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Limit Theorems for Default Contagion and Systemic Risk

Hamed Amini* Zhongyuan Cao † Agnès Sulem ‡

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Abstract

We consider a general tractable model for default contagion and systemic risk in a heterogeneous financial network, subject to an exogenous macroeconomic shock. We show that, under some regularity assumptions, the default cascade model could be transferred to a death process problem represented by balls-and-bins model. We also reduce the dimension of the problem by classifying banks according to different types, in an appropriate type space. These types may be calibrated to real-world data by using machine learning techniques. We then state various limit theorems regarding the final size of default cascade over different types. In particular, under suitable assumptions on the degree and threshold distributions, we show that the final size of default cascade has asymptotically Gaussian fluctuations. We next state limit theorems for different system-wide wealth aggregation functions and show how the systemic risk measure, in a given stress test scenario, could be related to the structure and heterogeneity of financial networks. We finally show how these results could be used by a social planner to optimally target interventions during a financial crisis, with a budget constraint and under partial information of the financial network.

Keywords: Systemic Risk, Default Contagion, Financial Networks, Random Graphs.

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1 Introduction

The financial crisis 2007-2009 has illustrated the significance of network structure on the amplification of initial shocks in the banking system to the level of the global financial system, leading to an economic recession.

This paper studies structural and dynamic models for loss propagation in the network of liabilities. This is in contrast to the well-known systemic risk indicators such as CoVAR [54] or SES [2], which are based on measuring losses in terms of market equity.

Empirical studies on network topology of banking systems show that we may have very different structures; from centralized networks as in [50] to core-periphery structures [23, 30, 47] and scale-free structures as in [17, 21]. The main focus of this paper is to use limit theorems to establish a link between the (final) size of default cascade to the structure and heterogeneity of financial networks. The paper also studies limit theorems to quantify the system-wide wealth and systemic risk in the financial system. We also show how these results could be used by a social planner to optimally target interventions during a financial crisis, under partial information of the financial network and under some budget constraint.

Dealing with missing data is another important aspect of this paper. While recent efforts have made available a tremendous amount of financial data, we are far from a global financial data collection firm-by-firm and there are systemic institutions which are not required to report their positions. As pointed out in [14, 33, 35, 53, 55], in general, there is only partial information available on the financial network, e.g., the total size of the assets and liabilities for each institution. Our probabilistic approaches allow us to deal with the incomplete observation of system connections. We also reduce the dimension of the problem by considering a classification of financial institutions according to different types (characteristics), in an appropriate type space $\mathcal{X}$. These characteristics may be calibrated to real-world data by using machine learning techniques for classification.

There is an emerging literature on systemic risk and financial networks, see e.g. [20, 37] for two recent surveys and references there. An extensive research in this area focuses on equilibrium approach, to derive recovery rates from an elegant fixed point equation [26, 27, 34, 52]. This relies on the assumption that all debts are instantaneously cleared, unlikely to hold in reality. Even in a given shock scenario, recovery rates are uncertain. For example, recovery rates after the failure of Lehman were around 8% ([49]). In this paper we model recovery rates as given. The model could be easily extended to a setup with random recovery rates satisfying some cash-flow consistency conditions, see e.g. [11].

Our work is related to the literature on network structure and contagion, see e.g., [3, 43, 46, 48, 59, 60] in the context of social networks and [1, 4, 13, 18, 25, 28, 31, 45, 51] for default contagion in financial networks.

To the best of our knowledge, our paper is the first to provide central limit theorems for default contagion and systemic risk in random financial networks. More closely related to our work, [9] study default contagion on configuration model and derive a criterion for the resilience of a financial network to insolvency contagion, based on connectivity and the

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structure of contagious links (i.e., those exposures of a bank larger than its capital). We propose multiple extensions of [9]:

- We slightly generalize the default contagion model and allow for more network heterogeneity by considering the type-dependent threshold model. These types may be calibrated to real-world data by using machine learning techniques for classification.
- We transfer the default cascade model to a death process problem represented by balls-and-bins model. This allows us to provide the limit theorems for the dynamic contagion model [1].
- The primary contribution of this paper is to provide central limit theorems for default contagion and systemic risk. We show that the final size of default cascade has asymptotically Gaussian fluctuations and state various theorems regarding the joint asymptotic normality between different contagion parameters, including the number of solvent banks, defaulted banks, healthy links (those initiated by solvent banks) and infected links (those initiated by defaulted banks) at any time t.
- Our next set of results concern the limit theorems for system-wide wealth aggregation functions, which could be used for measuring and quantifying systemic risk. This also provide an indicator for the health of financial system in different stress scenarios.
- Finally, we consider a social planner who seeks to optimally target interventions during a financial crisis, under partial information of the financial network and with a budget constraint. We show how limit theorems allows us to simplify the optimization problem. The complete information setup has been recently studied in [32, 36].

Aside from the application to default contagion and systemic risk in financial networks, our results contribute to the literature on diffusion processes on random graphs. Related problems are the k-core and bootstrap percolation. The k-core of any finite graph can be found by removing nodes with degree less than k, in any order, until no such nodes exist. The asymptotic normality of k-core has been studied in [41]. We also study the default contagion in a similar way as in [41], by transferring the process to a death process problem represented by balls-and-bins model. We generalize their results by allowing different types and threshold levels to each of nodes. The bootstrap percolation is a diffusion process that has been studied on a variety of graphs, see e.g., [5, 6, 39]. In bootstrap percolation process, for a fixed threshold $\theta \geqslant 2$, there is an initially subset of active nodes and in each round, each inactive node that has at least θ active neighbors becomes active and remains so forever. The asymptotic normality of bootstrap percolation has been recently studied in [8]. Our results generalize those of previous studies on bootstrap percolation in random graphs to the case of heterogeneous random directed networks with type-dependent random thresholds.

We end this introduction by the following remark. In the real world application, using limit theorems requires some caution. For example, in order for the asymptotic analysis

[1] The balls-and-bins model has been previously used in the economic literature; see e.g. [15] for a balls-and-bins model of international trade.

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to be relevant, the financial network should be sufficiently large. This could be true for example at the level of a large economic zone. Moreover, financial networks may have small cycles. Most existing literature on random networks future locally tree-like property. However, recent literature shows that the basic configuration model can be extended to incorporate clustering; see e.g., [22, 57]. Moreover, following the recent literature on portfolio compression in financial networks (see e.g., [10, 24, 58]), the study of default contagion and systemic risk in sparse financial networks regime becomes significantly important, as portfolio compression removes small cycles. In light of its tractability and interpretability, as well as its potential to be enriched with clustering, in this paper we use the configuration model as our base model. Note that the closed form interpretable limit theorems that we provide could also serve as a mandate for regulators to collect data on those specific network characteristics and assess systemic risk via more intensive computational methods.

Outline. The paper is organized as follows. Section 2 introduces a model for the network of financial counterparties and describe a mechanism for default cascade in such a network, after an exogenous macroeconomic shock. We also describe how the default contagion model could be transferred to a death process problem represented by balls-and-bins model. Section 3 gives our main results on limit theorems for the final size of default cascade. In particular, under some regularity assumptions, we show that different default contagion parameters have asymptotically Gaussian fluctuations. Section 4 states limit theorems for different financial system aggregation functions, which are used for measuring and quantifying systemic risk. Section 5 shows how these limit theorems could be used by a social planner to optimally target interventions during a financial crisis, with a budget constraint and under partial information of the financial network. Proof of main theorems are given in Section 6. Section 7 concludes. Proof of lemmas are provided in Appendix A.

Notation. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If $c \in \mathbb{R}$ is a constant, we write $X_n \xrightarrow{p} c$ to denote that X_n converges in probability to c. That is, for any $\epsilon > 0$, we have $\mathbb{P}(|X_n - c| > \epsilon) \to 0$ as $n \to \infty$. We write $X_n \xrightarrow{d} X$ to denote that X_n converges in distribution to X. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers that tends to infinity as $n \to \infty$. We write $X_n = o_p(a_n)$, if $|X_n|/a_n \xrightarrow{p} 0$. If $\mathcal{E}_n$ is a measurable subset of Ω, for any $n \in \mathbb{N}$, we say that the sequence $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ occurs with high probability (w.h.p.) or almost surely (a.s.) if $\mathbb{P}(\mathcal{E}_n) = 1 - o(1)$, as $n \to \infty$. Also, we denote by $\mathsf{Bin}(k, p)$ a binomial distribution corresponding to the number of successes of a sequence of k independent Bernoulli trials each having probability of success p. The notation $\mathbb{1}\{\mathcal{E}\}$ is used for the indicator of an event $\mathcal{E}$; this is 1 if $\mathcal{E}$ holds and 0 otherwise. We deonte by $\mathcal{D}[0, \infty)$ the standard space of right-continuous functions with left limits on $[0, \infty)$ equipped with the Skorokhod topology (see e.g. [38, 42]). We will suppress the dependence of parameters on the size of the network n, if it is clear from the context.

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2 Model

2.1 Financial Network

Consider an economy $\mathcal{E}_n$ consisting of n interlinked financial institutions (banks) denoted by $[n] := \{1, 2, \ldots, n\}$ that intermediate credit among end-users. Banks hold claims on each other. Interbank liabilities are represented by a matrix of nominal liabilities (ℓ_{ij}). For two financial institutions $i, j \in [n]$, $\ell_{ij} \geqslant 0$ denotes the cash-amount that bank i owes bank j. This also represents the maximum loss related to direct claims, incurred by bank j upon the default of bank i. The total nominal liabilities of bank i sum up to $\ell_i = \sum_{j\in[n]} \ell_{ij}$, while the total value of interbank assets sum up to $a_i = \sum_{j\in[n]} \ell_{ji}$. The total value of claims held by end-users on bank i (deposits) is given by d_i. The total value of claims held by bank i on end-users (external assets) is denoted by e_i.

In a stress testing framework, we apply a (fractional) shock ϵ_i to the external assets of bank i. Table 1 summarizes a stylized balance sheet of bank i after the shock ϵ_i. The capital of bank i after the shock denoted by $c_i = c_i(\epsilon_i)$ satisfies $c_i = (1-\epsilon_i)e_i + a_i - \ell_i - d_i$, which represents the capacity of bank i to absorb losses while remaining solvent.

<table>
<tr><td>External assets
e_i</td><td>Deposits
d_i</td></tr>
<tr><td>$\epsilon_i e_i$ - loss on assets</td><td rowspan="2">Interbank liabilities
$\ell_i = \sum_{j\in[n]} \ell_{ij}$</td></tr>
<tr><td rowspan="2">Interbank assets
$a_i = \sum_{j\in[n]} \ell_{ji}$</td></tr>
<tr><td>Capital
c_i</td></tr>
<tr><td></td><td>$\epsilon_i e_i$ - loss on capital</td></tr>
<tr><td>Assets</td><td>Liabilities</td></tr>
</table>

Table 1: Stylized balance sheet of bank i after shock.

A financial institution $i \in [n]$ is said to be *fundamentally insolvent* if its capital after the shock is negative, i.e. $c_i < 0$. For a given shock scenario $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_n) \in [0, 1]^n$, we let the set of fundamental defaults $\mathcal{D}_0(\boldsymbol{\epsilon}) = \{i \in [n] : c_i(\epsilon_i) < 0\}$. In the next section, we define the default cascade triggered by fundamentally insolvent institutions $\mathcal{D}_0(\boldsymbol{\epsilon})$.

