive fixed point of the Riccati equation

$$\mathcal{A}' \mathcal{Q}_\infty + \mathcal{Q}_\infty \mathcal{A} - \mathcal{Q}_\infty \mathcal{R} \mathcal{Q}_\infty + \mathcal{S} = 0.$$

In this case, we have $h_{s,t} = h$ for any $s \leq t$, and the R -process resumes to the h -process defined by the time homogeneous diffusion

$$dX_s^h(x) = (\mathcal{A} - \mathcal{R}Q_\infty) X_s^h(x) du + \mathcal{R}^{1/2} dW_s. \quad (66)$$

The Feynman-Kac path integrals associated with these models coincides with coupled quantum harmonic oscillators arising in physics. We refer the reader to [71] for a detailed discussion on this subject.

3 A brief review on integral operators

3.1 V -norms and Boltzmann-Gibbs estimates

For a given function $V \geq 1/2$, the V -oscillation of a function $f \in \mathcal{B}(E)$ is given by

$$\text{osc}_V(f) := \sup_{x,y} \frac{|f(x) - f(y)|}{V(x) + V(y)} \leq \|f\|_V$$

and with a slight abuse of notation, the V -norm on $\mathcal{M}(E)$ is given by

$$\|\mu\|_V = \sup\{|\mu(f)| : \|f\|_V \leq 1\} = \sup\{|\mu(f)| : \text{osc}_V(f) \leq 1\} = |\mu|(V).$$

In the above display, $|\mu| := \mu_+ + \mu_-$ stands for with the total variation measure associated with a Hahn-Jordan decomposition $\mu = \mu_+ - \mu_-$ of the measure. For a detailed proof of the equivalent formulations in the latter definition, we refer to Proposition 8.2.16 in [80]. The choice of condition $V \geq 1/2$ in the above two definitions is imposed only to recover the conventional total variation dual distance between probability measures when choosing $V = 1/2$ in the dual norms.

Recall that $\mathcal{M}_V(E) \subset \mathcal{M}(E)$ is the subset of Radon measures with $\|\mu\|_V < \infty$. A probability measure $\mu \in \mathcal{P}(E)$ is said to be strictly positive, and we write $\mu > 0$, whenever $\mu(A) > 0$ for any open set $A \subset E$. We finally recall that the integral map $(\mu, f) \mapsto \mu(f)$ gives the isometry

$$\mathcal{M}_V(E) \simeq \mathcal{C}_{0,V}(E)'.$$

The above assertion is a direct consequence of [170, Theorem 3.1] applied to the Nachbin family $\mathcal{V} := \{V_\alpha = \alpha/V : \alpha > 0\}$. See also [169, Theorem 3.26] as well as [23, Theorem 2.1].

As mentioned above, when $V = 1/2$ we recover the conventional total variation norm. For any $\mu_1, \mu_2 \in \mathcal{P}(E)$ we recall that

$$\|\mu_1 - \mu_2\|_{tv} \leq 1 - \epsilon \iff (\exists \nu \in \mathcal{P}(E) : \mu_1 \geq \epsilon \nu \text{ and } \mu_2 \geq \epsilon \nu). \quad (67)$$

We recall that the Boltzmann-Gibbs transformation Ψ_h associated with some bounded positive function $h > 0$ and defined in (2) satisfies the following Lipschitz type estimate

$$\|\Psi_h(\mu_1) - \Psi_h(\mu_2)\|_{tv} \leq \frac{\|h\|}{\mu_1(h) \vee \mu_2(h)} \|\mu_1 - \mu_2\|_{tv}.$$

For a detailed proof of the above assertion we refer to [123, lemma 9.5], or [83, appendix B], see also [69] as well as [143, proposition 3.1] and [64, proposition 12.1.7].

Finally, we present a technical lemma regarding the V -norm estimates of the Boltzmann-Gibbs operators; the proof is given in appendix A.2.

Lemma 3.1. *For any $0 < h \leq V$ and any $\mu_1, \mu_2 \in \mathcal{P}_{V/h}(E)$ we have the estimate*

$$\|\Psi_{1/h}(\mu_1) - \Psi_{1/h}(\mu_2)\|_V \leq \frac{1}{\mu_1(1/h)} \left(1 + \frac{\mu_2(V/h)}{\mu_2(1/h)}\right) \|\mu_1 - \mu_2\|_{V/h}. \quad (68)$$

In addition, for any $\mu_1, \mu_2 \in \mathcal{P}_V(E)$ we have

$$\|\Psi_h(\mu_1) - \Psi_h(\mu_2)\|_{V/h} \leq \frac{1}{\mu_1(h)} \left(1 + \frac{\mu_2(V)}{\mu_2(h)}\right) \|\mu_1 - \mu_2\|_V. \quad (69)$$

3.2 V -Dobrushin coefficient

Let us fix $0 \leq s \leq t$ and let V_s, V_t denote measurable functions such that $V_s, V_t \geq 1$. The V -Dobrushin coefficient of Markov integral operator $P_{s,t}$ from $\mathcal{B}_{V_t}(E)$ into $\mathcal{B}_{V_s}(E)$ is the norm operator defined by

$$\beta_{V_s, V_t}(P_{s,t}) = \sup_{\mu, \eta \in \mathcal{P}_{V_s}(E)} \|(\mu - \eta)P_{s,t}\|_{V_t} / \|\mu - \eta\|_{V_s}. \quad (70)$$

We also have the equivalent formulation

$$\begin{aligned} \beta_{V_s, V_t}(P_{s,t}) &= \sup \{ \text{osc}_{V_s}(P_{s,t}(f)) : \text{osc}_{V_t}(f) \leq 1 \} \\ &= \sup_{(x,y) \in E^2} \|\delta_x P_{s,t} - \delta_y P_{s,t}\|_{V_t} / (V_s(x) + V_s(y)). \end{aligned} \quad (71)$$

When $V_s = V_t = V$, we write $\beta_V(P_{s,t})$ instead of $\beta_{V,V}(P_{s,t})$. For a more thorough discussion on these contraction coefficients, we refer the reader to sections 8.2.5 - 8.2.7 in [80]. If in addition $V = 1/2$ we write $\beta(P_{s,t})$ instead of $\beta_{1/2}(P_{s,t})$, to denote the conventional Dobrushin ergodic coefficient with respect to the total variation distance

$$\beta(P_{s,t}) = \sup_{(x,y) \in E^2} \|\delta_x P_{s,t} - \delta_y P_{s,t}\|_{tv}.$$

By (67) the contraction condition $\beta(P_{s,t}) \leq (1 - \alpha)$ is satisfied for some parameter $\alpha \in]0, 1]$ if and only if the following minorisation condition holds

$$\forall (x, y) \in E^2 \quad \exists \nu \in \mathcal{P}(E) : \delta_x P_{s,t} \geq \alpha \nu \quad \text{and} \quad \delta_y P_{s,t} \geq \alpha \nu.$$

The next lemma provides some contraction conditions in terms of a Foster-Lyapunov inequality and a local minorisation condition on compact level sets. The proof is in appendix A.3.

Lemma 3.2. *Assume that there exist functions $V_s, V_t \geq 1$ with compact level sets, $\epsilon \in [0, 1[$ as well as a function $\alpha : r \in [r_0, \infty[\mapsto \alpha(r) \in]0, 1]$, for some $r_0 \geq 1$ such that for any $r \geq r_0$ we have*

$$P_{s,t}(V_t) \leq \epsilon V_s + 1 \quad \text{and} \quad \sup_{V_s(x) \vee V_s(y) \leq r} \|\delta_x P_{s,t} - \delta_y P_{s,t}\|_{tv} \leq 1 - \alpha(r). \quad (72)$$

Then, for any $0 < \delta \leq 1$ we have the contraction coefficient estimate

$$\beta_{V_s^{\epsilon,\delta}, V_t^{\epsilon,\delta}}(P_{s,t}) \leq 1 - \alpha(r_\epsilon(\delta))(1 - \delta/2) \quad \text{with} \quad V_t^{\epsilon,\delta} = 1/2 + \rho_\epsilon(\delta) V_t$$

with the parameters

$$r_\epsilon(\delta) := r_0 \vee \frac{2}{1 - \epsilon} \left(1 + \frac{1}{1 - \epsilon} \left(3 + 6 \left(\frac{1}{\delta} - 1 \right) \right) \right) \quad \text{and} \quad \rho_\epsilon(\delta) := \frac{\delta}{1 + \epsilon} \frac{\alpha(r_\epsilon(\delta))}{2r_\epsilon(\delta)}. \quad (73)$$

Now observe that

$$r_0(\delta) := r_0 \vee 4 \left(2 + 3 \left(\frac{1}{\delta} - 1 \right) \right) > 1 \quad \text{and} \quad \rho_0(\delta) := \frac{\delta}{2r_0(\delta)} \alpha(r_0(\delta)) \xrightarrow{\delta \rightarrow 0} 0.$$

Thus, whenever (72) is met with $V_s = V_t = r_0 = 1$, choosing $\epsilon = 0$ in (72) we have, for any $r > 1$,

$$\lim_{\delta \rightarrow 0} \beta_{V_s^{0,\delta}, V_t^{0,\delta}}(P_{s,t}) = \beta(P_{s,t}) \leq 1 - \alpha(r).$$

Whenever $P_{s,t}$ is a Markov semigroup and $V_t \geq V$ for some function $V \geq 1$ with compact level sets, the Foster-Lyapunov inequality in the left-hand side of (72) ensures that for any initial probability measure $\mu \in \mathcal{P}_{V_0}(E)$, the distribution flow $\mu_t := \mu P_{0,t}$ indexed by $t \in \mathcal{T}(\tau)$ for some $\tau > 0$ is tight. To check this claim, notice that

$$(72) \implies \mu_t(V) \leq \mu_t(V_t) \leq \epsilon^{t/\tau} \mu(V_0) + (1 - \epsilon)^{-1} \leq \mu(V_0) + (1 - \epsilon)^{-1}. \quad (74)$$

By the Markov inequality, for any $\delta > 0$ this implies that

$$K_\delta := \{x : \delta V(x) \leq \mu(V_0) + (1 - \epsilon)^{-1}\} \subset E \implies \sup_{t \in \mathcal{T}(\tau)} \mu_t(E - K_\delta) \leq \delta. \quad (75)$$

In this situation, if we set $V^{\epsilon,\delta} = 1/2 + \rho_\epsilon(\delta) V$, for any $s \leq t$ and $\mu, \eta \in \mathcal{P}_{V_s}(E)$ we also have the contraction estimate

$$\|\mu P_{s,t} - \eta P_{s,t}\|_{V^{\epsilon,\delta}} \leq (1 - \alpha(r_\epsilon(\delta))(1 - \delta/2))^{(t-s)/\tau} \|\mu - \eta\|_{V_s^{\epsilon,\delta}}. \quad (76)$$

For time homogeneous models, condition (72) clearly ensures the existence of a single invariant probability measure $\mu_\infty = \mu_\infty P_t \in \mathcal{P}_V(E)$. The analysis discussed above combines the ergodic theory for Markov operators presented in [114] with the V-Dobrushin contraction methodology developed in [80]. Alternative contraction approaches mainly based on [114] are discussed in [12, 42, 146, 149].

3.3 Lyapunov conditions

This section presents some practical ways of checking the Lyapunov condition stated on the left-hand side of (72). Given some $c > 0$, $\epsilon \in [0, 1[$, $K \in \mathcal{E}$ and functions $W_s, W_t \geq 0$, we have

$$\left. \begin{array}{l} P_{s,t}(W_t)(x) \leq \epsilon W_s(x) \quad \text{if } x \in E - K \\ P_{s,t}(W_t)(x) \leq c \quad \quad \quad \text{if } x \in K \end{array} \right\} \implies P_{s,t}(W_t/c) \leq \epsilon W_s/c + 1.$$

Moreover,

$$P_{s,t}(W_t) \leq \epsilon W_t + c \implies P_{s,t}(V_t) \leq \epsilon V_s + 1 \text{ with } V_u = 1 + \epsilon W_u/c \geq 1, \text{ for } u \in \{s, t\}.$$

For continuous and time homogenous semigroups $P_{t+s} = P_t P_s$ with generator L defined on some domain $D(L) \ni W$, for any $a > 0$ and $c < \infty$ we have the Lyapunov condition

$$L(W) \leq -aW + c \iff \forall \delta > 0 \quad P_\delta(W) \leq \frac{1}{1+a\delta} W + c\delta.$$

A detailed proof of the above assertion can be found in [80, Theorem 17.4.1].

In the case where $c = 0$, i.e. there exists some $K \subset E$ such that

$$\forall x \in E - K, \quad L(W)(x) \leq -aW(x) \quad \text{and} \quad \forall x \in K \quad L(W)(x) \leq c, \quad (77)$$

we readily check that

$$L(W) \leq -aW =: b.$$

For instance, when E is a normed space and $L(W)$ is continuous, it suffices to find a sufficient large radius r that

$$W(x) \geq r \implies W^{-1}(x)L(W)(x) \leq -a.$$

In this situation condition (77) is met with $K = \{W \leq r\}$ and $c = \sup_K \|L(W)\|$.

Whenever $c > 0$, by replacing W by W/c , there is no loss of generality by assuming that the above Lyapunov condition is met for $c = 1$. In the same vein, replacing W by $W_b = W + b$ for some $b \geq 0$ we find that

$$L(W_b) \leq -aW_b + c_b \quad \text{with} \quad c_b := c + ab.$$

Also notice that in this case, for any $\delta > 0$ we have

$$P_\delta(W_b) \leq \frac{1}{1+a\delta} W_b + c\delta \implies P_\delta(W) \leq \frac{1}{1+a\delta} W + \left(c - \frac{ab}{1+a\delta} \right) \delta.$$

Thus, there is no loss of generality to assume that $W \geq b$ satisfies the Lyapunov condition for some $b > 0$. For a more thorough discussion on these Foster-Lyapunov

techniques, including the design of Lyapunov functions for Ornstein-Uhlenbeck processes as well as for damped and over-damped Langevin diffusions; see [80, section 17.4 & section 17.5]. We end this section with some comments on the condition $Q_\tau(V)/V \in \mathcal{C}_0(E)$. Assume that

$$G_\tau := Q_\tau(1) \in \mathcal{C}_0(E) \quad \text{and} \quad 0 < G_\tau \leq 1.$$

In this case, we have the decomposition

$$Q_\tau(x, dy) = G_\tau(x) P_\tau(x, dy)$$

with the Markov transition

$$P_\tau(f) := Q_\tau(f)/Q_\tau(1).$$

Assume that $P_\tau(\mathcal{C}(E)) \subset \mathcal{C}(E)$ and $P_\tau(W) \leq aW + b$ for some non negative function $W \in \mathcal{C}(E)$ and for some $a, b \geq 0$. Now, one can check that $Q_\tau(V)/V \in \mathcal{C}_0(E)$ using

$$V := 1 + W \implies \frac{Q_\tau(V)}{V} = G_\tau \frac{(1+b) + aW}{1+W} \leq c G_\tau \quad \text{with} \quad c := ((1+b) \vee a) > 0.$$

We illustrate the above discussion in the context of Gaussian integral operators on Euclidian spaces. To avoid unnecessary vector and matrix-notation we restrict our attention to one dimensional models, that is $E = \mathbb{R}$. The analysis of multidimensional models follows along the same lines. Consider the following models

$$G_\tau(x) = e^{-U_0(x)} \in \mathcal{C}_0(\mathbb{R}) \quad \text{and} \quad P_\tau(x, dy) = \frac{1}{\sqrt{2\pi\sigma_1(x)^2}} e^{-\frac{(y-b(x))^2}{2\sigma_1(x)^2}} dy. \quad (78)$$

In the above display, $U_0 \geq 0$ is some function with compact level sets, $\sigma_1(x) > 0$ is a strictly positive function and $b(x)$ is the drift, both of which exhibit at most linear growth; that is

$$|b(x)| + \sigma_1(x) \leq c_1 + c_2 |x|$$

for some constant $c_1, c_2 \geq 0$. In this situation, we check that

$$W(x) = 1 + |x| \implies P_\tau(W) \leq a W \quad \text{with} \quad a = (1 + c_1) \vee c_2.$$

When $G_\tau \in \mathcal{C}_b(E)$ is not necessarily vanishing at infinity, the problem amounts to finding a function $V \geq 1$ with compact level set such that $P_\tau(V)/V \in \mathcal{C}_0(E)$. The design of such functions for a Markov transition P_τ is rather standard in applied probability literature (cf. for instance [91, 133, 178]). We further illustrate these constructions with the Gaussian transition discussed in (78). We assume that

$$\|\sigma_1\| < \infty, \quad |b(x)| \leq c + (1 - \delta) |x| \quad \text{for some} \quad 0 \leq c < \infty \quad \text{and} \quad 0 < \delta \leq 1 \quad (79)$$

and set $V(x) := e^{v|x|}$ for some $v > 0$.

