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***Abstract*— The problem of finite-time stabilization of linear systems under state and control input constraints in the presence of external disturbances is addressed. The proposed control is designed such that the closed-loop system is 1) superstable, when system trajectories risk to violate the state constraints, and 2) homogeneous with a negative degree, otherwise. While the former property ensures the fulfillment of the state and control constraints, the latter implies finite-time stability. The robustness of the control scheme with respect to disturbances is studied. Theoretical results are supported by numerical simulations.**

I. INTRODUCTION

In practice physical limitations and operational requirements impose restrictions on the functioning of dynamical systems. For example, state constraints require system trajectories always remain in some set. Besides, in many applications a control goal must be achieved within a prescribed time interval. Furthermore, only a limited amount of energy is available for control purposes. Clearly, neglecting the constraints at the control design stage may lead to performance degradation, system failures or safety risks.

Albeit numerous approaches have been developed to address these problems, the most popular ones are Lyapunov-like methods based on *control barrier functions* (CBFs) [15], [18] or *barrier Lyapunov functions* (BLFs) [5], [17]. The former method consists in introducing a continuously differentiable function 1) that is nonpositive for all admissible values of the state vector and 2) whose time derivative does not increase along trajectories of the system. However, this approach has limited use in practice due to the assumption on smoothness of CBFs. On the other hand, the latter method involves the construction of a Lyapunov function that grows to infinity whenever its arguments approach some limits. Though a wider class of state constraints can be considered, it is clear that BLFs have to be essentially nonlinear. Thus, stability analysis is a difficult task even for linear systems.

Nevertheless, if system trajectories have to stay in some hyperrectangle centered at the origin, then for a subclass of linear systems, so-called *superstable* [7], [9], control design admits a relatively simple solution. Superstability means strict diagonal dominance of the state matrix. Therefore, the control design can be easily formulated as a linear optimization problem [8]. However, superstabilization requires more severe restrictions on system matrices rather than just controllability. For example, being controllable, a chain of integrators cannot be superstabilized by any linear feedback. On the other hand, if the corresponding optimization problem is feasible, then the system is superstabilizable. Furthermore, the state constraints are fulfilled even in the presence of sufficiently small external disturbances [7].

However, superstability itself does not provide finite-time stability and, thus, additional nonlinear control has to be designed. One of the conventional ways to analyze finite-time stability is to consider *homogeneous* systems [12], [16], [21]. Such a property drastically simplifies stability analysis and control design: the rate of convergence of an asymptotically stable homogeneous system is completely defined by its degree of homogeneity. Moreover, selection of a canonical homogeneous norm as a Lyapunov function provides a constructive way to control design [13], [20]: all conditions to check have a linear matrix inequality (LMI) representation. Usually in control theory *weighted homogeneity* [21] is used. Nevertheless, it only can be applied if either the system itself or its block decomposition [14] has a special canonical form (a chain of integrators). While the former contradicts the necessary condition on superstabilizability, the latter can lead to significant computational errors. On the other hand, *generalized homogeneity* is free of these requirements [20] and, therefore, suitable for a larger class of systems.

Therefore, the main objective of this paper is to design a continuous control law such that the closed-loop system is 1) superstable, when system trajectories risk to violate the state constraints, and 2) homogeneous with a negative degree, otherwise. Such a choice allows us to use features of both approaches: the former ensures the fulfillment of the state and control constraints, while the latter implies finite-time stability. Furthermore, the closed-loop system is input-to-state stable (ISS) with respect to external disturbances. Another advantage of the proposed control scheme compared to [5], [17] is a simple tuning algorithm: all control parameters are obtained as solutions of linear matrix equations and inequalities with only two adjustable parameters.

The remainder of this paper is outlined as follows. Notation that will be used throughout the paper is given in Section II. The problem formulation and the state transformation are introduced in Section III. The problem of superstabilization of linear systems is discussed in Section IV. Definition of generalized homogeneity and its use for finite-time stabilization are presented in Section V. The control design is

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described in Section VI. Finally, results of the numerical simulation can be found in Section VII. All theorems are formulated without proofs due to the space limitation.