2.2 Default Cascade

In the given shock scenario $\boldsymbol{\epsilon}$, following the fundamentally insolvent institutions $\mathcal{D}_0(\boldsymbol{\epsilon})$, there will be a default contagion process. Let us denote by $R_{ij} = R_{ij}(\boldsymbol{\epsilon})$ the recovery rate of the liability of i to j. The matrix of recovery rates is denoted by $\mathcal{R} = (R_{ij})$. Since any bank i cannot pay more than its external assets $(1 - \epsilon_i)e_i$ plus what it recovered from its

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debtors, the recovery rates of i should satisfy the following cash-flow consistency constraints

$$(1-\epsilon_i)e_i + \sum_{j=1}^{n} R_{ji}\ell_{ji} \geqslant \sum_{j=1}^{n} R_{ij}\ell_{ij} + d_i.$$

Given the shock scenario $\boldsymbol{\epsilon}$ and the matrix of recovery rates $\mathcal{R}$, following the set of fundamental default $\mathcal{D}_0$, there is a default cascade that reaches the set $\mathcal{D}^\star$ in equilibrium. This represents the set of financial institutions whose capital is insufficient to absorb losses and should satisfy the following fixed point equation:

$$\mathcal{D}^\star = \mathcal{D}^\star(\boldsymbol{\epsilon}, \mathcal{R}) = \Big\{ i \in [n] : c_i(\epsilon_i) < \sum_{j \in \mathcal{D}^\star} (1 - R_{ji})\ell_{ji} \Big\}.$$

As stated in [11], the above fixed point default cascade set has in general multiple solutions. The smallest fixed point set which corresponds to smallest number of defaults can be obtained by starting from $\mathcal{D}_0$ and setting at step k:

$$\mathcal{D}_k = \mathcal{D}_k(\boldsymbol{\epsilon}, \mathcal{R}) = \big\{ i \in [n] : c_i(\epsilon_i) < \sum_{j \in \mathcal{D}_{k-1}} (1 - R_{ji})\ell_{ji} \big\}. \tag{1}$$

The cascade ends at the first time k such that $\mathcal{D}_k = \mathcal{D}_{k-1}$. Hence in a financial network of size n, the cascade will end after at most $n-1$ steps and $\mathcal{D}_{n-1} = \mathcal{D}_{n-1}(\boldsymbol{\epsilon}, \mathcal{R})$ represents the final set of insolvent institutions starting from the initial set of defaults $\mathcal{D}_0$.

2.3 Node Classification and Configuration Model

Under some regularity assumptions detailed below, as also shown in [9] for a similar setup, one can show that the information regarding assets, liabilities, capital after exogenous shocks and recovery rates (distributions) could all be encoded in a single probability threshold function. Namely, for a given shock scenario $\boldsymbol{\epsilon}$ and the matrix of recovery rates $\mathcal{R}$[2], we introduce the (random) threshold $\Theta_i = \Theta_i^{(n)}$ for any institution $i \in [n]$ which measures how many defaults i can tolerate before becoming insolvent, if its counterparties default in a uniformly at random order, i.e., when i's debtors default order environment is chosen uniformly at random among all possible permutations.

In the following, in order to reduce the dimensionality of the problem, we consider a classification of financial institutions into a countable (finite or infinite) possible set of characteristics $\mathcal{X}$. All (observable) characteristics for financial institution i is encoded in $x_i = (d_i^+, d_i^-, t_i, ...) \in \mathcal{X}$, where d_i^+ denotes the in-degree (number of institutions i is exposed to), d_i^- denotes the out-degree (number of institutions exposed to i) and t_i denotes any other institution's type specific (e.g., credit rating, seniority class, etc.).

As we are interested in limit theorems, we consider a sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$, indexed by the number of institutions. In particular, in the economy $\mathcal{E}_n$, the characteristic

[2]Our results can easily be extended to a framework with independent random recovery rates; see e.g. [11] for discussion.

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of any institution $i \in [n]$ will be denoted by

$$x_i^{(n)} = (d_i^{+(n)}, d_i^{-(n)}, t_i^{(n)}, ...) \in \mathcal{X}.$$

Note that, without loss of generality, the institutions in the same class $x \in \mathcal{X}$ assumed to have the same number of creditors (denoted by d_x^-) and the same number of debtors (denoted by d_x^+). Further, for tractability, we make the following assumption on the probability threshold functions.

Assumption 1. *We assume that there exists a classification of the financial institutions into a countable set of possible characteristics $\mathcal{X}$ such that, for each $n \in \mathbb{N}$, the institutions in the same characteristic class have the same threshold distribution function (denoted by $q_x^{(n)}$ for institutions in class $x \in \mathcal{X}$). Namely, for economy $\mathcal{E}_n, i \in [n]$ and for all $\theta \in \mathbb{N}$:*

$$\mathbb{P}(\Theta_i^{(n)} = \theta) = q_{x_i^{(n)}}^{(n)}(\theta).$$

In particular, in the network of size n, $q_x^{(n)}(0)$ represents the proportion of initially insolvent institutions with type $x \in \mathcal{X}$.

Let $\mu_x^{(n)}$ denote the fraction of institutions with characteristic $x \in \mathcal{X}$ in the economy $\mathcal{E}_n$. In order to study the asymptotics, it is natural to assume the following.

Assumption 2. *We assume that for some probability distribution functions μ and q over the set of characteristics $\mathcal{X}$ and independent of n, we have $\mu_x^{(n)} \to \mu_x$ and $q_x^{(n)}(\theta) \to q_x(\theta)$ as $n \to \infty$, for all $x \in \mathcal{X}$ and $\theta = 0, 1, \ldots, d_x^+$. Moreover, we assume that $\sum_{\theta=0}^{d_x^+} q_x(\theta) = 1$ for all $x \in \mathcal{X}$.*

In the following, we will suppress the dependence of parameters on the size of the network n, if it is clear from the context.

Given the degree sequences $\mathbf{d}_n^+ = (d_1^+, \ldots, d_n^+)$ and $\mathbf{d}_n^- = (d_1^-, \ldots, d_n^-)$ such that $\sum_{i\in[n]} d_i^+ = \sum_{i\in[n]} d_i^-$, we associate to each institution i two sets: $\mathcal{H}_i^+$ the set of incoming half-edges and $\mathcal{H}_i^-$ the set of outgoing half-edges, with $|\mathcal{H}_i^+| = d_i^+$ and $|\mathcal{H}_i^-| = d_i^-$. Let $\mathbb{H}^+ = \bigcup_{i=1}^n \mathcal{H}_i^+$ and $\mathbb{H}^- = \bigcup_{i=1}^n \mathcal{H}_i^-$. A *configuration* is a matching of $\mathbb{H}^+$ with $\mathbb{H}^-$. When an out-going half-edge of instituion i is matched with an in-coming half-edge of institution j, a directed edge from i to j appears in the graph. The configuration model is the random directed multigraph which is uniformly distributed across all configurations. The random graph constructed by configuration model will be denoted by $\mathcal{G}^{(n)}(\mathbf{d}_n^+, \mathbf{d}_n^-)$. It is then easy to show that conditional on the multigraph being a simple graph, we obtain a uniformly distributed random graph with these given degree sequences denoted by $\mathcal{G}_*^{(n)}(\mathbf{d}_n^+, \mathbf{d}_n^-)$. In particular, any property that holds with high probability on the configuration model also holds with high probability conditional on this random graph being simple (for the random graph $\mathcal{G}_*^{(n)}(\mathbf{d}_n^+, \mathbf{d}_n^-)$) provided $\liminf_{n\to\infty} \mathbb{P}(\mathcal{G}^{(n)}(\mathbf{d}_n^+, \mathbf{d}_n^-) \text{ simple}) > 0$, see e.g. [56].

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2.4 Death Process and Final Solvent Institutions

We consider the default contagion process in the random financial network $\mathcal{G}^{(n)}(\mathbf{d}_n^+, \mathbf{d}_n^-)$, initiated by the set of fundamentally insolvent institutions $\mathcal{D}_0$.

Recall that Θ_i denotes the random threshold of institution $i \in [n]$ which measures how many defaults i can tolerate before becoming insolvent in the uniformly chosen i's counterparties default order environment. By Assumption 1 and standard coupling arguments, as also proved in [9], one can assume that these thresholds are assigned initially to any institution $i \in [n]$ according to the distribution $q_{x_i}^{(n)}(\cdot)$.

Finding the final solvent institutions. We consider the above default contagion progress in the following way. At time 0 in the (random) graph $\mathcal{G}^{(n)}(\mathbf{d}_n^+, \mathbf{d}_n^-)$, all institutions with threshold 0 become defaulted. We remove all the initially defaulted institutions $\mathcal{D}_0$ from the network. Next, in order to find $\mathcal{D}_1$, we identify the partners of $\mathcal{D}_0$. Note that the out-degree and in-degree of each institution in the network induced by $[n]\backslash\mathcal{D}_0$ is less than or equal to those in the previous network. At step $k \in \mathbb{N}$, the default set $\mathcal{D}_k$ can be identified by

$$\mathcal{D}_k = \Big\{ i \in [n] : \sum_{j:j\to i} \mathbb{1}\{j \in \mathcal{D}_{k-1}\} \geqslant \Theta_i \Big\}, \tag{2}$$

where $\mathbb{1}\{\mathcal{E}\}$ denotes the indicator of an event $\mathcal{E}$, i.e., this is 1 if $\mathcal{E}$ holds and 0 otherwise. We denote the in-degree and out-degree of each institution i after k steps evolution by $d_i^+(k)$ and $d_i^-(k)$ respectively. Note that initially $d_i^+(0) = d_i^+$ and $d_i^-(0) = d_i^-$. At step k, we remove all institutions $i \in [n]$ with $d_i^+(k) < d_i^+ - \Theta_i$. At the end of the above procedure, all the removed institutions are defaulted and the remaining institutions are solvent.

Transferring to a death process problem represented by balls-and-bins. It is not hard to see that the calendar time does not take any important role in the above contagion process. We can define the time interval as we want. So instead of removing institutions, we can also remove the links and define a proper time interval between two successive removals. Namely, at each step, we only look at one removal (interaction) between two institutions, yielding at least one default.

In the following, we simultaneously run the default contagion process and construct the configuration model. We call all out half-edges and in half-edges that belong to a defaulted (solvent) institution the *infected* (*healthy*) half-edges. We consider all the institutions as bins and all the (in and out) half-edges as (in and out) balls. Consequently, the bins are called defaulted (**D** type) or solvent (**S** type) according to their states as institutions. Similarly the balls are called infected (**I** type) or healthy (**H** type) when they are infected or healthy as half-edges. Hence, all institutions are of two types and all balls are of four different types. For convenience, we denote them as **S** (solvent), **D** (defaulted) bins, and further $\mathbf{H}^+$ (healthy in), $\mathbf{H}^-$ (healthy out), $\mathbf{I}^+$ (infected in) and $\mathbf{I}^-$ (infected out) balls, respectively.

We start from the set of fundamental defaults $\mathcal{D}_0$, which gives the set of initially defaulted bins and infected balls. Consequently, at each step, we first remove a uniformly

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chosen ball of type $\mathbf{I}^-$ and then a uniformly chosen ball from $\mathbf{H}^+ \cup \mathbf{I}^+$. In this process $\mathbf{S}$ bins may change to $\mathbf{D}$ bins and, consequently, $\mathbf{H}$ balls may change to $\mathbf{I}$ balls. We continue the above process until there is no more $\mathbf{I}^-$ balls.

We now change the description a little by introducing colors for the $\mathbf{I}^-$ balls and life for all in balls from $\mathbf{H}^+ \cup \mathbf{I}^+$. We let all $\mathbf{I}^-$ balls are white and all in balls from $\mathbf{H}^+ \cup \mathbf{I}^+$ are initially alive. We begin by recoloring one random $\mathbf{I}^-$ ball red. Subsequently, in each removal step, we first kill a random in ball from $\mathbf{H}^+ \cup \mathbf{I}^+$ and at the same moment we also recolor a random white ball red. This is repeated until no more white $\mathbf{I}^-$ balls remain.

