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Approximate analytical coefficient of restitution formulation for single bead impact with external load, using nonlinear visco-elastic models

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Abstract

This work is largely concerned with the calculation of an approximate coefficient of restitution (CoR) associated with nonlinear spring/dashpot models, when a constant external load acts upon the colliding body. Some of the main applications concern well known visco-elastic models such as *Kuwabara-Kono*, *Simon-Hunt-Crosseley*, *Tsuji-Tanaka-Ishida*, etc, however the proposed approach to a very wide class of nonlinear visco-elastic contact models. It is shown that suitable expansions allow to derive a computable expression which provides accurate predictions in a valuable range of external loads and viscosity coefficients.

1 Introduction

Impacts in multibody systems are highly nonlinear phenomena whose modelling has attracted numerous researchers since a long time [1, 2, 3, 4], see the preface in [5] for a short historical summary. There are several ways to classify impact models within gross classes, as:

1. Single impacts, *i.e.*, a single collision occurs without any overlap with foregoing or next collisions.
2. Multiple impacts, *i.e.*, several collisions occur simultaneously in the system.

or as [6]:

1. Algebraic (zero-order) models which assume instantaneous collisions between perfectly rigid bodies, and relate post- and pre-impact velocities through a so-called restitution rule.
2. First-order dynamics following the Routh and the Darboux-Keller approaches, where the normal contact force impulse is used as a new time-scale.
3. Second-order dynamics which rely on compliant models of lumped flexibilities and dampings (linear or nonlinear spring-dashpot rheological models), through assemblies of various basic components: elastic, visco-elastic, frictional, fractional elastic, visco-plastic, *etc*.

In a Lagrangian framework, simultaneity of collisions means that the boundary of the admissible domain in the configuration space, is attained at a co-dimension ≥ 2 submanifold [5, Definition 6.1]. When contact compliance is taken into account, a multiple impact occurs each time several collisions overlap. In this work only single impacts are studied. Each of the above gross classes for single impacts can be refined into subclasses. One major issue is the modelling of kinetic energy dissipation. Tangential dissipation is usually modelled using extensions of set-valued Coulomb's friction. As far as normal effects are concerned, several approaches have been proposed so far to take dissipation into account (here we restrict ourselves to single impacts):

- 37 1. linear viscosity (damping),
- 38 2. nonlinear damping as in the well-known Simon-Hunt-Crossley, Kuwabara-Kono, Tsuji-Tanaka-Ishida
39 models or the Lankarani-Nikravesh approach [7, 8][5, section 2.2.2] [9],
- 40 3. phenomenological parameters like coefficients of restitution (CoR) for the zero and first-order dynamics
41 models, the most popular ones being kinematic (Newton), kinetic (Poisson) and energetic CoRs.
- 42 4. varying stiffnesses in compression and expansion phases (the bistiffness Crook’s approach [10] [5, section
43 4.2.1.2]) [11],
- 44 5. plasticity effects (à la Johnson [12, 13, 14] [5, section 4.2.1.1]), Masin, Persoz and viscoelasto-plastic
45 assemblies with set-valued frictional elements [15, 16, 17],
- 46 6. adhesive effects with Dugdale or Lennard-Jones potentials [18] [5, section 4.2.2], body bulk vibrations
47 (see [1, Chapter III] [13, Chapter 11] [5, section 4.2.4] for references) *etc.*

48 In addition, it should be remarked that these models and parameters, are not independent, nor used inde-
49 pendently one from each other. For instance it is known that parameters like CoRs in algebraic restitution
50 rules are related [5, section 4.3.3.1], and this is also the case for first-order Darboux-Keller impact dynamics
51 [19, 20, 21, 22] [5, sections 4.3.5.3 and 4.3.6]. Also CoR may be used in rheological models to tune the dissi-
52 pation parameters as in the Lankarani-Nikravesh approach and its many extensions and variants, see [7, 8] [5,
53 Chapter 2]. Another research direction, consists of determining the equivalent CoR to compliant rheological
54 models: the obtained CoR can be used in algebraic models, hence introducing useful informations in the
55 CoR (dependence on pre-impact velocity, on bodies’ masses, contact equivalent stiffness, contact equivalent
56 viscous friction, plasticity parameters, contact area, *etc*), while keeping an overall rigid-body instantaneous
57 collisions framework. This is usually not a straightforward task, and it has been the object of many investi-
58 gations for all types of models, linear spring/dashpots, nonlinear Hertz-plastic piecewise-smooth models [12]
59 [5, section 4.2.1.1] [13], and for nonlinear spring-dashpot Simon-Hunt-Crossley and Kuwabara-Kono models.
60 A common feature of (almost) all the models and analyses in the literature, is that it is assumed, as a
61 fundamental basic assumption, that all the forces which are applied on the colliding bodies at the impact
62 time, are negligible compared to the impact forces. Then exact or approximate expressions for the CoR can
63 be derived. This is the case in particular for the Simon-Hunt-Crossley (SHC) [23, 24] and the Kuwabara-
64 Kono (KK) [25] models, which are widely used in applications. Very few studies have analysed the same
65 problem, taking into account the effect of an external load on the impacting bodies (*i.e.*, calling into question
66 the fundamental assumption). Among these we may cite [26, 27, 28], which all conclude about the importance
67 of considering external loads in certain circumstances. One of the main goals in this work is to study the
68 effect of gravity or constant external force on the computed values of CoR, for the Kuwabara-Kono model,
69 hereby extending the fundamental results in [29, 30, 31]. The Kuwabara-Kono model stems from continuum
70 mechanics [32, 33, 34] and therefore occupies a particular place in the class of rheological compliant spring-
71 dashpot models. The provided theory applies, however, to different viscous nonlinearities than SHC and KK.
72 Approximate expressions for the CoR are given, and are shown through extensive numerical calculations to
73 fit very well with the exact numerical solution, for large range of external loads.

74 The report is organised as follows: the basic model is introduced in section 2. The algorithm for the exact
75 computation of the CoR through the integration of the dynamics is presented in section 3. Section 4 is devoted
76 to the calculation of approximate expressions for the CoR: Section 4.1 deals with the case of small equivalent
77 external load, section 4.2 deals with large external loads (and impacts with sticking), the accuracy of the
78 approximation is improved in section 4.3 by taking into account high-order terms in the expansions, finally
79 section 4.4 presents the case of any external loads (small or large). A recapitulatory section 5 summarizes
80 the main steps which are needed to calculate the CoR approximations. Conclusions are in section 6, and
81 useful developments and calculations can be found in Appendices A, B, C, and D.

82 2 Nonlinear Visco-Elastic Impact Model

83 Consider a system consisting of a single bead impacting the ground, as shown in Fig. 1. The mass of the
84 bead is m and the stiffness constant is k . The external load induced on the bead is given by the parameter

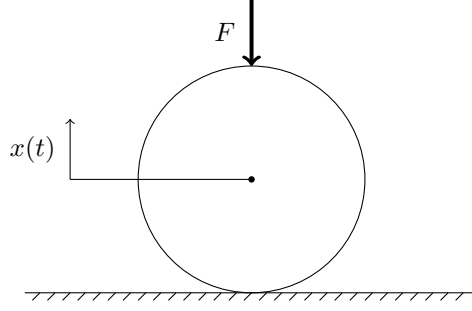


Figure 1: Single bead impacting ground, with external load

85 F , such that $F > 0$ produces a compressive (downward) force on the bead according to Fig. 1. The external
 86 load F is considered to be constant, which is a realistic assumption given that impact duration is typically
 87 very small. The displacement of the center of the bead in reference to the undeformed position is denoted
 88 by $x(t)$, such that the bead is in contact when $x(t) \leq 0$ and free when $x(t) > 0$. Then, based on a general
 89 visco-elastic model, the equation of motion of the bead is given by

$$m\ddot{x} = k \left[(-x)_+^\alpha + \gamma_0 \frac{d}{dt} (-x)_+^\beta \right] - mg - F \quad (1)$$

90 where $x_+ = \max(x, 0)$. The initial conditions for (1) are $x(0) = 0$ and $\frac{dx}{dt}(0) = -v_0$ with $v_0 > 0$. The
 91 parameters, $\alpha \geq 0$, $\beta \geq 0$ and $\gamma_0 \geq 0$ characterize the contact force behaviour: α is associated with the
 92 stiffness, whereas β and γ_0 control the dissipation during the impact. Nevertheless, for $\alpha < 1$ or $\beta < 1$, the
 93 uniqueness of the solution to the initial value problem associated with (1) is not established [35]. Hence, we
 94 make the following assumption in this study,

95 **Assumption 1.** $\alpha \geq 1$ and $\beta \geq 1$

If the parameters values are chosen to be $\alpha = \beta = \frac{3}{2}$, the visco-elastic model in (1) is known as the
 Kuwabara-Kono model [25], similarly if $\alpha = 3/2$ and $\beta = 5/2$, the Simon-Hunt-Crossley model [23, 24] is
 obtained. In this work we will pay particular attention to these cases. The dynamics of (1) can be re-scaled
 with respect to the time and length scales:

$$T = \left(\frac{m}{k}\right)^{\frac{1}{\alpha+1}} v_0^{\frac{1-\alpha}{\alpha+1}} \quad \text{and} \quad \delta = v_0 T$$

96 by setting $x(t) = -\delta u(\tau)$ with $\tau = \frac{t}{T}$. Then the scaled dynamics is given by,

$$\frac{d^2 u}{d\tau^2} + \gamma \frac{d}{d\tau} \left(u_+^\beta \right) + u_+^\alpha = \tilde{g} \quad (2)$$

where,

$$\gamma = \gamma_0 v_0^{\frac{2\beta}{\alpha+1}-1} \left(\frac{k}{m}\right)^{1-\frac{\beta}{\alpha+1}} \quad \text{and} \quad \tilde{g} = \left(g + \frac{F}{m}\right) \frac{T^2}{\delta} = \left(\frac{m}{k}\right)^{\frac{1}{\alpha+1}} v_0^{\frac{-2\alpha}{\alpha+1}} \left(g + \frac{F}{m}\right)$$

97 The initial conditions for the scaled equations of motion in (2) are $u(0) = 0$ and $\frac{du}{d\tau}(0) = 1$. As we
 98 can see, the new scaling parameters γ and \tilde{g} depend on v_0 and F . The dependence of these new scaling
 99 parameters to the actual physical quantities are demonstrated in Fig. 2, with an example of an impact under
 100 gravity and no external force exerted on the system ($g = 9.8$ m/s² and $F = 0$ N). Figure 2(a) shows the
 101 relationship between γ and v_0 , where we can see that the value of γ increases for higher values of initial
 102 velocity, v_0 . Similarly, Fig. 2(b) plots the \tilde{g} with respect to the constant external force, F , for different
 103 values of v_0 . Here we see that the external force F balances the gravitational constant $g = 9.8$ m/s²
 104 at a specific value, $F = -1.5092$ N, for all initial velocities v_0 . The values for mass, $m = 1.54 \times 10^{-1}$ kg
 105 and the stiffness constant, $k = 3.6138 \times 10^{10}$ N/m^{3/2} are based on the experimental result of the steel ball
 106 impact properties reported in [25], given $\alpha = \beta = 3/2$. The value of $\gamma_0 = 1.5237 \times 10^{-6}$ is related to
 107 the CoR value for steel ball, $e = 0.893$ reported in [25], and is calculated according to an approximation

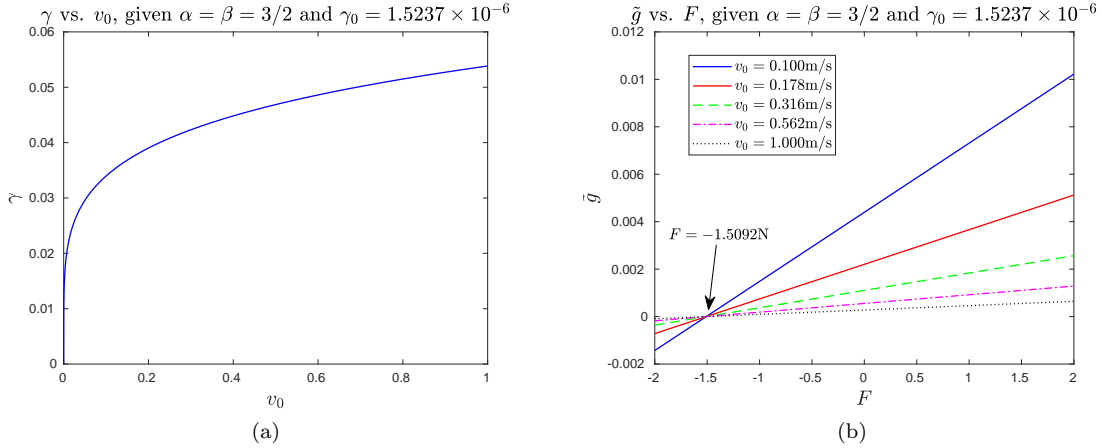


Figure 2: Values of the dimensionless scaling parameters, γ and \tilde{g} with respect to changes in (a) initial velocity, v_0 and (b) external force F . The gravitational constant is $g = 9.8\text{m/s}^2$. Values for mass, $m = 1.54 \times 10^{-1}\text{kg}$ and spring constant, $k = 3.6138 \times 10^{10}\text{N/m}^{3/2}$, are based on the experimental result on steel balls in [25]. $\alpha = \beta = 3/2$ and the value of $\gamma_0 = 1.5237 \times 10^{-6}$ is calculated using the CoR, $e = 0.893$ reported in [25], according to the method mentioned in [36].

108 mentioned in [36]. The parameter values explain the value $F = -1.5092\text{ N}$ at $\tilde{g} = 0$ intercept in Fig. 2(b),
 109 since $F = -mg = -1.54 \times 10^{-1}\text{ kg} \times 9.8\text{ m/s}^2 = -1.5092\text{ N}$. Also we observe, that an increase in the value
 110 of the initial velocity, v_0 diminishes the effects of external load, i.e., $g + F/m$, on \tilde{g} .

111 Let us assume that the impact terminates at $\tau = T_f$ with $T_f > 0$. The impact takes place for $\tau \in (0, T_f)$,
 112 so the scaled displacement $u(\tau)$ is positive while $\tau \in (0, T_f)$ and $u(T_f) = 0$ at impact termination. The CoR
 113 for the impact can be expressed in terms of the scaled velocities as,

$$e = -\frac{\frac{du}{d\tau}(T_f)}{\frac{du}{d\tau}(0)} = -\frac{du}{d\tau}(T_f) \quad (3)$$

114 Thus, the CoR for a given impact can be numerically evaluated by integrating the ordinary differential
 115 equation for the scaled dynamics given in (2) until $\tau = T_f$. The ultimate goal of this work is to obtain an
 116 analytical approximation for the CoR, for arbitrary parameter values (α, β, γ , and \tilde{g}). Nevertheless, in order
 117 to study the validity of analytical approximate CoR, we would always need to establish a reference on the
 118 “true” values of CoR. In this study, the values of CoR obtained through numerical integration of (2) are
 119 considered the “true” CoR values, and are treated as a reference to study the validity of the approximations
 120 later. The next section discusses the method used for numerically integrating (2) in this study.

121 3 Calculation of the CoR by Numerical Integration

122 The term $\frac{d}{d\tau} (u_+^\beta) = \beta u_+^{\beta-1} \frac{du}{d\tau}$ in (2) is not Lipschitz continuous, and therefore can lead to difficulty with
 123 the numerical integration. As discussed in [35, 36], these numerical issues can be circumvented with an
 124 appropriate choice of the dynamical variable. Hence, here we will consider the same change of variable, and
 125 rewrite (2) in terms of a new *generalized velocity* parameter w :

$$\begin{aligned} \frac{du}{d\tau} &= w - \gamma u_+^\beta \\ \frac{dw}{d\tau} &= -u_+^\alpha + \tilde{g} \end{aligned} \quad (4)$$

126 The state variables in (4) are $\mathbf{Y} = [u \quad w]^T$, such that the system in (4) can be concisely expressed as :
 127 $\dot{\mathbf{Y}} = \mathbf{F}(\mathbf{Y})$ with the initial conditions $\mathbf{Y}(0) = [0 \quad 1]^T$, which has to be numerically integrated till $\tau = T_f$

128 to obtain the CoR as $e = -\frac{du}{d\tau}(T_f)$. However, since the value of T_f is not a priori known, one would need to
 129 continue the numerical integration until the impact termination conditions are met. The impact termination
 130 can take place when the bead detaches at some T_f , such that $u(T_f) = 0$ and $\frac{du}{d\tau}(T_f) < 0$. Alternatively, the
 131 bead may converge to the equilibrium point without detachment, i.e., the bead doesn't rebound and sticks
 132 to the ground. The parameters values (α , β , γ and \tilde{g}) dictate whether the bead will detach or not, for a
 133 given impact.

The equilibrium point of (2), by setting $\left(\frac{d^2u}{d\tau^2} = \frac{du}{d\tau} = 0\right)$, is at $\bar{u} = \tilde{g}^{\frac{1}{\alpha}}$. We can determine the character-
 istics of the convergence towards equilibrium by linearizing (2) about the equilibrium point. If we consider
 the value of $u > 0$ in the neighbourhood of \bar{u} , as $u = \bar{u} + v$ for a small $v \rightarrow 0$, the dynamical equations
 linearized around $v = 0$ read

$$\frac{d^2v}{d\tau^2} + \underbrace{\gamma\beta\tilde{g}^{\frac{\beta-1}{\alpha}}}_{2\xi\omega_n} \frac{dv}{d\tau} + \underbrace{\alpha\tilde{g}^{\frac{\alpha-1}{\alpha}}}_{\omega_n^2} v = 0.$$

The damping ratio ξ , and the natural frequency ω_n , for the linearized second-order dynamics around the
 equilibrium point are expressed as,

$$\xi = \frac{\gamma\beta}{2\sqrt{\alpha}}\tilde{g}^{\frac{2\beta-\alpha-1}{2\alpha}} \quad \text{and} \quad \omega_n = \sqrt{\alpha}\tilde{g}^{\frac{\alpha-1}{2\alpha}}.$$

We know that the condition for an underdamped response is $0 < \xi < 1$, which leads to the equivalent
 inequality,

$$\gamma < \frac{2\sqrt{\alpha}}{\beta}\tilde{g}^{\frac{\alpha+1-2\beta}{2\alpha}}.$$

Thus, for any given value-pair of (α, β) , there exists a constraint on the values of γ and \tilde{g} that must be satisfied
 for an underdamped response. As an example, if we consider the Kuwabara-Kono model ($\alpha = \beta = 3/2$), the
 constraint on the parameter γ and \tilde{g} is given by,

$$\gamma < 2\sqrt{\frac{2}{3}}\tilde{g}^{-\frac{1}{6}} \quad \text{or} \quad \tilde{g} < \frac{512}{27\gamma^6}.$$

134 In this work, the numerical simulations will be mainly performed in the underdamped regime. If the bead
 135 does not rebound, the velocity $\frac{du}{d\tau}$ will converge to zero with oscillations, while $u(\tau)$ will always remain
 136 positive. Hence, time-integration will stop when $u(T_f) \leq \bar{u}$ and $\frac{du}{d\tau}(T_f) = 0$, where $\bar{u} = \tilde{g}^{\frac{1}{\alpha}}$ is the equilibrium
 137 point. On the other hand, the bead will rebound if $u(T_f) = 0$ and $\frac{du}{d\tau}(T_f) < 0$ for some $T_f > 0$. Consequently,
 138 in the underdamped regime, time integration will stop at $t = T_f > 0$ where $u(T_f)u'(T_f) = 0$ and $u(T_f) \leq \tilde{g}^{\frac{1}{\alpha}}$.

139 The numerical integration is performed in three separate intervals (a pseudo-code for this numerical
 140 scheme is presented in Algorithm 1.):

- 141 1 Starting from initial conditions, the system of equations (4) is integrated until τ_0 , such that $u'(\tau_0) = 0$.
 142 *This stops the integration at maximum compression, which guarantees that system enters the restitution*
 143 *phase, before the final termination.*
- 144 2 Next starting with the system states at τ_0 , the system of equations (4) is numerically integrated until
 145 τ_1 , such that $u(\tau_1) - \tilde{g}^{\frac{1}{\alpha}} = 0$. *After starting from the maximal compression, stopping the integration*
 146 *at τ_1 , such that $u(\tau_1) = \tilde{g}^{\frac{1}{\alpha}}$ and $u'(\tau_1) \leq 0$, guarantees that $u(T_f) \leq \tilde{g}^{\frac{1}{\alpha}}$ and $u'(T_f) \leq 0$.*
- 147 3 Lastly, starting with the system states at τ_1 , the system of equations (4) is integrated until a final time
 148 τ_2 , such that $u(\tau_2)u'(\tau_2) = 0$. *This condition finally stops the integration to yield $u(T_f) = u(\tau_2) \leq \tilde{g}^{\frac{1}{\alpha}}$*
 149 *and $u'(T_f) = u'(\tau_2) \leq 0$.*

150 In addition, time integration is stopped if $\|(u - \bar{u}, u')\|_\infty < \varepsilon$, for some small threshold ε (we fix $\varepsilon = 10^{-8}$),
 151 leading to $e = 0$.

152 Figure 3 shows an example of the evolution of u and $\frac{du}{d\tau}$, corresponding to the parameter values: $\alpha = 3/2$,
 153 $\beta = 3/2$, $\gamma = 0.006$ and $\tilde{g} = 0.1$, as obtained by numerically integrating (4). As one may notice from Fig. 3,
 154 in this example the bead detaches at the end of an impact, since the scaled displacement $u(\tau)$ becomes zero
 155 at the end of the impact. Fig. 4 shows the evolution of $u(\tau)$ and $\frac{du}{d\tau}$, for the parameters: $\alpha = 3/2$, $\beta = 3/2$,
 156 $\gamma = 0.06$ and $\tilde{g} = 4$. In this case, the time-integration terminates when $\frac{du}{d\tau} = 0$ and $0 < u(\tau) < \tilde{g}^{\frac{1}{\alpha}}$. Note
 157 that the CoR value shown in Fig. 4(a) is $e = 0$.

Algorithm 1 Numerical Integration Scheme

Set: $\mathbf{Y}_0 \leftarrow \mathbf{Y}(0)$, where $\mathbf{Y}(0) = [u(0) \quad w(0)]^T$
while $u'(\tau) = w(\tau) - \gamma u_+^\beta(\tau) > 0$ **do**
 Integrate: $\dot{\mathbf{Y}}(\tau) = \mathbf{F}(\mathbf{Y}(\tau))$, with initial condition \mathbf{Y}_0
 Update: $\tau, \mathbf{Y}(\tau)$
end while
 Set: $\mathbf{Y}_0 \leftarrow \mathbf{Y}(\tau)$
while $u - \tilde{g}^{\frac{1}{\alpha}} > 0$ **do**
 Integrate: $\dot{\mathbf{Y}}(\tau) = \mathbf{F}(\mathbf{Y}(\tau))$, with initial condition \mathbf{Y}_0
 Update: $\tau, \mathbf{Y}(\tau)$
end while
 Set: $\mathbf{Y}_0 \leftarrow \mathbf{Y}(\tau)$
while $u(\tau)u'(\tau) = u(\tau) [w(\tau) - \gamma u_+^\beta(\tau)] < 0$ **do**
 Integrate: $\dot{\mathbf{Y}}(\tau) = \mathbf{F}(\mathbf{Y}(\tau))$, with initial condition \mathbf{Y}_0
 Update: $\tau, \mathbf{Y}(\tau)$
end while

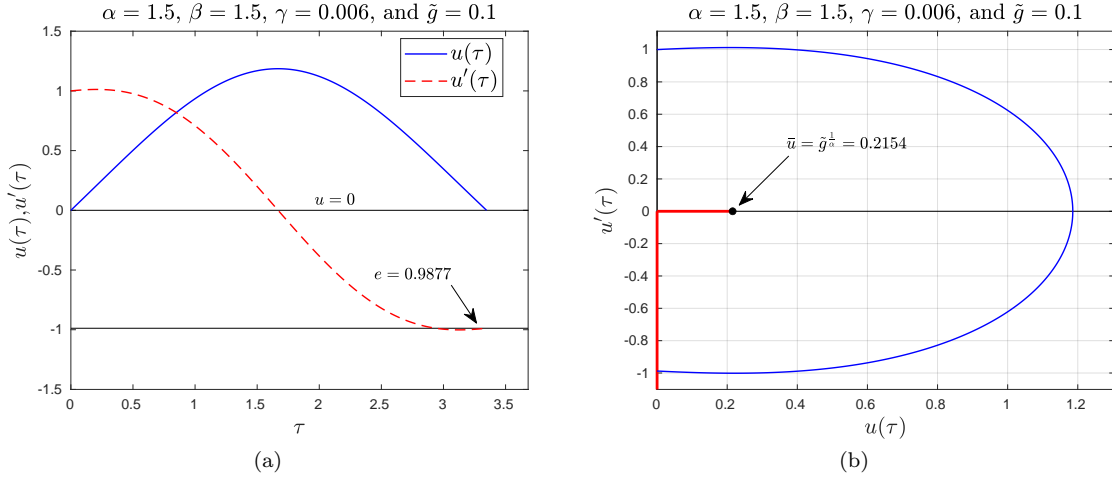


Figure 3: Numerical integration of scaled dynamics for the parameters $\alpha = \beta = \frac{3}{2}$, $\gamma = 0.006$ and $\tilde{g} = 0.1$: (a) Responses of $u(\tau)$ and $u'(\tau)$, and (b) Phase-space plot of $(u(\tau), u'(\tau))$ (the stopping condition is shown in red)

4 Analytical CoR Approximated Expression

The numerical integration scheme for CoR computation, discussed in the previous section, is mainly presented for the purpose of establishing a reference for the analytical CoR approximations that will be presented in this section. In order to obtain an analytical CoR approximation, let us begin by transforming the scaled equations of motion (2), in terms of e .

Firstly, multiplying equation (2) by $\frac{du}{d\tau}$ gives,

$$\frac{d}{d\tau} \left[\frac{1}{2} \left(\frac{du}{d\tau} \right)^2 \right] + \gamma \beta \left(\frac{du}{d\tau} \right)^2 (u)^{\beta-1} + \frac{d}{d\tau} \left(\frac{u^{\alpha+1}}{\alpha+1} \right) = \tilde{g} \frac{du}{d\tau} \quad (5)$$

which if integrated from $\tau = 0$ to $\tau = T_f$, yields.

$$e^2 = 1 - 2\gamma\beta \int_0^{T_f} \left(\frac{du}{d\tau} \right)^2 u^{\beta-1} d\tau \quad (6)$$

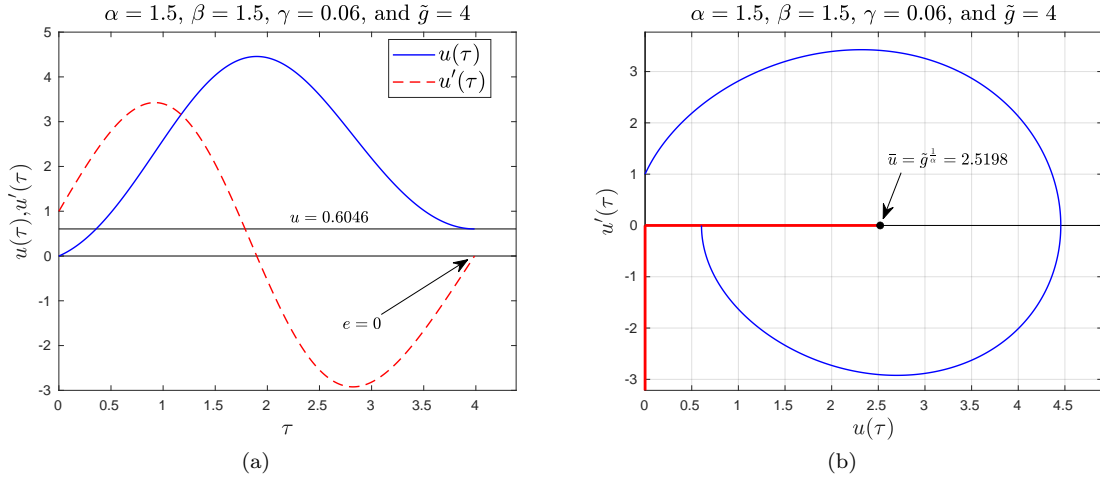


Figure 4: Numerical integration of scaled dynamics for the parameters $\alpha = \beta = \frac{3}{2}$, $\gamma = 0.06$ and $\tilde{g} = 4$: (a) Responses of $u(\tau)$ and $u'(\tau)$, and (c) Phase-space plot of $(u(\tau), u'(\tau))$ (the stopping condition is shown in red)

165 Let us now consider that the scaled displacement and time of detachment corresponding to the non-
 166 dissipative case ($\gamma = 0$) are $u_0(\tau)$ and T_0 , respectively. Therefore, the scaled displacement and time of
 167 detachment corresponding to the dissipative case, with very small γ values ($0 \leq \gamma \ll 1$) can be expanded as
 168 : $u(\tau) = u_0(\tau) + \mathcal{O}(\gamma)$ and $T_f = T_0 + \mathcal{O}(\gamma)$. Therefore, using equation (2) we get,

$$\frac{d^2 u_0}{d\tau^2} + u_0^\alpha = \tilde{g} \quad \text{where,} \quad u_0(0) = 0, \quad \frac{du_0}{d\tau}(0) = 1, \quad u_0(T_0) = 0 \quad \text{and,} \quad \frac{du_0}{d\tau}(T_0) = -1 \quad (7)$$

169 Similarly, (6) can be rewritten in terms of u_0 and rearranged as,

$$e^2 = 1 - 2\gamma\beta \int_0^{T_0} \left(\frac{du_0}{d\tau} \right)^2 u_0^{\beta-1} d\tau + \text{H.O.T.} \quad (8)$$

170 Also, noting that for small γ the value of e is close to one, which implies that the value of the second
 171 term, ϵ in equation (8) of the form: $e^2 = 1 - \epsilon$, should be close to zero. Therefore, using the expansion
 172 $(1 - \epsilon)^{\frac{1}{2}} = 1 - \frac{\epsilon}{2} + \mathcal{O}(\epsilon^2)$, an expression for e can be obtained as,

$$e = 1 - \gamma\beta \int_0^{T_0} \left(\frac{du_0}{d\tau} \right)^2 u_0^{\beta-1} d\tau + \text{H.O.T.} \quad (9)$$

173 In the non-dissipative case, using the symmetry of the force time-history over the impact duration, one finds
 174 that $\tau = \frac{T_0}{2}$ corresponds to the time of maximal compression and $u_0\left(\frac{T_0}{2} + \tau\right) = u_0\left(\frac{T_0}{2} - \tau\right)$, which allows
 175 for (9) to be rewritten as,

$$e = 1 - 2\gamma\beta \int_0^{\frac{T_0}{2}} \left(\frac{du_0}{d\tau} \right)^2 u_0^{\beta-1} d\tau + \text{H.O.T.} \quad (10)$$

Also, multiplying (7) by $\frac{du_0}{d\tau}$ and integrating from $\tau = 0$ yields,

$$\frac{1}{2} \left[\left(\frac{du_0}{d\tau}(\tau) \right)^2 - 1 \right] + \frac{u_0(\tau)^{\alpha+1}}{\alpha+1} = \tilde{g}u_0(\tau)$$

176 which can be used to solve for $\frac{du_0}{d\tau}$,

$$\frac{du_0}{d\tau} = \left[1 + 2 \left(\tilde{g}u_0(\tau) - \frac{u_0(\tau)^{\alpha+1}}{\alpha+1} \right) \right]^{\frac{1}{2}} \quad (11)$$

177 Thus we can express $d\tau$ in terms of du_0 ,

$$d\tau = \left[1 + 2 \left(\tilde{g}u_0 - \frac{u_0^{\alpha+1}}{\alpha+1} \right) \right]^{-\frac{1}{2}} du_0 \quad (12)$$

178 At the maximum compression, $\frac{du_0}{d\tau} = 0$ and therefore, the displacement given by $u_M = u_0 \left(\frac{T_0}{2} \right)$ can be
179 obtained using (11),

$$\left[1 + 2 \left(\tilde{g}u_M - \frac{u_M^{\alpha+1}}{\alpha+1} \right) \right]^{\frac{1}{2}} = 0 \quad \Rightarrow \quad \frac{1}{\alpha+1} u_M^{\alpha+1} - \tilde{g}u_M = \frac{1}{2} \quad (13)$$

180 Based on (13), we can observe that the solution of the displacement at the maximum compression, u_M
181 depends on \tilde{g} . Thus, in the development that follows, we will treat u_M , as a function of \tilde{g} , $u_M = u_M(\tilde{g})$.

