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# Remarks on the Radiative Transfer Equations for Climatology

Claude Bardos<sup>1</sup>, François Golse<sup>2</sup> and Olivier Pironneau<sup>3</sup>

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## Abstract

Using theoretical and numerical arguments we discuss some of the commonly accepted approximations for the radiative transfer equations in climatology.

*Keywords:* Radiative transfer, Electronic Temperature , Integral equation, Numerical analysis

AMS 3510, 35Q35, 35Q85, 80A21, 80M10

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## 1. Introduction

Satellite, atmospheric and terrestrial measurements are numerous to support the global warming. Various hypothesis are made to explain the disastrous effect of the Green-House Gases (GHG). However a rigorous proof derived from the fundamental equations of physics is not available and one of the problems is the complexity of the physical system of planet Earth in its astronomical setting around the Sun.

In many references such as [8],[9],[2] and [4] it is explained that the Sun radiates light with a heat flux  $Q = 1370 \text{ Watt/m}^2$ , in the frequency range  $(0.5, 20) \times 10^{14} \text{ Hz}$  corresponding approximately to a black body at temperature of 5800K; 70% of this light intensity reaches the ground because the atmosphere is almost transparent to this spectrum and about 30% is reflected back by the clouds or the ocean, snow, etc (albedo). The Earth behaves almost like a black body at temperature  $T_e = 288 \text{ K}$  and as such radiates rays of frequencies  $\nu$  in the infrared spectrum  $(0.03, 0.6) \times 10^{14} \text{ Hz}$ .

“The absorption coefficient variation with frequencies is such that the atmosphere is essentially transparent to solar radiation” ([4], p65).

“Solar radiation that is not absorbed or reflected by the atmosphere (for example by clouds) reaches the surface of the Earth. The Earth absorbs most of the energy reaching its surface, a small fraction is reflected<sup>4</sup>.”

Carbone dioxide renders the atmosphere opaque to infrared radiations around 20THz . Hence increasing its proportion in air increases the absorption coefficient in that range. It is believed to be one of the causes of global warming. However it must be more complex because some experimental measurements show that increasing opacity decreases the temperature at high altitude, above the clouds [3].

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<sup>4</sup> <https://public.wmo.int/en/sun%E2%80%99s-impact-earth>

In the framework of the radiative transfer equations, one can studies the conjectures cited above. To support theoretical claims, the numerical solver developed in [1], [10] and [7] is used. We will address 4 questions:

- What is the effect of increasing an altitude dependent absorption coefficient?
- Is it true that sunlight crosses the Earth atmosphere unaffected?
- Is 30% of albedo equivalent to 30% reduction of the Sun radiative energy ?
- What is the effect on the temperature of increasing the absorption coefficient in a specific frequency range?

## 2. The Radiative Transfer Equations for a Stratified Atmosphere

Finding the temperature  $T$  in a fluid heated by electromagnetic radiations is a complex problem because interactions of photons with atoms in the medium involve rather intricate quantum phenomena. Assuming local thermodynamic equilibrium leads to a well-defined electronic temperature. In that case, one can write a kinetic equation for the radiative intensity  $I_\nu(\mathbf{x}, \boldsymbol{\omega}, t)$  at time  $t$ , at position  $\mathbf{x}$  and in the direction  $\boldsymbol{\omega}$  for photons of frequency  $\nu$ , in terms of the temperature field  $T(\mathbf{x}, t)$ .

$$\begin{aligned} \frac{1}{c} \partial_t I_\nu + \boldsymbol{\omega} \cdot \nabla I_\nu + \rho \bar{\kappa}_\nu a_\nu \left[ I_\nu - \frac{1}{4\pi} \int_{\mathbb{S}^2} p(\boldsymbol{\omega}, \boldsymbol{\omega}') I_\nu(\boldsymbol{\omega}') d\boldsymbol{\omega}' \right] \\ = \rho \bar{\kappa}_\nu (1 - a_\nu) [B_\nu(T) - I_\nu]. \end{aligned} \quad (1)$$

In this equation  $c$  is the speed of light,  $\rho$  is the density of the fluid,  $\nabla$  designates the gradient with respect to the position  $\mathbf{x}$ , while

$$B_\nu(T) = \frac{2\hbar\nu^3}{c^2 [e^{\frac{\hbar\nu}{kT}} - 1]} \quad (2)$$

is the Planck function at temperature  $T$ , with  $\hbar$  the Planck constant and  $k$  the Boltzmann constant. Recall the Stefan-Boltzmann identity,

$$\int_0^\infty B_\nu(T) d\nu = \bar{\sigma} T^4, \quad \bar{\sigma} = \frac{2\pi^4 k^4}{15c^2 \hbar^3}, \quad (3)$$

where  $\pi\bar{\sigma}$  is the Stefan-Boltzmann constant.

The intricacy of the interaction of photons with the atoms of the medium is contained in the mass-absorption  $\bar{\kappa}_\nu$ , which is theoretically the result of vibration and rotation atomic resonance, but for practical purpose the fraction of radiative intensity at frequency  $\nu$  that is absorbed by fluid per unit length. The coefficient  $a_\nu \in (0, 1)$  is the scattering albedo, and  $\frac{1}{4\pi} p(\boldsymbol{\omega}, \boldsymbol{\omega}') d\boldsymbol{\omega}$  is the probability that an incident ray of light with direction  $\boldsymbol{\omega}'$  scatters in the infinitesimal element of solid angle  $d\boldsymbol{\omega}$  centered at  $\boldsymbol{\omega}$ .

The kinetic equation (1) is coupled to the temperature - or energy conservation - equations of the fluid . When thermal diffusion is small, the system decouples and energy balance becomes:

$$\int_0^\infty \rho \bar{\kappa}_\nu (1 - a_\nu) \left( \int_{\mathbb{S}^2} I_\nu(\boldsymbol{\omega}) d\boldsymbol{\omega} - 4\pi B_\nu(T) \right) d\nu = 0. \quad (4)$$

When the wave source is far in the direction  $z$  the problem becomes one-dimensional in  $z$  and the radiative intensity scattered in the direction  $\omega$  depends only on  $\mu$ , the cosine of the angle between  $\omega$  and  $Oz$ . Consequently the system becomes,

$$\begin{aligned} \mu \partial_z I_\nu + \rho \bar{\kappa}_\nu I_\nu &= \rho \bar{\kappa}_\nu (1 - a_\nu) B_\nu(T) + \frac{\rho \bar{\kappa}_\nu a_\nu}{2} \int_{-1}^1 p(\mu, \mu') I_\nu(z, \mu') d\mu', \\ \int_0^\infty \rho \bar{\kappa}_\nu (1 - a_\nu) \left( B_\nu(T) - \frac{1}{2} \int_{-1}^1 I_\nu d\mu \right) d\nu &= 0, \quad z \in (0, H), \quad |\mu| < 1, \quad \nu \in \mathbb{R}^+. \end{aligned} \quad (5)$$

For this system to be mathematically well posed, the radiation intensity  $I_\nu$  must be given on the domain boundary where radiation enters. For example,

$$I_\nu(H, -\mu) = Q^-(\mu), \quad I(0, \mu) = Q^+(\mu), \quad 0 < \mu < 1. \quad (6)$$

### 2.1. Semi-analytical Solution when the scattering is isotropic

Isotropic scattering is modelled by taking  $p \equiv 1$ . By introducing an optical depth (in [4] and others, the signs are changed),

$$\tau = \int_0^z \rho(\eta) d\eta, \quad \rho \text{ becomes } 1 \text{ and } z \text{ becomes } \tau \in (0, Z), \quad Z = \int_0^H \rho(\eta) d\eta. \quad (7)$$

Let us denote the exponential integral by

$$E_p(X) := \int_0^1 e^{-X/\mu} \mu^{p-2} d\mu. \quad (8)$$

Define

$$J_\nu(\tau) = \frac{1}{2} \int_{-1}^1 I_\nu d\mu, \quad S_\nu(\tau) = \frac{1}{2} \int_0^1 \left( e^{-\frac{\kappa_\nu \tau}{\mu}} Q_\nu^+(\mu) + e^{-\frac{\kappa_\nu (Z-\tau)}{\mu}} Q_\nu^-(\mu) \right) d\mu. \quad (9)$$

The problem is equivalent to a functional integral equation:

$$\left. \begin{aligned} J_\nu(\tau) &= S_\nu(\tau) + \frac{\kappa_\nu}{2} \int_0^Z E_1(\kappa_\nu |\tau - t|) ((1 - a_\nu) B_\nu(T(t)) + a_\nu J_\nu(t)) dt, \\ \int_0^\infty (1 - a_\nu) \kappa_\nu B_\nu(T(\tau)) d\nu &= \int_0^\infty (1 - a_\nu) \kappa_\nu J_\nu(\tau) d\nu, \end{aligned} \right\} \quad (10)$$

Once  $J_\nu, T$  are known,  $I_\nu$  can be recovered by

$$\begin{aligned} I_\nu(\tau, \mu) &= e^{-\frac{\kappa_\nu \tau}{\mu}} I_\nu(0, \mu) \mathbf{1}_{\mu > 0} + e^{-\frac{\kappa_\nu (Z-\tau)}{\mu}} I_\nu(Z, \mu) \mathbf{1}_{\mu < 0} \\ &\quad + \mathbf{1}_{\mu > 0} \int_0^\tau e^{-\frac{\kappa_\nu |\tau-t|}{|\mu|}} \frac{\kappa_\nu}{|\mu|} ((1 - a_\nu) \kappa_\nu B_\nu(T(t)) + a_\nu \kappa_\nu J_\nu(t)) dt \\ &\quad + \mathbf{1}_{\mu < 0} \int_\tau^Z e^{-\frac{\kappa_\nu |\tau-t|}{|\mu|}} \frac{\kappa_\nu}{|\mu|} ((1 - a_\nu) \kappa_\nu B_\nu(T(t)) + a_\nu \kappa_\nu J_\nu(t)) dt. \end{aligned} \quad (11)$$

**Remark 1.** Notice that two solutions  $(T, J_\nu), (T', J'_\nu)$  with  $J_\nu \equiv J'_\nu$  will have  $T \equiv T'$  but not necessarily  $I_\nu \equiv I'_\nu$ .