In this case, we check that

$$\begin{aligned}
P_\tau(V)(x)/V(x) &= e^{-v(|x|-b(x))+\frac{v^2\sigma_1(x)^2}{2}} \int_0^\infty \frac{e^{-\frac{(y-(b(x)+v\sigma_1(x)^2))^2}{2\sigma_1(x)^2}}}{\sqrt{2\pi\sigma_1(x)^2}} dy \\
&\quad + e^{-v(|x|+b(x))+\frac{v^2\sigma_1(x)^2}{2}} \int_{-\infty}^0 \frac{e^{-\frac{(y-(b(x)-v\sigma_1(x)^2))^2}{2\sigma_1(x)^2}}}{\sqrt{2\pi\sigma_1(x)^2}} dy.
\end{aligned}$$

In this situation, whenever $G_\tau \in \mathcal{C}_b(E)$ we have

$$Q_\tau(V)(x)/V(x) \leq 2e^{(v\|\sigma_1\|)^2/2} \|G_\tau\| e^{-v(|x|-|b(x)|)} \leq 2e^{vc+(v\|\sigma_1\|)^2/2} \|G_\tau\| V(x)^{-\delta},$$

which implies (18). Note that the above construction requires the sub-linear growth of the drift function $b(x)$ stated in (79). Consider the Gaussian density

$$G_\tau(x) = \exp\left(-\frac{x^2}{2\sigma_0^2(x)}\right). \quad (80)$$

In the above display, $\sigma_0(x) > 0$ stands for some strictly positive bounded function. In this case, we have

$$\begin{aligned}
|b(x)| &\leq c_0 + c_1 |x| \quad \text{for some } 0 \leq c_0, c_1 < \infty \\
\implies \frac{Q_\tau(V)(x)}{V(x)} &\leq 2e^{(v\|\sigma\|)^2/2} \exp\left(-v\left(|x| + \frac{|x|^2}{2v\|\sigma_0\|^2} |b(x)|\right)\right) \xrightarrow{|x| \rightarrow \infty} 0.
\end{aligned}$$

Observe that for any $0 < \delta \leq 1$ we have

$$|x| \geq r := (2v\|\sigma_0\|^2) c_1 \implies c_1 \leq (1 - \delta) + |x|/(2v\|\sigma_0\|^2).$$

This shows that

$$c_0 + c_1 |x| \leq (c_0 + c_1 r) + (1 - \delta) |x| + |x|^2/(2v\|\sigma_0\|^2) \quad \text{with } c_2 = c_0 + c_1 r$$

from which we conclude that

$$|b(x)| \leq c_0 + c_1 |x| \implies Q_\tau(V)(x)/V(x) \leq 2e^{vc_2+(v\|\sigma\|)^2/2} V(x)^{-\delta},$$

which yields (18).

3.4 Quasi-compact operators

Next, we recall some standard definitions and compactness principles from time homogeneous positive semigroup theory.

Let $(Q_t)_{t \geq 0}$ be a time homogenous positive semigroup from $\mathcal{B}_V(E)$ into $\mathcal{B}_V(E)$. Then Q_t is said to be irreducible (a.k.a. ideal irreducible) if there exists no closed

Q_t -invariant ideals distinct from $\{0\}$ and $\mathcal{B}_V(E)$ (cf. [157, Definition 4.2.1]). It is also well known (e.g. [92, Proposition 2.1] or [102, Proposition 4.1]) that this condition is met if and only if for any non zero function $f \geq 0$ on $\mathcal{B}_V(E)$ and any non zero positive measure μ on E , there exists some $t \in \mathcal{T}$ such that $\mu Q_t(f) > 0$.

The spectral radius of $(Q_t)_{t \geq 0}$ is defined as

$$r_V(Q) := \lim_{t \rightarrow \infty} \|\|Q_t\|\|_V^{1/t} = \lim_{t \rightarrow \infty} \|Q_t(V)\|_V^{1/t} = \inf_{t \geq 0} \|Q_t(V)\|_V^{1/t}. \quad (81)$$

In the above display, $\|\|\cdot\|\|_V$ stands for the operator norm.

The semigroup Q_t is said to be quasi-compact if its essential spectral radius

$$\bar{r}_V(Q) := \lim_{t \rightarrow \infty} (\inf \{\|\|Q_t - T\|\|_V : T \text{ compact}\})^{1/t}$$

satisfies $\bar{r}_V(Q) < r_V(Q)$. Recalling that the product of positive operators $Q_1 Q_2$ is compact as soon as the Q_1 is compact (cf. theorem VI.12 in [160]), the quasi-compactness property of the operator Q_t (for sufficiently large t) is clearly met as soon Q_τ is a compact operator for some $\tau \in \mathcal{T}$. Such one-parameter semigroups are sometimes called eventually compact semigroups (see for instance section 3 in the book [94]).

Now assume there exists some $\tau \in \mathcal{T}$ such that the discrete generation semigroup Q_t indexed by $t \in \mathcal{T}(\tau)$ is irreducible and quasi-compact on $\mathcal{B}_V(E)$. By a variant of the Krein-Rutman theorem (cf. for instance Theorem 1.1 in [93, 151]), $r_V(Q) > 0$ is an eigenvalue corresponding to a non null eigenfunction $h \in \mathcal{B}_V(E)$. More precisely, there exists some a non null function $h \in \mathcal{B}_V(E)$ such that

$$\forall t \in \mathcal{T}(\tau) \quad Q_t(h) = e^{\rho t} h \quad \text{with} \quad \rho := \log r_V(Q).$$

Note that since Q_τ is irreducible, for any $x \in E$ there exists some $t \in \mathcal{T}(\tau)$ such that

$$Q_t(h)(x) = e^{\rho t} h(x) > 0.$$

Remark 3.3. *Whenever $Q_\tau(V)/V \in \mathcal{B}_0(E)$, for any $t \geq \tau$ the operators Q_t map $\mathcal{B}_V(E)$ into $\mathcal{B}_{0,V}(E)$, and the eigen-function h belongs to $\mathcal{B}_{0,V}(E)$. If, in addition, Q_τ maps $\mathcal{B}_V(E)$ into $\mathcal{C}_V(E)$, then the operators Q_t map $\mathcal{B}_V(E)$ into $\mathcal{C}_{0,V}(E)$, and the eigen-function h belongs to $\mathcal{C}_{0,V}(E)$.*

Additionally, if the assumptions of Theorem 2.1 are met, Corollary 2.4 ensures that the existence of an eigenfunction $h \in \mathcal{C}_{0,V}(E)$.

Sufficient conditions in terms of the Lyapunov functions $V \geq 1$ ensuring the compactness of Q_t for some $t \in \mathcal{T}$ are discussed in [100] and [161]. For instance, we have the following lemma.

Lemma 3.4. *Assume that $Q_\tau(V)/V \in \mathcal{B}_0(E)$ for some $\tau > 0$. In addition, assume that for any compact set $K \subset E$ the operator $Q_\tau^K(f) := 1_K Q_\tau(1_K f)$ is compact on $\mathcal{B}_V(E)$. In this situation, for any $t \in \mathcal{T}$ with $t \geq \tau$, the operator Q_t is a compact operator from $\mathcal{B}_V(E)$ into itself.*

Proof. For any $t \in \mathcal{T}$, we set $R_t^K(f) := 1_K Q_t(f)$ so that

$$Q_t = R_t^K + R_t^{K^c} \quad \text{and} \quad Q_{t+\tau} - Q_\tau^K Q_t = Q_\tau R_t^{K^c} + R_\tau^{K^c} R_t^K.$$

This decomposition implies that

$$\begin{aligned} \left\| \|Q_{t+\tau} - Q_\tau^K Q_t\|_V \right\|_V &\leq (\|Q_\tau\|_V + \|R_t^K\|_V) \|R_\tau^{K^c}\|_V \\ &\leq (\|Q_\tau\|_V + \|Q_t\|_V) \|R_\tau^{K^c}\|_V. \end{aligned}$$

Also note that

$$\|R_\tau^{K^c}\|_V = \|1_{K^c} Q_\tau(V)/V\|.$$

Thus, choosing a sequence of compact sets K_n such that

$$1_{K_n^c} Q_\tau(V) \leq \frac{1}{n} V$$

implies that

$$\left\| \|Q_{t+\tau} - Q_\tau^{K_n} Q_t\|_V \right\|_V \leq \frac{1}{n} (\|Q_\tau\|_V + \|Q_t\|_V).$$

Since the product operator $Q_\tau^{K_n} Q_t$ is compact, $Q_{t+\tau}$ is the limit in norm of compact operators, hence it is compact. This ends the proof of the lemma. \blacksquare

Remark 3.5. *The condition $Q_\tau(V)/V \in \mathcal{B}_0(E)$ allows one to localise the operators on compact sets. The compactness condition of the semigroup Q_τ^K is readily checked for absolutely continuous operators of the form*

$$Q_\tau(x, dy) = q_\tau(x, y) \nu_\tau(dy). \tag{82}$$

In the above display $q_\tau(x, y)$ is a continuous density with respect to some Radon measure ν_τ on E . Indeed, in this case, for any collection of functions $(f_n)_{n \geq 0} \in \mathcal{B}_V(E)^\mathbb{N}$ in the unit ball $\{f \in \mathcal{B}_V(E) : \|f\|_V \leq 1\}$ we have

$$\sup_{n \geq 0} \|Q_\tau^K(f_n)\|_V \leq \|Q_\tau^K(V)/V\|.$$

In addition, for any $(x, y) \in K^2$ we have

$$|Q_\tau^K(f_n)(x) - Q_\tau^K(f_n)(y)| \leq \int |q_\tau(x, z) - q_\tau(y, z)| 1_K(z) \nu_\tau(dz).$$

Since q_τ is continuous on the compact set $(K \times K)$, it is uniformly continuous. Thus for any $\epsilon > 0$, there is some $\delta > 0$ such that $d(x, y) \leq \delta$ implies that for any $n \geq 0$ we have

$$|Q_\tau^K(f_n)(x) - Q_\tau^K(f_n)(y)| \leq \epsilon \nu_\tau(K).$$

By the Arzela-Ascoli theorem, $Q_\tau^K(f_n)$ converges uniformly to a continuous limit extended by 0 outside the set K . Recalling that $V \geq 1$ is also converges on $\mathcal{B}_V(E)$.

Q_τ^K is an irreducible and compact operator on $\mathcal{B}_V(E)$. The above models encapsulate Markov transitions restricted to a compact set $K \subset \mathbb{R}^r$, for some $r \geq 1$, defined by

$$Q_\tau^K(x, dy) := 1_K(x) p_\tau(x, y) 1_K(y) dy.$$

In the above, $p_\tau(x, y)$ is a continuous probability transition density with respect to the Lebesgue measure dy on \mathbb{R}^r . For a more thorough discussion on this class of compact integral operators we refer to [107] and the more recent article [33].

Remark 3.6. Now assume that E is a countable space. In this situation, whenever $V = 1$ we can also use the following equivalence principle (see for instance Theorem 2.1 in [184])

$$Q_\tau \text{ is compact on } \mathcal{B}(E) \iff \forall \epsilon > 0 \exists K_\epsilon \text{ finite s.t. } \sup_{x \in E} Q_\tau(1_{E-K_\epsilon})(x) \leq \epsilon.$$

More generally, assume that $Q_\tau(V)/V \in \mathcal{B}_0(E)$. In this situation, for any $\epsilon > 0$ the set

$$K_\epsilon := \{Q_\tau(V)/V \geq \epsilon\}$$

is finite, and following the proof of the above lemma, we have

$$\|Q_{t+\tau} - Q_\tau^{K_\epsilon} Q_t\|_V \leq \epsilon (\|Q_\tau\|_V + \|Q_t\|_V).$$

Since for any finite set K , the operator

$$Q_\tau^K(f)(x) = 1_K(x) \sum_{y \in K} Q_\tau(x, y) f(y)$$

is bounded and has finite range, it is compact and we thus conclude that Q_t is compact for any $t \geq \tau$ as soon as $Q_\tau(V)/V \in \mathcal{B}_0(E)$.

The set E can be written as the union $E = \cup_{n \geq 0} K_n$ of an increasing sequence of compact sets K_n . Whenever Q_t is compact on $\mathcal{B}_V(E)$, up to a subsequence extraction the functions $Q_t(V1_{K_n^c}) \in \mathcal{B}_V(E)$ forms a Cauchy sequence. Thus, for any $\epsilon > 0$ there exists some $n_\epsilon \geq 1$ such that for any $n_\epsilon \leq m \leq n$ we have

$$\|Q_t(V1_{K_n^c}) - Q_t(V1_{K_m^c})\|_V = \|Q_t(V1_{K_n - K_m})\|_V \leq \epsilon$$

and therefore

$$\|Q_t(V1_{K_m^c})/V\| \leq \epsilon.$$

Now, assume that for any $\epsilon > 0$ there exists some $n_\epsilon \geq 1$ such that

$$\forall n \geq n_\epsilon \quad \|Q_t(V1_{K_n^c})/V\| \leq \epsilon.$$

For any $\|f\|_V \leq 1$, we have

$$\|Q_t(f) - Q_t(1_{K_n} f)\|_V \leq \|Q_t(V1_{K_n^c})/V\| \leq \epsilon.$$

Arguing as above, using the fact that $f \in \mathcal{B}_V(E) \mapsto 1_{K_n} f \in \mathcal{B}_V(E)$ is a finite range compact operator we conclude that Q_t is compact on $\mathcal{B}_V(E)$. We conclude that

$$Q_t \text{ compact on } \mathcal{B}_V(E) \iff \forall \epsilon > 0 \exists K_\epsilon \text{ finite s.t. } \|Q_t(1_{K_\epsilon^c} V)\|_V \leq \epsilon.$$

Whenever E is compact, by a theorem of de Pagter (cf. [157, Theorem 4.2.2]), a compact and irreducible positive operator Q_t on $\mathcal{C}(E)$ has a positive spectral radius $r(Q_t) > 0$, while its essential spectral radius is null. Applying the Krein-Rutman theorem (cf. [93, 151, Theorem 1.1]), there exists some non-negative and non-zero measure $\nu_\infty \in \mathcal{M}(E) = \mathcal{C}(E)'$ and a non-negative and non-zero function $h \in \mathcal{C}(E)$ such that

$$\nu_\infty Q_t = e^{\rho t} \nu_\infty \quad \text{and} \quad Q_t(h) = e^{\rho t} h \quad \text{with} \quad \rho = \log r(Q_1). \quad (83)$$

This yields the fixed point equation

$$\eta_\infty := \nu_\infty / \nu_\infty(1) = \Phi_t(\eta_\infty).$$

More generally, when E is compact and Q_τ is a positive operator from $\mathcal{B}(E)$ into itself, the Schauder fixed-point theorem ensures the existence of a fixed point

$$\eta_\infty = \Phi_\tau(\eta_\infty) \in \mathcal{P}(E) \quad \text{and} \quad \forall t \in \mathcal{T}(\tau), \quad \eta_\infty(Q_t(f)) = \eta_\infty(f) \eta_\infty(Q_\tau(1))^{t/\tau}.$$

Note that the right-hand side assertion is a direct consequence of the product formula (6).