182 Hence, substituting (11) in (10) and then performing a change of the integration variable ($\tau \rightarrow u_0$), using
183 the expression of $d\tau$ in (12), as well as setting $u_0 \left(\frac{T_0}{2} \right) = u_M(\tilde{g})$ for the upper-bound, we can rewrite the
184 expression for e as,

$$e = 1 - 2\gamma\beta\mathcal{I}(\tilde{g}) + \text{H.O.T.} \quad (14)$$

185 with

$$\mathcal{I}(\tilde{g}) = \int_0^{u_M(\tilde{g})} \left[1 + 2 \left(\tilde{g}u_0 - \frac{u_0^{\alpha+1}}{\alpha+1} \right) \right]^{\frac{1}{2}} u_0^{\beta-1} du_0. \quad (15)$$

186 The exact closed-form solution cannot be obtained for the integral, however in the following sections we will
187 consider a number of approximations that would ultimately yield an analytical solution for the CoR.

188 4.1 Approximation for small \tilde{g}

189 We begin by considering approximations of e , for small \tilde{g} ($\tilde{g} \approx 0$). In order to do this, we will perform a
190 first-order Taylor expansion of (14) about $\tilde{g} = 0$,

$$e = 1 - 2\gamma\beta\mathcal{I}(0) - 2\gamma\beta\tilde{g}\frac{d\mathcal{I}}{d\tilde{g}}(0) + \text{H.O.T.} \quad (16)$$

191 Therefore, now we need to obtain expressions for $\mathcal{I}(0)$ and $\frac{d\mathcal{I}}{d\tilde{g}}(0)$. In order to do this, let us first consider
192 the derivative of $\mathcal{I}(\tilde{g})$,

$$\frac{d\mathcal{I}}{d\tilde{g}}(\tilde{g}) = \frac{d}{d\tilde{g}} \int_0^{u_M(\tilde{g})} h(\tilde{g}, u_0) du_0 \quad (17)$$

where,

$$h(\tilde{g}, u_0) = \left[1 + 2 \left(\tilde{g}u_0 - \frac{u_0^{\alpha+1}}{\alpha+1} \right) \right]^{\frac{1}{2}} u_0^{\beta-1}$$

Using Leibniz integral rule, we may rewrite (17) as,

$$\frac{d\mathcal{I}}{d\tilde{g}}(\tilde{g}) = h(\tilde{g}, u_M(\tilde{g})) \cdot \frac{du_M}{d\tilde{g}}(\tilde{g}) + \int_0^{u_M(\tilde{g})} \frac{\partial h}{\partial \tilde{g}}(\tilde{g}, u_0) du_0$$

193 The term $h(\tilde{g}, u_M(\tilde{g}))$ can be evaluated using (13) to be,

$$h(\tilde{g}, u_M(\tilde{g})) = \left[1 + 2 \left(\tilde{g}u_M - \frac{u_M^{\alpha+1}}{\alpha+1} \right) \right]^{\frac{1}{2}} u_M^{\beta-1} = [0]^{\frac{1}{2}} u_M^{\beta-1} = 0 \quad (18)$$

194 Hence (4.1) is given by,

$$\frac{d\mathcal{I}}{d\tilde{g}}(\tilde{g}) = \int_0^{u_M(\tilde{g})} \frac{\partial h}{\partial \tilde{g}}(\tilde{g}, u_0) du_0 = \int_0^{u_M(\tilde{g})} \left[1 + 2 \left(\tilde{g}u_0 - \frac{u_0^{\alpha+1}}{\alpha+1} \right) \right]^{-\frac{1}{2}} u_0^{\beta} du_0 \quad (19)$$

195 We note from (19), that the coefficient of $\frac{du_M}{d\tilde{g}}$ in (4.1), given by $h(\tilde{g}, u_M(\tilde{g}))$ vanishes. Therefore, we can
 196 conclude that a zeroth-order approximation of $u_M(0)$ would be sufficient to evaluate the values of $\mathcal{I}(0)$ and
 197 $\frac{d\mathcal{I}}{d\tilde{g}}(0)$ in (16). Using (13) the zeroth-order approximation of $u_M(0)$ is given by

$$u_M(0) = u_M^* = \left(\frac{\alpha + 1}{2}\right)^{\frac{1}{\alpha+1}} \quad (20)$$

198 Thus, evaluating the expressions for $\mathcal{I}(0)$ and $\frac{d\mathcal{I}}{d\tilde{g}}(0)$ using (20) and substituting them into (16), we obtain,

$$e = 1 - 2\gamma\beta \int_0^{u_M^*} \left[1 - 2\frac{u_0^{\alpha+1}}{\alpha+1}\right]^{\frac{1}{2}} u_0^{\beta-1} du_0 - 2\gamma\beta \int_0^{u_M^*} \left[1 - 2\frac{u_0^{\alpha+1}}{\alpha+1}\right]^{-\frac{1}{2}} u_0^\beta du_0 + \text{H.O.T.} \quad (21)$$

199 which may be rewritten as,

$$e = 1 - \gamma C_0 - \gamma \tilde{g} C_1 + \text{H.O.T.} \quad (22)$$

where,

$$C_0 = 2\beta \int_0^{u_M^*} \left[1 - 2\frac{u_0^{\alpha+1}}{\alpha+1}\right]^{\frac{1}{2}} u_0^{\beta-1} du_0 \quad \text{and} \quad C_1 = 2\beta \int_0^{u_M^*} \left[1 - 2\frac{u_0^{\alpha+1}}{\alpha+1}\right]^{-\frac{1}{2}} u_0^\beta du_0$$

200 We can notice that the definite integrals C_0 and C_1 can be rewritten in the form of *Euler's Beta Function*
 201 $B(z, w) = \int_0^1 t^{z-1}(1-t)^{w-1} dt$, with $w, z > 0$. In this work, the Euler's Beta function would be used several
 202 times to approximate integrals, therefore here we will define a more general expression for the Beta function
 203 that can be readily used to approximate integrals,

Definition 1. *If the scalars $x, y, \rho, \lambda \in \mathbb{R}$, with $\rho, \lambda > 0$ and $x, y > -1$, then the following relationship holds,*

$$\int_0^{(1/\rho)^{1/\lambda}} t^x (1 - \rho t^\lambda)^y dt = \frac{1}{\lambda \rho^{(x+1)/\lambda}} B\left(\frac{x+1}{\lambda}, y+1\right)$$

204 where B is Euler's Beta function.

The upper-bounds of the integrals C_0 and C_1 in (22) is u_M^* . According to (20) the maximum compression
 u_M^* is given by $u_M^* = \left(\frac{\alpha+1}{2}\right)^{\frac{1}{\alpha+1}}$. Therefore, Def. 1 can be used to express C_0 and C_1 in terms of Euler's Beta
 function,

$$C_0 = \beta \left(\frac{\alpha+1}{2}\right)^{\frac{\beta-\alpha-1}{\alpha+1}} B\left(\frac{\beta}{\alpha+1}, \frac{3}{2}\right) \quad \text{and} \quad C_1 = \beta \left(\frac{\alpha+1}{2}\right)^{\frac{\beta-\alpha}{\alpha+1}} B\left(\frac{\beta+1}{\alpha+1}, \frac{1}{2}\right)$$

205 Thus, having obtained the first-order analytical approximation of e in (22) for small \tilde{g} values, we will now
 206 study the validity of this approximation against numerically integrated results of e . Figures 5, 6, and
 207 7 compares the CoR values computed using the approximation in (22) and the values obtained through
 208 numerical integration of (4). In these figures the numerically integrated values of CoR are labelled as *Num.*
 209 *Integ.*, and the values obtained using the approximation in (22) is labeled *Small \tilde{g} Approx.*. Figure 5 shows
 210 the result corresponding to the parameter values $\alpha = \beta = 3/2$, and $\gamma = 0.0001$, with \tilde{g} varying in the range
 211 $[0, 10]$ in Fig. 5(a) and in the range $[0, 1]$ in Fig. 5(b). Fig. 5(b) also includes a zoomed-in plot near the value
 212 $\tilde{g} = \tilde{g}_0$. \tilde{g}_0 is the value of \tilde{g} computed using some physical parameters from the study [37], to show the realistic
 213 range of the value of \tilde{g} . The value of $\tilde{g}_0 = 5.1224 \times 10^4$ is computed using the parameters: $m = 2.05 \times 10^{-3} \text{kg}$,
 214 $k = 9.858 \times 10^9 \text{N/m}^{\frac{3}{2}}$, $\alpha = \beta = 3/2$, $g = 9.8 \text{ m/s}^2$, and $F = 0 \text{ N}$. These parameters represent the value of \tilde{g} ,
 215 when a steel bead of 8mm diameter impacts under gravity. We note that the error between the numerically
 216 integrated value of CoR and the approximation using (22) is very small, i.e. 8.6615×10^{-8} . Therefore, we
 217 can conclude that the approximation (22) can perform well for $\gamma = 0.0001$, when the net external forces is
 218 in the same order of magnitude as the force due to gravitational acceleration. Results for other values of
 219 γ , namely $\gamma = 0.001$ and $\gamma = 0.01$, are also evaluated in Fig. 6 and Fig. 7. Figures 6(a) and 6(b) compares
 220 the CoR values within the ranges $\tilde{g} \in [0, 10]$ and $\tilde{g} \in [0, 1]$, respectively with $\gamma = 0.001$, and similarly

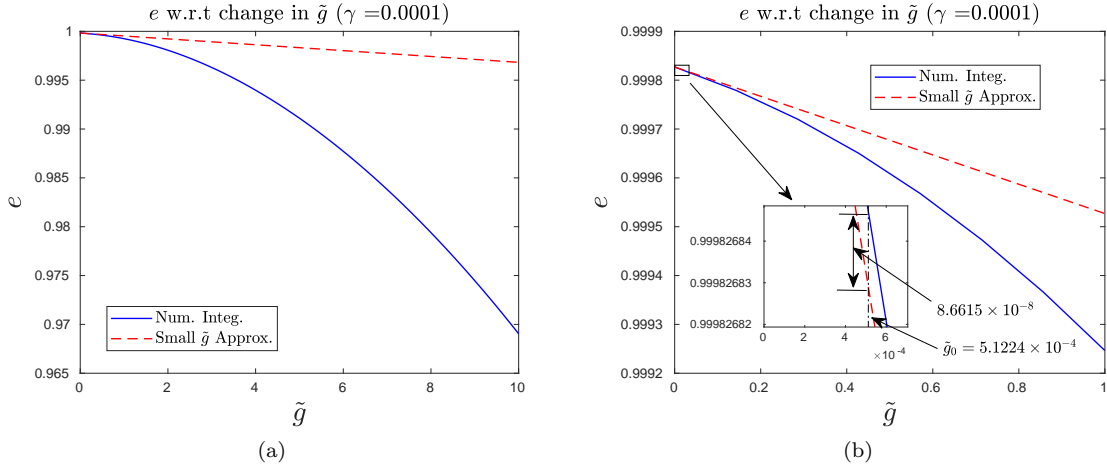


Figure 5: Change in CoR with respect to (a) $\tilde{g} \in [0, 10]$ and (b) $\tilde{g} \in [0, 1]$, and also contains a zoomed-in plot that shows the value of \tilde{g}_0 (which is the value of \tilde{g} calculated using the parameters in [37], for $g = 9.8 \text{ m/s}^2$ and $F = 0 \text{ N}$), with $\alpha = \frac{3}{2}$, $\beta = \frac{3}{2}$, and $\gamma = 0.0001$. The numerically integrated values of CoR are labeled as *Num. Integ.* and the values obtained using the approximation (22) are labeled *Small \tilde{g} Approx.*

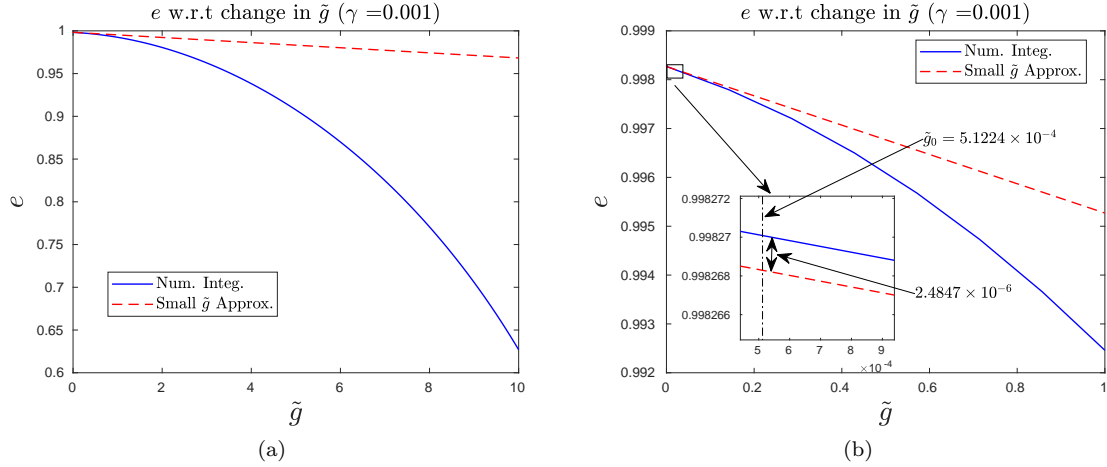


Figure 6: Change in CoR with respect to (a) $\tilde{g} \in [0, 10]$, (b) $\tilde{g} \in [0, 1]$, and also contains a zoomed-in plot that shows the value of \tilde{g}_0 (which is the value of \tilde{g} calculated using the parameters in [37], for $g = 9.8 \text{ m/s}^2$ and $F = 0 \text{ N}$), with $\alpha = \frac{3}{2}$, $\beta = \frac{3}{2}$, and $\gamma = 0.001$. The numerically integrated values of CoR are labeled as *Num. Integ.* and the values obtained using the approximation (22) are labeled *Small \tilde{g} Approx.*

221 Fig. 7(a) and Fig. 7(b) compares the CoR values in the ranges of $\tilde{g} \in [0, 10]$ and $\tilde{g} \in [0, 1]$, with $\gamma = 0.01$.
 222 Fig. 6(b) and Fig. 7(b) also includes zoomed-in plots to show the errors in CoR near the value of \tilde{g}_0 , which
 223 is independent to γ . Again we note that the approximate CoR using (22) works well for small enough
 224 value of \tilde{g} . Although, it appears that the approximation (22) works well when the \tilde{g} value is in the same order
 225 of magnitude as the value of \tilde{g}_0 , one needs a better approximation for larger values of \tilde{g} . Also in Fig. 5,
 226 Fig. 6, and Fig. 7, the numerically calculated CoR becomes zero for high \tilde{g} , corresponding to the bead
 227 impacting with no detachment. The linear approximation (22) is unable to represent this no-detachment
 228 case. Additionally, we observe that the \tilde{g} value beyond which the the impacting bead doesn't detach
 229 as γ increases. Therefore, in order to further understand the connection between the value of γ and the
 230 value of \tilde{g} beyond which the bead doesn't detach, next we will consider the problem of approximating CoR
 231 for large \tilde{g} . Appendix A derives a CoR approximation corresponding to large \tilde{g} , and also establishes an

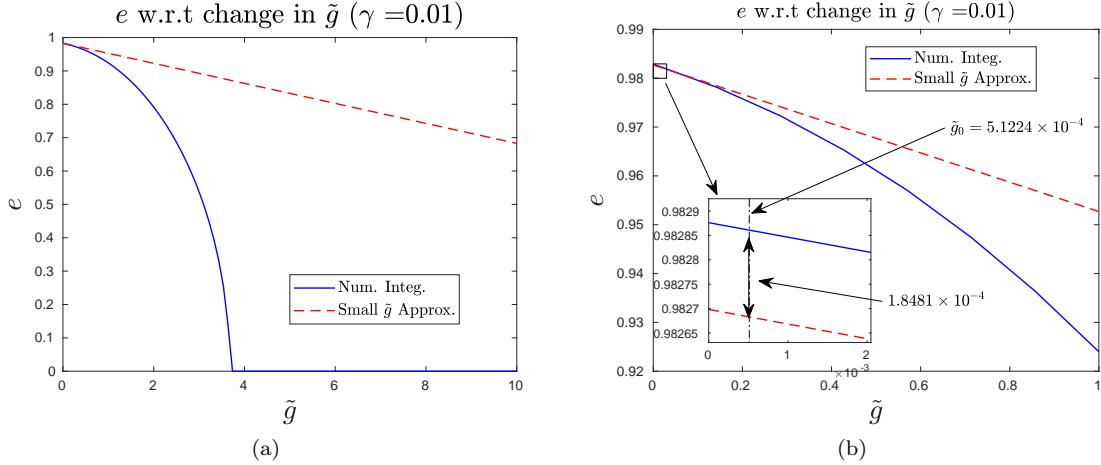


Figure 7: Change in CoR with respect to (a) $\tilde{g} \in [0, 10]$, (b) $\tilde{g} \in [0, 1]$, and also contains a zoomed-plot that shows the value of \tilde{g}_0 (which is the value of \tilde{g} calculated using the parameters in [37], for $g = 9.8$ m/s² and $F = 0$ N), with $\alpha = \frac{3}{2}$, $\beta = \frac{3}{2}$, and $\gamma = 0.01$. The numerically integrated values of CoR are labeled as *Num. Integ.* and the values obtained using the approximation (22) are labeled *Small \tilde{g} Approx.*

approximate relationship between the value of γ and the smallest value of \tilde{g} , beyond which the bead doesn't detach following the impact. The following section summarizes the main results of this approximation.

4.2 Approximation for large \tilde{g} and no-rebound condition

As seen in the previous section, it is relatively straightforward to derive an analytical approximation of CoR for small values of \tilde{g} . The results shown in Fig. 5, Fig. 6, and Fig. 7 show that these approximations work well when the value of \tilde{g} is small, but are highly inaccurate as \tilde{g} becomes large. Therefore a different of CoR approximation for large \tilde{g} is needed. This section presents the main results related to large \tilde{g} approximation of CoR; the complete derivation of the results in this section is presented in Appendix A.

Considering a large \tilde{g} , the squared CoR expansion in (6), can be approximated as (See Appendix A),

$$e^2 = 1 - 2\gamma C \tilde{g}^{\left(\frac{\beta}{\alpha} + \frac{1}{2\alpha} + \frac{1}{2}\right)} \quad \text{with} \quad C = 2\sqrt{2} \frac{\beta}{\alpha} (\alpha + 1)^{\left(\frac{\beta}{\alpha} + \frac{1}{2\alpha}\right)} \text{B}\left(\frac{\beta + 1/2}{\alpha}, \frac{3}{2}\right) \quad (23)$$

Hence, using (23), the CoR corresponding to large values of \tilde{g} can be approximated as $e = \sqrt{e_+^2}$, where $e_+^2 = \max(e^2, 0)$. The value of \tilde{g} is considered large if $\tilde{g} \approx \frac{u_M^\alpha}{\alpha + 1}$, which follows from the derivation in Appendix A. It is evident from (23), that the CoR e decreases as \tilde{g} increases. We know that when \tilde{g} becomes very large the bead does not detach after impact. The smallest value of \tilde{g} , after which the bead doesn't detach, can be determined by solving for $e = 0$, using (23), as described in the proposition below.

Proposition 1. For $\alpha, \beta \geq 1$ and small $\gamma > 0$, there exists $\tilde{g}_c \gg 1$ satisfying

$$\tilde{g}_c^{\left(\frac{\beta}{\alpha} + \frac{1}{2\alpha} + \frac{1}{2}\right)} \approx \frac{1}{2\gamma C} \quad \text{where,} \quad C = 2\sqrt{2} \frac{\beta}{\alpha} (\alpha + 1)^{\left(\frac{\beta}{\alpha} + \frac{1}{2\alpha}\right)} \text{B}\left(\frac{\beta + \frac{1}{2}}{\alpha}, \frac{3}{2}\right) \quad (24)$$

such that $e = 0$ for $\tilde{g} = \tilde{g}_c$ and $e > 0$ when $1 \ll \tilde{g} < \tilde{g}_c$.

Proof. See Appendix A □

Using (24), one can determine the smallest value of \tilde{g} , i.e. $\tilde{g}_c \equiv \tilde{g}$, beyond which the CoR will remain zero. Alternatively, for a given value of \tilde{g} , (24) can also be used to determine the smallest value of γ , i.e. γ_c , such that $e = 0$ for all $\gamma > \gamma_c$. In case of Kuwabara-Kono model ($\alpha = \beta = 3/2$), \tilde{g}_c can be determined by first calculating $C = 5\sqrt{2} \frac{5}{2}^{1/3} \text{B}(4/3, 3/2) = 4.403986$ which yields the expression $\tilde{g}_c = \frac{0.30522}{\gamma^{6/11}}$.

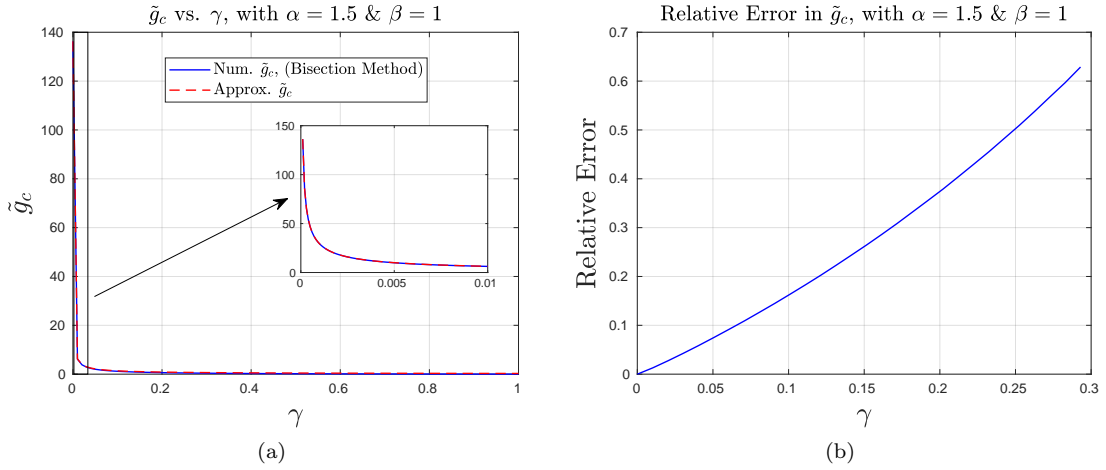


Figure 8: Plots comparing the \tilde{g}_c approximation with numerically computed \tilde{g}_c value using the Bisection Method, given $\alpha = 3/2$ & $\beta = 1$ (*Hertz stiffness & linear damping*). (a) \tilde{g}_c vs. γ , where the plot labeled *Num. \tilde{g}_c (Bisection Method)* refer to the numerical values and the plot labeled *Approx. \tilde{g}_c* refer to the approximate analytical value obtained using (24), (b) Relative error of the \tilde{g}_c between the analytical approximations and the numerically computed values, over the range of $\gamma \in [0, 0.3]$.

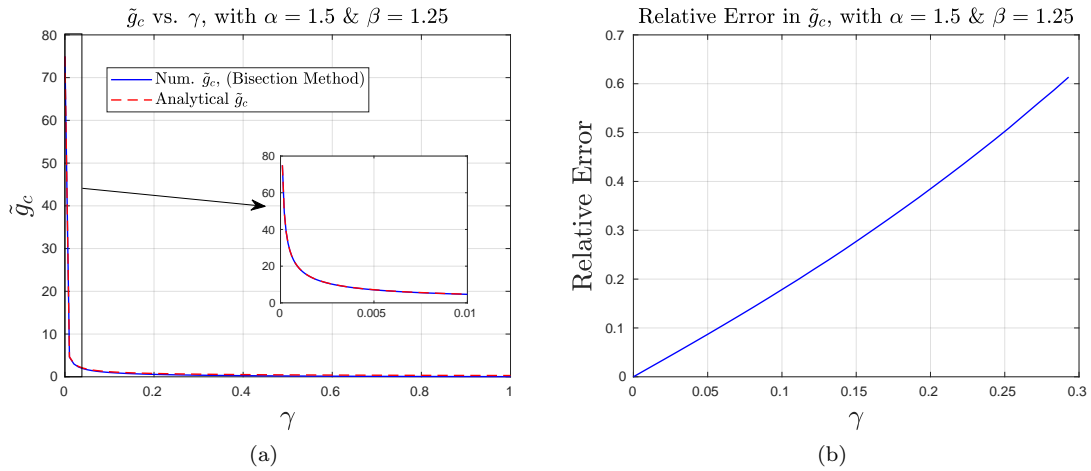


Figure 9: Plots comparing the \tilde{g}_c approximation with numerically computed \tilde{g}_c value using the Bisection Method, given $\alpha = 3/2$ & $\beta = 5/4$ (*Tsuji-Tanaka-Ishida model*). (a) \tilde{g}_c vs. γ , where the plot labeled *Num. \tilde{g}_c (Bisection Method)* refer to the numerical values and the plot labeled *Approx. \tilde{g}_c* refer to the approximate analytical value obtained using (24), (b) Relative error of the \tilde{g}_c between the analytical approximations and the numerically computed values, over the range of $\gamma \in [0, 0.3]$.

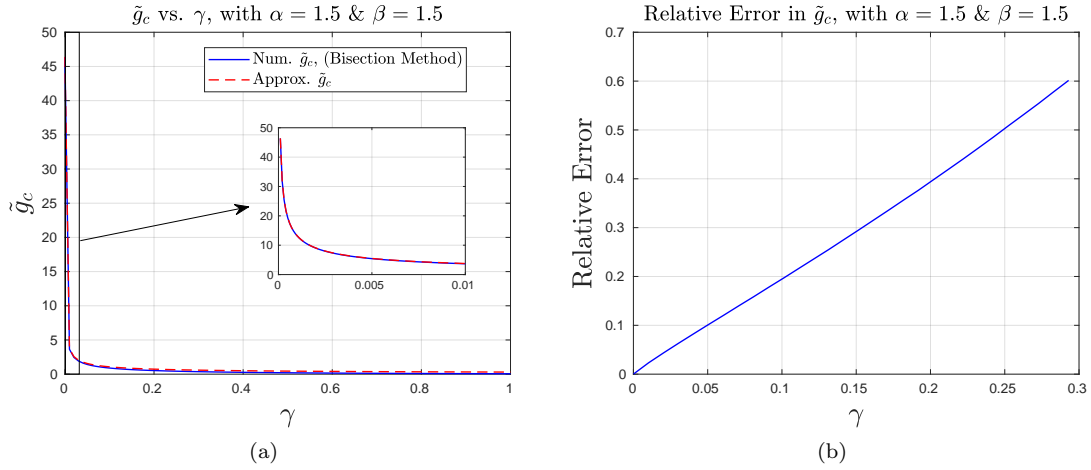


Figure 10: Plots comparing the \tilde{g}_c approximation with numerically computed \tilde{g}_c value using the Bisection Method, given $\alpha = \beta = 3/2$ (*Kuwabara-Kono model*). (a) \tilde{g}_c vs. γ , where the plot labeled *Num. \tilde{g}_c (Bisection Method)* refer to the numerical values and the plot labeled *Approx. \tilde{g}_c* refer to the approximate analytical value obtained using (24), (b) Relative error of the \tilde{g}_c between the analytical approximations and the numerically computed values, over the range of $\gamma \in [0, 0.3]$.

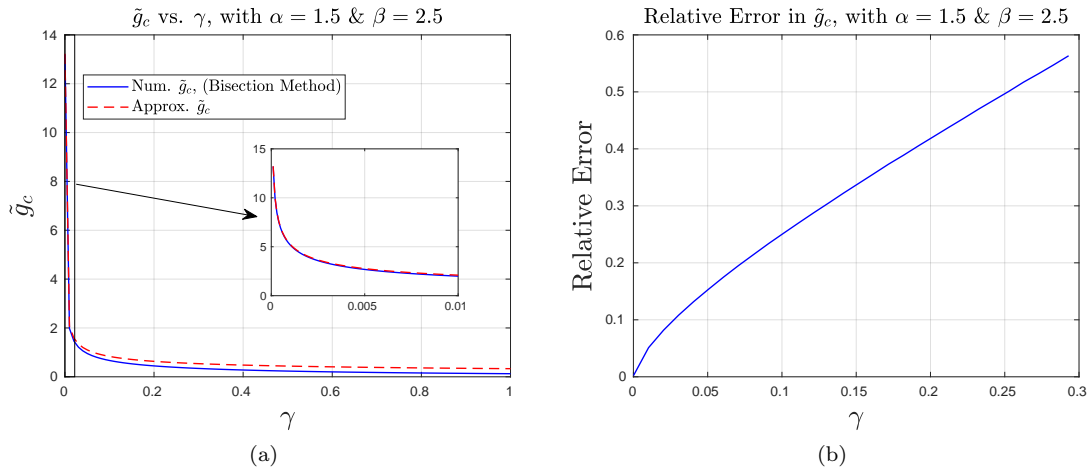


Figure 11: Plots comparing the \tilde{g}_c approximation with numerically computed \tilde{g}_c value using the Bisection Method, given $\alpha = 3/2$ & $\beta = 5/2$ (*Hunt-Crossley model*). (a) \tilde{g}_c vs. γ , where the plot labeled *Num. \tilde{g}_c (Bisection Method)* refer to the numerical values and the plot labeled *Approx. \tilde{g}_c* refer to the approximate analytical value obtained using (24), (b) Relative error of the \tilde{g}_c between the analytical approximations and the numerically computed values, over the range of $\gamma \in [0, 0.3]$.