### 3. Increasing an Altitude Dependent Absorption Coefficient Decreases the Temperature in the atmosphere

The following argument goes against a common belief which claims that increasing the absorption coefficient - which GHG do indeed - implies an increase of temperature in the atmosphere. Consider the case of an altitude dependent absorption coefficient which we write as  $r(z)\kappa$  with  $\kappa$  constant.

**Proposition 1.** *If  $\kappa$  is independent of  $\nu$  but function of the altitude and  $T$  is monotone decreasing with altitude, then, increasing  $\kappa$  anywhere will decrease the temperature.*

*Proof* The proof is a straightforward consequence of the concept of optical length (7). Here, let  $\tau = \int_0^z r(\zeta)d\zeta$ . Observe that  $\partial_z = \frac{\partial\tau}{\partial z}\partial_\tau = r(z)\partial_\tau$ ; consequently  $r$  disappears from (5),  $H$  is replaced by  $Z = \int_0^H r(\zeta)d\zeta$ , and  $z$  is replaced by  $\tau$ .

When  $\kappa$  and  $a$  are constant,  $\bar{I}_r = \int_0^\infty I_\nu d\nu$  is solution of the "grey model":

$$\begin{aligned} \mu\partial_\tau\bar{I}_1 + \kappa\bar{I}_1 &= \frac{1}{2} \int_{-1}^1 \left( \kappa(1-a)\bar{I}_1 + \kappa a \int_{-1}^1 p(\mu, \mu')\bar{I}_1(\mu, \mu')d\mu' \right) d\mu, \\ I_r(H, -\mu) &= 0, \quad \bar{I}_r(0, \mu) = Q^+(\mu), \quad 0 < \mu < 1. \end{aligned} \quad (12)$$

where the Stefan-Boltzmann identity has been used:  $\sigma T^4 = \frac{1}{2} \int_{-1}^1 \bar{I}_r d\mu$ .

Let  $\bar{I}_1(\tau, \mu)$  be the solution of (12). We want to compare  $I_{r+\delta r}$  with  $I_r$  when  $\delta r$  has a small support near altitude  $y$ . By (7),

$$\begin{aligned} \bar{I}_{r+\delta r}(z) &= \bar{I}_1\left(\int_0^z (r(\zeta) + \delta r(\zeta))d\zeta\right) \approx \bar{I}_1(\tau(z)) + \frac{d}{d\tau}\bar{I}_1(\tau(y))\delta r(y) \\ \implies \frac{1}{2} \int_{-1}^1 \bar{I}_{r+\delta r}(z)d\mu - \frac{1}{2} \int_{-1}^1 \bar{I}_r(z)d\mu &\approx \frac{d}{d\tau} \left( \frac{1}{2} \int_{-1}^1 \bar{I}_1(\tau)d\mu \right) |_{\tau(y)} \delta r(y) \end{aligned} \quad (13)$$

Finally from (3),

$$T_{r+\delta r}(z) - T_r(z) \approx 4\sigma T^3(y) \frac{\delta r(y)}{r(y)} \frac{dT_r}{dz}(y). \quad (14)$$

□

### 4. Sunlight Crosses the Earth Atmosphere Unaffected?

It is true that the absorption coefficient  $\kappa_\nu$  is small in the frequency range of visible light; but does it imply that the light source can be used as a boundary condition at altitude zero instead of altitude  $H$ , the top of the troposphere?

First eliminate the  $z$  dependency of the coefficients by using the optical length  $\tau$  defined in (7). The following result gives an interesting answer to this question.

**Proposition 2.** Assume  $a_\nu = 0$  and assume that for some  $\nu^*$ ,

$$\epsilon := \max_{\nu > \nu^*} \kappa_\nu < 1 \text{ and } \kappa_\nu Q_\nu^0|_{\nu > \nu^*} = O(1).$$

Let  $I_\nu, T$  and  $I'_\nu, T'$  be solutions of (5),(6) respectively with

$$\begin{aligned} Q_\nu^+(\mu) &= 0, \quad Q_\nu^-(\mu) = \mu Q_\nu^0, \quad 0 < \mu < 1. \\ Q_\nu'^+ &= \mu e^{-\frac{\kappa_\nu Z}{\mu}} Q_\nu^0, \quad Q_\nu'^-(\mu) = 0, \quad 0 < \mu < 1. \end{aligned} \quad (15)$$

Then  $T'(\tau) = T(\tau) + O(\epsilon)$  and

$$I'_\nu - I_\nu = \mu e^{-\frac{\kappa_\nu Z}{\mu}} Q_\nu^0 \left( \mathbf{1}_{\mu > 0} e^{-\frac{\kappa_\nu}{\mu} \tau} + \mathbf{1}_{\mu < 0} e^{\frac{\kappa_\nu}{\mu} \tau} \right) + O(\epsilon).$$

*Proof* Note that

$$\left. \begin{aligned} \mu \partial_\tau I_\nu + \kappa_\nu (I_\nu - B_\nu(T)) &= 0, \quad \nu < \nu^*, \\ I_\nu(0)|_{\mu > 0} &= 0, \quad I_\nu(Z)|_{\mu < 0} = Q_\nu^-(\mu), \quad \nu < \nu^*, \\ I_\nu &\approx I_\nu^*(\mu) \text{ independent of } \tau, \quad \nu > \nu^*. \\ I_\nu^*|_{\mu > 0} &= 0, \quad I_\nu^*|_{\mu < 0} = Q_\nu^-(\mu), \quad \nu > \nu^*. \\ \int_0^\infty \kappa_\nu B_\nu(T) d\nu &\approx \int_0^{\nu^*} \frac{\kappa_\nu}{2} \int_{-1}^1 I_\nu d\mu d\nu + \int_{\nu^*}^\infty \frac{\kappa_\nu}{2} \int_{-1}^0 Q_\nu^-(\mu) d\mu d\nu \end{aligned} \right\} \quad (16)$$

Now  $I'_\nu$  is defined by (5) but with

$$I'_\nu(0)|_{\mu > 0} = Q_\nu'^+(\mu), \quad I'_\nu(Z)|_{\mu < 0} = 0.$$

By the same argument, when  $\nu > \nu^*$ ,  $I'_\nu \approx I_\nu'^*(\mu)$ , independent of  $\tau$ , and

$$\left. \begin{aligned} \mu \partial_\tau I'_\nu + \kappa_\nu (I'_\nu - B_\nu(T')) &= 0, \quad \nu < \nu^*, \\ I'_\nu(0)|_{\mu > 0} &= Q_\nu'^+(\mu), \quad I'_\nu(Z)|_{\mu < 0} = 0, \quad \nu < \nu^*, \\ \int_0^\infty \kappa_\nu B_\nu(T') d\nu &\approx \int_0^{\nu^*} \frac{\kappa_\nu}{2} \int_{-1}^1 I'_\nu d\mu d\nu + \int_{\nu^*}^\infty \frac{\kappa_\nu}{2} \int_0^1 Q_\nu'^+(\mu) d\mu d\nu \end{aligned} \right\} \quad (17)$$

Let  $I''_\nu = I_\nu - I'_\nu$ . It holds

$$\left. \begin{aligned} \mu \partial_\tau I''_\nu + \kappa_\nu (I''_\nu - (B_\nu(T) - B_\nu(T'))) &= 0, \quad \nu < \nu^*, \\ \int_0^\infty \kappa_\nu (B_\nu(T) - B_\nu(T')) d\nu &= \int_0^{\nu^*} \frac{\kappa_\nu}{2} \int_{-1}^1 I''_\nu d\mu d\nu \\ &+ \int_{\nu^*}^\infty \left( \frac{\kappa_\nu}{2} \int_0^1 (Q_\nu^-(\mu) - Q_\nu'^+(\mu)) d\mu \right) d\nu \\ I''_\nu(0)|_{\mu > 0} &= -Q_\nu'^+(\mu), \quad I''_\nu(Z)|_{\mu < 0} = Q_\nu^-(\mu), \quad \nu < \nu^*. \end{aligned} \right\} \quad (18)$$

We notice that  $T = T'$  and  $Q_\nu^-(\mu) = e^{-\frac{\kappa_\nu}{\mu} Z} Q_\nu'^+(\mu)$  implies

$$\left. \begin{aligned} \mu \partial_\tau I''_\nu + \kappa_\nu I''_\nu &= 0, \quad \nu < \nu^*, \\ \int_0^{\nu^*} \left( \frac{\kappa_\nu}{2} \int_{-1}^1 I''_\nu d\mu \right) d\nu &= \int_{\nu^*}^\infty \left( \frac{\kappa_\nu}{2} \int_0^1 (1 - e^{-\frac{\kappa_\nu}{\mu} Z}) Q_\nu'^+(\mu) d\mu \right) d\nu \\ I''_\nu(0)|_{\mu > 0} &= -Q_\nu'^+(\mu), \quad I''_\nu(Z)|_{\mu < 0} = e^{\frac{\kappa_\nu}{\mu} Z} Q_\nu'^+(-\mu) \end{aligned} \right\} \quad (19)$$