We next run the above death process in continuous time. We assume that each ball from $\mathbf{H}^+ \cup \mathbf{I}^+$ has an exponentially distributed random lifetime with mean one, independent of all other balls. Namely, if there are ℓ alive in balls remaining, then we wait an exponential time with mean $1/\ell$ until the next pair of interactions. We stop when we should recolor a white ball red but there is no such ball.

Let us denote by $W_n(t)$ the number of white $\mathbf{I}^-$ balls at time t. Hence, the above death process ends at the stopping time $\tau_n^\star$ which is the first time when we need to recolor a white ball but there are no white balls left. However, we pretend that we recolor a (nonexistent) white ball at time $\tau_n^\star$ and write $W_n(\tau_n^\star) = -1$.

We denote by $I_n^+(t)$, $H_n^+(t)$ and $L_n(t)$ the number of alive (in) balls in $\mathbf{I}^+$, $\mathbf{H}^+$ and $\mathbf{H}^+ \cup \mathbf{I}^+$ at time t, respectively. For $x \in \mathcal{X}, \theta \in \mathbb{N}, \ell = 0, \ldots, \theta - 1$, we let $S_{x,\theta,\ell}^{(n)}(t)$ denote the number of solvent institutions (bins) with type x, threshold θ and ℓ defaulted neighbors at time t. Further, let $S_n(t)$ and $D_n(t)$ be the numbers of $\mathbf{S}$ bins and $\mathbf{D}$ bins at time t. Hence, $S_n(\tau_n^\star)$ denotes the final number of solvent institutions. Further, $D_n(\tau_n^\star) = n - S_n(\tau_n^\star) = |\mathcal{D}_{n-1}|$ will be the final number of defaulted institutions.

3 Limit Theorems

In this section we consider the above dynamic default contagion model (which is now transferred to a death process problem represented by balls-and-bins) and state our main results regarding the limit theorems in the random financial network $\mathcal{G}^{(n)}(\mathbf{d}_n^+, \mathbf{d}_n^-)$.

We first define some functions that will be used later. Let for $z \in [0, 1]$:

$$b(d, z, \ell) := \mathbb{P}(\mathsf{Bin}(d, z) = \ell) = \binom{d}{\ell} z^\ell (1 - z)^{d-\ell}, \tag{3}$$

$$\beta(d, z, \ell) := \mathbb{P}(\mathsf{Bin}(d, z) \geqslant \ell) = \sum_{r=\ell}^{d} \binom{d}{r} z^r (1 - z)^{d-r}, \tag{4}$$

and $\mathsf{Bin}(d, z)$ denotes the binomial distribution with parameters d and z.

In the following, for simplicity of the calculations, we assume that the thresholds distribution are such that $\sum_{\theta=0}^{d_x^+} q_x(\theta) = 1$ for all $x \in \mathcal{X}$. Otherwise, one could define $q_x(\infty) := 1 - \sum_{\theta=0}^{d_x^+} q_x(\theta)$ and remove initially all these institutions from the network (since they will not play any role in the default contagion process). Note that this removal process

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(site percolation) will affect the degree distributions.

3.1 Asymptotic Magnitude of Default Contagion

In this section we consider the random financial network $\mathcal{G}^{(n)}(\mathbf{d}_n^+, \mathbf{d}_n^-)$ and assume that the average degrees converges to a finite limit.

Assumption 3a. *We assume that, as $n \to \infty$, the average degrees converges and is finite:*

$$\lambda^{(n)} := \sum_{x\in\mathcal{X}} d_x^+ \mu_x^{(n)} = \sum_{x\in\mathcal{X}} d_x^- \mu_x^{(n)} \longrightarrow \lambda := \sum_{x\in\mathcal{X}} d_x^+ \mu_x \in (0,\infty).$$

For $z \in [0,1]$, we define the functions:

$$f_S(z) := \sum_{x\in\mathcal{X}} \mu_x \sum_{\theta=1}^{d_x^+} q_x(\theta)\beta\big(d_x^+, z, d_x^+ - \theta + 1\big), \quad f_D(z) = 1 - f_S(z),$$

$$f_{H^+}(z) := \sum_{x\in\mathcal{X}} \mu_x \sum_{\theta=1}^{d_x^+} q_x(\theta) \sum_{\ell=d_x^+-\theta+1}^{d_x^+} \ell b\big(d_x^+, z, \ell\big), \quad f_{I^+}(z) = \lambda z - f_{H^+}(z),$$

$$f_W(z) := \lambda z - \sum_{x\in\mathcal{X}} \mu_x d_x^- \sum_{\theta=1}^{d_x^+} q_x(\theta)\beta\big(d_x^+, z, d_x^+ - \theta + 1\big).$$

The following theorem states the law of large numbers for the number of solvent banks, defaulted banks, healthy links, infected links and the total number of existing white balls (remaining interactions yielding at least one default) at any time t in the economy $\mathcal{E}_n$ satisfying above regularity assumptions.

Theorem 3.1. *Suppose that Assumptions 1-3a hold. Let $\tau_n \leqslant \tau_n^\star$ be a stopping time such that $\tau_n \xrightarrow{p} t_0$ for some $t_0 > 0$. Then for all $x \in \mathcal{X}, \theta = 1, \ldots, d_x^+$ and $\ell = 0, \ldots, \theta - 1$, we have (as $n \to \infty$)*

$$\sup_{t\leqslant\tau_n} \Big| \frac{S_{x,\theta,\ell}^{(n)}(t)}{n} - \mu_x q_x(\theta) b\left(d_x^+, 1 - e^{-t}, \ell\right) \Big| \xrightarrow{p} 0.$$

Further, as $n \to \infty$,

$$\sup_{t\leqslant\tau_n} \Big| \frac{S_n(t)}{n} - f_S(e^{-t}) \Big| \xrightarrow{p} 0, \quad \sup_{t\leqslant\tau_n} \Big| \frac{D_n(t)}{n} - f_D(e^{-t}) \Big| \xrightarrow{p} 0,$$

$$\sup_{t\leqslant\tau_n} \Big| \frac{H_n^+(t)}{n} - f_{H^+}(e^{-t}) \Big| \xrightarrow{p} 0, \quad \sup_{t\leqslant\tau_n} \Big| \frac{I_n^+(t)}{n} - f_{I^+}(e^{-t}) \Big| \xrightarrow{p} 0,$$

and the number of white balls satisfies

$$\sup_{t\leqslant\tau_n} \Big| \frac{W_n(t)}{n} - f_W(e^{-t}) \Big| \xrightarrow{p} 0.$$

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The proof of the above theorem is provided in Section 6.3.

We consider now the stopping time $\tau_n^\star$ which is the first time such that $W_n(\tau_n^\star) = -1$ (becomes negative). Let us define

$$z^\star := \sup\{z \in [0,1] : f_W(z) = 0\}. \tag{5}$$

Then we have the following lemma.

Lemma 3.2. *Suppose that Assumptions 1-3a hold. We have (as $n \to \infty$):*

(i) If $z^\star = 0$ then $\tau_n^\star \xrightarrow{p} \infty$.

(ii) If $z^\star \in (0,1]$ and $z^\star$ is a stable solution, i.e. $f'_W(z^\star) > 0$, then $\tau_n^\star \xrightarrow{p} -\ln z^\star$.

The proof of lemma is provided in Appendix A.1.

As a corollary of Theorem 3.1 and Lemma 3.2, we next provide the law of large numbers for the final state of default contagion. Namely, under Assumption 3a, the following holds.

Theorem 3.3. *Suppose that Assumptions 1-3a hold. The final fraction of defaults satisfies:*

(i) If $z^\star = 0$ then asymptotically almost all institutions default during the cascade and

$$\left|\mathcal{D}_{n-1}\right| = n - o_p(n).$$

(ii) If $z^\star \in (0,1]$ and $z^\star$ is a stable solution, i.e. $f'_W(z^\star) > 0$, then

$$\frac{\left|\mathcal{D}_{n-1}\right|}{n} \xrightarrow{p} f_D(z^\star).$$

Further, in this case, for all $x \in \mathcal{X}, \theta = 1, \ldots, d_x^+$ and $\ell = 0, \ldots, \theta - 1$, the final fraction of solvent institutions with type x, threshold θ and ℓ defaulted neighbors satisfies

$$\frac{S_{x,\theta,\ell}^{(n)}}{n} \xrightarrow{p} \mu_x q_x(\theta) b\left(d_x^+, 1 - z^\star, \ell\right).$$

The proof of the above theorem is provided in Section 6.4. In particular, the above theorem (in a simpler setup) has been used in [9] to give a resilience condition for contagion in random financial networks. Namely, in the notations above, starting from a small fraction ϵ of institutions representing the fundamental defaults, i.e., $\sum_{x \in \mathcal{X}} \mu_x q_x(0) = \epsilon$, the financial network is said to be resilient if $\lim_{\epsilon \to 0} z^\star = 0$. We refer to [7, 9] for the resilience conditions.

3.2 Asymptotic Normality of Default Contagion

In order to study the central limit theorems, we need to restrict our attention to the sparse networks regime. Namely, we consider the random financial network $\mathcal{G}^{(n)}(\mathbf{d}_n^+, \mathbf{d}_n^-)$ and assume that degrees sequences satisfy the following moment condition.

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Assumption 3b. *We assume that for every constant $A > 1$, we have*

$$\sum_{i=1}^{n} A^{d_i^+} = n \sum_{x \in \mathcal{X}} \mu_x^{(n)} A^{d_x^+} = O(n) \quad \text{and} \quad \sum_{i=1}^{n} A^{d_i^-} = n \sum_{x \in \mathcal{X}} \mu_x^{(n)} A^{d_x^-} = O(n).$$

Remark 3.4. *Let (D_n^+, D_n^-) be random variables with joint distribution*

$$\mathbb{P}(D_n^+ = d^+, D_n^- = d^-) = \sum_{x \in \mathcal{X}} \mu_x^{(n)} \mathbb{1}\{d_x^+ = d^+, d_x^- = d^-\},$$

which is the joint distribution of in- and out- degrees for a random node in $\mathcal{G}^{(n)}(\mathbf{d}_n^+, \mathbf{d}_n^-)$. Let also (D^+, D^-) be random variables (over nonnegative integers) with joint distribution

$$\mathbb{P}(D^+ = d^+, D^- = d^-) = \sum_{x \in \mathcal{X}} \mu_x \mathbb{1}\{d_x^+ = d^+, d_x^- = d^-\}.$$

Then Assumption 3b can be rewritten as $\mathbb{E}[A^{D_n^+}] = O(1)$ and $\mathbb{E}[A^{D_n^-}] = O(1)$ for each $A > 1$, which in particular implies the uniform integrability of D_n^+ and D_n^-, so

$$\lambda^{(n)} := \sum_{x \in \mathcal{X}} d_x^+ \mu_x^{(n)} = \mathbb{E}[D_n^+] \longrightarrow \mathbb{E}[D^+] = \lambda \in (0, \infty).$$

Similarly, all higher moments converge.

By the construction of the balls-and-bins model, the independency exists between any two different types $x_1 \neq x_2$, at any time t before the final state is reached. Hence we study the asymptotic normality type by type. We first show the following joint convergence theorem for all $x \in \mathcal{X}, \theta = 1, \ldots, d_x^+, \ell = 0, \ldots, \theta - 1$.

Theorem 3.5. *Suppose that Assumptions 1-3b hold. Let $\tau_n \leqslant \tau_n^\star$ be a stopping time such that $\tau_n \xrightarrow{p} t_0$ for some $t_0 > 0$. For all $x \in \mathcal{X}, \theta = 1, \ldots, d_x^+, \ell = 0, \ldots, \theta - 1$ and jointly in $\mathcal{D}[0, \infty)$,*

$$n^{-1/2} \left(S_{x,\theta,\ell}^{(n)}(t \wedge \tau_n) - n\mu_x^{(n)} q_x^{(n)}(\theta) b\big(d_x^+, 1 - e^{-(t \wedge \tau_n)}, \ell\big) \right) \xrightarrow{d} \mathcal{Z}_{x,\theta,\ell}(t \wedge t_0),$$

where $\mathcal{Z}_{x,\theta,\ell}(t)$ is a Gaussian process with mean 0 and variance $\sigma_{x,\theta,\ell}(t)$ given by (14).