253 The condition (24) can work for any value of α and β . Figures 8, 9 10, and 11 show four different
254 cases, with constant $\alpha = 3/2$, but β value is changed in each figure. Figure 8 shows the dependence of γ
255 on \tilde{g}_c , when $\alpha = 3/2$ and $\beta = 1$ (which correspond to the visco-elastic model defined by *Hertz stiffness*
256 *& linear damping*). Figure 8(a) shows the value of \tilde{g}_c as a function of $\gamma \in (0, 1]$. It also includes the
257 zoomed-in plots of \tilde{g}_c with $\gamma \in (0, 0.01]$. The plots labeled *Num. \tilde{g}_c (Bisection Method)* are the results
258 obtained using a numerical scheme, whereas the plots with the label *Approx. \tilde{g}_c* correspond to the results
259 obtained using the \tilde{g}_c approximation given in (24). The numerical results, corresponding to the label *Num.*
260 *\tilde{g}_c (Bisection Method)*, is obtained by using the Bisection method to calculate the value of \tilde{g}_c such that
261 $e = 0$, for a given value of $\gamma \in (0, 1]$. The values of e needed for the bisection method implementation is
262 computed via numerical integration of (4), to yield numerically accurate values for \tilde{g}_c . We can see from
263 Fig. 8 that analytical approximation of \tilde{g}_c closely matches the numerically computed values of \tilde{g}_c . Fig.8(b)
264 shows the relative errors of the approximated \tilde{g}_c with respect to the numerically computed values. However
265 the range of γ values in Fig. 8(b) has been reduced to $\gamma \in [0, 0.3]$, since the relative errors for very high
266 γ values looses meaning as $\tilde{g}_c \rightarrow 0$ for high γ values. We see that the relative errors remain small in this
267 range. While the relative error appears to be increasing with γ , that is expected since \tilde{g}_c gets smaller as γ is
268 increased. Similar, comparisons can be carried out for other (α, β) value pairs. Figure 9, Fig. 10, and Fig. 11
269 present the comparisons of \tilde{g}_c for *Tsuji-Tanaka-Ishida model* ($\alpha = 3/2$ & $\beta = 5/4$), *Kuwabara-Kono model*
270 ($\alpha = \beta = 3/2$), and *Simon-Hunt-Crossley model* ($\alpha = 3/2$ & $\beta = 5/2$), respectively. In all these comparisons
271 it is evident that the analytical approximation of \tilde{g}_c in terms of γ given in (24), works well for all values of
272 α and β . This also shows that the analytical approximation of e for very large values of \tilde{g} given in (23) is
273 also correct. However, we know that for very large \tilde{g} , the value of $e \rightarrow 0$, so the approximation (23) doesn't
274 serve much purpose. It would be more interesting to have an approximation for intermediate values of \tilde{g} .
275 Recall that all approximations presented till this this point, are linearly dependent to γ . In order to obtain a
276 valid approximation in the intermediate range of \tilde{g} value, it would be useful to include higher order γ terms,
277 into the formulation for CoR. In the following section, we derive another approximation of e that includes a
278 $\mathcal{O}(\gamma^2)$ term.

279 4.3 Inclusion of an $\mathcal{O}(\gamma^2)$ term

280 In order to make the CoR approximation valid over larger ranges of γ and \tilde{g} , we need to include higher-order
281 terms in the expansion. This section deals with the inclusion of an $\mathcal{O}(\gamma^2)$ term into the CoR approximation.
282 A CoR expansion with an $\mathcal{O}(\gamma^2)$ term will have the following general form,

$$e = 1 - \gamma C_0 - \gamma \tilde{g} C_1 - \gamma^2 C_2 + \text{H.O.T.} \quad (25)$$

283 where C_0 and C_1 are known and identical to the coefficients obtained in(22). The coefficient C_2 is an
284 unknown that would contribute towards the $\mathcal{O}(\gamma^2)$ term. Here we will obtain the unknown coefficient C_2 ,
285 by first using a known expansion for the Kuwabara - Kono model ($\alpha = \beta = 3/2$), with no external force
286 ($\tilde{g} = 0$), presented by Schwager and Pöschel [31]. We will study whether the simple inclusion of the C_2
287 term based on the Schwager-Pöschel expansion coefficients can predict the CoR with adequate accuracy for
288 arbitrary values of \tilde{g} . Later, we will also derive a more general $\mathcal{O}(\gamma^2)$ expansion of e in terms of γ , without
289 linearizing it with respect to \tilde{g} , for arbitrary values of α and β .

290 4.3.1 Using Schwager-Pöschel CoR approximation

291 The CoR formulation presented by Schwager and Pöschel in [31] is a power-series solution of e , consisting
292 of arbitrary order powers of γ , assuming no external force ($\tilde{g} = 0$). However, we are only interested in the
293 expansion with terms up to $\mathcal{O}(\gamma^2)$, which as per [31] is given by,

$$e = 1 + c_1 \bar{\beta} v_0^{\frac{1}{5}} + c_2 \bar{\beta}^2 v_0^{\frac{2}{5}} + \text{H.O.T.} \quad (26)$$

294 where v_0 is the initial pre-impact velocity, and c_i are numerical coefficients derived from the solution of the
295 displacement trajectory. $\bar{\beta} = \bar{\gamma} \bar{k}^{-\frac{3}{5}}$, where $\bar{k} = \frac{k}{m}$ and $\bar{\gamma} = \frac{3}{2} \frac{k \gamma_0}{m}$. Thus, $\bar{\beta}$ is given by,

$$\bar{\beta} = \frac{3}{2} \gamma_0 \left(\frac{k}{m} \right)^{\frac{2}{5}} \quad (27)$$

296 Next, considering $\gamma = \gamma_0 v_0^{\frac{2\beta}{\alpha+1}-1} \frac{k}{m} 1 - \frac{\beta}{\alpha+1}$, which when substituted with $\alpha = \beta = \frac{3}{2}$ becomes $\gamma = \gamma_0 v_0^{\frac{1}{5}} \left(\frac{k}{m}\right)^{\frac{2}{5}}$.
 297 Thus, the expression for $\bar{\beta}$ can be rewritten as,

$$\bar{\beta} = \frac{3}{2} \gamma v_0^{-\frac{1}{5}} \quad (28)$$

298 Substituting this into (26) we get,

$$e = 1 + \frac{3}{2} c_1 \gamma + \frac{9}{4} c_2 \gamma^2 + \text{H.O.T.} \quad (29)$$

299 The authors of [31], refer to (26) or equivalently (29) as the naive formulation, since the impact termination
 300 criteria used to derive this expression is based on displacement. They also present an alternative formulation
 301 for e that is determined by evaluating when the contact force vanishes. However, since we do not use a
 302 contact force based criteria in this work, we are using the so-called naive CoR formulation from [31] to
 303 obtain the value of C_2 .

304 Comparing (29) with (25), we notice that the unknown coefficient C_2 in (25) can be expressed as,

$$C_2 = -\frac{9}{4} c_2 \quad (30)$$

305 The value of c_2 , as presented in [31] is $c_2 = 0.7982665553$. Thus, the numerical value of C_2 is $C_2 =$
 306 -1.7960997494 . Now, having estimated the value of C_2 in the expansion (25) with the help of the Schwager-
 307 Pöschel coefficient (corresponding to $\tilde{g} = 0$) [31], we can begin to study the effects of adding this $\mathcal{O}(\gamma^2)$
 308 term. Figures 12(a), 12(b), 12(c), and 12(d) show how e varies with respect of γ , given fixed values of $\tilde{g} = 0$,
 309 $\tilde{g} = 0.05$, $\tilde{g} = 1$, and $\tilde{g} = 10$, respectively. The plots in Fig. 12 are presented over the range $\gamma \in [0, 0.1]$
 310 and include close-up zoom near $\gamma \approx 0$. The corresponding absolute errors of $\mathcal{O}(\gamma)$ and $\mathcal{O}(\gamma^2)$ expansions
 311 of e , with respect to the numerical computation, are shown in Fig. 13. In these figures, *Num. Integ.* refers
 312 to the CoR results obtained via direct numerical integration of (4), *Small \tilde{g} Approx.* refers to the results
 313 obtained using the first-order approximation in (22), and *SP Coeff.* refers to the results computed using the
 314 $\mathcal{O}(\gamma^2)$ approximation given in (25) with the C_2 coefficient defined in terms of (30), which is equivalent to
 315 the coefficient found in [31] corresponding to $\tilde{g} = 0$. We observe from these results that the inclusion of the
 316 Schwager-Pöschel-based $\mathcal{O}(\gamma^2)$ term improves the accuracy of e for small values of \tilde{g} , as can be seen from
 317 Fig. 12(a) and Fig. 12(b), and their corresponding absolute errors in Fig. 13(a) and Fig. 13(b). However,
 318 as the magnitude of \tilde{g} is increased in Fig. 12(c) and Fig. 12(d), and their corresponding absolute error plots
 319 in Fig. 13(c) and Fig. 13(d), we see that the Schwager-Pöschel based $\mathcal{O}(\gamma^2)$ approximation fails for large
 320 γ . The coefficient C_2 obtained in (30), uses the coefficient value c_2 from [31]. Since [31] ignores external
 321 load and gravitational acceleration, the coefficient value of c_2 doesn't hold for higher value of \tilde{g} . Thus, we
 322 can conclude that the coefficients in the $\mathcal{O}(\gamma^2)$ expansion in (25) can not be independent of \tilde{g} , if it has to
 323 remain valid for cases with high external load, or \tilde{g} . Hence, in the following section we derive an alternate
 324 $\mathcal{O}(\gamma^2)$ expansion of e , with coefficients dependent on \tilde{g} , such that the CoR approximation can remain valid
 325 for arbitrary \tilde{g} .

326 4.3.2 Using the solution for $\frac{\partial u}{\partial \gamma}$

327 The $\mathcal{O}(\gamma^2)$ term obtained from the Schwager-Pöschel expansion in the preceding section, is only valid for
 328 the parameter values $\alpha = \beta = 3/2$. Moreover, we noticed from Figs. 12 and 13, that the inclusion of C_2
 329 terms does not significantly improve the results for high values of γ and \tilde{g} . Therefore, now we will consider
 330 an alternative approach for deriving an $\mathcal{O}(\gamma^2)$ series solution for e , that is valid for arbitrary values of $\alpha \geq 1$,
 331 $\beta \geq 1$ and $\tilde{g} \geq 0$. This approach is based on an expansion with respect to γ , that uses the solution for
 332 the variation of the displacement at zero-dissipation ($\gamma = 0$), $\frac{\partial u}{\partial \gamma}$ at $\gamma = 0$. Note that all the expansions
 333 presented up to this point were based on a small \tilde{g} approximation, therefore they did not predict well the
 334 CoR solutions when \tilde{g} was large. The expansion presented in this section avoids the first-order Taylor
 335 expansion with respect to \tilde{g} , which makes them accurate for larger values of \tilde{g} . Nevertheless, a consequence
 336 of not linearizing e with respect to \tilde{g} is that the coefficients in the resulting series solution are not simple
 337 functions, and therefore cannot be readily evaluated and would require some further approximations which
 338 are discussed in later sections.

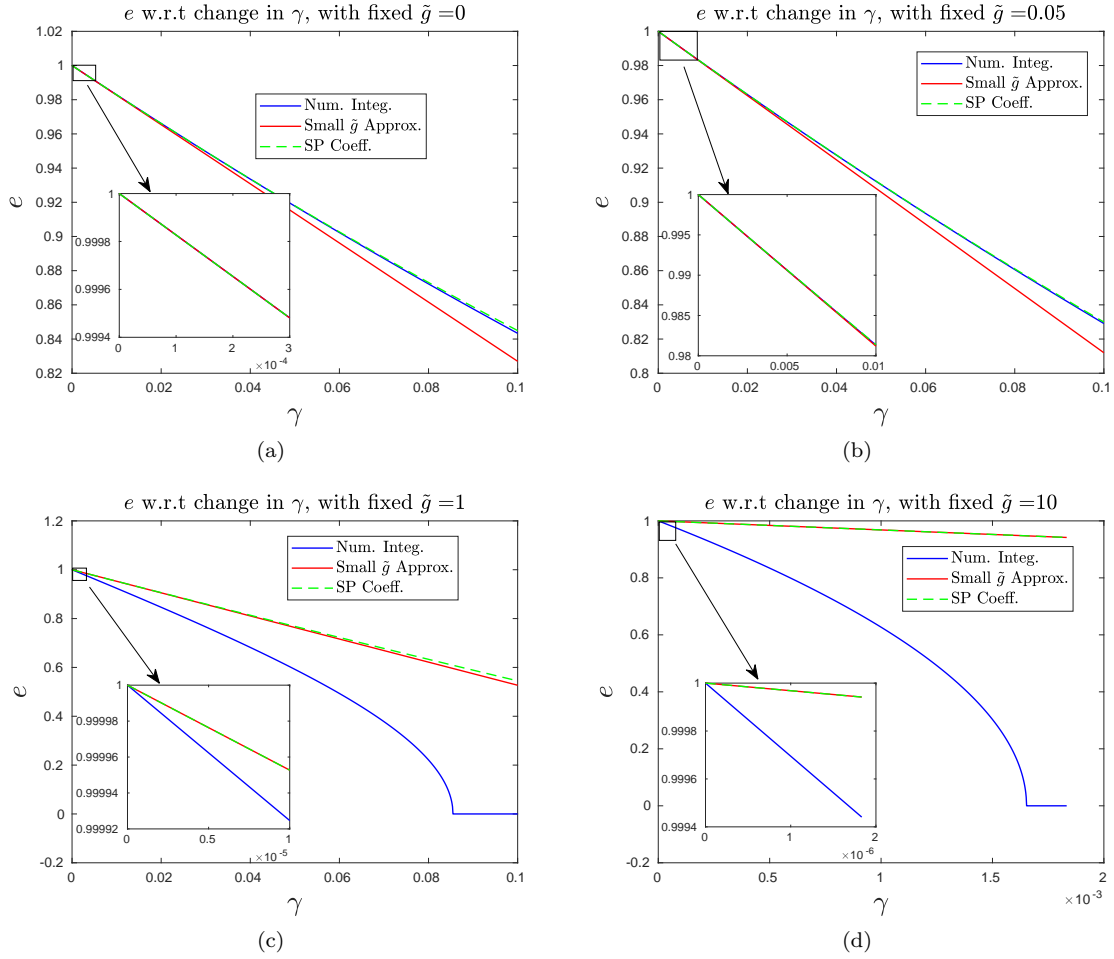


Figure 12: Comparison between the CoR values computed via numerical integration of (4) (plots labeled *Num. Integ.*), with the the $\mathcal{O}(\gamma)$ expansion in (22) (plots labeled *Small \tilde{g} Approx.*), and the Schwager-Pöschel [31] coefficient based $\mathcal{O}(\gamma^2)$ expansion in (25) & (30), (plots labeled *SP Coeff.*), for varying \tilde{g} values. (a) $\tilde{g} = 0$, (b) $\tilde{g} = 0.05$, (c) $\tilde{g} = 1$ and (d) $\tilde{g} = 10$. Zoomed-in plots included in the figure.

339 We will begin by defining u as a function of both γ and τ , of the form $u(\gamma, \tau)$ and rewriting the governing
340 differential equation in (2) as,

$$\frac{\partial^2 u}{\partial \tau^2} + \gamma \frac{\partial u}{\partial \tau} (u_+^\beta) + u_+^\alpha = \tilde{g} \quad (31)$$

$$\text{with, } u(\gamma, 0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial \tau}(\gamma, 0) = 1 \quad \text{where, } \tau \in [0, T_f]$$

341 T_f is the impact duration with $u(\gamma, T_f) = 0$ and $\frac{\partial u}{\partial \tau}(\gamma, T_f) = -e$. Now, the task at hand, is to expand the
342 CoR expression with respect to γ , so we consider the e^2 expression, given in (6), and rewrite it as,

$$e^2 = 1 - 2\gamma\beta\mathcal{I}(\gamma, \tilde{g}) \quad \text{where, } \mathcal{I}(\gamma, \tilde{g}) = \int_0^{T_f} \left(\frac{\partial u}{\partial \tau} \right)^2 u^{\beta-1} d\tau \quad (32)$$

343 The integral function $\mathcal{I}(\gamma, \tilde{g})$ can be simplified, by first performing an integration by parts and then substi-
344 tuting (31) for $\frac{\partial^2 u}{\partial \tau^2}$ (noting that $u(\gamma, 0) = u(\gamma, T_f) = 0$),

$$\mathcal{I}(\gamma, \tilde{g}) = \int_0^{T_f} \left(\frac{\partial u}{\partial \tau} \right)^2 u^{\beta-1} d\tau = -\frac{1}{\beta} \int_0^{T_f} \frac{\partial^2 u}{\partial \tau^2} u^\beta d\tau = \frac{1}{\beta} \int_0^{T_f} u^\beta (u^\alpha - \tilde{g}) d\tau \quad (33)$$

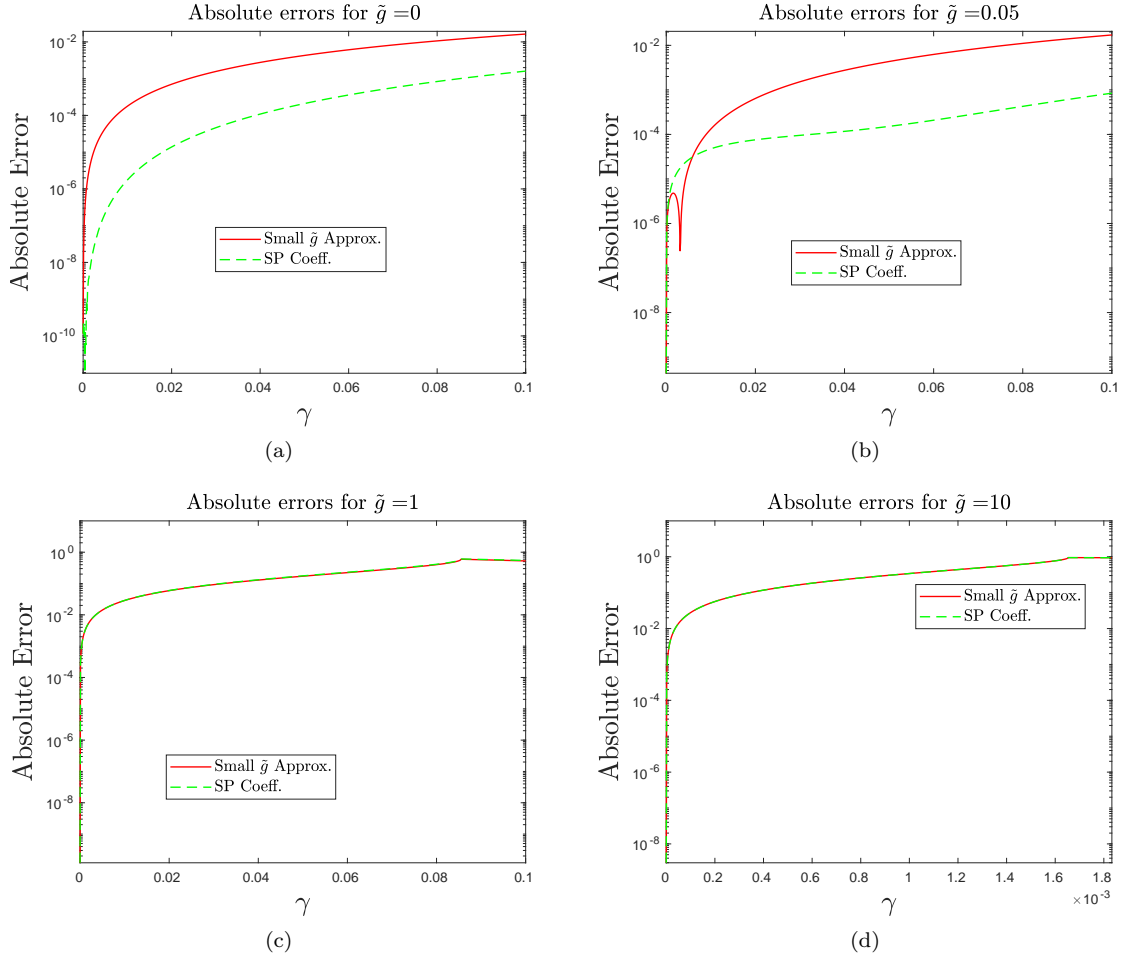


Figure 13: Absolute errors of e computed using the $\mathcal{O}(\gamma)$ expansion in (22) (plot labeled *Small \tilde{g} Approx.*) and the Schwager-Poschel [31] coefficient based $\mathcal{O}(\gamma^2)$ CoR expansion (25) & (30) (plots labeled *SP Coeff.*) with respect to the CoR values computed via direct numerical integration of (4), for varying \tilde{g} values. (a) $\tilde{g} = 0$, (b) $\tilde{g} = 0.05$, (c) $\tilde{g} = 1$ and (d) $\tilde{g} = 10$.

345 Differentiating $\mathcal{I}(\gamma, \tilde{g})$ with respect to γ using the Leibniz integral rule and then substituting $u(\gamma, T_f) = 0$
 346 leads to,

$$\frac{\partial \mathcal{I}}{\partial \gamma}(\gamma, \tilde{g}) = \frac{1}{\beta} \int_0^{T_f} \frac{\partial u}{\partial \gamma} [(\beta + \alpha)u^{\beta+\alpha-1} - \tilde{g}\beta u^{\beta-1}] d\tau \quad (34)$$

347 Thus, the expansion of (32) about $\gamma \approx 0$, can be expressed in terms of (33) and (34),

$$e^2 = 1 - 2\beta \mathcal{I}(0, \tilde{g})\gamma - 2\beta \frac{\partial \mathcal{I}}{\partial \gamma}(0, \tilde{g})\gamma^2 + \mathcal{O}(\gamma^3) \quad (35)$$

348 However, note that $\frac{\partial \mathcal{I}}{\partial \gamma}(\gamma, \tilde{g})$ depends on the variation of the displacement solution with respect to γ , i.e.,
 349 $\frac{\partial u}{\partial \gamma}$, which is unknown. Appendix B reformulates $\frac{\partial \mathcal{I}}{\partial \gamma}(\gamma, \tilde{g})$, based on a solution for $\frac{\partial u}{\partial \gamma}$ at $\gamma = 0$. This
 350 reformulation leads to an $\mathcal{O}(\gamma^2)$ expression for the CoR e , as defined below in Proposition 2.

351 **Proposition 2.** *The $\mathcal{O}(\gamma^2)$ order series solution for the coefficient of restitution, e is given by,*

$$e \approx \min(e_+, e_s), \quad (36)$$

352 where, $e_+^2 = \max(e^2, 0)$, and e^2 is given by the expansion,

$$e^2 = 1 - 2\beta\mathcal{I}(0, \tilde{g})\gamma + 2\beta^2\mathcal{I}(0, \tilde{g})\mathcal{Q}(0, \tilde{g})\gamma^2 \quad (37)$$

353 In (36), e_s is a linearization of e^2 that is valid for small \tilde{g} , and is given by,

$$e_s = \max\left(1 - \beta\mathcal{I}(0, \tilde{g})\gamma + \beta^2\mathcal{I}(0, \tilde{g})\left[\mathcal{Q}(0, \tilde{g}) - \frac{1}{2}\mathcal{I}(0, \tilde{g})\right]\gamma^2, 0\right). \quad (38)$$

The functions $\mathcal{I}(0, \tilde{g})$ and $\mathcal{Q}(0, \tilde{g})$ are integrals given by,

$$\mathcal{I}(0, \tilde{g}) = 2 \int_0^{u_M} \left[1 + 2\left(\tilde{g}u_0 - \frac{u_0^{\alpha+1}}{\alpha+1}\right)\right]^{\frac{1}{2}} u_0^{\beta-1} du_0$$

and,

$$\mathcal{Q}(0, \tilde{g}) = \int_0^{u_M} \left[1 + 2\left(\tilde{g}u_0 - \frac{u_0^{\alpha+1}}{\alpha+1}\right)\right]^{-\frac{1}{2}} u_0^{\beta-1} du_0.$$

354 *Proof.* See Appendix B. □

355 We can check the validity of Proposition 2 for the case $\tilde{g} = 0$, by comparing the coefficient values with
 356 the ones obtained using Schwager and Pöschel's results [31]. The integrals $\mathcal{I}(0, \tilde{g})$ can be evaluated at $\tilde{g} = 0$.
 357 Therefore, we may use the value of maximum compression as, $u_M = u_M^* = \left(\frac{\alpha+1}{2}\right)^{1/(\alpha+1)}$, and then use
 358 Definition 1 to yield,

$$\begin{aligned} \mathcal{I}(0, 0) &= 2 \int_0^{\left(\frac{\alpha+1}{2}\right)^{1/(\alpha+1)}} \left(1 - 2\frac{u_0^{\alpha+1}}{\alpha+1}\right) u_0^{\beta-1} du_0 \\ &= \left(\frac{\alpha+1}{2}\right)^{\frac{\beta}{\alpha+1}-1} \frac{\alpha+1}{\alpha+1+2\beta} B\left(\frac{1}{2}, \frac{\beta}{\alpha+1}\right) \\ &= \left(\frac{\alpha+1}{2}\right)^{\frac{\beta}{\alpha+1}-1} \frac{\alpha+1}{\alpha+1+2\beta} \sqrt{\pi} \frac{\Gamma\left(\frac{\beta}{\alpha+1}\right)}{\Gamma\left(\frac{1}{2} + \frac{\beta}{\alpha+1}\right)} \end{aligned} \quad (39)$$

359 Similarly, for the integral in $\mathcal{Q}(0, \tilde{g})$ at $\tilde{g} = 0$, we can use Definition 1, to compute $\mathcal{Q}(0, 0)$, in terms of the
 360 beta function,

$$\begin{aligned} \int_0^{T_0/2} u_0^{\beta-1} d\tau &= \int_0^{\left(\frac{\alpha+1}{2}\right)^{1/(\alpha+1)}} u_0^{\beta-1} \left(1 - 2\frac{u_0^{\alpha+1}}{\alpha+1}\right)^{-\frac{1}{2}} du_0 \\ &= \left(\frac{\alpha+1}{2}\right)^{\frac{\beta}{\alpha+1}-1} \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{\beta}{\alpha+1}\right)}{\Gamma\left(\frac{1}{2} + \frac{\beta}{\alpha+1}\right)} \end{aligned} \quad (40)$$

As defined in Proposition 2, for small \tilde{g} , $e = e_s$ which is given in (38). Hence, if we substitute (39) and (40) into (38), we obtain the CoR expression, given by,

$$e|_{\tilde{g}=0} = 1 - \gamma C_0 - \gamma^2 C_2 + \mathcal{O}(\gamma^3)$$

with,

$$\begin{aligned} C_0 &= \beta \left(\frac{\alpha+1}{2}\right)^{\frac{\beta}{\alpha+1}-1} \frac{\alpha+1}{\alpha+1+2\beta} \sqrt{\pi} \frac{\Gamma\left(\frac{\beta}{\alpha+1}\right)}{\Gamma\left(\frac{1}{2} + \frac{\beta}{\alpha+1}\right)} \\ C_2 &= -\frac{\beta^3(\alpha+1)^{\frac{2\beta}{\alpha+1}-1}}{2^{\frac{2\beta}{\alpha+1}-2}(\alpha+2\beta+1)^2} \left[B\left(\frac{\beta}{\alpha+1}, \frac{1}{2}\right)\right]^2. \end{aligned}$$

361 Evaluating C_0 and C_2 for the values $\alpha = \beta = \frac{3}{2}$, we obtain the values $C_0 = 1.7302$ and $C_2 = -1.7961$.
362 Comparing with (29), we know that $C_0 = -\frac{3}{2}c_1$ and $C_2 = -\frac{9}{4}c_2$, where c_1 and c_2 are the coefficients presented
363 in [31]. The values of these coefficients reported in [31] are $c_1 = -1.15448854$ and $c_2 = 0.7982665553$. So, we
364 can check $C_0 = -\frac{3}{2}(-1.15448854) = 1.7302$ and $C_2 = -\frac{9}{4}(0.7982665553) = -1.7961$ (which are identical
365 to the values obtained in Sec. 4.1 and Sec. 4.3.1) to conclude that the e expansion obtained above in (38)
366 agrees with the $\mathcal{O}(\gamma^2)$ expansion presented in [31], when $\tilde{g} = 0$. Additionally, one can also check the validity
367 of this coefficients when $\tilde{g} \neq 0$, by numerically computing the values of the integrals $\mathcal{I}(0, \tilde{g})$ and $\mathcal{Q}(0, \tilde{g})$ and
368 then use them to compute the e with the two approximations provided in (36), to see which approximation
369 works better in comparison to the CoR computed via direct numerical integration of (4). Appendix B
370 presents these comparisons, from which one can conclude that both approximations in Proposition 2 hold in
371 their respective range of validity which depends on the values of γ and \tilde{g} .

372 4.4 Approximate CoR for arbitrary \tilde{g}

From the previous sections it follows that we can obtain analytical CoRs corresponding to both small and large values of \tilde{g} . In (24) we obtained an expression for \tilde{g}_c , which is the large value of \tilde{g} when $e = 0$, or in other words the bead doesn't rebound. Also, we have previously noted that the inclusion of an $\mathcal{O}(\gamma^2)$ term improves the accuracy of analytical CoR. The CoR approximations in Proposition 2 that include $\mathcal{O}(\gamma^2)$ term consist of integral functions dependent on \tilde{g} . While we have good approximations of these integral functions when $\tilde{g} \approx 0$ and $\tilde{g} \approx \tilde{g}_c$, a valid approximation for the entire range of \tilde{g} remains to be computed. So, here we will consider (37), which can be rewritten (with the notation $\mathcal{I}(0, \tilde{g}) = \mathcal{I}(\tilde{g})$) as,

$$e^2 = 1 - 2\beta\mathcal{I}(\tilde{g})\gamma + 2\beta^2\mathcal{I}(\tilde{g})\mathcal{Q}(\tilde{g})\gamma^2 + \mathcal{O}(\gamma^3)$$

373 where,

$$\mathcal{I}(\tilde{g}) = 2 \int_0^{u_M} \left[1 + 2 \left(\tilde{g} - \frac{u_0^{\alpha+1}}{\alpha+1} \right) \right]^{\frac{1}{2}} u_0^{\beta-1} du_0 \quad (41)$$

374 and,

$$\mathcal{Q}(\tilde{g}) = \int_0^{T_0/2} u_0^{\beta-1} d\tau = \int_0^{u_M} \left[1 + 2 \left(\tilde{g} - \frac{u_0^{\alpha+1}}{\alpha+1} \right) \right]^{-\frac{1}{2}} u_0^{\beta-1} du_0 \quad (42)$$

375 The integral functions $\mathcal{I}(\tilde{g})$ and $\mathcal{Q}(\tilde{g})$ can not be explicitly evaluated for arbitrary values of \tilde{g} . However, we
376 can notice that the integrands of the two integral functions is closely linked to the integrand of Euler's Beta
377 function, such that we can evaluate these integrals exactly in terms of Euler's Beta function when $\tilde{g} = 0$.
378 Hence, to obtain approximations of $\mathcal{I}(\tilde{g})$ and $\mathcal{Q}(\tilde{g})$, we will rely on finding an equivalent approximation for
379 the integrands, such that the integrals could be evaluated in terms of Euler's Beta function for all values of
380 \tilde{g} .