For potentially non-compact spaces, condition (47) is met as soon as $Q_\tau(V)/V \in \mathcal{C}_0(E)$ and Q_τ is a quasi-compact operator from $\mathcal{B}_V(E)$ into itself and mapping $\mathcal{B}_V(E)$ into $\mathcal{C}_V(E)$. Several sufficient conditions in terms of the Lyapunov functions $V \geq 1$ ensuring the quasi-compactness properties of Q_τ are discussed in [100, 119, 161], see also Lemma 3.4 and Remark 3.5 in the present article. For a more detailed discussion on quasi-compact and compact operators we refer to [94, 151, 157, 160] and references therein.

For potentially non-compact space E we have the following Krylov-Bogolioubov type theorem, which is of interest in its own right.

Theorem 3.7. *Assume that $Q_\tau(\mathcal{C}_b(E)) \subset \mathcal{C}_b(E)$. Then the mapping Φ_τ has at least one invariant probability measure η_∞ if and only if there exists some probability measure η such that the flow $\Phi_t(\eta)$ indexed by $t \in \mathcal{T}(\tau)$ is tight and we have*

$$\beta_t(\eta) := \Phi_t(\eta)(Q_\tau(1)) \in]0, 1] \longrightarrow_{t \rightarrow \infty} \beta_\infty(\eta) > 0. \quad (84)$$

Whenever these conditions are met we have $\beta_\infty(\eta) = \eta_\infty(Q_\tau(1))$.

The proof of the above theorem is provided in section 4.4. Related equivalent conditions for the existence of quasi-invariant measures are discussed in [138] for models such that $Q_\tau(\mathcal{C}_0(E)) \subset \mathcal{C}_0(E)$, see also [97].

4 Proofs of the stability theorems

4.1 Proof of Theorem 2.5

This section is mainly concerned with the proof of Theorem 2.5. For any $\mu \in \mathcal{P}_V(E)$ and $n \geq 1$ we set

$$\epsilon_n(\mu) := \theta(\kappa_V(\mu) + n).$$

Applying Markov's inequality we have

$$\Phi_{s,t}(\mu)(\mathcal{V}_{\epsilon_n(\mu)}^c) = \Phi_{s,t}(\mu)(V \geq \theta^{-1}(\epsilon_n(\mu))) \leq \kappa_V(\mu)/(\kappa_V(\mu) + n) > 0.$$

We conclude that

$$\left\{ \begin{array}{l} \forall 0 < \epsilon \leq \epsilon_1(\mu) \quad \inf \Phi_{s,t}(\mu)(\mathcal{V}_{\epsilon_1(\mu)}) \geq 1 - \frac{1}{1 + 1/\kappa_V(\mu)} > 0 \\ \text{and} \quad \inf \Phi_{s,t}(\mu)(\mathcal{V}_{\epsilon_n(\mu)}) \geq \frac{1}{1 + \kappa_V(\mu)/n} \xrightarrow{n \rightarrow \infty} 1. \end{array} \right. \quad (85)$$

In the above display, the infimum are taken over all indices (s, t) such that $t \in \mathcal{T}(\tau)$ and $s \in \mathcal{T}_t(\tau)$.

Lemma 4.1. *Assume that (16) and (17) are satisfied some locally bounded function V with compact level sets. Then the Lyapunov estimate (36) is satisfied.*

Proof. For any $n \geq 1$ we set $\epsilon_n := \theta(\bar{\eta}(V) + n)$. For any $s \in \mathcal{T}(\tau)$ and $0 < \epsilon \leq \epsilon_1/\lambda^-$, thanks to (16), we have the estimate

$$Q_{s,s+\tau}(V)/V < \epsilon \lambda^- 1_{K_\epsilon(s)^c} + \|\theta(V)\| 1_{K_\epsilon(s)},$$

with the compact sets $K_\epsilon(s)$ are defined by

$$K_\epsilon(s) := \{Q_{s,s+\tau}(V)/V \geq \epsilon \lambda^-\} \subset \mathcal{K}_\epsilon^- := \{\theta(V) \geq \epsilon \lambda^-\}.$$

On the other hand, for any $s, s + \tau \in \mathcal{T}_t(\tau)$ we have

$$\frac{R_{s,s+\tau}^{(t)}(V/H_{s+\tau,t})}{(V/H_{s,t})} = \frac{1}{\lambda_{s,s+\tau}} \frac{Q_{s,s+\tau}(V)}{V}.$$

This yields the estimate

$$\begin{aligned} \frac{R_{s,s+\tau}^{(t)}(V/H_{s+\tau,t})}{(V/H_{s,t})} &\leq \frac{\lambda^-}{\lambda_{s,s+\tau}} \left(\epsilon 1_{K_\epsilon(s)^c} + \frac{\|\theta(V)\|}{\lambda^-} 1_{K_\epsilon(s)} \right) \\ &\leq \epsilon 1_{K_\epsilon(s)^c} + \frac{\|\theta(V)\|}{\lambda^-} 1_{K_\epsilon(s)}, \end{aligned}$$

from which we check that

$$R_{s,s+\tau}^{(t)}(V/H_{s+\tau,t}) \leq \epsilon (V/H_{s,t}) + \frac{\|\theta(V)\|}{\lambda^-} \sup_{\mathcal{K}_\epsilon^-}(V/H_{s,t}).$$

Now, choosing $n \geq 1$ sufficiently large such that

$$\epsilon_n \leq \epsilon \lambda^- \leq \epsilon_1 \vee \epsilon_H,$$

we have

$$\mathcal{K}_\epsilon^- = \{\theta(V) \geq \epsilon \lambda^-\} \subset \mathcal{V}_{\epsilon_n} := \{\theta(V) \geq \epsilon_n\}.$$

By (17) this yields the uniform estimate

$$\sup_{\mathcal{K}_\epsilon^-} (V/H_{s,t}) \leq \varsigma_\epsilon(H)^{-1} \sup_{\mathcal{V}_{\epsilon_n}} V < \infty.$$

■

Proof of Theorem 2.5. Observe that

$$\|H\|_V = 1 \implies H \leq V \quad \text{and} \quad H_{s,t} = \overline{Q}_{s,t}(H) \leq \overline{Q}_{s,t}(V).$$

This implies that

$$\lambda^- H_{s,t}/V \leq \lambda_{s,s+\tau} H_{s,t}/V = Q_{s,s+\tau}(H_{s+\tau,t})/V \leq Q_{s,s+\tau}(V)/V.$$

Combining this with (\mathcal{H}_1) , we obtain the estimate

$$\lambda^- V_\theta \leq V/H_{s,t} \quad \text{with} \quad V_\theta := 1/\theta(V).$$

Notice that V_θ has compact level sets and for any $r \geq 1$ we have

$$\{V/H_{s,t} \leq r\} \subset \{V_\theta \leq r/\lambda^-\}.$$

This implies that $V/H_{s,t}$ also has compact level sets. Under (\mathcal{H}_0) (which holds under (\mathcal{H}_1)), for any $r \geq r_1 := \lambda^- r_0$ we have

$$\sup_{(V/H_{s,t})(x) \vee (V/H_{s,t})(y) \leq r} \left\| \delta_x R_{s,s+\tau}^{(t)} - \delta_y R_{s,s+\tau}^{(t)} \right\|_{tv} \leq 1 - \alpha(r/\lambda^-) < 1.$$

Now, due to the previous lemma, (36) holds which in turn implies that

$$R_{s,s+\tau}^{(t)}(W_{s+\tau,t}) \leq \epsilon W_{s,t} + 1$$

with the collection of functions $W_{s,t} \geq 1$ defined by

$$\frac{\epsilon}{c} \frac{V}{H_{s,t}} \leq W_{s,t} := 1 + \frac{\epsilon}{c} \frac{V}{H_{s,t}} \leq \left(1 + \frac{\epsilon}{c}\right) \frac{V}{H_{s,t}}.$$

Then, for any $r \geq \epsilon r_1/c$ we have

$$\sup_{W_{s,t}(x) \vee W_{s,t}(y) \leq r} \left\| \delta_x R_{s,s+\tau}^{(t)} - \delta_y R_{s,s+\tau}^{(t)} \right\|_{tv} \leq 1 - \alpha\left(\frac{cr}{\epsilon\lambda^-}\right) < 1.$$

The estimate (35) is now a direct consequence of Lemma 3.2. This ends the proof of the theorem. ■

4.2 Proof of Theorem 2.9

Proof. Observe that

$$\lambda_{s,t} H_{s,t} R_{s,t}^{(t)}(f/H) = Q_{s,t}(f),$$

which implies that

$$\Psi_{H_{s,t}}(\mu) R_{s,t}^{(t)}(f/H) = \frac{\mu(Q_{s,t}(f))}{\mu(Q_{s,t}(H))} = \frac{\Phi_{s,t}(\mu)(f)}{\Phi_{s,t}(\mu)(H)}.$$

Combining this with (11) and the estimate (68) we have

$$\begin{aligned} \|\Phi_{s,t}(\mu) - \Phi_{s,t}(\eta)\|_V &= \|\Psi_{1/H}(\Psi_{H_{s,t}}(\mu)R_{s,t}^{(t)}) - \Psi_{1/H}(\Psi_{H_{s,t}}(\eta)R_{s,t}^{(t)})\|_V \\ &\leq \Phi_{s,t}(\mu)(H) (1 + \Phi_{s,t}(\eta)(V)) \times \|(\Psi_{H_{s,t}}(\mu) - \Psi_{H_{s,t}}(\eta)) R_{s,t}^{(t)}\|_{V/H}. \end{aligned}$$

Applying (35) to $u = t$ we find that

$$\|\Phi_{s,t}(\mu) - \Phi_{s,t}(\eta)\|_V \leq \kappa_H(\mu) (1 + \kappa_V(\eta)) a e^{-b(t-s)} \|\Psi_{H_{s,t}}(\mu) - \Psi_{H_{s,t}}(\eta)\|_{V/H_{s,t}}.$$

On the other hand, applying (69) we check that

$$\|\Psi_{H_{s,t}}(\mu) - \Psi_{H_{s,t}}(\eta)\|_{V/H_{s,t}} \leq \frac{1}{\mu(H_{s,t})} \left(1 + \frac{\eta(V)}{\eta(H_{s,t})}\right) \|\mu - \eta\|_V.$$

This ends the proof of the theorem. ■

4.3 Proof of Theorem 2.17

This section is mainly concerned with the proof of Theorem 2.17 on the stability of the time homogeneous models discussed in (47).

Lemma 4.2. *For any $\epsilon > 0$ we have the Foster-Lyapunov inequality*

$$P_\tau^h(V/h) < \epsilon (V/h) 1_{K_{\epsilon,\tau}^c} + c_{\epsilon,\tau} 1_{K_{\epsilon,\tau}} \quad (86)$$

with the parameter $c_{\epsilon,\tau}$ and the compact set $K_{\tau,\epsilon}$ given by

$$c_{\epsilon,\tau} := e^{-\rho\tau} \|Q_\tau(V)/V\| \sup_{K_{\epsilon,\tau}}(V/h) \quad \text{and} \quad K_{\tau,\epsilon} := \{Q_\tau(V)/V \geq \epsilon e^{\rho\tau}\}.$$

Proof. For any $\epsilon > 0$ we have

$$V^h := V/h \implies P_\tau^h(V^h)/V^h = e^{-\rho\tau} Q_\tau(V)/V < \epsilon 1_{K_{\epsilon,\tau}^c} + e^{-\rho\tau} \|Q_\tau(V)/V\| 1_{K_{\epsilon,\tau}}.$$

This readily yields the estimate (86). ■

Proposition 4.3. *For any $\mu \in \mathcal{P}_{V/h}(E)$ we have*

$$0 < \kappa(\mu) := \inf_{t \in \mathcal{T}(\tau)} \mu P_t^h(1/h) \leq \kappa_{V/h}^h(\mu) := \sup_{t \in \mathcal{T}(\tau)} \mu P_t^h(V/h) < \infty.$$

In addition, for any $\eta \in \mathcal{P}_V(E)$ we have the estimates

$$0 < \kappa_h^-(\eta) \leq \kappa_V(\eta) < \infty. \quad (87)$$

Proof. Following word-for-word the same arguments as the proof of (74) the Foster-Lyapunov estimate (86) implies that

$$\kappa_{V/h}^h(\mu) \leq \mu(V/h) + c_{\epsilon, \tau}(1 - \epsilon)^{-1} < \infty.$$

Now, we come to the proof of the left-hand side estimate. Since $h \in \mathcal{B}_{0,V}(E)$ and $\|h\|_V = 1$ the function $V/h \geq 1$ has compact level sets and h is bounded on compact sets. Since $h \in \mathcal{B}_V^+(E)$ we also have $\inf_K h > 0$ for any compact set $K \subset E$. Consider the compact sets $\mathcal{K}_{V/h}^h(\delta)$ indexed by $\delta \in]0, 1[$ and defined by

$$\mathcal{K}_{V/h}^h(\mu, \delta) := \{x : \delta V(x)/h(x) \leq \kappa_{V/h}^h(\mu)\}.$$

Arguing as in (75), we have

$$\inf_{t \in \mathcal{T}(\tau)} \mu P_t^h(\mathcal{K}_{V/h}^h(\mu, \delta)) \geq 1 - \delta \quad \text{and} \quad \kappa(\mu) \geq (1 - \delta) / \sup_{\mathcal{K}_{V/h}^h(\mu, \delta)} h.$$

Observe that

$$\sup_{t \in \mathcal{T}(\tau)} \Psi_h(\eta) P_t^h(V/h) \leq \eta(V)/\eta(h) + c_{\epsilon, \tau}(1 - \epsilon)^{-1}$$

and consider the compact sets

$$\mathcal{K}_V(\eta, \delta) := \{x : \delta V(x) \leq \eta(V)/\eta(h) + c_{\epsilon, \tau}(1 - \epsilon)^{-1}\} \subset E.$$