II. Preliminaries

A. Notation

- $\mathbb{R}$ is the field of real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and $\mathbb{R}_+^* = \mathbb{R}_+ \setminus \{0\}$;
- a series of integers $1, 2, \ldots, n$ is denoted by $\overline{1,n}$;
- $\mathrm{vec}(A) := [A_{11}, \ldots, A_{n1}, \ldots, A_{1m} \ldots, A_{nm}]^\top \in \mathbb{R}^{mn}$ is the vectorization of matrix $A \in \mathbb{R}^{n\times m}$;
- $\mathrm{diag}\{\lambda_i\}_{j=1}^n$ is the diagonal matrix with the elements $\lambda_j \in \mathbb{R}, j = \overline{1,n}$;
- $I_n \in \mathbb{R}^{n\times n}$ and $O_{n\times m} \in \mathbb{R}^{n\times m}$ are the identity and zero matrices, respectively;
- if $P = P^\top \in \mathbb{R}^{n\times n}$ then notations $P \succ 0$ ($P \prec 0$) and $P \succcurlyeq 0$ ($P \preccurlyeq 0$) mean that P is positive (negative) definite and semidefinite, respectively;
- $\|x\|_P := \sqrt{x^\top P x}$ is the weighted Euclidean norm of vector $x \in \mathbb{R}^n$, where $0 \prec P \in \mathbb{R}^{n\times n}$. The conventional Euclidean norm ($P = I_n$) is denoted by $\|x\|$;
- $\|A\|_P := \sup_{x\in\mathbb{R}^n\setminus\{0\}} \frac{\|Ax\|_P}{\|x\|_P}$ is the induced norm of matrix $A \in \mathbb{R}^{m\times n}$;
- $\|x\|_\infty := \max_{i=\overline{1,n}}\{|x_i|\}$ is the maximum norm;
- L^∞ is the space of Lebesgue measurable essentially bounded functions $\delta : \mathbb{R}_+ \to \mathbb{R}^n$ with the norm defined as $\|\delta\|_{L^\infty} := \mathrm{ess\,sup}_{t\in\mathbb{R}_+} \|\delta(t)\|_\infty < +\infty$;
- $\otimes$ is the Kronecker product;
- $\partial\Omega$ is the boundary of a closed set Ω;
- $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ are the minimal and maximal eigenvalues of a symmetric matrix $P \in \mathbb{R}^{n\times n}$, respectively;
- $\mathfrak{Re}(\lambda)$ denotes the real part of a complex number λ.

B. Comparison functions

A continuous function $w : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to *class* $\mathcal{K}$ if it is strictly increasing and $w(0) = 0$. A continuous function $w : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a *generalized class-*$\mathcal{K}$ *function* ($\mathcal{GK}$ function) if it is strictly increasing on $[s_0, +\infty)$ and $w(s) = 0$ for all $s \in [0, s_0]$ for some $s_0 \in \mathbb{R}_+$. A function $v : \mathbb{R}_+\times\mathbb{R}_+ \to \mathbb{R}_+$ is a *generalized class-*$\mathcal{KL}$ *function* ($\mathcal{GKL}$ function) if for each fixed $t \in \mathbb{R}_+$ the function $v(\cdot, t)$ is a class-$\mathcal{GK}$ function, and for each fixed $s \in \mathbb{R}_+$ the function $v(s, \cdot)$ is continuous, strictly decreasing and there exists some $T(s) \in \mathbb{R}_+$ such that $v(s, t) \to 0$ as $t \to T$.

III. Problem statement

Consider a linear system in the form

$$\dot{z}(t) = A_0 z(t) + B_0 u(t) + \delta(t), \quad z(0) = z_0, \qquad (1)$$

where $z(t) \in \mathbb{R}^n$ is the state vector available for measurement, $u(t) \in \mathbb{R}^m$ is the vector of control inputs, $\delta(t) \in \mathbb{R}^n$ is the external disturbance, $\delta \in \mathcal{D} := \{\delta \in L^\infty : \|\delta\|_{L^\infty} \leq \bar{\delta}\}$, $\bar{\delta} \in (0, +\infty)$ is known, $A_0 \in \mathbb{R}^{n\times n}$ and $B_0 \in \mathbb{R}^{n\times m}$ ($\mathrm{rank}(B_0) = m < n$) are the system matrices such that the pair (A_0, B_0) is controllable. Initial conditions z_0 belong to the set $\Pi_0 := \{z \in \mathbb{R}^n : |z_i| \leq \Delta_i, i = \overline{1,n}\}$, where $0 < \Delta_0 \leq \Delta_i < +\infty$, $i = \overline{1,n}$. A solution to system (1) with $u = u(z)$ is denoted as $z(t, z_0, \delta)$.

The goal is to design state feedback $u(z)$ such that for all $t \in \mathbb{R}_+$, $z_0 \in \Pi_0$ and $\delta \in \mathcal{D}$:

1) $|z_i(t, z_0, \delta)| \leq \Delta_i$, $i = \overline{1,n}$. Clearly, the ***state constraints*** can be expressed in the equivalent vector form:

$$\|Mz(t, z_0, \delta)\|_\infty \leq 1, \qquad (2)$$

where $M := \mathrm{diag}\{\Delta_i^{-1}\}_{i=1}^n$;

2) system (1) is ***finite-time ISS*** [4], i.e., there exist functions $v \in \mathcal{GKL}$ and $w \in \mathcal{K}$ such that:

$$\|Mz(t, z_0, \delta)\|_P \leq v(\|Mz_0\|_P, t) + w(\|\delta\|_{L^\infty}); \qquad (3)$$

3) control inputs are ***bounded***, i.e., for all $k = \overline{1,m}$

$$|u_k(t)| \leq \bar{u}_k < +\infty. \qquad (4)$$

To simplify the control design we will define new coordinates $x := Mz$. Therefore, system (1) is rewritten as follows:

$$\dot{x}(t) = Ax(t) + Bu(t) + M\delta(t), \quad x(0) = x_0, \qquad (5)$$

where $A := MA_0M^{-1}$, $B := MB_0$ and $x_0 := Mz_0$. The corresponding solution is $x(t, x_0, \delta) = Mz(t, z_0, \delta)$. Obviously, control constraints (4) and controllability property are invariant under the state transformation. Thus, further only system (5) will be considered.

