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# On the Stability of Positive Semigroups

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## Abstract

The stability and contraction properties of positive integral semigroups on locally compact Polish spaces are investigated. We provide a novel analysis based on an extension of  $V$ -norm, Dobrushin-type, contraction techniques on functionally weighted Banach spaces for Markov operators. These are applied to a general class of positive and possibly time-inhomogeneous bounded integral semigroups and their normalised versions. Under mild regularity conditions, the Lipschitz-type contraction analysis presented in this article simplifies and extends several exponential estimates developed in the literature. The spectral-type theorems that we develop can also be seen as an extension of Perron-Frobenius and Krein-Rutman theorems for positive operators to time-varying positive semigroups. We review and illustrate in detail the impact of these results in the context of positive semigroups arising in transport theory, physics, mathematical biology and advanced signal processing.

**Keywords:** Positive semigroups; Boltzmann-Gibbs transformations; Contraction inequalities; Dobrushin's ergodic coefficient; Spectral theorems; Foster-Lyapunov conditions.

**Mathematics Subject Classification:** Primary: 47D08, 47D06, 47D07, 47H07; Secondary: 47B65, 37A30, 37M25, 60J25.

# 1 Introduction

Positive semigroups arise in a variety of areas of applied mathematics, including non-linear filtering, rare event analysis, branching processes, mean field particle methods and sequential Monte Carlo methodologies, physics and molecular chemistry. In this article we will study the possibly time-inhomogeneous linear semigroup  $Q_{s,t}$  and its normalisation  $\Phi_{s,t}$ , where  $0 \leq s \leq t$  refers to a discrete or a continuous time parameter, which are formally introduced in (3) and (4) below. Their interpretation depends upon the application model area.

In order to guide the reader through the vast array of stability analysis results developed in these different disciplines, we give a brief overview of the literature in these fields and provide some precise reference points aide with the navigation between applications.

1. In signal processing, the normalised semigroup  $\Phi_{s,t}$  depends on a random observation process and describes the solution to the nonlinear filtering equations.
2. In the context of killed absorption processes,  $\Phi_{s,t}$  represents the evolution of the distribution of a given process conditioned on non-absorption (a.k.a. the  $Q$ -process). In this context, the fixed point probability measure  $\eta_\infty = \Phi_{0,t}(\eta_\infty)$  of time homogeneous semigroups is sometimes called the Yaglom or quasi-invariant measure.
3. In quantum physics and molecular chemistry the top of the spectrum of positive integral semigroups is related to ground state and free energy computations of Schrödinger operators and Feynman-Kac semigroups (see for instance [48, 45] and the relevant literature therein). These spectral objects are naturally related to the analysis of quasi-compact operators and Fredholm integral equations, see for instance [3, 67, 77, 58, 103].
4. In risk analysis, these semigroups represent the evolution of a process evolving in a rare event regime. The details of these application areas are considered in [39, 42, 51] and the relevant references therein.

## 1.1 Literature review

A unifying point of interest in the above applications is the study of the stability of the afore-mentioned semigroups. In the context of dynamic populations, the long time behavior of branching processes certainly goes back to the end of the 1940s with the pioneering work of Yaglom [130] on Galton-Watson processes. The stability analysis of time homogenous birth-and-death-type processes with absorption on finite or countable spaces dates back to the 1950s-1960s with the pioneering works [84, 116] and the later developments [19, 36, 37, 66, 122]. Sufficient conditions ensuring the existence of a quasi-invariant measure are also developed in [75, 91]. Spectral and  $h$ -process type techniques are also developed in [17, 65, 111], as well as in [18, 33, 65].

Stochastic approximation-type techniques are also presented in [6, 98]. Non asymptotic spectral techniques that apply to possibly transient and unstable mutation-type linear diffusions that do not necessarily have a gradient form and with a quadratic absorption rate are discussed in the more recent work [45].

One of the first well-founded results on the long time behavior of the nonlinear filtering equation is the seminal article by Ocone and Pardoux in the mid-1990s [106]. In this work the authors show that the optimal filter forgets the initial condition without giving a non-asymptotic rate. This latter analysis is critical in the study of, for instance, numerical algorithms such as the particle filter. The stability of the nonlinear filtering equation is also related to the Lyapunov spectrum and the asymptotic properties of products of random positive matrices [10, 11, 70, 76, 83, 62, 116]. In contrast with positive semigroups arising in physics and biology, the stability analysis of the nonlinear filtering equation involves the study of sophisticated stochastic semigroups that depend on the partial and noisy observations.

The systematic, non-asymptotic, stability analysis of non-homogeneous sub-Markovian and Feynman-Kac type semigroups on general state spaces has also been considered at the end of the 1990s, mainly in nonlinear filtering theory and rare event analysis. Several techniques have been adopted: Hilbert metric and robustification techniques [2, 14, 35, 85, 107, 92, 93, 97, 108, 100], Dobrushin's ergodic coefficients [39, 42, 44, 94, 95, 97, 100], bounded Lipschitz distances [94, 95] and relative entropy-like criteria [31, 39, 50, 96].

More recent local Doeblin-type minorisation conditions, including Foster-Lyapunov approaches and coupling, that apply to ergodic signal-observation filtering problems, including stable Gaussian-linear filtering models, are also discussed in the series of articles [55, 56, 61, 64, 69, 81, 82, 127, 129], see also [124] for related Lipschitz norm techniques as well as the more recent article [23] in the context of discrete time bounded positivity preserving operators. Functional inequalities, including Poincaré inequalities and Bakry-Emery criteria approaches are discussed in [105].

Spectral techniques, drift conditions and Wasserstein norm approaches for time-homogenous models are also discussed respectively in [45, 47, 48] and [45, 86] as well as in [128]. Further, two-sided minorisation conditions are discussed in [28, 29], see also [39, 40, 42], and truncation-type techniques are presented in [34, 61, 73, 127].

General asymptotic stability results are also provided in [125]. Stabilising changes of Feynman-Kac measures and related importance-sampling-type and  $h$ -process techniques that apply to possibly unstable killed processes on unbounded domains are also discussed in [61, 114], see also [45] and well as section B.1 in [7] and in section 7.1 in [46] in the context of discrete time models. The stability of the nonlinear filtering equations with deterministic hyperbolic signals is also developed in the recent article [107].

Despite the numerous references given in the introduction, a complete literature review is not possible. For a more detailed discussion on this subject, we refer the reader to [15, 30] for a review on the asymptotic stability of nonlinear filters, to the bibliographies and reviews [32, 99, 110, 123] on the theory of quasi-stationary

distributions, the articles [42, 61] as well as the bibliography in [45] in quantum physics, and the books [39, 40] for a detailed discussion on the long time stability of Feynman-Kac semigroups.

## 1.2 Contribution and structure

The main objective of this article is to further develop the stability analysis of positive semigroups for a general and abstract class of time inhomogeneous models, which, in general, are much more difficult to handle than their time homogeneous counterparts; this is because the operators may drastically change during the semigroup evolution. We will also tackle the problem of non-compact state spaces.

In the present article, we present a new unifying methodology that combines Dobrushin ergodic coefficient techniques developed in [39, 40, 42, 44, 49] with the powerful Foster-Lyapunov methodologies developed in the articles [56, 61, 90, 127] in the context of Feynman-Kac semigroups and nonlinear filtering. Our approach relies on the contraction analysis of a class of triangular arrays of Markov semigroups introduced in [42, 44, 49] in terms of the  $V$ -Dobrushin contraction-type ergodic theory for Markov operators presented in [71] and further developed in a systematic way in the book [51]. A brief review on this subject is provided in section 3.2. For a more thorough discussion on these  $V$ -contraction coefficient principles, we refer to section 8.2.5 in [51]. The triangular-array-type Markov semigroups discussed above are described in section 1.5.

The article is structured as follows. In section 1.3 we introduce the notation that will be used throughout and state our main results. Section 1.4 and section 1.5 are dedicated to the detailed description of the semigroups considered in the article. The regularity conditions used in our analysis are presented in section 1.6. The main results of the article are described in section 2. In particular, in section 2.2, we present the main results for time-inhomogeneous models, with a more refined analysis of time homogeneous models being given in section 2.3. In section 3, we give a review on integral operators, which connects to some of the major topics that have been considered in the modern literature. Section 4 is dedicated to the proofs of the main theorems presented in this article. The appendix houses most of our technical proofs.

## 1.3 Some basic notation

Let  $\mathcal{B}(E)$  be the space of measurable functions on a locally compact Polish metric space  $(E, d)$  equipped with its Borel  $\sigma$ -field  $\mathcal{E}$ . We denote by  $\mathcal{B}_b(E) \subset \mathcal{B}(E)$  the subspace of bounded measurable functions, and by  $\mathcal{B}_0(E) \subset \mathcal{B}_b(E)$  the subspace of functions  $f$  that vanish at infinity, that is for any  $\epsilon > 0$  the set  $\{f \geq \epsilon\}$  is compact. Let  $\mathcal{C}(E) \subset \mathcal{B}(E)$  be the subset of continuous functions,  $\mathcal{C}_b(E) \subset \mathcal{C}(E)$  the subset of bounded continuous functions and  $\mathcal{C}_0(E) \subset \mathcal{C}_b(E)$  the subspace of functions that vanish at infinity. One of our main reasons for working with locally compact Polish spaces is due to the fact that for infinite dimensional normed spaces, by Riesz's lemma

we have  $\mathcal{C}_0(E) = \{0\}$ .

Let  $V \in \mathcal{B}(E)$  be a continuous function such that  $V \geq 1$ . We let  $\mathcal{B}_V(E) \subset \mathcal{B}(E)$  be the Banach space of functions  $f$  with  $\|f\|_V := \|f/V\| < \infty$ . With a slight abuse of notation, we denote by 0 and 1 the null and unit scalars as well as the null and unit function on  $E$ . We also consider the sets

$$\begin{aligned}\mathcal{B}_{0,V}(E) &:= \{f \in \mathcal{B}(E) : f/V \in \mathcal{B}_0(E)\} \\ \mathcal{C}_V(E) &:= \mathcal{C}(E) \cap \mathcal{B}_V(E) \quad \text{and} \quad \mathcal{C}_{0,V}(E) := \mathcal{C}(E) \cap \mathcal{B}_{0,V}(E).\end{aligned}$$

We also denote by  $\mathcal{M}(E)$  the set of signed Radon measures  $\mu$  on  $E$  and by  $\mathcal{M}_V(E) \subset \mathcal{P}(E)$  the subset of measures  $\mu \in \mathcal{M}(E)$  such that  $|\mu|(V) < \infty$ , where  $|\mu| := \mu_+ + \mu_-$  stands for with the total variation measure associated with a Hahn-Jordan decomposition  $\mu = \mu_+ - \mu_-$  of the measure. We also let  $\mathcal{P}_V(E) \subset \mathcal{M}_V(E)$  the set of probability measures  $\mu \in \mathcal{P}(E)$  such that  $\mu(V) < \infty$ .

We define the duality map, as well as the right and dual left action of a bounded integral operator  $Q$  using the classical measure theoretic notation, as follows:

$$\begin{aligned}(\mu, f) \in (\mathcal{M}_V(E) \times \mathcal{B}_V(E)) &\mapsto \mu(f) \in \mathbb{R} \quad \text{with} \quad \mu(f) := \int f(x)\mu(dx) \\ f \in \mathcal{B}_V(E) &\mapsto Q(f) \in \mathcal{B}_V(E) \quad \text{with} \quad Q(f)(x) := \int Q(x, dy) f(y) \\ \mu \in \mathcal{M}_V(E) &\mapsto \mu Q \in \mathcal{M}_V(E) \quad \text{with} \quad \mu Q(dy) := \int \mu(dx) Q(x, dy).\end{aligned} \tag{1}$$

We denote by  $I$  the identity integral operator. We say that  $Q$  is positive if  $Q(f) \geq 0$  whenever  $f \geq 0$ . Whenever  $Q(1) \leq 1$  we say that  $Q$  is sub-Markovian, and  $Q$  is said to be Markovian when  $Q(1) = 1$ . We associate with a function  $H > 0$  the Boltzmann-Gibbs transformation

$$\Psi_H : \mu \in \mathcal{P}_V(E) \mapsto \Psi_H(\mu) \in \mathcal{P}_{V/H}(E) \quad \text{with} \quad \Psi_H(\mu)(dx) := \frac{1}{\mu(H)} H(x) \mu(dx). \tag{2}$$

We let  $\mathcal{T} = \mathbb{R}_+ := [0, \infty[$  or  $\mathcal{T} = \mathbb{N}$  be the continuous or discrete time space, respectively. For a given time horizon  $t \in \mathcal{T}$ , we set  $\mathcal{T}_t := \mathcal{T} \cap [0, t]$ . Given some  $\tau \in \mathcal{T}$  with  $\tau > 0$ , we consider the time mesh  $\mathcal{T}(\tau) := \{n\tau : n \in \mathbb{N}\}$  and we set  $\mathcal{T}_t(\tau) = \mathcal{T}(\tau) \cap [0, t]$ . We also consider for any  $s \leq t$  the  $\tau$ -intervals

$$[s, t]_\tau = \mathcal{T}(\tau) \cap [s, t] \quad \text{and} \quad [s, t[_\tau = \mathcal{T}(\tau) \cap [s, t[.$$

When  $f = 1_A$  is the indicator function of some set  $A \in \mathcal{E}$ , we will sometimes slightly abuse notation and write  $\mu(A)$  instead of  $\mu(1_A)$ . We also set  $a \wedge b = \min(a, b)$  and  $a \vee b = \max(a, b)$ , for  $a, b \in \mathbb{R}$  and we use the conventions  $(\sum_\emptyset, \prod_\emptyset) = (0, 1)$  and  $(\sup_\emptyset, \inf_\emptyset) = (-\infty, +\infty)$ .

## 1.4 Positive integral semigroups

Let  $Q_{s,t}$  be a collection of positive integral operators from  $\mathcal{B}_V(E)$  to  $\mathcal{B}_V(E)$ , indexed by parameters  $s, t \in \mathcal{T}$  with  $s \leq t$ , and satisfying for any  $s, u, t \in \mathcal{T}$ ,  $s \leq u \leq t$ , the semigroup property

$$Q_{s,u}Q_{u,t} = Q_{s,t} \quad \text{with} \quad Q_{s,s} = I. \quad (3)$$

Assume that  $Q_{t,t+\tau}(1) > 0$  for some  $\tau > 0$  and any  $t \in \mathcal{T}$ . This irreducibility condition ensures that  $Q_{t,t+\tau}(f) > 0$  for any  $f > 0$ . In the discrete time setting, this condition can be relaxed up to some modification of the state space; see for instance the construction in [39, section 4.4] as well as [41]. For instance, for sub-Markovian time homogeneous semigroups  $Q_{t,t+\tau} = Q_{0,\tau}$  the Borel set  $\{Q_{0,\tau}(1) = 0\}$  can be taken as part of an absorbing set. In this context, to analyse the behavior of a non absorbed particle there is no loss of generality in assuming that  $Q_{0,\tau}(1) > 0$ .

In discrete time settings, up to a time-rescaling, there is no loss of generality in assuming that  $\tau = 1$ . In addition, continuous time models satisfying  $Q_{t,t+\tau}(1) > 0$  for a particular  $\tau > 0$  can be encapsulated by the discrete time setting via the time discretisation  $\mathcal{T}(\tau)$ .

The continuous time models considered in the present article are assumed to satisfy  $Q_{t,t+s}(1) > 0$  for any  $s > 0$ . For both continuous and discrete time models, unless otherwise is stated, we shall denote by  $\tau > 0$  a given and fixed parameter belonging to the continuous or the discrete time index set  $\mathcal{T}$  such that  $Q_{t,t+\tau}(1) > 0$ . From our discussion above, this parameter can be chosen  $\tau = 1$  in discrete time settings, and by any parameter  $\tau > 0$  in the continuous time framework.

We emphasise that the analysis of continuous time models is sometimes based on a discrete time approach based on regularity conditions that depend on the parameter  $\tau$  chosen by the user. In this context, the analysis is performed at the level of the  $\tau$ -discretised model and the estimation constants presented in our results may depend on the parameter  $\tau$ .

For any  $s, t \in \mathcal{T}$  with  $s \leq t$  and for any measure  $\eta_s \in \mathcal{P}_V(E)$  we let  $\Phi_{s,t}(\eta_s) \in \mathcal{P}_V(E)$  be the normalised distribution defined for any  $f \in \mathcal{B}_V(E)$  by the formula

$$\Phi_{s,t}(\eta_s)(f) := \eta_s Q_{s,t}(f) / \eta_s Q_{s,t}(1). \quad (4)$$

The mapping  $\Phi_{s,t}$  is a well defined nonlinear map from  $\mathcal{P}_V(E)$  into  $\mathcal{P}_V(E)$  satisfying for any  $s, u, t \in \mathcal{T}$  with  $s \leq u \leq t$  the semigroup property

$$\Phi_{s,t} = \Phi_{u,t} \circ \Phi_{s,u} \quad \text{with} \quad \Phi_{s,s}(\mu) = \mu.$$

Unless otherwise stated, all the semigroups discussed in this article are indexed by conformal indices  $s \leq t$  in the set  $\mathcal{T}$ . To avoid repetition, we often write  $Q_{s,t}$  and  $\Phi_{s,t}$  without specifying the order  $s \leq t$  of the indices  $s, t \in \mathcal{T}$ .

For a given a function  $W \geq 0$  and some probability measure  $\eta$  on  $E$  we shall denote by  $\kappa_W^-(\eta), \lambda^-(\eta) \in [0, +\infty[$  and  $\kappa_W(\eta), \lambda(\eta) \in [0, +\infty]$  the parameters

$$\begin{aligned} \kappa_W^-(\eta) &:= \inf \Phi_{s,t}(\eta)(W) && \leq \kappa_W(\eta) &:= \sup \Phi_{s,t}(\eta)(W) \\ \lambda^-(\eta) &:= \inf \Phi_{s,t}(\eta)(Q_{t,t+\tau}(1)) && \leq \lambda(\eta) &:= \sup \Phi_{s,t}(\eta)(Q_{t,t+\tau}(1)). \end{aligned}$$

In the above display, the infimum and the supremum are taken over all indices  $(s, t)$  such that  $t \in \mathcal{T}(\tau)$  and  $s \in \mathcal{T}_t(\tau)$ . We also set

$$\lambda^-(\bar{\eta}_0) \leq \lambda^- := \inf_{t \in \mathcal{T}(\tau)} \bar{\eta}_t(Q_{t,t+\tau}(1)) \leq \lambda := \sup_{t \in \mathcal{T}(\tau)} \bar{\eta}_t(Q_{t,t+\tau}(1)) \leq \lambda(\bar{\eta}_0). \quad (5)$$

The unnormalised and the normalised semigroups  $Q_{s,t}$  and  $\Phi_{s,t}$  are connected for any  $\mu \in \mathcal{M}_V(E)$ ,  $f \in \mathcal{B}_V(E)$  and  $s, t \in \mathcal{T}$  by the formula

$$\mu Q_{s,t}(f) = \Phi_{s,t}(\mu)(f) \prod_{u \in [s,t]_\tau} \Phi_{s,u}(\mu)(Q_{u,u+\tau}(1)). \quad (6)$$

The above formula coincides with the product formula relating the unnormalised operators  $Q_{s,t}$  with the normalised semigroup  $\Phi_{s,t}$  discussed in [42, section 1.3.2], see also [39, proposition 2.3.1] and [40, section 12.2.1]. For time homogeneous models we use the notation

$$(V_t, \Phi_t, Q_t, q_t) := (V, \Phi_{0,t}, Q_{0,t}, q_{0,t}).$$

In this setting, the product formula (6) reduces to

$$\mu(Q_t(f)) = \Phi_t(\mu)(f) \prod_{s \in [0,t]_\tau} \Phi_s(\mu)(Q_\tau(1)). \quad (7)$$

The general abstract class of time varying semigroups discussed above encapsulates most of the regular discrete or continuous time-varying stochastic models on bounded spaces encountered in physics, biology and signal processing. In contrast with conventional Hilbert space approaches to the stability of reversible Markov semigroups (cf. for instance [12, 53, 54, 89]), the analysis of time varying models does not rely on a particular reversible measure. The framework of weighted spaces and  $V$ -norms considered in the article is a natural but non-unique framework to analyse time-varying positive semigroups.