Arguing as above, we check that

$$\inf_{t \in \mathcal{T}(\tau)} \Psi_h(\eta) P_t^h(\mathcal{K}_V(\eta, \delta)) \geq 1 - \delta \quad \text{and} \quad \inf_{t \in \mathcal{T}(\tau)} \Psi_h(\eta) P_t^h(1/h) \geq \frac{(1 - \delta)}{\sup_{\mathcal{K}_V(\eta, \delta)} h}.$$

This yields for any $t \in \mathcal{T}(\tau)$ and $\delta \in]0, 1[$ the uniform estimate

$$\Phi_t(\eta)(V) = \frac{\Psi_h(\eta) P_t^h(V/h)}{\Psi_h(\eta) P_t^h(1/h)} \leq \frac{\eta(V)/\eta(h) + c_{\epsilon, \tau}(1 - \epsilon)^{-1}}{(1 - \delta)/\sup_{\mathcal{K}_V(\eta, \delta)} h}.$$

This implies that $\kappa_V(\eta) < \infty$. This also ensures that the sequence of probability measures $\Phi_{s,t}(\eta)$ indexed by $s \leq t$ is tight. Choosing the compact set

$$K(\delta, \eta) := \{x \in E : \delta V(x) \leq \kappa_V(\eta)\} \quad (88)$$

we readily check that

$$\Phi_{s,t}(\eta)(h) \geq \left(\inf_{K(\delta, \eta)} h \right) \Phi_{s,t}(\eta)(K(\delta, \eta)) \geq (1 - \delta) \left(\inf_{K(\delta, \eta)} h \right) > 0.$$

We conclude that $\kappa_h^-(\eta) > 0$. ■

Proof of Theorem 2.17. The last assertion has been checked in (87). Observe that

$$(86) \implies P_\tau^h(W) \leq \epsilon W + 1$$

with the function $W \geq 1$ defined by

$$\frac{\epsilon}{c_{\epsilon,\tau}} \frac{V}{h} \leq W := 1 + \frac{\epsilon}{c_{\epsilon,\tau}} \frac{V}{h} \leq \left(1 + \frac{\epsilon}{c_{\epsilon,\tau}}\right) \frac{V}{h}, \quad (89)$$

which has compact level sets. On the other hand, by (48) we have

$$\sup_{W(x) \vee W(y) \leq r} \|\delta_x P_\tau^h - \delta_y P_\tau^h\|_{tv} \leq 1 - \alpha(c_{\epsilon,\tau} r / \epsilon).$$

The estimate (49) is now a direct consequence of (76) and the estimates (89). The proof of the theorem is now completed. \blacksquare

4.4 Proof of Theorem 3.7

Proof. One direction of the proof is obvious. Indeed, if Φ_τ has at least one invariant probability measure η_∞ choosing $\eta = \eta_\infty$ the measure $\Phi_t(\eta_\infty) = \eta_\infty$ is tight and $\beta_t(\eta_\infty) = \eta_\infty(Q_\tau(1))$.

Conversely, assume that for some η the sequence of probability measures $\Phi_t(\eta)$ is tight and (84) is satisfied. In this situation, for any $\epsilon > 0$ there exists a compact $K_\epsilon \subset E$ such that

$$\Phi_t(\eta)(Q_\tau(1)) \geq \Phi_t(\eta)(1_{K_\epsilon} Q_\tau(1)) \geq (1 - \epsilon) \inf_{K_\epsilon} Q_\tau(1) \quad \text{and} \quad \beta_\infty(\eta) > 0.$$

We simplify the notation and write β_n and β_∞ instead of $\beta_{n\tau}(\eta)$ and $\beta_\infty(\eta)$. Consider the probability measures

$$\eta_m := \frac{1}{m} \sum_{0 \leq k < m} \Phi_{k\tau}(\eta) \implies \eta_m(Q_\tau(1)) = \frac{1}{m} \sum_{0 \leq k < m} \beta_k \xrightarrow{m \rightarrow \infty} \beta_\infty.$$

There exists at least one probability measure $\eta_\infty := \varpi(\eta)$ and a sub-sequence $m_k \rightarrow_{k \rightarrow \infty} \infty$ such that η_{m_k} converges weakly to η_∞ as $k \rightarrow \infty$. Hence, η_{m_k} and $\Phi(\eta_{m_k})$ converge weakly to η_∞ and $\Phi(\eta_\infty)$, respectively, as $m \rightarrow \infty$. In addition, we have

$$\Phi_{n\tau}(\eta)(Q_\tau(f)) = \Phi_{n\tau}(\eta)(Q_\tau(1)) \Phi_{(n+1)\tau}(\mu_0)(f).$$

This yields the formula

$$\Phi_\tau(\eta_m) = \sum_{0 \leq k < m} \frac{\beta_k}{\sum_{0 \leq l < m} \beta_l} \Phi_{(k+1)\tau}(\mu_0)$$

from which we check that

$$\begin{aligned}\Phi_\tau(\eta_m) - \eta_m &= \frac{1}{m} \sum_{0 \leq k < m} \left(\frac{\beta_k}{\frac{1}{m} \sum_{0 \leq l < m} \beta_l} - 1 \right) \Phi_{(k+1)\tau}(\mu_0) \\ &\quad + \frac{1}{m} \sum_{0 \leq k < m} (\Phi_{(k+1)\tau}(\mu_0) - \Phi_{k\tau}(\eta)).\end{aligned}$$

As a result

$$\begin{aligned}\Phi_\tau(\eta_m) - \eta_m &= \frac{1}{m} \sum_{0 \leq k < m} \left(\frac{\beta_k}{\frac{1}{m} \sum_{0 \leq l < m} \beta_l} - 1 \right) \Phi_{(k+1)\tau}(\eta) + \frac{1}{m} (\Phi_{(k+1)\tau}(\eta) - \eta).\end{aligned}$$

On the other hand, we have

$$\frac{1}{m} \sum_{0 \leq k < m} \left| \beta_k - \frac{1}{m} \sum_{0 \leq l < m} \beta_l \right| \leq \frac{2}{m} \sum_{0 \leq k < m} |\beta_k - \beta_\infty| \xrightarrow{m \rightarrow \infty} 0.$$

For any $f \in \mathcal{C}_b(E)$ we conclude that

$$|(\Phi_\tau(\eta_{m_k}) - \eta_{m_k})(f)| \xrightarrow{k \rightarrow \infty} 0 \quad \text{and therefore} \quad \eta_\infty = \Phi_\tau(\eta_\infty).$$

The last assertion follows from the fact that

$$\eta_{m_k}(Q_\tau(1)) = \frac{1}{m_k} \sum_{l=0}^{m_k-1} \beta_l \xrightarrow{m \rightarrow \infty} \beta_\infty = \eta_\infty(Q_\tau(1)).$$

This ends the proof of the theorem. ■

A Proofs of the technical lemmas

A.1 Proof of Lemma 1.3

In this section, we will work under the assumptions imposed by (\mathcal{H}_1) . Recall that the function $H \in \mathcal{B}_V^+(E)$ is chosen such that $\|H\|_V = 1$ and we have $Q_{t,t+\tau}(H) > 0$ for any $t \in \mathcal{T}$ and $\tau > 0$.

Further recall that from section 1.5, for any $u, t \in \mathcal{T}$, we have

$$\lambda_{u,t} H_{u,t} = Q_{u,t}(H) =: h_{u,t} \implies Q_{s,u}(h_{u,t}) = h_{s,t} > 0. \quad (90)$$

We also recall that we assume (19) and (20) are satisfied. To simplify the notation, we write ν instead of ν_τ . In this notation, for any $0 < \epsilon < \epsilon_1$ there exist finite constants $0 < \iota_\epsilon^- \leq \iota_\epsilon < \infty$ such that for any $t \in \mathcal{T}$ we have

$$\iota_\epsilon^- 1_{\mathcal{V}_\epsilon}(x) \nu(dy) 1_{\mathcal{V}_\epsilon}(y) \leq 1_{\mathcal{V}_\epsilon}(x) Q_{t,t+\tau}(x, dy) 1_{\mathcal{V}_\epsilon}(y) \leq \iota_\epsilon 1_{\mathcal{V}_\epsilon}(x) \nu(dy) 1_{\mathcal{V}_\epsilon}(y). \quad (91)$$

The next lemma is a slight modification of [178, Proposition 1].

Lemma A.1. *There exists $\lambda_0 > 0$ and a finite constant $c_0 < \infty$ such that for any $s, t \in \mathcal{T}(\tau)$ and $\eta \in \mathcal{P}_V(E)$ we have*

$$R_{s,t}^{(t)} \left(\frac{V}{H} \right) \leq e^{-\lambda_0(t-s)} \frac{V}{h_{s,t}} + c_0 \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{e^{-\lambda_0(t-s)}}{\eta(h_{s,t})} = 0. \quad (92)$$

In addition, we have the uniform estimates stated in (21).

Proof. We set

$$h_{s,t}^V := h_{s,t} / \|h_{s,t}\|_V \quad \text{and} \quad \beta_{s,u}^{(t)} = \|h_{s,t}\|_V / \|h_{u,t}\|_V.$$

We have

$$R_{s,s+\tau}^{(t)} \left(\frac{V}{h_{s+\tau,t}^V} \right) := \frac{1}{\beta_{s,s+\tau}^{(t)}} \frac{V}{h_{s,t}^V} \frac{Q_{s,s+\tau}(V)}{V} \quad \text{and} \quad Q_{s,s+\tau}(h_{s+\tau,t}^V) = \beta_{s,s+\tau}^{(t)} h_{s,t}^V.$$

Using (14) and (91) and recalling that $Q_{t,t+\tau}(V)/V \leq \theta(V)$, for any $0 \leq \epsilon < \epsilon_1$ we check that

$$R_{s,s+\tau}^{(t)} \left(\frac{V}{h_{s+\tau,t}^V} \right) \leq \frac{\epsilon}{\beta_{s,s+\tau}^{(t)}} \frac{V}{h_{s,t}^V} + \frac{a_\epsilon}{\nu(1_{\mathcal{V}_\epsilon} h_{s+\tau}^V)} \quad \text{with} \quad a_\epsilon := \sup_{\mathcal{V}_\epsilon} (\theta(V)V) / \iota_\epsilon^-.$$

This implies that

$$R_{s,s+n\tau}^{(t)} \left(\frac{V}{h_{s+n\tau,t}^V} \right) \leq \frac{\epsilon}{\beta_{s+(n-1)\tau,s+n\tau}^{(t)}} R_{s,s+(n-1)\tau}^{(t)} \left(\frac{V}{h_{s+(n-1)\tau,t}^V} \right) + \frac{a_\epsilon}{\nu(1_{\mathcal{V}_\epsilon} h_{s+n\tau,t}^V)}.$$

Applying the above to $s + n\tau = t$ we obtain the formula

$$\begin{aligned} & R_{s,t}^{(t)} \left(\frac{V}{H} \right) \\ & \leq \frac{\epsilon^n}{\beta_{s,t}^{(t)}} \frac{V}{h_{s,t}^V} + \sum_{1 \leq k \leq n} \frac{\epsilon^{n-k}}{\beta_{s+k\tau,t}^{(t)}} \frac{a_\epsilon}{\nu(1_{\mathcal{V}_\epsilon} h_{s+k\tau,t}^V)} = \frac{\epsilon^n}{h_{s,t}} V + \sum_{1 \leq k \leq n} \epsilon^{n-k} \frac{a_\epsilon}{\nu(1_{\mathcal{V}_\epsilon} h_{s+k\tau,t}^V)}. \end{aligned}$$

Observe that for any $0 < \epsilon \leq \epsilon_2 < \epsilon_1$ we have $\mathcal{V}_{\epsilon_2} \subset \mathcal{V}_\epsilon$. Thus, for any $k \leq n$ and $t = s + n\tau$ we have the lower bound estimate

$$\begin{aligned} 1_{\mathcal{V}_\epsilon}(x) h_{s+k\tau,s+n\tau}(x) & \geq 1_{\mathcal{V}_{\epsilon_2}}(x) Q_{s+k\tau,s+(k+1)\tau} 1_{\mathcal{V}_{\epsilon_2}} \cdots Q_{s+(n-1)\tau,s+n\tau} (1_{\mathcal{V}_{\epsilon_2}} H)(x) \\ & \geq 1_{\mathcal{V}_{\epsilon_2}}(x) \left(\inf_{\mathcal{V}_{\epsilon_2}} H \right) (\iota_{\epsilon_2}^- \nu(\mathcal{V}_{\epsilon_2}))^{n-k}. \end{aligned}$$

Choosing $0 < \epsilon < \epsilon_3 := \epsilon_2 \wedge (\iota_{\epsilon_2}^- \nu(\mathcal{V}_{\epsilon_2})/2)$ we conclude that

$$R_{s,t}^{(t)} \left(\frac{V}{H} \right) \leq \frac{\epsilon^n}{h_{s,t}} V + \frac{2a_\epsilon}{\nu(\mathcal{V}_{\epsilon_2})} \frac{1}{\inf_{\mathcal{V}_{\epsilon_2}} H}.$$

In the same vein, for any $\eta \in \mathcal{P}_V(E)$ we have

$$\eta(h_{s,s+n\tau}) \geq \eta(\mathcal{V}_{\epsilon_2}) \left(\inf_{\mathcal{V}_{\epsilon_2}} H \right) (\iota_{\epsilon_2}^- \nu(\mathcal{V}_{\epsilon_2}))^n \geq \eta(\mathcal{V}_{\epsilon_2}) \left(\inf_{\mathcal{V}_{\epsilon_2}} H \right) (2\epsilon)^n.$$

This yields the estimate

$$\frac{\epsilon^{(t-s)/\tau}}{\eta(h_{s,t})} \leq 2^{-(t-s)/\tau} \frac{1}{\eta(\mathcal{V}_{\epsilon_2}) \inf_{\mathcal{V}_{\epsilon_2}} H}.$$

To summarise: there exists some $\epsilon_3 > 0$ such that for any $s, t \in \mathcal{T}(\tau)$ s.t. $s \leq t \in \mathcal{T}(\tau)$ and for any $0 < \epsilon < \epsilon_3$ we have

$$R_{s,t}^{(t)}(V/H) \leq \frac{\epsilon^{(t-s)/\tau}}{h_{s,t}} V + c_\epsilon \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{\epsilon^{(t-s)/\tau}}{\eta(h_{s,t})} = 0.$$

This ends the proof of (92). The last assertion follows from the fact that

$$\left(\Psi_{h_{s,t}}(\eta) R_{s,t}^{(t)} \right) (V/H) \leq \frac{e^{-\lambda_0(t-s)}}{\eta(h_{s,t})} \eta(V) + c_0 \leq c_1(\eta)$$

for some finite constant $c_1(\eta) < \infty$. This implies that the sequence of probability measures

$$\Psi_{h_{s,t}}(\eta) R_{s,t}^{(t)}$$

indexed by $s, t \in \mathcal{T}(\tau)$ is tight. More precisely, choosing the compact set

$$K_{\epsilon,\eta} := \{x \in E : \epsilon(V/H)(x) \leq c_1(\eta)\}$$

we have

$$\left(\Psi_{h_{s,t}}(\eta) R_{s,t}^{(t)} \right) (K_{\epsilon,\eta}^c) \leq \epsilon \quad \text{so that} \quad \left(\Psi_{h_{s,t}}(\eta) R_{s,t}^{(t)} \right) (1/H) \geq \frac{1-\epsilon}{\sup_{K_{\epsilon,\eta}} H}.$$

On the other hand, we have

$$\Phi_{s,t}(\eta)(V) = \Psi_{1/H} \left(\Psi_{h_{s,t}}(\eta) R_{s,t}^{(t)} \right) (V) \leq c_1(\eta) \frac{\sup_{K_{\epsilon,\eta}} H}{1-\epsilon}.$$

We conclude that $\kappa_V(\eta) < \infty$. This also shows that the sequence of probability measures $\Phi_{s,t}(\eta)$ indexed by $s \leq t$ is tight. Choosing the compact set

$$K_{\epsilon,\eta} := \{x \in E : \epsilon V(x) \leq \kappa_V(\eta)\} \tag{93}$$

we readily check that

$$\Phi_{s,t}(\eta)(H) \geq \left(\inf_{K_{\epsilon,\eta}} H \right) \Phi_{s,t}(\eta)(K_{\epsilon,\eta}) \geq (1-\epsilon) \left(\inf_{K_{\epsilon,\eta}} H \right) > 0.$$

We conclude that $\kappa_H^-(\eta) > 0$. This ends the proof of the left-hand side estimate in (21).