381 4.4.1 Approximate calculation of $\mathcal{I}(\tilde{g})$ and $\mathcal{Q}(\tilde{g})$

382 Before approximating $\mathcal{I}(\tilde{g})$ and $\mathcal{Q}(\tilde{g})$, we will consider a change of variable. Since we wish to obtain this
383 interpolation for arbitrarily large $\tilde{g} \in [0, \infty]$, we will redefine it in terms of a new variable $\theta \in [0, 1]$, which is
384 given by,

$$\frac{u_M^{\alpha+1}}{\alpha+1} = \frac{1}{2\theta} \quad \text{such that,} \quad \tilde{g}(\theta) = \left(\frac{\alpha+1}{2\theta} \right)^{\frac{-1}{\alpha+1}} \left(\frac{1}{2\theta} - \frac{1}{2} \right) \quad (43)$$

385 Thus, \tilde{g} in terms of θ correspond to,

$$\theta = 1 \implies \tilde{g} = 0 \quad \text{and,} \quad \theta \rightarrow 0 \implies \tilde{g} \rightarrow \infty \quad (44)$$

386 Finally, with the substitution $u_0 = u_M x$, we may write the integrals as,

$$\mathcal{I}(\tilde{g}(\theta)) = \frac{2}{\sqrt{\theta}} \left(\frac{\alpha+1}{2\theta} \right)^{\frac{\beta}{\alpha+1}} \int_0^1 [\theta + (1-\theta)x - x^{\alpha+1}]^{\frac{1}{2}} x^{\beta-1} dx = K(\theta) J_{\alpha, \beta}(\theta) \quad (45)$$

387 and,

$$Q(\tilde{g}(\theta)) = \sqrt{\theta} \left(\frac{\alpha+1}{2\theta} \right)^{\frac{\beta}{\alpha+1}} \int_0^1 [\theta + (1-\theta)x - x^{\alpha+1}]^{-\frac{1}{2}} x^{\beta-1} dx = L(\theta) M_{\alpha,\beta}(\theta) \quad (46)$$

388 where,

$$\begin{aligned} K(\theta) &= \frac{2}{\sqrt{\theta}} \left(\frac{\alpha+1}{2\theta} \right)^{\frac{\beta}{\alpha+1}} & J_{\alpha,\beta}(\theta) &= \int_0^1 [\theta + (1-\theta)x - x^{\alpha+1}]^{\frac{1}{2}} x^{\beta-1} dx \\ L(\theta) &= \sqrt{\theta} \left(\frac{\alpha+1}{2\theta} \right)^{\frac{\beta}{\alpha+1}} & M_{\alpha,\beta}(\theta) &= \int_0^1 [\theta + (1-\theta)x - x^{\alpha+1}]^{-\frac{1}{2}} x^{\beta-1} dx \end{aligned} \quad (47)$$

389 The functions $K(\theta)$ and $L(\theta)$ can be computed directly given the values of α , β and θ . Whereas, the functions
390 $J_{\alpha,\beta}(\theta)$ and $M_{\alpha,\beta}(\theta)$ are integrals that can be evaluated in terms of the Euler's Beta function when $\theta = 0$
391 and $\theta = 1$, but do not have a closed-form solution when $\theta \in (0, 1)$. Hence, we seek an approximation for
392 the integrals $J_{\alpha,\beta}(\theta)$ and $M_{\alpha,\beta}(\theta)$ for the all values of $\theta \in [0, 1]$. In order to proceed further with such an
393 approximation, we first note that the integrals $J_{\alpha,\beta}(\theta)$ and $M_{\alpha,\beta}(\theta)$ can be related, due the similarity in
394 the structure of their integrands. The relationship between the two integral functions is defined below in
395 Proposition 3, the proof for which can be found in Appendix C.

396 **Proposition 3.** *The integrals $J_{\alpha,\beta}(\theta)$ and $M_{\alpha,\beta}(\theta)$ are related according to the following relationship,*

$$J_{\alpha,\beta}(\theta) = \frac{\alpha+1}{2\beta} M_{\alpha,\alpha+\beta+1}(\theta) + \frac{\theta-1}{2\beta} M_{\alpha,\beta+1}(\theta) \quad (48)$$

397 *Proof.* See Appendix C □

398 Therefore, now we only need to find an approximation for $M_{\alpha,\beta}(\theta)$, and then use the approximating
399 function to compute $J_{\alpha,\beta}(\theta)$ using (48).

400 In order to compute $M_{\alpha,\beta}(\theta)$ for all values of $\theta \in [0, 1]$, we follow the approach of approximating the
401 integrand of $M_{\alpha,\beta}(\theta)$, such that the integral $M_{\alpha,\beta}(\theta)$ could be represented as a summation of Beta functions
402 with coefficients. The integrand is approximated with the help of Chebyshev polynomials of arbitrary order
403 n , which leads to some recurrence relations that are used for the approximation of $M_{\alpha,\beta}(\theta)$. The details of
404 this development can be found in Appendix C, and here we present the final result of this approximation in
405 Proposition 4.

406 **Proposition 4.** *The n -th (n being an odd number) order approximation of $M_{\alpha,\beta}(\theta)$ is given by,*

$$M_{\alpha,\beta}(\theta) \simeq \sum_{j=0}^{n-1} c_j a_{j,0}, \quad \text{with, } j \text{ even} \quad (49)$$

407 where coefficients $a_{j,k}$ and c_j are defined below. The coefficients c_j can be computed as,

$$\begin{cases} c_0 &= \frac{2}{n+1} \sum_{k=(n+1)/2}^n h_{\alpha,\theta}(x_k) \\ c_j &= \frac{4}{n+1} \sum_{k=(n+1)/2}^n h_{\alpha,\theta}(x_k) \cos\left(j\pi \frac{2k+1}{2n+2}\right), & \text{if } j \geq 2 \text{ and is even} \\ c_j &= 0, & \text{if } j \geq 2 \text{ and is odd} \end{cases} \quad (50)$$

408 where $h_{\alpha,\theta}(x)$ is a function related to the approximated and the actual integrand of $M_{\alpha,\beta}(\theta)$, and is given by,

$$h_{\alpha,\theta}(x) = \left(\frac{x^{1-\theta} - x^{\alpha+1}}{\theta + (1-\theta)x - x^{\alpha+1}} \right)^{\frac{1}{2}} \quad (51)$$

410 x_k are Chebyshev nodes in the range $[0, 2]$ and are given by,

$$x_k = \cos\left(\pi \frac{2k+1}{2n+2}\right) + 1, \quad k \in \mathbb{N} \quad (52)$$

j \ k	0	1	2	3	4	...	n-1
0	$a_{0,0}$	$a_{0,1}$	$a_{0,2}$	$a_{0,3}$	$a_{0,4}$	\dots	$a_{0,n-1}$
1	$a_{1,0}$	$a_{1,1}$	$a_{1,2}$	$a_{1,3}$	\dots	$a_{1,n-2}$	
2	$a_{2,0}$	$a_{2,1}$	$a_{2,2}$	\dots	$a_{2,n-3}$		
3	$a_{3,0}$	$a_{3,1}$	\dots	$a_{3,n-4}$			
4	$a_{4,0}$	\dots	$a_{4,n-4}$				
\vdots	\vdots	\dots					
n-1	$a_{n-1,0}$						

Table 1: Tabular computation of $a_{i,j}$, based on the recurrence relations in (53). The values $a_{j,0}$ in column $k = 0$ (highlighted in gray), are required for the final computation of $M_{\alpha,\beta}(\theta)$.

411 Due to the relationship of the coefficients $a_{j,k}$ to Chebyshev polynomials, they can be calculated using recur-
412 rence relations given by,

$$a_{j+2,k} = 2a_{j+1,k+1} - 2a_{j+1,k} - a_k \quad \text{for} \quad j \geq 0 \quad (53)$$

$$a_{0,k} = \frac{1}{\alpha+\theta} B\left(\frac{\beta+k-1/2+\theta/2}{\alpha+\theta}, \frac{1}{2}\right) \quad \text{and} \quad a_{1,k} = a_{0,k+1} - a_{0,k}$$

413 *Proof.* See Appendix C. □

414 As seen in (49), the evaluation of the approximation $M_{\alpha,\beta}(\theta)$ for a given order n (n being an odd number),
415 requires computation of the coefficients $a_{j,k}$, using the recurrence relations in (53). These calculations may
416 be summarized via the tabular computation scheme demonstrated in Table 1.

417 Note that the order n for the approximation of $M_{\alpha,\beta}(\theta)$ in (49) needs to be carefully selected. While a
418 lower order of n corresponds to large overall errors, very high order of n leads to lower aggregate errors, but
419 high local errors due to oscillations in the response of $M_{\alpha,\beta}(\theta)$ with respect to θ . It was found that an order
420 $n = 21$ works best for the given approximations (See Appendix C for more details). Thus the numerical
421 results presented henceforth were computed using the order $n = 21$.

4.4.2 Numerical Results

423 In this section we will primarily study the validity of the $\mathcal{O}(\gamma^2)$ approximation e , as given in Proposition 2,
424 against numerical computation of CoR, based on (4). The main goal is to study how the parameter values
425 γ and \tilde{g} affect the prediction of e using the $\mathcal{O}(\gamma^2)$ approximation. Since, the approximations can work with
426 arbitrary values of $\alpha \geq 1$ and $\beta \geq 1$, here we will study the validity of the $\mathcal{O}(\gamma^2)$ CoR approximation in the
427 context of four different visco-elastic models: 1) *Hertz stiffness with linear damping*, 2) *Tsuji-Tanaka-Ishida*
428 *model*, 3) *Kuwabara-Kono model*, and 4) *Simon-Hunt-Crossley model*. The four figures, Fig. 14, Fig. 16,
429 Fig. 18, and, Fig. 20, correspond to four visco-elastic models considered in this study, namely *Hertz stiffness*
430 *with linear damping* ($\alpha = 3/2$ & $\beta = 1$), *Tsuji-Tanaka-Ishida model* ($\alpha = 3/2$ & $\beta = 5/4$), *Kuwabara-*
431 *Kono model* ($\alpha = \beta = 3/2$), and *Simon-Hunt-Crossley model* ($\alpha = 3/2$ & $\beta = 5/2$), respectively. Each
432 of these figures consists of four sub-figures, corresponding to the values $\tilde{g} = 0, 0.05, 1$, and 10, that show
433 the dependence of e with respect to γ . The plots labeled *Numer. Integ.* show the results of e that are
434 computed via numerical integration of (4). These numerically integrated results form the reference against
435 which other approximations are compared. The figures corresponding to the value of $\tilde{g} = 0$ contain plots

436 labeled *Exact Coeffs.* which show the results of e computed using the $\mathcal{O}(\gamma^2)$ approximation using the exact
 437 values of $\mathcal{I}(\tilde{g})$ and $\mathcal{Q}(\tilde{g})$ using (39) and (40). In all other figures, corresponding to values of $\tilde{g} \neq 0$, the plots
 438 labeled *Num. Coeffs.* show the results of e with the $\mathcal{O}(\gamma^2)$ approximation, where the integrals $\mathcal{I}(\tilde{g})$ and
 439 $\mathcal{Q}(\tilde{g})$ are computed via numerical integration of (41) and (42). Lastly, in all figures within Fig. 14, Fig. 16,
 440 Fig. 18, and Fig. 20, there are plots labeled *Approx. Coeffs.*. These plots refer to the results of e using the
 441 $\mathcal{O}(\gamma^2)$ approximation according to (36), where the integrals are approximated as $\mathcal{I}(\tilde{g}(\theta)) = K(\theta)J_{\alpha,\beta}(\theta)$ and
 442 $\mathcal{Q}(\tilde{g}(\theta)) = L(\theta)M_{\alpha,\beta}(\theta)$, with the functions $J_{\alpha,\beta}(\theta)$ and $M_{\alpha,\beta}(\theta)$ further approximated using (48) and (49),
 443 respectively. We can see from the figures that these results closely match the results where the integrals
 444 are computed numerically. The CoR results based on Schwager-Pöschel coefficient, which we have already
 445 studied in Sec. 4.3.1, are also included in Fig. 18, corresponding to $\alpha = \beta = 3/2$ for completeness. The
 446 absolute errors associated with each approximation of e presented in Fig. 14, Fig. 16, Fig. 18, and Fig. 20,
 447 with respect to the numerically integrated CoR values are shown in Fig. 15, Fig. 17, Fig. 19, and Fig. 21.
 448 Additionally, the maximum absolute errors within certain interesting ranges of γ are also reported in Tables 2,
 449 3, 4, and 5. The absolute error values for the various approximations presented in Fig. 15, Fig. 17, Fig. 19,
 450 Fig. 21, Tab. 2, Tab. 3, Tab. 4, and Tab. 5 are all computed with respect to the CoR values computed
 451 via numerical integration of (4), which are labeled *Num. Integ.* in Fig. 14, Fig. 16, Fig. 18, and Fig. 20.
 452 However, for some specific values of the parameters α , β , and \tilde{g} , the CoR has an explicit analytical solution.
 453 The proposed $\mathcal{O}(\gamma^2)$ approximations in (36), is also studied against these known analytical CoR solutions.
 454 These results are presented in Appendix D.

455 The general conclusion that can be drawn by studying the results in Fig. 14, Fig. 15, Fig. 16, Fig. 17,
 456 Fig. 18, Fig. 19, Fig. 20, Fig. 21, Tab. 2, Tab. 3, Tab. 4, and Tab. 5 is that the $\mathcal{O}(\gamma^2)$ approximation of e
 457 using (36) gives very accurate CoR results for small and large \tilde{g} , and particularly when γ is also small. Based
 458 on Fig. 15, Fig. 17, Fig. 19, and Fig. 21, we can observe that the absolute errors corresponding to small γ
 459 are very small. The order of magnitude for these errors in the small range of γ values can be read off from
 460 the set of maximum absolute error values reported in Tables 2, 3, 4, and 5. The absolute error values
 461 increases with the γ value until attachment (no rebound, corresponding to $e = 0$) takes place. In particular,
 462 one can notice the large errors corresponding to higher values of γ when $\tilde{g} = 1$ in Fig. 14(c), Fig. 16(c),
 463 Fig. 18(c), Fig. 20(c), and the corresponding maximum absolute error values for $\gamma \geq 0.02$ in Tab. 2(a),
 464 Tab. 3(a), Tab. 4(a), and Tab. 5(a). The $\mathcal{O}(\gamma^2)$ expansion of e given in (36) relies on the expansion with
 465 respect to γ about $\gamma = 0$ (associated with the non-dissipative impacts). Therefore, the absolute error of the
 466 predicted CoR based on $\mathcal{O}(\gamma^2)$ expansion with respect to the numerically integrated e , increases with γ . Also,
 467 comparing Fig. 14(c), Fig. 16(c), Fig. 18(c), and Fig. 20, we can notice the pattern that the magnitude of the
 468 absolute error at high γ also increases with the value of β . Since, the parameter β in the equation of motion
 469 (1) refers to the power on the dissipation term, it affects the amount of dissipation that takes place during
 470 the impact, and thereby its influence is also scaled by γ , which is a coefficient to the dissipation term in (1).
 471 Hence, given that the $\mathcal{O}(\gamma^2)$ expansion of e is considered about $\gamma = 0$, the error increases with the β values.
 472 In Fig. 14(d), Fig. 14(d), Fig. 18(d), and Fig. 20(d), corresponding to $\tilde{g} = 10$, we see that the prediction of e
 473 using the $\mathcal{O}(\gamma^2)$ expansion is very accurate in the range of γ values for which attachment takes place. This
 474 is to be expected since both the numerically integrated e and the analytical $\mathcal{O}(\gamma^2)$ expansion, following the
 475 threshold $e_+^2 = \max(e^2, 0)$ in (36), becomes zero. Tables 2(b), 4(b), and 5(b), which examines this range
 476 separately, reports maximum absolute error values close to machine precision, for high γ values. While the e
 477 predictions look very accurate for all γ values in Fig. 14(d), Fig. 18(d), and Fig. 20(d), a closer examination
 478 of the maximum absolute error values in Tab. 2(b), Tab. 4(b), and Tab. 5(b) reveals that, similar to all other
 479 cases of \tilde{g} values, the errors increase with the value γ until the impact attaches or doesn't rebound. Another
 480 observation one can make based on the reported maximum absolute errors in the intermediate γ ranges
 481 (corresponding to the second columns) in Tab. 2(b), Tab. 4(b), and Tab. 5(b), is that the errors associated
 482 for the $\mathcal{O}(\gamma^2)$ expansion with the approximated integrals ($\mathcal{I}(\tilde{g}(\theta)) = K(\theta)J_{\alpha,\beta}(\theta)$ & $\mathcal{Q}(\tilde{g}(\theta)) = L(\theta)M_{\alpha,\beta}(\theta)$)
 483 are higher than the errors from the same expansion using numerically integrated integrals ($\mathcal{I}(\tilde{g})$ and $\mathcal{Q}(\tilde{g})$).
 484 This is a consequence of the approximations (based on Chebyshev nodes) of the integral $M_{\alpha,\beta}(\theta)$, which is
 485 very accurate at $\theta = 0$ and $\theta = 1$, and less accurate if $\theta \in (0, 1)$. Nevertheless, despite the aforementioned
 486 inaccuracies for high γ values, one notices that the absolute error values are generally small, while being very
 487 accurate for small γ and small \tilde{g} values. As noted in [35, 36], the γ values calculated using the experimental
 488 values of e reported in [25] associated with various materials using the *Kuwabara-Kono model* ($\alpha = \beta = 3/2$)
 489 have a maximum order of 10^{-1} . Also from the examples based on experimental results presented in Fig. 2(b),

490 5(b), 6(b), and 7(b), we can see that the value of \tilde{g} remains very small for a gravitational acceleration value
 491 of $g = 9.8 \text{ m/s}^2$, with $F = 0 \text{ N}$. Therefore, based on these physically relevant parameter values, we can
 492 conclude that the $\mathcal{O}(\gamma^2)$ approximation of e will remain very accurate for most materials within a reasonable
 γ range, and with an external load up to several orders of magnitude higher than the weight of the bead.

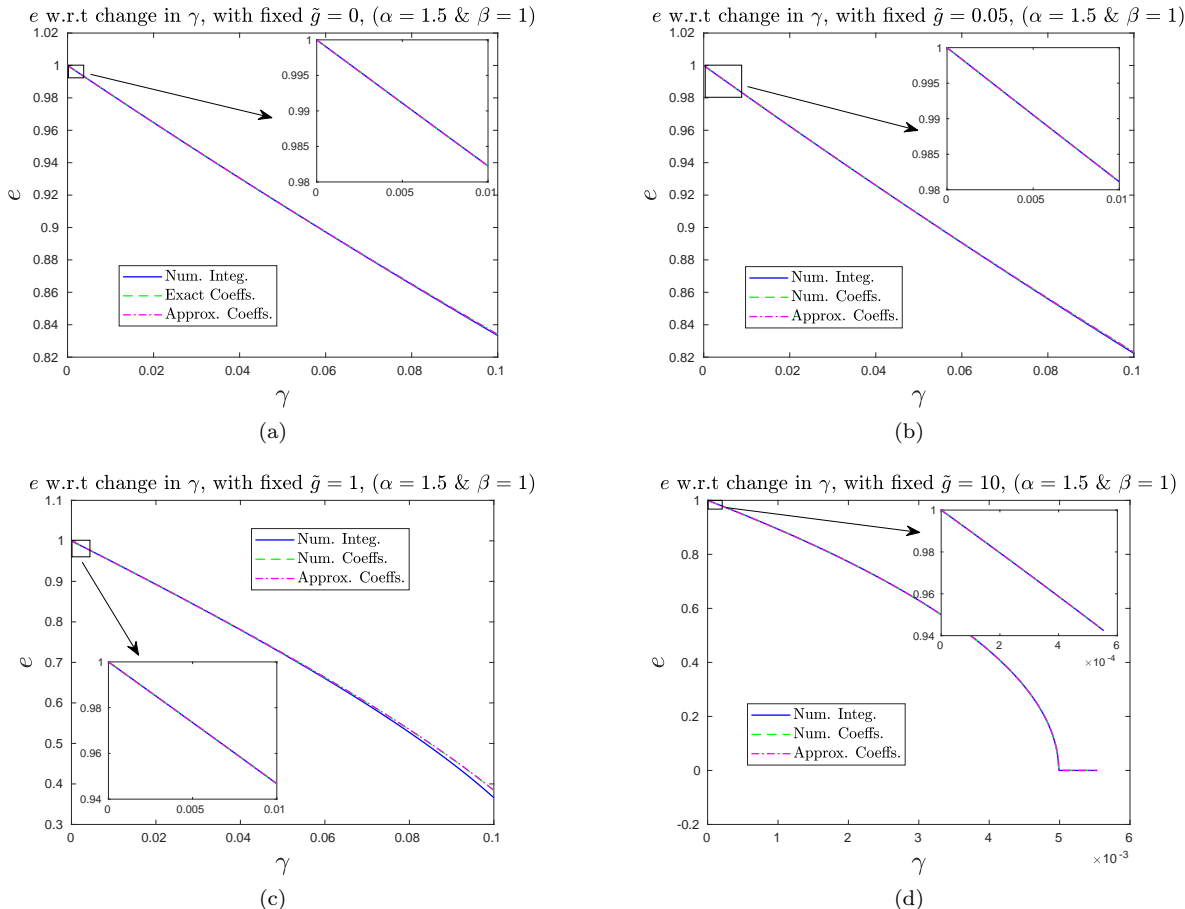


Figure 14: Comparison between the CoR values computed via numerical integration of (4), and the $\mathcal{O}(\gamma^2)$ approximations given by (36) (labeled *Num. Integ.*), using the numerically integrated functions $\mathcal{I}(\tilde{g})$ and $\mathcal{Q}(\tilde{g})$ from (41) and (42) (labeled *Num. Coeffs.*), and the approximated functions $\mathcal{I}(\tilde{g}(\theta)) = K(\theta)J_{\alpha,\beta}(\theta)$ and $\mathcal{Q}(\tilde{g}(\theta)) = L(\theta)M_{\alpha,\beta}(\theta)$ using (48) and (49) (labeled *Approx. Coeffs.*). When $\tilde{g} = 0$, the values $\mathcal{I}(0)$ and $\mathcal{Q}(0)$ are calculated using (39) and (40). Results are shown for the visco-elastic model with $\alpha = 3/2$ & $\beta = 1$ (*Hertz stiffness with linear damping*), for the cases: (a) $\tilde{g} = 0$, (b) $\tilde{g} = 0.05$, (c) $\tilde{g} = 1$, and (d) $\tilde{g} = 10$.

493
 494

495 5 Recapitulation

496 This section is dedicated to present a brief summary of the method for computing the analytical approxi-
 497 mation of the CoR for single spherical bead undergoing impact in the presence of a constant external load.
 498 The fundamental set of quantities needed for this computation is given by:

499 m : Mass of the spherical bead.

500 k : Stiffness constant used in the general nonlinear visco-elastic contact model

Types of \mathcal{I} & \mathcal{Q}	Exact[(39),(40)]/Num.[(41),(42)]		Approx.[(48),(49)]	
\tilde{g}	$\gamma < 0.02$	$\gamma \geq 0.02$	$\gamma < 0.02$	$\gamma \geq 0.02$
0	$5.39 \times 10^{-6*}$	$6.57 \times 10^{-4*}$	5.39×10^{-6}	6.57×10^{-4}
0.05	5.63×10^{-6}	6.89×10^{-4}	1.32×10^{-5}	7.26×10^{-4}
1	6.82×10^{-5}	1.77×10^{-2}	1.32×10^{-4}	1.85×10^{-2}

(a) Max. absolute errors within different γ ranges for $\tilde{g} = 0, 0.05, \text{ and } 1$

Type of \mathcal{I} & \mathcal{Q}	$\gamma < 0.0011$	$0.0011 \leq \gamma \leq 0.0051$	$\gamma > 0.0051$
Num.[(41),(42)]	4.39×10^{-7}	4.25×10^{-3}	5.63×10^{-13}
Approx.[(48),(49)]	8.03×10^{-5}	1.33×10^{-2}	5.63×10^{-13}

(b) Max. absolute errors within different γ ranges for $\tilde{g} = 10$

Table 2: Maximum Absolute Errors on the $\mathcal{O}(\gamma^2)$ expansion of e using a visco-elastic model with $\alpha = 3/2$ and $\beta = 1$ (*Hertz Stiffness with linear damping*), for $\tilde{g} = 0, 0.05, 1, \text{ and } 10$, in various γ ranges. *Errors on $\mathcal{O}(\gamma^2)$ expansions of e while using the exact values of $\mathcal{I}(\tilde{g} = 0)$ and $\mathcal{Q}(\tilde{g} = 0)$, according to (39) and (40).

Types of \mathcal{I} & \mathcal{Q}	Exact[(39),(40)]/Num.[(41),(42)]		Approx.[(48),(49)]	
\tilde{g}	$\gamma < 0.02$	$\gamma \geq 0.02$	$\gamma < 0.02$	$\gamma \geq 0.02$
0	$9.17 \times 10^{-6*}$	$1.11 \times 10^{-3*}$	9.17×10^{-6}	1.11×10^{-3}
0.05	9.88×10^{-6}	1.12×10^{-3}	1.77×10^{-5}	1.12×10^{-3}
1	1.65×10^{-4}	1.10×10^{-1}	2.40×10^{-4}	1.11×10^{-1}

(a) Max. absolute errors within different γ ranges for $\tilde{g} = 0, 0.05, \text{ and } 1$

Types of \mathcal{I} & \mathcal{Q}	$\gamma < 0.00063$	$0.00063 \leq \gamma \leq 0.0029$	$\gamma > 0.0029$
Num.[(41),(42)]	5.56×10^{-8}	3.32×10^{-4}	2.84×10^{-13}
Approx.[(48),(49)]	8.07×10^{-5}	1.76×10^{-2}	2.84×10^{-13}

(b) Max. absolute errors within different γ ranges for $\tilde{g} = 10$

Table 3: Maximum Absolute Errors on the $\mathcal{O}(\gamma^2)$ expansion of e using a visco-elastic model with $\alpha = 3/2$ and $\beta = 5/4$ (*Tsuji-Tanaka-Ishida model*), for $\tilde{g} = 0, 0.05, 1, \text{ and } 10$, in various γ ranges. *Errors on $\mathcal{O}(\gamma^2)$ expansions of e while using the exact values of $\mathcal{I}(\tilde{g} = 0)$ and $\mathcal{Q}(\tilde{g} = 0)$, according to (39) and (40).

Types of \mathcal{I} & \mathcal{Q}	Exact[(39),(40)]/Num.[(41),(42)]		Approx.[(48),(49)]	
\tilde{g}	$\gamma < 0.02$	$\gamma \geq 0.02$	$\gamma < 0.02$	$\gamma \geq 0.02$
0	$1.37 \times 10^{-5*}$	$1.60 \times 10^{-3*}$	1.37×10^{-5}	1.60×10^{-3}
0.05	1.52×10^{-5}	1.78×10^{-3}	2.33×10^{-5}	1.82×10^{-3}
1	3.67×10^{-4}	2.08×10^{-1}	4.58×10^{-4}	2.09×10^{-1}

(a) Max. absolute errors within different γ ranges for $\tilde{g} = 0, 0.05, \text{ and } 1$

Types of \mathcal{I} & \mathcal{Q}	$\gamma < 0.000367$	$0.000367 \leq \gamma \leq 0.00169$	$\gamma > 0.00169$
Num.[(41),(42)]	1.06×10^{-7}	2.38×10^{-4}	2.84×10^{-13}
Approx.[(48),(49)]	8.11×10^{-5}	2.11×10^{-2}	2.84×10^{-13}

(b) Max. absolute errors within different γ for $\tilde{g} = 10$

Table 4: Maximum Absolute Errors on the $\mathcal{O}(\gamma^2)$ expansion of e using a visco-elastic model with $\alpha = \beta = 3/2$ (*Kuwabara-Kono model*), with $\tilde{g} = 0, 0.05, 1, \text{ and } 10$, for various γ ranges. *Errors on $\mathcal{O}(\gamma^2)$ expansions of e while using the exact values of $\mathcal{I}(\tilde{g} = 0)$ and $\mathcal{Q}(\tilde{g} = 0)$, according to (39) and (40).

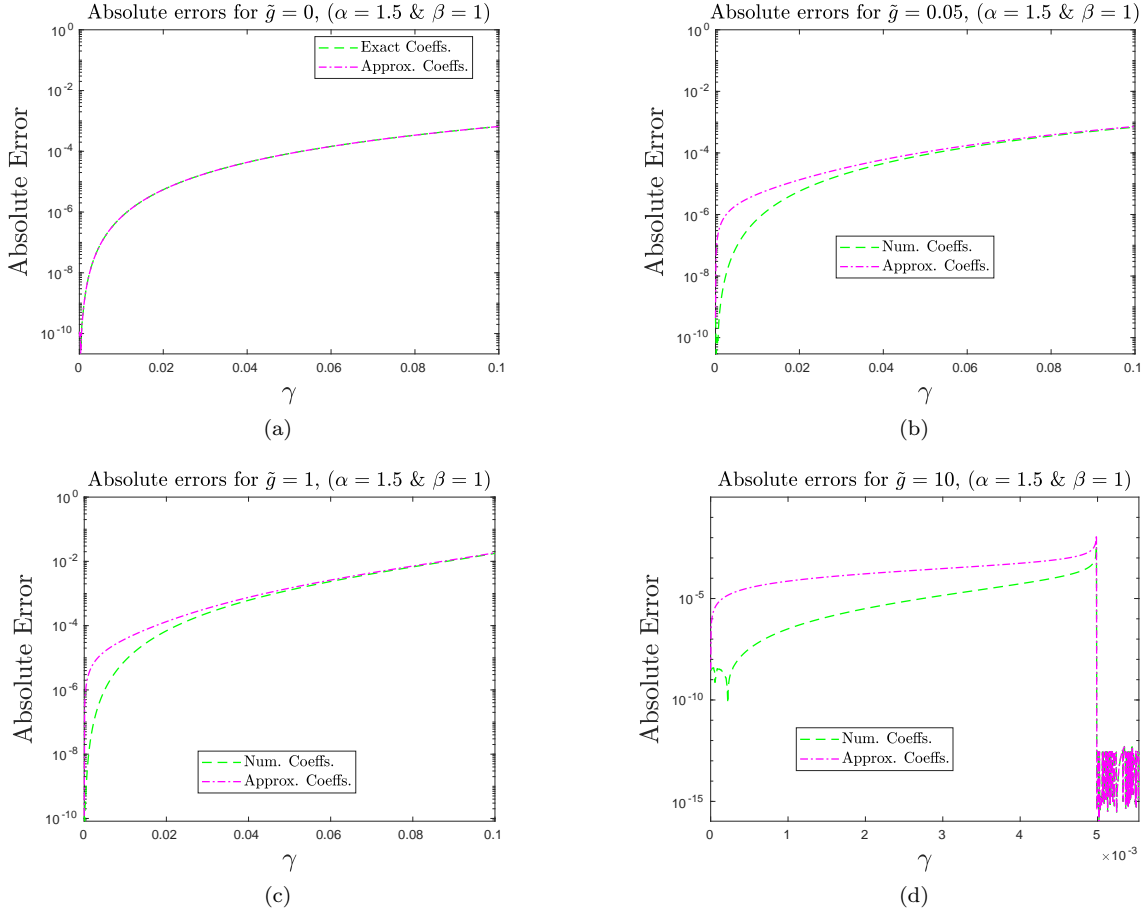


Figure 15: Absolute errors of the $\mathcal{O}(\gamma^2)$ CoR approximations (36) using the numerically integrated functions $\mathcal{I}(\tilde{g})$ and $\mathcal{Q}(\tilde{g})$ from (41) and (42) (labeled *Num. Coeffs.*), and the approximated functions $\mathcal{I}(\tilde{g}(\theta)) = K(\theta)J_{\alpha,\beta}(\theta)$ and $\mathcal{Q}(\tilde{g}(\theta)) = L(\theta)M_{\alpha,\beta}(\theta)$ from (48) and (49) (labeled *Approx. Coeffs.*). When $\tilde{g} = 0$, the results with the numerical results are replaced with the absolute error of $\mathcal{O}(\gamma^2)$ CoR computed with $\mathcal{I}(0)$ and $\mathcal{Q}(0)$ using (39) and (40) (labeled *Exact Coeffs.*). The absolute errors are computed in reference to the CoR values computed through numerical integration of (4). Results are shown for the visco-elastic model with $\alpha = 3/2$ & $\beta = 1$ (*Hertz stiffness with linear damping*), for the cases: (a) $\tilde{g} = 0$, (b) $\tilde{g} = 0.05$, (c) $\tilde{g} = 1$, and (d) $\tilde{g} = 10$.