As it has been mentioned in the Introduction, it is suggested to split the problem into two consecutive tasks: 1) fulfillment of state (2) and control input (4) constraints, and 2) finite-time stabilization (3). In the next sections it will be shown that both subproblems can be effectively solved by exploiting such properties as superstability [7], [9] and generalized homogeneity [10], [20].

IV. Superstability

A. Definition and input-to-state stability

Definition 1 [7]. *A matrix A is said to be superstable if it is strictly diagonally dominant, i.e.*

$$\sigma(A) = \sigma := \min_{i=\overline{1,n}} \Big\{ -A_{ii} - \sum_{j\neq i} |A_{ij}| \Big\} > 0, \qquad (6)$$

where parameter $\sigma > 0$ is called the degree of superstability.

Definition 2 [7]. *System* (5) *is called superstable if matrix A is superstable.*

For superstable systems the maximum norm $\|x\|_\infty$ always can be chosen as a Lyapunov function [19]. By analyzing its generalized gradient [2], the main property of superstable systems can be formulated in the following theorem.

Theorem 1 [7]. *If system* (5) *is superstable, then for $u(t) \equiv 0$ the following estimate holds:*

$$\|x(t, x_0, \delta)\|_\infty \leq \gamma + e^{-\sigma t}\chi(\|x_0\|_\infty - \gamma), \quad t \in \mathbb{R}_+, \qquad (7)$$

where $\gamma := \|M\delta\|_{L^\infty}/\sigma$, $\chi(\cdot) := \max\{0, \cdot\}$.

In other words, Theorem 1 states that for superstable systems the maximum norm of the state $\|x(t)\|_\infty$ decreases monotonically. Therefore, state constraints (2) are fulfilled if matrix A is superstable and $\|M\delta\|_{L^\infty} \leq \sigma$.

B. Superstabilization

Obviously, if matrix A is not superstable, there is still a possibility (not always, see Remark 1) to superstabilize system (5) by applying conventional linear feedback [8]

$$u(x) = u_1(x) := Kx, \tag{8}$$

where $K \in \mathbb{R}^{m\times n}$.

The next theorem presents a simple algorithm of calculating matrix K such that the closed-loop system (5), (8) is superstable of maximum possible degree σ.

Theorem 2. *If for given $\bar\delta > 0$ and $\bar u_k < +\infty$, $k = \overline{1,m}$, the following linear optimization problem*

$$\textbf{Maximize} \quad \sigma \quad \textbf{subject to}$$

$$\begin{cases} -D_{ii}(K) - \sum_{j\neq i} N_{ij} \geq \sigma, & i = \overline{1,n}, \\ -N_{ij} \leq D_{ij}(K) \leq N_{ij}, & i,j = \overline{1,n},\ j \neq i, \end{cases} \tag{9a}$$

$$\begin{cases} \sum_{j=1}^{n} Q_{kj} \leq \bar u_k, & k = \overline{1,m}, \\ -Q_{kj} \leq K_{kj} \leq Q_{kj}, & k = \overline{1,m},\ j = \overline{1,n}, \end{cases} \tag{9b}$$

$$\sigma > \bar\delta/\Delta_0, \tag{9c}$$

where $D_{ij}(K) := A_{ij} + \sum_{r=1}^{n} B_{ir}K_{rj}$, has a solution for some scalars σ, N_{ij}, $i,j = \overline{1,n}$, $j \neq i$, and matrices $K, Q \in \mathbb{R}^{m\times n}$, then:

1) the closed-loop system (5), (8) *is superstable;*

2) state (2) *and control* (4) *constraints are fulfilled.*

Remark 1. *It follows from* (9c) *and* (9b) *that if optimization problem* (9) *has a solution for $\bar\delta = 0$ and $\bar u_k = +\infty$, $k = \overline{1,m}$, i.e., $\sigma > 0$, then it is also feasible for sufficiently small $\bar\delta > 0$ and sufficiently large $\bar u_k < +\infty$, $k = \overline{1,m}$. Otherwise, system* (5) *cannot be superstabilized by any linear feedback* (8)*. Moreover, for some systems, e.g., a chain of integrators, infeasibility of* (9) *can be shown beforehand [8].*

Therefore, linear inequalities (9) *provide necessary and sufficient conditions on superstabilization of system* (5) *by using linear feedback* (8).

V. Generalized Homogeneity

A. Linear dilation and its properties

Homogeneity is a symmetry of an object with respect to a group of transformations, which is usually called dilation. The generalized homogeneity deals with the groups of linear transformations.

