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# Sparse recovery for inverse potential problems in divergence form

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**Abstract.** We discuss recent results from [10] on sparse recovery for inverse potential problem with source term in divergence form. The notion of sparsity which is set forth is measure-theoretic, namely pure 1-unrectifiability of the support. The theory applies when a superset of the support is known to be slender, meaning it has measure zero and all connected components of its complement has infinite measure in  $\mathbb{R}^3$ . We also discuss open issues in the non-slender case.

## 1. Introduction

Inverse potential problems with source term in divergence form consist in recovering a  $\mathbb{R}^3$ -valued distribution  $\mu$ , knowing the potential  $\Phi$  of  $\operatorname{div} \mu$  which is the solution to the Poisson-Hodge equation  $\Delta \Phi = \operatorname{div} \mu$  having “least growth” at infinity. In practice, a superset  $S$  of the support of  $\mu$  is known *a priori*, and sensors will measure the field  $\nabla \Phi$  rather than the potential  $\Phi$  itself.

Issues of this kind typically arise in source identification from field measurements for Maxwell’s equations, in the quasi-static regime. They occur for instance in electroencephalography (EEG), magneto-encephalography (MEG), geomagnetism and paleomagnetism, as well as in several non-destructive testing problems, see *e.g.* [1, 2, 3, 4, 5] and their bibliographies. A model problem of our particular interest is inverse scanning magnetic microscopy, as considered for instance in [9, 6, 7, 8] to recover magnetization distributions of thin rock samples, but the considerations below are of a rather general and mathematical nature.

Such problems are known to be difficult, for they are not only ill-posed but the forward operator, mapping  $\mu$  to the field, is not even injective. Recently, in the preprint [10], notions of sparsity have been introduced concerning  $\mu$ , when the latter is a finite  $\mathbb{R}^3$ -valued measure. They justify the use of Tikhonov-like regularization schemes that minimize the residuals while penalizing the total variation norm, in order to asymptotically reconstruct a sparse measure  $\mu$  when the regularization parameter goes to zero, under a specific assumption on  $S$ : it should be *slender*, meaning it has measure zero and each connected component of  $\mathbb{R}^3 \setminus S$  has infinite measure.

This situation is reminiscent of compressive sensing, where sparse solutions to underdetermined systems of linear equations in  $\mathbb{R}^n$  (*i.e.* solutions having a large number of zero components) are sought by minimizing the residuals while penalizing the  $l^1$ -norm; the gist of this approach is that, for “most” large underdetermined systems, the solution with minimal  $l^1$ -norm is also the sparsest solution, see *e.g.* [11].

However, in the present, infinite-dimensional context, it is unclear which assumption on  $\mu$  will ensure that it has minimum total variation among all solutions to the (noise-free) inverse problem, and why such an assumption should connect with some kind of sparsity. In fact, the answer to such questions will much depend on the null-space of the forward operator. In [10], the assumption that  $S$  is slender is to the effect that this kernel consists exactly of divergence-free  $\mathbb{R}^2$ -valued measures, also known as *solenoids*. Then, using a characterization of solenoids as integrals of elementary ones supported on curves [12], a natural notion of sparsity is found that ensures a sparse measure is mutually singular to all solenoids. This notion of sparsity which pertains to geometric measure theory, namely the support of  $\mu$  should be purely 1-unrectifiable; roughly speaking this means it contains no rectifiable arc. For instance, a countable sum of Dirac masses will satisfy this condition, but other, more complicated supports would also qualify.

The goal of this paper is to present main results from [10], and to discuss new issues arising when  $S$  is not slender.

We mention that a general Tikhonov-like regularization theory was developed in [14, 15, 16] for linear equations whose unknown is a  $\mathbb{R}^n$ -valued measure, by minimizing the residuals while penalizing the total variation. As expected from the non-reflexive character of spaces of measures, consistency estimates hold in the sense of weak-\* convergence of subsequences to solutions of minimum total variation, or convergence in the Bregman distance when the so-called source condition holds (which is, by the way, not the case here). In principle, such methods yield algorithms to approximate a solution of minimum total variation to the initial equation by a sequence of discrete measures, but imply nothing about the nature of limit points as discrete measures are weak-\* dense in the space of measures supported on an open subset of  $\mathbb{R}^n$ .