## 1.5 A class of $R$ -semigroups

Consider some measure  $\bar{\eta} \in \mathcal{P}_V(E)$  and some positive function  $H \in \mathcal{C}_V(E)$ . We associate with these objects the functions  $H_{s,t}$  and the normalised semigroups  $\bar{Q}_{s,t}$  defined for any  $s, t \in \mathcal{T}$  by the formulae

$$H_{s,t} := \bar{Q}_{s,t}(H) \quad \text{with} \quad \bar{Q}_{s,t}(f) := Q_{s,t}(f)/\bar{\eta}_s Q_{s,t}(1) \quad \text{and} \quad \bar{\eta}_s = \Phi_{0,s}(\bar{\eta}). \quad (8)$$

From the definitions, it follows that

$$\bar{\eta}_s \bar{Q}_{s,t} = \bar{\eta}_t \quad \text{and} \quad \bar{Q}_{s,u} \bar{Q}_{u,t} = \bar{Q}_{s,t} \quad \text{as well as} \quad \bar{\eta}_s(H_{s,t}) = \bar{\eta}_t(H) \quad \text{and} \quad H_{t,t} = H, \quad (9)$$

and

$$Q_{s,v}(H_{v,t}) = \lambda_{s,v} H_{s,t} \quad \text{with} \quad \lambda_{s,v} := \bar{\eta}_s Q_{s,v}(1) = \lambda_{s,u} \lambda_{u,v}.$$

**Definition 1.1.** We define the triangular array of Markov operators  $R_{u,v}^{(t)}$ , indexed by the parameter  $0 \leq u \leq v \leq t$  and  $t \in \mathcal{T}$ , for any  $f \in \mathcal{B}_b(E)$  by

$$R_{u,v}^{(t)}(f) := \frac{Q_{u,v}(f Q_{v,t}(H))}{Q_{u,v}(Q_{v,t}(H))} = \frac{1}{\lambda_{u,v} H_{u,t}} Q_{u,v}(H_{v,t}f). \quad (10)$$

For any given time horizon  $t \in \mathcal{T}$ , we have the semigroup property

$$R_{s,v}^{(t)} = R_{s,u}^{(t)} R_{u,v}^{(t)} \quad \text{and} \quad R_{s,s}^{(t)} = I.$$

This shows that  $R_{u,v}^{(t)}(x, dy)$  can be interpreted as the Markov transition of a stochastic flow  $X_{u,v}^{(t)}(x)$  on the time interval  $\mathcal{T}_t$ . In the further development of the article we use the terminology *R-semigroup* and *R-process* to refer to these objects.

Several remarks are of interest here:

1. For time homogeneous semigroups, the *R*-semigroups associated with the positive eigenstate of the semigroup  $Q_t$  coincides with the semigroup of the so-called *h*-process, see for instance (38) and (42).
2. The stability properties of these stochastic models plays a crucial role in the analysis of positive operators. An illustration in the context of coupled harmonic oscillators arising in physics is discussed in (54).
3. The use of these triangular arrays of Markov semigroups in the stability analysis of the time varying positive semigroups has been considered in [42, 44, 49] in the context of Feynman-Kac semigroups and when  $H = 1$ . We also refer the reader to chapter 4 in [39], and chapter 12 in [40] for a systematic analysis of the contraction properties of these semigroups with respect to the total variation norm.

We now provide some important Markov transport formulae relating the semigroups introduced so far. The first formula connecting the *R*-semigroup with the flow of measures  $\eta_u = \Phi_{s,u}(\eta_s)$  is given by

$$\Psi_{H_{s,t}}(\eta_s) R_{s,u}^{(t)}(f) = \frac{\eta_s(Q_{s,u}(H_{u,t}f))}{\eta_s(Q_{s,u}(H_{u,t}))} = \frac{\eta_u(H_{u,t}f)}{\eta_u(H_{u,t})}$$

This yields the formula

$$\Psi_{H_{s,t}}(\eta_s) R_{s,u}^{(t)} = \Psi_{H_{u,t}}(\eta_u).$$

Choosing  $u = t$  and  $\eta_s = \eta$  so that  $\eta_t = \Phi_{s,t}(\eta)$  in the above display, we obtain the following lemma.

**Lemma 1.2.** For any  $t \in \mathcal{T}$  and  $s \in \mathcal{T}_t$  and any  $\eta \in \mathcal{P}_V(E)$  we have

$$\Psi_H(\Phi_{s,t}(\eta)) = \Psi_{H_{s,t}}(\eta) R_{s,t}^{(t)} \quad \text{and} \quad \eta_s(H_{s,t}) \Psi_{H_{s,t}}(\eta_s) (1/H_{s,t}) = 1. \quad (11)$$

## 1.6 Regularity conditions

Let us begin with our first regularity condition.

( $\mathcal{H}_0$ ) *There exists a non-increasing function  $\theta : [1, \infty[ \rightarrow \mathbb{R}_+$ , a constant  $r_0 \geq 0$  and a function  $\alpha : r \in [r_0, \infty[ \mapsto \alpha(r) \in ]0, 1]$  such that for any  $t \in \mathcal{T}$ ,  $s + \tau \in \mathcal{T}_t$  and  $r \geq r_0$  we have*

$$\sup_{V_\theta(x) \vee V_\theta(y) \leq r} \left\| \delta_x R_{s, s+\tau}^{(t)} - \delta_y R_{s, s+\tau}^{(t)} \right\|_{tv} \leq 1 - \alpha(r) \quad \text{with} \quad V_\theta := 1/\theta(V). \quad (12)$$

The condition (12) is satisfied for some parameter  $\alpha(r) \in ]0, 1]$  *if and only if* the following local minorisation condition holds

$$\forall t \in \mathcal{T} \quad \forall s + \tau \in \mathcal{T}_t \quad \forall r \geq r_0 \quad \forall (x, y) \in E^2 : V_\theta(x) \vee V_\theta(y) \leq r$$

$$\exists \nu \in \mathcal{P}(E) \quad \text{such that} \quad \delta_x R_{s, s+\tau}^{(t)} \geq \alpha(r) \nu \quad \text{and} \quad \delta_y R_{s, s+\tau}^{(t)} \geq \alpha(r) \nu.$$

Note that the measure  $\nu$  in the above display may depend upon the parameters  $(x, y)$  as well as  $(r, s, t, \tau)$ . This rather mild Doeblin-type local minorisation condition is rather standard in the stability analysis of Markov semigroups (see for instance [102, Theorem 6.15], [121, Proposition 2], and [101, Theorem 16.2.3]).

Apart from the fact that the uniform minorisation condition (20) is met for some particular classes of operators on non-compact spaces (including for ad-hoc truncated drift Gaussian transitions or Laplace-type transitions on non-compact spaces - see [42, section 5] and [39, exercise 3.5.2]) it is rarely satisfied for non-compact state spaces and thus our starting point is to localise the uniform minorisation condition (20).

Consider a function  $V$  with compact level sets. Note that in this case the function  $\theta(V)$  vanishes at infinity and  $V_\theta$  also has compact level sets defined by

$$\forall 0 < \epsilon < \epsilon_0 := \theta(\inf_E V)$$

$$\mathcal{V}_\epsilon := \{V \leq r_\epsilon\} = \{\theta(V) \geq \epsilon\} = \{V_\theta \leq 1/\epsilon\} \neq \emptyset \quad \text{with} \quad \inf_E V < r_\epsilon := \theta^{-1}(\epsilon). \quad (13)$$

In this notation, condition (12) is satisfied for some  $r_0 \geq \theta^{-1}(\epsilon_0)$  *if and only if* the following local minorisation condition holds

$$\forall 0 < \epsilon \leq \theta(r_0) \quad \forall t \in \mathcal{T} \quad \forall s + \tau \in \mathcal{T}_t \quad \forall (x, y) \in \mathcal{V}_\epsilon^2$$

$$\exists \nu \in \mathcal{P}(E) \quad \text{such that} \quad \delta_x R_{s, s+\tau}^{(t)} \geq \alpha(r_\epsilon) \nu \quad \text{and} \quad \delta_y R_{s, s+\tau}^{(t)} \geq \alpha(r_\epsilon) \nu.$$

Note that the measure  $\nu$  in the above display may depend upon the parameters  $(x, y)$  and  $(r_0, s, t, \tau)$  as well as on the level set parameter  $0 < \epsilon \leq \theta(r_0)$ .

We strengthen this local minorisation property with the following condition.

( $\mathcal{H}_1$ ) Condition ( $\mathcal{H}_0$ ) is met for some  $(\alpha, \theta)$  and some continuous function  $V \geq 1$  with compact level sets. For any  $t \in \mathcal{T}$  and  $\mu \in \mathcal{P}_V(E)$  we have

$$Q_{t,t+\tau}(V)/V \leq \theta(V) \quad \text{as well as} \quad \kappa_V(\mu) < \infty \quad \text{and} \quad \lambda^- > 0. \quad (14)$$

In addition, there exists some  $\epsilon_H > 0$  such that for any  $0 < \epsilon \leq \epsilon_H$  we have

$$\varsigma_\epsilon(H) := \inf_{t \in \mathcal{T}(\tau)} \inf_{s \in \mathcal{T}_t(\tau)} \inf_{\mathcal{V}_\epsilon} H_{s,t} > 0 \quad \text{and} \quad \|H\|_V := \sup_{t \in \mathcal{T}(\tau)} \sup_{s \in \mathcal{T}_t(\tau)} \|H_{s,t}\|_V < \infty. \quad (15)$$

Whenever ( $\mathcal{H}_1$ ) is satisfied, by replacing  $H$  with  $H/\|H\|_V$ , there is no loss of generality if we assume that  $\|H_{s,t}\|_V \leq 1$ , so that  $H_{s,t} \leq V$  for any  $s, t \in \mathcal{T}(\tau)$  and  $\|H\|_V \leq 1$ . Note that the growth condition stated in the left-hand side of (14) is satisfied with  $\theta(v) = v^{-\delta}$  as soon as

$$Q_\tau(V)/V^{1-\delta} \in \mathcal{B}_b(E) \quad \text{for some} \quad 0 < \delta < 1. \quad (16)$$

Several ways to design functions  $V$  satisfying the above condition are discussed in section 3.3.

Checking the estimates (15) may involve delicate calculations. Thus, we now present a stronger but simpler and more tractable condition that applies to absolutely continuous semigroups.

( $\mathcal{H}_2$ ) : For any  $t \in \mathcal{T}$ , the positive operator  $Q_{t,t+\tau}$  has the form

$$Q_{t,t+\tau}(x, dy) = q_{t,t+\tau}(x, y) \nu_\tau(dy) \quad \text{and} \quad Q_{t,t+\tau}(V)/V \leq \theta(V) \quad (17)$$

for some non-increasing function  $\theta : [1, \infty[ \rightarrow \mathbb{R}_+$ . In the above display,  $q_{t,t+\tau}$  is a density with respect to some Radon positive measure  $\nu_\tau$  on  $E$ , such that for some  $\epsilon_1 > 0$  and any  $0 < \epsilon \leq \epsilon_1$  we have

$$0 < \iota_\epsilon^- = \inf_{t \in \mathcal{T}} \inf_{\mathcal{V}_\epsilon^2} q_{t,t+\tau} \leq \iota_\epsilon := \sup_{t \in \mathcal{T}} \sup_{\mathcal{V}_\epsilon^2} q_{t,t+\tau} < \infty \quad \text{and} \quad 0 < \nu_\tau(\mathcal{V}_\epsilon) < \infty. \quad (18)$$

Absolutely continuous integral operators arise in natural way in discrete time settings [39, 42] as well as for continuous time elliptic diffusion-type absorption models [59, 60]. In connection to this, two-sided estimates for stable-like processes are provided in [118, 126], see also [9, 88]. Jump type models can be handled using the methodology and minorisation conditions discussed in sections 2.1 and 2.3.

The following lemma will help us to establish the link between the hypotheses ( $\mathcal{H}_0$ ), ( $\mathcal{H}_1$ ) and ( $\mathcal{H}_2$ ). It is a slight modification of [127, Proposition 1 & Lemma 10], based on technical approaches from [56] in the context of nonlinear filtering stability.

**Lemma 1.3.** *Assume that (17) and (18) are satisfied. In this situation, (15) is satisfied with  $\epsilon_H := \theta(\bar{\eta}(V) + 1)$ . In addition, for any  $\eta \in \mathcal{P}_V(E)$  we have*

$$0 < \kappa_H^-(\eta) \leq \kappa_V(\eta) < \infty \quad \text{and} \quad 0 < \lambda^-(\eta) \leq \lambda(\eta) < \infty. \quad (19)$$

For the convenience of the reader a detailed proof in our context is provided in the appendix, see section A.1. The above lemma also ensures for any bounded  $f \geq 0$ ,  $x \in \mathcal{V}_\epsilon$  and  $0 < \epsilon \leq \epsilon_H := \theta(\bar{\eta}(V) + 1)$  we have

$$R_{s,s+\tau}^{(t)}(f)(x) \geq \frac{\iota_\epsilon^-}{\lambda \| \| H \| \|_V} \nu_\tau(1_{\mathcal{V}_\epsilon} H_{s+\tau,t} f) \geq \frac{\iota_\epsilon^- \varsigma_\epsilon(H) \nu_\tau(\mathcal{V}_\epsilon)}{\lambda \| \| H \| \|_V} \frac{\nu_\tau(1_{\mathcal{V}_\epsilon} H_{s+\tau,t} f)}{\nu_\tau(1_{\mathcal{V}_\epsilon} H_{s+\tau,t})},$$

which implies (12). Thus, the above lemma ensures that

$$(\mathcal{H}_2) \implies ((\mathcal{H}_1) \quad \text{with} \quad \epsilon_H = \theta(\bar{\eta}(V) + 1)) \implies (\mathcal{H}_0).$$

## 2 Statement of the main results

### 2.1 A uniform stability theorem

In this section, we discuss some conditions under which we have uniform stability of the non-linear semigroup. We remark that (12) is satisfied for the unit function  $V = 1 = r_0$  and the function  $\theta(v) = 1/v$  if and only if the following uniform minorisation condition holds

$$\begin{aligned} \forall t \in \mathcal{T} \quad \forall s + \tau \in \mathcal{T}_t \quad \forall (x, y) \in E^2 \\ \exists \nu \in \mathcal{P}(E) \quad \text{such that} \quad \delta_x R_{s,s+\tau}^{(t)} \geq \alpha(1) \nu \quad \text{and} \quad \delta_y R_{s,s+\tau}^{(t)} \geq \alpha(1) \nu. \end{aligned} \tag{20}$$

The above minorisation condition provides a practical and natural way to check condition  $(\mathcal{H}_0)$ . For Markovian semigroups, this condition reduces to the well-known Dobrushin's condition. In addition, when the measure  $\nu$  in (20) does not depend on the state variables  $(x_1, x_2)$ , condition (20) coincides with Doeblin's condition.

Dobrushin's and Doeblin's conditions are popular conditions in probability theory. They are not always easily checked but are met for a large class of discrete or continuous time irreducible stochastic processes mainly on compact domains. For instance, in the context of continuous time sub-Markovian semigroups this condition is satisfied for elliptic diffusions on compact manifolds killed at a bounded rate as well as elliptic diffusions killed at the boundary of a bounded domain, cf. [49, Proposition 3.1] and [38, Lemma 3]. Combining the proof of [49, Proposition 3.1] with the two-sided estimates presented in [5, Lemma 3.3], one can check that (20) is also met for reflected one-dimensional diffusions killed at a bounded rate on some compact interval.

Condition (20) is also met for absorbed time homogenous neutron transport processes for sufficiently smooth absorption domains as well as for time varying multidimensional birth and death processes with mass extinction, see [20, sections 4.1 & 4.2]. Further examples of sub-Markovian semigroups satisfying (20) are discussed in [39, 40, 42], see also [41, 47] as well as [25]. Sufficient conditions and further examples are discussed in section 2.4.

In this context, we now recall the rather well known strong stability theorem.

**Theorem 2.1** ([42, 44]). *Assume that (20) is satisfied and  $H = 1$ . Then for any  $s, t \in \mathcal{T}$  and any probability measures  $\mu, \eta \in \mathcal{P}(E)$  we have the uniform stability estimate*

$$\|\Phi_{s,t}(\mu) - \Phi_{s,t}(\eta)\|_{tv} \leq (1 - \alpha(1))^{(t-s)/\tau}. \quad (21)$$

*In addition, for any  $\mu_1, \mu_2 \in \mathcal{P}(E)$  we have the local Lipschitz estimate*

$$\|\Phi_{s,t}(\mu_1) - \Phi_{s,t}(\mu_2)\|_{tv} \leq \frac{\|Q_{s,t}(1)\|}{\mu_1(Q_{s,t}(1)) \vee \mu_2(Q_{s,t}(1))} (1 - \alpha(1))^{(t-s)/\tau} \|\mu_1 - \mu_2\|_{tv}. \quad (22)$$

A detailed and remarkably simple proof of the above theorem based on the nonlinear transport formula (11) is provided in [42] (cf. for instance Theorem 2.3), see also [44, Lemma 2.1 & Lemma 2.3], [49, Lemma 2.1], [40, section 12.2], [42, section 2.1.2 & section 3.1.3] and [39, section 4]. An extended version of the above theorem for general relative entropy criteria is provided in [39, section 4.3.1].

For the rest of this section, we place ourselves in the time homogeneous setting. In this case, a variety of results follows almost immediately from the estimates obtained in this theorem. Before stating them, we first discuss some relevant properties of time homogeneous models.

The uniform estimate (21) implies that  $\Phi_t(\mu)$  is a Cauchy sequence in the complete set of probability measures  $\mathcal{P}(E)$  equipped with the total variation distance. Thus, for any  $\mu$ , the flow  $\Phi_t(\mu)$  converges, as  $t \rightarrow \infty$ , exponentially fast to a single probability measure  $\eta_\infty \in \mathcal{P}(E)$ . Using the fixed point equation, for any  $s, t \in \mathcal{T}$  we readily check that

$$\eta_\infty(Q_{s+t}(1)) = \eta_\infty(Q_s(1)) \eta_\infty(Q_t(1)).$$

Thus for any  $t \in \mathcal{T}$  we have the exponential formula

$$\eta_\infty(Q_t(1)) = e^{\rho t} \quad \text{for some } \rho \in \mathbb{R}.$$

Choosing  $\bar{\eta} = \eta_\infty$  in (8) and using (7) for any  $t \in \mathcal{T}(\tau)$  we check that

$$\mu \bar{Q}_t(1) = \prod_{s \in [0, t]_\tau} \{1 + [\Phi_s(\mu)(\bar{Q}_\tau(1)) - \Phi_s(\eta_\infty)(\bar{Q}_\tau(1))]\}. \quad (23)$$

Conversely, by (21) we have

$$\sum_{s \in \mathcal{T}(\tau)} |\Phi_s(\mu)(\bar{Q}_\tau(1)) - \Phi_s(\eta_\infty)(\bar{Q}_\tau(1))| < \infty$$

as well as

$$\|\bar{Q}_t(1)\| \leq \prod_{s \in \mathcal{T}(\tau)} (1 + (1 - \alpha(1))^{s/\tau} \|Q_\tau(1)\| / \eta_\infty(Q_\tau(1))) < \infty.$$

The two estimates discussed above ensure that  $\mu \overline{Q}_t(1)$  converges, as  $\mathcal{T}(\tau) \ni t \rightarrow \infty$ , to a non zero number and we have

$$0 < \inf_{t \in \mathcal{T}(\tau)} \mu(\overline{Q}_t(1)) \leq \sup_{t \in \mathcal{T}(\tau)} \|\overline{Q}_t(1)\| < \infty. \quad (24)$$

Note that the convergence of the series ensures that  $\Phi_t(\mu)(\overline{Q}_\tau(1)) \xrightarrow{t \rightarrow \infty} 1$ . Also observe that (24) implies that

$$\forall \mu \in \mathcal{P}(E) \quad q(\mu) := \sup_{t \in \mathcal{T}(\tau)} \|Q_t(1)\| / \mu(Q_t(1)) < \infty.$$

We are now in a position to state our first corollary of Theorem 2.1.

**Corollary 2.2.** *Under the assumptions of Theorem 2.1, for any  $\mu, \eta \in \mathcal{P}(E)$  and  $t \in \mathcal{T}$  we have the local contraction estimate*

$$\|\Phi_t(\mu) - \Phi_t(\eta)\|_{tv} \leq (q(\mu) \wedge q(\eta)) (1 - \alpha(1))^{t/\tau} \|\mu - \eta\|_{tv}. \quad (25)$$

Choosing  $\mu = \delta_x$  in (23), we also readily check that there exists  $h \in \mathcal{B}_b(E)$  such that  $\eta_\infty(h) = 1$  and

$$\lim_{t \rightarrow \infty} \overline{Q}_t(1)(x) = \lim_{t \rightarrow \infty} Q_t(1)(x) = h(x) > 0.$$

By the dominated convergence theorem, for any  $s \in \mathcal{T}$  this implies that

$$\overline{Q}_{s+t}(1)(x) = \overline{Q}_s(\overline{Q}_t(1))(x) \xrightarrow{t \rightarrow \infty} h(x) = \overline{Q}_s(h)(x).$$

Choosing  $H = h$  in (10), formula (11) reads

$$\Psi_h(\Phi_t(\eta)) = \Psi_h(\eta) P_t^h \quad \text{with} \quad P_t^h(f) := Q_t(hf) / Q_t(h) = R_{0,t}^{(t)}(f). \quad (26)$$

Note that  $h \in \mathcal{C}_b(E)$  as soon as  $Q_\tau(\mathcal{B}_b(E)) \subset \mathcal{C}_b(E)$ . Also observe that

$$\delta_x P_t^h = \Psi_h(\Phi_t(\delta_x)).$$

The second corollary is now a direct consequence of Theorem 2.1.