Definition 3 [11], [12]. *A map $\mathbf{d} : \mathbb{R} \to \mathbb{R}^{n\times n}$ is called dilation in the space $\mathbb{R}^n$ if it satisfies:*

- ***group property:*** *$\mathbf{d}(0) = I_n$ and $\mathbf{d}(t+s) = \mathbf{d}(t)\mathbf{d}(s) = \mathbf{d}(s)\mathbf{d}(t)$ for any $t, s \in \mathbb{R}$;*
- ***continuity property:*** *$\mathbf{d}$ is a continuous map;*
- ***limit property:*** *$\lim_{s\to-\infty} \|\mathbf{d}(s)x\| = 0$ and $\lim_{s\to+\infty} \|\mathbf{d}(s)x\| = +\infty$ uniformly on the unit sphere $\mathbb{S} := \{x \in \mathbb{R}^n : \|x\| = 1\}$.*

It follows from Definition 3 that dilation $\mathbf{d}$ is a uniformly continuous group. A matrix $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$ defined as $G_{\mathbf{d}} := \lim_{s\to 0} \frac{\mathbf{d}(s)-I_n}{s}$ is called the generator of dilation $\mathbf{d}$ and it satisfies the following properties [6]:

$$\tfrac{d}{ds}\mathbf{d}(s) = G_{\mathbf{d}}\mathbf{d}(s) = \mathbf{d}(s)G_{\mathbf{d}},$$

$$\mathbf{d}(s) = e^{G_{\mathbf{d}}s} = \textstyle\sum_{i=0}^{+\infty} \frac{s^i G_{\mathbf{d}}^i}{i!}, \quad s \in \mathbb{R}.$$

Clearly, uniform [16] (or standard) and weighted [21] dilations, which are widely used in control theory, satisfy Definition 3 with generators $G_{\mathbf{d}} = I_n$ and $G_{\mathbf{d}} = \operatorname{diag}\{r_i\}_{i=1}^n$, $r_i > 0$, $i = \overline{1,n}$, respectively.

Definition 4 [12]. *Dilation $\mathbf{d}$ is said to be **strictly monotone** if there exist $\beta > 0$ and $0 \prec P \in \mathbb{R}^{n\times n}$ such that $\|\mathbf{d}(s)\|_P \leq e^{\beta s}$, $\forall s \leq 0$.*

Thus, monotonicity means that $\mathbf{d}(s)$ is a strong contraction (expansion) for $s < 0$ ($s > 0$) and implies that for any $x \in \mathbb{R}\setminus\{0\}$ there exists a unique pair $(s_0, x_0) \in \mathbb{R}\times\mathbb{S}$ such that $x = \mathbf{d}(s_0)x_0$.

Obviously, monotonicity of dilation $\mathbf{d}$ depends on matrix P. The next theorem shows how any dilation in $\mathbb{R}^n$ always can be made strictly monotone.

Theorem 3 [10]. *If $\mathbf{d}$ is a dilation in $\mathbb{R}^n$, then:*

- *the generator matrix $G_{\mathbf{d}}$ is anti-Hurwitz, i.e., all eigenvalues $\lambda_i(G_{\mathbf{d}})$, $i = \overline{1,n}$, have strictly positive real part: $\mathfrak{Re}(\lambda_i(G_{\mathbf{d}})) > 0$, $i = \overline{1,n}$. Therefore, there exists a matrix $0 \prec P \in \mathbb{R}^{n\times n}$ such that*

$$PG_{\mathbf{d}} + G_{\mathbf{d}}^\top P \succ 0; \tag{10}$$

- *dilation $\mathbf{d}$ is strictly monotone with respect to the weighted Euclidean norm $\|\cdot\|_P$ with P satisfying* (10)*, i.e.*

$$\begin{aligned} e^{\alpha s} \leq \|\mathbf{d}(s)\|_P \leq e^{\beta s}, \quad \text{if } s \leq 0, \\ e^{\beta s} \leq \|\mathbf{d}(s)\|_P \leq e^{\alpha s}, \quad \text{if } s \geq 0, \end{aligned}$$

where $\alpha := 0.5\lambda_{\max}(P^{1/2}G_{\mathbf{d}}P^{-1/2} + P^{-1/2}G_{\mathbf{d}}^\top P^{1/2})$ and $\beta := 0.5\lambda_{\min}(P^{1/2}G_{\mathbf{d}}P^{-1/2} + P^{-1/2}G_{\mathbf{d}}^\top P^{1/2})$.

B. Generalized homogeneous functions and vectors fields

Definition 5 [12]. *A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ (a function $g : \mathbb{R}^n \to \mathbb{R}$) is said to be $\mathbf{d}$-homogeneous of degree $\nu \in \mathbb{R}$ if*

$$\begin{aligned} &f(\mathbf{d}(s)x) = e^{\nu s}\mathbf{d}(s)f(x), \quad \forall x \in \mathbb{R}^n\setminus\{0\},\ \forall s \in \mathbb{R}, \\ &\big(g(\mathbf{d}(s)x) = e^{\nu s}g(x), \quad \forall x \in \mathbb{R}^n\setminus\{0\},\ \forall s \in \mathbb{R}\big). \end{aligned}$$

For linear vector maps the following theorem is valid.