501 α : Exponent parameter on the displacement (deformation), according to the general nonlinear visco-
502 elastic contact model ($\alpha = 3/2$ for Kuwabara-Kono model).

503 β : Exponent parameter on the speed (deformation-rate), according to the general nonlinear visco-
504 elastic contact model ($\beta = 3/2$ for Kuwabara-Kono model; $\beta = 5/2$ for Simon-Hunt-Crossley).

505 γ_0 : A coefficient to the speed (deformation-rate), according to the general nonlinear visco-elastic
506 contact model.

507 F : Constant external load on the bead.

508 g : Gravitational acceleration constant.

509 v_0 : Impact velocity of the bead.

The first step is to convert the aforementioned fundamental parameters into two re-scaled parameters that are necessary for the computation of the approximate CoR. These two re-scaled parameters, \tilde{g} and γ can be

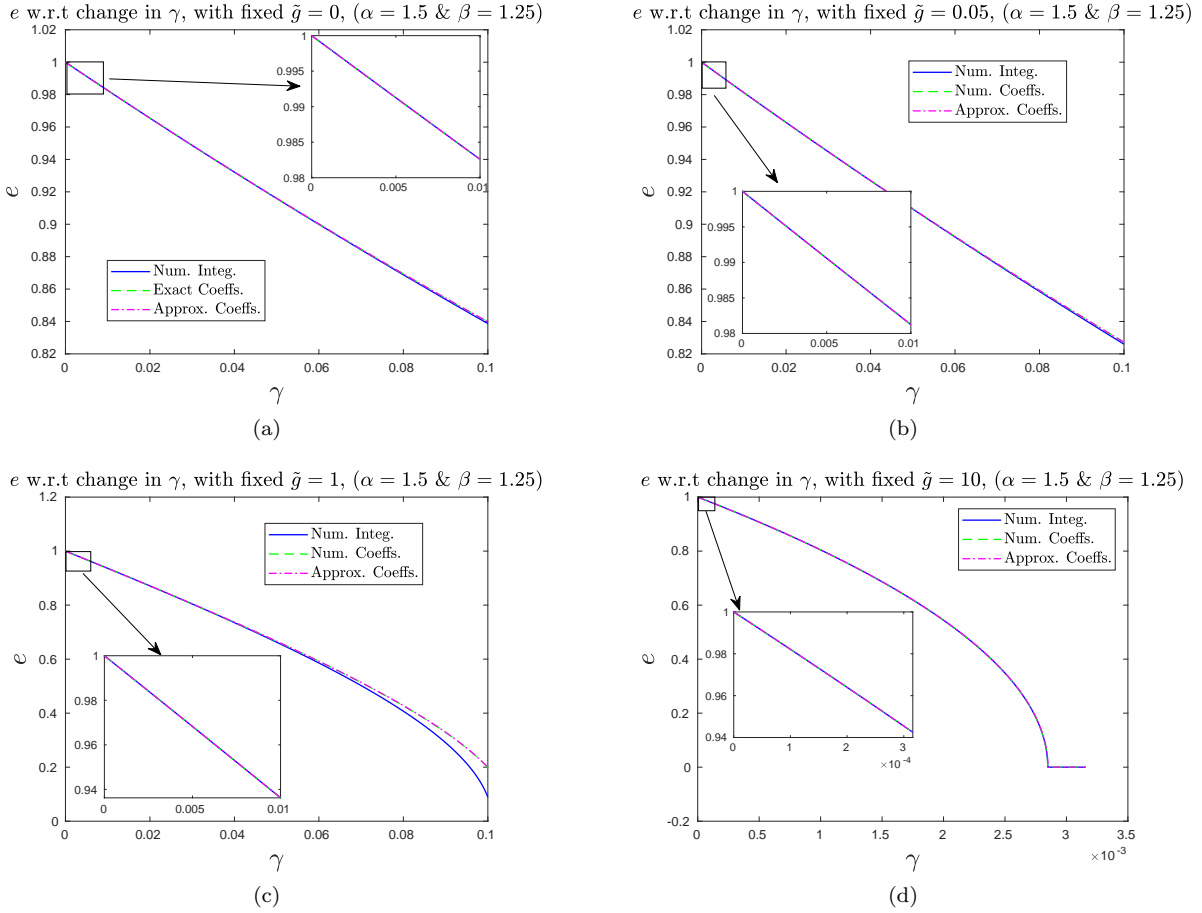


Figure 16: Comparison between the CoR values computed via numerical integration of (4), and the $\mathcal{O}(\gamma^2)$ approximations given by (36) (labeled *Num. Integ.*), using the numerically integrated functions $\mathcal{I}(\tilde{g})$ and $\mathcal{Q}(\tilde{g})$ from (41) and (42) (labeled *Num. Coeffs.*), and the approximated functions $\mathcal{I}(\tilde{g}(\theta)) = K(\theta)J_{\alpha,\beta}(\theta)$ and $\mathcal{Q}(\tilde{g}(\theta)) = L(\theta)M_{\alpha,\beta}(\theta)$ using (48) and (49) (labeled *Approx. Coeffs.*). When $\tilde{g} = 0$, the values $\mathcal{I}(0)$ and $\mathcal{Q}(0)$ are calculated using (39) and (40). Results are shown for the visco-elastic model with $\alpha = 3/2$ & $\beta = 5/4$ (*Tsuji-Tanaka-Ishida model*), for the cases: (a) $\tilde{g} = 0$, (b) $\tilde{g} = 0.05$, (c) $\tilde{g} = 1$, and (d) $\tilde{g} = 10$.

computed as:

$$\gamma = \gamma_0 v_0^{\frac{2\beta}{\alpha+1}-1} \left(\frac{k}{m}\right)^{1-\frac{\beta}{\alpha+1}}$$

$$\text{and, } \tilde{g} = \left(\frac{m}{k}\right)^{\frac{1}{\alpha+1}} v_0^{\frac{-2\alpha}{\alpha+1}} \left(g + \frac{F}{m}\right)$$

510 After the re-scaled parameters of \tilde{g} and γ are obtained, one may proceed to calculate the integrals $\mathcal{I}(\tilde{g})$ and
 511 $\mathcal{Q}(\tilde{g})$. These integrals can be calculated either numerically or analytically. Numerical integration would lead
 512 to more accurate results when \tilde{g} is neither too large nor too small. However, small and large values of \tilde{g} the
 513 analytical approximation is very accurate. The steps for computing the analytical approximation of $\mathcal{I}(\tilde{g})$
 514 and $\mathcal{Q}(\tilde{g})$ are listed below:

- 515 1 Solve for $\theta \in [0, 1]$, such that $\tilde{g}(\theta) = \left(\frac{\alpha+1}{2\theta}\right)^{-\frac{1}{\alpha+1}} \left(\frac{1}{2\theta} - \frac{1}{2}\right)$. This can be obtained using the *Bisection*
 516 *Method*.
- 517 2 Using the value of θ directly evaluate the functions $K(\theta) = \frac{2}{\sqrt{\theta}} \left(\frac{\alpha+1}{2\theta}\right)^{\frac{\beta}{\alpha+1}}$ and $L(\theta) = \sqrt{\theta} \left(\frac{\alpha+1}{2\theta}\right)^{\frac{\beta}{\alpha+1}}$.
- 518 3 Next to evaluate the function $M_{\alpha,\beta}(\theta)$, first choose an order of approximation n (ideally $n = 21$).

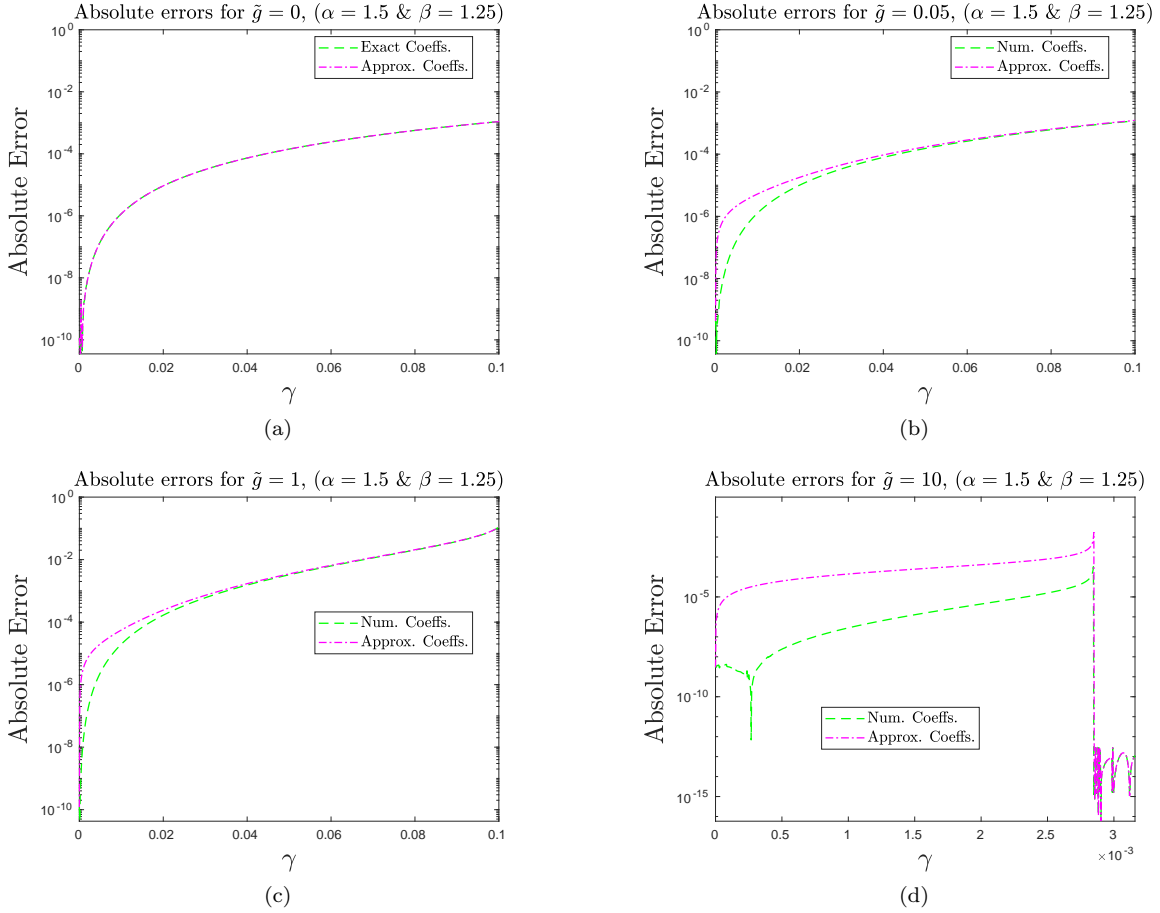


Figure 17: Absolute errors of the $\mathcal{O}(\gamma^2)$ CoR approximations (36) using the numerically integrated functions $\mathcal{I}(\tilde{g})$ and $\mathcal{Q}(\tilde{g})$ from (41) and (42) (labeled *Num. Coeffs.*), and the approximated functions $\mathcal{I}(\tilde{g}(\theta)) = K(\theta)J_{\alpha,\beta}(\theta)$ and $\mathcal{Q}(\tilde{g}(\theta)) = L(\theta)M_{\alpha,\beta}(\theta)$ from (48) and (49) (labeled *Approx. Coeffs.*). When $\tilde{g} = 0$, the results with the numerical results are replaced with the absolute error of $\mathcal{O}(\gamma^2)$ CoR computed with $\mathcal{I}(0)$ and $\mathcal{Q}(0)$ using (39) and (40) (labeled *Exact Coeffs.*). The absolute errors are computed in reference to the CoR values computed through numerical integration of (4). Results are shown for the visco-elastic model with $\alpha = 3/2$ & $\beta = 5/4$ (*Tsuji-Tanaka-Ishida model*), for the cases: (a) $\tilde{g} = 0$, (b) $\tilde{g} = 0.05$, (c) $\tilde{g} = 1$, and (d) $\tilde{g} = 10$.

519 Based on the choice of n the coefficients c_j and $a_{j,k}$ can be calculated using the relations in (50) and
520 (53). Finally compute the $M_{\alpha,\beta}(\theta)$ approximation using (49).

521 4 Following the same procedure as in the previous step, compute $M_{\alpha,\alpha+\beta+1}(\theta)$ and $M_{\alpha,\beta+1}(\theta)$. Then
522 use these values to compute $J_{\alpha,\beta}(\theta)$ according to (48).

523 5 Compute $\mathcal{I}(\tilde{g}(\theta)) = K(\theta)J_{\alpha,\beta}(\theta)$ and $\mathcal{Q}(\tilde{g}(\theta)) = L(\theta)M_{\alpha,\beta}(\theta)$, using (47).

524 Lastly, select the appropriate expansion from (36) to calculate the analytical approximation of e , using the
525 values of $\mathcal{I}(\tilde{g}(\theta))$ and $\mathcal{Q}(\tilde{g}(\theta))$. The MATLAB implementation of the aforementioned method along with all
526 numerical results presented in this report is available at the GitHub repository [38].

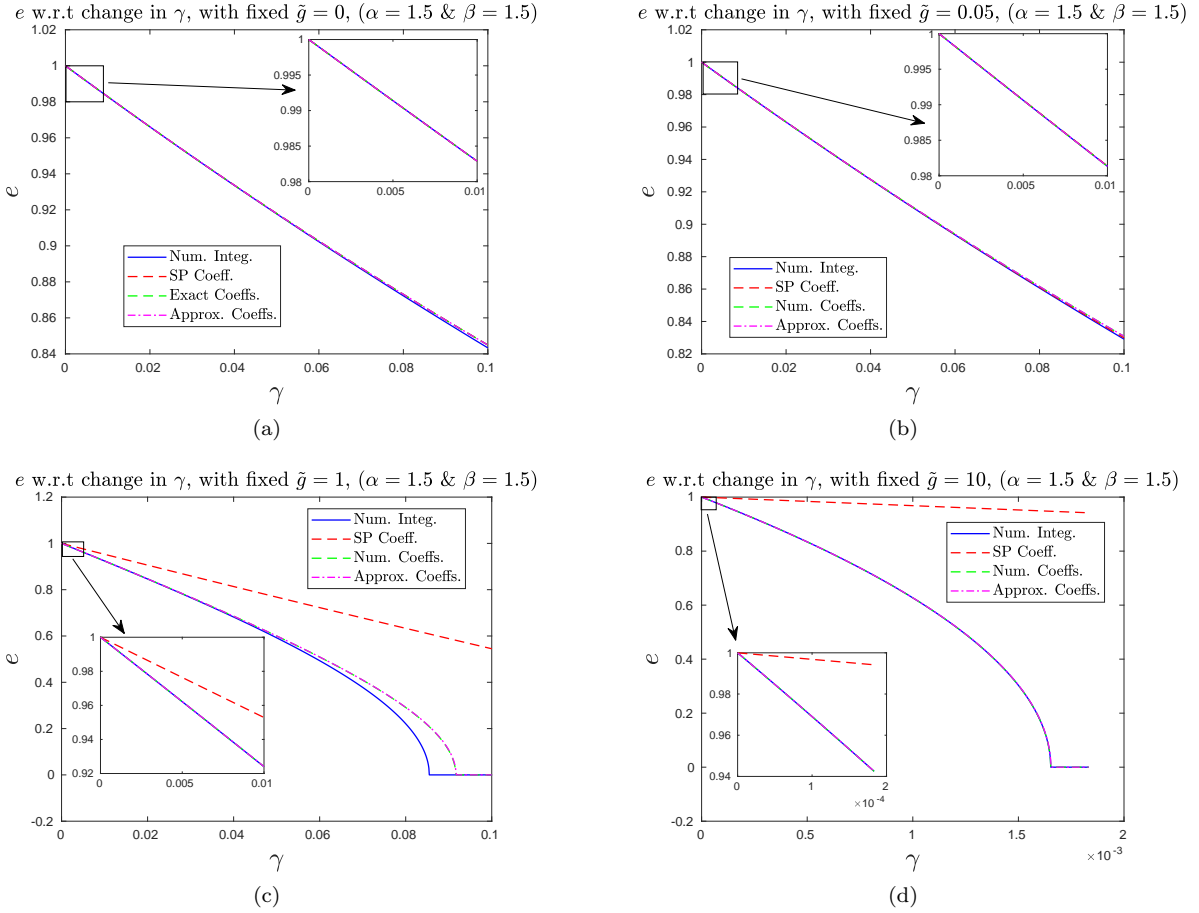


Figure 18: Comparison between the CoR values computed via numerical integration of (4), and the $\mathcal{O}(\gamma^2)$ approximations given by (36) (labeled *Num. Integ.*), using the numerically integrated functions $\mathcal{I}(\tilde{g})$ and $\mathcal{Q}(\tilde{g})$ from (41) and (42) (labeled *Num. Coeffs.*), and the approximated functions $\mathcal{I}(\tilde{g}(\theta)) = K(\theta)J_{\alpha,\beta}(\theta)$ and $\mathcal{Q}(\tilde{g}(\theta)) = L(\theta)M_{\alpha,\beta}(\theta)$ using (48) and (49) (labeled *Approx. Coeffs.*). When $\tilde{g} = 0$, the values $\mathcal{I}(0)$ and $\mathcal{Q}(0)$ are calculated using (39) and (40). Results are shown for the visco-elastic model with $\alpha = \beta = 3/2$ (*Kuwabara-Kono model*), for the cases: (a) $\tilde{g} = 0$, (b) $\tilde{g} = 0.05$, (c) $\tilde{g} = 1$, and (d) $\tilde{g} = 10$. The Schwager-Pöschel coefficient based $\mathcal{O}(\gamma^2)$ approximation from (25) (30) is also included (labeled *SP Coeff.*).

6 Conclusion

527

528 The goal of this study was to derive an approximate analytical solution of Coefficient of Restitution (CoR)
529 for a single bead impact, while considering the effects of external load. In this study, we derived several
530 series expansions of the CoR based on a general nonlinear visco-elastic model and studied their accuracy with
531 respect to calculations of CoR from numerical integration. First, a first-order expansion of CoR was obtained
532 that linearly depended on a scaled parameter representing the external load and force due to gravity. We
533 saw that this first-order approximation was only valid for small magnitudes of external loads and gravity.
534 We then derived a second-order approximation, which while still linearly depending on gravity and external
535 load, has a second-order dependence to the scaled parameter representing the damping (dissipation). To
536 compute the unknown coefficient for this second-order term, we used a coefficient value from another study
537 that assumed no external load or gravity in the derivation. The second-order CoR approximation that we
538 obtained based on this coefficient value improved the CoR approximation for small magnitudes of small
539 external load and gravitational force, but didn't perform well for higher forces. Hence, lastly we derived our
540 final second-order approximation of the CoR with respect to the dissipation-dependent scaled parameter.

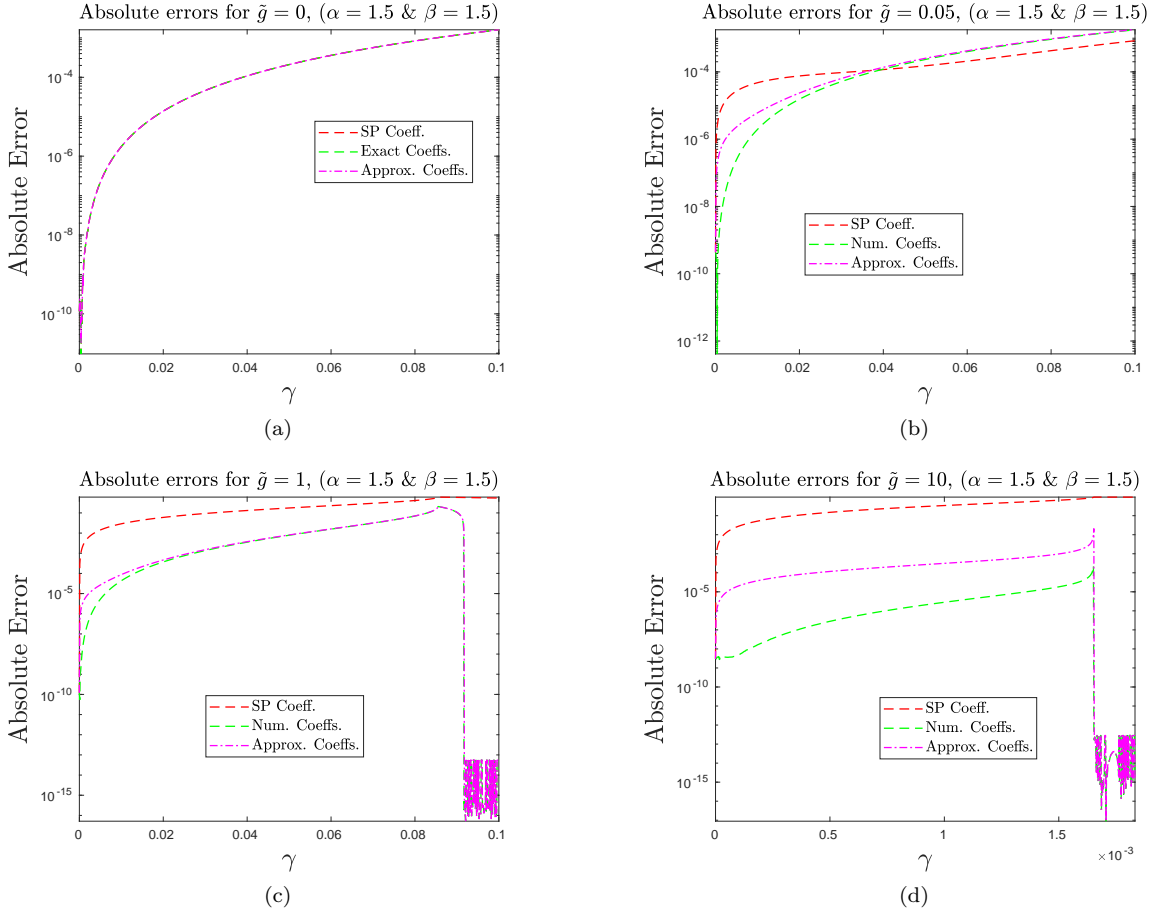


Figure 19: Absolute errors of the $\mathcal{O}(\gamma^2)$ CoR approximations (36) using the numerically integrated functions $\mathcal{I}(\tilde{g})$ and $\mathcal{Q}(\tilde{g})$ from (41) and (42) (labeled *Num. Coeffs.*), and the approximated functions $\mathcal{I}(\tilde{g}(\theta)) = K(\theta)J_{\alpha,\beta}(\theta)$ and $\mathcal{Q}(\tilde{g}(\theta)) = L(\theta)M_{\alpha,\beta}(\theta)$ from (48) and (49) (labeled *Approx. Coeffs.*). When $\tilde{g} = 0$, the results with the numerical results are replaced with the absolute error of $\mathcal{O}(\gamma^2)$ CoR computed with $\mathcal{I}(0)$ and $\mathcal{Q}(0)$ using (39) and (40) (labeled *Exact Coeffs.*). The absolute errors are computed in reference to the CoR values computed through numerical integration of (4). Results are shown for the visco-elastic model with $\alpha = \beta = 3/2$ (*Kuwabara-Kono model*), for the cases: (a) $\tilde{g} = 0$, (b) $\tilde{g} = 0.05$, (c) $\tilde{g} = 1$, and (d) $\tilde{g} = 10$. The absolute errors of Schwager-Pöschel coefficient based $\mathcal{O}(\gamma^2)$ approximation from (25) (30) are also included.

541 In this final second-order approximation, the coefficients obtained were functions of the applied force, hence
 542 this approximation was found to be very accurate for bot large and small magnitudes of external load and
 543 gravitational force.

544 A Condition for no detachment after impact for large \tilde{g}

545 The goal of this appendix is to approximate the solution of e for large \tilde{g} , using the expansion (8), and
 546 determine the value of \tilde{g}_c , such that $e = 0, \forall \tilde{g} \geq \tilde{g}_c$. Let us begin by rewriting (8) as,

$$e^2 = 1 - 2\gamma\beta\mathcal{I}(\tilde{g}) + \text{H.O.T.} \quad (54)$$

547 where,

$$\mathcal{I}(\tilde{g}) = \int_0^{T_0} \left(\frac{du_0}{d\tau} \right)^2 u_0^{\beta-1} d\tau \quad (55)$$

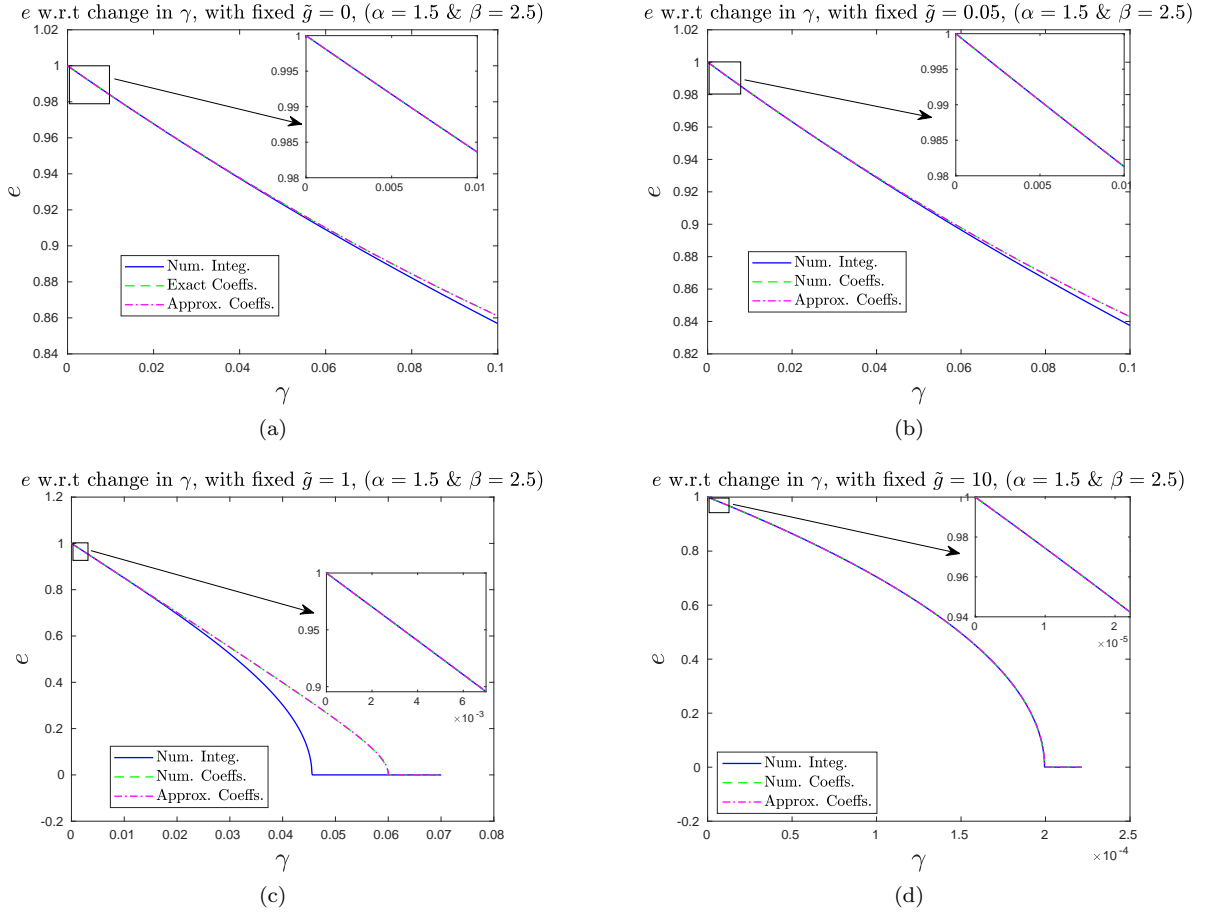


Figure 20: Comparison between the CoR values computed via numerical integration of (4), and the $\mathcal{O}(\gamma^2)$ approximations given by (36) (labeled *Num. Integ.*), using the numerically integrated functions $\mathcal{I}(\tilde{g})$ and $\mathcal{Q}(\tilde{g})$ from (41) and (42) (labeled *Num. Coeffs.*), and the approximated functions $\mathcal{I}(\tilde{g}(\theta)) = K(\theta)J_{\alpha,\beta}(\theta)$ and $\mathcal{Q}(\tilde{g}(\theta)) = L(\theta)M_{\alpha,\beta}(\theta)$ using (48) and (49) (labeled *Approx. Coeffs.*). When $\tilde{g} = 0$, the values $\mathcal{I}(0)$ and $\mathcal{Q}(0)$ are calculated using (39) and (40). Results are shown for the visco-elastic model with $\alpha = 3/2$ & $\beta = 5/2$ (*Simon-Hunt-Crossley model*), for the cases: (a) $\tilde{g} = 0$, (b) $\tilde{g} = 0.05$, (c) $\tilde{g} = 1$, and (d) $\tilde{g} = 10$.