**Corollary 2.3.** *Under the assumptions of Theorem 2.1, there exists a positive function  $h \in \mathcal{B}_b(E)$  and a constant  $\rho \in \mathbb{R}$  such that for any  $t \in \mathcal{T}$  we have*

$$Q_t(h) = e^{\rho t} h \quad \text{and} \quad \eta_\infty^h P_t^h = \eta_\infty^h \quad \text{with} \quad \eta_\infty^h := \Psi_h(\eta_\infty). \quad (27)$$

In addition, we have the total variation exponential decays

$$\|\delta_x P_t^h - \eta_\infty^h\|_{tv} \leq \frac{\|h\|}{\eta_\infty(h)} (1 - \alpha(1))^{t/\tau}. \quad (28)$$

Using (24) and applying the uniform estimate (21) to  $\mu = \delta_x$  and  $\eta = \eta_\infty$  for any  $s, t \in \mathcal{T}$  and  $x \in E$  we have

$$|\overline{Q}_{t+s}(1)(x)/\overline{Q}_t(1)(x) - 1| = |\Phi_t(\delta_x)(\overline{Q}_s(1)) - 1| \leq 2 q(\eta_\infty) (1 - \alpha(1))^{t/\tau}.$$

This implies that

$$\|\overline{Q}_t(1) - h/\eta_\infty(h)\| = \lim_{s \rightarrow \infty} \|\overline{Q}_t(1) - \overline{Q}_{t+s}(1)\| \leq 2 q(\eta_\infty)^2 (1 - \alpha(1))^{t/\tau}.$$

On the other hand, for any  $f \in \mathcal{B}_b(E)$  and  $x \in E$  we have

$$|\overline{Q}_t(f)(x) - \frac{h(x)}{\eta_\infty(h)} \eta_\infty(f)| \leq \|f\| |\overline{Q}_t(1)(x) - h(x)/\eta_\infty(h)| + \frac{\|h\|}{\eta_\infty(h)} \|\Phi_t(\delta_x)(f) - \eta_\infty(f)\|,$$

which leads us to our final corollary of the section.

**Corollary 2.4.** *Under the assumptions of Corollary 2.3, we have the operator norm exponential decays*

$$\sup_{\|f\| \leq 1} \left\| \overline{Q}_t(f) - \frac{h}{\eta_\infty(h)} \eta_\infty(f) \right\| \leq 2 (1 - \alpha(1))^{t/\tau} (\|h\|/\eta_\infty(h) + q(\eta_\infty)^2), \quad (29)$$

as well as for any  $s, t \in \mathcal{T}$  and  $\eta \in \mathcal{P}(E)$  the uniform total variation estimates

$$\begin{aligned} & \|\Psi_{Q_t(1)}(\Phi_s(\eta)) - \Psi_h(\eta_\infty)\|_{tv} \\ & \leq q(\eta_\infty) (1 - \alpha(1))^{s/\tau} + 2 (1 - \alpha(1))^{t/\tau} (\|h\|/\eta_\infty(h) + q(\eta_\infty)^2). \end{aligned} \quad (30)$$

The last assertion comes from the decomposition

$$\begin{aligned} \overline{\eta} = \eta_\infty & \implies \Psi_{Q_t(1)}(\mu)(f) - \Psi_h(\eta_\infty)(f) \\ & = [\Psi_{\overline{Q}_t(1)}(\mu) - \Psi_{\overline{Q}_t(1)}(\eta_\infty)](f) + \eta_\infty(f(\overline{Q}_t(1) - h/\eta_\infty(h))). \end{aligned}$$

The above corollary can be interpreted as an extended version of the Krein-Rutman theorem to positive semigroups satisfying the uniform minorisation condition (20). The estimate (30) is often used in the analysis of the ergodic properties of particle absorption models, see for instance [22, 72] as well as equation (51) in the present article.

A more refined analysis under weaker conditions, including exponential stability theorems and contraction properties of time homogeneous semigroups is provided in section 2.3.

## 2.2 Contraction of $R$ -semigroups

We now present the first main result of the article, the proof of which can be found in section 4.1.

**Theorem 2.5.** *Assume that  $(\mathcal{H}_1)$  is satisfied. Then there exist constants  $a < \infty$  and  $b > 0$  such that for any  $t \in \mathcal{T}$ ,  $s, u \in \mathcal{T}_t$  and  $\mu, \eta \in \mathcal{P}_{V/H_{s,t}}$ , we have the uniform contraction estimate*

$$\|\mu R_{s,u}^{(t)} - \eta R_{s,u}^{(t)}\|_{V/H_{u,t}} \leq a e^{-b(u-s)} \|\mu - \eta\|_{V/H_{s,t}}. \quad (31)$$

**Remark 2.6.** *As we shall see in Lemma 4.1, condition  $(\mathcal{H}_1)$  also ensures the existence of some  $0 < \epsilon < 1$  and some constant  $c > 0$  such that for any time horizon  $t \in \mathcal{T}$  and any  $s + \tau \in \mathcal{T}_t$  we have the Lyapunov estimate*

$$R_{s,s+\tau}^{(t)}(V/H_{s+\tau,t}) \leq \epsilon V/H_{s,t} + c. \quad (32)$$

*In this direction, we also emphasise that the main ingredient of the proof of Theorem 2.5 is the  $V$ -contraction for Markov operators discussed in Lemma 3.2.*

The estimate on the left-hand side of (15) allows one to control, uniformly with respect to the time parameter, the quantities  $\mu(H_{s,t})$  in terms of the parameters  $\mu(V)$ , for any  $\mu \in \mathcal{P}_V(E)$ . Since these uniform estimates will be used several times in the sequel, we present them here in a general form. Applying the Markov inequality, for any  $\mu \in \mathcal{P}_V(E)$  the left-hand side condition in (15) ensures the existence of some  $n \geq 1$  and some  $\delta_n > 0$  such that

$$\epsilon_n = \theta(\mu(V) + n) \leq \epsilon_H \quad \text{and} \quad \mu(H_{s,t}) \geq \delta_n \mu(\mathcal{V}_{\epsilon_n}) \geq \frac{\delta_n}{1 + \mu(V)/n} > 0. \quad (33)$$

We conclude that

$$(\mathcal{H}_1) \implies \mu \in \mathcal{P}_V(E) \quad 0 < \omega_H(\mu) := \inf_{t \in \mathcal{T}(\tau)} \inf_{s \in \mathcal{T}_t(\tau)} \mu(H_{s,t}) \leq \mu(V). \quad (34)$$

Similarly, we check that the condition  $\kappa_V(\mu) < \infty$  ensures the tightness of sequence of measures  $\Phi_{s,t}(\mu)$  indexed by  $s, t \in \mathcal{T}$ . In the same vein, by (5), condition  $(\mathcal{H}_1)$  ensures that the flow of measures  $\bar{\eta}_t$  is tight. Thus, choosing

$$\epsilon_n = \theta(\bar{\eta}(V) + n) \leq \epsilon_H \quad \text{with} \quad \bar{\eta}(V) := \sup_{t \in \mathcal{T}(\tau)} \bar{\eta}_t(V)$$

arguing as above we check that

$$\inf_{t \in \mathcal{T}(\tau)} \inf_{s \in \mathcal{T}_t(\tau)} \bar{\eta}_s(H_{s,t}) \geq \varsigma_{\epsilon_n}(H) \inf_{s \in \mathcal{T}(\tau)} \bar{\eta}_s(\mathcal{V}_{\epsilon_n}) \geq \varsigma_{\epsilon_n}(H)/(1 + \bar{\eta}(V)/n) > 0.$$

Arguing as above we present a simple way to check that  $\lambda^- > 0$ . Assume that  $\inf_{t \in \mathcal{T}} Q_{t,t+\tau}(1) \geq g_\tau > 0$  for some continuous function  $g_\tau > 0$ . As a result, we have that

$$\bar{\eta}(V) := \sup_{t \in \mathcal{T}(\tau)} \bar{\eta}_t(V) \leq \kappa_V(\bar{\eta}_0) < \infty \implies \lambda^- > 0.$$

The right-hand side assertion follows from the fact that

$$\epsilon_n = \theta(\bar{\eta}(V) + n) \implies \bar{\eta}_t(Q_{t,t+\tau}(1)) \geq \frac{\inf_{V_{\epsilon_n}} g_\tau}{1 + \bar{\eta}(V)/n} > 0.$$

We are now in position to discuss some direct consequences of the above theorem. The first corollary and its time homogeneous version discussed in Corollary 2.15 can be interpreted as an extended version of the Krein-Rutman theorem to time varying positive semigroups.

**Corollary 2.7.** *Assume that  $(\mathcal{H}_1)$  is satisfied. Then there exists some finite constant  $c < \infty$  such that for any  $t \in \mathcal{T}$  and  $s \in \mathcal{T}_s$  we have the operator norm exponential decay*

$$\sup_{\|f\|_V \leq 1} \left\| \bar{Q}_{s,t}(f) - \frac{H_{s,t}}{\bar{\eta}_s(H_{s,t})} \bar{\eta}_t(f) \right\|_V \leq c e^{-bt},$$

where  $b$  was defined in (31).

*Proof.* Using (9) and (11) we check that

$$\Psi_{H_{s,t}}(\bar{\eta}_s) R_{s,t}^{(t)}(f/H) = \Phi_{s,t}(\bar{\eta}_s)(f)/\Phi_{s,t}(\bar{\eta}_s)(H) = \bar{\eta}_t(f)/\bar{\eta}_t(H).$$

This yields the decomposition

$$\bar{Q}_{s,t}(f)(x) - \frac{H_{s,t}}{\bar{\eta}_s(H_{s,t})} \bar{\eta}_t(f) = H_{s,t}(x) \left( R_{s,t}^{(t)}(f/H) - \Psi_{H_{s,t}}(\bar{\eta}_s) R_{s,t}^{(t)}(f/H) \right)$$

Applying (31) to  $u = t$ ,  $\mu = \delta_x$  and  $\eta = \Psi_{H_{s,t}}(\bar{\eta}_s)$ , for any  $\|f\|_V = \|f/H\|_{V/H} \leq 1$  we check the estimate

$$\left| \bar{Q}_{s,t}(f) - \frac{H_{s,t}}{\bar{\eta}_s(H_{s,t})} \bar{\eta}_t(f) \right| \leq a e^{-b(t-s)} \left( V + \frac{H_{s,t}}{\bar{\eta}_s(H_{s,t})} \bar{\eta}_s(V) \right)$$

and this concludes the proof. ■

When  $H = 1$ , the extended version of the above corollary in the context of random semigroups arising in filtering is provided in [127]. The proof in [127] relies on rather sophisticated coupling and decomposition techniques given in [87], which were further developed in [56]. The time homogeneous version of the above result is provided in Corollary 2.15. In this context, condition  $(\mathcal{H}_1)$  is readily checked by choosing  $H = h$ , where  $h$  is the positive eigenstate of the positive semigroup, see for instance (38) and (42).

Our second main result transfers the stability of the  $R$ -semigroup to that of the normalised semigroup  $\Phi$ .

**Theorem 2.8.** *Under the assumptions of Theorem 2.5, for any  $t \in \mathcal{T}$  and  $s \in \mathcal{T}_t$  and any  $\mu, \eta \in \mathcal{P}_V(E)$  we have the local contraction estimate*

$$\|\Phi_{s,t}(\mu) - \Phi_{s,t}(\eta)\|_V \leq a \kappa(\eta, \mu) e^{-b(t-s)} \|\mu - \eta\|_V \quad (35)$$

with  $(a, b)$  as in (31) and  $\kappa(\eta, \mu)$  given by

$$\kappa(\eta, \mu) := \kappa_H(\mu) (1 + \kappa_V(\eta)) (1 + \eta(V)/\omega_H(\eta)) / \omega_H(\mu).$$

This result is a direct consequence of the  $V$ -contraction estimates (31) stated in Theorem 2.5 and rather elementary Boltzmann-Gibbs estimates (57) and (58) presented in section 3.1. The full proof is provided in section 4.2.

The  $V$ -norm stability of the semigroup  $\Phi_{s,t}$  is also discussed in [127] (for instance [127, Corollary 1]). The proof in [127] is based on Corollary 2.7 and it does not provide local Lipschitz contraction estimates.

Choosing  $H = 1$  in (11) we have

$$[\Phi_{s,t}(\mu) - \Phi_{s,t}(\eta)](f) = \frac{1}{\mu(H_{s,t})} (\mu - \eta) (D_\eta \Phi_{s,t}(f)) \quad (36)$$

with the first order operator  $D_{\mu_2} \Phi_{s,t}$  defined by the formula

$$D_\eta \Phi_{s,t}(f)(x) := H_{s,t}(x) (\Phi_{s,t}(\delta_x) - \Phi_{s,t}(\eta))(f).$$

Taylor-type expansions of higher order are also discussed in [1]. A weak version of the total variation estimate (35) is now easily obtained from the above perturbation formula.

**Corollary 2.9.** *Consider the  $R$ -semigroup (11) associated to the unit function  $H = 1$ . Under the assumptions of Theorem 2.5, for any  $t \in \mathcal{T}$ ,  $s \in \mathcal{T}_t$  and  $\eta \in \mathcal{P}_V(E)$  we have*

$$\sup_{\|f\|_{V/H_{s,t}} \leq 1} \|D_\eta \Phi_{s,t}(f)\|_V \leq a e^{-b(t-s)} (1 + \eta(V)/\omega_H(\eta)). \quad (37)$$

*Proof.* We have

$$D_\eta \Phi_{s,t}(f)(x) := H_{s,t}(x) \int \eta(dy) \frac{H_{s,t}(y)}{\eta(H_{s,t})} (\delta_x R_{s,t}^{(t)} - \delta_y R_{s,t}^{(t)})(f).$$

Using (31) we check that

$$\|\delta_x R_{s,t}^{(t)} - \delta_y R_{s,t}^{(t)}\|_{V/H_{s,t}} \leq a e^{-b(t-s)} ((V/H_{s,t})(x) + (V/H_{s,t})(y)).$$

This implies that

$$\sup_{\|f\|_{V/H_{s,t}} \leq 1} \|D_\eta \Phi_{s,t}(f)\|_V \leq a e^{-b(t-s)} (1 + \eta(V) \|H_{s,t}\|_V / \eta(H_{s,t}))$$

and we can now conclude. ■

**Remark 2.10.** *These weak form estimates are particularly useful in the convergence analysis of the mean field particle models associated with sub-Markovian integral operators. Taylor-type expansions at any order are discussed in section 3.1.3 and chapter 10 in [40], see also section 2.3 in the more recent article [1].*

## 2.3 Time homogenous models

This section is concerned with a more refined analysis of time homogenous semigroups satisfying the following condition

$$\begin{cases} Q_\tau(V)/V \in \mathcal{C}_0(E) \\ Q_\tau(h) = e^{\rho\tau} h > 0 \text{ for some } \rho \in \mathbb{R}, \text{ and } h \in \mathcal{C}_{0,V}(E) \text{ s.t. } \|h\|_V = 1. \end{cases} \quad (38)$$

We recall that the parameter  $\tau$  in the above display stands for positive parameter such that  $Q_\tau(1) > 0$ . Illustrations and practical ways of checking the Lyapunov condition stated above and in (38) are provided in section 3.3. Illustrations and sufficient conditions ensuring the existence of the leading eigen-pair  $(\rho, h)$  satisfying (38) are discussed in section 3.4 dedicated to quasi-compact positive operators, see also Corollary 2.3.

In this context, choosing  $H = h$  in (10), for any  $\eta \in \mathcal{P}_V(E)$  we have the formula (26). It should be noted that the Doob h-transform,  $P_t^h$  defined in (26) is a semigroup of Markov operators from  $\mathcal{B}_{V/h}(E)$  to itself.

We now introduce another condition.

$(\mathcal{H}^h)$  *There exists some  $r_0 > 0$  and some  $\alpha : r \in [r_0, \infty[ \mapsto \alpha(r) \in ]0, 1]$  such that for any  $r \geq r_0$  we have*

$$\sup_{V^h(x) \vee V^h(y) \leq r} \|\delta_x P_\tau^h - \delta_y P_\tau^h\|_{tv} \leq 1 - \alpha(r) \quad \text{with } V^h := V/h. \quad (39)$$

The above condition is clearly met as soon as the following minorisation condition is met

$$\forall x \in K_r := \{x : V^h(x) \leq r\}, \quad Q_\tau(x, dz) \geq \iota_r \nu_r(dz) \quad (40)$$

for some probability measure  $\nu_r$  on  $K_r$  and some  $\iota_r > 0$ . To check this claim, observe that for any  $x \in K_r$  we have

$$\begin{aligned} P_\tau^h(x, dz) &\geq \frac{1}{e^{\rho\tau} \sup_{K_r} h} Q_\tau(x, dz) 1_{K_r}(z) h(z) \\ &\geq \alpha(r) \nu_r^h(dz) \quad \text{with } \alpha(r) = \iota_r e^{-\rho\tau} \frac{\nu_r(h)}{\sup_{K_r} h} \quad \text{and } \nu_r^h := \Psi_h(\nu_r). \end{aligned}$$

This clearly implies (39). We also recall that (40) is satisfied if there exists a locally bounded measure  $\chi_\tau$  such that

$$Q_\tau(x_1, dx_2) \geq q_\tau(x_1, x_2) \chi_\tau(dx_2),$$

with a lower semi-continuous function  $q_\tau(x_1, x_2)$  with respect to the first variable, and upper-semicontinuous with respect to the second (see for instance [51, Proposition 17.4.2]).

We are now in position to state the main result of this section, the proof of which can be found in section 4.3.

**Theorem 2.11.** *Assume that  $(\mathcal{H}^h)$  is satisfied. Then the Markov semigroup  $P_t^h$  has a single invariant measure  $\mu_\infty^h \in \mathcal{P}_{V/h}(E)$ . In addition, there also exists some finite constant  $a_h < \infty$  and some parameter  $b_h > 0$ , such that for any  $\mu, \eta \in \mathcal{P}_{V/h}(E)$  and  $t \in \mathcal{T}$  we have the contraction estimate*

$$\|\mu P_t^h - \eta P_t^h\|_{V/h} \leq a_h e^{-b_h t} \|\mu - \eta\|_{V/h}. \quad (41)$$

**Remark 2.12.** *Choosing  $H = h$  in (10) the R-semigroup coincides with the semigroup obtained from the Doob  $h$ -transform of  $Q$ . As we shall see in Lemma 4.2, the condition  $Q_\tau(V)/V \in \mathcal{C}_0(E)$  ensures the existence of some  $0 < \epsilon < 1$  and some constant  $c > 0$  such that*

$$P_\tau^h(V/h) \leq \epsilon V/h + c.$$

*This property also ensures that for any  $\eta \in \mathcal{P}_V(E)$  we have  $\kappa_V(\eta) < \infty$ . For a more thorough discussion on the consequences of the Foster-Lyapunov inequality we refer the reader to Proposition 4.3.*

*In this context, as with Theorem 2.5, the main ingredient of the proof is the  $V$ -contraction for Markov operators discussed in Lemma 3.2.*

**Remark 2.13.** *The above theorem ensures the uniqueness of the invariant measure  $\eta_\infty^h = \eta_\infty^h P_t^h \in \mathcal{P}_{V/h}(E)$  and exponential decay to equilibrium of the  $h$ -process. Choosing  $\bar{\eta} = \eta_\infty := \Psi_{1/h}(\eta_\infty^h)$  we have  $\bar{\eta} = \Phi_\tau(\bar{\eta})$ . In this scenario, we readily check (15) using the fact that*

$$\lambda_{t,t+\tau} := \bar{\eta} Q_{t,t+\tau}(1) = e^{\rho\tau} \quad \text{and} \quad H_{s,t} = h \quad (42)$$

*and thus, (16) and  $(\mathcal{H}^h)$  imply that  $(\mathcal{H}_1)$  holds with  $H = h$ .*

We illustrate the impact of the above theorem with some direct corollaries.

**Corollary 2.14.** *Under condition  $(\mathcal{H}^h)$ , the measure  $\eta_\infty := \Psi_{1/h}(\eta_\infty^h) \in \mathcal{P}_V(E)$  is the unique invariant measure of the semigroup  $\Phi_t$ . In addition, for any  $\mu, \eta \in \mathcal{P}_V(E)$  and  $t \in \mathcal{T}$  we have  $\kappa_V(\mu) < \infty$  and*

$$\|\Phi_t(\mu) - \Phi_t(\eta)\|_V \leq a_h \kappa(\mu, \eta) e^{-b_h t} \|\mu - \eta\|_V, \quad (43)$$

*with  $(a_h, b_h)$  as in (41) and the parameters  $\kappa(\mu, \eta)$  defined by*

$$\kappa(\mu, \eta) := \kappa_h(\mu) (1 + \kappa_V(\eta)) (1 + \eta(V)/\eta(h)) / \mu(h).$$

*Proof.* Combining (26) with the Boltzman-Gibbs estimate (57) we readily check the estimate

$$\|\Phi_t(\mu) - \Phi_t(\eta)\|_V \leq \kappa_h(\mu) (1 + \kappa_V(\eta)) \|(\Psi_h(\mu) - \Psi_h(\eta))P_t^h\|_{V/h}.$$

The contraction estimate (41) now implies that

$$\|(\Psi_h(\mu) - \Psi_h(\eta))P_t^h\|_{V/h} \leq a_h \kappa_h(\mu) (1 + \kappa_V(\eta)) e^{-b_h t} \|\Psi_h(\mu) - \Psi_h(\eta)\|_{V/h}.$$

Using the Boltzman-Gibbs estimate (58) we also have

$$\|\Psi_h(\mu) - \Psi_h(\eta)\|_{V/h} \leq \frac{1}{\mu(h)} \left(1 + \frac{\eta(V)}{\eta(h)}\right) \|\mu - \eta\|_V.$$

This ends the proof of the corollary. ■

Corollary 2.14 is closely related to the exponential decay estimates stated in Theorem 1 of [61] under different regularity conditions. In contrast with [61], our approach is based on the  $V$ -contraction properties of the  $h$ -semigroups and it allows one to derive local contraction estimates. The proof of the next corollary follows word-for-word the same arguments as the proof of Corollary 2.7, thus it is skipped.