Let  $Q'_\nu(\mu) = \mu Q_\nu^0$ . Then

$$I''_\nu(0)|_{\mu>0} = -\mu Q_\nu^0, \quad I''_\nu(Z)|_{\mu<0} = -\mu e^{\frac{\kappa_\nu}{\mu}Z} Q_\nu^0.$$

Consequently, from (19) and (11), the solution is

$$\begin{aligned} I''_\nu(\tau, \mu) &= -\mu Q_\nu^0 \left( \mathbf{1}_{\mu>0} e^{-\frac{\kappa_\nu}{\mu}\tau} + \mathbf{1}_{\mu<0} e^{\frac{\kappa_\nu}{\mu}\tau} \right). \\ \implies \int_{-1}^1 I''_\nu(\tau) d\mu &= -Q_\nu^0 E_3(\kappa_\nu \tau) + Q_\nu^0 E_3(\kappa_\nu \tau) = 0, \end{aligned} \quad (20)$$

We have satisfied (19) with precision  $O(\kappa_\nu^2)$  because

$$\kappa_\nu(1 - e^{-\frac{\kappa_\nu}{\mu}Z})\mu Q_\nu^0 = \kappa_\nu^2 Z Q_\nu^0 + O(\kappa_\nu^3) \approx 0, \quad \nu > \nu^*$$

□

**Remark 2.** It is easy to see that applying  $e^{-\frac{\kappa_\nu Z}{\tau}} \mu Q_\nu^0$  at  $\tau = 0$ , leads to change  $E_3(\kappa_\nu \tau)$  in (11) into  $E_3(\kappa_\nu(Z + \tau))$ .

The change in temperature is shown on Figure 1. All variables are rescaled as in [6]§7.1. In particular temperatures are in Kelvin and divided by 4780. The air density is  $1.22e^{-z}$ . Here  $Q^0 = 5.66 \cdot 10^{-5}$  and  $T_{sun} = 1.2$ .

*Conclusion:* For the specific case of sunlight on Earth It is a gross approximation to assume that sunlight crosses the atmosphere unaffected.

However the theory works when  $T_{sun} = 2.4$  and  $\nu^* = 0.2$ , probably because  $\max_\nu B_\nu(T_{sun} \approx 18$  against 2 with  $T_{sun} = 1.2$ .

## 5. Is the 30% Earth Albedo Equivalent to a 30% reduction of the Sun Radiative Energy?

Note that (11) can be used to find the solution of (5)-a with generalized albedo (inspired by the model proposed in [3]):

$$I_\nu(0, \mu) = \sum_k \alpha_k I_\nu(\tau_k, -\mu) + Q_\nu^0 \mu, \quad I_\nu(Z, -\mu) = 0, \quad 0 < \mu < 1.$$

with  $k = 1, 2, \dots, K$ . Indeed by (11)

$$\begin{aligned} I_\nu(\tau, \mu) &= e^{-\frac{\kappa_\nu \tau}{\mu}} \left( \sum_k \alpha_k I_\nu(\tau_k, -\mu) + Q_\nu^0 \mu \right) \mathbf{1}_{\mu>0} \\ &+ \mathbf{1}_{\mu>0} \int_0^\tau e^{-\frac{\kappa_\nu |\tau-t|}{|\mu|}} \frac{\kappa_\nu}{|\mu|} ((1 - a_\nu) \kappa_\nu B_\nu(T(t)) + a_\nu \kappa_\nu J_\nu(t)) dt \\ &+ \mathbf{1}_{\mu<0} \int_\tau^Z e^{-\frac{\kappa_\nu |\tau-t|}{|\mu|}} \frac{\kappa_\nu}{|\mu|} ((1 - a_\nu) \kappa_\nu B_\nu(T(t)) + a_\nu \kappa_\nu J_\nu(t)) dt \end{aligned} \quad (21)$$

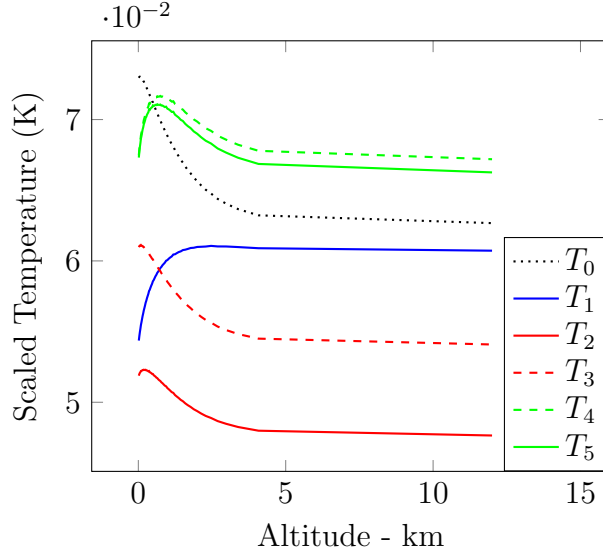


Figure 1: Scaled temperatures  $z \rightarrow T(z)$  versus altitude, computed with  $0.5(0.1 + \mathbf{1}_{\nu < 3})$ . The 3 curves  $T_1, T_2, T_3$  corresponds to **1**:  $Q_\nu^- = Q_\nu^0$ ,  $Q_\nu^+ = 0$ , **2**:  $Q_\nu^- = 0$ ,  $I_\nu(0, \mu)|_{\mu > 0} = \mu e^{-\frac{\kappa_\nu Z}{\mu}} Q_\nu^0$  and to **3**:  $Q_\nu^- = 0$ ,  $Q_\nu^+ = Q_\nu^0$ . These can be compared to the grey case  $T_0$  with  $\kappa = 0.5$ . Note that  $T_1$  is very different from  $T_2$ . However if the Sun temperature is multiplied by 2 and  $\kappa_\nu = 0.5(0.1 + \mathbf{1}_{\nu < 0.2})$ , then the conditions in Proposition 2 are met and confirmed by the numerical simulations:  $T_4$  is computed like  $T_1$  and  $T_5$  is computed like  $T_2$ .

$$\begin{aligned}
&= \mathbf{1}_{\mu > 0} e^{-\frac{\kappa_\nu \tau}{\mu}} \sum_k \alpha_k \int_{\tau_k}^Z e^{-\frac{\kappa_\nu(t-\tau_k)}{\mu}} \frac{\kappa_\nu}{\mu} ((1-a_\nu)\kappa_\nu B_\nu(T(t)) + a_\nu \kappa_\nu J_\nu(t)) dt \\
&+ \mathbf{1}_{\mu > 0} \left( e^{-\frac{\kappa_\nu \tau}{\mu}} Q_\nu^0 \mu + \int_0^\tau e^{-\frac{\kappa_\nu|\tau-t|}{|\mu|}} \frac{\kappa_\nu}{|\mu|} ((1-a_\nu)\kappa_\nu B_\nu(T(t)) + a_\nu \kappa_\nu J_\nu(t)) dt \right) \\
&+ \mathbf{1}_{\mu < 0} \int_\tau^Z e^{-\frac{\kappa_\nu|\tau-t|}{|\mu|}} \frac{\kappa_\nu}{|\mu|} ((1-a_\nu)\kappa_\nu B_\nu(T(t)) + a_\nu \kappa_\nu J_\nu(t)) dt \\
&\implies \\
&J_\nu(\tau) = \frac{1}{2} Q_\nu^0 E_3(\kappa_\nu \tau) + \frac{1}{2} \int_0^Z E_1(\kappa_\nu |\tau-t|) [(1-a_\nu)B_\nu(T(t)) + a_\nu \kappa_\nu J_\nu(t)] dt, \\
&+ \frac{1}{2} \sum_k \alpha_k \kappa_\nu \int_{\tau_k}^Z E_1(\kappa_\nu(t+\tau-\tau_k)) [(1-a_\nu)\kappa_\nu B_\nu(T(t)) + a_\nu \kappa_\nu J_\nu(t)] dt.
\end{aligned} \tag{22}$$

### 5.1. Earth Albedo

Consider the case

$$I_\nu(Z, \mu)|_{\mu < 0} = 0, \quad I_\nu(0, \mu)|_{\mu > 0} = Q_\nu^0 + \alpha I_\nu(0, -\mu).$$

Assume no scattering,  $\tau_1 = 0$ ,  $K = 1$ . As in [7], the following iterative scheme is considered:

$$\begin{aligned}
J_\nu^{n+1}(\tau) &= \frac{1}{2} Q_\nu^0 E_3(\kappa_\nu \tau) + \frac{\kappa_\nu}{2} \int_0^Z (E_1(\kappa_\nu |\tau-t|) + \alpha E_1(\kappa_\nu(t+\tau))) B_\nu(T^n(t)) dt, \\
\int_0^\infty \kappa_\nu B_\nu(T_\tau^{n+1}) d\nu &= \int_0^\infty \kappa_\nu J_\nu^{n+1}(\tau) d\nu.
\end{aligned} \tag{23}$$