Lemma 4 [20]. *A vector field $f(x) = Fx$, $x \in \mathbb{R}^n$, $F \in \mathbb{R}^{n\times n}$, is $\mathbf{d}$-homogeneous of degree $\nu \in \mathbb{R}$ if and only if*

$$FG_{\mathbf{d}} - G_{\mathbf{d}}F = \nu F, \tag{11}$$

where $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$ is the generator of dilation $\mathbf{d}$.

A special case of homogeneous function $g(x)$ is a homogeneous norm [10]: a continuous positive definite $\mathbf{d}$-homogeneous function of degree $\nu = 1$.

Definition 6 [10]. *For a monotone dilation $\mathbf{d}$, the canonical homogeneous norm $\|x\|_{\mathbf{d}} : \mathbb{R}^n \to \mathbb{R}_+$ is defined as:*

$$\|x\|_{\mathbf{d}} := \begin{cases} e^{s_x}, \ \text{where } s_x \in \mathbb{R} : \|\mathbf{d}(-s_x)x\|_P = 1, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Definition 6 implies that the canonical homogeneous norm $\|x\|_{\mathbf{d}}$ is implicitly defined by the following equation:

$$\|\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x\|_P = 1, \quad \forall x \in \mathbb{R}^n \setminus \{0\}. \tag{12}$$

To solve equation (12) with respect to homogeneous norm $\|x\|_{\mathbf{d}}$ one can use, for example, the bisection method [14]. On the other hand, for the uniform dilation $\mathbf{d}(s) = e^{I_n s}$ the homogeneous norm is calculated explicitly as $\|x\|_{\mathbf{d}} = \|x\|_P$.

Theorem 5. *For a given strictly monotone dilation* $\mathbf{d}$*, its canonical homogeneous norm* $\|x\|_{\mathbf{d}}$ *admits the following estimates:*

$$\begin{aligned} \|x\|_P^{1/\beta} \leq \|x\|_{\mathbf{d}} \leq \|x\|_P^{1/\alpha}, \quad \text{if } \|x\|_{\mathbf{d}} \leq 1, \\ \|x\|_P^{1/\alpha} \leq \|x\|_{\mathbf{d}} \leq \|x\|_P^{1/\beta}, \quad \text{if } \|x\|_{\mathbf{d}} \geq 1, \end{aligned}$$

where α *and* β *are defined in Theorem 3.*

It is worth stressing that for a given strictly monotone dilation $\mathbf{d}$, its canonical homogeneous norm $\|x\|_{\mathbf{d}}$ is continuous on $\mathbb{R}^n$ and Lipschitz continuous on $\mathbb{R}^n \setminus \{0\}$ (see [10]). These properties motivate to use the canonical homogeneous norm as a Lyapunov function candidate for stability analysis of generalized homogeneous systems.

C. Generalized Homogeneous Finite-time Stabilization

The main feature of homogeneous systems is a quite simple criterion of finite-time stabilization that is formulated in the following theorem.

Theorem 6 [10]. *An asymptotically stable* $\mathbf{d}$*-homogeneous system* $\dot{x} = f(x)$*,* $f : \mathbb{R}^n \to \mathbb{R}^n$ *of degree* $\nu \in \mathbb{R}$ *is finite-time stable if and only if* $\nu < 0$*.*

For example, in [20] it is shown that for $\delta(t) \equiv 0$ system (5) can be stabilized in finite time by using the following continuous control law:

$$u(x) = u_2(x) := K_1 x + \|x\|_{\mathbf{d}}^{\nu+\epsilon} K_2 \mathbf{d}(-\ln \|x\|_{\mathbf{d}})x, \quad (13)$$

where $\nu < 0$, $\epsilon > |\nu|$ and $K_1, K_2 \in \mathbb{R}^{m\times n}$.

Let us comment on the selection of the control parameters.

Firstly, dilation $\mathbf{d}$ has to be strictly monotone. According to Theorem 3, this requirement is fulfilled for any matrix P satisfying Lyapunov inequality (10).

Secondly, feedback gains K_2 are selected such that matrix $A + B(K_1 + K_2)$ is Hurwitz. By introducing a Lyapunov function candidate $V(x) := \|x\|_{\mathbf{d}}$ and analyzing its derivative it can be proven that such a choice ensures asymptotic stability of the closed-loop system (5), (13). In this paper we will set $K_2 := K - K_1$, where K is obtained from (9).

Thirdly, feedback gains K_1 and generator $G_{\mathbf{d}}$ are chosen such that matrix $F := A + BK_1$ satisfies (11) and

$$G_{\mathbf{d}} B = \epsilon B. \quad (14)$$

Note that the latter implies $\|x\|_{\mathbf{d}}^{\epsilon} B = \mathbf{d}(\ln \|x\|_{\mathbf{d}})B$. Indeed, taking $s = \ln \|x\|_{\mathbf{d}}$, one can check that

$$\begin{aligned} &\big(\|x\|_{\mathbf{d}}^{\epsilon} I_n - \mathbf{d}(\ln \|x\|_{\mathbf{d}})\big)B = \mathbf{d}(s)\big(e^{\epsilon s}\mathbf{d}(-s) - I_n\big)B \\ &= \mathbf{d}(s) \textstyle\sum_{i=1}^{+\infty} \frac{s^i (\epsilon I_n - G_{\mathbf{d}})^{i-1}}{i!} \big(\epsilon B - G_{\mathbf{d}} B\big) = O_{n\times m}. \end{aligned}$$

Taking this and (11) into account, it is easy to show that the closed-loop system (5), (13) with $\delta(t) \equiv 0$ is $\mathbf{d}$-homogeneous of degree $\nu < 0$.