Also observe that for any $0 < \epsilon < \epsilon_0$ we have

$$(91) \implies \inf_{t \in \mathcal{T}(\tau)} \inf_{\mathcal{V}_\epsilon} Q_{t,t+\tau}(1) \geq \nu(\mathcal{V}_\epsilon) > 0. \quad (94)$$

Thus, for any $t \in \mathcal{T}(\tau)$ and $0 < \epsilon < \epsilon_0$ we have

$$\Phi_{s,t}(\eta)(Q_{t,t+\tau}(1)) \geq \Phi_{s,t}(\eta)(1_{\mathcal{V}_\epsilon} Q_{t,t+\tau}(1)) \geq \Phi_{s,t}(\eta)(\mathcal{V}_\epsilon) \nu(\mathcal{V}_\epsilon).$$

By choosing $\epsilon = \epsilon_n := \theta(\kappa_V(\eta) + n)$ for some $n \geq 1$ we check that

$$\mathcal{V}_{\epsilon_n} = \{x \in E : V(x) \leq \kappa_V(\eta) + n\} \quad \text{and} \quad \Phi_{s,t}(\eta)(\mathcal{V}_{\epsilon_n}) \geq \frac{1}{1 + \kappa_V(\eta)/n} > 0.$$

Similarly, we have

$$\Phi_{s,t}(\eta)(Q_{t,t+\tau}(1)) \leq \Phi_{s,t}(\eta)(V Q_{t,t+\tau}(V)/V) \leq \|\theta(V)\| \kappa_V(\eta).$$

This ends the proof of the left-hand side estimate in (21). The proof of the lemma is completed. \blacksquare

The next result is a variation of [178, lemma 10].

Lemma A.2. *We have the estimates (17). In addition, for any $0 < \delta < 1$ there exists some $0 < \epsilon \leq \epsilon_1$ such that the following uniform compact-approximation estimate holds*

$$\sup_{t \in \mathcal{T}(\tau)} \left\| \overline{Q}_{t,t+\tau} - 1_{\mathcal{V}_\epsilon} \overline{Q}_{t,t+\tau} \right\|_V < \delta \quad \text{and} \quad \sup_{t \in \mathcal{T}(\tau)} \sup_{s \in \mathcal{T}_t(\tau)} \left\| \overline{Q}_{s,t} \right\|_V < \infty. \quad (95)$$

Proof. We start by proving the estimates given in (95). We use the same notation as in the proof of Lemma 4.1. For any $n \geq 1$ such that $\epsilon_n^- := \epsilon_n/\lambda^- < 1$ we have

$$\begin{aligned} & \left\| 1_{\mathcal{V}_{\epsilon_n}^c} \overline{Q}_{s,s+\tau} \right\|_V \\ &= \left\| \overline{Q}_{s,s+\tau} - 1_{\mathcal{V}_{\epsilon_n}} \overline{Q}_{s,s+\tau} \right\|_V = \frac{1}{\lambda_{s,s+\tau}} \left\| (Q_{s,s+\tau}(V)/V) 1_{\mathcal{V}_{\epsilon_n}^c} \right\| < \epsilon_n^-. \end{aligned}$$

This yields the left-hand side estimate in (95).

For the right-hand side, for any $i \in \{0, 1\}^n$ and $1 \leq k \leq n$, we set

$$l(i) := \inf \{1 \leq k < n : (i_k, i_{k+1}) = (1, 1)\}$$

and

$$\{0, 1\}_k^n = \{i \in \{0, 1\}^n : l(i) = k\}$$

with the convention that

$$\{0, 1\}_n^n = \{i \in \{0, 1\}^n : \forall 1 \leq k < n : (i_k, i_{k+1}) \neq (1, 1)\}.$$

Further, set

$$\mathcal{V}_\epsilon(0) = \mathcal{V}_\epsilon^c \quad \text{and} \quad \mathcal{V}_\epsilon(1) = \mathcal{V}_\epsilon.$$

Then we may decompose \bar{Q} as follows:

$$\begin{aligned} \bar{Q}_{s_0, s_n} &= \sum_{i \in \{0,1\}^n} 1_{\mathcal{V}_\epsilon(i_1)} \bar{Q}_{s_0, s_1} \cdots 1_{\mathcal{V}_\epsilon(i_n)} \bar{Q}_{s_0, s_n} \\ &= \sum_{1 \leq k \leq n} \sum_{i \in \{0,1\}_k^n} 1_{\mathcal{V}_\epsilon(i_1)} \bar{Q}_{s_0, s_1} \cdots 1_{\mathcal{V}_\epsilon(i_n)} \bar{Q}_{s_{n-1}, s_n} \end{aligned}$$

Our aim is to obtain suitable bounds for the summands in the above decomposition. To this end, set for any $0 < \epsilon \leq \epsilon_n$

$$\nu_\epsilon(dx) := \frac{\nu(dx) 1_{\mathcal{V}_\epsilon}(x)}{\nu(\mathcal{V}_\epsilon)}.$$

Using this and the notation introduced in (91), we have

$$\begin{aligned} &1_{\mathcal{V}_\epsilon}(x) \bar{Q}_{s, s+\tau} (1_{\mathcal{V}_\epsilon} \bar{Q}_{s+\tau, t}(V)) (x) \\ &\leq \iota_\epsilon \frac{\nu(1_{\mathcal{V}_\epsilon} Q_{s+\tau, t}(V))}{\bar{\eta}_s(Q_{s, s+\tau} Q_{s+\tau, t}(1))} \leq \iota_\epsilon \frac{\nu(1_{\mathcal{V}_\epsilon} Q_{s+\tau, t}(V))}{\bar{\eta}_s(1_{\mathcal{V}_\epsilon} Q_{s, s+\tau} 1_{\mathcal{V}_\epsilon} Q_{s+\tau, t}(1))} \\ &\leq \frac{\iota_\epsilon}{\bar{\eta}_s(\mathcal{V}_\epsilon)} \frac{\nu(1_{\mathcal{V}_\epsilon} Q_{s+\tau, t}(V))}{\nu(1_{\mathcal{V}_\epsilon} Q_{s+\tau, t}(1))} = \frac{\iota_\epsilon / \iota_\epsilon^-}{\bar{\eta}_s(\mathcal{V}_\epsilon)} \Phi_{s+\tau, t}(\nu_\epsilon)(V) \leq \kappa_V(\nu_\epsilon) \frac{\iota_\epsilon / \iota_\epsilon^-}{\bar{\eta}_s(\mathcal{V}_\epsilon)}. \end{aligned}$$

This yields the uniform estimate

$$1_{\mathcal{V}_{\epsilon_n}}(x) \bar{Q}_{s, s+\tau} (1_{\mathcal{V}_{\epsilon_n}} \bar{Q}_{s+\tau, t}(V)) (x) \leq \iota'_{\epsilon_n} := \left(\frac{\iota_{\epsilon_n}}{\iota_{\epsilon_n}^-} \right) \kappa_V(\nu_{\epsilon_n}) \left(1 + \frac{\bar{\eta}(V)}{n} \right).$$

Next fix $1 \leq k < n$. Then one has

$$\begin{aligned} \left\| 1_{\mathcal{V}_\epsilon(i_1)} \bar{Q}_{s_0, s_1} \cdots 1_{\mathcal{V}_\epsilon(i_n)} \bar{Q}_{s_{n-1}, s_n} \right\|_V &\leq \iota'_{\epsilon_n} \left\| 1_{\mathcal{V}_\epsilon(i_1)} \bar{Q}_{s_0, s_1} \cdots 1_{\mathcal{V}_\epsilon(i_k)} \bar{Q}_{s_{k-1}, s_k} \right\|_V \\ &\leq \iota'_{\epsilon_n} \prod_{1 \leq m \leq k} \left\| 1_{\mathcal{V}_\epsilon(i_m)} \bar{Q}_{s_{m-1}, s_m} \right\|_V \end{aligned} \quad (96)$$

$$\begin{aligned} &\leq \iota'_{\epsilon_n} \prod_{1 \leq m \leq k} \|\theta(V)\|^{1_{i_m=1}} (\epsilon_n^-)^{1_{i_m=0}} \\ &= \iota'_{\epsilon_n} \|\theta(V)\|^{\sum_{m=1}^k 1_{i_m=1}} (\epsilon_n^-)^{\sum_{m=1}^k 1_{i_m=0}} \end{aligned} \quad (97)$$

Similarly, for the case $k = n$, we have

$$\left\| 1_{\mathcal{V}_\epsilon(i_1)} \bar{Q}_{s_0, s_1} \cdots 1_{\mathcal{V}_\epsilon(i_n)} \bar{Q}_{s_{n-1}, s_n} \right\|_V \leq \|\theta(V)\|^{\sum_{m=1}^n 1_{i_m=1}} (\epsilon_n^-)^{\sum_{m=1}^n 1_{i_m=0}}. \quad (98)$$

Now note that for any $i \in \{0, 1\}^n$ we have

$$\sum_{1 \leq k \leq n} 1_{i_k=1} \leq \frac{n+1}{2} + \frac{1}{2} \sum_{1 \leq k < n} 1_{(i_k, i_{k+1})=(1,1)}.$$

This follows from the fact that

$$\begin{aligned} n-1 &\geq \sum_{1 \leq k < n} (1_{(i_k, i_{k+1})=(1,0)} + 1_{(i_k, i_{k+1})=(0,1)}) \\ &= \left(2 \sum_{1 \leq k \leq n} 1_{i_k=1} - (1_{i_n=1} + 1_{i_1=1}) \right) - \sum_{1 \leq k < n} 1_{(i_k, i_{k+1})=(1,1)} \\ &\geq 2 \left(\sum_{1 \leq k \leq n} 1_{i_k=1} - 1 \right) - \sum_{1 \leq k < n} 1_{(i_k, i_{k+1})=(1,1)}. \end{aligned}$$

Equivalently, we have

$$\sum_{1 \leq k \leq n} 1_{i_k=0} \geq \frac{n-1}{2} - \frac{1}{2} \sum_{1 \leq k < n} 1_{(i_k, i_{k+1})=(1,1)}.$$

Further observe that for any $i \in \{0, 1\}_k^n$ with $1 \leq k \leq n$ we have

$$\sum_{m=1}^k 1_{(i_m, i_{m+1})=(1,1)} = 0 \implies \sum_{m=1}^k 1_{i_m=1} \leq \frac{k+1}{2} \quad \text{and} \quad \sum_{m=1}^k 1_{i_m=0} \geq \frac{k-1}{2}.$$

From (97) and (98), respectively, we conclude that for any $i \in \{0, 1\}_k^n$ with $1 \leq k < n$ we have

$$\|1_{\mathcal{V}_\epsilon(i_1)} \overline{Q}_{s_0, s_1} \cdots 1_{\mathcal{V}_\epsilon(i_n)} \overline{Q}_{s_{n-1}, s_n}\|_V \leq \iota'_{\epsilon_n} (1 \vee \|\theta(V)\|)^{(k+1)/2} (\epsilon_n^-)^{(k-1)/2}.$$

and for $k = n$, we have

$$\|1_{\mathcal{V}_\epsilon(i_1)} \overline{Q}_{s_0, s_1} \cdots 1_{\mathcal{V}_\epsilon(i_n)} \overline{Q}_{s_{n-1}, s_n}\|_V \leq (1 \vee \|\theta(V)\|)^{(n+1)/2} (\epsilon_n^-)^{(n-1)/2}.$$

We end the proof of the right-hand side estimate in (95) by choosing $n \geq 1$ such that

$$\epsilon_n^- < 1 \wedge (1/\|\theta(V)\|).$$

Now for the estimates in (17), observe that

$$H \leq V \implies \|H\|_V := \sup_{t \in \mathcal{T}(\tau)} \sup_{s \in \mathcal{T}_t(\tau)} \|\overline{Q}_{s,t}(H)\|_V \leq \sup_{t \in \mathcal{T}(\tau)} \sup_{s \in \mathcal{T}_t(\tau)} \|\overline{Q}_{s,t}\|_V < \infty.$$

This yields right-hand side estimate in (17).