We note also that an infinite-dimensional recovery result for sparse measures, in the sense of being a sum of Dirac masses, was established in [17] for 1-D deconvolution issues, where a train of spikes is to be recovered from filtered observation thereof. Thus, [10] does not state the first sparse recovery result in an infinite-dimensional setting. It seems however, that [10] produces the first sparse recovery result for convolution operators in space-dimension greater than 1. Moreover, we should stress in our case that the convolution kernel is singular and the null-space of the forward operator has infinite dimension.

## 2. The inverse problem

Without loss of generality, we consider the issue of recovering a magnetization distribution from a collection of measurements of the magnetic field the magnetization generates. For a closed subset  $S \subset \mathbb{R}^3$ , let  $\mathcal{M}(S)$  denote the space of finite signed Borel measures on  $\mathbb{R}^3$  whose support lies in  $S$ . We model *magnetization distributions* supported in  $S$  as  $\mathbb{R}^3$ -valued measures  $\mu \in \mathcal{M}(S)^3$ . Hereafter, we often call a member of  $\mathcal{M}(S)^3$  a magnetization supported on  $S$ , as this terminology is suggestive of the problems we address.

The magnetic field  $b(\mu)$  generated by a magnetization  $\mu \in \mathcal{M}(S)^3$  is defined, at a point  $x$  not in the support of  $\mu$ , by the formula [18]:

$$b(\mu)(x) := -c \left( \int \frac{1}{|x-y|^3} d\mu(y) - 3 \int (x-y) \frac{(x-y) \cdot d\mu(y)}{|x-y|^5} \right) \quad x \notin \text{supp } \mu, \quad (1)$$

where  $c = 10^{-7} Hm^{-1}$  and for  $x, y \in \mathbb{R}^3$ ,  $x \cdot y$  and  $|x|$  denote the Euclidean scalar product and norm. Equivalently:

$$b(\mu)(x) = -\mu_0 \nabla \Phi(\mu)(x), \quad x \notin \text{supp } \mu, \quad (2)$$

with  $\mu_0 = 4\pi \times c$  and  $\Phi(\mu)$  is the *scalar magnetic potential* defined by

$$\Phi(\mu)(x) := \frac{1}{4\pi} \int \nabla_y \frac{1}{|x-y|} \cdot d\mu(y) = \frac{1}{4\pi} \int \frac{x-y}{|x-y|^3} \cdot d\mu(y). \quad (3)$$

and  $\nabla_y$  denotes the gradient with respect to  $y$

Generally speaking, the inverse magnetization problem is to recover  $\mu \in \mathcal{M}(S)^3$ , knowing  $b(\mu)$  in  $\mathbb{R}^3 \setminus S$ . However  $b(\mu)$  is usually measured on a rather small subset  $Q \subset \mathbb{R}^3 \setminus S$ , typically a compact surface patch. Also, in most cases, only one component of  $b(\mu)$  can be measured, because coils are oriented. For the sake of simplicity, we shall assume that  $S$  has connected complement and is contained in the closed lower half-space  $H := \{x = (x_1, x_2, x_3)^T \in \mathbb{R}^3, x_3 \leq 0\}$ , while  $Q$  is a compact set of Hausdorff dimension greater than 1, contained in a horizontal plane  $\Pi_h$  at strictly positive height  $h$ . Also, the component of  $b(\mu)$  which is measured will be  $b_3(\mu)$ , the third (vertical) one. This is the setting adopted in scanning magnetic microscopy [9, 6, 7, 8]. One could also consider the case where  $S$  is a surface disconnecting the space (e.g. a plane), in which case  $Q$  should intersect each of component of  $\mathbb{R}^3 \setminus S$  and be contained in a union of real analytic surfaces positively separated from  $S$  and satisfying mild conditions. We refer to [10] for this more exhaustive framework.

Letting  $\{e_j, 1 \leq j \leq 3\}$  indicate the canonical basis of  $\mathbb{R}^3$ , we get from (1) that

$$b_3(\mu)(x) := -\frac{\mu_0}{4\pi} \int \mathbf{K}(x-y) \cdot d\mu(y), \quad (4)$$

where

$$K(x) = \frac{e_3}{|x|^3} - 3x \frac{x_3}{|x|^5} = \nabla \left( \frac{x_3}{|x|^3} \right). \quad (5)$$

We define the *forward operator*  $A : \mathcal{M}(S)^3 \rightarrow L^2(Q)$  by

$$A(\mu)(x) := b_3(\mu)(x), \quad x \in Q. \quad (6)$$

Now, the inverse problem consists in recovering  $\mu$  knowing  $A(\mu)$ .