The proof of the above theorem is provided in Section 6.5. Indeed, we prove a stronger version of above theorem, providing also the covariance between $\mathcal{Z}^*_{x_1,\theta_1,\ell_1}$ and $\mathcal{Z}^*_{x_2,\theta_2,\ell_2}$, for any two triplets (x_1, θ_1, ℓ_1) and (x_2, θ_2, ℓ_2); the covariance is given by (13).

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For $z \in [0,1]$, we define the functions:

$$f_S^{(n)}(z) := \sum_{x\in\mathcal{X}} \mu_x^{(n)} \sum_{\theta=1}^{d_x^+} q_x^{(n)}(\theta)\beta\big(d_x^+, z, d_x^+ - \theta + 1\big), \quad f_D^{(n)}(z) = 1 - f_S^{(n)}(z),$$

$$f_{H^+}^{(n)}(z) := \sum_{x\in\mathcal{X}} \mu_x^{(n)} \sum_{\theta=1}^{d_x^+} q_x^{(n)}(\theta) \sum_{\ell=d_x^+-\theta+1}^{d_x^+} \ell b\big(d_x^+, z, \ell\big), \quad f_{I^+}^{(n)}(z) = \lambda z - f_{H^+}^{(n)}(z),$$

$$f_W^{(n)}(z) := \lambda^{(n)} z - \sum_{x\in\mathcal{X}} \mu_x^{(n)} d_x^- \sum_{\theta=1}^{d_x^+} q_x^{(n)}(\theta)\beta\big(d_x^+, z, d_x^+ - \theta + 1\big).$$

In the following theorem, we show the joint asymptotic normality between the total number of solvent institutions, number of defaulted institutions, number of infected and healthy links, and the total number of white balls (controlling the default contagion stopping time) at any time t before the end of default cascade.

For convenience, we set

$$\widehat{f}_{\clubsuit}^{(n)}(t) = f_{\clubsuit}^{(n)}(e^{-t}),$$

for $\clubsuit \in \{S, D, H^+, I^+, W\}$.

Theorem 3.6. *Suppose that Assumptions 1-3b hold. Let $\tau_n \leqslant \tau_n^\star$ be a stopping time such that $\tau_n \xrightarrow{p} t_0$ for some $t_0 > 0$. Then, under the above assumptions and jointly in $\mathcal{D}[0,\infty)$, as $n \to \infty$,*

$$n^{-1/2}\left(\clubsuit_n(t\wedge\tau_n) - n\widehat{f}_{\clubsuit}^{(n)}(t\wedge\tau_n)\right) \xrightarrow{d} \mathcal{Z}_{\clubsuit}(t\wedge t_0) \tag{6}$$

for $\clubsuit \in \{S, D, H^+, I^+, W\}$, where $\{Z_{\clubsuit}\}$ are continuous Gaussian processes on $[0, t_0]$ with mean 0 and covariances that satisfy, for $0 \leqslant t \leqslant t_0$ and $\clubsuit, \spadesuit \in \{S, D, H^+, I^+, W\}$,

$$\mathrm{Cov}\big(\mathcal{Z}_{\clubsuit}(t), \mathcal{Z}_{\spadesuit}(t)\big) = \sigma_{\clubsuit,\spadesuit}(e^{-t}),$$

where the form of $\sigma_{\clubsuit,\spadesuit}(x)$ are given by (51)-(55) in Appendix A.6.

The proof of the theorem is provided in Section 6.6. Note that since $D_n(t) = n - S_n(t)$, it would be easy to transfer the result to D by setting $\sigma_{D,D} = \sigma_{S,S}$ and $\sigma_{D,\clubsuit} = -\sigma_{S,\clubsuit}$. Further, since $I_n(t) = L_n(t) - H_n^+(t)$, calculating the covariances of I^+ is similar to H^+ and is omitted from the proof. We only provide the covariances for $\clubsuit, \spadesuit \in \{S, H^+, W\}$.

We further define

$$s_{x,\theta,\ell}(z) := \mu_x q_x(\theta) b\big(d_x^+, 1-z, \ell\big), \quad s_{x,\theta,\ell}^{(n)}(z) := \mu_x^{(n)} q_x^{(n)}(\theta) b\big(d_x^+, 1-z, \ell\big).$$

As a corollary of Theorem 3.6 and Lemma 3.2, we have the following theorem regarding the final state of default contagion.

Theorem 3.7. *Suppose that Assumptions 1-3b hold. Let $t^\star = -\ln z^\star$ and $\widehat{z}_n$ be the largest $z \in [0,1]$ such that $f_W^{(n)}(z) = 0$. Under the above assumptions, we have:*

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(i) If $z^\star = 0$ then asymptotically almost all institutions default during the cascade and

$$\left|\mathcal{D}_{n-1}\right| = n - o_p(n).$$

(ii) If $z^\star \in (0,1]$ and $z^\star$ is a stable solution, i.e. $\alpha := f'_W(z^\star) > 0$, then we have

$$n^{-1/2}(\clubsuit_n(\tau_n^\star) - nf_\clubsuit^{(n)}(\widehat{z}_n)) \xrightarrow{d} \mathcal{Z}_\clubsuit(t^\star) - \alpha^{-1} f'_\clubsuit(z^\star)\mathcal{Z}_W(t^\star), \tag{7}$$

for $\clubsuit \in \{S, D, H^+, I^+, W\}$, where the limit distributions compose a Gaussian vector. Furthermore, $\widehat{z}_n \to z^\star$ and, for all $x \in \mathcal{X}$, $0 \leqslant \ell < \theta \leqslant d_x^+$,

$$n^{-1/2}(S_{x,\theta,\ell}^{(n)}(\tau_n^\star) - ns_{x,\theta,\ell}^{(n)}(\widehat{z}_n)) \xrightarrow{d} \mathcal{Z}^*_{x,\theta,\ell}(t^\star) - \alpha^{-1} s'_{x,\theta,\ell}(z^\star)\mathcal{Z}_W(t^\star). \tag{8}$$

The proof of the theorem is provided in Section 6.7.

4 Quantifying Systemic Risk

In order to determine the health of the financial network, we consider in this section a *systemic risk measure* applied to the (random) financial network, introduced in previous sections. These measures are decomposed as $\rho \circ \Gamma$ for a stand-alone risk measure (usually assumed convex) ρ and an aggregation function $\Gamma = \Gamma(\boldsymbol{\epsilon})$ for losses under the stress scenario $\boldsymbol{\epsilon}$. This was first introduced in [19, 44]; see also [10, 29].

The following three aggregation functions has been considered in the literature. At time t, for the economy $\mathcal{E}_n$ and given shock scenario $\boldsymbol{\epsilon}$, we let:

- **Number of solvent banks**: $\Gamma_n^\#(t) := S_n(t) = n - D_n(t)$.
- **External wealth**: Let $\bar{\Gamma}_n^\odot$ denote the total external wealth to society if there is no default in the financial system (small shock regime). We define the external wealth (societal) aggregation function as

$$\Gamma_n^\odot(t) := \bar{\Gamma}_n^\odot - \sum_{x\in\mathcal{X}} \bar{L}_x^\odot D_x^{(n)}(t),$$

  where $D_x^{(n)}(t) = n\mu_x^{(n)} - \sum_\theta \sum_{\ell=0}^{\theta-1} S_{x,\theta,\ell}^{(n)}(t)$ denotes the total number of defaulted institutions with type $x \in \mathcal{X}$ at time t. Note that (for simplicity) we assume a bounded constant type-dependent societal loss $\bar{L}_x^\odot$ over each defaulted institution.
- **System-wide wealth** : Let $\bar{\Gamma}_n^\Diamond$ denote the total wealth in the financial system if there is no default in the system. We define the system-wide aggregation function as

$$\Gamma_n^\Diamond(t) := \bar{\Gamma}_n^\Diamond - \sum_{x\in\mathcal{X}} \bar{L}_x^\odot D_x^{(n)}(t) - \sum_{x\in\mathcal{X}} \bar{L}_x^\Diamond \sum_{\theta=1}^{d_x^+} \sum_{\ell=1}^{\theta-1} \ell S_{x,\theta,\ell}^{(n)}(t).$$

  For each type $x \in \mathcal{X}$, we consider a bounded fixed (type-dependent) societal cost $\bar{L}_x^\odot$

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for defaulted institutions and a bounded fixed (host institutions' type-dependent) cost $\bar{L}_x^{\Diamond}$ over each defaulted links.

For the aggregation function $\Gamma_n^{\#}(t)$, we already stated the limit theorems in Section 3. Since the societal aggregation function $\Gamma_n^{\odot}$ can be seen as a particular case of system-wide aggregation function $\Gamma_n^{\Diamond}$ (by setting $\bar{L}_x^{\Diamond} = 0$), we only state limit theorems for $\Gamma_n^{\Diamond}$.

In order to state limit theorems, it is natural to assume that $\bar{\Gamma}_n^{\Diamond}/n \to \bar{\Gamma}^{\Diamond}$ when the size of network $n \to \infty$. Let us define

$$f_{\Diamond}^{(n)}(z) := \bar{\Gamma}_n^{\Diamond}/n - \sum_{x\in\mathcal{X}} \bar{L}_x^{\odot} f_D^{(n)}(z) - \sum_{x\in\mathcal{X}} \bar{L}_x^{\Diamond} \sum_{\theta=1}^{d_x^+} \sum_{\ell=1}^{\theta-1} \ell s_{x,\theta,\ell}^{(n)}(z),$$

$$f_{\Diamond}(z) := \bar{\Gamma}^{\Diamond} - \sum_{x\in\mathcal{X}} \bar{L}_x^{\odot} f_D(z) - \sum_{x\in\mathcal{X}} \bar{L}_x^{\Diamond} \sum_{\theta=1}^{d_x^+} \sum_{\ell=1}^{\theta-1} \ell s_{x,\theta,\ell}(z).$$

Similarly we also set

$$\widehat{f}_{\Diamond}^{(n)}(t) = f_{\Diamond}^{(n)}(e^{-t}), \qquad \widehat{f}_{\Diamond}(t) = f_{\Diamond}(e^{-t}).$$

By applying Theorem 3.1 and Theorem 3.3, under Assumption 3a, the following holds.

Theorem 4.1. *Suppose that Assumptions 1-3a hold. Let $\tau_n \leqslant \tau_n^{\star}$ be a stopping time such that $\tau_n \xrightarrow{p} t_0$ for some $t_0 > 0$. Then, as $n \to \infty$,*

$$\sup_{t\leqslant\tau_n} \left| \frac{\Gamma_n^{\Diamond}(t)}{n} - f_{\Diamond}(e^{-t}) \right| \xrightarrow{p} 0. \tag{9}$$

Further, the final (system-wide) aggregation functions satisfy:

(i) If $z^{\star} = 0$ then asymptotically almost all institutions default during the cascade and

$$\frac{\Gamma_n^{\Diamond}(\tau_n^{\star})}{n} \xrightarrow{p} \bar{\Gamma}^{\Diamond} - \sum_{x\in\mathcal{X}} \mu_x \bar{L}_x^{\odot}.$$

(ii) If $z^{\star} \in (0,1]$ and $z^{\star}$ is a stable solution, i.e. $f_W'(z^{\star}) > 0$, then

$$\frac{\Gamma_n^{\Diamond}(\tau_n^{\star})}{n} \xrightarrow{p} f_{\Diamond}(z^{\star}).$$

The proof of the above theorem is provided in Section 6.8.