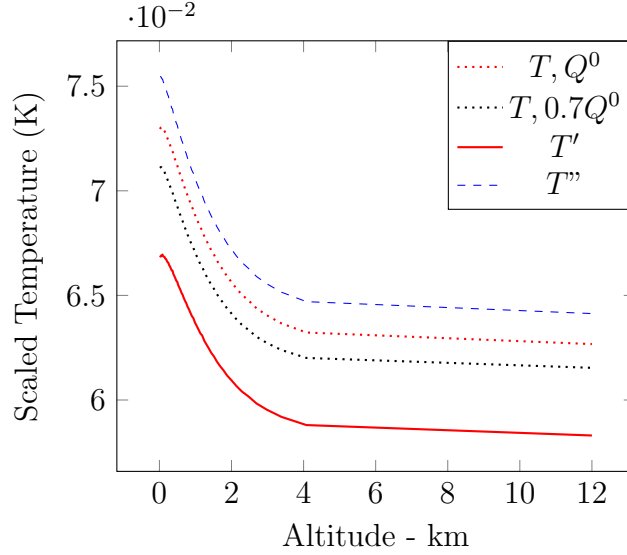


Figure 2: Scaled temperatures  $z \rightarrow T(z)$  versus altitude. In all cases  $Q_\nu^- = 0$ . The dotted curves are  $T$  when  $Q_\nu^+ = \lambda Q_\nu^0$  with  $\lambda = 1$  (top dotted curve) and  $\lambda = 0.7$  (lower dotted curve); the dotted line is the solution  $T'$  when  $I_\nu(0, \mu)|_{\mu>0} = 0.3I_\nu(0, -\mu) + \mu e^{-\frac{\kappa_\nu z}{\mu}} Q_\nu^0/0.7$ . The dashed line is when  $I_\nu(0, \mu)|_{\mu>0} = 0.3I_\nu(0, -\mu) + \mu Q_\nu^0$ .

### 5.2. Numerical Example

Assume  $a_\nu = 0$ . Figure 2 compares the numerical solutions of (5) with  $\kappa_\nu = 0.5[\mathbf{1}_{\nu<6} + 0.1]$  and

$$Q_\nu^- = 0, \quad Q_\nu^+ = \lambda Q_\nu^0, \quad \lambda = 1 \text{ or } 0.7.$$

with the solution of (5) with  $\beta = 0$  or 1 in

$$Q_\nu'^- = 0, \quad I_\nu'(0, \mu) = 0.3I_\nu'(0, -\mu) + \mu Q_\nu^0 \left[ \frac{1}{0.7} e^{-\frac{\kappa_\nu z}{\mu}} \beta + 1 - \beta \right], \quad 0 < \mu < 1.$$

This numerical simulation is an attempt to replace the radiative source at  $\tau = H$  by a radiative source at  $\tau = 0$  which gives the same result.

Notice alongside that adding 0.3 albedo induces a heating of the atmosphere (i.e. comparing both computations with  $\lambda = 1, \beta = 1$  above).

## 6. Earth Albedo, General Statement, Accommodation Coefficient

As above one considers a stratified atmosphere (5) with  $(z, \mu, \nu)$  in  $(0, H) \times (-1, 1) \times \mathbb{R}^+$ ; the radiation intensity  $I_\nu(z, \mu)$  satisfies on at  $z = H$  an incoming condition

$$I_\nu(H, -\mu) = Q_\nu^-(\mu), \quad 0 < \mu < 1, \quad (24)$$

and on Earth, at  $z = 0$ , some “albedo” condition, denoted:

$$I_\nu(0, \mu) = \mathcal{A}_\nu(I_\nu(0, -\mu)), \quad 0 < \mu < 1, \quad (25)$$

with  $\mathcal{A}$  representing the albedo effect of the earth, hence being an operator from the space of outgoing intensities into the space of incoming intensities.

Below will be given, based on energy estimates, some sufficient conditions on  $\mathcal{A}$  for existence, stability – hence uniqueness – of the solution, beginning with the grey model and then considering a natural operator which involves the effective temperature of Earth (see [4] page 66).

### 6.1. The energy type estimate for the grey problem

For clarity we present the case without scattering ( $a_\nu = 0$ ) and with a constant  $\kappa_\nu$  independent of the frequency  $\nu$ . By rescaling  $z$  with  $\kappa$ , the problem can be formulated in term of  $\bar{I}(z, \mu) = \int_0^\infty I_\nu(z, \mu, \nu) d\nu$  as,

$$\mu \partial_z \bar{I}(z, \mu) + \bar{I}(z, \mu) - \frac{1}{2} \int_{-1}^1 \bar{I}(z, \mu') d\mu' = 0, \quad (26a)$$

$$\bar{I}(H, -\mu) = Q(\mu) := \bar{Q}_\nu^-(\mu), \quad \bar{I}(0, \mu) = \mathcal{A}(\bar{I}(0, -\mu)), \quad 0 < \mu < 1. \quad (26b)$$

Note that we have assumed that  $\mathcal{A}$  commutes with the integration in  $\nu$ . First observe that the operator  $\bar{I} \mapsto \bar{I} - \frac{1}{2} \int_{-1}^1 \bar{I}(\mu') d\mu'$  is, in the space  $L^2(-1, 1)$  the orthogonal projection on functions of mean value 0, because

$$\int_{-1}^1 (\bar{I} - \frac{1}{2} \int_{-1}^1 \bar{I}(\mu') d\mu') \bar{I}(\mu) d\mu = \int_{-1}^1 (\bar{I} - \frac{1}{2} \int_{-1}^1 \bar{I}(\mu') d\mu')^2 d\mu. \quad (27)$$

This implies that the left hand side of (27) is nonnegative and equal to 0 iff  $\bar{I} - \frac{1}{2} \int_{-1}^1 \bar{I}(\mu') d\mu'$  is independent of  $\mu$ . Multiplying the equation (26a) by  $\bar{I}(z, \mu)$ , integrating over  $(0, H) \times (-1, 1)$ , using Green's formula and inserting in the computations the conditions (26b) one obtains:

$$\begin{aligned} & \int_0^H dz \int_{-1}^1 (\bar{I} - \frac{1}{2} \int_{-1}^1 \bar{I}(\mu') d\mu')^2 d\mu + \frac{1}{2} \int_0^1 \mu (\bar{I}(H, -\mu))^2 d\mu \\ & + \frac{1}{2} \int_0^1 \mu ((\bar{I}(0, \mu))^2 - (\mathcal{A}(\bar{I}(0, \mu)))^2) d\mu = \frac{1}{2} \int_0^1 \mu (Q(\mu))^2 d\mu \end{aligned} \quad (28)$$

This estimate indicates that any property which implies the positivity of

$$\int_0^1 \mu (\bar{I}(0, \mu)^2 - (\mathcal{A}(\bar{I}(0, \mu)))^2) d\mu$$

will lead to a well posed problem with a unique nonnegative solution; for instance:

**Theorem 1.** *Assume that the operator  $\bar{I} \mapsto \mathcal{A}(\bar{I})$  restricted to the positive cone  $C^+(\bar{I}(\mu) \geq 0)$  on the space  $(L^2(-1, 1), \mu, d\mu)$  then for any positive  $\mu \mapsto Q(\mu)$  there is a unique non-negative solution of the (26a), (26b).*

**Remark 3.** *Making use of the maximum principle one could prove similar results when the albedo operator is a contraction in the positive cone of  $L^\infty(0, 1)$ .*

**Remark 4.** The albedo condition which is a relation, for  $0 < \mu < 1$ , between  $\bar{I}(0, \mu)|_{\mu>0}$  and  $\bar{I}(0, \mu)|_{\mu<0}$  can also be expressed in term of  $\bar{I}(0, \mu)|_{\mu>0}$  only.

Indeed, define the operator  $\mathcal{T} : \bar{I}_+(\mu) \mapsto \mathcal{T}(\bar{I}_+(\mu)) = I(0, -\mu)|_{\mu>0}$  (defined for instance in  $L^\infty(0, 1)$ ) by solving (26a) with

$$\bar{I}(H, -\mu) = Q(\mu), \quad \bar{I}(0, \mu) = \bar{I}_+(\mu), \quad 0 < \mu < 1. \quad (29)$$

Then one has:

$$\bar{I}(0, \mu)|_{\mu>0} = \mathcal{A}(\bar{I}(0, -\mu)) \Leftrightarrow \bar{I}(0, \mu)|_{\mu>0} = \mathcal{A}(\mathcal{T}(\bar{I}(0, \mu)|_{\mu>0})). \quad (30)$$

With  $Q(\mu) > 0$  in  $L^\infty(0, 1)$ ,  $\mathcal{T}$  is an affine operator which preserves the positivity and the monotonicity. and this leads to the simple but useful proposition, which, in turn, is the key to the proof of Theorem 1.