548 where, T_0 corresponds to the final time at the end of the non-rebounding impact, when e reaches 0. Assuming
 549 symmetry about the maximal compression, we can rewrite the integral in (55) as,

$$\mathcal{I}(\tilde{g}) = 2 \int_0^{\frac{T_0}{2}} \left(\frac{du_0}{d\tau} \right)^2 u_0^{\beta-1} d\tau \quad (56)$$

Since, we are considering the case when γ is small ($\gamma \ll 1$), the derivation of $\frac{du_0}{d\tau}$ given in (11) remains valid here, where it is given by,

$$\frac{du_0}{d\tau} = \left[1 + 2 \left(\tilde{g}u_0 - \frac{u_0^{\alpha+1}}{\alpha+1} \right) \right]^{\frac{1}{2}}$$

550 The maximal deformation u_M , as shown earlier in (13), is given by the solution to the equation,

$$\frac{1}{\alpha+1} u_M^{\alpha+1} - \tilde{g}u_M = \frac{1}{2} \quad (57)$$

551 Since, here we consider a large \tilde{g} , (57) can be approximated as,

$$\frac{1}{\alpha+1} u_M^{\alpha+1} - \tilde{g}u_M \approx 0 \quad \rightarrow \quad u_M = \tilde{g}^{\frac{1}{\alpha}} y \quad (58)$$

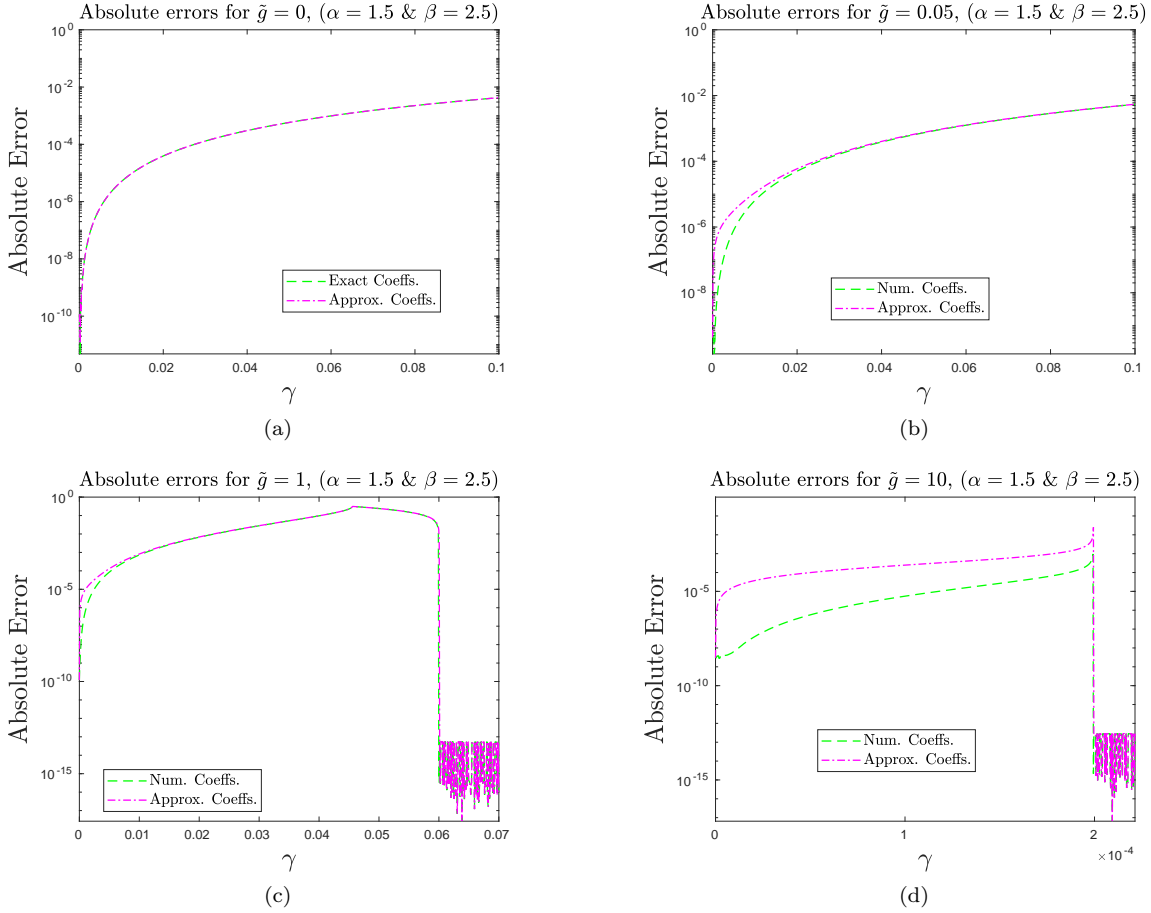


Figure 21: Absolute errors of the $\mathcal{O}(\gamma^2)$ CoR approximations (36) using the numerically integrated functions $\mathcal{I}(\tilde{g})$ and $\mathcal{Q}(\tilde{g})$ from (41) and (42) (labeled *Num. Coeffs.*), and the approximated functions $\mathcal{I}(\tilde{g}(\theta)) = K(\theta)J_{\alpha,\beta}(\theta)$ and $\mathcal{Q}(\tilde{g}(\theta)) = L(\theta)M_{\alpha,\beta}(\theta)$ from (48) and (49) (labeled *Approx. Coeffs.*). When $\tilde{g} = 0$, the results with the numerical results are replaced with the absolute error of $\mathcal{O}(\gamma^2)$ CoR computed with $\mathcal{I}(0)$ and $\mathcal{Q}(0)$ using (39) and (40) (labeled *Exact Coeffs.*). The absolute errors are computed in reference to the CoR values computed through numerical integration of (4). Results are shown for the visco-elastic model with $\alpha = 3/2$ & $\beta = 5/2$ (*Simon-Hunt-Crossley model*), for the cases: (a) $\tilde{g} = 0$, (b) $\tilde{g} = 0.05$, (c) $\tilde{g} = 1$, and (d) $\tilde{g} = 10$.

552 where $y = (1 + \alpha)^{\frac{1}{\alpha}} + \mathcal{O}(\tilde{g}, \alpha)$ and is treated as an unknown at this point. Equation (58) intuitively
 553 approximates that the left-hand side of (57) becomes negligible, when $\tilde{g} \gg 1$; a more rigorous approximation
 554 is presented below. Next, substituting u_M from the approximation in (58) back into (57), we obtain,

$$\frac{1}{\alpha + 1} y^{\alpha+1} \tilde{g}^{1+\frac{1}{\alpha}} - y \tilde{g}^{1+\frac{1}{\alpha}} = \frac{1}{2} \quad (59)$$

555 Now if we consider a parameter $\epsilon = \frac{1}{\tilde{g}^{1+\frac{1}{\alpha}}}$, which is very small since $\tilde{g} \gg 1$, then (59) can be rewritten as,

$$\frac{1}{\alpha + 1} y^{\alpha+1} - y = \frac{\epsilon}{2} \quad (60)$$

556 The left-hand side of (60) depends upon the small value of ϵ , and thus validates the approximation shown
 557 earlier in (58). Since ϵ already depends on \tilde{g} and α , we can rewrite the expansion of y as $y = (1 + \alpha)^{\frac{1}{\alpha}} + \mathcal{O}(\epsilon)$,
 558 and also u_M as $u_M = [\tilde{g}(1 + \alpha)]^{\frac{1}{\alpha}} + \mathcal{O}(\frac{1}{\tilde{g}})$. Now, considering the integral in (56), changing integration variable

Types of \mathcal{I} & \mathcal{Q}	Exact[(39),(40)]/Num.[(41),(42)]		Approx. [(48),(49)]	
\tilde{g}	$\gamma < 0.02$	$\gamma \geq 0.02$	$\gamma < 0.02$	$\gamma \geq 0.02$
0	$3.87 \times 10^{-5*}$	$4.26 \times 10^{-3*}$	3.87×10^{-5}	4.26×10^{-3}
0.05	4.94×10^{-5}	5.39×10^{-3}	5.84×10^{-5}	5.43×10^{-3}
1	2.06×10^{-3}	3.11×10^{-1}	2.20×10^{-3}	3.11×10^{-1}

(a) Max. absolute errors within different γ ranges for $\tilde{g} = 0, 0.05, \text{ and } 1$

Types of \mathcal{I} & \mathcal{Q}	$\gamma < 0.00004$	$0.00004 \leq \gamma \leq 0.00020$	$\gamma > 0.00020$
Num. [(41),(42)]	3.83×10^{-7}	9.67×10^{-4}	2.84×10^{-13}
Approx. [(48),(49)]	8.57×10^{-5}	2.50×10^{-1}	2.84×10^{-13}

(b) Max. absolute errors within different γ for $\tilde{g} = 10$

Table 5: Maximum Absolute Errors on the $\mathcal{O}(\gamma^2)$ expansion of e using a visco-elastic model with $\alpha = 3/2$ and $\beta = 5/2$ (*Simon-Hunt-Crossley model*), with $\tilde{g} = 0, 0.05, 1, \text{ and } 10$, for various γ ranges. *Errors on $\mathcal{O}(\gamma^2)$ expansions of e while using the exact values of $\mathcal{I}(\tilde{g} = 0)$ and $\mathcal{Q}(\tilde{g} = 0)$, according to (39) and (40).

559 $(\tau \rightarrow u_0)$, and substituting the upper-bound as $u_0(\frac{T_0}{2}) = u_M = [\tilde{g}(1 + \alpha)]^{\frac{1}{\alpha}}$, we get,

$$\mathcal{I}(\tilde{g}) = 2 \int_0^{\tilde{g}^{\frac{1}{\alpha}} y} \left[1 + 2 \left(\tilde{g} u_0 - \frac{u_0^{\alpha+1}}{\alpha + 1} \right) \right]^{\frac{1}{2}} u_0^{\beta-1} du_0 \quad (61)$$

560 Let us now introduce a new variable y_0 and consider $u_0 = \tilde{g}^{\frac{1}{\alpha}} y_0$, which also implies $du_0 = \tilde{g}^{\frac{1}{\alpha}} dy_0$. Considering
561 these substitution in (61) we obtain,

$$\mathcal{I}(\tilde{g}) = 2 \int_0^y \left[1 + 2\tilde{g}^{1+\frac{1}{\alpha}} \left(y_0 - \frac{y_0^{\alpha+1}}{\alpha + 1} \right) \right]^{\frac{1}{2}} \tilde{g}^{\frac{\beta}{\alpha}} y_0^{\beta-1} dy_0 = 2\tilde{g}^{\frac{\beta}{\alpha} + \frac{1}{2\alpha} + \frac{1}{2}} \int_0^y \left[\epsilon + 2 \left(y_0 - \frac{y_0^{\alpha+1}}{\alpha + 1} \right) \right]^{\frac{1}{2}} y_0^{\beta-1} dy_0 \quad (62)$$

562 Since, $\epsilon \approx 0$, the integral in (62) can be approximated as,

$$\mathcal{I}(\tilde{g}) = 2\sqrt{2}\tilde{g}^{\frac{\beta}{\alpha} + \frac{1}{2\alpha} + \frac{1}{2}} \int_0^{(1+\alpha)^{\frac{1}{\alpha}}} \left(1 - \frac{y_0^\alpha}{\alpha + 1} \right)^{\frac{1}{2}} y_0^{\beta-\frac{1}{2}} dy_0 + \text{H.O.T.} \quad (63)$$

563 Now we can consider another change of variable with $t = \frac{y_0^\alpha}{\alpha+1}$, which also yields $dt = \frac{\alpha y_0^{\alpha-1}}{\alpha+1} dy_0$. Note that
564 with this variable, $y_0 = y$, we get $t = \frac{y^\alpha}{\alpha+1} = \frac{[(1+\alpha)^{\frac{1}{\alpha}}]^\alpha}{\alpha+1} = 1$. Hence, after making these substitutions in (63)
565 and simplifying, we obtain,

$$\mathcal{I}(\tilde{g}) = 2\sqrt{2} \left(\frac{\alpha + 1}{\alpha} \right) (\alpha + 1)^{\frac{\beta-\alpha+\frac{1}{2}}{\alpha}} \tilde{g}^{\left(\frac{\beta}{\alpha} + \frac{1}{2\alpha} + \frac{1}{2}\right)} \int_0^1 (1-t)^{\frac{1}{2}} t^{\frac{\beta-\alpha+\frac{1}{2}}{\alpha}} dt \quad (64)$$

566 The integral in (64), is in the form of Euler's beta function, which can be evaluated. Thus, substituting (64)
567 into (54), and expressing the integral in terms of Euler's beta function, according to Def. 1, yields,

$$e^2 \approx 1 - 2\gamma C \tilde{g}^{\left(\frac{\beta}{\alpha} + \frac{1}{2\alpha} + \frac{1}{2}\right)} \quad (65)$$

where,

$$C = 2\sqrt{2} \frac{\beta}{\alpha} (\alpha + 1)^{\left(\frac{\beta}{\alpha} + \frac{1}{2\alpha}\right)} \text{B} \left(\frac{\beta + \frac{1}{2}}{\alpha}, \frac{3}{2} \right)$$

568 Thus, (65) presents an approximation for the expansion (54), corresponding to large \tilde{g} . Hence, (65) can be
569 used to solve for $\tilde{g}_c \equiv \tilde{g}$, such that, $e = 0$,

$$\tilde{g}_c^{\left(\frac{\beta}{\alpha} + \frac{1}{2\alpha} + \frac{1}{2}\right)} \approx \frac{1}{2\gamma C} \quad (66)$$

570 The relationship (66), establishes a relationship between \tilde{g}_c and the other parameters, namely α, β , and γ .
571 The value of CoR remains $e = 0$ for any $\tilde{g} \geq \tilde{g}_c$.

572 B Order 2 expansion of e^2 around $\gamma \approx 0$

573 The main objective of this appendix is to formulate the quantity $\frac{\partial \mathcal{I}}{\partial \gamma}(\gamma, \tilde{g})$ at $\gamma = 0$, which follows from
 574 the development presented in Sec. 4.2 on the $\mathcal{O}(\gamma^2)$ expansion of e^2 . Let the solution of (31) for the non-
 575 dissipative case, $\gamma = 0$ be u_0 , with the corresponding impact duration as T_0 . We know that u_0 is *symmetric*
 576 around $\frac{T_0}{2}$ with $u_0(\frac{T_0}{2} + h) = u_0(\frac{T_0}{2} - h)$, $\forall h \in [0, \frac{T_0}{2}]$, and from (13), that $u_M = u_0(\frac{T_0}{2})$, satisfies
 577 the constraint $\frac{1}{\alpha+1}u_M^{\alpha+1} - \tilde{g}u_M = \frac{1}{2}$. Also, we note that the scaled velocity $\frac{\partial u_0}{\partial \tau}$, is *anti-symmetric*, since
 578 $\frac{\partial u_0}{\partial \tau}(\frac{T_0}{2} + h) = -\frac{\partial u_0}{\partial \tau}(\frac{T_0}{2} - h)$, $\forall h \in [0, \frac{T_0}{2}]$. In (35), we need to evaluate \mathcal{I} and $\frac{\partial \mathcal{I}}{\partial \gamma}$ for $\gamma = 0$, so using the
 579 symmetric property we can rewrite the integral at $\gamma = 0$ as,

$$\mathcal{I}(0, \tilde{g}) = 2 \int_0^{\frac{T_0}{2}} \left(\frac{\partial u_0}{\partial \tau} \right)^2 u_0^{\beta-1} d\tau \quad (67)$$

580 Using (11), we can rewrite the above expression as,

$$\mathcal{I}(0, \tilde{g}) = 2 \int_0^{u_M} \left[1 + 2 \left(\tilde{g}u_0 - \frac{u_0^{\alpha+1}}{\alpha+1} \right) \right]^{\frac{1}{2}} u_0^{\beta-1} du_0 \quad (68)$$

581 Similarly, when $\gamma = 0$, $\frac{\partial \mathcal{I}}{\partial \gamma}$ can be expressed as,

$$\frac{\partial \mathcal{I}}{\partial \gamma}(0, \tilde{g}) = \frac{1}{\beta} \int_0^{T_0} \frac{\partial u}{\partial \gamma}(0, \tau) \left[(\beta + \alpha)u_0^{\beta+\alpha-1} - \tilde{g}\beta u_0^{\beta-1} \right] d\tau \quad (69)$$

582 We need to find the expression $\frac{\partial u}{\partial \gamma}(0, \tau)$, to further evaluate (69). We may proceed towards this goal by
 583 differentiating (31) with respect to γ and then setting $\gamma = 0$,

$$\frac{\partial^2}{\partial \tau^2} \left(\frac{\partial u}{\partial \gamma}(0, \tau) \right) + \frac{\partial}{\partial \tau}(u_0^\beta) + \alpha u_0^{\alpha-1} \frac{\partial u}{\partial \tau}(0, \tau) = 0 \quad (70)$$

584 Now, if we consider the variable $u_\gamma^0 = \frac{\partial u}{\partial \gamma}(0, \tau)$, we obtain a linear non-homogeneous ordinary differential
 585 equation of the form,

$$\frac{\partial^2 u_\gamma^0}{\partial \tau^2} + \alpha u_0^{\alpha-1} u_\gamma^0 + \frac{\partial}{\partial \tau}(u_0^\beta) = 0, \quad (71)$$

$$u_\gamma^0(0) = 0 \quad \text{and} \quad \frac{du_\gamma^0}{d\tau}(0) = 0.$$

If we consider the homogeneous part of (71), we can deduce that one of the solutions is $\frac{\partial u_0}{\partial \tau}$. We may
 check this by setting $\gamma = 0$ in (31) and differentiating it with respect to τ ,

$$\frac{\partial^2}{\partial \tau^2} \left(\frac{\partial u_0}{\partial \tau} \right) + \alpha u_0^{\alpha-1} \left(\frac{\partial u_0}{\partial \tau} \right) = 0$$

586 Comparing the above equation with (71), we see that indeed $\frac{\partial u_0}{\partial \tau}$ is one of the solutions for the homoge-
 587 neous ODE. Next we need to find a solution for the homogeneous system independent from $\frac{\partial u_0}{\partial \tau}$. Let this
 588 complementary solution be u_1 , such that it satisfies the differential equation,

$$\frac{\partial^2 u_1}{\partial \tau^2} + \alpha u_0^{\alpha-1} u_1 = 0 \quad (72)$$

589 and the boundary condition,

$$\frac{\partial u_1}{\partial \tau} \left(\frac{T_0}{2} \right) = 0 \quad (73)$$

590 In order for u_1 to be independent of $\frac{\partial u_0}{\partial \tau}$, the Wronskian of these two solution, must be non-zero. Additionally,
 591 we note that the differential equation in (71) does not have a dissipative term, which implies that the

592 Wronskian must be a constant. Thus, we may obtain the other boundary condition for the complementary
593 solution u_1 by first setting the Wronskian to a non-zero constant and then solving for the $u_1(\frac{T_0}{2})$. The
594 Wronskian for the complementary solutions, u_1 and $\frac{\partial u_0}{\partial \tau}$, is given by,

$$W = \begin{vmatrix} u_1 & \frac{\partial u_0}{\partial \tau} \\ \frac{\partial u_1}{\partial \tau} & \frac{\partial^2 u_0}{\partial \tau^2} \end{vmatrix} = u_1 \frac{\partial^2 u_0}{\partial \tau^2} - \frac{\partial u_0}{\partial \tau} \frac{\partial u_1}{\partial \tau} = \text{constant} \quad (74)$$

595 Thus, having defined the complementary solutions u_1 and $\frac{\partial u_0}{\partial \tau}$ and their associated Wronskian, W , we can
596 define the particular solution of (71), via. the variation of parameter formulation as,

$$u_\gamma^0 = -\frac{\partial u_0}{\partial \tau} \int_0^\tau \frac{1}{W} \frac{\partial u_0^\beta}{\partial \tau} u_1 d\tau + u_1 \int_0^\tau \frac{1}{W} \frac{\partial u_0^\beta}{\partial \tau} \frac{\partial u_0}{\partial \tau} d\tau \quad (75)$$

597 Note that in (75), the particular solution is scaled by the reciprocal of W . Thus, we are allowed to choose
598 any constant value for W . Here we choose the value of the Wronskian to be $W = 1$. Since, the Wronskian
599 is a constant, now we can evaluate it at $\tau = \frac{T_0}{2}$,

$$\begin{aligned} W = W\left(\frac{T_0}{2}\right) &= u_1\left(\frac{T_0}{2}\right) \frac{\partial^2 u_0}{\partial \tau^2}\left(\frac{T_0}{2}\right) - \frac{\partial u_0}{\partial \tau}\left(\frac{T_0}{2}\right) \frac{\partial u_1}{\partial \tau}\left(\frac{T_0}{2}\right) \\ &= (\tilde{g} - u_M^\alpha) u_1\left(\frac{T_0}{2}\right) = 1 \end{aligned} \quad (76)$$

600 Thus, we can obtain the other boundary condition for u_1 , as,

$$u_1\left(\frac{T_0}{2}\right) = \frac{1}{\tilde{g} - u_M^\alpha} \quad (77)$$

601 Additionally, we can also deduce the relationship,

$$\frac{\partial}{\partial \tau} \left(\frac{u_1}{\partial u_0 / \partial \tau} \right) = -\frac{W}{(\partial u_0 / \partial \tau)^2} = -\frac{1}{(\partial u_0 / \partial \tau)^2} \quad (78)$$

602 Lastly, given that $W = 1$, the particular solution of u_γ^0 simplifies to,

$$u_\gamma^0 = -\frac{\partial u_0}{\partial \tau} \int_0^\tau \frac{\partial u_0^\beta}{\partial \tau} u_1 d\tau + u_1 \int_0^\tau \frac{\partial u_0^\beta}{\partial \tau} \frac{\partial u_0}{\partial \tau} d\tau \quad (79)$$

603 Since, the particular solution u_γ^0 is to be used in the evaluation $\frac{\partial \mathcal{I}}{\partial \gamma}(0, \tilde{g})$ in (69), which requires an integration
604 over the domain $[0, T_0/2]$, we can further simplify (79) by considering the symmetry. We proceed by splitting
605 u_γ^0 as,

$$u_\gamma^0 = y_a + z_a + y_s + w_a$$

where,

$$\begin{aligned} y_a &= -\frac{\partial u_0}{\partial \tau} \int_0^{T_0/2} \frac{\partial u_0^\beta}{\partial \tau} u_1 d\tau && \text{is anti-symmetric} \\ z_a &= -\frac{\partial u_0}{\partial \tau} \int_{T_0/2}^\tau \frac{\partial u_0^\beta}{\partial \tau} u_1 d\tau && \text{is anti-symmetric} \\ y_s &= u_1 \int_0^{T_0/2} \frac{\partial u_0^\beta}{\partial \tau} \frac{\partial u_0}{\partial \tau} d\tau && \text{is symmetric} \\ w_a &= u_1 \int_{T_0/2}^\tau \frac{\partial u_0^\beta}{\partial \tau} \frac{\partial u_0}{\partial \tau} d\tau && \text{is anti-symmetric} \end{aligned} \quad (80)$$

606 Since the function $(\beta + \alpha)u_0^{\beta+\alpha-1} - \tilde{g}\beta u_0^{\beta-1}$ is symmetric, the only non-vanishing term in (69) is given by,

$$\begin{aligned} \frac{\partial \mathcal{I}}{\partial \gamma}(0, \tilde{g}) &= \frac{1}{\beta} \int_0^{T_0} y_s \left[(\beta + \alpha)u_0^{\beta+\alpha-1} - \tilde{g}\beta u_0^{\beta-1} \right] d\tau \\ &= 2 \left[\int_0^{T_0/2} u_0^{\beta-1} \left(\frac{\partial u_0}{\partial \tau} \right)^2 d\tau \right] \left[\int_0^{T_0/2} u_1 \left((\beta + \alpha)u_0^{\alpha+\beta-1} - \tilde{g}\beta u_0^{\beta-1} \right) d\tau \right] \end{aligned} \quad (81)$$

607 In what follows, we will see that the second integral in (81) will become singular at upper-bound, $\frac{T_0}{2}$. Hence,
608 to avoid this singularity, we proceed by integrating the second term up to $\tau_1 = \frac{T_0}{2} - \eta$, with $\eta > 0$. So the
609 integration of the second term up to τ_1 can be performed via integration by parts,

$$\begin{aligned} J &= \int_0^{\tau_1} u_1 \left[(\alpha + \beta)u_0^{\alpha+\beta-1} - \tilde{g}\beta u_0^{\beta-1} \right] d\tau \\ &= R(\tau_1) + \int_0^{\tau_1} \left(u_0^{\alpha+\beta} - \tilde{g}u_0^\beta \right) \frac{1}{(\partial u_0 / \partial \tau)^2} d\tau \end{aligned} \quad (82)$$

where,

$$R(\tau_1) = \frac{u_1(\tau_1)}{\partial u_0 / \partial \tau(\tau_1)} u_0^\beta(\tau_1) (u_0^\alpha(\tau_1) - \tilde{g})$$

610 Next, using (31) for $\gamma = 0$, we can rewrite (82) as,

$$\begin{aligned} J &= R(\tau_1) - \int_0^{\tau_1} u_0^\beta \frac{\partial^2 u_0 / \partial \tau^2}{(\partial u_0 / \partial \tau)^2} d\tau \\ &= R(\tau_1) - \int_0^{\tau_1} u_0^\beta \frac{\partial}{\partial \tau} \left(\frac{1}{(\partial u_0 / \partial \tau)^2} \right) d\tau \\ &= R(\tau_1) + \frac{u_0^\beta(\tau_1)}{\partial u_0 / \partial \tau(\tau_1)} - \beta \int_0^{\tau_1} u_0^{\beta-1} d\tau \end{aligned} \quad (83)$$

611 Since, $\tau_1 = \frac{T_0}{2} - \eta$, the first term in (83) can be evaluated as,

$$\begin{aligned} R(\tau_1) + \frac{u_0^\beta(\tau_1)}{\partial u_0 / \partial \tau(\tau_1)} &= \frac{u_0^\beta(\tau_1)}{\partial u_0 / \partial \tau(\tau_1)} [u_1(\tau_1) (u_0^\alpha(\tau_1) - \tilde{g}) + 1] \\ &= \frac{u_0^\beta(\tau_1)}{\partial u_0 / \partial \tau(\tau_1)} \left[1 - u_1(\tau_1) \frac{\partial^2 u_0}{\partial \tau} \right] \end{aligned} \quad (84)$$

612 Note that, given the denominator of $\partial u_0 / \partial \tau(\tau_1)$ would cause (84) to be singular, if $\tau_1 \rightarrow T_0/2$. Thus, to
613 circumvent this issue, we replace the 1 in (84) with the Wronskian W , (since we had chosen $W = 1$), and
614 then compensate for the singularity to yield,

$$R(\tau_1) + \frac{u_0^\beta(\tau_1)}{\partial u_0 / \partial \tau(\tau_1)} = \frac{u_0^\beta(\tau_1)}{\partial u_0 / \partial \tau(\tau_1)} \left[W - u_1(\tau_1) \frac{\partial^2 u_0}{\partial \tau} \right] = -u_0^\beta(\tau_1) \frac{\partial u_1}{\partial \tau}(\tau_1) \quad (85)$$

615 Since, we have the boundary condition $\frac{\partial u_1}{\partial \tau}(T_0/2) = 0$, the right-hand side of the above equation limits to
616 zero,

$$\lim_{\tau_1 \rightarrow \frac{T_0}{2}} R(\tau_1) + \frac{u_0^\beta(\tau_1)}{\partial u_0 / \partial \tau(\tau_1)} = 0 \quad (86)$$

617 Thus, using the results of (83) and (86) for $\tau_1 \rightarrow \frac{T_0}{2}$ in (82) we obtain,

$$J = -\beta \int_0^{T_0/2} u_0^{\beta-1} d\tau \quad (87)$$

618 Substituting (87) into (81), we get the final expression for $\frac{\partial \mathcal{I}}{\partial \tau}(0, \tilde{g})$,

$$\frac{\partial \mathcal{I}}{\partial \gamma}(0, \tilde{g}) = -2\beta \left[\int_0^{T_0/2} u_0^{\beta-1} \left(\frac{\partial u_0}{\partial \tau} \right)^2 d\tau \right] \left[\int_0^{T_0/2} u_0^{\beta-1} d\tau \right] = -\beta \mathcal{I}(0, \tilde{g}) \int_0^{\tau_1} u_0^{\beta-1} d\tau \quad (88)$$

619 Now, we can substitute (88) into (35) to obtain an expression for e^2 ,

$$e^2 = 1 - 2\beta \mathcal{I}(0, \tilde{g}) \gamma + 2\beta^2 \mathcal{I}(0, \tilde{g}) \mathcal{Q}(0, \tilde{g}) \gamma^2 + \mathcal{O}(\gamma^3) \quad (89)$$

with,

$$\mathcal{Q}(0, \tilde{g}) = \int_0^{T_0/2} u_0^{\beta-1} d\tau$$

620 Using (89) we can obtain the value of CoR as $e = \sqrt{e_+^2}$, where $e_+^2 = \max(0, e^2)$. Note that the evaluating
621 (89) can lead to $e^2 < 0$, therefore the square-root needs to be performed on e_+^2 . We may also obtain an
622 approximate expression for e , by using the expansion $(1 + \epsilon)^{\frac{1}{2}} = 1 + \frac{\epsilon}{2} - \frac{\epsilon^2}{8} + \mathcal{O}(\epsilon^3)$ on (89),

$$e = 1 - \beta \mathcal{I}(0, \tilde{g}) \gamma + \beta^2 \mathcal{I}(0, \tilde{g}) \left[\mathcal{Q}(0, \tilde{g}) - \frac{1}{2} \mathcal{I}(0, \tilde{g}) \right] \gamma^2 + \mathcal{O}(\gamma^3) \quad (90)$$

623 Hence, we have now obtained two approximations for the CoR. The first one depends on computing e^2 using
624 (89) followed by evaluating the CoR as $e = \sqrt{e_+^2}$, where $e_+^2 = \max(0, e^2)$. The second approximation is
625 based on the linearization of e^2 obtained from (89), and is given by (90). Both of these approximations
626 depend on the integral quantities $\mathcal{I}(\tilde{g})$ and $\mathcal{Q}(\tilde{g})$ that do not have explicit solutions when $\tilde{g} \neq 0$. One needs
627 further approximations for the integrals $\mathcal{I}(\tilde{g})$ and $\mathcal{Q}(\tilde{g})$ to be able to analytically evaluate the CoR for a
628 given impact. Appendix C presents further approximations of $\mathcal{I}(\tilde{g})$ and $\mathcal{Q}(\tilde{g})$ that will help us analytically
629 evaluate the value for e . However before proceeding further, since we have obtained two approximations for e
630 in this section, it is important to study the validity of these approximations. Figure 22 presents comparisons
631 for e using the two approximations in (89) and (35) with the *Kuwabara-Kono model* ($\alpha = \beta = 3/2$) for
632 $\tilde{g} = 0, 0.05, 1$, and 10. The plots labeled *Num. Integ.* represent the values of e computed via the numerical
633 integration of (4), whereas the plots labeled $e = \sqrt{e_+^2}$, $\mathcal{O}(\gamma^2)$ (*Num. Coeffs.*) and *Lin. e*, $\mathcal{O}(\gamma^2)$ (*Num.*
634 *Coeffs.*) are the approximations of e using (89) and (35) respectively, with the integrals $\mathcal{I}(\tilde{g})$ and $\mathcal{Q}(\tilde{g})$
635 computed via numerical integration. Figure 22(a) corresponding to the case when $\tilde{g} = 0$, has the plots
636 labeled $e = \sqrt{e_+^2}$, $\mathcal{O}(\gamma^2)$ (*Exact Coeffs.*) and *Lin. e*, $\mathcal{O}(\gamma^2)$ (*Exact Coeffs.*) instead, denoting that the
637 integrals $\mathcal{I}(\tilde{g} = 0)$ and $\mathcal{Q}(\tilde{g} = 0)$ can be computed exactly in terms of Euler's Beta function, as shown in (39)
638 and (40). The Fig. 22 also includes the results obtained using the Schwager-Pöschel based approximations,
639 which are labeled as *SP Coeff.*. The absolute errors of all approximations in Fig. 22 with respect to the
640 reference of the values obtained via numerical integration of (4), are reported in Fig. 23.