We next consider the central limit theorems for the societal and system-wide aggregation functions. By applying Theorem 3.6, under Assumption 3b, the following holds.

Theorem 4.2. *Suppose that Assumptions 1-3b hold. Let $\tau_n \leqslant \tau_n^{\star}$ be a stopping time such that $\tau_n \xrightarrow{p} t_0$ for some $t_0 > 0$. Then jointly in $\mathcal{D}[0,\infty)$,*

$$n^{-1/2}\left(\Gamma_n^{\Diamond}(t\wedge\tau_n) - n\widehat{f}_{\Diamond}^{(n)}(t\wedge\tau_n)\right) \xrightarrow{d} \mathcal{Z}_{\Diamond}(t\wedge t_0),$$

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where $\mathcal{Z}_\Diamond$ is a continuous Gaussian process on $[0, t_0]$ with mean 0 and variance $\sigma_\Diamond(t)$ given by (56) in Appendix A.7.

Moreover, the final system-wide aggregation function satisfies:

(i) If $z^\star = 0$ then asymptotically almost all institutions default during the cascade and

$$\frac{\Gamma_n^\Diamond(\tau_n^\star)}{n} \xrightarrow{p} \bar{\Gamma}^\Diamond - \sum_{x\in\mathcal{X}} \mu_x \bar{L}_x^\odot.$$

(ii) If $z^\star \in (0, 1]$ and $z^\star$ is a stable solution, i.e. $\alpha := f'_W(z^\star) > 0$, then we have

$$n^{-1/2}\big(\Gamma_n^\Diamond(\tau_n^\star) - n\widehat{f}_\Diamond^{(n)}(\widehat{z}_n)\big) \xrightarrow{d} \mathcal{Z}_\Diamond^\star,$$

where $\mathcal{Z}_\Diamond^\star$ is a centered Gaussian random variable with variance $\sigma_\Diamond^\star$ given by (57) in Appendix A.7.

The proof of the above theorem is provided in Section 6.9.

5 Targeting Interventions in Financial Networks

In this section we consider a planner (lender of last resort or government) who seeks to minimize the systemic risk at the beginning of the financial contagion, after an exogenous macroeconomic shock $\boldsymbol{\epsilon}$, subject to a budget constraint. As discussed in Section 4, we assume that the systemic risk is represented by $\rho(\Gamma_n^\Diamond)$, for some convex function ρ applied to system-wide wealth. Note that since we study the interventions for a given shock $\boldsymbol{\epsilon}$, the uncertainty (in stress scenario) for the risk measure ρ is only on the network structure (which is assumed to be uniformly at random). The planner only has information regarding the type of each institution and, consequently, the institutions' threshold distributions. Hence, the planner' decision is only based on the type of each institution.

The timeline is as follows. At time $t = 0$, the financial network is subject to an economic shock $\boldsymbol{\epsilon}$. At time $t = 1$, the planer (observing the external shock $\boldsymbol{\epsilon}$) calculates the threshold distribution $q_x(.)$ for each $x \in \mathcal{X}$. Then she makes decisions, under some budget constraint, on the number (fraction) of interventions over all defaulted links leading to any institutions with any type $x \in \mathcal{X}$. When the planner intervenes on a defaulting bank, its threshold (distance to default) increases by 1. These interventions will be type-dependent and at random over all defaulted links leading to the same type institutions.

For $x \in \mathcal{X}$, let us denote by $\alpha_x^{(n)}$ the planner intervention decision on the fraction of the saved links leading to any institution of type $x \in \mathcal{X}$. We assume that $\alpha_x^{(n)} \to \alpha_x$ for all $x \in \mathcal{X}$, and some constants α_x independent of n. Let $\boldsymbol{\alpha}_n = \left\{\alpha_x^{(n)}\right\}_{x\in\mathcal{X}}$, and, $\Gamma_n^\Diamond(\boldsymbol{\alpha}_n)$ denote the system-wide wealth under the intervention decision $\boldsymbol{\alpha}_n$. Further, $S_{x,\theta,\ell}^{(n)}(\boldsymbol{\alpha}_n)$ denotes the number of solvent banks with type x, threshold θ and ℓ defaulted neighbors under the

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intervention decision $\boldsymbol{\alpha}_n$. Similarly, $D_n(\boldsymbol{\alpha}_n)$ denotes the total number of defaults under intervention $\boldsymbol{\alpha}_n$.

Let $C_x \in \mathbb{R}^+$ denote the cost associated to saving any defaulted link leading to an institution of type $x \in \mathcal{X}$. We assume that C_x is a bounded function. We denote by $\Phi_n(\boldsymbol{\alpha}_n)$ the total cost associated to the planner for the intervention strategy $\boldsymbol{\alpha}_n$.

We next state a limit theorem regarding the number of solvent institutions, defaulted institutions, the total aggregate wealth of the financial system and the total cost of intervention for the planner, under the intervention decision $\boldsymbol{\alpha}_n$.

Let us define

$$f_W^{(\boldsymbol{\alpha})}(z) := \lambda z - \sum_{x\in\mathcal{X}} \mu_x d_x^- \sum_{\theta=1}^{d_x^+} q_x(\theta)\beta\big(d_x^+, \alpha_x + (1-\alpha_x)z, d_x^+ - \theta + 1\big),$$

and,

$$z_{\boldsymbol{\alpha}}^\star := \sup\big\{z \in [0,1] : f_W^{(\boldsymbol{\alpha})}(z) = 0\big\}. \tag{10}$$

Theorem 5.1. *Suppose that Assumptions 1-3a hold. Let $\boldsymbol{\alpha}_n \to \boldsymbol{\alpha}$ as $n \to \infty$. If $z_{\boldsymbol{\alpha}}^\star$ is a stable solution, then as $n \to \infty$:*

(i) For all $x \in \mathcal{X}, \theta = 1, \ldots, d_x^+$ and $\ell = 0, \ldots, \theta - 1$, the final fraction of solvent institutions with type x, threshold θ and ℓ defaulted neighbors under intervention $\boldsymbol{\alpha}_n$ converges to

$$\frac{S_{x,\theta,\ell}^{(n)}(\boldsymbol{\alpha}_n)}{n} \xrightarrow{p} s_{x,\theta,\ell}^{(\boldsymbol{\alpha})}(z_{\boldsymbol{\alpha}}^\star) := \mu_x q_x(\theta) b\left(d_x^+, (1-\alpha_x)(1-z_{\boldsymbol{\alpha}}^\star), \ell\right).$$

(ii) The total number of defaulted institutions under intervention $\boldsymbol{\alpha}_n$ converges to:

$$\frac{D_n(\boldsymbol{\alpha}_n)}{n} \xrightarrow{p} f_D^{(\boldsymbol{\alpha})}(z_{\boldsymbol{\alpha}}^\star) := 1 - \sum_{x\in\mathcal{X}} \mu_x \sum_{\theta=1}^{d_x^+} q_x(\theta)\beta\big(d_x^+, \alpha_x + (1-\alpha_x)z_{\boldsymbol{\alpha}}^\star, d_x^+ - \theta + 1\big).$$

(iii) The system-wide wealth under the intervention decision $\boldsymbol{\alpha}_n$ converges to

$$\frac{\Gamma_n^\diamondsuit(\boldsymbol{\alpha}_n)}{n} \xrightarrow{p} f_\diamondsuit^{(\boldsymbol{\alpha})}(z_{\boldsymbol{\alpha}}^\star) := \bar{\Gamma}^\diamondsuit - \sum_{x\in\mathcal{X}} \bar{L}_x^\odot f_D^{(\boldsymbol{\alpha})}(z_{\boldsymbol{\alpha}}^\star) - \sum_{x\in\mathcal{X}} \bar{L}_x^\diamondsuit \sum_{\theta=1}^{d_x^+} \sum_{\ell=1}^{\theta-1} \ell s_{x,\theta,\ell}^{(\boldsymbol{\alpha})}(z_{\boldsymbol{\alpha}}^\star).$$

(iv) The total cost of interventions $\boldsymbol{\alpha}_n$ for the planner converges to

$$\frac{\Phi_n(\boldsymbol{\alpha}_n)}{n} \xrightarrow{p} \phi(z_{\boldsymbol{\alpha}}^\star) := \sum_{x\in\mathcal{X}} \mu_x \alpha_x C_x \sum_{\ell=1}^{d_x^+} \ell b\left(d_x^+, 1 - z_{\boldsymbol{\alpha}}^\star, \ell\right).$$

The proof of theorem is provided in Section 6.10.

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We conclude that, as $n \to \infty$, the planner optimal decision problem simplifies to

$$\max_{\boldsymbol{\alpha}} f_{\Diamond}^{(\boldsymbol{\alpha})}(z_{\boldsymbol{\alpha}}^{\star}) := \bar{\Gamma}^{\Diamond} - \sum_{x \in \mathcal{X}} \bar{L}_x^{\odot} f_D^{(\boldsymbol{\alpha})}(z_{\boldsymbol{\alpha}}^{\star}) - \sum_{x \in \mathcal{X}} \bar{L}_x^{\Diamond} \sum_{\theta=1}^{d_x^+} \sum_{\ell=1}^{\theta-1} \ell s_{x,\theta,\ell}^{(\boldsymbol{\alpha})}(z_{\boldsymbol{\alpha}}^{\star}),$$

$$\text{subject to} \quad \phi(z_{\boldsymbol{\alpha}}^{\star}) := \sum_{x \in \mathcal{X}} \mu_x \alpha_x C_x \sum_{\ell=1}^{d_x^+} \ell b\left(d_x^+, 1 - z_{\boldsymbol{\alpha}}^{\star}, \ell\right) \leqslant C,$$

for some budget constraint $C > 0$ and $z_{\boldsymbol{\alpha}}^{\star}$ given by (10).

It would be interesting to extend the model to a continuous-time Markov decision process by the planner, where the links (starting from fundamentally defaulted institutions) are revealed one by one, and consequently, the planner should decide at any time to intervene or not. This would lead to a Markov decision problem and one could solve it (under some regularity assumptions) by using a dynamic programming approach. We refer to [12] for a similar model in a simpler setup with a core-periphery network structure. We leave this and some related issues to a future work.

6 Proof of Main Theorems

This section contains the proofs of all the theorems in previous sections. We first provide some preliminary results on death processes and martingale theory that will be used in our proofs. Proofs of lemmas are provided in Appendix A.

6.1 Some Death Processes

The studies of death processes in the balls-and-bins problem rely on the following classical result, see e.g. [42, Proposition 4.24].

Lemma 6.1. *(The Glivenko-Cantelli Theorem) Let $T_1, \ldots, T_n$ be i.i.d random variables with distribution function $F(t) := \mathbb{P}(T_i \leqslant t)$. Let $X_n(t)$ be their empirical distribution function $X_n(t) := \#\{i \leqslant n : T_i \leqslant t\}/n$. Then $\sup_t |X_n(t) - F(t)| \xrightarrow{p} 0$ as $n \to \infty$.*

Since in the balls-and-bins model described in Section 2.4, every in ball dies independently after an exponential time with parameter 1, this is a pure death process starting with some number of balls whose lifetimes are i.i.d exp(1).