**Corollary 2.15.** *Under condition  $(\mathcal{H}^h)$ , for any  $t \in \mathcal{T}$  we have the operator norm exponential decay*

$$\sup_{\|f\|_V \leq 1} \left\| e^{-t\rho} Q_t(f) - \frac{h}{\eta_\infty(h)} \eta_\infty(f) \right\|_V \leq a_h e^{-b_h t} (1 + \eta_\infty(V)/\eta_\infty(h)) \quad (44)$$

with the same parameters  $(a_h, b_h)$  as in (41).

## 2.4 Illustrations and comparisons

In this section, we highlight some of the comparisons between the models and the regularity conditions discussed in the present article and conditions often used in the literature for positive integral operators.

### Comparison of regularity conditions

We begin by remarking that the class of positive semigroups discussed in this article encapsulates discrete generation Feynman-Kac semigroups defined for any  $t \in \mathcal{T}(\tau)$  by the formula

$$Q_{t,t+\tau}(x, dy) = G_t(x) P_{t,t+\tau}(x, dy)$$

with the potential function  $G_t$  and the Markov transition  $P_{t,t+\tau}$  defined by

$$G_t := Q_{t,t+\tau}(1) \quad \text{and} \quad P_{t,t+\tau}(f) := Q_{t,t+\tau}(f)/Q_{t,t+\tau}(1). \quad (45)$$

This class of probabilistic models arises in a variety of disciplines including statistical physics, biology, signal processing, rare event analysis, and many others; see [39, 40, 42, 51] and the relevant references therein. In this context, the uniform minorization condition is a well known strong condition ensuring the stability of the semigroups  $\Phi_{s,t}$  and the existence of fixed point invariant measures for time homogeneous semigroups; see for instance [42, Theorem 2.3], [44, Lemma 2.1 & Lemma 2.3], [49, Lemma 2.1], [40, section 12.2], as well as [42, section 2.1.2 & section 3.1.3] and [39, section 4].

The next condition is taken from [20], see also conditions (A1) and (A2) in the proof of [21, Theorem 2.1], as well as in [22]. In terms of the Markov transition  $P_{s,s+\tau}$  discussed in (45) it takes the following form:

( $\mathcal{P}$ ) : For any  $s \in \mathcal{T}$  and  $(x_1, x_2) \in E^2$  there exists some  $\nu \in \mathcal{P}(E)$  such that for  $i = 1, 2$  and any  $t \geq s + \tau$  we have the estimates

$$\delta_{x_i} P_{s,s+\tau} \geq \epsilon_1 \nu \quad \text{and} \quad \epsilon_2 \|Q_{s+\tau,t}(1)\| \leq \nu(Q_{s+\tau,t}(1)) \quad (46)$$

for some parameters  $0 < \epsilon_i \leq 1$  whose values don't depend on  $x_i$ , nor on  $s \in \mathcal{T}$ . The time homogeneous version of the above condition is also used in [78, Theorem 10] and [79, Theorem 7.1], see also the design of admissible coupling constants (a.k.a. generalised Doeblin's conditions) discussed in [5, section 2.2], see also [72, section 2]. Condition ( $\mathcal{P}$ ) is also the main ingredient of the proof of Theorem 3.1 in the article [26] on multi-type birth-and-death processes.

When ( $\mathcal{P}$ ) is met, for any  $f \geq 0$  and  $i = 1, 2$  we have the lower bound estimate

$$R_{s,s+\tau}^{(t)}(f)(x_i) \geq \epsilon_1 \frac{\nu(Q_{s+\tau,t}(1) f)}{P_{s,s+\tau}(Q_{s+\tau,t}(1))(x_i)} \geq \epsilon_1 \epsilon_2 \Psi_{Q_{s+\tau,t}(1)}(\nu_s)(f).$$

This shows that

$$(\mathcal{P}) \implies (20) \quad \text{with} \quad \alpha(1) := \epsilon_1 \epsilon_2 \quad \text{and} \quad \nu = \Psi_{Q_{s+\tau,t}(1)}(\nu_s).$$

As previously mentioned, the uniform minorisation condition (20) is difficult to check in practice; several sufficient conditions are discussed in [39, 40, 42, 44, 48, 49].

The next condition is a slight extension of [4, condition (3.24)].

( $\mathcal{Q}$ ) : There exists a positive measurable function  $\varsigma_s(x) > 0$ , a constant  $\rho > 0$  and probability measures  $\nu_s$  such that

$$\varsigma_s(x) \nu_s(dy) \leq \delta_x Q_{s,s+\tau}(dy) \leq \rho \varsigma_s(x) \nu_s(dy). \quad (47)$$

This condition implies that for any  $f \geq 0$  we have

$$R_{s,u}^{(t)}(f) := \frac{Q_{s,u}(q_{u,t}f)}{Q_{s,u}(q_{u,t})} \geq \rho^{-1} \nu_{s,u}^{(t)}(f) \quad \text{with} \quad \nu_s^{(t)}(f) := \frac{\nu_s(Q_{s+\tau,t}(1)f)}{\nu_s(Q_{s+\tau,t}(1))}.$$

This shows that

$$(\mathcal{Q}) \implies (20) \quad \text{with} \quad \alpha(1) := \rho^{-1} \quad \text{and} \quad \nu = \nu_s^{(t)}.$$

Condition ( $\mathcal{Q}$ ) with  $s(x) = 1$  is also discussed in [39, section 4.3.2], [92] and [61, section 3.3].

## Absorbed processes with bounded potentials

Whenever the operator  $Q_{s,t}$  is sub-Markovian, we have the nonlinear transport equation

$$\Phi_{s,t}(\mu) = \mu M_{s,t}^\mu$$

with the collection of Markov transition  $M_{s,t}^\mu$  indexed by  $\mu \in \mathcal{P}(E)$  given by the formula

$$M_{s,t}^\mu(f)(x) = Q_{s,t}(1)(x) \frac{Q_{s,t}(f)(x)}{Q_{s,t}(1)(x)} + (1 - Q_{s,t}(1)(x)) \Phi_{s,t}(\mu)(f).$$

We also recall that for any  $s \leq u \leq t$  we have the nonlinear semigroup equation

$$M_{s,t}^\mu = M_{s,u}^\mu M_{u,t}^{\Phi_{s,u}(\mu)}.$$

This shows that the normalised semigroup  $\Phi_{s,t}$  is the semigroup of a nonlinear Markov process sometimes called process conditioned to non-absorption at every time step. For time homogeneous models, unless  $\mu$  coincides with the quasi-invariant measure  $\Phi_t(\eta_\infty) = \eta_\infty$ , the process is a nonlinear interacting jump process. In this interpretation, the distribution on path-space of the nonlinear process coincides with the McKean-measure associated with a jump type process whose jumps intensity depends on the distribution of the random states.

For a more thorough discussion on the nonlinear interacting jump processes associated with these nonlinear Markov semigroups we refer to section 12.3 in [40] and the articles [1, 42, 43]. Next proposition is a direct consequence of Theorem 2.8.

**Proposition 2.16.** *Under the assumptions of Theorem 2.5, for any  $t \in \mathcal{T}$  and  $s \in \mathcal{T}_t$  and any  $\mu, \eta \in \mathcal{P}_V(E)$  we have the local Lipschitz operator norm estimate*

$$\|\delta_x M_{s,t}^\mu - \delta_x M_{s,t}^\eta\|_V \leq a \kappa(\eta, \mu) e^{-b(t-s)} \|\mu - \eta\|_V$$

with  $(a, b, \kappa(\eta, \mu))$  as in (35).

Sub-Markovian operators are naturally associated with killed or absorbed stochastic processes. Consider a stochastic flow  $t \in [s, \infty[ \mapsto X_{s,t}^c(x)$  starting at  $X_{s,s}^c(x) = x \in E$  when  $t = s$  and absorbed in a cemetery state  $c$  at some random time  $T_s^c(x)$ . For instance, suppose we are given an auxiliary stochastic flow  $X_{s,t}(x)$  evolving on  $E$ , which is sent to the cemetery at some uniformly bounded rate  $U_t(y) \geq 0$  when at  $y \in E$ . In this situation, we have the so-called Feynman-Kac propagator formulae

$$Q_{s,t}(f)(x) = \mathbb{E} \left( f(X_{s,t}^c(x)) 1_{T_s^c(x) > t} \right) = \mathbb{E} \left( f(X_{s,t}(x)) \exp \left( - \int_s^t U_u(X_{s,u}(x)) du \right) \right). \quad (48)$$

In this context, it is readily checked that the normalised and unnormalised semigroups are connected by the formula

$$\mu Q_{s,t}(f) = \Phi_{s,t}(\mu)(f) \exp \left( \int_s^t \Phi_{s,u}(\mu)(U_u) du \right). \quad (49)$$

Applying the above to  $f = 1$  for any  $\mu_1, \mu_2 \in \mathcal{P}(E)$  and  $s \leq t$  we check that

$$\mu_1 Q_{s,t}(1) / \mu_2 Q_{s,t}(1) = \exp \left( \int_s^t (\Phi_{s,u}(\mu_1)(U_u) - \Phi_{s,u}(\mu_2)(U_u)) du \right).$$

The discrete time version of the above formula coincides with (6). Under the assumptions of Theorem 2.1 the norm of the first order operator introduced in (36) decays exponentially. That is for any  $f$  such that  $\text{osc}(f) := \sup_{(x,y) \in E^2} |f(x) - f(y)| = 1$ , we have

$$\|D_{\mu_2} \Phi_{s,t}(f)\| \leq q (1 - \alpha(1))^{(t-s)/\tau} \quad \text{with} \quad q := \sup \frac{\mu_1 Q_{s,t}(1)}{\mu_2 Q_{s,t}(1)} < \infty. \quad (50)$$

In the above display, the supremum is taken over all indices  $(s, t)$  such that  $t \in \mathcal{T}(\tau)$ ,  $s \in \mathcal{T}_t(\tau)$  and  $\mu_1, \mu_2 \in \mathcal{P}(E)$ .

We now state some results pertaining to the limiting behaviour of the occupation measure of the time homogeneous stochastic flow. These results are direct consequences of our semigroup analysis that build on the analysis developed in [22, 27, 72].

Observe that for any  $s \leq u \leq t$  we have

$$\mathbb{E} (f(X_{s,u}^c(x)) \mid T_s^c(x) > t) = \Psi_{Q_{u,t}(1)} (\Phi_{s,u}(\delta_x)) (f).$$

Setting  $X_u^c(x) := X_{0,u}^c(x)$  and  $T^c(x) := T_0^c(x)$ , due to (30) we obtain the following corollary.

**Corollary 2.17.** *Under the assumptions of Corollary 2.2, we have*

$$\begin{aligned} & \sup_{\|f\| \leq 1} \sup_{x \in E} \left| \mathbb{E} \left( \frac{1}{t} \int_0^t f(X_u^c(x)) du \mid T^c(x) > t \right) - \Psi_h(\eta_\infty)(f) \right| \\ & \leq \frac{\tau}{t} \frac{1}{|\log(1 - \alpha(1))|} (q(\eta_\infty) + 2(\|h\|/\eta_\infty(h) + q(\eta_\infty)^2)) \end{aligned} \quad (51)$$

In the above display,  $h$  stands for the eigenfunction defined in Corollary 2.4. The measure  $\Psi_h(\eta_\infty)$  which coincides with the invariant measure of the  $h$ -process is sometimes called the quasi-ergodic measure of the non absorbed process.

In the same vein, we also obtain a uniform bound on the  $L^2$  distance.

**Proposition 2.18.** *Under the assumptions of Corollary 2.2, for any  $f \in \mathcal{B}_b(E)$  with  $\text{osc}(f) \leq 1$  and any  $t \in \mathcal{T}$  we have the uniform estimate*

$$\begin{aligned} & \left| \mathbb{E} \left( \left( \frac{1}{t} \int_0^t f(X_u^c(x)) du - \Psi_h(\eta_\infty)(f) \right)^2 \mid T^c(x) > t \right) \right| \\ & \leq \frac{8\tau}{t} \frac{(\|h\|/\eta_\infty(h) + (1 \vee q(\eta_\infty))^2)}{|\log(1 - \alpha(1))|}. \end{aligned}$$

*Proof.* First observe that for any  $s \leq u \leq v \leq t$  we have

$$\begin{aligned} & \mathbb{E} \left( f(X_{s,u}^c(x)) f(X_{s,v}^c(x)) \mid T_s^c(x) > t \right) \\ &= \frac{\delta_x Q_{s,u} \left( f Q_{u,t}(1) \frac{Q_{u,v}(f Q_{v,t}(1))}{Q_{u,v}(Q_{v,t}(1))} \right)}{\delta_x Q_{s,u}(Q_{u,t}(1))} \\ &= \int \Psi_{Q_{u,t}(1)}(\Phi_{s,u}(\delta_x))(dy) f(y) \int \Psi_{Q_{v,t}(1)}(\Phi_{u,v}(\delta_y))(dz) f(z). \end{aligned}$$

Replacing  $f$  by  $(f - \Psi_h(\eta_\infty)(f))$  there is no loss of generality to assume that  $\Psi_h(\eta_\infty)(f) = 0$ . In this situation, thanks again to (30), we have

$$\begin{aligned} & \left| \mathbb{E} \left( \left( \frac{1}{t} \int_0^t f(X_u^c(x)) du \right)^2 \mid T^c(x) > t \right) \right| \\ & \leq \frac{4}{t^2} (\|h\|/\eta_\infty(h) + (1 \vee q(\eta_\infty))^2) \int_{0 \leq u \leq v \leq t} (e^{-a(v-u)} + e^{-a(t-v)}) dv du \end{aligned}$$

with

$$a := \frac{|\log(1 - \alpha(1))|}{\tau}.$$

This implies that

$$\left| \mathbb{E} \left( \left( \frac{1}{t} \int_0^t f(X_u^c(x)) du \right)^2 \mid T^c(x) > t \right) \right| \leq \frac{8}{at} (\|h\|/\eta_\infty(h) + (1 \vee q(\eta_\infty))^2).$$

■

**Remark 2.19.** We end our discussion of sub-Markov operators with bounded potentials by noting that we may also extend Proposition 2.18 to also obtain bounds in  $L^p$  of the order  $t^{-1/2}$ . The idea behind the proof is to write the averages in terms of the  $h$ -process (in particular, the trajectorial version) and then apply  $L^p$  bounds for occupation measures of a Markov process, using the Poisson equation associated with the  $h$ -semigroup (see for instance Lemma 8.4.11 in [51] in discrete time settings).

## Unbounded potentials

The analysis of unbounded potential functions  $U_t(x)$  requires more care. The prototype solvable model is the  $\mathbb{R}^{d_0}$ -valued linear Gaussian diffusion

$$dX_{s,t}(x) = \mathcal{A}_t X_{s,t}(x) dt + \mathcal{R}_t^{1/2} d\mathcal{W}_t \quad \text{with quadratic potential} \quad U_t(x) = \frac{1}{2} x' \mathcal{S}_t x \quad (52)$$

for some multidimensional Brownian motion  $\mathcal{W}_t \in \mathbb{R}^{d_1}$  with  $d_1 \geq d_0 \geq 1$  and some conformable matrices  $(\mathcal{A}_t, \mathcal{R}_t, \mathcal{S}_t)$  such that  $\mathcal{R}_t, \mathcal{S}_t > 0$ . Solving the Riccati time reversal equation

$$-\partial_s \mathcal{Q}_s := \mathcal{A}'_s \mathcal{Q}_s + \mathcal{Q}_s \mathcal{A}_s - \mathcal{Q}_s \mathcal{R}_s \mathcal{Q}_s + \mathcal{S}_s,$$

for  $s \in [0, t]$  with terminal condition  $\mathcal{Q}_t = \mathcal{Q}$ , we check that

$$\begin{aligned} h_{s,t}(x) &:= \exp\left(-\frac{1}{2} x' \mathcal{Q}_s x\right) \quad \text{and} \quad h_{t,t}(x) = h(x) := \exp\left(-\frac{1}{2} x' \mathcal{Q} x\right) \\ \implies h_{s,t}^{-1}(\partial_s + L_s)(h_{s,t})(x) - U_s(x) &= \lambda_s := -\text{Tr}(\mathcal{R}_s \mathcal{Q}_s). \end{aligned}$$

In the above display,  $L_s$  stands for the generator of the stochastic process  $X_{0,s}(x)$ . This yields the formula

$$Q_{s,t}(h)(x) = \exp\left(-\int_s^t \text{Tr}(\mathcal{R}_u \mathcal{Q}_u) du\right) h_{s,t}(x).$$

On the other hand, for any  $s \leq t$  we have the exponential change of measure formula

$$\begin{aligned} Q_{s,t}(f)(x) &= e^{\int_s^t \lambda_u du} \mathbb{E}\left(f(X_{s,t}^{(t)}(x)) h_{s,t}(x)/h(X_{s,t}^{(t)}(x))\right) \\ &= e^{\int_s^t \lambda_u du} h_{s,t}(x) R_{s,t}^{(t)}(f/h)(x), \end{aligned} \tag{53}$$

where  $(X_{s,u}^{(t)}(x))_{s \leq u \leq t}$  is the stochastic flow with generator

$$L_u^{(t)}(f) := L_u(f) + \Gamma_{L_u}(h_{u,t}, f)/h_{u,t} \quad \text{and} \quad R_{s,u}^{(t)}(x, dy) := \mathbb{P}(X_{s,u}^{(t)}(x) \in dy).$$

Equivalently, we have

$$R_{s,t}^{(t)}(f) = \frac{Q_{s,t}(h f)}{Q_{s,t}(h)} = \frac{Q_{s,u}(Q_{u,t}(h f))}{Q_{s,u}(Q_{u,t}(h))} = \frac{Q_{s,u}(h_{u,t} P_{u,t}^h(f))}{Q_{s,u}(h_{u,t})} = R_{s,u}^{(t)} R_{u,t}^{(t)}(f).$$

In this situation, the stochastic flow  $X_{s,u}^{(t)}(x)$  of time varying  $R$ -process is given for any  $0 \leq s \leq u \leq t$  by

$$dX_{s,u}^{(t)}(x) = (\mathcal{A}_u - \mathcal{R}_u \mathcal{Q}_u) X_{s,u}^{(t)}(x) du + \mathcal{R}_u^{1/2} dW_u. \tag{54}$$

For time homogenous models  $(\mathcal{A}_t, \mathcal{R}_t, \mathcal{S}_t) = (\mathcal{A}, \mathcal{R}, \mathcal{S})$  it is more judicious to choose  $\mathcal{Q} = \mathcal{Q}_\infty$  where  $\mathcal{Q}_\infty$  stands for the unique positive fixed point of the Riccati equation

$$\mathcal{A}' \mathcal{Q}_\infty + \mathcal{Q}_\infty \mathcal{A} - \mathcal{Q}_\infty \mathcal{R} \mathcal{Q}_\infty + \mathcal{S} = 0.$$

In this case, we have  $h_{s,t} = h$  for any  $s \leq t$ , and the  $R$ -process resumes to the  $h$ -process defined by the time homogeneous diffusion

$$dX_s^h(x) = (\mathcal{A} - \mathcal{R} \mathcal{Q}_\infty) X_s^h(x) du + \mathcal{R}^{1/2} dW_s. \tag{55}$$

The Feynman-Kac path integrals associated with these models coincides with coupled quantum harmonic oscillators arising in physics. We refer the reader to [45] for a detailed discussion on this subject.

### 3 A brief review on integral operators

#### 3.1 $V$ -norms and Boltzmann-Gibbs estimates

For a given function  $V \geq 1/2$ , the  $V$ -oscillation of a function  $f \in \mathcal{B}(E)$  is given by

$$\text{osc}_V(f) := \sup_{x,y} \frac{|f(x) - f(y)|}{V(x) + V(y)} \leq \|f\|_V$$

and with a slight abuse of notation, the  $V$ -norm on  $\mathcal{M}(E)$  is given by

$$\|\mu\|_V = \sup\{|\mu(f)| : \|f\|_V \leq 1\} = \sup\{|\mu(f)| : \text{osc}_V(f) \leq 1\} = |\mu|(V).$$

In the above display,  $|\mu| := \mu_+ + \mu_-$  stands for with the total variation measure associated with a Hahn-Jordan decomposition  $\mu = \mu_+ - \mu_-$  of the measure. For a detailed proof of the equivalent formulations in the latter definition, we refer to Proposition 8.2.16 in [51]. The choice of condition  $V \geq 1/2$  in the above two definitions is imposed only to recover the conventional total variation dual distance between probability measures when choosing  $V = 1/2$  in the dual norms.

Recall that  $\mathcal{M}_V(E) \subset \mathcal{M}(E)$  is the subset of Radon measures with  $\|\mu\|_V < \infty$ . A probability measure  $\mu \in \mathcal{P}(E)$  is said to be strictly positive, and we write  $\mu > 0$ , whenever  $\mu(A) > 0$  for any open set  $A \subset E$ . We finally recall that the integral map  $(\mu, f) \mapsto \mu(f)$  gives the isometry

$$\mathcal{M}_V(E) \simeq \mathcal{C}_{0,V}(E)'.$$

The above assertion is a direct consequence of [120, Theorem 3.1] applied to the Nachbin family  $\mathcal{V} := \{V_\alpha = \alpha/V : \alpha > 0\}$ . See also [119, Theorem 3.26] as well as [13, Theorem 2.1].