Note that in practice, only pointwise values of  $b_3(\mu)$  can be estimated, whereas we assume here knowledge of  $b_3(\mu)$  at each point of  $Q$ . We shall ignore this important issue, as it pertains to a numerical approach of the the problem which is beyond the scope of the present paper, devoted to basic principles.

### 2.1. Slenderness and null-space of the forward operator

Let  $\mathcal{L}_3$  denote Lebesgue measure on  $\mathbb{R}^3$ . We say that  $E \subset \mathbb{R}^3$  is *slender* if  $\mathcal{L}_3(E) = 0$  and any connected component  $C$  of  $\mathbb{R}^3 \setminus E$  has  $\mathcal{L}_3(C) = +\infty$ .

Clearly, the potential  $\Phi(\mu)$  defined by (3) is harmonic in  $\mathbb{R}^3 \setminus S$ , and so are the components of  $b(\mu)$ . It is easy to check that  $\Phi(\mu)$  extends to a function in  $L^2(\mathbb{R}^3) + L^1(\mathbb{R}^3)$  (still denoted  $\Phi(\mu)$ ), and that  $b(\mu)$  extends to a  $\mathbb{R}^3$ -valued divergence-free distribution [10, Prop. 2.1], with

$$\Delta \Phi = \operatorname{div} \mu \quad \text{and} \quad b(\mu) = \mu_0 (\mu - \nabla \Phi(\mu)). \quad (7)$$

It is not difficult to check that  $A(\mu)$  characterizes  $b(\mu)$  completely, and we explain this in the simple case where  $S$  has connected complement: if  $A(\mu) = 0$ , then  $b_3(\mu) = 0$  on  $Q$  and consequently on the entire plane  $\Pi_h$ , because it is real analytic in  $\{x_3 > 0\}$ , being harmonic in the upper half-space. Then,  $\Phi$  is a harmonic function in the upper half-space which solves a Neumann problem in  $\{x_3 > h\}$  with vanishing normal derivative, hence it is constant and since it lies in  $L^2(\mathbb{R}^3) + L^1(\mathbb{R}^3)$  it must be zero.

Still,  $A$  will generally have nontrivial null-space, because the mapping  $\mu \rightarrow b(\mu)$  is typically not injective. We say that  $\mu, \nu \in \mathcal{M}(S)^3$  are *S-equivalent* if  $b(\mu)$  and  $b(\nu)$  agree on  $\mathbb{R}^3 \setminus S$ , in which case we write  $\mu \equiv \nu[S]$ . A magnetization  $\mu$  is said to be *S-silent* if  $\mu \equiv 0[S]$ ; i.e., if  $b(\mu)$  vanishes on  $\mathbb{R}^3 \setminus S$ .

Since no nonzero harmonic function can lie in  $L^2(\mathbb{R}^3) + L^1(\mathbb{R}^3)$ , by the mean value property and Liouville's theorem, it follows from (7) that a divergence  $\mu$  belongs to the kernel of  $A$ . The converse needs not hold in general, but it does if  $S$  is slender:

**Theorem.** *If  $S$  is a slender set and  $\mu$  is  $S$ -silent, then  $\operatorname{div} \mu = 0$ .*

For a proof, we refer to [10, Thm. 2.2].

**Corollary** *If  $S$  is slender, the null-space of  $A$  consists of all divergence-free  $\mathbb{R}^3$ -valued measures on  $\mathbb{R}^3$  that are supported on  $S$ .*

## 2.2. Divergence-free measures, pure 1-unrectifiability and total variation

We let  $\mathcal{H}_1$  indicate 1-dimensional Hausdorff measure, see [13] for a definition. A set  $E \subset \mathbb{R}^3$  is said to be *1-rectifiable* if there exist Lipschitz maps  $f_i : \mathbb{R} \rightarrow \mathbb{R}^3$ ,  $i = 1, 2, \dots$ , such that

$$\mathcal{H}_1 \left( E \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R}) \right) = 0.$$

A set  $B \subset \mathbb{R}^3$  is *purely 1-unrectifiable* if  $\mathcal{H}_1(E \cap B) = 0$  for every 1-rectifiable set  $E$ , see [19] for these definitions. Clearly a set of  $\mathcal{H}_1$ -measure zero is purely 1-unrectifiable.