The proof of the left-hand side estimate in (17) follows the same lines of arguments as those given for (95), thus it is only sketched. Using the same notation as above, we have

$$\bar{\eta}_{s_0} Q_{s_0, s_n}(1) = \sum_{1 \leq k \leq n} \sum_{i \in \{0,1\}_k^n} \bar{\eta}_{s_0} 1_{\mathcal{V}_\epsilon(i_1)} Q_{s_0, s_1} \cdots 1_{\mathcal{V}_\epsilon(i_n)} Q_{s_{n-1}, s_n}(1).$$

For any $i \in \{0,1\}_k^n$ for some $1 \leq k < n$, we also have

$$\begin{aligned} & \bar{\eta}_{s_0} 1_{\mathcal{V}_\epsilon(i_1)} Q_{s_0, s_1} \cdots 1_{\mathcal{V}_\epsilon(i_n)} Q_{s_{n-1}, s_n}(1) \\ & \leq \iota_\epsilon \bar{\eta}_{s_0} (V 1_{\mathcal{V}_\epsilon(i_1)} Q_{s_0, s_1}^V \cdots 1_{\mathcal{V}_\epsilon(i_k)} Q_{s_{k-1}, s_k}^V (1_{\mathcal{V}_\epsilon})) \nu(1_{\mathcal{V}_\epsilon} Q_{s_{k+1}, s_n}(1)) \\ & \leq \iota_\epsilon \bar{\eta}(V) (1 \vee \|\theta(V)\|)^{(k+1)/2} \epsilon^{(k-1)/2} \nu(1_{\mathcal{V}_\epsilon} Q_{s_{k+1}, s_n}(1)), \end{aligned}$$

with $Q_{s,t}^V(f) := Q_{s,t}(fV)/V$. For any $\epsilon \leq \epsilon_1$ we have

$$\begin{aligned} & 1_{\mathcal{V}_\epsilon}(x) Q_{s_0, s_n}(H)(x) \\ & \geq 1_{\mathcal{V}_\epsilon}(x) 1_{\mathcal{V}_{\epsilon_1}}(x) (Q_{s_0, s_1} 1_{\mathcal{V}_{\epsilon_1}} Q_{s_1, s_n}(H))(x) \geq \iota_\epsilon^- 1_{\mathcal{V}_\epsilon}(x) \nu(1_{\mathcal{V}_{\epsilon_1}} Q_{s_1, s_n}(H)). \end{aligned}$$

In addition, we have

$$\begin{aligned} & \nu(1_{\mathcal{V}_{\epsilon_1}} Q_{s_1, s_n}(H)) \\ & \geq \nu(1_{\mathcal{V}_{\epsilon_1}} Q_{s_1, s_2} 1_{\mathcal{V}_\epsilon} \cdots Q_{s_{k-1}, s_k} 1_{\mathcal{V}_\epsilon} Q_{s_k, s_{k+1}} 1_{\mathcal{V}_\epsilon} Q_{s_{k+1}, s_n}(H)) \\ & \geq \iota_\epsilon^- \nu(1_{\mathcal{V}_{\epsilon_1}} Q_{s_1, s_2} 1_{\mathcal{V}_\epsilon} \cdots Q_{s_{k-1}, s_k} 1_{\mathcal{V}_\epsilon}) \nu(1_{\mathcal{V}_\epsilon} Q_{s_{k+1}, s_n}(H)). \end{aligned}$$

This yields the estimate

$$1_{\mathcal{V}_\epsilon}(x) Q_{s_0, s_n}(H) \geq 1_{\mathcal{V}_\epsilon}(x) (\iota_\epsilon^-)^2 (\iota_{\epsilon_1}^- \nu(\mathcal{V}_{\epsilon_1}))^k \nu(1_{\mathcal{V}_\epsilon} Q_{s_{k+1}, s_n}(H)).$$

from which we check that

$$\begin{aligned} & 1_{\mathcal{V}_\epsilon}(x) (\bar{\eta}_{s_0} 1_{\mathcal{V}_\epsilon(i_1)} Q_{s_0, s_1} \cdots 1_{\mathcal{V}_\epsilon(i_n)} Q_{s_{n-1}, s_n}(1)) / Q_{s_0, s_n}(H)(x) \\ & \leq (\iota_\epsilon / \epsilon) (\bar{\eta}(V) / \kappa_H^-(\nu_\epsilon)) (\epsilon (1 \vee \|\theta(V)\|) / (\iota_{\epsilon_1}^- \nu(\mathcal{V}_{\epsilon_1}))^2)^{k/2}. \end{aligned}$$

The end of the proof of the left-hand side assertion in (17) now follows word-for-word the same lines of arguments as the proof of the right-hand side estimate in (17), thus it is skipped. The proof of the lemma is now completed. \blacksquare

A.2 Proof of Lemma 3.1

Proof. Observe that for any f such that $\|f\|_V \leq 1$, we have

$$(\Psi_{1/h}(\mu_1) - \Psi_{1/h}(\mu_2))(f) = \frac{1}{\mu_1(1/h)} (\mu_1 - \mu_2)(g)$$

with the function

$$g := (1/h)(f - \Psi_{1/h}(\mu_2)(f)) \in \mathcal{B}_{V/h}(E).$$

Note that

$$\frac{g}{V/h} = \frac{f - \Psi_{1/h}(\mu_2)(f)}{V} = \frac{f}{V} - \frac{\Psi_{1/h}(\mu_2)(Vf/V)}{V} \implies \|g\|_{V/h} \leq 1 + \frac{\mu_2(V/h)}{\mu_2(1/h)}.$$

This ends the proof of the first assertion. Now, for any $\|f\|_{V/h} \leq 1$ we have

$$(\Psi_h(\mu_1) - \Psi_h(\mu_2))(f) = \frac{1}{\mu_1(h)} (\mu_1 - \mu_2)(g)$$

with the function

$$\begin{aligned} g &:= h(f - \Psi_h(\mu_2)(f)) \\ \implies \frac{|g|}{V} &\leq 1 + \frac{h}{V} \frac{\mu_2(V)}{\mu_2(h)} \leq 1 + \frac{\mu_2(V)}{\mu_2(h)} \implies \|g\|_V \leq 1 + \frac{\mu_2(V)}{\mu_2(h)}. \end{aligned}$$

This ends the proof of the lemma. ■

A.3 Proof of Lemma 3.2

Proof. We set $(s, t) = (1, 2)$ and $P = P_{1,2}$, and $V_i^\rho := 1/2 + \rho V_i$, with $i \in \{1, 2\}$ and $\rho \in]0, 1[$. We also consider the function Δ_ρ on E_1^2 defined for any $(x, y) \in E_1^2$ by

$$\begin{aligned} \Delta_\rho(x, y) &:= \frac{\|\delta_x P - \delta_y P\|_{V_2^\rho}}{\|\delta_x - \delta_y\|_{V_1^\rho}} \\ &\leq \frac{\|\delta_x P - \delta_y P\|_{tv}}{1 + \rho(V_1(x) + V_1(y))} + \frac{\rho(P(V_2)(x) + P(V_2)(y))}{1 + \rho(V_1(x) + V_1(y))}. \end{aligned}$$

Using the left-hand side estimate in (72), we have

$$\begin{aligned} P(V_2)(x) + P(V_2)(y) &\leq \epsilon(V_1(x) + V_1(y)) + 2 \\ &= (V_1(x) + V_1(y)) \left(\epsilon + \frac{2}{(V_1(x) + V_1(y))} \right). \end{aligned}$$

When $V_1(x) + V_1(y) \geq r$ this yields the estimate

$$\begin{aligned} \Delta_\rho(x, y) &\leq \frac{1}{1 + \rho(V_1(x) + V_1(y))} + \frac{\rho(V_1(x) + V_1(y))}{1 + \rho(V_1(x) + V_1(y))} \left(\epsilon + \frac{2}{r} \right) \\ &= 1 - \left(1 - \frac{1}{1 + \rho(V_1(x) + V_1(y))} \right) \left(1 - \left(\epsilon + \frac{2}{r} \right) \right) \leq 1 - d_\rho^1, \end{aligned}$$

with

$$d_\rho^1 := \left(1 - \frac{1}{1 + 2\rho r}\right) \left(1 - \left(\epsilon + \frac{2}{r}\right)\right).$$

Recalling that $V_i \geq 1$ we have

$$P(V_2)/V_1 \leq 1 + \epsilon.$$

This implies that for any (x, y) such that $V_1(x) + V_1(y) \leq r$ we have

$$\frac{\rho V_1(x)(P(V_2)(x)/V_1(x)) + \rho V_1(y)(P(V_2)(y)/V_1(y))}{1 + \rho(V_1(x) + V_1(y))} \leq (1 + \epsilon) \frac{\rho r}{1 + \rho},$$

where we have also used the fact that $\rho < 1$. This yields the estimate

$$\Delta_\rho(x, y) \leq 1 - d_\rho^2 := \frac{1 - \alpha(r)}{1 + 2\rho} + \frac{\rho r}{1 + 2\rho}(1 + \epsilon).$$

Observe that

$$\rho \leq \frac{1}{1 + \epsilon} \frac{\alpha(r)}{r} \implies 1 - d_\rho^2 \leq \frac{1}{1 + 2\rho} < 1.$$

Recalling that $\alpha(r) > 0$ for some $r > r_\epsilon := 2/(1 - \epsilon)$ we have

$$\frac{2}{r} < 1 - \epsilon \implies 0 < d_\rho^1 = \left(1 - \frac{1}{1 + 2\rho r}\right) \left(1 - \left(\epsilon + \frac{2}{r}\right)\right) < 1.$$

Choosing

$$r = r_\epsilon(1 + \delta_1) \quad \text{and} \quad \rho = \frac{\delta_2}{1 + \epsilon} \frac{\alpha(r)}{2r}$$

for some $0 < \delta_1 \leq 1$ and $0 < \delta_2 \leq 1$ we have

$$0 < \frac{\delta_2}{3} \alpha(r) (1 - \epsilon) \frac{\delta_1}{2} < d_\rho^1 = \frac{\alpha(r)\delta_2}{(1 + \epsilon) + \alpha(r)\delta_2} (1 - \epsilon) \frac{\delta_1}{1 + \delta_1} < 1,$$

which implies that

$$1 - d_\rho^1 \leq 1 - \alpha(r) (1 - \epsilon) \frac{\delta_1 \delta_2}{6}$$

and

$$1 - d_\rho^2 = \frac{1 - \alpha(r) + \rho r(1 + \epsilon)}{1 + 2\rho} = \frac{1 - \alpha(r)(1 - \delta_2/2)}{1 + 2\rho} \leq 1 - \alpha(r)(1 - \delta_2/2).$$

Then setting

$$\delta_1 = \frac{6}{1 - \epsilon} \frac{1 - \delta_2/2}{\delta_2}$$

we can check that

$$(1 - d_\rho^1) \vee (1 - d_\rho^2) \leq 1 - \alpha(r)(1 - \delta_2/2).$$

For any $0 < \delta \leq 1$ we set

$$r_\epsilon(\delta) := r_0 \vee r_\epsilon \left(1 + \frac{1}{1-\epsilon} \left(3 + 6 \left(\frac{1}{\delta} - 1 \right) \right) \right) \quad \rho_\epsilon(\delta) := \frac{\delta}{1+\epsilon} \frac{\alpha(r_\epsilon(\delta))}{2r_\epsilon(\delta)}.$$

We conclude that

$$\begin{aligned} \beta_{V_1^{\delta,\epsilon}, V_2^{\delta,\epsilon}}(P) &= \sup_{x,y \in E} \frac{\|\delta_x P - \delta_y P\|_{V_2^{\delta,\epsilon}}}{V_1^{\delta,\epsilon}(x) + V_1^{\delta,\epsilon}(y)} \leq 1 - (d_{\rho_\epsilon(\delta)}^1 \wedge d_{\rho_\epsilon(\delta)}^2) \\ &\leq 1 - \alpha(r_\epsilon(\delta))(1 - \delta/2). \end{aligned}$$

This ends the proof of the lemma. ■

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References

- [1] D. Aldous and U. Vazirani. Go with the winners algorithms; in Proc. 35th IEEE Sympos. on Foundations of Computer Science (1994)
- [2] A. Arlotto, J.M. Steele. A central limit theorem for temporally nonhomogenous markov chains with applications to dynamic programming. *Mathematics of Operations Research*, vol. 41, no. 4, pp. 1448–1468 (2016).
- [3] M. Arnaudon, P. Del Moral. A duality formula and a particle Gibbs sampler for continuous time Feynman-Kac measures on path spaces. *Electronic Journal of Probability*, vol. 25, pp. 1–54 (2020).
- [4] A. Asselah, P.A. Ferrari, P. Groisman, P. Quasistationary distributions and Fleming-Viot processes in finite spaces”. *Journal of Applied Probability*, vol. 48, no. 2, pp. 322–332 (2011).
- [5] A. Asselah, P.A. Ferrari, P. Groisman, M. Jonckheere, Fleming-Viot selects the minimal quasi-stationary distribution: the Galton-Watson case, *Ann. Inst. Henri Poincaré, Probab. Stat.*, vol. 52, no. 2, pp. 647–668 (2016).
- [6] D. G. Aronson. Bounds for the fundamental solution of a parabolic equation. *Bull. Amer. Math. Soc.* 73, pp. 890–896 (1967).
- [7] R. Assaraf, M. Caffarel, and A. Khelif. Diffusion Monte Carlo methods with a fixed number of walkers. *Physical Review E*, vol. 61, no. 4-4566, (2000).

- [8] R. Atar, O. Zeitouni. Exponential stability for nonlinear filtering. *Annales de l’Institut Henri Poincaré (B) Probability and Statistics*. Vol. 33. No. 6 (1997).
- [9] K. E. Atkinson. *A Survey of Numerical Methods for the Solution of Fredholm Integral Equations of the Second Kind*. SIAM, Philadelphia (1976).
- [10] M. Baudel, A. Guyader, T. Lelièvre. On the Hill relation and the mean reaction time for metastable processes. *Arxiv no:2008.09790*, August (2020).
- [11] V. Bansaye, B. Cloez, P. Gabriel. Ergodic behavior of non-conservative semi-groups via generalized Doeblin’s conditions. *Acta Applicandae Mathematicae*, vol. 166, no. 1, pp. 29–72 (2020).
- [12] V. Bansaye, B. Cloez, P. Gabriel, A. Marguet. A non-conservative Harris ergodic theorem. *arXiv preprint arXiv:1903.03946* (2019).
- [13] V. Bansaye. Ancestral lineages and limit theorems for branching Markov chains in varying environment. *Journal of Theoretical Probability*, vol. 32, no. 1, pp. 249–281 (2019).
- [14] M. Benaim, B. Cloez, F. Panloup. Stochastic approximation of quasi-stationary distributions on compact spaces and applications. *The Annals of Applied Probability*, vol. 28, no. 4, pp. 2370–2416 (2018).
- [15] G. Birkhoff. Extensions of Jentzsch’s theorem. *Transactions of the American Mathematical Society*, vol. 85, no.1, pp. 219–227 (1957).
- [16] A.N. Bishop, P. Del Moral, K. Kamatani, B. Remillard. On one-dimensional Riccati diffusions. *Annals of Applied Probability*. volume 29, Number 2, pp. 1127–1187 (2019).
- [17] J. Blanchet, P. Glynn, and S. Zheng. Analysis of a stochastic approximation algorithm for computing quasi-stationary distributions. *Advances in Applied Probability*, vol. 48, no. 3, pp. 792–811 (2016).
- [18] V.I. Bogachev. *Weak convergence of measures*. Providence, RI: American Mathematical Society (2018).
- [19] K. Bogdan, K. Szczypkowski. Gaussian estimates for Schrödinger perturbations. *arXiv preprint arXiv:1301.4627* (2013).
- [20] P. Bougerol. Comparaison des exposants de Lyapounov des processus markoviens multiplicatifs. *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques*, vol. 24, no. 4, pp. 439–489 (1988).
- [21] P. Bougerol. Théorèmes limites pour les systèmes linéaires à coefficients markoviens. *Probability Theory and Related Fields*, vol. 78, no. 2, pp. 193–221, (1988).