641 Examining the results in Fig. 22 and Fig. 23, we can firstly observe that all approximations work well
642 when both γ and \tilde{g} are small. However, as we had observed earlier, the approximation using the Schwager-
643 Pöschel coefficient doesn't work well as γ and \tilde{g} are increased. Among the two approximations derived in
644 this section, it appears that while both approximation appear to work well for small \tilde{g} , as \tilde{g} value increase
645 the linearized approximation in (35) performs poorly compared to (89) for high γ values. This shows that
646 the approximation based on $e = \sqrt{e_+^2}$ with (89) is best suited for high \tilde{g} . However, careful observation of the
647 absolute error in Fig. 23 also reveals that the linearized expansion in (35) is slightly more accurate than the
648 approximation using $e = \sqrt{e_+^2}$ for small values of \tilde{g} , as can be seen from Fig. 23(a) and Fig. 23(b). Hence,
649 based on these results we can conclude that both these approximations are valid within certain ranges of
650 values of γ and \tilde{g} .

651 We have seen that the linearized expression in (90) is more accurate when \tilde{g} is small, and contrarily
652 the approximation $e = \sqrt{e_+^2}$ is more accurate when \tilde{g} is large. Therefore, a condition may be used to
653 select between (89) and (90), depending upon the value of \tilde{g} and γ . In numerical computations we observe
654 that the two approximations tend to overestimate the CoR, hence it is interesting to choose the smallest
655 approximation.

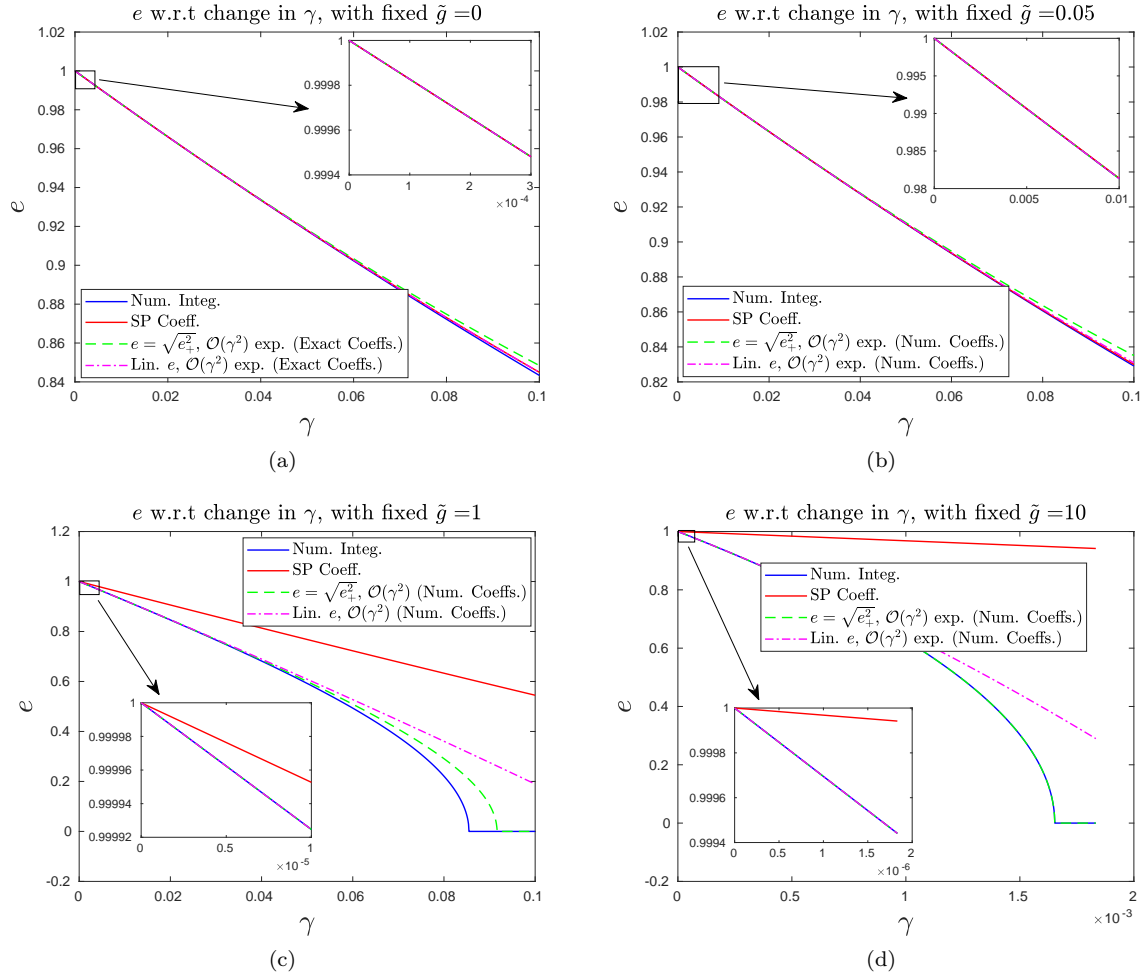


Figure 22: Comparison between the CoR values computed via numerical integration of (4), and the two CoR $\mathcal{O}(\gamma^2)$ approximations presented in (89) (labeled $e = \sqrt{e_+^2}, \mathcal{O}(\gamma^2) (Num. Coeffs.)$) and (90) (labeled $Lin. e, \mathcal{O}(\gamma^2) (Num. Coeffs.)$), with the functions $\mathcal{I}(\tilde{g})$ and $\mathcal{Q}(\tilde{g})$ computed via numerical integration of (41) and (42). In case of $\tilde{g} = 0$, the values of $\mathcal{I}(0)$ and $\mathcal{Q}(0)$ are explicitly calculated using (39) and (40), so the corresponding labels have been replaced by $e = \sqrt{e_+^2}, \mathcal{O}(\gamma^2) (Exact Coeffs.)$ and $Lin. e, \mathcal{O}(\gamma^2) (Exact Coeffs.)$. Results are shown for the visco-elastic model with $\alpha = \beta = 3/2$ (*Kuwabara-Kono model*), for the cases: (a) $\tilde{g} = 0$, (b) $\tilde{g} = 0.05$, (c) $\tilde{g} = 1$, and (d) $\tilde{g} = 10$. Additionally the Schwager-Pöschel coefficient based $\mathcal{O}(\gamma^2)$ approximation [using (30)] results are also shown.

C Approximation of $J_{\alpha,\beta}(\theta)$ and $M_{\alpha,\beta}(\theta)$

656

657 The goal of the development presented in this appendix is to find approximations for the functions $J_{\alpha,\beta}(\theta)$ and
658 $M_{\alpha,\beta}(\theta)$, which are related to the the integral quantities $\mathcal{I}(\tilde{g}(\theta)) = K(\theta)J_{\alpha,\beta}(\theta)$ and $\mathcal{Q}(\tilde{g}(\theta)) = L(\theta)M_{\alpha,\beta}(\theta)$,
659 which are required for the $\mathcal{O}(\gamma^2)$ expansion of CoR in (36). The parameter θ for these functions maps the
660 value of \tilde{g} such that: $\theta = 1 \implies \tilde{g} = 0$ and $\theta \rightarrow 0 \implies \tilde{g} \rightarrow \infty$. The functions $J_{\alpha,\beta}(\theta)$ and $M_{\alpha,\beta}(\theta)$
661 themselves are integrals, and are given by,

$$J_{\alpha,\beta}(\theta) = \int_0^1 [\theta + (1-\theta)x - x^{\alpha+1}]^{\frac{1}{2}} x^{\beta-1} dx \quad \text{and} \quad M_{\alpha,\beta}(\theta) = \int_0^1 [\theta + (1-\theta)x - x^{\alpha+1}]^{-\frac{1}{2}} x^{\beta-1} dx \quad (91)$$

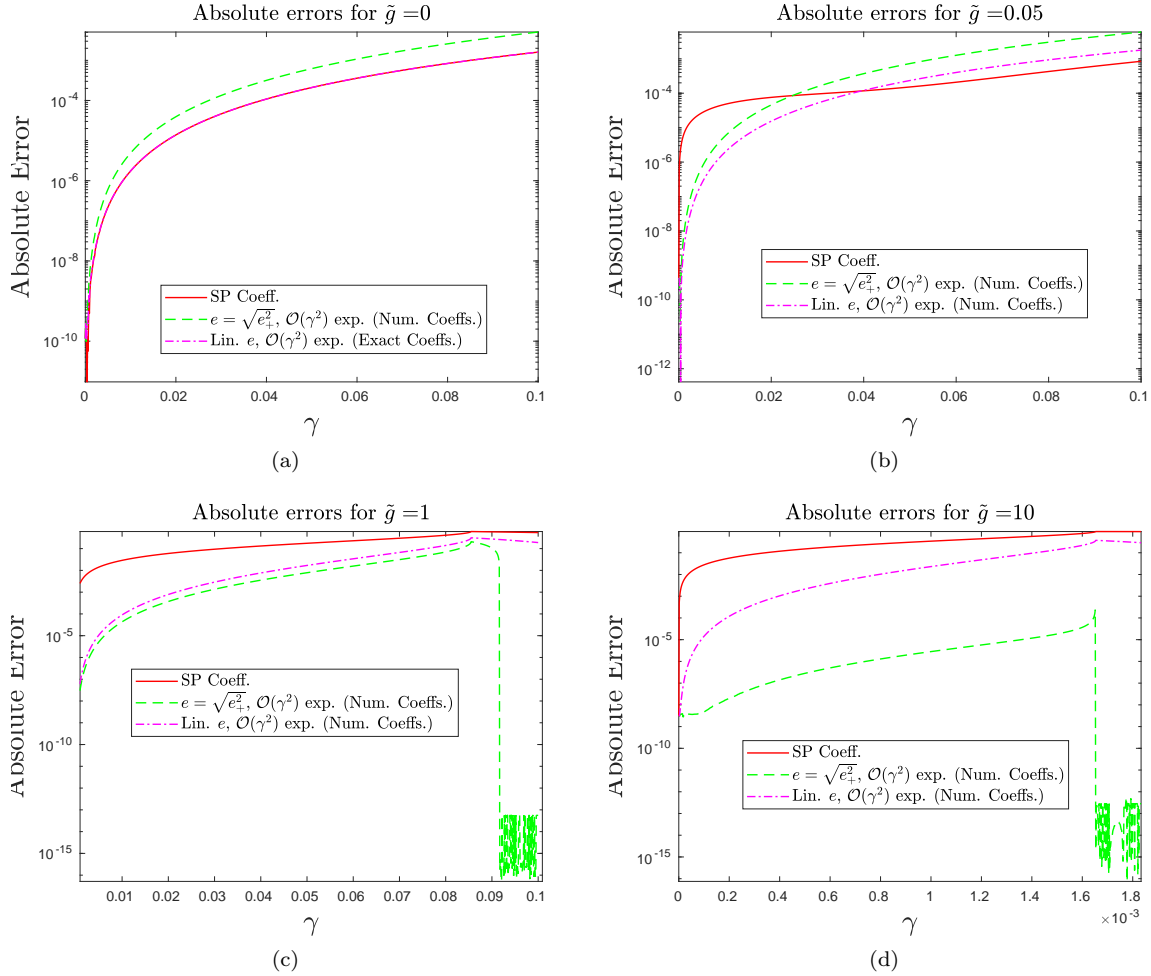


Figure 23: Absolute errors of the two $\mathcal{O}(\gamma^2)$ CoR approximations given in (89) (labeled $e = \sqrt{e_+^2}$, $\mathcal{O}(\gamma^2)$ (Num. Coeffs.)) and (90) (labeled *Lin. e*, $\mathcal{O}(\gamma^2)$ (Num. Coeffs.)), with the functions $\mathcal{I}(\tilde{g})$ and $\mathcal{Q}(\tilde{g})$ computed via numerical integration of (41) and (42). In case of $\tilde{g} = 0$, the values of $\mathcal{I}(0)$ and $\mathcal{Q}(0)$ are explicitly calculated using (39) and (40), so the corresponding labels have been replaced by $e = \sqrt{e_+^2}$, $\mathcal{O}(\gamma^2)$ (Exact Coeffs.) and *Lin. e*, $\mathcal{O}(\gamma^2)$ (Exact Coeffs.). The absolute errors are calculated in reference to CoR values computed through numerical integration of (4). Results are shown for the visco-elastic model with $\alpha = \beta = 3/2$ (*Kuwabara-Kono model*), for the cases: (a) $\tilde{g} = 0$, (b) $\tilde{g} = 0.05$, (c) $\tilde{g} = 1$, and (d) $\tilde{g} = 10$. The absolute errors of Schwager-Pöschel coefficient based $\mathcal{O}(\gamma^2)$ approximation [using (30)] with respect to the numerically integrated CoRs are also included for comparison.

To approximate the integrals $J_{\alpha,\beta}(\theta)$ and $M_{\alpha,\beta}(\theta)$, we will follow the approach of finding equivalent approximations for the integrands of these functions that can be integrated in closed-form. However, observing the structure of the two integrals $J_{\alpha,\beta}(\theta)$ and $M_{\alpha,\beta}(\theta)$, it appears that they can be related to one another. If we multiply $M_{\alpha,\beta}$ with $1 - \theta$, we obtain the relationship,

$$\begin{aligned}
 (1 - \theta)M_{\alpha,\beta}(\theta) &= \int_0^1 [\theta + (1 - \theta)x - x^{\alpha+1}]^{-\frac{1}{2}} (1 - \theta)x^{\beta-1} dx \\
 \implies (1 - \theta)M_{\alpha,\beta}(\theta) &= \int_0^1 [\theta + (1 - \theta)x - x^{\alpha+1}]^{-\frac{1}{2}} [(1 - \theta) - (\alpha + 1)x^\alpha + (\alpha + 1)x^{\alpha+1}] x^{\beta-1} dx
 \end{aligned}$$

662 which yields the relationship,

$$(1 - \theta)M_{\alpha,\beta}(\theta) = -2(\beta - 1)J_{\alpha,\beta-1}(\theta) + (\alpha + 1)M_{\alpha,\alpha+\beta}(\theta) \quad (92)$$

663 Thus, through a substitution of $\beta = \beta + 1$ in the relationship (92), we can obtain a definition of the integral
664 $J_{\alpha,\beta}$ in terms of the integral $M_{\alpha,\beta}(\theta)$,

$$J_{\alpha,\beta}(\theta) = \frac{\alpha + 1}{2\beta}M_{\alpha,\alpha+\beta+1}(\theta) + \frac{\theta - 1}{2\beta}M_{\alpha,\beta+1}(\theta) \quad (93)$$

665 Hence, we only need to find the approximation for $M_{\alpha,\beta}(\theta)$ to obtain a complete solution. Let us rewrite
666 the integral $M_{\alpha,\beta}(\theta)$ as,

$$M_{\alpha,\beta}(\theta) = \int_0^1 f_{\alpha,\theta}(x)x^{\beta-1}dx \quad (94)$$

667 where,

$$f_{\alpha,\theta}(x) = [\theta + (1 - \theta)x - x^{\alpha+1}]^{-\frac{1}{2}} \quad (95)$$

668 Using Definition 1, the values of $M_{\alpha,\beta}(\theta)$ at $\theta = 0$ and $\theta = 1$, can be directly evaluated as,

$$\begin{aligned} M_{\alpha,\beta}(0) &= \int_0^1 (1 - x^\alpha)^{-1/2} x^{\beta-3/2} dx = \frac{1}{\alpha} B\left(\frac{\beta - 1/2}{\alpha}, \frac{1}{2}\right) \\ M_{\alpha,\beta}(1) &= \int_0^1 (1 - x^{\alpha+1})^{-1/2} x^{\beta-1} dx = \frac{1}{\alpha + 1} B\left(\frac{\beta}{\alpha + 1}, \frac{1}{2}\right) \end{aligned} \quad (96)$$

669 Based on the integrand evaluations at the end-points, we can approximate the function $f_{\alpha,\theta}(x)$ in the integral
670 as,

$$f_{\alpha,\theta}(x) = [\theta + (1 - \theta)x - x^{\alpha+1}]^{-\frac{1}{2}} \simeq (x^{1-\theta} - x^{\alpha+1})^{-1/2} p_n(x) \quad (97)$$

671 where $p_n(x)$ is a polynomial of degree n , such that $p_n(x) = 1$ when $\theta = 0$ and $\theta = 1$ to satisfy the end-point
672 integral expressions of $M_{\alpha,\beta}(0)$ and $M_{\alpha,\beta}(1)$. Now to obtain the desired polynomial $p_n(x)$, we rewrite (97)
673 as,

$$h_{\alpha,\theta}(x) = \left(\frac{x^{1-\theta} - x^{\alpha+1}}{\theta + (1 - \theta)x - x^{\alpha+1}} \right)^{\frac{1}{2}} \simeq p_n(x) \quad (98)$$

674 Hence, our goal now is to determine the polynomial $p_n(x)$ that approximates $h_{\alpha,\theta}(x)$ when $\theta \in [0, 1]$. We
675 can achieve by interpolating the function $h_{\alpha,\theta}(x)$ at a number of interpolation points in $x \in [0, 1]$. However,
676 in order to approximate the function $h_{\alpha,\theta}(x)$ with a polynomial, a finer mesh of interpolation points are
677 required near $x \rightarrow 0$, compared to $x \rightarrow 1$. Thus, we will use Chebyshev nodes for this interpolation. For this
678 purpose, we will first symmetrize the function over the domain $x \in [0, 2]$, by defining,

$$\tilde{h}_{\alpha,\theta}(x) = \begin{cases} h_{\alpha,\theta}(x) & \text{if } x \in [0, 1] \\ h_{\alpha,\theta}(2 - x) & \text{if } x \in [1, 2] \end{cases} \quad (99)$$

679 The Chebyshev nodes over the range $[0, 2]$ are defined as,

$$x_k = \cos\left(\pi \frac{2k + 1}{2n + 2}\right) + 1, \quad k \in \mathbb{N} \quad (100)$$

680 Thus, the interpolation points for the polynomial $p_n(x)$ over the domain $[0, 1]$ are defined as $(x_k, \tilde{h}_{\alpha,\beta}(x_k))$,
681 with $k \in \mathbb{N}$ and $(n + 1)/2 \leq k \leq n$ (Assuming n to be an odd number). Our approach would be to define
682 the polynomial $p_n(x)$ using the Chebyshev polynomial basis, and then use some properties of Chebyshev
683 polynomials to obtain a series solution for $M_{\alpha,\beta}(\theta)$. In this approach, we define the polynomial $p_n(x)$ as,

$$p_n(x) = \sum_{j=0}^n c_j T_j(x - 1) \quad (101)$$

684 where $T_j(y) = \cos(j \arccos y)$, for $y \in [-1, 1]$ are the Chebyshev polynomials of the first kind. Next using
 685 the orthogonality property of Chebyshev polynomials the coefficients c_j can be solved as,

$$\begin{cases} c_0 &= \frac{1}{n+1} \sum_{k=0}^n \tilde{h}_{\alpha,\theta}(x_k) \\ c_j &= \frac{2}{n+1} \sum_{k=0}^n \tilde{h}_{\alpha,\theta}(x_k) \cos\left(j\pi \frac{2k+1}{2n+2}\right), \quad 1 \leq j \leq n \end{cases} \quad (102)$$

686 Assuming n odd, and using the symmetry of $\tilde{h}_{\alpha,\theta}(x)$ (such that, $x_{n-k} = 2 - x_k$ and $\tilde{h}_{\alpha,\theta}(x_{n-k}) = \tilde{h}_{\alpha,\theta}(x_k)$),
 687 the coefficients may be collected as,

$$\begin{cases} c_0 &= \frac{2}{n+1} \sum_{k=(n+1)/2}^n h_{\alpha,\theta}(x_k) \\ c_j &= \frac{4}{n+1} \sum_{k=(n+1)/2}^n h_{\alpha,\theta}(x_k) \cos\left(j\pi \frac{2k+1}{2n+2}\right), \quad \text{if } j \geq 2 \text{ and is even} \\ c_j &= 0, \quad \text{if } j \geq 2 \text{ and is odd} \end{cases} \quad (103)$$

688 Thus, with the values of c_j we can approximate $h_{\alpha,\beta}(x) \simeq p_n(x)$. The order n used for the polynomial
 689 approximation is important. Figures 24 and 25 show the approximations $h_{\alpha,\beta}(x) \simeq p_n(x)$ and their errors
 690 for different values of the polynomial order n . As we can note that small values of n lead to larger aggregate
 691 errors, whereas large n correspond to small aggregate errors but large local errors between interpolation
 692 points. Thus an intermediate value of n is appropriate. Based on the results in Figures 24 and 25 we can
 693 conclude that $n = 21$ gives the best overall approximation results. Therefore, we choose $n = 21$ for the
 remaining of the computations.

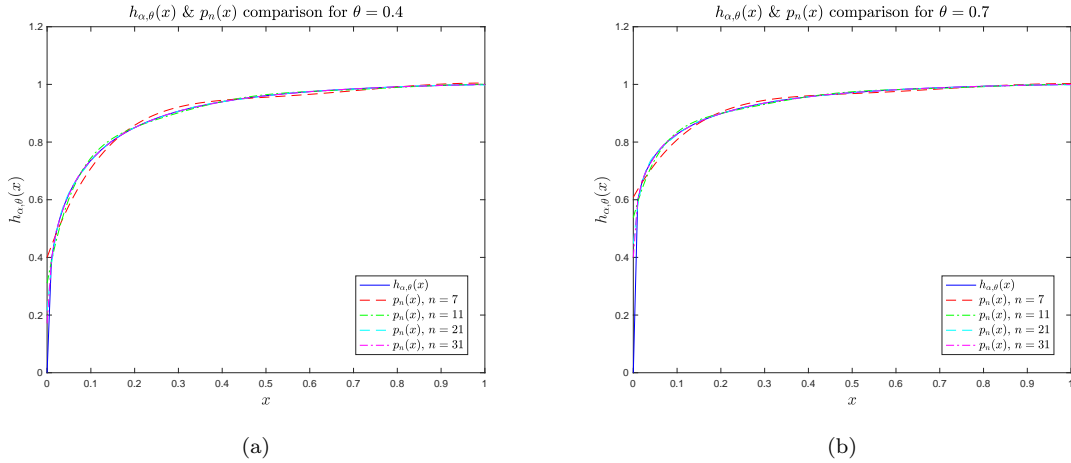


Figure 24: Polynomial approximation of $h_{\alpha,\theta}(x)$, (a) $\theta = 0.4$ and (b) $\theta = 0.7$

694 Now having determined the coefficients c_j , we are now ready to derive the approximation for the integral
 695 $M_{\alpha,\beta}(\theta)$,
 696

$$M_{\alpha,\beta}(\theta) = \int_0^1 h_{\alpha,\beta}(\theta) (x^{1-\theta} - x^{\alpha+1})^{-\frac{1}{2}} x^{\beta-1} dx \simeq \int_0^1 p_n(x) (1 - x^{\alpha+\theta})^{-\frac{1}{2}} x^{\beta+\theta/2-3/2} dx \quad (104)$$

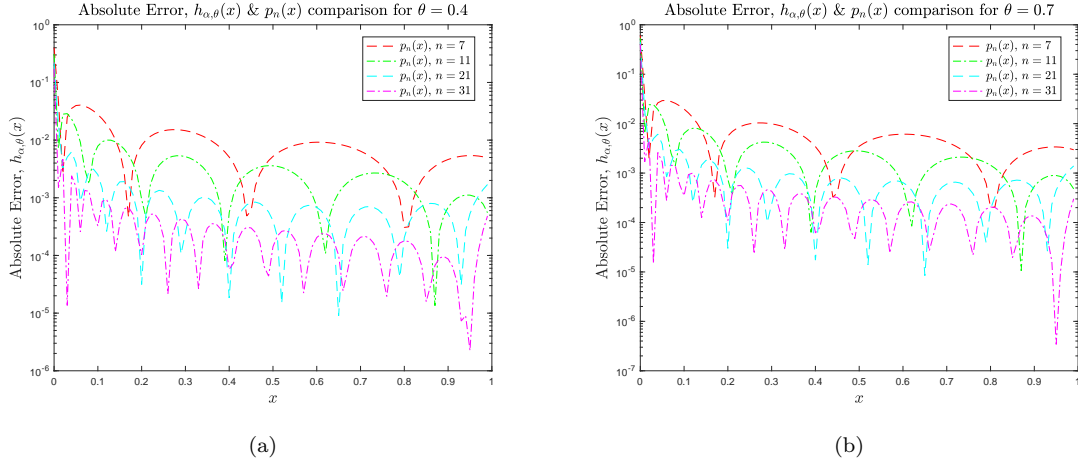


Figure 25: Errors in Polynomial approximation of $h_{\alpha,\theta}(x)$, (a) $\theta = 0.4$ and (b) $\theta = 0.7$

697 Thus, using the definition of $p_n(x)$ from (101), we may write the approximation of $M_{\alpha,\beta}(\theta)$ as,

$$M_{\alpha,\beta}(\theta) \simeq \sum_{j=0}^{n-1} c_j a_{j,0}, \quad \text{with, } j \text{ even} \quad (105)$$

where, $a_{j,k} = \int_0^1 T_j(x-1)(1-x^{\alpha+\theta})^{-\frac{1}{2}} x^{\beta+\theta/2-3/2+k} dx$

698 Now we can use the recurrence relation properties on the Chebyshev polynomials of the first kind $T_j(x)$ to
 699 derive a sequential solution for $a_{j,k}$. The recurrence relation for the Chebyshev polynomials of the first kind
 700 is given by,

$$T_{j+2} = 2xT_{j+1}(x) - T_j(x) \quad \text{for } j \geq 0 \quad (106)$$

with, $T_0(x) = 1$ and $T_1(x) = x$

701 Therefore, the recurrence relation of the Chebyshev polynomials in (106) can be used to yield a sequential
 702 solution for $a_{j,k}$, when $j \geq 0$

$$a_{j+2,k} = \int_0^1 2(x-1)T_{j+1}(1-x)^{-1/2} x^{\beta+\theta/2-3/2+k} dx - a_{j,k} \quad (107)$$

$$\implies a_{j+2,k} = 2a_{j+1,k+1} - 2a_{j+1,k} - a_{j,k}$$

703 Similarly, due to the the first two Chebyshev polynomials $T_1(x) = x$ and $T_0(x) = 1$, we obtain the values,

$$a_{0,k} = \frac{1}{\alpha + \theta} B\left(\frac{\beta + k - 1/2 + \theta/2}{\alpha + \theta}, \frac{1}{2}\right) \quad \text{and} \quad a_{1,k} = a_{0,k+1} - a_{0,k} \quad (108)$$

704 Thus, the quantities $a_{j,k}$ can be determined recurrently. After obtaining all values of $a_{j,k}$, one can use
 705 the $a_{j,0}$ values and the c_j values to obtain an approximation for $M_{\alpha,\beta}(\theta)$. Again, using the relation (48),
 706 $J_{\alpha,\beta}(\theta)$ can also be evaluated by computing $M_{\alpha,\alpha+\beta+1}(\theta)$ and $M_{\alpha,\beta+1}(\theta)$. Thus, both $J_{\alpha,\beta}(\theta)$ and $M_{\alpha,\beta}(\theta)$
 707 approximations are now known and can be easily computed based on the aforementioned recurrence relations.
 708 Also, having obtained these approximations the functions $\mathcal{I}(\tilde{g}(\theta))$ and $\mathcal{Q}(\tilde{g}(\theta))$ can be easily computed as
 709 well.

710 We can now check the validity of these approximations by comparing them against numerical integration
 711 results of $J_{\alpha,\beta}(\theta)$, $M_{\alpha,\beta}(\theta)$, $\mathcal{I}(g(\theta))$, and $\mathcal{Q}(g(\theta))$. Figure 26 shows the comparison of the approximated and
 712 numerically integrated values of $J_{\alpha,\beta}(\theta)$ and $\mathcal{I}(\tilde{g})$. Figure 26(a) compares the approximated and numerically
 713 integrated values of $J_{\alpha,\beta}(\theta)$ for the specific parameter values of $\alpha = \beta = 3/2$ (corresponding to Kuwabara-
 714 Kono model), and the relative errors for the same is shown in Fig. 26(b). The integral $\mathcal{I}(\tilde{g}(\theta))$ based on the

715 computation of $J_{\alpha,\beta}(\theta)$ is shown in Fig. 26(c), and their relative errors shown in Fig. 26(d). Lastly, Fig. 26(e)
 716 and Fig. 26(d) are showing the maximum relative error of $J_{\alpha,\beta}(\theta)$ and $\mathcal{I}(\tilde{g}(\theta))$ within $\theta \in [0, 1]$, for all values
 717 of $\beta \in [0, 4]$, while keeping α fixed. We note that the maximum value of the relative errors is in the order of
 718 10^{-4} for both $J_{\alpha,\beta}(\theta)$ and $\mathcal{I}(\tilde{g}(\theta))$.

719 Similarly, Fig. 27 compares the approximated and the numerically integrated results of $M_{\alpha,\beta}(\theta)$ and
 720 $\mathcal{Q}(\tilde{g}(\theta))$. Figure 27(a) and Fig. 27(b) show the result comparisons of $M_{\alpha,\beta}(\theta)$ for the Kuwabara-Kono model
 721 ($\alpha = \beta = 3/2$). Figure 27(c) and Fig. 27(d) compares the approximated and numerically integrated results
 722 of $\mathcal{Q}(\tilde{g}(\theta))$. Lastly, Fig.27(e) and Fig.27(f) show the maximum relative errors of $M_{\alpha,\beta}(\theta)$ and $\mathcal{I}(\tilde{g}(\theta))$, as
 723 β is varied in the range $\beta \in [0, 4]$. The maximum relative error in these computation is in the order of
 724 10^{-3} . Therefore, we can conclude that the approximations are in good agreement with numerical integration
 725 computation results. Hence, we can expect to see good analytical CoR approximation results, if we use the
 726 $J_{\alpha,\beta}(\theta)$ and $M_{\alpha,\beta}(\theta)$ integral approximations that we have derived in this section.