Following [20], we consider a special class of linear dilations $\mathbf{d}$ for which matrix equations (11) and (14) are always fulfilled if the pair (A, B) is controllable.

Theorem 7. *Let the pair of matrices* $A \in \mathbb{R}^{n\times n}$ *and* $B \in \mathbb{R}^{n\times m}$ *(*$\text{rank}(B) = m < n$*) be controllable. Then matrix equations* (11) *and* (14) *are fulfilled for* $G_{\mathbf{d}} = \nu L + \epsilon I_n$*,* $\nu, \epsilon \in \mathbb{R}$*, and* $F = A + BK_1$*, where matrices* $L \in \mathbb{R}^{n\times n}$ *and* $K_1 \in \mathbb{R}^{m\times n}$ *are solutions to the following system of matrix equations:*

$$(A + BK_1)L - LA = A + BK_1, \quad (15a)$$

$$LB = O_{n\times m}. \quad (15b)$$

Remark 2. *Note that system* (15) *always has a solution for the controllable pair* (A, B)*. Furthermore, for some special cases (e.g.,* $m = 1$*) the solution is unique.*

It also can be proven that matrix $L - I_n$ *is invertible. Hence, using the Kronecker product notation and the vectorization operator* $\text{vec}(\cdot)$*, the system of matrix equations* (15) *can be rewritten in a suitable for solving form, i.e.*

$$\begin{bmatrix} I_n \otimes A - A^\top \otimes I_n & I_n \otimes B \\ B^\top \otimes I_n & O_{nm\times nm} \end{bmatrix} \begin{bmatrix} \text{vec}(L) \\ \text{vec}(Y) \end{bmatrix} = \begin{bmatrix} \text{vec}(A) \\ O_{nm\times 1} \end{bmatrix}, \quad (16)$$

where $Y := K_1(L - I_n)$*.*

It is worth mentioning that the closed-loop system (5), (13) also remains stable for the disturbed case, i.e., $\delta(t) \neq 0$, due to input-to-state property of homogeneous systems [1].

VI. Main result

Taking into account advantages of superstable and homogeneous systems, we will define control law in the form

$$u(x) = \begin{cases} u_1(x), & \text{if } \|x\|_P > 1, \\ u_2(x), & \text{if } \|x\|_P \leq 1, \end{cases} \quad (17)$$

where $u_1(x)$ and $u_2(x)$ are defined in (8) and (13), respectively. Matrix P is chosen such that it satisfies (10) and, additionally, $\Omega := \{x \in \mathbb{R}^n : \|x\|_P \leq 1\} \subset \Pi := \{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\}$ (see Fig. 1). Hence, control (17) is continuous. Indeed, it follows from (12) that $\|x\|_P = 1 \Leftrightarrow \|x\|_{\mathbf{d}} = 1$ and $u_2(x) = K_1 x + K_2 x = Kx = u_1(x)$ for all $x \in \partial\Omega$.

Thus, one can see that the closed-loop system (5), (17) is:

1) ***superstable*** for all $x \in \Pi \setminus \Omega$ (blue area in Fig. 1). Therefore, Theorem 2 implies that state (2) and control (4) constraints are fulfilled;

2) ***finite-time ISS*** for all $x \in \Omega$ (red area in Fig. 1). Since $\Omega \subset \Pi$, then state constraints (2) cannot be violated.

Besides, the transition time from the set Π to the set Ω can be reduced if the latter is chosen as big as possible. Since $\partial\Omega$ defines a hyperellipsoid, then it can be done by maximizing its semi-minor axis, which is equal to $\lambda_{\max}^{-1/2}(P)$ (see Fig. 1). The next theorem presents an algorithm of calculating matrix P satisfying all the above-mentioned restrictions.

Theorem 8. *Let the pair* (A, B) *be controllable and linear optimization problem* (9) *be feasible for given* $\bar{\delta} > 0$ *and* $\bar{u}_k < +\infty$*,* $k = \overline{1, m}$*. If there exist* $\nu \in (-1, 0)$ *and* $\epsilon > |\nu|$ *such that the following linear optimization problem*

$$\textbf{Minimize} \quad \eta \quad \textbf{subject to}:$$

$$(A + BK)^\top P + P(A + BK) + \tfrac{3}{2}\sigma P \preccurlyeq 0, \quad (18a)$$