As mentioned above, when  $V = 1/2$  we recover the conventional total variation norm. For any  $\mu_1, \mu_2 \in \mathcal{P}(E)$  we recall that

$$\|\mu_1 - \mu_2\|_{tv} \leq 1 - \epsilon \iff (\exists \nu \in \mathcal{P}(E) : \mu_1 \geq \epsilon \nu \text{ and } \mu_2 \geq \epsilon \nu). \quad (56)$$

We recall that the Boltzmann-Gibbs transformation  $\Psi_h$  associated with some bounded positive function  $h > 0$  and defined in (2) satisfies the following Lipschitz type estimate

$$\|\Psi_h(\mu_1) - \Psi_h(\mu_2)\|_{tv} \leq \frac{\|h\|}{\mu_1(h) \vee \mu_2(h)} \|\mu_1 - \mu_2\|_{tv}.$$

For a detailed proof of the above assertion we refer to [80, lemma 9.5], or [52, appendix B], see also [44] as well as [96, proposition 3.1] and [40, proposition 12.1.7].

Finally, we present a technical lemma regarding the  $V$ -norm estimates of the Boltzmann-Gibbs operators; the proof is given in appendix A.2.

**Lemma 3.1.** For any  $0 < h \leq V$  and any  $\mu_1, \mu_2 \in \mathcal{P}_{V/h}(E)$  we have the estimate

$$\|\Psi_{1/h}(\mu_1) - \Psi_{1/h}(\mu_2)\|_V \leq \frac{1}{\mu_1(1/h)} \left(1 + \frac{\mu_2(V/h)}{\mu_2(1/h)}\right) \|\mu_1 - \mu_2\|_{V/h}. \quad (57)$$

In addition, for any  $\mu_1, \mu_2 \in \mathcal{P}_V(E)$  we have

$$\|\Psi_h(\mu_1) - \Psi_h(\mu_2)\|_{V/h} \leq \frac{1}{\mu_1(h)} \left(1 + \frac{\mu_2(V)}{\mu_2(h)}\right) \|\mu_1 - \mu_2\|_V. \quad (58)$$

### 3.2 $V$ -Dobrushin coefficient

Let us fix  $0 \leq s \leq t$  and let  $V_s, V_t$  denote bounded, continuous functions such that  $V_s, V_t \geq 1$ . The  $V$ -Dobrushin coefficient of Markov integral operator  $P_{s,t}$  from  $\mathcal{B}_{V_t}(E)$  into  $\mathcal{B}_{V_s}(E)$  is the norm operator defined by

$$\beta_{V_s, V_t}(P_{s,t}) = \sup_{\mu, \eta \in \mathcal{P}_{V_s}(E)} \|(\mu - \eta)P_{s,t}\|_{V_t} / \|(\mu - \eta)\|_{V_s}. \quad (59)$$

We also have the equivalent formulation

$$\begin{aligned} \beta_{V_s, V_t}(P_{s,t}) &= \sup \{ \text{osc}_{V_s}(P_{s,t}(f)) : \text{osc}_{V_t}(f) \leq 1 \} \\ &= \sup_{(x,y) \in E^2} \|\delta_x P_{s,t} - \delta_y P_{s,t}\|_{V_t} / (V_s(x) + V_s(y)). \end{aligned} \quad (60)$$

When  $V_s = V_t = V$ , we write  $\beta_V(P_{s,t})$  instead of  $\beta_{V,V}(P_{s,t})$ . For a more thorough discussion on these contraction coefficients, we refer the reader to sections 8.2.5 - 8.2.7 in [51]. If in addition  $V = 1/2$  we write  $\beta(P_{s,t})$  instead of  $\beta_{1/2}(P_{s,t})$ , to denote the conventional Dobrushin ergodic coefficient with respect to the total variation distance

$$\beta(P_{s,t}) = \sup_{(x,y) \in E^2} \|\delta_x P_{s,t} - \delta_y P_{s,t}\|_{tv}.$$

By (56) the contraction condition  $\beta(P_{s,t}) \leq (1 - \alpha)$  is satisfied for some parameter  $\alpha \in ]0, 1]$  if and only if the following minorisation condition holds

$$\forall (x, y) \in E^2 \quad \exists \nu \in \mathcal{P}(E) : \delta_x P_{s,t} \geq \alpha \nu \quad \text{and} \quad \delta_y P_{s,t} \geq \alpha \nu.$$

The next lemma provides some contraction conditions in terms of a Foster-Lyapunov inequality and a local minorisation condition on compact level sets. The proof is in appendix A.3.

**Lemma 3.2.** Assume that there exist continuous, bounded functions  $V_s, V_t \geq 1$  with compact level sets,  $\epsilon \in [0, 1[$  as well as a function  $\alpha : r \in [r_0, \infty[ \mapsto \alpha(r) \in ]0, 1]$ , for some  $r_0 \geq 1$  such that for any  $r \geq r_0$  we have

$$P_{s,t}(V_t) \leq \epsilon V_s + 1 \quad \text{and} \quad \sup_{V_s(x) \vee V_s(y) \leq r} \|\delta_x P_{s,t} - \delta_y P_{s,t}\|_{tv} \leq 1 - \alpha(r). \quad (61)$$

Then, for any  $0 < \delta \leq 1$  we have the contraction coefficient estimate

$$\beta_{V_s^{\epsilon,\delta}, V_t^{\epsilon,\delta}}(P_{s,t}) \leq 1 - \alpha(r_\epsilon(\delta))(1 - \delta/2) \quad \text{with} \quad V_t^{\epsilon,\delta} = 1/2 + \rho_\epsilon(\delta) V_t$$

with the parameters

$$r_\epsilon(\delta) := r_0 \vee \frac{2}{1 - \epsilon} \left( 1 + \frac{1}{1 - \epsilon} \left( 3 + 6 \left( \frac{1}{\delta} - 1 \right) \right) \right) \quad \text{and} \quad \rho_\epsilon(\delta) := \frac{\delta}{1 + \epsilon} \frac{\alpha(r_\epsilon(\delta))}{2r_\epsilon(\delta)}. \quad (62)$$

Again,  $\epsilon$  and  $\alpha$  may depend on  $s$  and  $t$ . Now observe that

$$r_0(\delta) := r_0 \vee 4 \left( 2 + 3 \left( \frac{1}{\delta} - 1 \right) \right) > 1 \quad \text{and} \quad \rho_0(\delta) := \frac{\delta}{2r_0(\delta)} (1 - \beta(P)) \xrightarrow{\delta \rightarrow 0} 0.$$

Thus, whenever (61) is met with  $V_s = V_t = r_0 = 1$ , choosing  $\epsilon = 0$  in (61) we have, for any  $r > 1$ ,

$$\lim_{\delta \rightarrow 0} \beta_{V_s^{0,\delta}, V_t^{0,\delta}}(P_{s,t}) = \beta(P_{s,t}) \leq 1 - \alpha(r).$$

Whenever  $P_{s,t}$  is a Markov semigroup and  $V_t \geq V$  for some function  $V \geq 1$  with compact level sets, the Foster-Lyapunov inequality in the left-hand side of (61) ensures that for any initial probability measure  $\mu \in \mathcal{P}_{V_0}(E)$ , the distribution flow  $\mu_t := \mu P_{0,t}$  indexed by  $t \in \mathcal{T}(\tau)$  for some  $\tau > 0$  is tight. To check this claim, notice that

$$(61) \implies \mu_t(V) \leq \mu_t(V_t) \leq \epsilon^{t/\tau} \mu(V_0) + (1 - \epsilon)^{-1} \leq \mu(V_0) + (1 - \epsilon)^{-1}. \quad (63)$$

By the Markov inequality, for any  $\delta > 0$  this implies that

$$K_\delta := \{x : \delta V(x) \leq \mu(V_0) + (1 - \epsilon)^{-1}\} \subset\subset E \implies \sup_{t \in \mathcal{T}(\tau)} \mu_t(E - K_\delta) \leq \delta. \quad (64)$$

In this situation, if we set  $V^{\epsilon,\delta} = 1/2 + \rho_\epsilon(\delta) V$ , for any  $s \leq t$  and  $\mu, \eta \in \mathcal{P}_{V_s}(E)$  we also have the contraction estimate

$$\|\mu P_{s,t} - \eta P_{s,t}\|_{V^{\epsilon,\delta}} \leq (1 - \alpha(r_\epsilon(\delta))(1 - \delta/2))^{(t-s)/\tau} \|\mu - \eta\|_{V_s^{\epsilon,\delta}}. \quad (65)$$

For time homogeneous models, condition (61) clearly ensures the existence of a single invariant probability measure  $\mu_\infty = \mu_\infty P_t \in \mathcal{P}_V(E)$ .

### 3.3 Lyapunov conditions

This section presents some practical ways of checking the Lyapunov condition stated on the left-hand side of (61). Given some  $c > 0$ ,  $\epsilon \in [0, 1[$ ,  $K \in \mathcal{E}$  and functions  $W_s, W_t \geq 0$ , we have

$$\left. \begin{array}{l} P_{s,t}(W_t)(x) \leq \epsilon W_s(x) \quad \text{if } x \in E - K \\ P_{s,t}(W_t)(x) \leq c \quad \quad \quad \text{if } x \in K \end{array} \right\} \implies P_{s,t}(W_t/c) \leq \epsilon W_s/c + 1.$$

Moreover,

$$P_{s,t}(W_t) \leq \epsilon W_t + c \implies P_{s,t}(V_t) \leq \epsilon V_s + 1 \quad \text{with} \quad V_u = 1 + \epsilon W_u/c \geq 1, \text{ for } u = s, t.$$

For continuous and time homogenous semigroups  $P_{t+s} = P_t P_s$  with generator  $L$  defined on some domain  $D(L) \ni W$ , for any  $a > 0$  and  $c < \infty$  we have the Lyapunov condition

$$L(W) \leq -aW + c \iff \forall \delta > 0 \quad P_\delta(W) \leq \frac{1}{1+a\delta} W + c\delta.$$

A detailed proof of the above assertion can be found in [51, Theorem 17.4.1].

In the case where  $c = 0$ , i.e. there exists some  $K \subset E$  such that

$$\forall x \in E - K, \quad L(W)(x) \leq -aW(x) \quad \text{and} \quad \forall x \in K \quad L(W)(x) \leq c, \quad (66)$$

we readily check that

$$L(W) \leq -aW =: b.$$

For instance, when  $E$  is a normed space and  $L(W)$  is continuous, it suffices to find a sufficient large radius  $r$  that

$$\forall \|W\| \geq r \quad W^{-1}(x)L(W)(x) \leq -a.$$

In this situation condition (66) is met with  $K = \{\|W\| \leq r\}$  and  $c = \sup_K \|L(W)\|$ .

Whenever  $c > 0$ , by replacing  $W$  by  $W/c$ , there is no loss of generality by assuming that the above Lyapunov condition is met for  $c = 1$ . In the same vein, replacing  $W$  by  $W_b = W + b$  for some  $b \geq 0$  we find that

$$L(W_b) \leq -aW_b + c_b \quad \text{with} \quad c_b := c + ab.$$

Also notice that in this case, for any  $\delta > 0$  we have

$$P_\delta(W_b) \leq \frac{1}{1+a\delta} W_b + c\delta \implies P_\delta(W) \leq \frac{1}{1+a\delta} W + \left(c - \frac{ab}{1+a\delta}\right) \delta.$$

Thus, there is no loss of generality to assume that  $W \geq b$  satisfies the Lyapunov condition for some  $b > 0$ . For a more thorough discussion on these Foster-Lyapunov techniques, including the design of Lyapunov functions for Ornstein-Uhlenbeck processes as well as for damped and over-damped Langevin diffusions; see [51, section 17.4 & section 17.5]. We end this section with some comments on the condition  $Q_\tau(V)/V \in \mathcal{C}_0(E)$ . Assume that

$$G_\tau := Q_\tau(1) \in \mathcal{C}_0(E) \quad \text{and} \quad 0 < G_\tau \leq 1.$$

In this case, we have the decomposition

$$Q_\tau(x, dy) = G_\tau(x) P_\tau(x, dy)$$

with the Markov transition

$$P_\tau(f) := Q_\tau(f)/Q_\tau(1).$$

Assume that  $P_\tau(\mathcal{C}(E)) \subset \mathcal{C}(E)$  and  $P_\tau(W) \leq aW + b$  for some non negative function  $W \in \mathcal{C}(E)$  and for some  $a, b \geq 0$ . Now, one can check that  $Q_\tau(V)/V \in \mathcal{C}_0(E)$  using

$$V := 1 + W \implies \frac{Q_\tau(V)}{V} = G_\tau \frac{(1+b) + aW}{1+W} \leq c G_\tau \quad \text{with} \quad c := ((1+b) \vee a) > 0.$$

We illustrate the above discussion in the context of Gaussian integral operators on Euclidian spaces. To avoid unnecessary vector and matrix-notation we restrict our attention to one dimensional models, that is  $E = \mathbb{R}$ . The analysis of multidimensional models follows along the same lines. Consider the following models

$$G_\tau(x) = e^{-U_0(x)} \in \mathcal{C}_0(\mathbb{R}) \quad \text{and} \quad P_\tau(x, dy) = \frac{1}{\sqrt{2\pi\sigma_1(x)^2}} e^{-\frac{(y-b(x))^2}{2\sigma_1(x)^2}} dy. \quad (67)$$

In the above display,  $U_0 \geq 0$  is some function with compact level sets,  $\sigma_1(x) > 0$  is a strictly positive function and  $b(x)$  is the drift, both of which exhibit at most linear growth; that is

$$|b(x)| + \sigma_1(x) \leq c_1 + c_2 |x|$$

for some constant  $c_1, c_2 \geq 0$ . In this situation, we check that

$$W(x) = 1 + |x| \implies P_\tau(W) \leq a W \quad \text{with} \quad a = (1 + c_1) \vee c_2.$$

When  $G_\tau \in \mathcal{C}_b(E)$  is not necessarily vanishing at infinity, the problem amounts to finding a function  $V \geq 1$  with compact level set such that  $P_\tau(V)/V \in \mathcal{C}_0(E)$ . The design of such functions for a Markov transition  $P_\tau$  is rather standard in applied probability literature (cf. for instance [56, 87, 127]). We further illustrate these constructions with the Gaussian transition discussed in (67). We assume that

$$\|\sigma_1\| < \infty, \quad |b(x)| \leq c + (1 - \delta) |x| \quad \text{for some} \quad 0 \leq c < \infty \quad \text{and} \quad 0 < \delta \leq 1 \quad (68)$$

and set  $V(x) := e^{v|x|}$  for some  $v > 0$ .

In this case, we check that

$$\begin{aligned} P_\tau(V)(x)/V(x) &= e^{-v(|x|-b(x))+\frac{v^2\sigma_1(x)^2}{2}} \int_0^\infty \frac{e^{-\frac{(y-(b(x)+v\sigma_1(x)^2))^2}{2\sigma_1(x)^2}}}{\sqrt{2\pi\sigma_1(x)^2}} dy \\ &\quad + e^{-v(|x|+b(x))+\frac{v^2\sigma_1(x)^2}{2}} \int_{-\infty}^0 \frac{e^{-\frac{(y-(b(x)-v\sigma_1(x)^2))^2}{2\sigma_1(x)^2}}}{\sqrt{2\pi\sigma_1(x)^2}} dy. \end{aligned}$$

In this situation, whenever  $G_\tau \in \mathcal{C}_b(E)$  we have

$$Q_\tau(V)(x)/V(x) \leq 2e^{(v\|\sigma_1\|)^2/2} \|G_\tau\| e^{-v(|x|-|b(x)|)} \leq 2e^{vc+(v\|\sigma_1\|)^2/2} \|G_\tau\| V(x)^{-\delta},$$

which implies (16). Note that the above construction requires the sub-linear growth of the drift function  $b(x)$  stated in (68). Consider the Gaussian density

$$G_\tau(x) = \exp\left(-\frac{x^2}{2\sigma_0^2(x)}\right). \quad (69)$$

In the above display,  $\sigma_0(x) > 0$  stands for some strictly positive bounded function. In this case, we have

$$\begin{aligned} |b(x)| &\leq c_0 + c_1 |x| \quad \text{for some } 0 \leq c_0, c_1 < \infty \\ \implies Q_\tau(V)(x)/V(x) &\leq 2e^{(v\|\sigma\|)^2/2} \exp\left(-v\left(|x| + \frac{|x|^2}{2v\|\sigma_0\|^2} - |b(x)|\right)\right) \xrightarrow{|x| \rightarrow \infty} 0. \end{aligned}$$

Observe that for any  $0 < \delta \leq 1$  we have

$$|x| \geq r := (2v\|\sigma_0\|^2) c_1 \implies c_1 \leq (1 - \delta) + |x|/(2v\|\sigma_0\|^2).$$

This shows that

$$c_0 + c_1 |x| \leq (c_0 + c_1 r) + (1 - \delta) |x| + |x|^2/(2v\|\sigma_0\|^2) \quad \text{with } c_2 = c_0 + c_1 r$$

from which we conclude that

$$|b(x)| \leq c_0 + c_1 |x| \implies Q_\tau(V)(x)/V(x) \leq 2e^{vc_2+(v\|\sigma\|)^2/2} V(x)^{-\delta},$$

which yields (16).

### 3.4 Quasi-compact operators

Next, we recall some standard definitions and compactness principles from time homogeneous positive semigroup theory.

Let  $(Q_t)_{t \geq 0}$  be a time homogenous positive semigroup from  $\mathcal{B}_V(E)$  into  $\mathcal{B}_V(E)$ . Then  $Q_t$  is said to be irreducible (a.k.a. ideal irreducible) if there exists no closed  $Q_t$ -invariant ideals distinct from  $\{0\}$  and  $\mathcal{B}_V(E)$  (cf. [109, Definition 4.2.1]). It is also well known (e.g. [57, Proposition 2.1] or [63, Proposition 4.1]) that this condition is met if and only if for any non zero function  $f \geq 0$  on  $\mathcal{B}_V(E)$  and any non zero positive measure  $\mu$  on  $E$ , there exists some  $t \in \mathcal{T}$  such that  $\mu Q_t(f) > 0$ .

The spectral radius of  $(Q_t)_{t \geq 0}$  is defined as

$$r_V(Q) := \lim_{t \rightarrow \infty} \|Q_t\|_V^{1/t} = \lim_{t \rightarrow \infty} \|Q_t(V)\|_V^{1/t} = \inf_{t \geq 0} \|Q_t(V)\|_V^{1/t}. \quad (70)$$

In the above display,  $\|\cdot\|_V$  stands for the operator norm.

The semigroup  $Q_t$  is said to be quasi-compact if its essential spectral radius

$$\bar{r}_V(Q) := \lim_{t \rightarrow \infty} (\inf \{\|Q_t - T\|_V : T \text{ compact}\})^{1/t}$$

satisfies  $\bar{r}_V(Q) < r_V(Q)$ . Recalling that the product of positive operators  $Q_1 Q_2$  is compact as soon as the  $Q_1$  is compact (cf. theorem VI.12 in [112]), the quasi-compactness property of the operator  $Q_t$  (for sufficiently large  $t$ ) is clearly met as soon  $Q_\tau$  is a compact operator for some  $\tau \in \mathcal{T}$ . Such one-parameter semigroups are sometimes called eventually compact semigroups (see for instance section 3 in the book [74]).

Now assume there exists some  $\tau \in \mathcal{T}$  such that the discrete generation semigroup  $Q_t$  indexed by  $t \in \mathcal{T}(\tau)$  is irreducible and quasi-compact on  $\mathcal{B}_V(E)$ . By a variant of the Krein-Rutman theorem (cf. for instance Theorem 1.1 in [58, 103]),  $r_V(Q) > 0$  is an eigenvalue corresponding to a non null eigenfunction  $h \in \mathcal{B}_V(E)$ . More precisely, there exists some a non null function  $h \in \mathcal{B}_V(E)$  such that

$$\forall t \in \mathcal{T}(\tau) \quad Q_t(h) = e^{\rho t} h \quad \text{with} \quad \rho := \log r_V(Q).$$

Note that since  $Q_\tau$  is irreducible, for any  $x \in E$  there exists some  $t \in \mathcal{T}(\tau)$  such that

$$Q_t(h)(x) = e^{\rho t} h(x) > 0.$$

**Remark 3.3.** *Whenever  $Q_\tau(V)/V \in \mathcal{B}_0(E)$ , for any  $t \geq \tau$  the operators  $Q_t$  map  $\mathcal{B}_V(E)$  into  $\mathcal{B}_{0,V}(E)$ , and the eigen-function  $h$  belongs to  $\mathcal{B}_{0,V}(E)$ . If, in addition,  $Q_\tau$  maps  $\mathcal{B}_V(E)$  into  $\mathcal{C}_V(E)$ , then the operators  $Q_t$  map  $\mathcal{B}_V(E)$  into  $\mathcal{C}_{0,V}(E)$ , and the eigen-function  $h$  belongs to  $\mathcal{C}_{0,V}(E)$ .*

*Additionally, if the assumptions of Theorem 2.1 are met, Corollary 2.4 ensures that the existence of an eigenfunction  $h \in \mathcal{C}_{0,V}(E)$ .*

Sufficient conditions in terms of the Lyapunov functions  $V \geq 1$  ensuring the compactness of  $Q_t$  for some  $t \in \mathcal{T}$  are discussed in [61] and [113]. For instance, we have the following lemma.