**Proposition 3.** Consider two solutions  $\{I_i\}_{i=1,2}$  of the (26a) with (26b) with the same  $Q(\mu)$  but with two different albedo operators  $\mathcal{A}_i$ . Assume that both  $\mathcal{A}_i$  are linear contractions and one of them is a strict linear contraction, which preserve positivity. If

$$\forall f \geq 0 \in L^\infty(0, 1) \quad \forall \mu \in (0, 1), \quad \mathcal{A}_2(f)(\mu) \leq \mathcal{A}_1(f)(\mu) \quad (31)$$

then one has the same ordering for the corresponding solutions :

$$\forall (z, \mu) \in (0, 1) \times (0, H) \quad I_1(z, \mu) \leq I_2(z, \mu). \quad (32)$$

*Proof* Using the linearity of the operators  $\mathcal{A}_i$  one observes that  $R = I_1 - I_2$  solves (26a) with  $R(H, -\mu) = 0$  and, for all  $0 < \mu < 1$ ,

$$R(0, \mu) - \frac{1}{2}(\mathcal{A}_1 + \mathcal{A}_2)(\mathcal{T}(R(0, \mu))) = \frac{1}{2}(\mathcal{A}_1 - \mathcal{A}_2)(\mathcal{T}((I_1(0, \mu) + I_2(0, \mu)))) \quad (33)$$

Moreover, restricted to solutions which satisfy  $R(H, -\mu) = 0$ ,  $\mu > 0$  the operator  $\mathcal{T}$  is a strict monotonicity preserving linear contraction; the same observation holds for  $R \mapsto \frac{1}{2}(\mathcal{A}_1 + \mathcal{A}_2)(\mathcal{T}(R))$ . As a consequence the operator  $R \mapsto (I - \frac{1}{2}(\mathcal{A}_1 + \mathcal{A}_2)(\mathcal{T}))$  is invertible with inverse given by the Neumann series (also preserving the positivity):

$$R = \sum_{k \geq 0} \left( \frac{1}{2}(\mathcal{A}_1 + \mathcal{A}_2)(\mathcal{T}) \right)^k \frac{1}{2}(\mathcal{A}_1 - \mathcal{A}_2)(\mathcal{T}((I_1 + I_2)|_{z=0, \mu>0})). \quad (34)$$

$I_1$  and  $I_2$  are positive intensities and for  $\frac{1}{2}(\mathcal{A}_1 - \mathcal{A}_2)(\mathcal{T}((I_1 + I_2)|_{z=0, \mu>0}))$  and the righthand-side of (34), (31) is also true.  $\square$

## 6.2. The frequency dependent case

Let us come back to (5) with (26b). Let us denote  $f_+ = \max(f, 0)$ . When not ambiguous, let us write  $I_\nu(\mu)$  instead of  $I_\nu(0, \mu)$ .

**Definition 1.** An operator  $\mathcal{A}$  defined in  $L_\mu^1(0, 1)$  is non-accretive if

$$\forall I^1, I^2 \in L_\mu^1(0, 1), \quad \int_0^1 \mu \left( (I^2 - I^1)_+ - (\mathcal{A}(I^2) - \mathcal{A}(I^1))_+ \right) d\mu \geq 0. \quad (35)$$

**Remark 5.** Obviously (35) follows from the same pointwise property:

$$\forall (I^1, I^2) \quad ((\mathcal{A}(I^2) - \mathcal{A}(I^1))_+ \leq (I^2 - I^1)_+, \quad \mu > 0. \quad (36)$$

Let us use the symbol  $\langle f \rangle = \int_0^\infty \frac{1}{2} \int_{-1}^1 f_\nu(\mu) d\mu d\nu$ .

Then for two solutions  $(I_\nu^2, T^2)$ ,  $(I_\nu^1, T^1)$ , by (5), according to Theorem 4.1 in [7] one has:

$$\begin{aligned} & -\frac{d}{dz} \langle \mu(I_\nu^2 - I_\nu^1)_+ \rangle = \\ & = \langle \rho \bar{\kappa}_\nu (1 - a_\nu) ((I_\nu^2 - I_\nu^1) - (B_\nu(T^2) - B_\nu(T^1))) \mathbf{1}_{I_\nu^2 > I_\nu^1} \rangle \\ & + \left\langle \rho \bar{\kappa}_\nu a_\nu \left( (I_\nu^2 - I_\nu^1) - \frac{1}{2} \int_{-1}^1 p(\mu, \mu') (I_\nu^2 - I_\nu^1)(z, \mu') d\mu' \right) \mathbf{1}_{I_\nu^2 > I_\nu^1} \right\rangle \geq 0 \end{aligned} \quad (37)$$

and therefore  $\frac{d}{dz} \langle \mu(I_\nu^2(z, \mu) - I_\nu^1(z, \mu))_+ \rangle \leq 0$ . On the other hand

$$I_\nu^2(H, -\mu) \leq I_\nu^1(H, -\mu), \quad \mu > 0 \quad \implies \quad \langle \mu(I_\nu^2(H, \mu) - I_\nu^1(H, \mu))_+ \rangle = 0$$

while the non accretivity of  $\mathcal{A}$  gives

$$\begin{aligned} & \langle \mu(I_\nu^2(0, \mu) - I_\nu^1(0, \mu))_+ \rangle \\ & = \frac{1}{2} \int_0^\infty d\nu \int_0^1 \mu (\mathcal{A}(I_\nu^2(0, -\mu)) - \mathcal{A}(I_\nu^1(0, -\mu)))_+ - (I_\nu^2 - I_\nu^1)_+(0, -\mu) d\mu \leq 0 \end{aligned} \quad (38)$$

Therefore, unless  $\langle \mu(I_\nu^2(0, \mu) - I_\nu^1(0, \mu))_+ \rangle = 0$  there is a contradiction because it is a function which is decreasing, negative at  $z = 0$  and zero at  $z = H$ . Then consider the difference of equations (5) with arguments  $(I_\nu^i, T^i)$   $i = 1, 2$  multiplied by  $\frac{\mu}{\bar{\kappa}_\nu} \mathbf{1}_{I_\nu^2 > I_\nu^1}$ . Following the end of the proof of Theorem 4.1 in [7] (see also [5]), one finds by the same method that existence and uniqueness derives from

$$\frac{d}{dz} \left\langle \frac{\mu^2}{\bar{\kappa}_\nu} (I_\nu^2 - I_\nu^1)_+ \right\rangle = 0, \quad 0 < z < H.$$

□

Eventually one has:

**Theorem 2.** Consider (5) with incoming data  $0 \leq I_\nu(H, -\mu) \leq B_\nu(T_M)$ , depending on a given (but possibly large) temperature  $T_M$ , and with outgoing data given by an accretive operator, of the form  $I_\nu(0, \mu)|_{\mu>0} = \mathcal{A}(I_\nu(0, -\mu))|_{\mu>0}$ . The problem has a unique well defined nonnegative solution.

### 6.3. Examples of Albedo operators

Below are given some examples of albedo operators which combine an accommodation parameter  $\alpha$  and the reflection and the thermalisation effects of the Earth.

The simplest example would be given (cf. Remark 3) with  $0 \leq \alpha \leq 1$  by the formula,  $p > 0$ :

$$\mathcal{A}(I(0, \mu)) = \alpha I(0, -\mu) + (1 - \alpha) \int_0^1 \mu^p I(0, -\mu') d\mu'. \quad (39)$$

For the more realistic frequency dependent cases, one should compare the reflection with the emission from the Earth as global black body under an effective temperature  $T_e \simeq 288$  and this leads to try the albedo operator

$$\mathcal{A}(I_\nu(0, -\mu)) = \alpha I_\nu(0, -\mu) + (1 - \alpha) B_\nu(T_e), \quad \mu > 0. \quad (40)$$

This obviously satisfies the hypothesis of non accretivity.

## 7. Calculus of Variations for the $\nu$ -dependent Case

The following argument sheds some light on the conditions needed to obtain a cooling (resp. heating) from a local increase of the absorption coefficient.

**Conjecture 1.** *Let  $I_\nu^\varepsilon(\tau, \mu), T^\varepsilon(\tau)$  be the solution of (5) when*

$$\kappa_\nu^\varepsilon := \kappa + \varepsilon \delta \kappa_\nu, \quad (41)$$

*Let  $I_\nu^0(\tau, \mu), T^0(\tau)$  be the solution of (5) with  $\kappa_\nu = \kappa$  constant. When  $\delta \kappa_\nu \approx \delta_{\nu^*}$ , the Dirac mass at  $\nu^*$ , and  $\kappa$  is small, then the sign of  $\frac{dT}{d\varepsilon}(\tau)|_{\varepsilon=0}$  is governed by the sign of*

$$\frac{1}{2} \int_{-1}^1 I_{\nu^*}^0(\tau) d\mu - B_{\nu^*}(T^0(\tau)). \quad (42)$$

### 7.1. Calculus of Variations Support of the Conjecture

It will be convenient to define the Planck function in terms of the quantity given by the Stefan-Boltzmann law  $\Phi := \sigma T^4$ . Henceforth we denote

$$B_\nu(T) = b_\nu(\Phi).$$

We seek to study how the average radiation intensity or the temperature is altered if the absorption is modified on various intervals in the frequency variable. We expect that the simplest situation corresponds to absorption of the form (41) with  $\kappa > 0$  independent of  $\nu$ , while  $0 < \varepsilon \ll 1$ . This is of course an extremely general formulation, but in practice one could think of

$$\delta \kappa_\nu := \kappa \mathbf{1}_{\nu_1 < \nu < \nu_2}.$$

This corresponds to multiplying the absorption by  $(1 + \varepsilon)$  in the frequency interval  $(\nu_1, \nu_2)$  only, and leaving it invariant for all the other frequencies.

We assume that the scattering is isotropic with constant rate  $\lambda := \kappa a \geq 0$ .