Lemma 6.2. *(Death Process Lemma) Let $N^{(n)}(t)$ be the number of balls alive at time t and all balls have i.i.d. lifetime* exp(1)*, starting with initial number $N^{(n)}(0) = n$. Then*

$$\sup_{t \geqslant 0} |N^{(n)}(t)/n - e^{-t}| \xrightarrow{p} 0 \qquad \text{as} \quad n \to \infty.$$

Proof. Since $1 - N^{(n)}(t)/n$ is the empirical distribution function of the n i.i.d. random variables of distribution exp(1), whose distribution function is $1 - e^{-t}$, thus the result follows by using the Glivenko-Cantelli theorem. □

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6.2 Some Martingale Theory

Our proof of the asymptotic normality for the default contagion is based on a martingale limit theorem by [38]. Let X be a martingale defined on $[0,\infty)$. We denote its quadratic variation by $[X,X]_t$. We also denote the (bilinear) covariation of two martingales X and Y by $[X,Y]_t$. In particular, if X and Y are two martingales with path-wise finite variation, then $[X,Y]_t := \sum_{0<s\leqslant t} \Delta X(s)\Delta Y(s)$, where $\Delta X(s) := X(s)-X(s-)$ is the jump of X at s and similarly $\Delta Y(s) := Y(s)-Y(s-)$. Note that in this paper, the considered martingales are always RCLL (right continuous and with left limit) and have only finite number of jumps. Hence, the quadratic variation will be finite. We also set $[X,Y]_0 = 0$. For two vector-valued martingales $\mathbf{X} = (X_i)_{i=1}^n$ and $\mathbf{Y} = (Y_j)_{j=1}^m$, we define $[\mathbf{X},\mathbf{Y}]$ to be the $n\times m$ real matrix with every entry being $([\mathbf{X},\mathbf{Y}])_{i,j} = [X_i, Y_j]$.

We will use the following martingale limit theorem from [38].

Theorem 6.3 (Martingale Limit Theorem)**.** *Assume that for each n, $\mathbf{M}^{(n)}(t) = (\mathbf{M}_i^{(n)}(t))_{i=1}^q$ is a q-dimensional martingale on $[0,\infty)$ with $\mathbf{M}^{(n)}(0) = \mathbf{0}$, and that $\Sigma(t)$ is a (nonrandom) continuous matrix-valued function satisfying, for every fixed $t \geqslant 0$,*

(i) $[\mathbf{M}^{(n)},\mathbf{M}^{(n)}]_t \xrightarrow{p} \Sigma(t)$ *as* $n\to\infty$,

(ii) $\sup_n \mathbb{E}[M_i^{(n)}, M_i^{(n)}]_t < \infty$, *for all* $i = 1,\dots,q$.

Then $\mathbf{M}^{(n)} \xrightarrow{d} \mathbf{Z}$ as $n\to\infty$, in $\mathcal{D}[0,\infty)$, where $\mathbf{Z}$ is a continuous q-dimensional Gaussian martingale with $\mathbb{E}[\mathbf{Z}(t)] = \mathbf{0}$ and covariances $\mathbb{E}[\mathbf{Z}(t)\mathbf{Z}'(s)] = \Sigma(t)$, for $0\leqslant t\leqslant s<\infty$.

Hence, the above theorem yields joint convergence of the processes $\{M_i^{(n)}\}_{i=1}^q$ and could be extended immediately to infinitely many processes (i.e., for the case $q=\infty$). Indeed, by definition, an infinite family of stochastic processes converge jointly if every finite subfamily does. We will use the above theorem for stopped martingales.

6.3 Proof of Theorem 3.1

We denote by $U_{x,\theta,s}^{(n)}(t)$ the number of bins (institutions) with type $x\in\mathcal{X}$, threshold θ and s alive (in-) balls at time t. Let $N_{x,\theta}^{(n)}$ denote the (random) number of bins with type x and threshold θ. Let also $N_x^{(n)} = \sum_\theta N_{x,\theta}^{(n)}$ denote the number of bins with type x. We first state the following lemma on the convergences of $U_{x,\theta,s}^{(n)}(t)$.

Lemma 6.4. *Let $\tau_n \leqslant \tau_n^\star$ be a stopping time such that $\tau_n \xrightarrow{p} t_0$ for some $t_0>0$. Under Assumption 3a, for all $x\in\mathcal{X}, \theta = 1,\dots,d_x^+$ and $\ell = 0,\dots,\theta-1$, we have (as $n\to\infty$)*

$$\sup_{t\leqslant\tau_n}\Big|\frac{U_{x,\theta,\ell}^{(n)}(t)}{n} - \mu_x q_x(\theta) b\left(d_x^+, e^{-t}, \ell\right)\Big| \xrightarrow{p} 0. \tag{11}$$

Further,

$$\sup_{t\leqslant\tau_n}\sum_{x\in\mathcal{X}}(d_x^+ + d_x^-)\sum_{\theta=1}^{d_x^+}\sum_{s=d_x^+-\theta+1}^{d_x^+}\left|U_{x,\theta,s}^{(n)}(t)/n - \mu_x q_x(\theta) b(d_x^+, e^{-t}, s)\right| \xrightarrow{p} 0. \tag{12}$$

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The proof of above lemma is based on the death process Lemma 6.2 and provided in Section A.2. We now continue the proof of Theorem 3.1.

The proof of above lemma is based on the death process Lemma 6.2 and provided in Section A.2. We now continue the proof of Theorem 3.1.

Consider now $S^{(n)}_{x,\theta,\ell}$, the number of solvent institutions with type x, threshold θ and $\ell = 0, \ldots, \theta-1$ defaulted neighbors at time t. By definition, $S^{(n)}_{x,\theta,\ell}(t) = U^{(n)}_{x,\theta,d^+_x-\ell}(t)$. Hence, by (11), we obtain that

$$\sup_{t\leqslant\tau_n}\Big|\frac{S^{(n)}_{x,\theta,\ell}(t)}{n} - \mu_x q_x(\theta) b\left(d^+_x, 1-e^{-t}, \ell\right)\Big| \xrightarrow{p} 0, \quad \text{as} \quad n\to\infty.$$

The total number of solvent institutions at time t satisfies

$$S_n(t) = \sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d^+_x}\sum_{s=d^+_x-\theta+1}^{d^+_x} U^{(n)}_{x,\theta,s}(t),$$

which is dominated by $\sum_{x\in\mathcal{X}}(d^+_x+d^-_x)\sum_{\theta=1}^{d^+_x}\sum_{s=d^+_x-\theta+1}^{d^+_x} U^{(n)}_{x,\theta,s}(t)$. Then, by Lemma 6.4 and the convergence result (12), we obtain

$$\sup_{t\leqslant\tau_n}\Big|\frac{S_n(t)}{n} - f_S(e^{-t})\Big| \xrightarrow{p} 0.$$

Further, from $D_n(t) = n - S_n(t)$, the number of defaulted institutions at time t satisfies

$$\sup_{t\leqslant\tau_n}\Big|\frac{D_n(t)}{n} - f_D(e^{-t})\Big| \xrightarrow{p} 0.$$

Observe also that the total number of healthy in links at time t is given by

$$H^+_n(t) = \sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d^+_x}\sum_{s=d^+_x-\theta+1}^{d^+_x} sU^{(n)}_{x,\theta,s}(t),$$

which is also dominated by $\sum_{x\in\mathcal{X}}(d^+_x+d^-_x)\sum_{\theta=1}^{d^+_x}\sum_{s=d^+_x-\theta+1}^{d^+_x} U^{(n)}_{x,\theta,s}(t)$ and again by Lemma 6.4, we obtain

$$\sup_{t\leqslant\tau_n}\Big|\frac{H^+_n(t)}{n} - f_{H^+}(e^{-t})\Big| \xrightarrow{p} 0.$$

Moreover, the number of in balls from $\mathbf{I}^+$ at time t satisfies $I^+_n(t) = L_n(t) - H^+_n(t)$. By the construction of the balls-and-bins model, it is easily seen that $L_n(t)$ is a pure death process. It follows by Lemma 6.2 and Assumption 3a that $\sup_{t\geqslant 0}\big|L_n(t)/n - \lambda e^{-t}\big| \xrightarrow{p} 0$. We therefore obtain (by definition $f_{I^+}(z) = \lambda z - f_{H^+}(z)$)

$$\sup_{t\leqslant\tau_n}\Big|\frac{I^+_n(t)}{n} - f_{I^+}(e^{-t})\Big| \xrightarrow{p} 0.$$

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Finally, the total number of white (out) balls at time t satisfies $W_n(t) = L_n(t) - H_n^-(t)$, where $H_n^-(t)$ denotes the total number of healthy (out) balls at time t, given by

$$H_n^-(t) = \sum_{x\in\mathcal{X}} \sum_{\theta=1}^{d_x^+} \sum_{s=d_x^+-\theta+1}^{d_x^+} d_x^- U_{x,\theta,s}^{(n)}(t).$$

This is again dominated by $\sum_{x\in\mathcal{X}}(d_x^+ + d_x^-)\sum_{\theta=1}^{d_x^+}\sum_{s=d_x^+-\theta+1}^{d_x^+} U_{x,\theta,s}^{(n)}(t)$. Then Lemma 6.4 and Assumption 3a implies that the number of white balls satisfies

$$\sup_{t\leqslant\tau_n}\left|\frac{W_n(t)}{n} - f_W(e^{-t})\right| \xrightarrow{p} 0.$$

This completes the proof of Theorem 3.1.

6.4 Proof of Theorem 3.3

By Theorem 3.1, it follows that

$$\frac{S_n(\tau_n^\star)}{n} = f_S(e^{-\tau_n^\star}) + o_p(1).$$

If $z^\star = 0$ then, by Lemma 3.2, $\tau_n^\star \xrightarrow{p} \infty$. So $e^{-\tau_n^\star} \xrightarrow{p} 0$ and, since $f_S(0) = 0$, it follows by the continuity of f_S that $f_S(e^{-\tau_n^\star}) \xrightarrow{p} 0$. We therefore have $S_n(\tau_n^\star) = o_p(n)$. This implies that $|\mathcal{D}_{n-1}| = n - S_n(\tau_n^\star) = n - o_p(n)$ and, as desired, asymptotically almost all institutions default.

If $z^\star \in (0,1]$ and $f_W'(z^\star) > 0$, then by Lemma 3.2, we have $e^{-\tau_n^\star} \xrightarrow{p} z^\star$. Moreover, the continuity of f_D implies that $f_D(e^{-\tau_n^\star}) \xrightarrow{p} f_D(z^\star)$. Hence, we have by Theorem 3.1 that

$$\frac{|\mathcal{D}_{n-1}|}{n} = \frac{D_n(\tau_n^\star)}{n} \xrightarrow{p} f_D(z^\star).$$

Now using the first statement of Theorem 3.1 and the continuity of $\mu_x q_x(\theta) b(d_x^+, 1-z, \ell)$ on z, we obtain for all $x \in \mathcal{X}, \theta = 1, \ldots, d_x^+$ and $\ell = 0, \ldots, \theta - 1$, the final fraction of solvent institutions with type x, threshold θ and ℓ defaulted neighbors satisfy

$$\frac{S_{x,\theta,\ell}^{(n)}}{n} \xrightarrow{p} \mu_x q_x(\theta) b\left(d_x^+, 1-z^\star, \ell\right).$$

This completes the proof of Theorem 3.3.

6.5 Proof of Theorem 3.5

Recall that $U_{x,\theta,s}^{(n)}(t)$ denotes the number of bins (institutions) with type $x \in \mathcal{X}$, threshold θ and s alive (in-) balls at time t. Further, we let $V_{x,\theta,s}^{(n)}(t)$ denote the number of bins

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(institutions) with type $x \in \mathcal{X}$, threshold θ and at least s alive balls at time t, so that $V^{(n)}_{x,\theta,s}(t) = \sum_{\ell \geqslant s} U^{(n)}_{x,\theta,\ell}(t)$.

We first study the stochastic process $V^{(n)}_{x,\theta,s}$ for a given $x \in \mathcal{X}$ and integers θ, s. This stochastic process takes an important role in the proof of Theorem 3.5. For all possible triple (x, θ, s), we define

$$V^{*(n)}_{x,\theta,s}(t) := n^{-1/2}\big(V^{(n)}_{x,\theta,s}(t) - n\mu^{(n)}_x q^{(n)}_x(\theta)\beta(d^+_x, e^{-t}, s)\big).$$

We then have the following lemma.