**Lemma 3.4.** *Assume that  $Q_\tau(V)/V \in \mathcal{B}_0(E)$  for some  $\tau > 0$ . In addition, assume that for any compact set  $K \subset\subset E$  the operator  $Q_\tau^K(f) := 1_K Q_\tau(1_K f)$  is compact on  $\mathcal{B}_V(E)$ . In this situation, for any  $t \in \mathcal{T}$  with  $t \geq \tau$ , the operator  $Q_t$  is a compact operator from  $\mathcal{B}_V(E)$  into itself.*

*Proof.* For any  $t \in \mathcal{T}$ , we set  $R_t^K(f) := 1_K Q_t(f)$  so that

$$Q_t = R_t^K + R_t^{K^c} \quad \text{and} \quad Q_{t+\tau} - Q_\tau^K Q_t = Q_\tau R_t^{K^c} + R_\tau^{K^c} R_t^K.$$

This decomposition implies that

$$\|Q_{t+\tau} - Q_\tau^K Q_t\|_V \leq (\|Q_\tau\|_V + \|R_t^K\|_V) \|R_\tau^{K^c}\|_V \leq (\|Q_\tau\|_V + \|Q_t\|_V) \|R_\tau^{K^c}\|_V.$$

Also note that

$$\|R_\tau^{K^c}\|_V = \|1_{K^c} Q_\tau(V)/V\|.$$

Thus, choosing a sequence of compact sets  $K_n$  such that

$$1_{K_n^c} Q_\tau(V) \leq \frac{1}{n} V$$

implies that

$$\|Q_{t+\tau} - Q_\tau^{K_n} Q_t\|_V \leq \frac{1}{n} (\|Q_\tau\|_V + \|Q_t\|_V).$$

Since the product operator  $Q_\tau^{K_n} Q_t$  is compact,  $Q_{t+\tau}$  is the limit in norm of compact operators, hence it is compact. This ends the proof of the lemma.  $\blacksquare$

**Remark 3.5.** *The condition  $Q_\tau(V)/V \in \mathcal{B}_0(E)$  allows one to localise the operators on compact sets. The compactness condition of the semigroup  $Q_\tau^K$  is readily checked for absolutely continuous operators of the form*

$$Q_\tau(x, dy) = q_\tau(x, y) \nu_\tau(dy). \quad (71)$$

*In the above display  $q_\tau(x, y)$  is a continuous density with respect to some Radon measure  $\nu_\tau$  on  $E$ . Indeed, in this case, for any collection of functions  $(f_n)_{n \geq 0} \in \mathcal{B}_V(E)^\mathbb{N}$  in the unit ball  $\{f \in \mathcal{B}_V(E) : \|f\|_V \leq 1\}$  we have*

$$\sup_{n \geq 0} \|Q_\tau^K(f_n)\|_V \leq \|Q_\tau^K(V)/V\|.$$

*In addition, for any  $(x, y) \in K^2$  we have*

$$|Q_\tau^K(f_n)(x) - Q_\tau^K(f_n)(y)| \leq \int |q_\tau(x, z) - q_\tau(y, z)| 1_K(z) \nu_\tau(dz).$$

*Since  $q_\tau$  is continuous on the compact set  $(K \times K)$ , it is uniformly continuous. Thus for any  $\epsilon > 0$ , there is some  $\delta > 0$  such that  $d(x, y) \leq \delta$  implies that for any  $n \geq 0$  we have*

$$|Q_\tau^K(f_n)(x) - Q_\tau^K(f_n)(y)| \leq \epsilon \nu_\tau(K).$$

*By the Arzela-Ascoli theorem,  $Q_\tau^K(f_n)$  converges uniformly to a continuous limit extended by 0 outside the set  $K$ . Recalling that  $V \geq 1$  is also converges on  $\mathcal{B}_V(E)$ .  $Q_\tau^K$  is an irreducible and compact operator on  $\mathcal{B}_V(E)$ . The above models encapsulate Markov transitions restricted to a compact set  $K \subset \mathbb{R}^r$ , for some  $r \geq 1$ , defined by*

$$Q_\tau^K(x, dy) := 1_K(x) p_\tau(x, y) 1_K(y) dy.$$

*In the above,  $p_\tau(x, y)$  is a continuous probability transition density with respect to the Lebesgue measure  $dy$  on  $\mathbb{R}^r$ . For a more thorough discussion on this class of compact integral operators we refer to [67] and the more recent article [16].*

**Remark 3.6.** Now assume that  $E$  is a countable space. In this situation, whenever  $V = 1$  we can also use the following equivalence principle (see for instance Theorem 2.1 in [131])

$$Q_\tau \text{ is compact on } \mathcal{B}(E) \iff \forall \epsilon > 0 \exists K_\epsilon \text{ finite s.t. } \sup_{x \in E} Q_\tau(1_{E-K_\epsilon})(x) \leq \epsilon.$$

More generally, assume that  $Q_\tau(V)/V \in \mathcal{B}_0(E)$ . In this situation, for any  $\epsilon > 0$  the set

$$K_\epsilon := \{Q_\tau(V)/V \geq \epsilon\}$$

is finite, and following the proof of the above lemma, we have

$$\|Q_{t+\tau} - Q_\tau^{K_\epsilon} Q_t\|_V \leq \epsilon (\|Q_\tau\|_V + \|Q_t\|_V).$$

Since for any finite set  $K$ , the operator

$$Q_\tau^K(f)(x) = 1_K(x) \sum_{y \in K} Q_\tau(x, y) f(y)$$

is bounded and has finite range, it is compact and we thus conclude that  $Q_t$  is compact for any  $t \geq \tau$  as soon as  $Q_\tau(V)/V \in \mathcal{B}_0(E)$ .

The set  $E$  can be written as the union  $E = \cup_{n \geq 0} K_n$  of an increasing sequence of compact sets  $K_n$ . Whenever  $Q_t$  is compact on  $\mathcal{B}_V(E)$ , up to a subsequence extraction the functions  $Q_t(V1_{K_n^c}) \in \mathcal{B}_V(E)$  forms a Cauchy sequence. Thus, for any  $\epsilon > 0$  there exists some  $n_\epsilon \geq 1$  such that for any  $n_\epsilon \leq m \leq n$  we have

$$\|Q_t(V1_{K_n^c}) - Q_t(V1_{K_m^c})\|_V = \|Q_t(V1_{K_n - K_m})\|_V \leq \epsilon$$

and therefore

$$\|Q_t(V1_{K_m^c})/V\| \leq \epsilon.$$

Now, assume that for any  $\epsilon > 0$  there exists some  $n_\epsilon \geq 1$  such that

$$\forall n \geq n_\epsilon \quad \|Q_t(V1_{K_n^c})/V\| \leq \epsilon.$$

For any  $\|f\|_V \leq 1$ , we have

$$\|Q_t(f) - Q_t(1_{K_n} f)\|_V \leq \|Q_t(V1_{K_n^c})/V\| \leq \epsilon.$$

Arguing as above, using the fact that  $f \in \mathcal{B}_V(E) \mapsto 1_{K_n} f \in \mathcal{B}_V(E)$  is a finite range compact operator we conclude that  $Q_t$  is compact on  $\mathcal{B}_V(E)$ . We conclude that

$$Q_t \text{ compact on } \mathcal{B}_V(E) \iff \forall \epsilon > 0 \exists K_\epsilon \text{ finite s.t. } \|Q_t(1_{K_\epsilon^c} V)\|_V \leq \epsilon.$$

Whenever  $E$  is compact, by a theorem of de Pagter (cf. [109, Theorem 4.2.2]), a compact and irreducible positive operator  $Q_t$  on  $\mathcal{C}(E)$  has a positive spectral radius  $r(Q_t) > 0$ , while its essential spectral radius is null. Applying the Krein-Rutman theorem (cf. [58, 103, Theorem 1.1]), there exists some non-negative and non-zero measure  $\nu_\infty \in \mathcal{M}(E) = \mathcal{C}(E)'$  and a non-negative and non-zero function  $h \in \mathcal{C}(E)$  such that

$$\nu_\infty Q_t = e^{\rho t} \nu_\infty \quad \text{and} \quad Q_t(h) = e^{\rho t} h \quad \text{with} \quad \rho = \log r(Q_1). \quad (72)$$

This yields the fixed point equation

$$\eta_\infty := \nu_\infty / \nu_\infty(1) = \Phi_t(\eta_\infty).$$

More generally, when  $E$  is compact and  $Q_\tau$  is a positive operator from  $\mathcal{B}(E)$  into itself, the Schauder fixed-point theorem ensures the existence of a fixed point

$$\eta_\infty = \Phi_\tau(\eta_\infty) \in \mathcal{P}(E) \quad \text{and} \quad \forall t \in \mathcal{T}(\tau), \quad \eta_\infty(Q_t(f)) = \eta_\infty(f) \eta_\infty(Q_\tau(1))^{t/\tau}.$$

Note that the right-hand side assertion is a direct consequence of the product formula (7).

For potentially non-compact spaces, condition (38) is met as soon as  $Q_\tau(V)/V \in \mathcal{C}_0(E)$  and  $Q_\tau$  is a quasi-compact operator from  $\mathcal{B}_V(E)$  into itself and mapping  $\mathcal{B}_V(E)$  into  $\mathcal{C}_V(E)$ . Several sufficient conditions in terms of the Lyapunov functions  $V \geq 1$  ensuring the quasi-compactness properties of  $Q_\tau$  are discussed in [61, 77, 113], see also Lemma 3.4 and Remark 3.5 in the present article. For a more detailed discussion on quasi-compact and compact operators we refer to [74, 103, 109, 112] and references therein.

For potentially non-compact space  $E$  we have the following Krylov-Bogolioubov-type theorem, which is of interest in its own right.

**Theorem 3.7.** *Assume that  $Q_\tau(\mathcal{C}_b(E)) \subset \mathcal{C}_b(E)$ . Then the mapping  $\Phi_\tau$  has at least one invariant probability measure  $\eta_\infty$  if and only if there exists some probability measure  $\eta$  such that the flow  $\Phi_t(\eta)$  indexed by  $t \in \mathcal{T}(\tau)$  is tight and we have*

$$\beta_t(\eta) := \Phi_t(\eta)(Q_\tau(1)) \in ]0, 1] \longrightarrow_{t \rightarrow \infty} \beta_\infty(\eta) > 0. \quad (73)$$

*Whenever these conditions are met we have  $\beta_\infty(\eta) = \eta_\infty(Q_\tau(1))$ .*

The proof of the above theorem is provided in section 4.4. Related equivalent conditions for the existence of quasi-invariant measures are discussed in [91] for models such that  $Q_\tau(\mathcal{C}_0(E)) \subset \mathcal{C}_0(E)$ , see also [75].

## 4 Proofs of the stability theorems

### 4.1 Proof of Theorem 2.5

This section is mainly concerned with the proof of Theorem 2.5. For any  $\mu \in \mathcal{P}_V(E)$  and  $n \geq 1$  we set

$$\epsilon_n(\mu) := \theta(\kappa_V(\mu) + n).$$

Applying Markov's inequality we have

$$\Phi_{s,t}(\mu)(\mathcal{V}_{\epsilon_n(\mu)}^c) = \Phi_{s,t}(\mu)(V \geq \theta^{-1}(\epsilon_n(\mu))) \leq \kappa_V(\mu)/(\kappa_V(\mu) + n) > 0.$$

We conclude that

$$\left\{ \begin{array}{l} \forall 0 < \epsilon \leq \epsilon_1(\mu) \quad \inf \Phi_{s,t}(\mu)(\mathcal{V}_{\epsilon_1(\mu)}) \geq 1 - \frac{1}{1 + 1/\kappa_V(\mu)} > 0 \\ \text{and} \quad \inf \Phi_{s,t}(\mu)(\mathcal{V}_{\epsilon_n(\mu)}) \geq \frac{1}{1 + \kappa_V(\mu)/n} \xrightarrow{n \rightarrow \infty} 1. \end{array} \right. \quad (74)$$

In the above display, the infimum are taken over all indices  $(s, t)$  such that  $t \in \mathcal{T}(\tau)$  and  $s \in \mathcal{T}_t(\tau)$ .

**Lemma 4.1.** *Assume that (14) and (15) are satisfied some bounded, continuous function  $V$  with compact level sets. Then the Lyapunov estimate (32) is satisfied.*

*Proof.* For any  $n \geq 1$  we set  $\epsilon_n := \theta(\bar{\eta}(V) + n)$  for  $\bar{\eta} \in \mathcal{P}_V(E)$ . For any  $s \in \mathcal{T}(\tau)$  and  $0 < \epsilon \leq \epsilon_1/\lambda^-$ , thanks to (14), we have the estimate

$$Q_{s,s+\tau}(V)/V < \epsilon \lambda^- 1_{K_\epsilon(s)^c} + \|\theta(V)\| 1_{K_\epsilon(s)},$$

with the compact sets  $K_\epsilon(s)$  are defined by

$$K_\epsilon(s) := \{x \in E : Q_{s,s+\tau}(V)/V \geq \epsilon \lambda^-\} \subset \mathcal{K}_\epsilon^- := \{x \in E : \theta(V(x)) \geq \epsilon \lambda^-\}.$$

On the other hand, for any  $s, s + \tau \in \mathcal{T}_t(\tau)$  we have

$$\frac{R_{s,s+\tau}^{(t)}(V/H_{s+\tau,t})}{(V/H_{s,t})} = \frac{1}{\lambda_{s,s+\tau}} \frac{Q_{s,s+\tau}(V)}{V}.$$

This yields the estimate

$$\begin{aligned} \frac{R_{s,s+\tau}^{(t)}(V/H_{s+\tau,t})}{(V/H_{s,t})} &\leq \frac{\lambda^-}{\lambda_{s,s+\tau}} \left( \epsilon 1_{K_\epsilon(s)^c} + \frac{\|\theta(V)\|}{\lambda^-} 1_{K_\epsilon(s)} \right) \\ &\leq \epsilon 1_{K_\epsilon(s)^c} + \frac{\|\theta(V)\|}{\lambda^-} 1_{K_\epsilon(s)}, \end{aligned}$$

from which we check that

$$R_{s,s+\tau}^{(t)}(V/H_{s+\tau,t}) \leq \epsilon (V/H_{s,t}) + \frac{\|\theta(V)\|}{\lambda^-} \sup_{\mathcal{K}_\epsilon^-} (V/H_{s,t}).$$

Now, choosing  $n \geq 1$  sufficiently large such that

$$\epsilon_n \leq \epsilon \lambda^- \leq \epsilon_1 \vee \epsilon_H,$$

we have

$$\mathcal{K}_\epsilon^- = \{x \in E : \theta(V(x)) \geq \epsilon \lambda^-\} \subset \mathcal{V}_{\epsilon_n} := \{x \in E : \theta(V(x)) \geq \epsilon_n\}.$$

By (15) this yields the uniform estimate

$$\sup_{\mathcal{K}_\epsilon^-} (V/H_{s,t}) \leq \zeta_\epsilon(H)^{-1} \sup_{\mathcal{V}_{\epsilon_n}} V < \infty.$$

■

**Proof of Theorem 2.5:** Observe that

$$\|H\|_V = 1 \implies H \leq V \quad \text{and} \quad H_{s,t} = \overline{Q}_{s,t}(H) \leq \overline{Q}_{s,t}(V).$$

This implies that

$$\lambda^- H_{s,t}/V \leq \lambda_{s,s+\tau} H_{s,t}/V = Q_{s,s+\tau}(H_{s+\tau,t})/V \leq Q_{s,s+\tau}(V)/V.$$

Combining this with  $(\mathcal{H}_1)$ , we obtain the estimate

$$\lambda^- V_\theta \leq V/H_{s,t} \quad \text{with} \quad V_\theta := 1/\theta(V).$$

Notice that  $V_\theta$  is continuous with compact level sets and for any  $r \geq 1$  we have

$$\{V/H_{s,t} \leq r\} \subset \{V_\theta \leq r/\lambda^-\}.$$

This implies that  $V/H_{s,t}$  also has compact level sets. Under  $(\mathcal{H}_0)$  (which holds under  $(\mathcal{H}_1)$ ), for any  $r \geq r_1 := \lambda^- r_0$  we have

$$\sup_{(V/H_{s,t})(x) \vee (V/H_{s,t})(y) \leq r} \left\| \delta_x R_{s,s+\tau}^{(t)} - \delta_y R_{s,s+\tau}^{(t)} \right\|_{tv} \leq 1 - \alpha(r/\lambda^-) < 1.$$

Now, due to the previous lemma, (32) holds which in turn implies that

$$R_{s,s+\tau}^{(t)}(W_{s+\tau,t}) \leq \epsilon W_{s,t} + 1$$

with the collection of functions  $W_{s,t} \geq 1$  defined by

$$\frac{\epsilon}{c} \frac{V}{H_{s,t}} \leq W_{s,t} := 1 + \frac{\epsilon}{c} \frac{V}{H_{s,t}} \leq \left(1 + \frac{\epsilon}{c}\right) \frac{V}{H_{s,t}}.$$

Then, for any  $r \geq \epsilon r_1/c$  we have

$$\sup_{W_{s,t}(x) \vee W_{s,t}(y) \leq r} \left\| \delta_x R_{s,s+\tau}^{(t)} - \delta_y R_{s,s+\tau}^{(t)} \right\|_{tv} \leq 1 - \alpha\left(\frac{cr}{\epsilon \lambda^-}\right) < 1.$$

The estimate (31) is now a direct consequence of Lemma 3.2. This ends the proof of the theorem. ■

## 4.2 Proof of Theorem 2.8

Observe that

$$\lambda_{s,t} H_{s,t} R_{s,t}^{(t)}(f/H) = Q_{s,t}(f),$$

which implies that

$$\Psi_{H_{s,t}}(\mu) R_{s,t}^{(t)}(f/H) = \frac{\mu(Q_{s,t}(f))}{\mu(Q_{s,t}(H))} = \frac{\Phi_{s,t}(\mu)(f)}{\Phi_{s,t}(\mu)(H)}.$$

Combining this with (11) and the estimate (57) we have

$$\begin{aligned} \|\Phi_{s,t}(\mu) - \Phi_{s,t}(\eta)\|_V &= \|\Psi_{1/H}(\Psi_{H_{s,t}}(\mu) R_{s,t}^{(t)}) - \Psi_{1/H}(\Psi_{H_{s,t}}(\eta) R_{s,t}^{(t)})\|_V \\ &\leq \Phi_{s,t}(\mu)(H) (1 + \Phi_{s,t}(\eta)(V)) \|\Psi_{H_{s,t}}(\mu) - \Psi_{H_{s,t}}(\eta)\|_{V/H_{s,t}}. \end{aligned}$$

Applying (31) to  $u = t$  we find that

$$\|\Phi_{s,t}(\mu) - \Phi_{s,t}(\eta)\|_V \leq \kappa_H(\mu) (1 + \kappa_V(\eta)) a e^{-b(t-s)} \|\Psi_{H_{s,t}}(\mu) - \Psi_{H_{s,t}}(\eta)\|_{V/H_{s,t}}.$$

On the other hand, applying (58) we check that

$$\|\Psi_{H_{s,t}}(\mu) - \Psi_{H_{s,t}}(\eta)\|_{V/H_{s,t}} \leq \frac{1}{\mu(H_{s,t})} \left(1 + \frac{\eta(V)}{\eta(H_{s,t})}\right) \|\mu - \eta\|_V.$$

This ends the proof of the theorem. ■

## 4.3 Proof of Theorem 2.11

This section is mainly concerned with the proof of Theorem 2.11 on the stability of the time homogeneous models discussed in (38).