Henceforth, we consider the radiative transfer equation on  $(0, Z) \times (-1, 1) \times \mathbb{R}$ ,

$$\mu \partial_\tau I_\nu^\epsilon + \kappa_\nu^\epsilon (I_\nu^\epsilon - b_\nu(\Phi^\epsilon)) + \lambda(I_\nu^\epsilon - \tilde{I}_\nu^\epsilon) = 0, \quad \left\langle \kappa_\nu^\epsilon (\tilde{I}_\nu^\epsilon - b_\nu(\Phi^\epsilon)) \right\rangle = 0, \quad (43)$$

with the boundary conditions

$$I_\nu^\epsilon(0, \mu) = \mu Q_\nu^0, \quad I_\nu^\epsilon(Z, -\mu) = 0, \quad 0 < \mu < 1, \quad (44)$$

and the notations

$$\tilde{\phi} := \frac{1}{2} \int_{-1}^1 \phi(\mu) d\mu, \quad \langle \psi \rangle := \int_0^\infty \psi(\nu) d\nu.$$

First we study the case  $\epsilon = 0$ : since  $\kappa$  is constant, one can average in frequency, and set

$$\bar{I}^0(\tau, \mu) := \langle I_\nu^0(\tau, \mu) \rangle, \quad \tilde{\bar{I}}^0(\tau) = \langle b_\nu(\Phi^0) \rangle = \Phi^0(\tau).$$

One finds

$$\mu \partial_\tau \bar{I}^0(\tau, \mu) + \kappa(\bar{I}^0(\tau, \mu) - \Phi^0) + \lambda(\bar{I}^0(\tau, \mu) - \tilde{\bar{I}}^0(\tau)) = 0,$$

or, equivalently

$$\begin{aligned} \mu \partial_\tau \bar{I}^0(\tau, \mu) + (\kappa + \lambda)(\bar{I}^0(\tau, \mu) - \tilde{\bar{I}}^0(\tau)) &= 0, \quad \tau > 0, \\ \bar{I}^0(0, \mu) &= \mu \langle Q_\nu^0 \rangle, \quad 0 < \mu < 1. \end{aligned}$$

Once  $\tilde{\bar{I}}^0$  is known, one recovers  $I_\nu^0$  by using the semi-analytical formula (11), which in this case is

$$\begin{aligned} I_\nu^0(\tau, \mu) &= \mathbf{1}_{0 < \mu < 1} \mu e^{-\frac{(\kappa+\lambda)\tau}{\mu}} Q_\nu^0 + \mathbf{1}_{0 < \mu < 1} \int_0^\tau e^{-\frac{(\kappa+\lambda)(\tau-y)}{\mu}} \frac{\kappa b_\nu(\tilde{\bar{I}}^0(y)) + \lambda \tilde{\bar{I}}_\nu^0(y)}{\mu} dy \\ &\quad + \mathbf{1}_{-1 < \mu < 0} \int_\tau^\infty e^{-\frac{(\kappa+\lambda)(y-\tau)}{|\mu|}} \frac{\kappa b_\nu(\tilde{\bar{I}}^0(y)) + \lambda \tilde{\bar{I}}_\nu^0(y)}{|\mu|} dy. \end{aligned} \quad (45)$$

This is precisely the solution of the Milne problem in the case of radiative transfer in a grey atmosphere.

In order to study the effect of the perturbation of the absorption defined above, we seek to compute

$$I'_\nu(\tau, \mu) = \left. \frac{\partial I_\nu^\epsilon(\tau, \mu)}{\partial \epsilon} \right|_{\epsilon=0}, \quad \Phi'(\tau) = \left. \frac{\partial \Phi^\epsilon(\tau)}{\partial \epsilon} \right|_{\epsilon=0}, \quad u'(\tau, \mu) = \left. \frac{\partial u^\epsilon(\tau, \mu)}{\partial \epsilon} \right|_{\epsilon=0}.$$

Let  $\dot{b}_\nu(\Phi) = \partial_\Phi b_\nu(\Phi)$ . Let us differentiate in  $\epsilon$  both sides of the equations (43) at  $\epsilon = 0$ : one easily finds that

$$\begin{aligned} \mu \partial_\tau I'_\nu + \kappa(I'_\nu - \dot{b}_\nu(\Phi^0)\Phi'_0) + \lambda(I'_\nu - \tilde{I}'_\nu) + \delta\kappa_\nu(I_\nu^0 - b_\nu(\Phi^0)) &= 0, \\ \kappa \langle \tilde{I}'_\nu \rangle - \underbrace{\kappa \langle \dot{b}_\nu(\Phi^0) \rangle}_{=1} \Phi' + \langle \delta\kappa_\nu(\tilde{I}_\nu^0 - b_\nu(\Phi^0)) \rangle &= 0, \end{aligned}$$

and we can recast the second equality as

$$\Phi'(\tau) = \tilde{I}^0'(\tau) + \frac{1}{\kappa} \left\langle \delta\kappa_\nu(\tilde{I}_\nu^0(\tau) - b_\nu(\Phi^0(\tau))) \right\rangle = \tilde{I}^0'(\tau) + \frac{1}{\kappa} \left\langle \delta\kappa_\nu(\tilde{I}_\nu^0(\tau) - b_\nu(\tilde{I}^0(\tau))) \right\rangle. \quad (46)$$

Averaging in frequency the RT equation, we find that

$$\mu \partial_\tau \bar{I}^0'(\tau, \mu) + \kappa (\bar{I}^0'(\tau, \mu) - \underbrace{\langle \dot{b}_\nu(\Phi^0(\tau)) \rangle}_{=1} \Phi'(\tau)) + \left\langle \delta\kappa_\nu(I_\nu^0(\tau, \mu) - b_\nu(\Phi^0(\tau))) \right\rangle = 0,$$

and we can further eliminate  $\Phi'(\tau)$  between the last two equations, to find,

$$\mu \partial_\tau \bar{I}^0'(\tau, \mu) + (\kappa + \lambda)(\bar{I}^0'(\tau, \mu) - \tilde{I}^0'(\tau)) = - \left\langle \delta\kappa_\nu(I_\nu^0(\tau, \mu) - \tilde{I}_\nu^0(\tau)) \right\rangle. \quad (47)$$

If  $\delta\kappa$  is small everywhere but large near  $\nu^*$  and if  $\kappa + \lambda \ll 1$  so that the righthandside dominates the second term on the left, then (47) becomes

$$\mu \partial_\tau \bar{I}^0'(\tau, \mu) = -\delta\kappa(I_{\nu^*}^0(\tau, \mu) - \tilde{I}_{\nu^*}^0(\tau)).$$

Together with (46) it gives the sign of  $T'(z)$  by,

$$\Phi'(\tau) = -\delta\kappa \left| \int^\tau \frac{\widetilde{I_{\nu^*}^0(t, \mu)}}{\mu} dt \right| + \frac{\delta\kappa}{\kappa} (\tilde{I}_{\nu^*}^0(\tau) - b_{\nu^*}(\tilde{I}^0(\tau))). \quad (48)$$

when  $\kappa \ll 1$  the second terms dominate the first one and it is possible to obtain a positive sign when  $\tilde{I}_{\nu^*}^0(\tau) > b_{\nu^*}(\tilde{I}^0(\tau))$ .

## 7.2. Numerical Simulations

We study the sign of (42). As it is a complex surface of  $\nu$  and  $\tau$ , we look at the frequencies and altitudes for which (42) changes sign when one of the two variable varies. Figure 3 shows the frequencies and altitudes where there is a change of sign, for the grey case  $\kappa = 0.5$ .

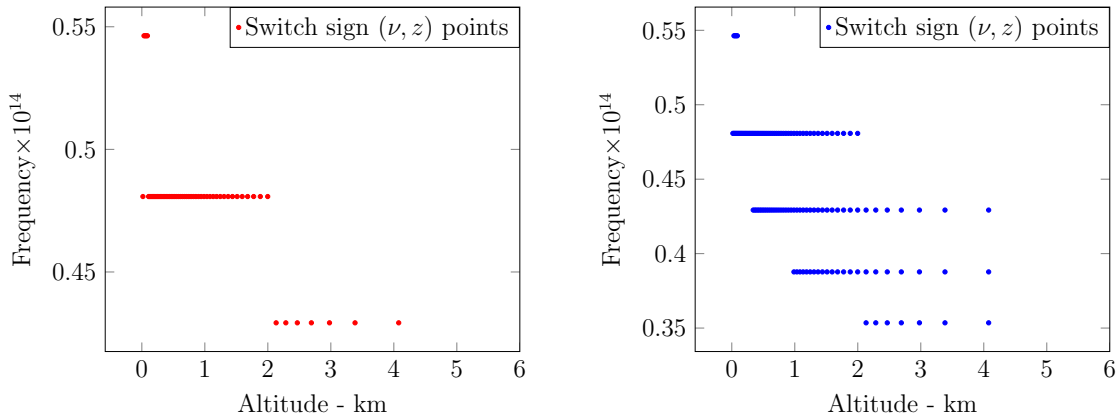


Figure 3: Switch sign points: left when  $\nu$  is increased at fixed  $z$ . Right : when altitude is increase at fixed  $\nu$ .

Consider now  $\kappa_\nu = 0.5 + \delta\kappa \mathbf{1}_{(\nu^1, \nu^2)}$ . We use an AD (automatic differentiation) extension of the computer program for (45) and display the sensitivity of  $T$  with respect to  $\delta\kappa$  for 3 ranges of support  $(\nu^1, \nu^2)$ .