727 D Comparison with known analytical solutions for specific values 728 of α and β

729 In this final appendix, we will study how well the $\mathcal{O}(\gamma^2)$ CoR approximation presented in this work performs
 730 in comparison to known analytical solutions of e for specific values of α and β . Two of the known solutions
 731 of e correspond to the *Linear Spring Dashpot* ($\alpha = \beta = 1$) and the *Tsuji-Tanaka-Ishida* ($\alpha = 3/2$ and
 732 $\beta = 5/4$). In both cases the solution is known exactly only when $\tilde{g} = 0$. Nevertheless, in the case of the
 733 *Linear Spring Dashpot* with $\tilde{g} \neq 0$, one can use a mixed approach by combining and numerical root-finding
 734 approach to obtain e accurate to arbitrary precision. Next, we will derive this (partially) analytical solution
 735 for the *Linear Spring Dashpot* model to compare with the predictions from the $\mathcal{O}(\gamma^2)$ CoR approximation
 736 presented in Proposition 2.

737 D.1 Liner Spring-Dashpot Model, $\alpha = \beta = 1$

738 We consider the scaled dynamics given in (2) with $\alpha = \beta = 1$,

$$\frac{d^2 u}{d\tau^2} + \gamma \frac{d}{d\tau}(u_+) + u_+ = \tilde{g}; \quad u(0) = 0 \quad \text{and} \quad u'(0) = 1 \quad (109)$$

739 The linear non-homogeneous linear ordinary differential equation that governs the dynamics can be solved
 740 exactly to yield solutions for the displacement $u(\tau)$ and velocity $u'(\tau)$ (assuming an underdamped response,
 741 $0 \leq \gamma \leq 2$),

$$\begin{aligned} u(\tau) &= -\tilde{g}e^{-\xi\tau} \cos(\omega_n\tau) + \left(\frac{1-\xi\tilde{g}}{\omega_n}\right) e^{-\xi\tau} + \tilde{g} \\ u'(\tau) &= e^{-\xi\tau} \cos(\omega_n\tau) + \left[\omega_n\tilde{g} - \frac{\xi(1-\xi\tilde{g})}{\omega_n}\right] e^{-\xi\tau} \sin(\omega_n\tau) \end{aligned} \quad (110)$$

where,

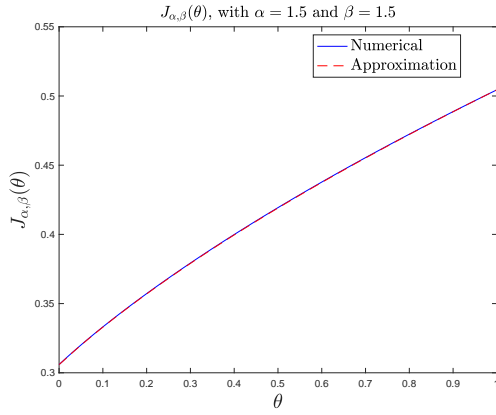
$$\xi = \frac{\gamma}{2} \quad \text{and} \quad \omega_n = \frac{\sqrt{4-\gamma^2}}{2},$$

742 The CoR e , based on the linear scaled spring-dashpot dynamics from (109) is given by,

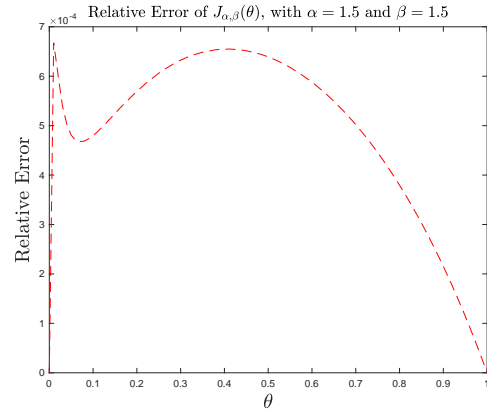
$$e = -\frac{u'(T_f)}{u'(0)} = -u'(T_f) \quad (111)$$

743 where T_f is the final time of impact. The impact ends when $u(\tau = T_f) = 0$, after compression. If there is no
 744 external load on the system, $\tilde{g} = 0$, the solutions for the scaled displacement $u(\tau)|_{\tilde{g}=0} \equiv u_{lh}(\tau)$, and velocity
 745 $u'(\tau)|_{\tilde{g}=0} \equiv u'_{lh}(\tau)$, are given by,

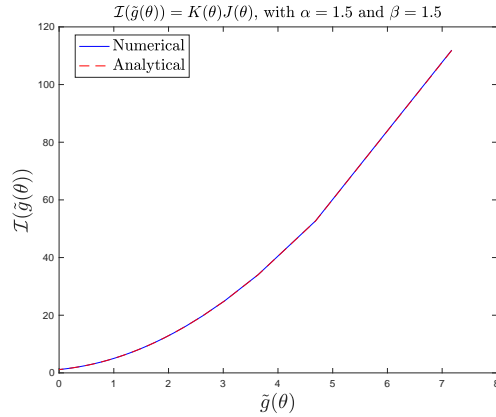
$$\begin{aligned} u_{lh}(\tau) &= \frac{1}{\omega_n} e^{-\xi\tau} \sin(\omega_n\tau) \\ u'_{lh}(\tau) &= e^{-\xi\tau} \cos(\omega_n\tau) - \frac{\xi}{\omega_n} e^{-\xi\tau} \sin(\omega_n\tau) \end{aligned} \quad (112)$$



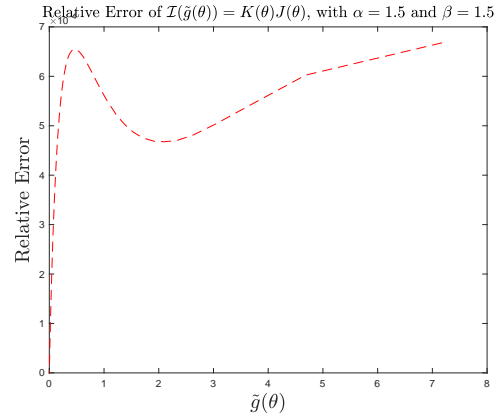
(a)



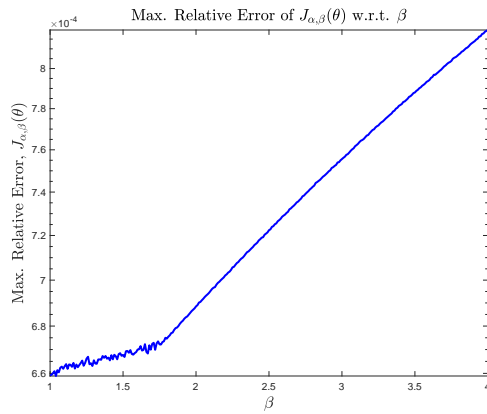
(b)



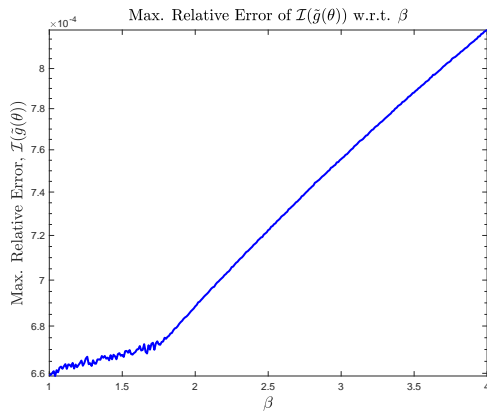
(c)



(d)



(e)



(f)

Figure 26: (a) Comparison between numerical and analytical approximation of $J_{\alpha,\beta}(\theta)$ when, $\alpha = \beta = 3/2$, (b) Relative error of $J_{\alpha,\beta}(\theta)$ approximation when, $\alpha = \beta = 3/2$, (c) Comparison between numerical and analytical approximation of $\mathcal{I}(\tilde{g}(\theta))$, when $\alpha = \beta = 3/2$, (d) Relative error of $\mathcal{I}(\tilde{g}(\theta))$ $\alpha = \beta = 3/2$, (e) Maximum relative error of $J_{\alpha,\beta}(\theta)$ approximation for $\alpha = 3/2$ & $\beta \in [1, 4]$, and (f) Maximum relative error of $\mathcal{I}(\tilde{g}(\theta))$ approximation for $\alpha = 3/2$ & $\beta \in [1, 4]$

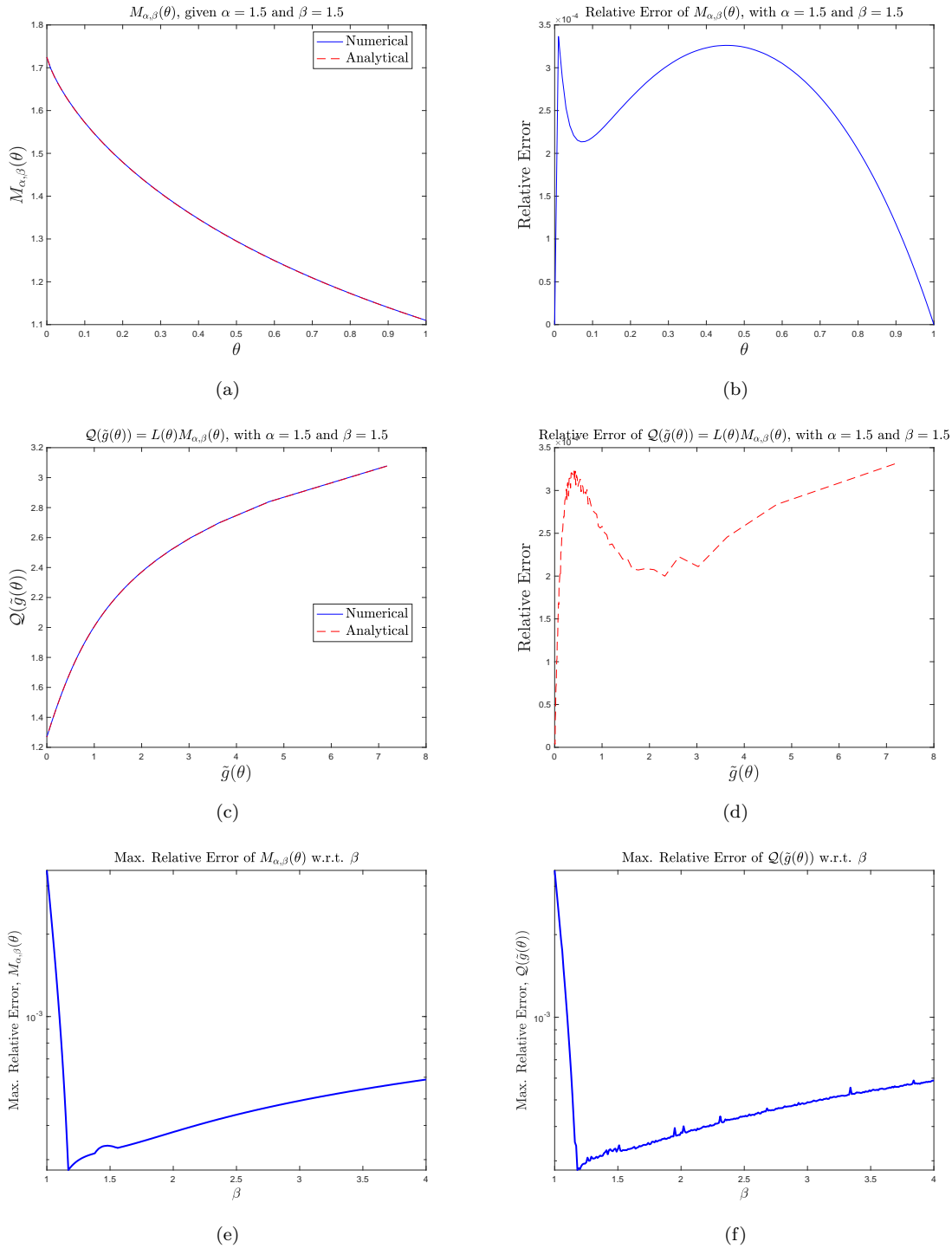


Figure 27: (a) Comparison between numerical and analytical approximation of $M_{\alpha,\beta}(\theta)$ when, $\alpha = \beta = 3/2$, (b) Relative error of $M_{\alpha,\beta}(\theta)$ approximation when, $\alpha = \beta = 3/2$, (c) Comparison between numerical and analytical approximation of $Q(\tilde{g}(\theta))$, when $\alpha = \beta = 3/2$, (d) Relative error of $Q(\tilde{g}(\theta))$ $\alpha = \beta = 3/2$, (e) Maximum relative error of $M_{\alpha,\beta}(\theta)$ approximation for $\alpha = 3/2$ & $\beta \in [1, 4]$, and (f) Maximum relative error of $Q(\tilde{g}(\theta))$ approximation for $\alpha = 3/2$ & $\beta \in [1, 4]$

746 The time of collision T_f , for the case when $\tilde{g} = 0$, such that $u_{lh}(\tau = T_f) = 0$, can be explicitly calculated as
 747 $T_f = \frac{\pi}{\omega_n}$, which also yields the explicit solution for the CoR given by,

$$e = \exp\left(\frac{-\xi\pi}{\omega_n}\right) \quad (113)$$

748 When $\tilde{g} \neq 0$, the final time of impact T_f , cannot be explicitly solved for using $u(\tau = T_f) = 0$, from (110).
 749 Instead, we may proceed by first an approximate value of T_f that is close to the true solution of $u(\tau) = 0$,
 750 and then use this approximate value as initial guess for Newton's method to obtain a more accurate value
 751 of T_f . The approximate value of T_f close to the solution of $u(\tau) = 0$ can be found by using the symmetric
 752 property of the first half-period of the non-dissipative $u(\tau)$ response (γ). The symmetric center of the
 753 first half-period, in case of a non-dissipative impact, is at the maximum displacement. The time τ at the
 754 maximum displacement can be found by obtaining the stationary points in the response of $u(\tau)$. The set of
 755 stationary points τ_{sk} with $k \in \mathbb{Z}$, can be solved for by setting $u'(\tau_{sk}) = 0$,

$$\begin{aligned} u'(\tau_{sk}) = 0 &= e^{-\xi\tau_{sk}} \cos(\omega_n\tau_{sk}) + \left(\frac{\omega_n^2\tilde{g}-\xi(1-\xi\tilde{g})}{\omega_n}\right) e^{-\xi\tau_{sk}} \sin(\omega_n\tau_{sk}) \\ \implies 0 &= e^{-\xi\tau_{sk}} \left(\frac{\omega_n^2+(\xi(1-\xi\tilde{g}))^2}{\omega_n}\right) \cos\left(\omega_n\tau_{sk} - \tan^{-1}\left(\frac{\omega_n^2\tilde{g}-\xi(1-\xi\tilde{g})}{\omega_n}\right)\right) \\ \implies \tau_{sk} &= \frac{1}{\omega_n} \left[\tan^{-1}\left(\frac{\omega_n^2\tilde{g}-\xi(1-\xi\tilde{g})}{\omega_n}\right) + \frac{2k+1}{2}\pi\right], \text{ for } k \in \mathbb{Z} \end{aligned} \quad (114)$$

756 The maximum displacement within the first half-period of the $u(\tau)$ response takes place at $k = 0$, so the time
 757 corresponding to the maximum displacement in the first half-period is τ_{s0} . Similarly, the stationary point in
 758 the second half-period (trough), is τ_{s1} . The stationary point within the second half-period is important as
 759 it could help us determine whether the impact will result in detachment ($0 < e \leq 1$) or sticking ($e = 0$). We
 760 can use the criteria,

$$\begin{cases} \text{if } u(\tau_{s1}) < 0 & e = -u'(T_f) \quad (\text{Detachment}) \\ \text{if } u(\tau_{s1}) \geq 0 & e = 0 \quad (\text{Sticking}) \end{cases} \quad (115)$$

761 to determine if the impact will result in detachment. Lastly, we need the impact time T_f to calculate the
 762 CoR, when there is detachment. Since, the first half-period is approximately symmetric about the maximum
 763 deformation, when γ is small, the final impact time can be approximated as $T_f \approx 2\tau_{s0}$. Nevertheless, this
 764 approximation becomes very inaccurate as γ becomes large. Thus we use Newton's method to obtain better
 765 approximation,

$$T_f^{i+1} = T_f^i - \frac{u(T_f^i)}{u'(T_f^i)} \quad \text{with, } T_f^0 = 2\tau_{s0} \quad (116)$$

766 where T_f^i is the value of T_f at an iteration i . Therefore, by combining Newton's method and the analytical
 767 solution of $u'(\tau)$ given in (110), one can obtain the value of e to an arbitrary precision.

768 Thus, now we can study the $O(\gamma^2)$ CoR approximation in reference to the aforementioned analytical
 769 solution of e for the *Linear Spring Dashpot* model. Figure 28 compares the two $O(\gamma^2)$ approximations and
 770 the numerically integrated CoR values to the analytical solution when $\alpha = 1$ and $\beta = 1$. Figures 28(a),
 771 28(b), 28(c), and 28(d) show the results corresponding to $\tilde{g} = 0, 0.05, 1, \text{ and } 10$. In these examples, the known
 772 analytical solution of the *Linear Spring Dashpot* model is treated as the reference, while the numerically
 773 integrated results are just included for completeness. Hence, the corresponding absolute errors shown in
 774 Fig. 29, are calculated with respect to the analytical solutions. We can see from these figures that the $O(\gamma^2)$
 775 approximations agrees very well with the known analytical solution for small and large \tilde{g} , while being slightly
 776 inaccurate for intermediate \tilde{g} when γ value is high. Similar to what we have observed for other cases, the
 777 $O(\gamma^2)$ approximation is less accurate for high γ and intermediate values of \tilde{g} . As discussed earlier, this is a
 778 consequence of the expansion of e around $\gamma \approx 0$ that was used to obtain the $O(\gamma^2)$ approximation in (36).
 779 Nevertheless as we can see, in this case magnitudes of these errors are very small.

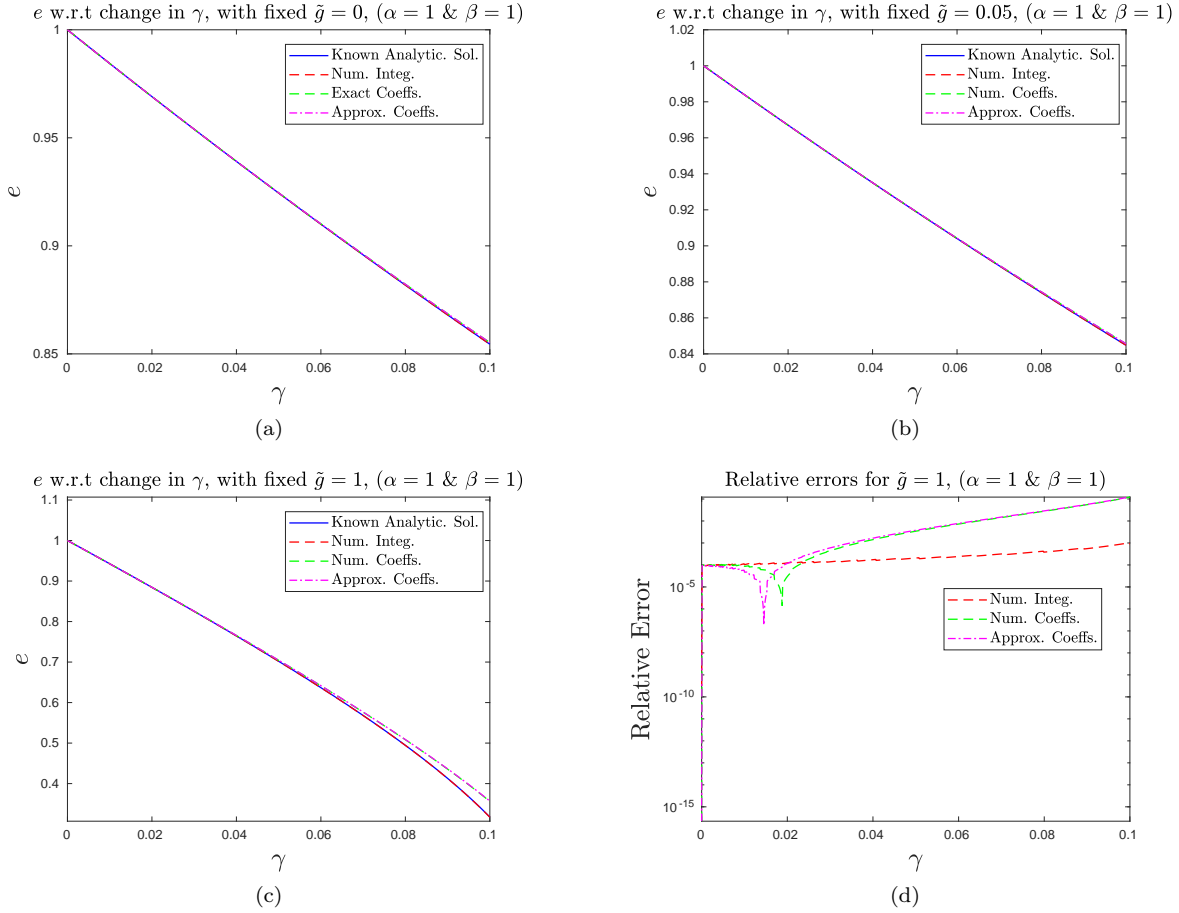


Figure 28: Comparison between the CoR values computed based on the known analytical solution for $\alpha = \beta = 1$ (*Linear Spring-Dashpot model*) given in (110), and the $\mathcal{O}(\gamma^2)$ approximations given by (36), using the numerically integrated functions $\mathcal{I}(\tilde{g})$ and $\mathcal{Q}(\tilde{g})$ from (41) and (42), and the approximated functions $\mathcal{I}(\tilde{g}(\theta)) = K(\theta)J_{\alpha,\beta}(\theta)$ and $\mathcal{Q}(\tilde{g}(\theta)) = L(\theta)M_{\alpha,\beta}(\theta)$ using (48) and (49). Results are shown for the cases: (a) $\tilde{g} = 0$, (b) $\tilde{g} = 0.05$, (c) $\tilde{g} = 1$, and (d) $\tilde{g} = 10$. The CoR results computed via numerical integration of (4) are also included for completeness.

D.2 Antypov-Elliott Solution for $\alpha = 3/2$ & $\beta = 5/4$

Another known analytical solution that exists in the literature, is for the case when $\alpha = 3/2$ & $\beta = 5/4$, which corresponds to the *Tsuji-Tanaka-Ishida* model [9]. Antypov and Elliott presented a mapping between the displacement solutions of *Tsuji-Tanaka-Ishida* model ($\alpha = 3/2$ & $\beta = 5/4$) [39] and *Linear Spring-Dashpot* model ($\alpha = \beta = 1$). However, this mapping is only valid for systems with no external load, which means $\tilde{g} = 0$. The mapping presented by Antypov and Elliot in [39], substitute the displacement δ into the equation of motion, by a function $\delta = Ax^n$. They find that by choosing the parameter values $A = (5/4)^{2/5}$ and $n = 4/5$, the equivalent phase-space equation becomes identical to the differential equation corresponding to linear spring-damper model, with no external load. This leads to a mapping of the solutions with the solutions of a linear-spring damper solution,

$$u(\tau) = \left(\frac{5}{4}\right)^{2/5} (u_{lh}(\tau))^{4/5} \quad \text{and} \quad u'(\tau) = u_{lh}(\tau) \quad (117)$$

where, $u(\tau)$ and $u'(\tau)$ are solutions to the displacement and velocity for the model corresponding to the parameter values $\alpha = 3/2$, $\beta = 5/4$, and $\tilde{g} = 0$, whereas $u_{lh}(\tau)$ and $u'_{lh}(\tau)$ are the velocity solutions for linear spring-damper model, with the parameter values $\alpha = \beta = 1$ and $\tilde{g} = 0$. It can be readily seen from

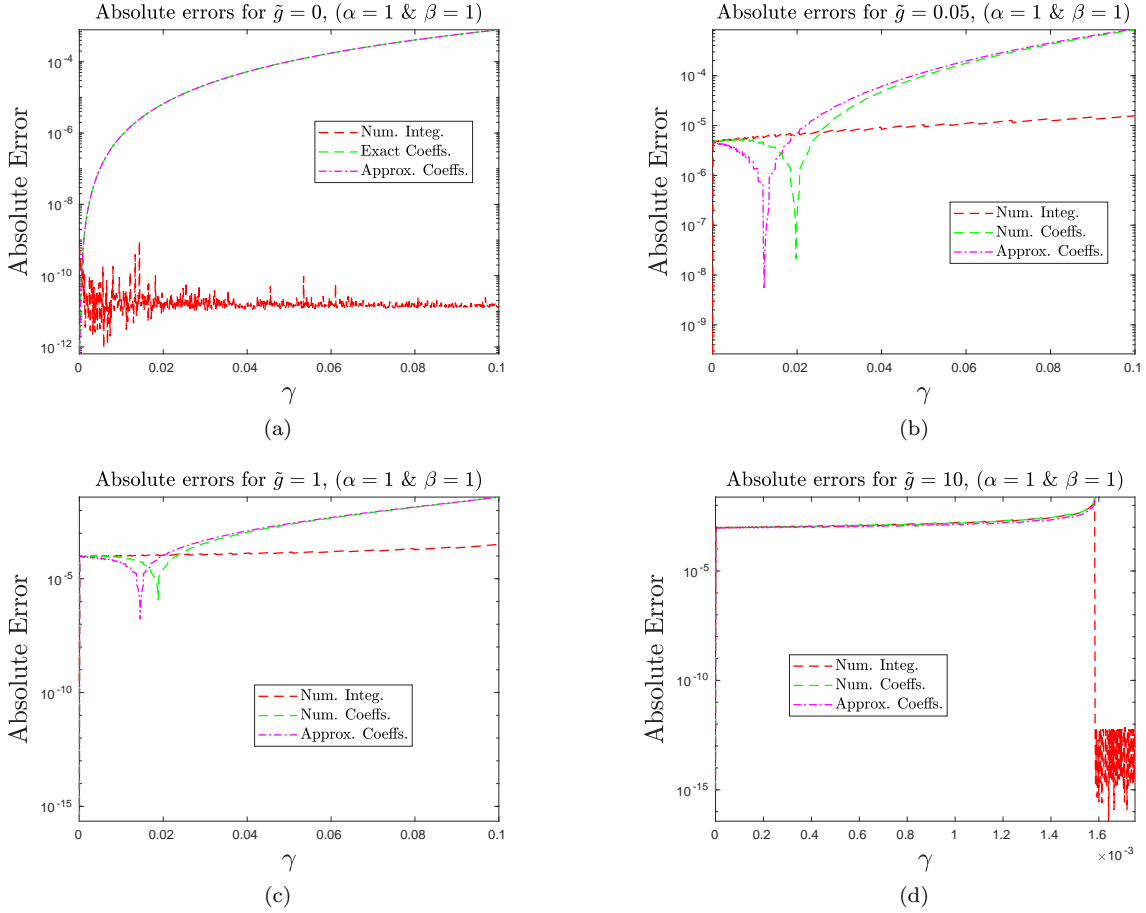


Figure 29: Absolute errors of the $\mathcal{O}(\gamma^2)$ CoR approximations (36), for the case $\alpha = \beta = 1$ (*Linear Spring-Dashpot model*), based on the numerically integrated functions $\mathcal{I}(\tilde{g})$ and $\mathcal{Q}(\tilde{g})$ from (41) and (42), and the approximated functions $\mathcal{I}(\tilde{g}(\theta)) = K(\theta)J_{\alpha,\beta}(\theta)$ and $\mathcal{Q}(\tilde{g}(\theta)) = L(\theta)M_{\alpha,\beta}(\theta)$ from (48) and (49). The absolute errors are computed in reference to the CoR values obtained based on the known analytical solution for the *Linear Spring-Dashpot model* in (110). Results are shown for the cases: (a) $\tilde{g} = 0$, (b) $\tilde{g} = 0.05$, (c) $\tilde{g} = 1$, and (d) $\tilde{g} = 10$. The absolute errors of the numerically integrated CoR results are also shown for completeness.

793 the mapping in (117), that the root for $u(\tau) = 0$, which gives the value of impact time T_f , is identical to
794 the roots of $u_{lh}(\tau)$. Therefore, the time of impact is given by $T_f = \frac{\pi}{\omega_n}$, and the coefficient of restitution is
795 identical to linear spring-damper model, i.e., $e = \exp(-\xi\pi/\omega_n)$.

796 Since, the mapping presented by Antypov and Elliott [39] is only valid for $\tilde{g} = 0$, we can only compare
797 the results of the $\mathcal{O}(\gamma^2)$ approximation from (36) with the analytical solution from (117), for the same case.
798 Figure 30(a) compares the two $\mathcal{O}(\gamma^2)$ approximation (with numerically integrated and approximate analytical
799 coefficients) with the analytical solution using the mapping (117). It also includes a plot corresponding to the
800 numerically integrated value of e . Once again here, the errors associated with these results are computed
801 in reference to the known exact analytical solution based on (117). These are presented in Fig. 30(b). As we
802 have seen before, the $\mathcal{O}(\gamma^2)$ CoR approximation tends to be very accurate when $\tilde{g} = 0$, and based on Fig. 30
803 we can confirm that the same is true for *Tsuji-Tanaka-Ishida* model with \tilde{g} when the $\mathcal{O}(\gamma^2)$ approximations
804 are compared against the exact known analytical solution based on [39].

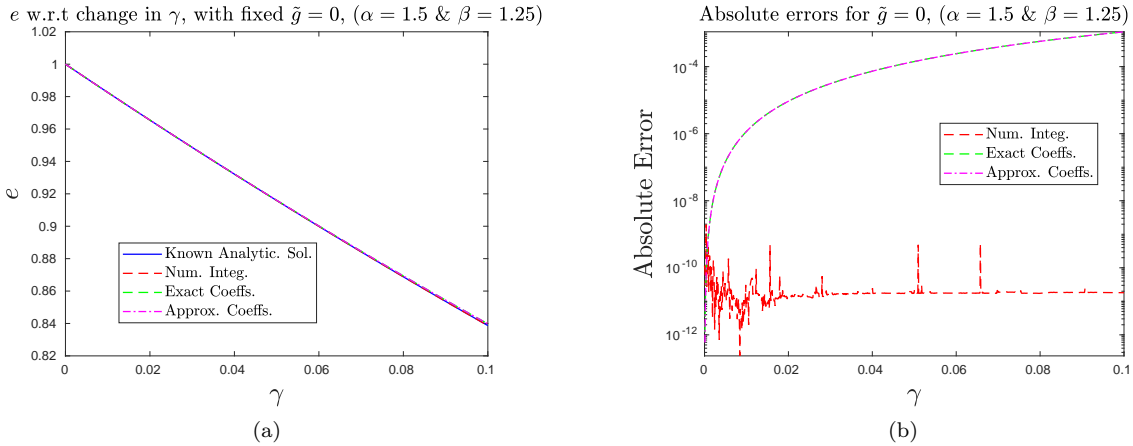


Figure 30: (a) Comparison of the $\mathcal{O}(\gamma^2)$ CoR approximations for the case $\alpha = 3/2$, $\beta = 5/4$ (*Tsuji-Tanaka-Ishida model*), and $\tilde{g} = 0$ (non-dissipative impact), with respect to the Antypov-Elliott [39] based analytical solution (117). (b) The corresponding absolute errors with respect to the Antypov-Elliott [39] based analytical solution (117).

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