For  $\mu \in \mathcal{M}(\mathbb{R}^3)$  the *total variation measure*  $|\mu|$  is defined on Borel sets  $B \subset \mathbb{R}^3$  by

$$|\mu|(B) := \sup_{\mathcal{P}} \sum_{P \in \mathcal{P}} |\mu(P)|, \quad (8)$$

where the supremum is taken over all finite Borel partitions  $\mathcal{P}$  of  $B$ . The total variation norm  $\|\mu\|_{TV}$  is then defined as  $|\mu|(\mathbb{R}^3)$ .

The theorem below, which is crucial to the present approach of the inverse problem, entails that a magnetization with purely 1-unrectifiable support is the unique element of minimal total variation norm in its coset modulo the null-space of  $A$ .

**Theorem** *Suppose  $S$  is slender. If  $\mu \in \mathcal{M}(S)^3$  has support that is purely 1-unrectifiable and  $\nu \in \mathcal{M}(S)^3$  is  $S$ -equivalent to  $\mu$ , then  $\|\nu\|_{TV} > \|\mu\|_{TV}$  unless  $\nu = \mu$ .*

The proof rests on [12, Thm. A] which represents divergence-free measures as integrals of elementary measures of the form  $R_\gamma(B) = \int_B \tau d(\mathcal{H}_1 \llcorner \gamma)$ , where  $\gamma$  is an oriented Lipschitz arc and  $\tau$  its unit tangent, with  $\mathcal{H}_1 \llcorner \gamma$  to mean the restriction of  $\mathcal{H}_1$  to the image of  $\gamma$ .

## 3. Total variation regularization and consistency of sparse recovery

For  $\mu \in \mathcal{M}(S)^3$ ,  $f \in L^2(Q)$ , and  $\lambda > 0$ , define

$$\mathcal{F}_{f,\lambda}(\mu) := \|f - A\mu\|_{L^2(Q)}^2 + \lambda \|\mu\|_{TV}. \quad (9)$$

To recover a magnetization  $\mu_0$  from measurements  $f$  of  $A\mu_0$ , we pick  $\lambda > 0$  and consider the following regularization scheme: *to find  $\mu_\lambda \in \mathcal{M}(S)^3$  such that*

$$\mathcal{F}_{f,\lambda}(\mu_\lambda) = \inf_{\mu \in \mathcal{M}(S)^3} \mathcal{F}_{f,\lambda}(\mu). \quad (10)$$

More precisely, to account for measurement noise, we assume that  $f = f_e = A\mu_0 + e$  and we call  $\mu_{\lambda,e}$  a minimizer of (10). It is easy to see that such a minimizer exists, and it follows from [14, Thms. 2&5] or [15, Thm. 3.5&4.4] that any weak-\* limit point of the family  $\mu_{\lambda,e}$  as both  $\lambda$  and  $\|e\lambda^{-1/2}\|_{L^2(Q)}$  tend to 0 is a magnetization  $\nu$  such that  $A\nu = A\mu_0$  of minimum total variation under this condition. In particular, if there is a unique such magnetization, we can easily formulate a weak-\* sequential consistency result in the zero-noise limit, by letting  $\lambda_n$  go to zero more slowly than  $e_n\|_{L^2(Q)}^2$  (the so-called Morozov discrepancy principle).

The theorem below dwells on this and on the previous theorem, but goes a little further in that the convergence holds not only for  $\mu_{\lambda,e}$  but also for  $|\mu_{\lambda,e}|$ , which is important for recovery (think of an oscillating density like  $e^{in\theta}$  on the unit circle, which goes weak-\* to 0 as  $n \rightarrow \infty$  but still has total variation  $2\pi$  for each  $n$ ). Moreover, convergence takes place in the narrow sense (test functions should be continuous and bounded but need not have compact support)

**Theorem** *Let  $S$  be slender and  $\mu_0 \in \mathcal{M}(S)^3$  have purely 1-unrectifiable support. Then,  $\mu_{\lambda,e}$  converges narrowly sequentially to  $\mu_0$  and  $|\mu_{\lambda,e}|$  converges narrowly sequentially to  $|\mu_0|$  as  $\lambda \rightarrow 0$  and  $\|e\|_{L^2(Q)}/\sqrt{\lambda} \rightarrow 0$ .*

#### 4. Issues in the non slender case

A typical slender set is two-dimensional. It could be a piece of plane, or the entire plane, or it could be a piece of sphere but not the entire sphere, nor a ball.