- [22] S.P. Brooks and G.O. Roberts. Convergence assessment techniques for Markov chain Monte Carlo. *Statistics and Computing*, vol. 8, pp. 319–335 (1998).
- [23] I. Bucur and G. Paltineanu. De Branges Type Lemma and Approximation in Weighted Spaces *Mediterr. J. Math.*, vol. 18, no. 120 (2021)
- [24] A. Budhiraja, D. Ocone. Exponential stability of discrete-time filters for bounded observation noise. *Systems & Control Letters*, vol. 30, no.4, pp. 185–193 (1997).
- [25] A. Budhiraja. Asymptotic stability, ergodicity and other asymptotic properties of the nonlinear filter. *Annales de l’IHP Probabilités et statistiques*. Vol. 39. No. 6. (2003).
- [26] M.C. Buonaura, S. Sorella. Numerical study of the two-dimensional Heisenberg model using a Green function Monte Carlo technique with a fixed number of walkers. *Physical Review B*, vol. 57, no. 18, 11446 (1998).
- [27] K. Burdzy, R. Holyst, and P. March. A Fleming-Viot particle representation of the Dirichlet Laplacian. *Comm. Math. Phys.*, vol. 214, no. 3, pp. 679–703 (2000).
- [28] K. Burdzy, R. Holyst, D. Ingerman, P. March. Configurational transition in a Fleming-Viot-type model and probabilistic interpretation of laplacian eigenfunctions. *J. Phys. A*, vol. 29,no. 29, pp. 2633–2642 (1996).
- [29] M. Caffarel, P. Claverie. Treatment of the Schrödinger equation through a Monte Carlo method based upon the generalized Feynman-Kac formula. *Journal of Statistical Physics*, vol. 43, no. 5, pp. 797–801 (1986).
- [30] M. Caffarel, P. Claverie. Development of a pure diffusion quantum Monte Carlo method using a full generalized Feynman-Kac formula. I. Formalism & II Applications to simple systems. *The Journal of chemical physics*, vol. 88, no. 2, pp. 1088–1099 and pp. 1100–1109 (1988).
- [31] M. Caffarel. Stochastic methods in quantum mechanics. In *Numerical Determination of the Electronic Structure of Atoms, Diatomic and Polyatomic Molecules* (pp. 85-105). Springer, Dordrecht (1989).
- [32] M. Caffarel, R. Assaraf. A pedagogical introduction to quantum Monte Carlo. In *Mathematical models and methods for ab initio Quantum Chemistry in Lecture Notes in Chemistry*, eds. M. Defranceschi and C.Le Bris, Springer p.45 (2000).
- [33] M. M. Castro, J.S.W. Lamb, G.O. Mendez, M. Rasmussen. Existence and uniqueness of quasi-stationary and quasi-ergodic measures for absorbing Markov processes. A Banach space approach. [Arxiv-2111.13791](https://arxiv.org/abs/2111.13791) November (2021)
- [34] P. Cattiaux, P. Collet, A. Lambert, S. Martinez, S. Méléard, and J. San Martin. Quasi-stationary distributions and diffusion models in population dynamics. *Ann. Probab.*, vol. 37, no. 5, pp. 1926–1969 (2009).

- [35] P. Cattiaux and S. Méléard. Competitive or weak cooperative stochastic Lotka-Volterra systems conditioned to non-extinction. *J. Math. Biology*, vol. 60, no. 6, pp. 797–829 (2010).
- [36] J. A. Cavender. Quasi-stationary distributions of birth-and-death processes. *Adv. Appl. Probab.*, vol. 10, no. 3, pp. 570–586 (1978).
- [37] N. Champagnat, K.A. Coulibaly-Pasquier, D. Villemonais. Criteria for exponential convergence to quasi-stationary distributions and applications to multi-dimensional diffusions. In *Séminaire de Probabilités XLIX* (pp. 165–182). Springer, (2018).
- [38] N. Champagnat, D. Villemonais. Exponential convergence to quasi-stationary distribution and Q -process. *Probability Theory and Related Fields*, Springer Verlag, vol. 164, no. 1, pp. 243–283 (2016).
- [39] N. Champagnat, D. Villemonais. Population processes with unbounded extinction rate conditioned to non-extinction. *ArXiv:1611.03010* (2016).
- [40] N. Champagnat, D. Villemonais. Uniform convergence to the Q -process. *Electronic Communications in Probability*, vol. 22, pp. 1–7 (2017).
- [41] N. Champagnat, D. Villemonais. Practical criteria for R -positive recurrence of unbounded semigroups. *Electronic Communications in Probability*, vol. 25, pp. 1–11 (2020).
- [42] N. Champagnat, D. Villemonais. General criteria for the study of quasi-stationarity. *Arxiv:1712.08092* (2018).
- [43] N. Champagnat, D. Villemonais. Lyapunov criteria for uniform convergence of conditional distributions of absorbed Markov processes. *Stochastic Processes and their Applications*, (Arxiv) vol. 135, pp 51-74 (2021).
- [44] N. Champagnat, D. Villemonais. Uniform convergence of penalized time-inhomogeneous Markov processes. *ESAIM: Probability and Statistics*, vol. 22, pp 129–162 (2018).
- [45] N. Champagnat and D. Villemonais, Convergence of the Fleming-Viot process toward the minimal quasi-stationary distribution, *ALEA*, *Lat. Am. J. Probab. Math. Stat.*, vol. 18, no. 1, pp. 1–15 (2021).
- [46] J.R. Chazottes, P. Collet, S. Méléard. On time scales and quasi-stationary distributions for multitype birth-and-death processes. *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques*, vol. 55, no. 4 (2019).
- [47] J. Chen, L. Haitao, J. Siqu. Some limit theorems for absorbing Markov processes.” *Journal of Physics A: Mathematical and Theoretical*, vol. 45, no. 34, 345003 (2012).

- [48] P. Chigansky, R. Liptser. Stability of nonlinear filters in nonmixing case. *The Annals of Applied Probability*, vol. 14, no. 4, pp. 2038–2056 (2004).
- [49] P. Chigansky, R. Liptser, and R. Van Handel. Intrinsic methods in filter stability. *Handbook of Nonlinear Filtering* (2009).
- [50] P. Chigansky. Stability of nonlinear filters: A survey. Lecture notes, Petropolis, Brazil (2006).
- [51] J.M.C. Clark, D. L. Ocone, and C. Coumarbatch. Relative entropy and error bounds for filtering of Markov processes. *Mathematics of Control, Signals and Systems*, vol. 12, no. 4, pp. 346–360 (1999).
- [52] B. Cloez, J. Corujo. Uniform in time propagation of chaos for a Moran model. arXiv preprint arXiv:2107.10794 (2021).
- [53] B. Cloez and M.-N. Thai, Quantitative results for the Fleming-Viot particle system and quasi-stationary distributions in discrete space, *Stochastic Process. Appl.*, vol. 126, no. 3, pp. 680–702 (2016).
- [54] B. Cloez and M.-N. Thai, Fleming-Viot processes: two explicit examples, *ALEA Lat. Am. J. Probab. Math. Stat.*, vol. 13, no. 1, pp. 337–356 (2016).
- [55] P. Collet, S. Martinez, J. San Martin. Quasi-stationary distributions: Markov chains, diffusions and dynamical systems. Springer Science & Business Media (2012).
- [56] P. Collet, S. Martinez, S. Méléard, J. San Martin. Quasi-stationary distributions for structured birth and death processes with mutations. *Probability Theory and Related Fields*, vol. 151, pp. 191–231 (2011).
- [57] D. Crisan, K. Heine. Stability of the discrete time filter in terms of the tails of noise distributions. *Journal of the London Mathematical Society*, vol. 78, no. 2, pp. 441–458 (2008).
- [58] G. Da Prato, M. Fuhrman, P. Malliavin. Asymptotic ergodicity of the process of conditional law in some problem of nonlinear filtering. *Journal of Functional Analysis*, vol. 164, no. 2, pp. 356–377 (1999).
- [59] J. N. Darroch, E. Seneta. On quasi-stationary distributions in absorbing discrete-time finite Markov chains. *J. Appl. Probability*, vol. 2, pp. 88–100, (1965).
- [60] J.N.Darroch, E.Seneta. On quasi-stationary distributions in absorbing continuous-time finite Markov chains. *J. Appl. Probab.*, vol.4, pp. 192–196 (1967).
- [61] P. Del Moral, D. Villemonais. Exponential mixing properties for time inhomogeneous diffusion processes with killing. 2014. hal-01083297v1. *Bernoulli*, vol. 24, no. 2, pp. 1010–1032 (2018).

- [62] P. Del Moral. Nonlinear Filtering: Interacting Particle Solution. *Markov Processes and Related Fields*, vol. 2, no. 4, pp. 555–580 (1996).
- [63] P. Del Moral. *Feynman-Kac formulae*. Springer, New York (2004).
- [64] P. Del Moral. Mean field simulation for Monte Carlo integration. CRC press (2013).
- [65] P. Del Moral, A. Doucet. Particle Motions in Absorbing Medium with Hard and Soft Obstacles. *Stochastic Analysis and Applications*. vol. 22, no. 5, pp. 1175–1207 (2004).
- [66] P. Del Moral, A. Doucet, A. Jasra. Sequential monte carlo samplers. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, vol. 68, no. 3, pp. 411–436 (2006).
- [67] P. Del Moral and L. Miclo. Branching and interacting particle systems approximations of Feynman-Kac formulae with applications to non-linear filtering. In *Séminaire de Probabilités, XXXIV*, volume 1729, pp. 1–145 (2000).
- [68] P. Del Moral, L. Miclo. A Moran particle system approximation of Feynman-Kac formulae. *Stochastic processes and their applications*, vol. 86, no. 2, pp. 193–216 (2000).
- [69] P. Del Moral, A. Guionnet. On the stability of interacting processes with applications to filtering and genetic algorithms. *Publication du Laboratoire de Statistique et Probabilitb 3-98*, Université Paul Sabatier, Toulouse (1998). *Annales de l’Institut Henri Poincaré Probab. & Statist.* vol. 37, No. 2, pp. 155–194 (2001).
- [70] P. Del Moral, A. Guionnet. On the stability of measure valued processes with applications to filtering. *Comptes Rendus de l’Académie des Sciences-Series I-Mathematics*, vol. 329, no. 5, pp. 429–434 (1999).
- [71] P. Del Moral, E. Horton. Quantum harmonic oscillators and Feynman-Kac path integrals for linear diffusive particles. arXiv preprint arxiv:2106.14592 (2021).
- [72] P. Del Moral, E. Horton. A theoretical analysis of one-dimensional discrete generation ensemble Kalman particle filters. arXiv preprint arxiv:2107.01855 (2021).
- [73] P. Del Moral, A. Jasra. A Note on Random Walks with Absorbing barriers and Sequential Monte Carlo Methods. *Stochastic Analysis and Applications* vol.36, no. 3, pp. 413–442 (2018).
- [74] P. Del Moral, L. Miclo. Genealogies and increasing propagation of chaos for Feynman-Kac and genetic models. *The Annals of Applied Probability*, vol. 11, no. 4, pp. 1166–1198 (2001).

- [75] P. Del Moral, L. Miclo. Particle approximations of Lyapunov exponents connected to Schrödinger operators and Feynman-Kac semigroups. *ESAIM: Probability and Statistics*, no. 7, pp. 171–208 (2003).
- [76] P. Del Moral, L. Miclo. On the stability of nonlinear Feynman-Kac semigroups. *Annales de la Faculté des sciences de Toulouse: Mathématiques, serie 6*, t. 11, no.2, pp. 135–175 (2002).
- [77] P. Del Moral, L. Miclo. Self-interacting Markov chains. *Stochastic Analysis and Applications*, vol. 24, no. 3, pp. 615–660 (2006).
- [78] P. Del Moral, L. Miclo. On convergence of chains with occupational self-interactions. *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, 460(2041), pp. 325–346 (2004).
- [79] P. Del Moral, M. Ledoux, L. Miclo. On contraction properties of Markov kernels. *Probability theory and related fields*, vol. 126, no. 3, pp. 395–420 (2003).
- [80] P. Del Moral, S. Penev. *Stochastic Processes: From Applications to Theory*. CRC Press, (2017).
- [81] P. Del Moral, D. Villemonais. Exponential mixing properties for time inhomogeneous diffusion processes with killing. *Bernoulli*, vol. 24, no. 2, pp. 1010–1032 (2018).
- [82] A. Dembo and O. Zeitouni. *Large Deviations Techniques and Applications*, volume 38 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, Berlin (2010)
- [83] P. Diaconis and D. Freedman. On the consistency of Bayes estimates (with discussion). *The Annals of Statistics*, vol. 14, no. 1, pp. 1–67, (1986).
- [84] P. Diaconis and L. Saloff-Coste. What do we know about the Metropolis algorithm? *27th Annual ACM Symposium on the Theory of Computing. J. Comput. System Sci.*, vol. 57, no. 1, pp. 20–36 (1998).
- [85] P. Diaconis, K. Khare, and L. Saloff-Coste. Gibbs sampling, exponential families and orthogonal polynomials. *Statist. Sci.*, vol. 23, no. 2, pp. 151–200 (2008).
- [86] W. Doeblin, Sur les propriétés asymptotiques de mouvements régis par certains types de chaînes simples. *Bull. Soc. Math. Roumaines des Sciences*, vol. 39, no. 1, pp. 57–115, vol. 39, no. 2, pp. 3–61 (1937).
- [87] R.L. Dobrushin, Central limit theorem for nonstationary Markov chains, I,II, *Theory Probab. Appl.*, vol. 1, no. 1,4, pp. 66–80, pp. 330–385 (1956).
- [88] R.L. Dobrushin, Prescribing a system of random variables by conditional distributions, *Theor. Prob. Appl.*, vol. 15, no. 3 (1970).

- [89] M. D. Donsker and S. R. S. Varadhan. On a variational formula for the principal eigenvalue for operators with maximum principle. *Proc. Natl. Acad. Sci.*, vol. 72, no. 3, pp. 780–783 (1975).
- [90] R. Douc, E. Moulines, J. Olsson. Long-term stability of sequential Monte Carlo methods under verifiable conditions. *The Annals of Applied Probability*, vol. 24, no. 5, pp. 1767–1802 (2014).
- [91] R. Douc, E. Moulines, Y. Ritov. Forgetting of the initial condition for the filter in general state-space hidden Markov chain: a coupling approach. *Electronic Journal of Probability*, vol. 14, pp. 27–49 (2009).
- [92] P. Drnovsek, M. Kandic. Ideal-triangularizability of semigroups of positive operators. *Integral Equations and Operator Theory*, vol. 64, no. 4, pp. 539–552 (2009).
- [93] Y. Du. Order structure and topological methods in nonlinear partial differential equations: Vol. 1: Maximum principles and applications (Vol. 2). World Scientific (2006).
- [94] K.J. Engel, R. Nagel, One-parameter semigroups for linear evolution equations. In *Semigroup forum Springer-Verlag*, vol. 63, no. 2, pp. 278–280 (2001).
- [95] J. Feng and T. G. Kurtz. Large Deviations for Stochastic Processes, volume 131 of *Mathematical Surveys and Monographs*. American Mathematical Soc. (2006).
- [96] P.A. Ferrari, N. Maric. Quasi stationary distributions and Fleming-Viot processes in countable spaces. *Electron. J. Probab.*, vol. 12, pp. 684–702 (2007).
- [97] P.A. Ferrari, H. Kesten, S. Martinez, D. Picco. Existence of quasi-stationary distributions: a renewal dynamical approach. *Ann. Probab.*, vol.23, pp. 501–521 (1995).
- [98] G. Ferré. Large Deviations Theory in Statistical Physics: Some Theoretical and Numerical Aspects. Diss. Université Marne La Vallée (2019).
- [99] G. Ferré, M. Rousset, and G. Stoltz. Large deviations of empirical measures of diffusions in weighted topologies.” *Electronic Journal of Probability*, vol. 25, pp. 1–52 (2020).
- [100] G. Ferré, M. Rousset, and G. Stoltz. More on the long time stability of Feynman-Kac semigroups. *Stochastics and Partial Differential Equations: Analysis and Computations*, vol. 9, no. 3, pp. 630–673 (2021).
- [101] H. Furstenberg and H. Kesten. Product of random matrices. *The Annals of Mathematical Statistics*, vol. 31, no. 2, pp. 457–469 (1960).