**Lemma 4.2.** *For any  $\epsilon > 0$  we have the Foster-Lyapunov inequality*

$$P_\tau^h(V/h) < \epsilon (V/h) 1_{K_{\epsilon,\tau}^c} + c_{\epsilon,\tau} 1_{K_{\epsilon,\tau}} \quad (75)$$

with the parameter  $c_{\epsilon,\tau}$  and the compact set  $K_{\tau,\epsilon}$  given by

$$c_{\epsilon,\tau} := e^{-\rho\tau} \|Q_\tau(V)/V\| \sup_{K_{\epsilon,\tau}}(V/h) \quad \text{and} \quad K_{\tau,\epsilon} := \{Q_\tau(V)/V \geq \epsilon e^{\rho\tau}\}.$$

*Proof.* For any  $\epsilon > 0$  we have

$$V^h := V/h \implies P_\tau^h(V^h)/V^h = e^{-\rho\tau} Q_\tau(V)/V < \epsilon 1_{K_\epsilon^c} + e^{-\rho\tau} \|Q_\tau(V)/V\| 1_{K_{\epsilon,\tau}}.$$

This readily yields the estimate (75). ■

**Proposition 4.3.** *For any  $\mu \in \mathcal{P}_{V/h}(E)$  we have*

$$0 < \kappa(\mu) := \inf_{t \in \mathcal{T}(\tau)} \mu P_t^h(1/h) \leq \kappa_{V/h}^h(\mu) := \sup_{t \in \mathcal{T}(\tau)} \mu P_t^h(V/h) < \infty.$$

*In addition, we have the estimates*

$$0 < \kappa_h^-(\eta) \leq \kappa_V(\eta) < \infty. \quad (76)$$

*Proof.* Following word-for-word the same arguments as the proof of (63) the Foster-Lyapunov estimate (75) implies that

$$\kappa_{V/h}^h(\mu) \leq \mu(V/h) + c_{\epsilon, \tau}(1 - \epsilon)^{-1} < \infty.$$

Now, we come to the proof of the left-hand side estimate. Since  $h \in \mathcal{C}_{0,V}(E)$  and  $\|h\|_V = 1$  the function  $V/h \geq 1$  has compact level sets. Consider the compact sets  $\mathcal{K}_{V/h}^h(\delta)$  indexed by  $\delta \in ]0, 1[$  and defined by

$$\mathcal{K}_{V/h}^h(\mu, \delta) := \{x : \delta V(x)/h(x) \leq \kappa_{V/h}^h(\mu)\}.$$

Arguing as in (64), we have

$$\inf_{t \in \mathcal{T}(\tau)} \mu P_t^h(\mathcal{K}_{V/h}^h(\mu, \delta)) \geq 1 - \delta \quad \text{and} \quad \kappa(\mu) \geq (1 - \delta) / \sup_{\mathcal{K}_{V/h}^h(\mu, \delta)} h.$$

Observe that

$$\sup_{t \in \mathcal{T}(\tau)} \Psi_h(\eta) P_t^h(V/h) \leq \eta(V)/\eta(h) + c_{\epsilon, \tau}(1 - \epsilon)^{-1}$$

and consider the compact sets

$$\mathcal{K}_V(\eta, \delta) := \{x : \delta V(x) \leq \eta(V)/\eta(h) + c_{\epsilon, \tau}(1 - \epsilon)^{-1}\} \subset\subset E.$$

Arguing as above, we check that

$$\inf_{t \in \mathcal{T}(\tau)} \Psi_h(\eta) P_t^h(\mathcal{K}_V(\eta, \delta)) \geq 1 - \delta \quad \text{and} \quad \inf_{t \in \mathcal{T}(\tau)} \Psi_h(\eta) P_t^h(1/h) \geq (1 - \delta) / \sup_{\mathcal{K}_V(\eta, \delta)} h.$$

This yields for any  $t \in \mathcal{T}(\tau)$  and  $\delta \in ]0, 1[$  the uniform estimate

$$\Phi_t(\eta)(V) = \frac{\Psi_h(\eta) P_t^h(V/h)}{\Psi_h(\eta) P_t^h(1/h)} \leq \frac{\eta(V)/\eta(h) + c_{\epsilon, \tau}(1 - \epsilon)^{-1}}{(1 - \delta) / \sup_{\mathcal{K}_V(\eta, \delta)} h}.$$

This implies that  $\kappa_V(\eta) < \infty$ . This also ensures that the sequence of probability measures  $\Phi_{s,t}(\eta)$  indexed by  $s \leq t$  is tight. Choosing the compact set

$$K(\delta, \eta) := \{x \in E : \delta V(x) \leq \kappa_V(\eta)\} \quad (77)$$

we readily check that

$$\Phi_{s,t}(\eta)(h) \geq \left( \inf_{K(\delta,\eta)} h \right) \Phi_{s,t}(\eta)(K(\delta,\eta)) \geq (1-\delta) \left( \inf_{K(\delta,\eta)} h \right) > 0.$$

We conclude that  $\kappa_h^-(\eta) > 0$ . ■

**Proof of Theorem 2.11:** Observe that

$$(75) \implies P_\tau^h(W) \leq \epsilon W + 1$$

with the function  $W \geq 1$  defined by

$$\frac{\epsilon}{c_{\epsilon,\tau}} \frac{V}{h} \leq W := 1 + \frac{\epsilon}{c_{\epsilon,\tau}} \frac{V}{h} \leq \left( 1 + \frac{\epsilon}{c_{\epsilon,\tau}} \right) \frac{V}{h}, \quad (78)$$

which has compact level sets. On the other hand, by (39) we have

$$\sup_{W(x) \vee W(y) \leq r} \|\delta_x P_\tau^h - \delta_y P_\tau^h\|_{tv} \leq 1 - \alpha(c_{\epsilon,\tau} r / \epsilon).$$

The estimate (41) is now a direct consequence of (65) and the estimates (78). The proof of the theorem is now completed. ■

## 4.4 Proof of Theorem 3.7

One direction of the proof is obvious. Indeed, if  $\Phi_\tau$  has at least one invariant probability measure  $\eta_\infty$  choosing  $\eta = \eta_\infty$  the measure  $\Phi_t(\eta_\infty) = \eta_\infty$  is tight and  $\beta_t(\eta_\infty) = \eta_\infty(Q_\tau(1))$ .

Conversely, assume that for some  $\eta$  the sequence of probability measures  $\Phi_t(\eta)$  is tight and (73) is satisfied. In this situation, for any  $\epsilon > 0$  there exists a compact  $K_\epsilon \subset E$  such that

$$\Phi_t(\eta)(Q_\tau(1)) \geq \Phi_t(\eta)(1_{K_\epsilon} Q_\tau(1)) \geq (1-\epsilon) \inf_{K_\epsilon} Q_\tau(1) \quad \text{and} \quad \beta_\infty(\eta) > 0.$$

We simplify the notation and write  $\beta_n$  and  $\beta_\infty$  instead of  $\beta_{n\tau}(\eta)$  and  $\beta_\infty(\eta)$ . Consider the probability measures

$$\eta_m := \frac{1}{m} \sum_{0 \leq k < m} \Phi_{k\tau}(\eta) \implies \eta_m(Q_\tau(1)) = \frac{1}{m} \sum_{0 \leq k < m} \beta_k \longrightarrow_{m \rightarrow \infty} \beta_\infty.$$

There exists at least one probability measure  $\eta_\infty := \varpi(\eta)$  and a sub-sequence  $m_k \rightarrow_{k \rightarrow \infty} \infty$  such that  $\eta_{m_k}$  converges weakly to  $\eta_\infty$  as  $k \rightarrow \infty$ . Hence,  $\eta_{m_k}$  and  $\Phi(\eta_{m_k})$  converge weakly to  $\eta_\infty$  and  $\Phi(\eta_\infty)$ , respectively, as  $m \rightarrow \infty$ . In addition, we have

$$\Phi_{n\tau}(\eta)(Q_\tau(f)) = \Phi_{n\tau}(\eta)(Q_\tau(1)) \Phi_{(n+1)\tau}(\mu_0)(f).$$

This yields the formula

$$\Phi_\tau(\eta_m) = \sum_{0 \leq k < m} \frac{\beta_k}{\sum_{0 \leq l < m} \beta_l} \Phi_{(k+1)\tau}(\mu_0)$$

from which we check that

$$\begin{aligned} & \Phi_\tau(\eta_m) - \eta_m \\ &= \frac{1}{m} \sum_{0 \leq k < m} \left( \frac{\beta_k}{\frac{1}{m} \sum_{0 \leq l < m} \beta_l} - 1 \right) \Phi_{(k+1)\tau}(\mu_0) + \frac{1}{m} \sum_{0 \leq k < m} (\Phi_{(k+1)\tau}(\mu_0) - \Phi_{k\tau}(\eta)). \end{aligned}$$

As a result

$$\begin{aligned} & \Phi_\tau(\eta_m) - \eta_m \\ &= \frac{1}{m} \sum_{0 \leq k < m} \left( \frac{\beta_k}{\frac{1}{m} \sum_{0 \leq l < m} \beta_l} - 1 \right) \Phi_{(k+1)\tau}(\eta) + \frac{1}{m} (\Phi_{(k+1)\tau}(\eta) - \eta). \end{aligned}$$

On the other hand, we have

$$\frac{1}{m} \sum_{0 \leq k < m} \left| \beta_k - \frac{1}{m} \sum_{0 \leq l < m} \beta_l \right| \leq \frac{2}{m} \sum_{0 \leq k < m} |\beta_k - \beta_\infty| \xrightarrow{m \rightarrow \infty} 0.$$

For any  $f \in \mathcal{C}_b(E)$  we conclude that

$$|(\Phi_\tau(\eta_{m_k}) - \eta_{m_k})(f)| \xrightarrow{k \rightarrow \infty} 0 \quad \text{and therefore} \quad \eta_\infty = \Phi_\tau(\eta_\infty).$$

The last assertion follows from the fact that

$$\eta_{m_k}(Q_\tau(1)) = \frac{1}{m_k} \sum_{l=0}^{m_k-1} \beta_l \xrightarrow{m \rightarrow \infty} \beta_\infty = \eta_\infty(Q_\tau(1)).$$

This ends the proof of the theorem. ■

## A Proofs of the technical lemmas

### A.1 Proof of Lemma 1.3

In this section, we will work under the assumptions imposed by  $(\mathcal{H}_1)$ . Recall that the function  $H$  is chosen such that  $\|H\|_V = 1$  and  $Q_{t,t+\tau}(H) > 0$  for any  $t \in \mathcal{T}$  and  $\tau > 0$ .

Further recall that from section 1.5, for any  $u, t \in \mathcal{T}$ , we have

$$\lambda_{u,t} H_{u,t} = Q_{u,t}(H) =: h_{u,t} \implies Q_{s,u}(h_{u,t}) = h_{s,t} > 0. \quad (79)$$

We also recall that we assume (17) and (18) are satisfied. To simplify the notation, we write  $\nu$  instead of  $\nu_\tau$ . In this notation, for any  $0 < \epsilon < \epsilon_1$  there exist finite constants  $0 < \iota_\epsilon^- \leq \iota_\epsilon < \infty$  such that for any  $t \in \mathcal{T}$  we have

$$\iota_\epsilon^- 1_{\mathcal{V}_\epsilon}(x) \nu(dy) 1_{\mathcal{V}_\epsilon}(y) \leq 1_{\mathcal{V}_\epsilon}(x) Q_{t,t+\tau}(x, dy) 1_{\mathcal{V}_\epsilon}(y) \leq \iota_\epsilon 1_{\mathcal{V}_\epsilon}(x) \nu(dy) 1_{\mathcal{V}_\epsilon}(y). \quad (80)$$

The next lemma is a slight modification of [127, Proposition 1].

**Lemma A.1.** *There exists  $\lambda_0 > 0$  and a finite constant  $c_0 < \infty$  such that for any  $s, t \in \mathcal{T}(\tau)$  and  $\eta \in \mathcal{P}_V(E)$  we have*

$$R_{s,t}^{(t)} \left( \frac{V}{H} \right) \leq e^{-\lambda_0(t-s)} \frac{V}{h_{s,t}} + c_0 \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{e^{-\lambda_0(t-s)}}{\eta(h_{s,t})} = 0. \quad (81)$$

In addition, we have the uniform estimates stated in (19).

*Proof.* We set

$$h_{s,t}^V := h_{s,t} / \|h_{s,t}\|_V \quad \text{and} \quad \beta_{s,u}^{(t)} = \|h_{s,t}\|_V / \|h_{u,t}\|_V.$$

We have

$$R_{s,s+\tau}^{(t)} \left( \frac{V}{h_{s+\tau,t}^V} \right) := \frac{1}{\beta_{s,s+\tau}^{(t)}} \frac{V}{h_{s,t}^V} \frac{Q_{s,s+\tau}(V)}{V} \quad \text{and} \quad Q_{s,s+\tau}(h_{s+\tau,t}^V) = \beta_{s,s+\tau}^{(t)} h_{s,t}^V.$$

Using (13) and (80) and recalling that  $Q_{t,t+\tau}(V)/V \leq \theta(V)$ , for any  $0 \leq \epsilon < \epsilon_1$  we check that

$$R_{s,s+\tau}^{(t)} \left( \frac{V}{h_{s+\tau,t}^V} \right) \leq \frac{\epsilon}{\beta_{s,s+\tau}^{(t)}} \frac{V}{h_{s,t}^V} + \frac{a_\epsilon}{\nu(1_{\mathcal{V}_\epsilon} h_{s+\tau}^V)} \quad \text{with} \quad a_\epsilon := \sup_{\mathcal{V}_\epsilon} (\theta(V)V) / \iota_\epsilon^-.$$

This implies that

$$R_{s,s+n\tau}^{(t)} \left( \frac{V}{h_{s+n\tau,t}^V} \right) \leq \frac{\epsilon}{\beta_{s+(n-1)\tau,s+n\tau}^{(t)}} R_{s,s+(n-1)\tau}^{(t)} \left( \frac{V}{h_{s+(n-1)\tau,t}^V} \right) + \frac{a_\epsilon}{\nu(1_{\mathcal{V}_\epsilon} h_{s+n\tau}^V)}.$$

Applying the above to  $s + n\tau = t$  we obtain the formula

$$\begin{aligned} & R_{s,t}^{(t)} \left( \frac{V}{H} \right) \\ & \leq \frac{\epsilon^n}{\beta_{s,t}^{(t)}} \frac{V}{h_{s,t}^V} + \sum_{1 \leq k \leq n} \frac{\epsilon^{n-k}}{\beta_{s+k\tau,t}^{(t)}} \frac{a_\epsilon}{\nu(1_{\mathcal{V}_\epsilon} h_{s+k\tau}^V)} = \frac{\epsilon^n}{h_{s,t}} V + \sum_{1 \leq k \leq n} \epsilon^{n-k} \frac{a_\epsilon}{\nu(1_{\mathcal{V}_\epsilon} h_{s+k\tau}^V)}. \end{aligned}$$

Observe that for any  $0 < \epsilon \leq \epsilon_2 < \epsilon_1$  we have  $\mathcal{V}_{\epsilon_2} \subset \subset \mathcal{V}_\epsilon$ . Thus, for any  $k \leq n$  and  $t = s + n\tau$  we have the lower bound estimate

$$\begin{aligned} 1_{\mathcal{V}_\epsilon}(x) h_{s+k\tau,s+n\tau}(x) & \geq 1_{\mathcal{V}_{\epsilon_2}}(x) Q_{s+k\tau,s+(k+1)\tau} 1_{\mathcal{V}_{\epsilon_2}} \cdots Q_{s+(n-1)\tau,s+n\tau} (1_{\mathcal{V}_{\epsilon_2}} H)(x) \\ & \geq 1_{\mathcal{V}_{\epsilon_2}}(x) \left( \inf_{\mathcal{V}_{\epsilon_2}} H \right) (\iota_{\epsilon_2}^- \nu(\mathcal{V}_{\epsilon_2}))^{n-k}. \end{aligned}$$

Choosing  $0 < \epsilon < \epsilon_3 := \epsilon_2 \wedge (\iota_{\epsilon_2}^- \nu(\mathcal{V}_{\epsilon_2})/2)$  we conclude that

$$R_{s,t}^{(t)} \left( \frac{V}{H} \right) \leq \frac{\epsilon^n}{h_{s,t}} V + \frac{2a_\epsilon}{\nu(\mathcal{V}_{\epsilon_2})} \frac{1}{\inf_{\mathcal{V}_{\epsilon_2}} H}.$$

In the same vein, for any  $\eta \in \mathcal{P}(V)$  we have

$$\eta(h_{s,s+n\tau}) \geq \eta(\mathcal{V}_{\epsilon_2}) \left( \inf_{\mathcal{V}_{\epsilon_2}} h \right) (\iota_{\epsilon_2}^- \nu(\mathcal{V}_{\epsilon_2}))^n \geq \eta(\mathcal{V}_{\epsilon_2}) \left( \inf_{\mathcal{V}_{\epsilon_2}} H \right) (2\epsilon)^n.$$

This yields the estimate

$$\frac{\epsilon^{(t-s)/\tau}}{\eta(h_{s,t})} \leq 2^{-(t-s)/\tau} \frac{1}{\eta(\mathcal{V}_{\epsilon_2}) \inf_{\mathcal{V}_{\epsilon_2}} H}.$$

To summarise: there exists some  $\epsilon_3 > 0$  such that for any  $s, t \in \mathcal{T}(\tau)$  s.t.  $s \leq t \in \mathcal{T}(\tau)$  and for any  $0 < \epsilon < \epsilon_3$  we have

$$R_{s,t}^{(t)} \left( \frac{V}{H} \right) \leq \frac{\epsilon^{(t-s)/\tau}}{h_{s,t}} V + c_\epsilon \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{\epsilon^{(t-s)/\tau}}{\eta(h_{s,t})} = 0.$$

This ends the proof of (81). The last assertion follows from the fact that

$$\left( \Psi_{h_{s,t}}(\eta) R_{s,t}^{(t)} \right) (V/H) \leq \frac{e^{-\lambda_0(t-s)}}{\eta(h_{s,t})} \eta(V) + c_0 \leq c_1(\eta)$$

for some finite constant  $c_1(\eta) < \infty$ . This implies that the sequence of probability measures

$$\Psi_{h_{s,t}}(\eta) R_{s,t}^{(t)}$$

indexed by  $s, t \in \mathcal{T}(\tau)$  is tight. More precisely, choosing the compact set

$$K_{\epsilon,\eta} := \{x \in E : \epsilon (V/H)(x) \leq c_1(\eta)\}$$

we have

$$\left( \Psi_{h_{s,t}}(\eta) R_{s,t}^{(t)} \right) (K_{\epsilon,\eta}^c) \leq \epsilon \quad \text{so that} \quad \left( \Psi_{h_{s,t}}(\eta) R_{s,t}^{(t)} \right) (1/H) \geq \frac{1 - \epsilon}{\sup_{K_{\epsilon,\eta}} H}.$$

On the other hand, we have

$$\Phi_{s,t}(\eta)(V) = \Psi_{1/H} \left( \Psi_{h_{s,t}}(\eta) R_{s,t}^{(t)} \right) (V) \leq c_1(\eta) \frac{\sup_{K_{\epsilon,\eta}} H}{1 - \epsilon}.$$

We conclude that  $\kappa_V(\eta) < \infty$ . This also shows that the sequence of probability measures  $\Phi_{s,t}(\eta)$  indexed by  $s \leq t$  is tight. Choosing the compact set

$$K_{\epsilon,\eta} := \{x \in E : \epsilon V(x) \leq \kappa_V(\eta)\} \tag{82}$$

we readily check that

$$\Phi_{s,t}(\eta)(H) \geq \left( \inf_{K_{\epsilon,\eta}} H \right) \Phi_{s,t}(\eta)(K_{\epsilon,\eta}) \geq (1 - \epsilon) \left( \inf_{K_{\epsilon,\eta}} H \right) > 0.$$

We conclude that  $\kappa_H^-(\eta) > 0$ . This ends the proof of the left-hand side estimate in (19).

Also observe that for any  $0 < \epsilon < \epsilon_0$  we have

$$(80) \implies \inf_{t \in \mathcal{T}(\tau)} \inf_{\mathcal{V}_\epsilon} Q_{t,t+\tau}(1) \geq \nu(\mathcal{V}_\epsilon) > 0. \quad (83)$$

Thus, for any  $t \in \mathcal{T}(\tau)$  and  $0 < \epsilon < \epsilon_0$  we have

$$\Phi_{s,t}(\eta)(Q_{t,t+\tau}(1)) \geq \Phi_{s,t}(\eta)(1_{\mathcal{V}_\epsilon} Q_{t,t+\tau}(1)) \geq \Phi_{s,t}(\eta)(\mathcal{V}_\epsilon) \nu(\mathcal{V}_\epsilon).$$

By choosing  $\epsilon = \epsilon_n := \theta(\kappa_V(\eta) + n)$  for some  $n \geq 1$  we check that

$$\mathcal{V}_{\epsilon_n} = \{x \in E : V(x) \leq \kappa_V(\eta) + n\} \quad \text{and} \quad \Phi_{s,t}(\eta)(\mathcal{V}_{\epsilon_n}) \geq \frac{1}{1 + \kappa_V(\eta)/n} > 0.$$

Similarly, we have

$$\Phi_{s,t}(\eta)(Q_{t,t+\tau}(1)) \leq \Phi_{s,t}(\eta)(V Q_{t,t+\tau}(V)/V) \leq \|\theta(V)\| \kappa_V(\eta).$$

This ends the proof of the left-hand side estimate in (19). The proof of the lemma is completed.  $\blacksquare$

The next result is a variation of [127, lemma 10].

**Lemma A.2.** *We have the estimates (15). In addition, for any  $0 < \delta < 1$  there exists some  $0 < \epsilon \leq \epsilon_1$  such that the following uniform compact-approximation estimate holds*

$$\sup_{t \in \mathcal{T}(\tau)} \left\| \overline{Q}_{t,t+\tau} - 1_{\mathcal{V}_\epsilon} \overline{Q}_{t,t+\tau} \right\|_V < \delta \quad \text{and} \quad \sup_{t \in \mathcal{T}(\tau)} \sup_{s \in \mathcal{T}_t(\tau)} \left\| \overline{Q}_{s,t} \right\|_V < \infty. \quad (84)$$

*Proof.* We start by proving the estimates given in (84). We use the same notation as in the proof of Lemma 4.1. For any  $n \geq 1$  such that  $\epsilon_n^- := \epsilon_n/\lambda^- < 1$  we have

$$\begin{aligned} & \left\| 1_{\mathcal{V}_{\epsilon_n}^c} \overline{Q}_{s,s+\tau} \right\|_V \\ &= \left\| \overline{Q}_{s,s+\tau} - 1_{\mathcal{V}_{\epsilon_n}^c} \overline{Q}_{s,s+\tau} \right\|_V = \frac{1}{\lambda_{s,s+\tau}} \left\| (Q_{s,s+\tau}(V)/V) 1_{\mathcal{V}_{\epsilon_n}^c} \right\| < \epsilon_n^-. \end{aligned}$$

This yields the left-hand side estimate in (84).