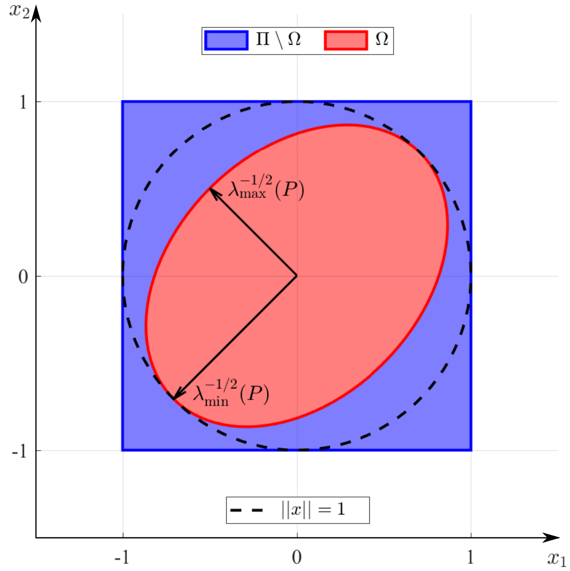

Fig. 1. Illustration of sets $\Pi \setminus \Omega$ and Ω for $n = 2$

$$P \succcurlyeq \nu(L^\top P + PL) + 2\epsilon P \succ 0, \tag{18b}$$

$$\eta I_n \succcurlyeq P \succcurlyeq I_n, \tag{18c}$$

where $\sigma > 0$, *K and L are obtained from* (9) *and* (16)*, has a solution for some η and $P = P^\top \in \mathbb{R}^{n\times n}$, then:*

1) the closed-loop system (5)*,* (17) *is finite-time ISS, i.e., relation* (3) *holds with*

$$w(p) = \max\left\{\tfrac{\sqrt{n\eta}}{\sigma\Delta_0}p, \left(\tfrac{2\sqrt{\eta}}{\sigma\Delta_0}p\right)^{\beta/(\nu+\alpha)}\right\},$$

$$v(s,t) = \begin{cases} \sqrt{n\eta}e^{-\sigma t}s, & \text{if } t \in [0, T_0(s)], \\ [\frac{\sigma|\nu|}{2}(T(s)-t)]^{\beta/|\nu|}, & \text{if } t \in [T_0(s), T(s)], \\ 0, & \text{if } t \geq T(s), \end{cases}$$

where α and β are defined in Theorem 3,

$$T_0(s) := \begin{cases} (\ln s\sqrt{n\eta})/\sigma, & \text{if } s > 1, \\ 0, & \text{if } s \leq 1, \end{cases}$$

$$T(s) = T_0(s) + \tfrac{2}{\sigma|\nu|}(\min\{1, s\})^{|\nu|/\alpha};$$

2) state (2) *and control* (4) *constraints are fulfilled.*

Remark 3. *Clearly, LMIs* (18) *are always feasible for sufficiently small ν and ϵ. Indeed, inequality* (18a) *can be rewritten as $(A+BK+\frac{3}{4}\sigma I_n)^\top P + P(A+BK+\frac{3}{4}\sigma I_n) \preccurlyeq 0$. Since matrix $A+BK$ is superstable of degree σ, then matrix $A+BK+\frac{3}{4}\sigma I_n$ is superstable of degree $\frac{1}{4}\sigma$ and, therefore, Hurwitz. Hence, there always exists positive definite matrix P such that* (18a) *holds.*

Taking $\nu = 0$ and $\epsilon \in (0, 0.5]$, one can see that (18b) *is fulfilled for any positive definite matrix P. Obviously, there exist sufficiently small ν and ϵ such that* (18b) *holds for some positive definite matrix P satisfying* (18c).

VII. Example

Consider system (1) for $n = 3$ and $m = 1$, where matrices A_0 and B_0 are of the form:

$$A_0 = \begin{bmatrix} 1 & -1 & 4 \\ -2 & 0 & -2 \\ 0 & -0.25 & -1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 2 \\ -2 \\ 0.5 \end{bmatrix}.$$

Let $\Delta_1 = 1$, $\Delta_2 = 2$, $\Delta_3 = 0.5$, $\bar{\delta} = 0.33$ and $\bar{u} = 3$. Thus, matrices A, B and M are equal to:

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 0 & -0.5 \\ 0 & -1 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \ M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

One can check that the pair (A, B) is controllable and linear optimization problem (9) is feasible for given $\bar{\delta}$ and $\bar{u}$. Hence, Theorem 8 can be applied.

A. Calculation of K, σ, L, K_1, P and η

Solutions to linear optimization problem (9) and linear matrix equation (16) are the following:

$$\begin{aligned} \sigma &= 0.67, \\ K &= \begin{bmatrix} -1.02 & 1.03 & -0.85 \end{bmatrix}, \\ K_1 &= \begin{bmatrix} -0.58 & 0 & 1.17 \end{bmatrix}, \end{aligned} \quad L = \begin{bmatrix} -0.17 & 2 & 2.33 \\ -0.63 & -2.5 & -1.25 \\ -0.83 & -2 & -0.33 \end{bmatrix}.$$