- [102] N. Gao, V. G. Troitsky Irreducible Semigroups of Positive Operators on Banach Lattices. *Linear and Multilinear Algebra*, vol. 62, no. 1, pp. 74–95 (2014).
- [103] M. Gerber, N. Whiteley. Stability with respect to initial conditions in V-norm for nonlinear filters with ergodic observations. *Journal of Applied Probability*, vol. 54; no. 1, pp. 118–133 (2017).
- [104] G. L. Gong, M. P. Qian, and Z. X. Zhao. Killed diffusions and their conditioning. *Probab. Theory Related Fields*, vol. 80, no. 1, pp. 151–167 (1988).
- [105] P. Good. The limiting behavior of transient birth and death processes conditioned on survival. *J. Austral. Math. Soc.*, vol. 8, pp. 716–722 (1968).
- [106] N. J. Gordon, D. J. Salmond, A. F. M. Smith. Novel approach to nonlinear/non-Gaussian Bayesian state estimation. *IEE PROCEEDINGS-*, Vol. 140, pp. 107–113 (1993).
- [107] I.G. Graham, I. H. Sloan. On the compactness of certain integral operators. *Journal of Mathematical Analysis and Applications*, vol. 68, no. 2, pp. 580–594 (1979).
- [108] P. Grassberger. Pruned-enriched Rosenbluth method: Simulations of polymers of chain length up to 1000000, *Physical Review E*, Vol. 56, pp. 3682–3693 (1997).
- [109] P. Grassberger, H. Frauenkron, W. Nadler. PERM: A Monte Carlo strategy for simulating polymers and other things, *Monte Carlo Approach to Biopolymers and Protein Folding*, ed. P. Grassberger, G. T. Barkema and W. Nadler (World Scientific, Singapore, 1998) pp. 301–315 (1998).
- [110] P. Grassberger, Go with the winners: A general Monte Carlo strategy. *Computer Physics Communications*, vol. 147, no.1–2, pp. 64–70 (2002).
- [111] D. Guibourg, L. Hervé and J. Ledoux. Quasi-compactness of Markov kernels on weighted-supremum spaces and geometrical ergodicity. *arXiv:1110.3240* (2011).
- [112] A. Guillin, B. Nectoux, and L. Wu, Quasi-stationary distribution for strongly Feller Markov processes by Lyapunov functions and applications to hypoelliptic Hamiltonian systems, working paper or preprint, December (2020).
- [113] Y. Guivarc’h. Exposants caractéristiques des produits de matrices aléatoires en dépendance markovienne. In H. Heyer, editor, *Probability Measures on Groups VII*, Oberwolfach 1983, pages 161–181. Volume 1064 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin (1984).
- [114] M. Hairer, J. C. Mattingly. Yet another look at Harris’ ergodic theorem for Markov chains. *Seminar on Stochastic Analysis, Random Fields and Applications VI*. Springer, Basel (2011).

- [115] G. He, Hanjun Zhang, and Yixia Zhu. On the quasi-ergodic distribution of absorbing Markov processes. *Statistics & Probability Letters*, vol. 149, pp. 116–123 (2019).
- [116] K. Heine. Stability of the discrete time filter in terms of the tails of noise distributions. *ESAIM: Proceedings*. Vol. 19. EDP Sciences (2007).
- [117] A. Hening, W. Qi, Z. Shen, Y. Yi . Quasi-stationary distributions of multi-dimensional diffusion processes. arXiv preprint arXiv:2102.05785 (2021).
- [118] H. Hennion. Limit theorems for products of positive random matrices. *The Annals of Probability*, vol. 25, no. 4, pp. 1545–1587, (1997).
- [119] H. Hennion. Quasi-compactness and absolutely continuous kernels. *Probability theory and related fields*, vol. 139, no. 3–4, pp. 451–471 (2007).
- [120] J. H. Hetherington, Observations on the statistical iteration of matrices. *Physical Review A*, 30(5), 2713 (1984).
- [121] E. Horton. Stochastic Analysis of the neutron transport equation. PhD Uni. Bath, November (2019).
- [122] E. Horton, A. Kyprianou, D. Villemonais. Stochastic methods for the neutron transport equation I: Linear semigroup asymptotics. *The Annals of Applied Probability* vol. 30, no. 6, pp. 2573–2612 (2020).
- [123] M. Hürzeler. Statistical Methods for General State-Space Models. Ph.D. Thesis, Department of Mathematics, ETH Zurich, Zürich (1998).
- [124] A. Jasra. On the behaviour of the backward interpretation of Feynman-Kac formulae under verifiable conditions. *J. Applied Probability*. vol. 52, no. 2, pp. 339–359 (2015).
- [125] A. Jasra, A. Doucet. Stability of sequential Monte Carlo samplers via the Foster-Lyapunov condition. *Statistics & probability letters*, vol. 78, no. 17, pp. 3062–3069 (2008).
- [126] L. Journal, P. Monmarché. Convergence of a particle approximation for the quasi-stationary distribution of a diffusion process: uniform estimates in a compact soft case. arXiv preprint arXiv:1910.05060 (2019).
- [127] T. Kaijser. A limit theorem for partially observed Markov chains. *The Annals of Probability*, vol. 3, no. 4, pp. 677–696 (1975).
- [128] S. Karlin and J. L. McGregor. The differential equations of birth-and- death processes, and the Stieltjes moment problem. *Trans. Amer. Math. Soc.*, vol. 85, pp. 489–546, (1957).

- [129] G. Kitagawa. Monte Carlo filter and smoother for non-Gaussian nonlinear state space models. *Journal of Computational and Graphical Statistics*, Vol. 5, pp. 1–25 (1996).
- [130] G. Kitagawa. A self-organizing state-space model. *Journal of American Statistical Association*, Vol. 93, pp. 1203–1215 (1998).
- [131] M.L. Kleptsyna, A.Y. Veretennikov. On robustness of discrete time optimal filters. *Mathematical Methods of Statistics*, vol. 25, no. 3, pp. 207–218 (2016).
- [132] Kleptsyna, M. L., and A. Yu Veretennikov. On continuous time ergodic filters with wrong initial data. *Theory of Probability & Its Applications* vol. 53, no. 2, pp. 269–300 (2009).
- [133] Kleptsyna, M. L., and A. Yu Veretennikov. On discrete time ergodic filters with wrong initial data. *Probab. Theory Related Fields*, vol. 141, pp. 411–444 (2008).
- [134] V. Knopova. On the Feynman-Kac semigroup for some Markov processes.” *Modern Stochastics: Theory and Applications*, vol. 2, n.2 pp. 107–129 (2015).
- [135] V. Konakov, S. Menozzi, S. Molchanov. Explicit parametrix and local limit theorems for some degenerate diffusion processes. *Annales de l’IHP Probabilités et statistiques*. Vol. 46. No. 4., pp. 908–923 (2010).
- [136] I. Kontoyiannis and S.P. Meyn. Spectral theory and limit theorems for geometrically ergodic Markov processes. *Ann. Appl. Probab.*, vol. 13, pp. 304–362 (2003).
- [137] I. Kontoyiannis, S. P. Meyn. Geometric ergodicity and the spectral gap of non-reversible Markov chains. *Probability Theory and Related Fields*, vol. 154, no. 1–2, pp. 327–339 (2012).
- [138] J.B. Lasserre, C.E.M. Pearce. On the existence of a quasistationary measure for a Markov chain. Technical Report no. 98109, LAAS-CNRS, Toulouse, France (1998). *The Annals of Probability*, vol. 29, no. 1, pp. 437–446 (2001).
- [139] F. Le Gland, N. Oudjane. Stability and uniform approximation of nonlinear filters using the Hilbert metric and application to particle filters. *The Annals of Applied Probability*, vol. 14, no. 1, pp. 144–187 (2001).
- [140] F. Le Gland, N. Oudjane. A robustification approach to stability and to uniform particle approximation of nonlinear filters: the example of pseudo-mixing signals. *Stochastic Processes and their Applications*, vol. 106, no. 2, pp. 279–316 (2003).
- [141] F. Le Gland, L. Mevel. Exponential forgetting and geometric ergodicity in hidden Markov models. *Mathematics of Control, Signals and Systems*, vol. 13, no. 1, pp. 63–93 (2000).

- [142] F. Le Gland, L. Mevel. Exponential forgetting and geometric ergodicity in HMM's. Proceedings of the 36th Conference on Decision and Control, San Diego. vol. 1. (1997).
- [143] F. Le Gland, L. Mevel. Stability and Approximation of nonlinear filters: an information theoretic approach. Proceedings of the 38th Conference on Decision and Control, Phoenix Arizona (1999).
- [144] F. Le Gland, L. Mevel. Basic properties of the projective product with application to products of column-allowable nonnegative matrices. Mathematics of Control, Signals and Systems, vol. 13, no. 1 pp. 41–62 (2000).
- [145] C. Mailler, D. Villemonais. Stochastic approximation on non-compact measure spaces and application to measure-valued Pólya processes. The Annals of Applied Probability 30, no. 5, pp. 2393-2438 (2020).
- [146] A. Marguet. A law of large numbers for branching Markov processes by the ergodicity of ancestral lineages. ESAIM: Probability and Statistics, 23, pp. 638–661 (2019).
- [147] S. Méléard, D. Villemonais. Quasi-stationary distributions and population processes. Probab. Surv., vol. 9, pp. 340–410 (2012).
- [148] C. McDonald, S. Yüksel. Exponential filter stability via Dobrushin's coefficient. Electronic Communications in Probability, vol. 25, pp. 1–13 (2020).
- [149] S.P. Meyn, R.L. Tweedie. Markov Chains and Stochastic Stability. Springer-Verlag, New York (1993).
- [150] E. Nummelin. A splitting technique for Harris recurrent Markov chains, Z. Wahsch. verw. Grebeite, vol. 43, pp. 309–318 (1978).
- [151] R. D. Nussbaum. Eigenvectors of order-preserving linear operators. J. London Math. Soc., vol. 58, no. 2, pp. 480–496 (1998).
- [152] R. D. Nussbaum. The radius of the essential spectral radius. Duke Math. J., vol. 37, pp. 473–478 (1970)
- [153] W. Oçafrain. Convergence to quasi-stationarity through Poincaré inequalities and Bakry-Emery criteria. Electronic Journal of Probability, vol. 26, pp. 1–30 (2021).
- [154] D.L. Ocone, E. Pardoux. Asymptotic stability of the optimal filter wit respect to its initial condition. SIAM Control and Optimization, vol. 34, no. 1, pp. 226-243 (1996).

- [155] L. Oljaca, T. Kuna, J. Bröcker. Exponential stability and asymptotic properties of the optimal filter for signals with deterministic hyperbolic dynamics. arXiv preprint arXiv:2103.01190 (2021).
- [156] N. Oudjane, S. Rubenthaler. Stability and uniform particle approximation of nonlinear filters in case of non ergodic signals. *Stochastic Analysis and applications*, vol. 23, no. 3, pp. 421–448 (2005).
- [157] P. Meyer-Nieberg. *Banach lattice*, Springer-Verlag (1991).
- [158] P. Pollett. Quasi-stationary distributions : a bibliography available at <http://www.maths.uq.edu.au/~pkp/papers/qsds/qsds.html> (2015).
- [159] R. G. Pinsky. On the convergence of diffusion processes conditioned to remain in a bounded region for large time to limiting positive recurrent diffusion processes. *Ann. Probab.*, vol.13, no. 2, pp. 363–378 (1985).
- [160] M. Reed and B. Simon. *Methods of Modern Mathematical Physics I: Functional Analysis*. Academic Press, San Diego (1980).
- [161] L. Rey-Bellet. Ergodic properties of Markov processes, volume 1881 of *Lecture Notes in Mathematics*, pages 1–39. Springer (2006).
- [162] M. N. Rosenbluth and A. W. Rosenbluth. Monte Carlo calculation of the average extension of molecular chains. *J. Chem. Phys.*, vol. 23, no. 2, pp. 356–359 (1955).
- [163] M. Rousset. On the control of an interacting particle estimation of Schrödinger ground states *SIAM journal on mathematical analysis*, vol. 38, no. 3, pp. 824–844 (2006).
- [164] H. H. Schaefer. *Topological vector spaces*. Springer-Verlag, New York. Third printing corrected (1971).
- [165] E. Seneta and D. Vere-Jones. On quasi-stationary distributions in discrete-time Markov chains with a denumerable infinity of states. *J. Appl. Probab.*, vol. 3, pp. 403–434 (1966).
- [166] E. Seneta. *Non-Negative Matrices and Markov Chains*, 2nd edition. Springer Series in Statistics. Springer-Verlag, New York (1981).
- [167] R. Song. Two-sided estimates on the density of the Feynman-Kac semigroups of stable-like processes. *Electronic Journal of Probability*, vol. 11, pp. 146–161 (2006).
- [168] D. W. Stroock. Diffusion semigroups corresponding to uniformly elliptic divergence form operators. *Séminaire de Probabilités, XXII*, pp. 316– 347. Springer, Berlin (1988).

- [169] W.H. Summers. Weighted Locally Convex Spaces of Continuous Functions. LSU Historical Dissertations and Theses, no. 1520 (1968)
- [170] W.H. Summers. Dual spaces of weighted spaces. *Trans. Americal Soc.*, vol. 151, pp. 323–333 (1970).
- [171] L. Tierney. Markov chains for exploring posterior distributions *Ann. of Statist.*, vol. 22, no. 4, pp. 1701–1762 (1994).
- [172] E. A. van Doorn. Quasi-stationary distributions and convergence to quasi-stationarity of birth-death processes. *Adv. in Appl. Probab.*, vol. 23, no. 4, pp. 683–700 (1991).
- [173] E.A. van Doorn and P.K. Pollett. Quasi-stationary distributions for discrete-state models. *European J. Oper. Res.*, vol. 230, pp. 1–14 (2013).
- [174] R. Van Handel. The stability of conditional Markov processes and Markov chains in random environments. *Ann. Probab.*, vol. 37, pp. 1876–1925 (2009).
- [175] R. Van Handel. Discrete time nonlinear filters with informative observations are stable. *Electronic Communications in Probability*, vol. 13, pp. 562–575 (2008).
- [176] D. Villemonais, General approximation method for the distribution of Markov processes conditioned not to be killed, *ESAIM Probab. Stat.*, vol. 18, pp. 441–467 (2014).
- [177] C. Wang. On estimates of the density of Feynman-Kac semigroups of α stable-like processes. *Journal of mathematical analysis and applications*, vol. 348, no.2, pp. 938–970 (2008).
- [178] N. Whiteley. Stability properties of some particle filters. *The Annals of Applied Probability*, vol. 23, no. 6, pp. 2500–2537 (2013).
- [179] N. Whiteley. Dimension-free Wasserstein contraction of nonlinear filters. *Stochastic Processes and their Applications*, vol. 135(C), pp. 31–50 (2021).
- [180] N. Whiteley, N. Kantas, A. Jasra. Linear variance bounds for particle approximations of time-homogeneous Feynman-Kac formulae. *Stochastic Processes and their Applications*, vol. 122, pp. 1840–1865 (2012).
- [181] N. Whiteley, N. Kantas,. Calculating principal eigen-functions of non-negative integral kernels: particle approximations and applications. *Mathematics of Operations Research*, vol. 42, no. 4, pp. 1007–1034 (2017).
- [182] L. Wu. Large and moderate deviations and exponential convergence for stochastic damping Hamiltonian systems. *Stoch. Process. Appl.*, vol. 91, no. 2, pp. 205–238 (2001).

- [183] A.M. Yaglom. Certain limit theorems of the theory of branching random processes. Dokl. Akad. Nauk SSSR (N.S.), vol. 56, pp. 795–798 (1947).
- [184] F. Zucca. A note on compact Markov operators. ArXiv preprint math/0402302 (2004).