On Figure 4 it is seen that when  $\nu^1 = 0.2, \nu^2 = 0.3$  the sensitivity of  $T$  is negative, meaning that increasing  $\kappa_\nu$  in the range  $(\nu^1, \nu^2)$  triggers a decrease of temperature at all altitudes. Conversely with  $\nu^1 = 0.6, \nu^2 = 0.8$  the sensitivity is positive above 800m meaning that an increase of  $\kappa_\nu$  in that range triggers a temperature decrease below 800m and an increase above 800m. The same is true for  $T'_1$  but at 1500m instead of 800m.

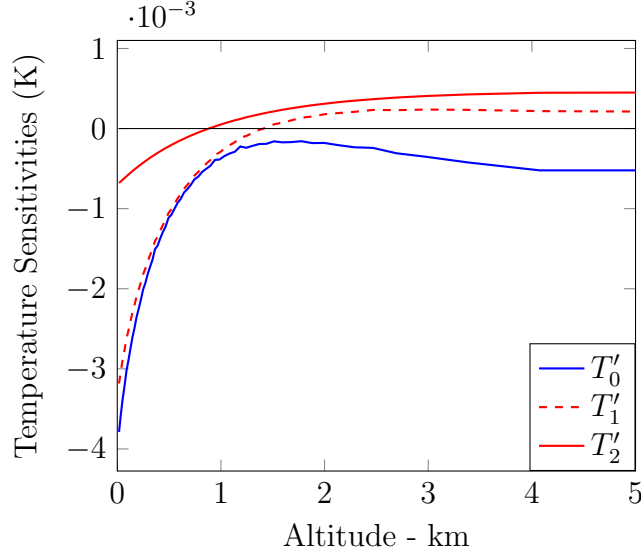


Figure 4: Scaled temperatures sensitivities  $z \rightarrow T'(z)$  versus altitude, computed with  $\kappa_\nu = 0.5 + \delta\kappa \mathbf{1}_{(\nu^1, \nu^2)}$  and  $\delta\kappa$ , at  $\delta\kappa = 0$  when case 0:  $\nu^1 = 0.2, \nu^2 = 0.3$ . Case 1:  $\nu^1 = 0.3, \nu^2 = 0.4$ . Case 2 is with  $\nu^1 = 0.6, \nu^2 = 0.8$ . Notice the change of signs in the sensitivity  $T'_2$ .

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```

1
2 // main.cpp : solves the radiative trnsfer equations
3 // for any absorption coefficient kappa
4 // does automatic differentiation if _NOAUTODIF is undefined
5 // Created by Olivier Pironneau on 10/10/2021.
6 //
7
8 #include <iostream>
9 #include <fstream>
10 #include <cmath>
11 #include <string>
12 #include <time.h>
13 using namespace std;
14 #define sqr(x) (x*x)
15
16 #define _NOAUTODIF
17
18 #if defined(_NOAUTODIF)
19     #define ddouble double
20     #define ffabs(x) (fabs(x))
21     const double dknu=0;
22 #else
23     #include "ddouble.h"
24     #define ffabs(x) (fabs(x.val[0]))
25     const ddouble dknu(0.,1.); // means dknu=0.1 and AD computes d.T/d.dknu
26 #endif
27 const bool greycase=true; // see beginning of main() for parameters. If
    false use below
28 const string kappa012="kappa0f"; // defines 3 runs with kappa vs nu in
    columns 2,3,4
29 const bool verbose=false;
30 const int Ntau=60; // nb points in tau
31 const int kmax=16; // nb fixed point iterations
32 const double Z=1-exp(-12.); // max tau after change of var
33 const double SBsun =2.03e-5*(2*sqr(2.))/0.7; // scaled sunlight power
34 const double Tsun = 1.209; // scaled sun temperature
35 const int jmaxmax=600; // max of max nb of points for integration in nu
    range
36 const int newton = 50; // to compute the temperature from int k*Botlzmann=
    int k*Imean
37 const double epsdycho=0.01, epsnewton=1.e-12; // precision for dychotomy
    before Newton
38 const double tmin=1.e-10; // min t in ExpInt(t)
39 const double dtt = 0.005; //min integration step size in ExpInt integrals

```

```

40 const double kappamin=0.001; // if kappa read is too small max it with
    kappamin
41 const int nt = 5; // min nb of integration step in anal formula
42 const double ealb=0.3; // and Earth albedo
43 const double alpha=1; // proportion of light affected at altitude 0 vs Z
44 const double ais=0., ars=0.; // max isotropic and Rayleigh scattering
45 const double tm1 = Z*0.6, tm2=Z*0.9; // altitudes for scattering
46 double nu[jmaxmax], // uneven discretization of [numin,numax]
47     ai[jmaxmax], ar[jmaxmax], aux[jmaxmax]; // isotropic and Rayleigh
    scattering
48 ddouble Inut[Ntau], // mu integral of I_nu
49     Snut[Ntau], // mu integral of mu^2 I_nu
50     G[jmaxmax][Ntau], // J(nu,tau)
51     S[jmaxmax][Ntau], // K(nu,tau)
52     T[Ntau], kappanu[jmaxmax]; // T0 with kappanu0[]
53 string basedir("/Users/pironneau/Dropbox/afaire/Golse-Bardos/greenhouse4/")
    ;
54 string myresulttemperature("temperature");
55 string myresultmeanintensity("imean");
56 string mykappafile(basedir+kappa012+".txt"); // has nu, kappa0, kappa1,
    kappa2
57 int jmax;
58
59 struct ddouble2{ ddouble d1,d2;};
60
61 // don't use if kappa >18/Z
62 ddouble expint_E1(const ddouble t=1){
63     // if your compiler has it or if you can link to gsl you may adapt this
    function
64     // it computes E1(t)*B when t<18
65     double abst=ffabs(t);
66     const int K = 9+(abst-1)*4; // precision in the exponential integral
    function E1
67     const double gaNtaua =0.577215664901533; // special integration for
    log(t)
68     if(abst<tmin) return 0.; // because integral_0^t(logx)dx ~0
69     if(abst>18) {cout << "value of E1 is incorrect with "<<t<<">18"<<endl;
    return 0;}
70     ddouble ak=(t<0)?-t: t, soNtaue=-gaNtaua - log(abst)+ak;
71     for(int k=2;k<K;k++){
72         ak *= -abst*(k-1)/sqr(k);
73         soNtaue += ak;
74     }
75     return soNtaue;
76 }
77 ddouble expint_E2(const ddouble t=1){
78     ddouble aux = exp(-t) - t*expint_E1(t);
79     return aux;
80 }
81 ddouble expint_E3(const ddouble t=1){
82     return (exp(-t) - t*expint_E2(t))/2;
83 }
84 ddouble expint_E4(const ddouble t=1){
85     return (exp(-t) - t*expint_E3(t))/3;
86 }

```

```

87 ddouble expint_E5(const ddouble t=1){
88     return (exp(-t) - t*expint_E4(t))/4;
89 }
90
91 int readkappa(string mykappafile, int which=0){
92     // on each line of kappafile: nu kappa0 & optional kappa1 kappa2
93     ifstream kappafile(mykappafile);
94     int j=-1;
95     double kappaux=0, dummy, nuj=0.01;
96     while((j++<=jmaxmax)&&!kappafile.eof()){
97         if(which==0) kappafile >> nuj >>kappaux >> dummy >> dummy >> dummy;
98         else if(which==1) kappafile >> nuj >> dummy >>kappaux >> dummy >>
99         dummy;
100        else if(which==2) kappafile >> nuj >> dummy >> dummy >>kappaux >>
101        dummy;
102        kappanu[j] = fmax(kappaux,kappamin);
103        nu[j]=nuj;
104    }
105    kappafile.close();
106    jmax=j;
107    return 0;
108 }
109
110 double Bsun(const double nu){ return SBsun*sqr(nu)*nu/(exp(nu/Tsun) -1);}
111 // Boltzmann
112
113 ddouble BB(const double nu, const ddouble T){
114     if(T<1.e-7) return 0;
115     if(nu<1e-10) return T*sqr(nu);
116     return sqr(nu)*nu/(exp(nu/T) -1);} // Boltzmann
117
118 ddouble dBB(const double nu, const ddouble T){
119     if(T<1.e-7) return 0;
120     if(nu<1.e-10) return sqr(nu);
121     ddouble a = exp(nu/T); return a*sqr(nu*nu/(a -1)/T);
122 }
123
124 ddouble2 intBS(const int jnu, const double tau, const double tmin, const
125 double tmax){
126     // returns the convolution t-integral for the mean in mu of kappa*I_nu
127     // and mu^2 kappa*I_nu
128     ddouble Imean=0, Imu2mean=0;
129     ddouble kappa=kappanu[jnu];
130     const double dt=fmin(dtt, nt/(tmax-tmin));
131     for(double t=tmin; t<tmax; t+=dt){
132         int it = int((Ntau-1)*t/Z); // parabolic length for frequency step
133         double arnu4=ar[it]*sqr(nu[jnu]-0.8)*sqr(nu[jnu]-1.2)*(nu[jnu]>0.8)
134         *(nu[jnu]<1.2)*40;
135         ddouble H0 = kappa*( BB(nu[jnu],T[it])*(1-arnu4) +
136             (ai[it]+1.125*arnu4)*G[jnu][it] -1.125*arnu4*S[jnu]
137             [it]) ;
138         ddouble H2 = -0.375*arnu4*kappa*(G[jnu][it] -3*S[jnu][it]);
139         if(kappa*(t-tau)!=0){
140             Imean += dt*H0*(expint_E1(kappa*fabs(tau-t))+ealb*expint_E1(
141             kappa*(tau+t)))/2;
142             Imu2mean += dt*H0*(expint_E3(kappa*fabs(tau-t))+ealb*expint_E3(

```