Setting $\nu = -0.14$ and $\epsilon = 1.01|\nu|$ in (18) yields:

$$P = \begin{bmatrix} 1.98 & 1.58 & 0.73 \\ 1.58 & 4.07 & 0.39 \\ 0.73 & 0.39 & 2.73 \end{bmatrix}, \quad \eta = 5.13.$$

B. Numerical simulation

The numerical simulation of the closed-loop system (5), (17) has been done in MATLAB Simulink by using the explicit Euler method with a state-dependent step [3]. The basic discretization step, simulation time and the maximum number of iterations have been taken as $h_0 = 10^{-2}$, $t_{\max} = 10$ and $N_{\max} = 2 \cdot 10^4$, respectively. Finally, the homogeneous norm of the closed-loop system (5), (17) has been defined as follows:

$$\|x\|_s := \begin{cases} \|x\|_P, & \text{if } \|x\|_P > 1, \\ \|x\|_{\mathbf{d}}, & \text{if } \|x\|_P \leq 1. \end{cases}$$

Besides, the bisection method [14] has been used to solve equation (12) with respect to $\|x\|_{\mathbf{d}}$. The minimum value of the norm has been chosen as $\|x\|_{\mathbf{d}\,\min} = 10^{-10}$.

Both disturbance-free ($\delta(t) \equiv 0$) and disturbed cases have been considered. In the latter case disturbance has been modeled as $\delta(t) = \bar{\delta} \cdot [\sin(0.1t), \cos(t), (1+\sin(2t))/2]^\top$.

Figures 2, 3 and 4 depict the time evolution of norms $\|x\|_\infty$, $\|x\|_s$ and control $u(x)$, respectively, for different initial conditions. Solid lines correspond to the disturbance-free case, dashed ones represent the disturbed case. Figure 3 is given in the logarithmic scale to better show finite-time convergence of the norm. One can see in Figures 2 and 4 that state (2) and control (4) constraints are indeed fulfilled.

VIII. Conclusion

The paper addresses the problem of finite-time stabilization of linear systems under state and control input constraints in the presence of external bounded disturbances. The problem has been divided into two consecutive tasks: 1) fulfillment of the state and control input constraints, and 2) finite-time stabilization. To this end, the control law has been designed such that the closed-loop system is 1) superstable, when system trajectories risk to violate the

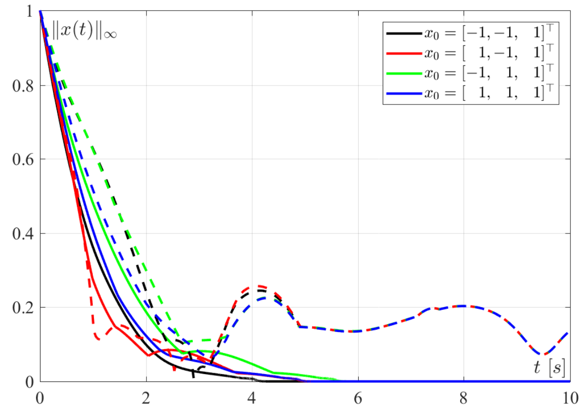

Fig. 2. Time evolution of norm $\|x\|_\infty$

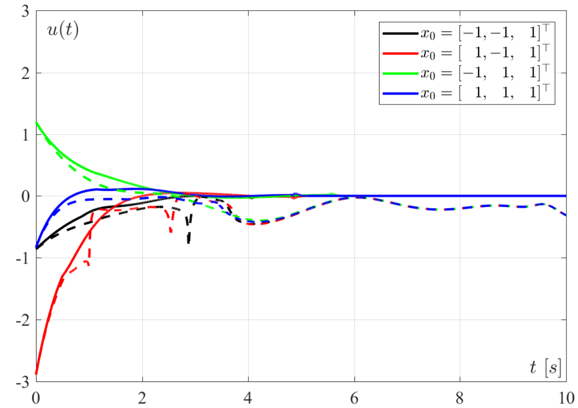

Fig. 4. Time evolution of control $u(x)$

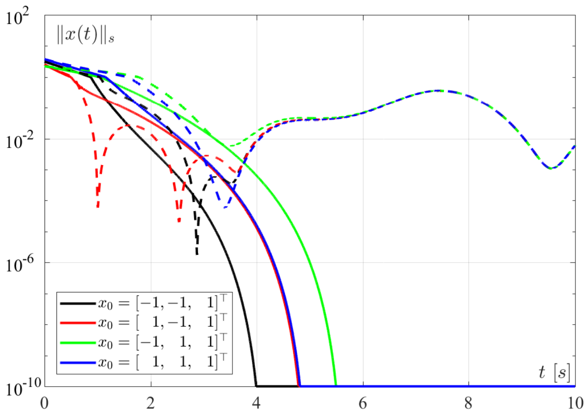

Fig. 3. Time evolution of norm $\|x\|_s$

state constraints, and 2) homogeneous with a negative degree, otherwise. The robustness of the proposed control scheme with respect to disturbances has been proven. Differently from existing approaches [5], [17], all control parameters can be easily obtained by solving linear matrix equations and inequalities. Numerical example has confirmed theoretical results.

IX. ACKNOWLEDGMENTS

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