If  $S$  is a genuine 3-D object, like the interior of a compact surface  $\Sigma$ , then it is not slender and the previous analysis fails. In fact, there are in this case magnetizations which are  $S$ -silent but not divergence-free: an example when  $\Sigma$  is Lipschitz is given by the gradient of a function of bounded variation inside  $\Sigma$  which has constant trace on  $\Sigma$ . Still, such a magnetization turns out to be singular with measures with pure 1-unrectifiable support. It would be most interesting to describe all silent magnetizations supported inside  $\Sigma$ . In this connection, we mention that such magnetizations, if they have  $L^p$  density, must be the sum of a divergence-free field and a gradient as above. For finite measures, however, no characterization is known.

- [1] S. Baillet, J.C. Mosher and R. M. Leahy 2001, Electromagnetic brain mapping, *IEEE Signal Processing Magazine*.
- [2] H. T. Banks and F. Kojima 2002, Identification of material damage in two-dimensional domains using the SQUID-based nondestructive evaluation system, *Inverse Problems* **18** 1831-55.
- [3] R. J. Blakely 1995, *Potential Theory in Gravity and Magnetic Applications*, Cambridge University Press.
- [4] R. L. Parker 1994, *Geophysical inverse theory*, Princeton University Press.
- [5] R. Kress, L. Kühn and R. Potthast 2002, Reconstruction of a current distribution from its magnetic field, *Inverse Problems* **18**, 1127-46.
- [6] L. Baratchart and D.P. Hardin and E.A. Lima and E.B. Saff and B.P. Weiss 2013, Characterizing kernels of operators related to thin-plate magnetizations via generalizations of Hodge decompositions, *Inverse Problems* **29**, 15004.
- [7] E. A. Lima, B. P. Weiss, L. Baratchart, D. P. Hardin and E. B. Saff 2013, Fast inversion of magnetic field maps of unidirectional planar geological magnetization, *Journal of Geophysical Research: Solid Earth* **118**, 2723-52.
- [8] L. Baratchart and S. Chevillard and J. Leblond 2017, Silent and equivalent magnetic distributions on thin plates, *Harmonic Analysis, Function Theory, Operator Theory, and Their Applications*, Theta series in advanced mathematics **18**.
- [9] B. P. Weiss, E. A. Lima, L. E. Fong, F. J. Baudenbacher 2007, Paleomagnetic analysis using SQUID microscopy, *Journal of Geophysical Research: Solid Earth* **112**.
- [10] L. Baratchart, C. Villalobos-Guillen, D. P. Hardin, M. C. Northington and E.B. Saff 2018, Inverse Potential Problems for Divergence of Measures with Total Variation Regularization, *Preprint* <https://arxiv.org/abs/1809.08334>.
- [11] S. Foucart and H. Rauhut 2013, *A Mathematical Introduction to Compressive Sensing*, Birkhäuser.
- [12] S. K. Smirnov 1994, Decomposition of solenoidal vector charges into elementary solenoids, and the structure of normal one-dimensional flows, *St. Petersburg Math. J.* **5**, 841-67.
- [13] L. C. Evans and R. F. Gariepy 1992, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press.
- [14] M. Burger and S. Osher 2004, Convergence rates of convex variational regularization, *Inverse Problems* **20**, 1411-21.
- [15] B. Hoffmann, B. Kaltenbacher, C. Pöschl and O. Scherzer 2007, *Inverse Problems* **23**, 987-10.
- [16] K. Bredies and H. K. Pikkarainen 2013, Inverse problems in spaces of measures, *ESAIM COCV* **19**, 190-08.
- [17] V. Duval and G. Peyré 2015, Exact Support Recovery for Sparse Spikes Deconvolution, *FoCM* **15**, 1315-55.
- [18] J. D. Jackson 1975, *Classical electrodynamics*, John Wiley & Sons.
- [19] P. Mattila 1995, *Geometry of sets and measures in Euclidean spaces*, Cambridge University Press.