```

134     kappa*(tau+t)))/2;
135         if(H2 !=0){
136             Imean += dt*H2*(expint_E3(kappa*fabs(tau-t))+ealb*expint_E3
137             (kappa*(tau+t)))/2;
138             Imu2mean += dt*H2*(expint_E5(kappa*fabs(tau-t))+ealb*
139             expint_E5(kappa*(tau+t)))/2;
140         }
141     }
142     }
143     }
144     ddouble2 mean; mean.d1=Imean; mean.d2 = Imu2mean;
145     return mean;
146 }
147
148 ddouble thefunc(const ddouble rhs, const ddouble T0){
149     ddouble myeq=-rhs;
150     for(int j=1; j<jmax;j++){
151         myeq += kappanu[j]*BB(nu[j],T0)*(nu[j]-nu[j-1]);
152     }
153     return myeq;
154 }
155
156 ddouble getTbydycho(const int i, const ddouble Tstart){
157     ddouble T0=Tstart;
158     if(Tstart<0.1) T0=0.1;
159     ddouble Taux, myeq0=1,myeq1=-1, T1=T0, rhs=0;
160     for(int j=1; j<jmax;j++){
161         rhs+=kappanu[j]*G[j][i]*(nu[j]-nu[j-1]);
162     }
163     myeq0 =-rhs;myeq1=-rhs;
164     while (myeq0>0){
165         T0=T0/2;
166         myeq0=thefunc(rhs,T0);
167         if(verbose) cout<<T0<<" down "<<myeq0<<endl;
168     }
169     while (myeq1<0){
170         T1=2*T1; myeq1=thefunc(rhs,T1);
171         if(verbose) cout<<T1<<" up "<<myeq1<<endl;
172     }
173     while (T1-T0 > epsdycho){
174         Taux=(T1+T0)/2;
175         myeq0=thefunc(rhs,Taux);
176         if(myeq0>0) T1=Taux; else T0=Taux;
177         if(verbose) cout<<T0<<" middle "<<T1<<" "<<myeq0 << endl;
178     }
179     return (T1+T0)/2;
180 }
181
182 int genT(){
183     for(int i=0;i<Ntau;i++){
184         T[i] = getTbydycho(i,T[i]);
185         ddouble presfunc = 1;
186         int inewton =0;
187         while(inewton++<newton && fabs(presfunc) > epsnewton){
188             ddouble T0 = T[i];
189             ddouble left=0, rhs=0, deriv=0;
190             double nu1=0;

```

```

186         for(int j=1; j<jmax; j++){
187             double dnu=nu[j]-nu[j-1]; // variable integral increment
188             nu1=(nu[j]+nu[j-1])/2;
189             rhs+=kappanu[j]*G[j][i]*dnu;
190             left += kappanu[j]*BB(nu1,T0)*dnu;
191             deriv += kappanu[j]*dBB(nu1,T0)*dnu;
192         }
193         presfunc = rhs-left;
194         if(ffabs(deriv)>1e-10) T[i] = T0+presfunc/deriv;
195         if(verbose) cout<<T[i]<<" newton "<<presfunc<<endl;
196     }
197     if(inewton>=newton)cout << "Newton precision doubtful "<<endl;
198 }
199 return 0;
200 }
201
202 int getT(){ // return temperature by Kirchhof's law when kappa is constant
203     const double pi = 4*atan(1.);
204     for(int i=0; i<jmax; i++){
205         ddouble rhs=0;
206         for(int j=1; j<jmax; j++){
207             rhs+=G[j][i]*(nu[j]-nu[j-1]);
208             T[i]=sqrt(sqrt(15*rhs))/pi;
209         }
210         return 0;
211     }
212
213 int getISnu(const int j){ // returns mean I_nu & mean mu^2 I_nu
214     double nu1 = nu[j];
215     ddouble kappa=kappanu[j];
216     if(kappa<0.01) kappa=0.01;
217     for(int i=0; i<Ntau; i++){
218         double x=i*Z/(Ntau-1);
219         ddouble2 aux = intBS(j,x,0,Z);
220         Inut[i] = aux.d1 + Bsun(nu1)*(alpha*expint_E3(kappa*(x+Z)) + (1-
221             alpha)*expint_E3(kappa*(Z-x))
222             )/2; // +ealb*alpha*
223             expint_E3(kappa*(Z+x)))/2;
224         Snut[i] = aux.d2 + Bsun(nu1)*(alpha*expint_E5(kappa*x) +(1-alpha)*
225             expint_E5(kappa*(Z-x))
226             )/2; // +ealb*alpha*(expint_E5(
227             kappa*(Z+x)))/2;
228     }
229     return 0;
230 }
231
232 int multiBlock(double initT)
233 {
234     for(int i=0; i<Ntau; i++){ T[i]=initT; // initialize
235         for(int j=0; j<=jmax; j++){ G[j][i]=0; S[j][i]=0; }
236     }
237     for(int k=0; k<kmax; k++){ // fixed point loop: first update F
238         for(int i=0; i<Ntau; i++){
239             Inut[i]=0; Snut[i]=0;
240         }
241     }

```

```

237         for(int j=0; j<jmax;j++){
238             getISnu(j);
239             for(int i=0;i<Ntau;i++){
240                 G[j][i]=Inut[i];
241                 S[j][i]=Snut[i];
242             }
243         }
244         if( greycase )
245 #if defined(_NOAUTODIF)
246             genT(); // replace by getT() to speed-up but know what you do!
247 #else
248             genT();// Then update T. AD need genT() and false with getT() because
                kappa varies
249 #endif
250         else
251             genT();
252             ddouble normG=0, normS=0;
253             for(int j=0; j<jmax;j++){
254                 for(int i=0;i<Ntau;i++){ normG+=ffabs(G[j][i]); normS+=ffabs(S[j][i]); }
255             cout << "k=" <<k <<" " <<T[2]<<" " <<normG<<" " <<normS<<endl;
256             // ofstream tempfile(basedir+"tempHistScatLow"+std::to_string(k)+".txt
                "); //study convergence
257             // for(int i=1;i<Ntau;i++) tempfile<<-log(1-i*Z/(Ntau-1))<<"\t" <<T[
                i] <<endl;
258             }
259             return 0;
260     }
261
262 int main(int argc, const char * argv[]) {
263     for(int it=0;it<Ntau;it++){ // defines scattering values vs tau
264         double tm = it*Z/Ntau;
265         ai[it] = ais*fmax(tm-tm1,0.)*fmax(tm2-tm,0.)*4/sqr(tm2-tm1);
266         ar[it] = ars*fmax(tm-tm2,0.)/(Z-tm2);
267     }
268     for(int which=0;which<1;which++){ // change to which<1 for a single
        computation
269         if(greycase){
270             const double kappa0=0.5, numin = 0.05, numax = 15;
271             jmax = 400;
272             for(int j=0;j<jmax;j++){
273                 nu[j]= numax/(1+(jmax-j)*(numax-numin)/numin/jmax); // uniform
                in wavelength
274             // kappanu[j]=kappa0;//+dknu*(nu[j]>0.2)*(nu[j]<0.3);
275                 kappanu[j]=kappa0*(nu[j]<6) + 0.1;
276             }
277         } else{
278             readkappa(mykappafile, which);
279             // for(int j=0;j<jmax;j++) if((nu[j]<3./4)*(nu[j]>3./7)) kappanu[j]
                =fmax(0.5,1.5*kappanu[j]);
280         }
281         ofstream resultfile =
282             ofstream(basedir+myresulttemperature+to_string(which+10*(ais>0))+".
                txt");
283         cout<<"\n iterations \t [T] near earth [T] far ||G|| and ||S||\n";

```

```

284     double t0 = clock();
285     multiBlock(0.); //,"tempmin");
286     printf( " Time CPU = %10.6f\n", (clock() - t0)/CLOCKS_PER_SEC);
287     cout<<"\n tau\t [T]:"<<endl;
288     for(int i=1;i<Ntau;i++){
289         cout << -log(1-i*Z/(Ntau-1))<<"\t"<<T[i] <<endl;
290         resultfile << -log(1-i*Z/(Ntau-1))<<"\t" // altitude
291             <<T[i]<<endl; // T(kappa) Milne by multigroup
292     }
293     ofstream imeanz0 =
294         ofstream(basedir+myresultmeanintensity+to_string(10*(ais>0)+
which)+"0.txt");
295     ofstream imeanzZ =
296         ofstream(basedir+myresultmeanintensity+to_string(10*(ais>0)
+which)+"Z.txt");
297     for(int j=10; j<jmax;j++){
298         imeanz0<< 3/nu[j] <<" "<<1e5*G[j][1]<<endl; // mean intensity
near the ground
299         imeanzZ<< 3/nu[j] <<" "<<1e5*G[j][Ntau-1]<<endl; // mean
intensity at max altitude
300     }
301     resultfile.close();
302     imeanz0.close();
303 }
304 return 0;
305 }

```