For the right-hand side, for any  $i \in \{0, 1\}^n$  and  $1 \leq k \leq n$ , we set

$$l(i) := \inf \{1 \leq k < n : (i_k, i_{k+1}) = (1, 1)\} \quad \text{and} \quad \{0, 1\}_k^n = \{i \in \{0, 1\}^n : l(i) = k\}$$

with the convention that

$$\{0, 1\}_n^n = \{i \in \{0, 1\}^n : \forall 1 \leq k < n : (i_k, i_{k+1}) \neq (1, 1)\}.$$

Further, set

$$\mathcal{V}_\epsilon(0) = \mathcal{V}_\epsilon^c \quad \text{and} \quad \mathcal{V}_\epsilon(1) = \mathcal{V}_\epsilon.$$

Then we may decompose  $\bar{Q}$  as follows:

$$\begin{aligned} \bar{Q}_{s_0, s_n} &= \sum_{i \in \{0, 1\}^n} 1_{\mathcal{V}_\epsilon(i_1)} \bar{Q}_{s_0, s_1} \cdots 1_{\mathcal{V}_\epsilon(i_n)} \bar{Q}_{s_0, s_n} \\ &= \sum_{1 \leq k \leq n} \sum_{i \in \{0, 1\}_k^n} 1_{\mathcal{V}_\epsilon(i_1)} \bar{Q}_{s_0, s_1} \cdots 1_{\mathcal{V}_\epsilon(i_n)} \bar{Q}_{s_{n-1}, s_n} \end{aligned}$$

Our aim is to obtain suitable bounds for the summands in the above decomposition. To this end, set for any  $0 < \epsilon \leq \epsilon_n$

$$\nu_\epsilon(dx) := \frac{\nu(dx) 1_{\mathcal{V}_\epsilon}(x)}{\nu(\mathcal{V}_\epsilon)}.$$

Using this and the notation introduced in (80), we have

$$\begin{aligned} &1_{\mathcal{V}_\epsilon}(x) \bar{Q}_{s, s+\tau} (1_{\mathcal{V}_\epsilon} \bar{Q}_{s+\tau, t}(V)) (x) \\ &\leq \iota_\epsilon \frac{\nu(1_{\mathcal{V}_\epsilon} Q_{s+\tau, t}(V))}{\bar{\eta}_s(Q_{s, s+\tau} Q_{s+\tau, t}(1))} \leq \iota_\epsilon \frac{\nu(1_{\mathcal{V}_\epsilon} Q_{s+\tau, t}(V))}{\bar{\eta}_s(1_{\mathcal{V}_\epsilon} Q_{s, s+\tau} 1_{\mathcal{V}_\epsilon} Q_{s+\tau, t}(1))} \\ &\leq \frac{\iota_\epsilon}{\bar{\eta}_s(\mathcal{V}_\epsilon)} \frac{\nu(1_{\mathcal{V}_\epsilon} Q_{s+\tau, t}(V))}{\nu(1_{\mathcal{V}_\epsilon} Q_{s+\tau, t}(1))} = \frac{\iota_\epsilon / \iota_\epsilon^-}{\bar{\eta}_s(\mathcal{V}_\epsilon)} \Phi_{s+\tau, t}(\nu_\epsilon)(V) \leq \kappa_V(\nu_\epsilon) \frac{\iota_\epsilon / \iota_\epsilon^-}{\bar{\eta}_s(\mathcal{V}_\epsilon)}. \end{aligned}$$

This yields the uniform estimate

$$1_{\mathcal{V}_{\epsilon_n}}(x) \bar{Q}_{s, s+\tau} (1_{\mathcal{V}_{\epsilon_n}} \bar{Q}_{s+\tau, t}(V)) (x) \leq \iota'_{\epsilon_n} := (\iota_{\epsilon_n} / \iota_{\epsilon_n}^-) \kappa_V(\nu_{\epsilon_n}) (1 + \bar{\eta}(V)/n).$$

Next fix  $1 \leq k < n$  Then one has

$$\begin{aligned} \left\| 1_{\mathcal{V}_\epsilon(i_1)} \bar{Q}_{s_0, s_1} \cdots 1_{\mathcal{V}_\epsilon(i_n)} \bar{Q}_{s_{n-1}, s_n} \right\|_V &\leq \iota'_{\epsilon_n} \left\| 1_{\mathcal{V}_\epsilon(i_1)} \bar{Q}_{s_0, s_1} \cdots 1_{\mathcal{V}_\epsilon(i_k)} \bar{Q}_{s_{k-1}, s_k} \right\|_V \\ &\leq \iota'_{\epsilon_n} \prod_{1 \leq m \leq k} \left\| 1_{\mathcal{V}_\epsilon(i_m)} \bar{Q}_{s_{m-1}, s_m} \right\|_V \quad (85) \\ &\leq \iota'_{\epsilon_n} \prod_{1 \leq m \leq k} \|\theta(V)\|^{1_{i_m=1}} (\epsilon_n^-)^{1_{i_m=0}} \\ &= \iota'_{\epsilon_n} \|\theta(V)\|^{\sum_{1 \leq m \leq k} 1_{i_m=1}} (\epsilon_n^-)^{\sum_{1 \leq m \leq k} 1_{i_m=0}} \quad (86) \end{aligned}$$

Similarly, for the case  $k = n$ , we have

$$\left\| 1_{\mathcal{V}_\epsilon(i_1)} \bar{Q}_{s_0, s_1} \cdots 1_{\mathcal{V}_\epsilon(i_n)} \bar{Q}_{s_{n-1}, s_n} \right\|_V \leq \|\theta(V)\|^{\sum_{1 \leq m \leq n} 1_{i_m=1}} (\epsilon_n^-)^{\sum_{1 \leq m \leq n} 1_{i_m=0}}. \quad (87)$$

Now note that for any  $i \in \{0, 1\}^n$  we have

$$\sum_{1 \leq k \leq n} 1_{i_k=1} \leq \frac{n+1}{2} + \frac{1}{2} \sum_{1 \leq k < n} 1_{(i_k, i_{k+1})=(1,1)}.$$

This follows from the fact that

$$\begin{aligned} n-1 &\geq \sum_{1 \leq k < n} (1_{(i_k, i_{k+1})=(1,0)} + 1_{(i_k, i_{k+1})=(0,1)}) \\ &= \left( 2 \sum_{1 \leq k \leq n} 1_{i_k=1} - (1_{i_n=1} + 1_{i_1=1}) \right) - \sum_{1 \leq k < n} 1_{(i_k, i_{k+1})=(1,1)} \\ &\geq 2 \left( \sum_{1 \leq k \leq n} 1_{i_k=1} - 1 \right) - \sum_{1 \leq k < n} 1_{(i_k, i_{k+1})=(1,1)}. \end{aligned}$$

Equivalently, we have

$$\sum_{1 \leq k \leq n} 1_{i_k=0} \geq \frac{n-1}{2} - \frac{1}{2} \sum_{1 \leq k < n} 1_{(i_k, i_{k+1})=(1,1)}.$$

Further observe that for any  $i \in \{0, 1\}_k^n$  with  $1 \leq k \leq n$  we have

$$\sum_{1 \leq m \leq k} 1_{(i_m, i_{m+1})=(1,1)} = 0 \implies \sum_{1 \leq m \leq k} 1_{i_m=1} \leq \frac{k+1}{2} \quad \text{and} \quad \sum_{1 \leq m \leq k} 1_{i_m=0} \geq \frac{k-1}{2}.$$

From (86) and (87), respectively, we conclude that for any  $i \in \{0, 1\}_k^n$  with  $1 \leq k < n$  we have

$$\|1_{\mathcal{V}_\epsilon(i_1)} \bar{Q}_{s_0, s_1} \cdots 1_{\mathcal{V}_\epsilon(i_n)} \bar{Q}_{s_{n-1}, s_n}\|_V \leq \iota'_{\epsilon_n} (1 \vee \|\theta(V)\|)^{(k+1)/2} (\epsilon_n^-)^{(k-1)/2}.$$

and for  $k = n$ , we have

$$\|1_{\mathcal{V}_\epsilon(i_1)} \bar{Q}_{s_0, s_1} \cdots 1_{\mathcal{V}_\epsilon(i_n)} \bar{Q}_{s_{n-1}, s_n}\|_V \leq (1 \vee \|\theta(V)\|)^{(n+1)/2} (\epsilon_n^-)^{(n-1)/2}.$$

We end the proof of the right-hand side estimate in (84) by choosing  $n \geq 1$  such that

$$\epsilon_n^- < 1 \wedge (1/\|\theta(V)\|).$$

Now for the estimates in (15), observe that

$$H \leq V \implies \|H\|_V := \sup_{t \in \mathcal{T}(\tau)} \sup_{s \in \mathcal{T}_i(\tau)} \|\bar{Q}_{s,t}(H)\|_V \leq \sup_{t \in \mathcal{T}(\tau)} \sup_{s \in \mathcal{T}_i(\tau)} \|\bar{Q}_{s,t}\|_V < \infty.$$

This yields right-hand side estimate in (15).

The proof of the left-hand side estimate in (15) follows the same lines of arguments as those given for (84), thus it is only sketched. Using the same notation as above, we have

$$\bar{\eta}_{s_0} Q_{s_0, s_n}(1) = \sum_{1 \leq k \leq n} \sum_{i \in \{0, 1\}_k^n} \bar{\eta}_{s_0} 1_{\mathcal{V}_\epsilon(i_1)} Q_{s_0, s_1} \cdots 1_{\mathcal{V}_\epsilon(i_n)} Q_{s_{n-1}, s_n}(1).$$

For any  $i \in \{0, 1\}_k^n$  for some  $1 \leq k < n$ , we also have

$$\begin{aligned} & \bar{\eta}_{s_0} 1_{\mathcal{V}_\epsilon(i_1)} Q_{s_0, s_1} \cdots 1_{\mathcal{V}_\epsilon(i_n)} Q_{s_{n-1}, s_n}(1) \\ & \leq \iota_\epsilon \bar{\eta}_{s_0} (V 1_{\mathcal{V}_\epsilon(i_1)} Q_{s_0, s_1}^V \cdots 1_{\mathcal{V}_\epsilon(i_k)} Q_{s_{k-1}, s_k}^V(1_{\mathcal{V}_\epsilon})) \nu(1_{\mathcal{V}_\epsilon} Q_{s_{k+1}, s_n}(1)) \\ & \leq \iota_\epsilon \bar{\eta}(V) (1 \vee \|\theta(V)\|)^{(k+1)/2} \epsilon^{(k-1)/2} \nu(1_{\mathcal{V}_\epsilon} Q_{s_{k+1}, s_n}(1)), \end{aligned}$$

with  $Q_{s,t}^V(f) := Q_{s,t}(fV)/V$ . For any  $\epsilon \leq \epsilon_1$  we have

$$\begin{aligned} & 1_{\mathcal{V}_\epsilon}(x) Q_{s_0, s_n}(H)(x) \\ & \geq 1_{\mathcal{V}_\epsilon}(x) 1_{\mathcal{V}_{\epsilon_1}}(x) (Q_{s_0, s_1} 1_{\mathcal{V}_{\epsilon_1}} Q_{s_1, s_n}(H))(x) \geq \iota_\epsilon^- 1_{\mathcal{V}_\epsilon}(x) \nu(1_{\mathcal{V}_{\epsilon_1}} Q_{s_1, s_n}(H)). \end{aligned}$$

In addition, we have

$$\begin{aligned} & \nu(1_{\mathcal{V}_{\epsilon_1}} Q_{s_1, s_n}(H)) \\ & \geq \nu(1_{\mathcal{V}_{\epsilon_1}} Q_{s_1, s_2} 1_{\mathcal{V}_\epsilon} \cdots Q_{s_{k-1}, s_k} 1_{\mathcal{V}_\epsilon} Q_{s_k, s_{k+1}} 1_{\mathcal{V}_\epsilon} Q_{s_{k+1}, s_n}(H)) \\ & \geq \iota_\epsilon^- \nu(1_{\mathcal{V}_{\epsilon_1}} Q_{s_1, s_2} 1_{\mathcal{V}_\epsilon} \cdots Q_{s_{k-1}, s_k} 1_{\mathcal{V}_\epsilon}) \nu(1_{\mathcal{V}_\epsilon} Q_{s_{k+1}, s_n}(H)). \end{aligned}$$

This yields the estimate

$$1_{\mathcal{V}_\epsilon}(x) Q_{s_0, s_n}(H) \geq 1_{\mathcal{V}_\epsilon}(x) (\iota_\epsilon^-)^2 (\iota_{\epsilon_1}^- \nu(\mathcal{V}_{\epsilon_1}))^k \nu(1_{\mathcal{V}_\epsilon} Q_{s_{k+1}, s_n}(H)).$$

from which we check that

$$\begin{aligned} & 1_{\mathcal{V}_\epsilon}(x) (\bar{\eta}_{s_0} 1_{\mathcal{V}_\epsilon(i_1)} Q_{s_0, s_1} \cdots 1_{\mathcal{V}_\epsilon(i_n)} Q_{s_{n-1}, s_n}(1)) / Q_{s_0, s_n}(H)(x) \\ & \leq (\iota_\epsilon / \epsilon) (\bar{\eta}(V) / \kappa_H^-(\nu_\epsilon)) (\epsilon (1 \vee \|\theta(V)\|) / (\iota_{\epsilon_1}^- \nu(\mathcal{V}_{\epsilon_1}))^2)^{k/2}. \end{aligned}$$

The end of the proof of the left-hand side assertion in (15) now follows word-for-word the same lines of arguments as the proof of the right-hand side estimate in (15), thus it is skipped. The proof of the lemma is now completed.  $\blacksquare$

## A.2 Proof of Lemma 3.1

Observe that for any  $f$  such that  $\|f\|_V \leq 1$ ,

$$(\Psi_{1/h}(\mu_1) - \Psi_{1/h}(\mu_2))(f) = \frac{1}{\mu_1(1/h)} (\mu_1 - \mu_2)(g)$$

with the function

$$g := (1/h)(f - \Psi_{1/h}(\mu_2)(f)) \in \mathcal{B}_V^h(E).$$

Note that

$$\frac{g}{V/h} = \frac{f - \Psi_{1/h}(\mu_2)(f)}{V} = \frac{f}{V} - \frac{\Psi_{1/h}(\mu_2)(Vf/V)}{V} \implies \|g\|_{V/h} \leq 1 + \frac{\mu_2(V/h)}{\mu_2(1/h)}.$$

This ends the proof of the first assertion. Now, for any  $\|f\|_{V/h} \leq 1$  we have

$$(\Psi_h(\mu_1) - \Psi_h(\mu_2))(f) = \frac{1}{\mu_1(h)} (\mu_1 - \mu_2)(g)$$

with the function

$$\begin{aligned} g &:= h(f - \Psi_h(\mu_2)(f)) \\ \implies \frac{|g|}{V} &\leq 1 + \frac{h}{V} \frac{\mu_2(V)}{\mu_2(h)} \leq 1 + \frac{\mu_2(V)}{\mu_2(h)} \implies \|g\|_V \leq 1 + \frac{\mu_2(V)}{\mu_2(h)}. \end{aligned}$$

This ends the proof of the lemma. ■

## A.3 Proof of Lemma 3.2

We set  $(s, t) = (1, 2)$  and  $P = P_{1,2}$ , and  $V_i^\rho := 1/2 + \rho V_i$ , with  $i \in \{1, 2\}$  and  $\rho \in ]0, 1[$ .

We also consider the function  $\Delta_\rho$  on  $E_1^2$  defined for any  $(x, y) \in E_1^2$  by

$$\begin{aligned} \Delta_\rho(x, y) &:= \frac{\|\delta_x P - \delta_y P\|_{V_2^\rho}}{\|\delta_x - \delta_y\|_{V_1^\rho}} \\ &\leq \frac{1}{1 + \rho(V_1(x) + V_1(y))} \|\delta_x P - \delta_y P\|_{tv} + \frac{\rho(P(V_2)(x) + P(V_2)(y))}{1 + \rho(V_1(x) + V_1(y))}. \end{aligned}$$

Using the left-hand side estimate in (61), we have

$$P(V_2)(x) + P(V_2)(y) \leq \epsilon(V_1(x) + V_1(y)) + 2 = (V_1(x) + V_1(y)) \left( \epsilon + \frac{2}{(V_1(x) + V_1(y))} \right).$$

When  $V_1(x) + V_1(y) \geq r$  this yields the estimate

$$\begin{aligned} \Delta_\rho(x, y) &\leq \frac{1}{1 + \rho(V_1(x) + V_1(y))} + \frac{\rho(V_1(x) + V_1(y))}{1 + \rho(V_1(x) + V_1(y))} \left( \epsilon + \frac{2}{r} \right) \\ &= 1 - \left( 1 - \frac{1}{1 + \rho(V_1(x) + V_1(y))} \right) \left( 1 - \left( \epsilon + \frac{2}{r} \right) \right) \\ &\leq 1 - d_\rho^1, \end{aligned}$$

with

$$d_\rho^1 := \left( 1 - \frac{1}{1 + 2\rho r} \right) \left( 1 - \left( \epsilon + \frac{2}{r} \right) \right).$$

Recalling that  $V_i \geq 1$  we have

$$P(V_2)/V_1 \leq 1 + \epsilon.$$

This implies that for any  $(x, y)$  such that  $V_1(x) + V_1(y) \leq r$  we have

$$\frac{\rho V_1(x)(P(V_2)(x)/V_1(x)) + \rho V_1(y)(P(V_2)(y)/V_1(y))}{1 + \rho(V_1(x) + V_1(y))} \leq (1 + \epsilon) \frac{\rho r}{1 + \rho},$$

where we have also used the fact that  $\rho < 1$ . This yields the estimate

$$\Delta_\rho(x, y) \leq 1 - d_\rho^2 := \frac{1 - \alpha(r)}{1 + 2\rho} + \frac{\rho r}{1 + 2\rho} (1 + \epsilon).$$

Observe that

$$\rho \leq \frac{1}{1 + \epsilon} \frac{\alpha(r)}{r} \implies 1 - d_\rho^2 \leq \frac{1}{1 + 2\rho} < 1.$$

Recalling that  $\alpha(r) > 0$  for some  $r > r_\epsilon := 2/(1 - \epsilon)$  we have

$$\frac{2}{r} < 1 - \epsilon \implies 0 < d_\rho^1 = \left( 1 - \frac{1}{1 + 2\rho r} \right) \left( 1 - \left( \epsilon + \frac{2}{r} \right) \right) < 1.$$

Choosing

$$r = r_\epsilon(1 + \delta_1) \quad \text{and} \quad \rho = \frac{\delta_2}{1 + \epsilon} \frac{\alpha(r)}{2r}$$

for some  $0 < \delta_1 \leq 1$  and  $0 < \delta_2 \leq 1$  we have

$$0 < \frac{\delta_2}{3} \alpha(r) (1 - \epsilon) \frac{\delta_1}{2} < d_\rho^1 = \frac{\alpha(r)\delta_2}{(1 + \epsilon) + \alpha(r)\delta_2} (1 - \epsilon) \frac{\delta_1}{1 + \delta_1} < 1,$$

which implies that

$$1 - d_\rho^1 \leq 1 - \alpha(r) (1 - \epsilon) \frac{\delta_1 \delta_2}{6}$$

and

$$1 - d_\rho^2 = \frac{1 - \alpha(r) + \rho r(1 + \epsilon)}{1 + 2\rho} = \frac{1 - \alpha(r)(1 - \delta_2/2)}{1 + 2\rho} \leq 1 - \alpha(r)(1 - \delta_2/2).$$

Then setting

$$\delta_1 = \frac{6}{1-\epsilon} \frac{1-\delta_2/2}{\delta_2}$$

we can check that

$$(1-d_\rho^1) \vee (1-d_\rho^2) \leq 1 - \alpha(r)(1-\delta_2/2).$$

For any  $0 < \delta \leq 1$  we set

$$r_\epsilon(\delta) := r_0 \vee r_\epsilon \left( 1 + \frac{1}{1-\epsilon} \left( 3 + 6 \left( \frac{1}{\delta} - 1 \right) \right) \right) \quad \rho_\epsilon(\delta) := \frac{\delta}{1+\epsilon} \frac{\alpha(r_\epsilon(\delta))}{2r_\epsilon(\delta)}.$$

We conclude that

$$\beta_{V_1^{\delta,\epsilon}, V_2^{\delta,\epsilon}}(P) = \sup_{x,y \in E} \frac{\|\delta_x P - \delta_y P\|_{V_2^{\delta,\epsilon}}}{V_1^{\delta,\epsilon}(x) + V_1^{\delta,\epsilon}(y)} \leq 1 - (d_{\rho_\epsilon(\delta)}^1 \wedge d_{\rho_\epsilon(\delta)}^2) \leq 1 - \alpha(r_\epsilon(\delta))(1-\delta/2).$$

This ends the proof of the lemma. ■

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