Lemma 6.5. *Let $\tau_n \leqslant \tau^\star_n$ be a stopping time such that $\tau_n \xrightarrow{p} t_0$ for some $t_0 > 0$. Under Assumption 3b, we have that for all couple (x, θ), jointly as $n \to \infty$,*

$$N^{*(n)}_{x,\theta} \xrightarrow{d} \mathcal{Y}^*_{x,\theta},$$

where all $\mathcal{Y}^*_{x,\theta}$ are Gaussian random variables with mean 0 and covariances

$$\mathrm{Cov}(\mathcal{Y}^*_{x_1,\theta_1}, \mathcal{Y}^*_{x_2,\theta_2}) = \psi_{x,\theta_1,\theta_2}\mathbb{1}\{x_1 = x_2\},$$

where

$$\psi_{x,\theta,\theta} := \mu_x q_x(\theta)(1 - q_x(\theta)), \quad \psi_{x,\theta_1,\theta_2} := -\mu_x q_x(\theta_1)q_x(\theta_2) \quad \textit{for all} \quad \theta_1 \neq \theta_2.$$

Further for all triple (x, θ, s), jointly in $\mathcal{D}[0, \infty)$, as $n \to \infty$,

$$V^{*(n)}_{x,\theta,s}(t \wedge \tau_n) \xrightarrow{d} \mathcal{Z}^*_{x,\theta,s}(t \wedge t_0),$$

where all $\mathcal{Z}^*_{x,\theta,s}(t)$ are continuous Gaussian processes with mean 0 and covariances

$$\begin{aligned}
\mathrm{Cov}\big(\mathcal{Z}^*_{x_1,\theta_1,s_1}(t), \mathcal{Z}^*_{x_2,\theta_2,s_2}(t)\big) =& 0, \quad \textit{for all} \quad x_1 \neq x_2,\\
\mathrm{Cov}\big(\mathcal{Z}^*_{x,\theta_1,s_1}(t), \mathcal{Z}^*_{x,\theta_2,s_2}(t)\big) =& \widehat{\sigma}_{x,\theta_1,\theta_2,s_1,s_2}(e^{-t}), \quad \textit{for all} \quad \theta_1 \neq \theta_2,\\
\mathrm{Cov}\big(\mathcal{Z}^*_{x,\theta,s_1}(t), \mathcal{Z}^*_{x,\theta,s_2}(t)\big) =& \widehat{\sigma}_{x,\theta,\theta,s_1,s_2}(e^{-t}) + \widetilde{\sigma}_{x,\theta,s_1,s_2}(e^{-t}),
\end{aligned}$$

where

$$\widehat{\sigma}_{x,\theta_1,\theta_2,s_1,s_2}(e^{-t}) := \beta(d^+_x, e^{-t}, s_1)\beta(d^+_x, e^{-t}, s_2)\psi_{x,\theta_1,\theta_2},$$

and $\widetilde{\sigma}_{x,\theta,s_1,s_2} = \widetilde{\sigma}_{x,\theta,s_2,s_1}$ with

$$\widetilde{\sigma}_{x,\theta,s,s+k}(y) := \frac{1}{2}y^{2s+k}\sum_{j=s+k}^{d^+_x}\binom{j-1}{s-1}\binom{j-1}{s+k-1}\int_y^1 (v-y)^{2j-2s-k}v^{-2j}d\varphi_{x,\theta,j}(v),$$

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with $\varphi_{x,\theta,j}(y) := \mu_x q_x(\theta)\beta(d_x^+, y, j)$.

Moreover, the covariance between $\mathcal{Z}^*_{x_1,\theta_1,s}(t)$ *and* $\mathcal{Y}^*_{x_2,\theta_2}$ *is given by*

$$\mathrm{Cov}\big(\mathcal{Z}^*_{x_1,\theta_1,s}(t), \mathcal{Y}^*_{x_2,\theta_2}\big) = \beta(d^+_{x_1}, e^{-t}, s)\psi_{x_1,\theta_1,\theta_2}\mathbb{1}\{x_1 = x_2\}.$$

The proof of Lemma is in Appendix A.3.

We now return back to the proof of Theorem 3.5. For the number of solvent institutions with type $x \in \mathcal{X}$, threshold $\theta = 1, \ldots, d_x^+ - 1$ and $\ell = 1, \ldots, \theta - 1$ defaulted neighbors at time t, we have

$$S^{(n)}_{x,\theta,\ell}(t) = V^{(n)}_{x,\theta,d_x^+-\ell} - V^{(n)}_{x,\theta,d_x^+-\ell+1}.$$

Moreover, for $\ell = 0$, $S^{(n)}_{x,\theta,0}(t) = V^{(n)}_{x,\theta,d_x^+}$. Using the joint asymptotic normality of $V^{*(n)}_{x,\theta,d_x^+-\ell}$ and $V^{*(n)}_{x,\theta,d_x^+-\ell+1}$, we obtain that in $\mathcal{D}[0,\infty)$,

$$n^{-1/2}\left(S^{(n)}_{x,\theta,\ell}(t\wedge\tau_n) - n\mu^{(n)}_x q^{(n)}_x(\theta) b\big(d_x^+, 1 - e^{-(t\wedge\tau_n)}, \ell\big)\right) \xrightarrow{d} \mathcal{Z}_{x,\theta,\ell}(t\wedge t_0),$$

where $\mathcal{Z}_{x,\theta,\ell} = \mathcal{Z}^*_{x,\theta,d_x^+-\ell} - \mathcal{Z}^*_{x,\theta,d_x^+-\ell+1}$ for $\ell \geqslant 1$ and $\mathcal{Z}_{x,\theta,0} = \mathcal{Z}^*_{x,\theta,d_x^+}$. Further, for convenience, we set $\mathcal{Z}^*_{x,\theta,s} = 0$ for all $s > d_x^+$.

Thus for any two triple (x_1, θ_1, ℓ_1) and (x_2, θ_2, ℓ_2),

$$\begin{aligned}
&\mathrm{Cov}\big(\mathcal{Z}_{x_1,\theta_1,\ell_1}(t), \mathcal{Z}_{x_2,\theta_2,\ell_2}(t)\big)\\
=&\mathrm{Cov}\big(\mathcal{Z}^*_{x_1,\theta_1,d^+_{x_1}-\ell_1}(t), \mathcal{Z}^*_{x_2,\theta_2,d^+_{x_2}-\ell_2}(t)\big) + \mathrm{Cov}\big(\mathcal{Z}^*_{x_1,\theta_1,d^+_{x_1}-\ell_1+1}(t), \mathcal{Z}^*_{x_2,\theta_2,d^+_{x_2}-\ell_2+1}(t)\big)\\
&- \mathrm{Cov}\big(\mathcal{Z}^*_{x_1,\theta_1,d^+_{x_1}-\ell_1}(t), \mathcal{Z}^*_{x_2,\theta_2,d^+_{x_2}-\ell_2+1}(t)\big) - \mathrm{Cov}\big(\mathcal{Z}^*_{x_1,\theta_1,d^+_{x_1}-\ell_1+1}(t), \mathcal{Z}^*_{x_2,\theta_2,d^+_{x_2}-\ell_2}(t)\big),
\end{aligned} \tag{13}$$

where the covariances on the right hand side could be calculated by using Lemma 6.5.

In particular, the variance of $\mathcal{Z}_{x,\theta,\ell}(t)$ is given by

$$\begin{aligned}
\sigma_{x,\theta,\ell}(t) =& \tilde{\sigma}_{x,\theta,d_x^+-\ell,d_x^+-\ell}(e^{-t}) + \tilde{\sigma}_{x,\theta,d_x^+-\ell+1,d_x^+-\ell+1}(e^{-t})\\
&- 2\tilde{\sigma}_{x,\theta,d_x^+-\ell,d_x^+-\ell+1}(e^{-t}) + b^2(d_x^+, e^{-t}, d_x^+ - \ell)\psi_{x,\theta,\theta},
\end{aligned} \tag{14}$$

where $\psi_{x,\theta,\theta} = \mu_x q_x(\theta)(1 - q_x(\theta))$ as in Lemma 6.5.

6.6 Proof of Theorem 3.6

We first show that the moment regularity condition (i.e., Assumption 3b) guarantees that, as $n \to \infty$, not only $f^{(n)}_{\clubsuit}(z) \to f_{\clubsuit}(z)$ for all $\clubsuit \in \{S, D, H^+, I^+, W\}$, but we also have these convergences together with all their derivatives, uniformly on $[0, 1]$.

Lemma 6.6. *The Assumption 3b guarantees that, as* $n \to \infty$, $f^{(n)}_{\clubsuit}(z) \to f_{\clubsuit}(z)$, *for all* $\clubsuit \in \{S, D, H^+, I^+, W\}$, *together with all their derivatives, uniformly on* $[0, 1]$.

The proof of Lemma is in Appendix A.4. The above lemma allows us to extend the

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convergence results to some infinite sums of $V^{*(n)}_{x,\theta,s}(t)$. We denote by $\mathcal{X}^+_\ell$ and $\mathcal{X}^-_\ell$ the set of characteristics $x \in \mathcal{X}$ with in-degree $d^+_x \geqslant \ell$ and out-degree $d^-_x \geqslant \ell$, respectively.

Lemma 6.7. *Let $\tau_n \leqslant \tau^\star_n$ be a stopping time such that $\tau_n \xrightarrow{p} t_0$ for some $t_0 > 0$. Under Assumption 3b and for n large enough, we have as $\ell \to \infty$,*

$$\mathbb{E}\big[\sup_{t\leqslant T}\big|\sum_{x\in\mathcal{X}^+_\ell}\sum_{\theta=1}^{d^+_x}\sum_{s=d^+_x-\theta+2}^{d^+_x} V^{*(n)}_{x,\theta,s}(t)\big|\big]\to 0,$$

Further as $n \to \infty$,

$$\sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d^+_x}\sum_{s=d^+_x-\theta+2}^{d^+_x} V^{*(n)}_{x,\theta,s}(t\wedge\tau_n) \xrightarrow{d} \sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d^+_x}\sum_{s=d^+_x-\theta+2}^{d^+_x} \mathcal{Z}^*_{x,\theta,s}(t\wedge t_0). \tag{15}$$

Similarly, as $n \to \infty$,

$$\sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d^+_x} V^{*(n)}_{x,\theta,d^+_x-\theta+1}(t\wedge\tau_n) \xrightarrow{d} \sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d^+_x} \mathcal{Z}^*_{x,\theta,d^+_x-\theta+1}(t\wedge t_0),$$

$$\sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d^+_x}(d^+_x-\theta+1)V^{*(n)}_{x,\theta,d^+_x-\theta+1}(t\wedge\tau_n) \xrightarrow{d} \sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d^+_x}(d^+_x-\theta+1)\mathcal{Z}^*_{x,\theta,d^+_x-\theta+1}(t\wedge t_0),$$

and

$$\sum_{x\in\mathcal{X}} d^-_x\sum_{\theta=1}^{d^+_x} V^{*(n)}_{x,\theta,d^+_x-\theta+1}(t\wedge\tau_n) \xrightarrow{d} \sum_{x\in\mathcal{X}} d^-_x\sum_{\theta=1}^{d^+_x} \mathcal{Z}^*_{x,\theta,d^+_x-\theta+1}(t\wedge t_0).$$

Moreover, all the above limit processes on the right hand side are continuous.

The proof of Lemma is in Appendix A.5.

We come back to the proof of Theorem 3.6. Recall that $L_n(t)$ denotes the total number of alive in balls at time t. At the initial time, $L_n(0) = n\lambda^{(n)}$ and $L_n(t)$ decreases by 1 each time a (in) ball dies. Since the death happens after an exponential time with rate 1 independently, therefore on $[0, \tau^\star_n]$, writing in differential form, we have

$$dL_n(t) = -L_n(t)dt + d\mathcal{M}_t,$$

where $\mathcal{M}$ is a martingale. Then by Ito's lemma we have

$$d(e^t L_n(t)) = e^t dL_n(t) + e^t L_n(t)dt = e^t d\mathcal{M}_t,$$

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which implies $\widehat{L}_n(t) := e^t L_n(t)$ is another martingale. The quadratic variation of $\widehat{L}_n(t)$ is

$$\begin{aligned}
[\widehat{L}_n, \widehat{L}_n]_t &= \sum_{0<s\leqslant t} |\Delta \widehat{L}_n(s)|^2 = \sum_{0<s\leqslant t} e^{2s} |\Delta L_n(s)|^2 \\
&= \sum_{0<s\leqslant t} e^{2s}(L_n(s) - L_n(s-)) = \int_0^t e^s d(-L_n(s)) \\
&= -e^{2t} L_n(t) + L_n(0) + \int_0^t 2e^{2s} L_n(s) ds.
\end{aligned}$$

In particular,

$$[\widehat{L}_n, \widehat{L}_n]_t \leqslant e^{2t} \int_0^t d(-L_n(s)) \leqslant n\lambda^{(n)} e^{2t}.$$

In addition, as shwon in the proof of Lemma 3.1, uniformly on $[0, \tau_n]$ we have

$$\sup_{t\leqslant \tau_n} \Big| \frac{L_n(t)}{n} - \lambda e^{-t} \Big| \xrightarrow{p} 0.$$

Going back to the quadratic variation, for every $t \in [0, T \wedge \tau_n]$ with T fixed,

$$\begin{aligned}
[\widehat{L}_n, \widehat{L}_n]_t &= -e^{2t} L_n(0) e^{-t} + L_n(0) + \int_0^t 2e^{2s} L_n(0) e^{-s} ds + o_p(n) \\
&= L_n(0)(-e^t + 1 + 2e^t - 2) + o_p(n) \\
&= L_n(0)(e^t - 1) + o_p(n) = \frac{\lambda n}{2}(e^t - 1) + o_p(n).
\end{aligned}$$

Next we define

$$\widetilde{L}_n(t) := n^{-1/2}\big(\widehat{L}_n(t) - \widehat{L}_n(0)\big) = n^{-1/2}\big(e^t L_n(t) - n\lambda^{(n)}\big),$$

and the quadratic variation is given by

$$[\widetilde{L}_n, \widetilde{L}_n]_t = n^{-1}[\widehat{L}_n, \widehat{L}_n]_t = \frac{\lambda}{2}(e^t - 1) + o_p(1).$$

Let us stop the process at $\tau_n \leqslant \tau_n^\star$. By assumption $\tau_n \xrightarrow{p} t_0$, the quadratic variation of the stopped process converges in probability to $\lambda(e^{t\wedge t_0} - 1)/2$. Then for any fixed $t \geqslant 0$, there exist some constant C such that

$$[\widetilde{L}_n, \widetilde{L}_n]_{t\wedge\tau_n} = n^{-1}[\widehat{L}_n, \widehat{L}_n]_{t\wedge\tau_n} \leqslant \lambda^{(n)} e^{2t} \leqslant C e^{2t},$$

which holds uniformly for n. Thus, from Theorem 6.3 for the stopped process, we have

$$\widetilde{L}_n(t \wedge \tau_n) \xrightarrow{d} \widetilde{\mathcal{Z}}_L(t \wedge t_0) \qquad \text{in} \quad \mathcal{D}[0, \infty),$$

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where $\widetilde{\mathcal{Z}}_L$ is a continuous Gaussian process with $\mathbb{E}[\widetilde{\mathcal{Z}}_L(t)] = 0$ and covariances

$$\mathbb{E}\big[\widetilde{\mathcal{Z}}_L(t)\widetilde{\mathcal{Z}}_L(u)\big] = \lambda(e^t - 1)/2, \qquad 0 \leqslant t \leqslant u < \infty.$$

We further define

$$\widetilde{\widehat{L}}_n(t) := e^{-t}\widetilde{L}_n(t), \quad \mathcal{Z}_L(t) := e^{-t}\widetilde{\mathcal{Z}}_L(t).$$

Thus it follows that

$$\widetilde{\widehat{L}}_n(t \wedge \tau_n) \xrightarrow{d} \mathcal{Z}_L(t \wedge t_0) \qquad \text{in} \quad \mathcal{D}\left[0, \infty\right).$$

Let $U^{(n)}_{x,\theta,s}(t)$ and $V^{(n)}_{x,\theta,s}(t)$ be as defined in the proof of Theorem 6.5. Note that the followings hold:

$$S_n(t) = \sum_{x\in\mathcal{X}} \sum_{\theta=1}^{d_x^+} V^{(n)}_{x,\theta,d_x^+-\theta+1}(t), \quad D_n(t) = n - S_n(t), \tag{16}$$

and,

$$\begin{aligned} H_n^+(t) &= \sum_{x\in\mathcal{X}} \sum_{\theta=1}^{d_x^+} \sum_{s=d_x^+-\theta+1}^{d_x^+} sU^{(n)}_{x,\theta,s}(t) \\ &= \sum_{x\in\mathcal{X}} \sum_{\theta=1}^{d_x^+} \big[(d_x^+ - \theta + 1)V^{(n)}_{x,\theta,d_x^+-\theta+1}(t) + \sum_{s=d_x^+-\theta+2}^{d_x^+} V^{(n)}_{x,\theta,s}(t)\big]. \end{aligned} \tag{17}$$

Further, $I_n^+(t) = L_n(t) - H_n^+(t)$, and,

$$W_n(t) = L_n(t) - \sum_{x\in\mathcal{X}} d_x^- \sum_{\theta=1}^{d_x^+} V^{(n)}_{x,\theta,d_x^+-\theta+1}(t). \tag{18}$$

Then by Lemma 6.5, combining (16), (17), (18) and the convergences results for the infinite sums in lemma 6.7, it yields that for $\clubsuit \in \{S, H^+, I^+, W\}$,

$$n^{-1/2}\left(\clubsuit_n(t \wedge \tau_n) - n\widehat{f}^{(n)}_{\clubsuit}(t \wedge \tau_n)\right) \xrightarrow{d} \mathcal{Z}_{\clubsuit}(t \wedge t_0),$$

with

$$\mathcal{Z}_S := \sum_{x\in\mathcal{X}} \sum_{\theta=1}^{d_x^+} \mathcal{Z}^*_{x,\theta,d_x^+-\theta+1}, \tag{19}$$

$$\mathcal{Z}_{H^+} := \sum_{x\in\mathcal{X}} \sum_{\theta=1}^{d_x^+} (d_x^+ - \theta + 1)\mathcal{Z}^*_{x,\theta,d_x^+-\theta+1} + \sum_{x\in\mathcal{X}} \sum_{\theta=1}^{d_x^+} \sum_{s=d_x^+-\theta+2}^{d_x^+} \mathcal{Z}^*_{x,\theta,s}, \tag{20}$$

$$\mathcal{Z}_{I^+} := \mathcal{Z}_L - \mathcal{Z}_{H^+}, \tag{21}$$

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and

$$\mathcal{Z}_W := \mathcal{Z}_L - \sum_{x\in\mathcal{X}} d_x^- \sum_{\theta=1}^{d_x^+} \mathcal{Z}^*_{x,\theta,d_x^+-\theta+1}. \tag{22}$$

Hence we have proved the asymptotic normality in Theorem 3.5. The covariances between all the processes are given by (50)-(55) in Appendix A.6.

6.7 Proof of Theorem 3.7

As discussed in Remark 3.4, the Assumption 3b implies Assumption 3a. Hence, the case $z^\star = 0$ follows from Theorem 3.3. We now consider the case $z^\star \in (0,1]$, where $z^\star$ is a stable solution and $\alpha := f'_W(z^\star) > 0$.

Since $\alpha > 0$, for a positive constant ε small enough, we have $f_W(z^\star - \varepsilon) < 0$ and $f_W(z^\star + \varepsilon) > 0$. By Lemma 6.6, we have $f_W^{(n)} \to f_W$ uniformly on $[0,1]$. For n large enough, we also have $f_W^{(n)}(z^\star - \varepsilon) < 0$ and $f_W^{(n)}(z^\star + \varepsilon) > 0$. Hence for n large enough, there exists a sequence $\widehat{z}_n$ in $(z^\star - \varepsilon, z^\star + \varepsilon)$ such that $f_W^{(n)}(\widehat{z}_n) = 0$ and $f_W^{(n)} > 0$ on $[z^\star + \varepsilon, 1]$. Since ε can be arbitrarily small, we obtain $\widehat{z}_n \to z^\star$.

Define $\widehat{t}_n := -\ln \widehat{z}_n$. Consequently we also have $\widehat{t}_n \to t^\star$.

Next, we use the Skorokhod coupling theorem [42, Theorem 3.30] which shows that one can change the probability space where all the random variables are well defined and all the convergence results of Theorem 3.5 and $\tau_n^\star \to t^\star$ (from Lemma 3.2) hold a.s..

Taking $t = \tau_n^\star$ and $t_0 = t^\star$, we get

$$\begin{aligned} W_n(\tau_n^\star) &= n\widehat{f}_W^{(n)}(\tau_n^\star) + n^{1/2}\mathcal{Z}_W(\tau_n^\star \wedge t^\star) + o(n^{1/2}) \\ &= n\widehat{f}_W^{(n)}(\tau_n^\star) + n^{1/2}\mathcal{Z}_W(t^\star) + o(n^{1/2}), \end{aligned}$$

by the continuity of $\mathcal{Z}_W$. Since $W_n(\tau_n^\star) = -1$, then

$$\widehat{f}_W^{(n)}(\tau_n^\star) = -n^{-1/2}\mathcal{Z}_W\big(t^\star\big) + o(n^{-1/2}).$$

Since, as $n \to \infty$, $\tau_n^\star \to t^\star$ and $\widehat{t}_n \to t^\star$ hold a.s., there exists some ξ_n in the interval between $\widehat{t}_n$ and $\tau_n^\star$ such that $\xi_n \to t^\star$. Further, by Lemma 6.6,

$$(\widehat{f}_W^{(n)})'(\xi_n) \to \widehat{f}'_W(t^\star) = -z^\star\alpha.$$

It follows then by Mean-Value theorem that

$$\widehat{f}_W^{(n)}(\tau_n^\star) = \widehat{f}_W^{(n)}(\tau_n^\star) - \widehat{f}_W^{(n)}(\widehat{t}_n) = (\widehat{f}_W^{(n)})'(\xi_n)(\tau_n^\star - \widehat{t}_n) = (-z^\star\alpha + o(1))(\tau_n^\star - \widehat{t}_n).$$

Hence we have

$$\tau_n^\star - \widehat{t}_n = \Big(-\frac{1}{z^\star\alpha} + o(1)\Big)\widehat{f}_W^{(n)}(\tau_n^\star) = n^{-1/2}\frac{1}{z^\star\alpha}(\mathcal{Z}_W(t^\star) + o(1)).$$

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Then, by a similar argument for $S_n(\tau_n^\star)$, combining the above formula and Lemma 6.6, we have a.s. for some $\xi_n' \to t^\star$,

$$\begin{aligned} n^{-1/2}S_n(\tau_n^\star) &= n^{1/2}\widehat{f}_S^{(n)}(\tau_n^\star) + \mathcal{Z}_S(t^\star) + o(1) \\ &= n^{1/2}\widehat{f}_S^{(n)}(\tau_n^\star) + n^{1/2}(\widehat{f}_S^{(n)})'(\xi_n')(\tau_n^\star - \widehat{t}_n) + \mathcal{Z}_S(t^\star) + o(1) \\ &= n^{1/2}f_S^{(n)}(\widehat{z}_n) + \frac{\widehat{f}_S'(t^\star)}{\alpha z^\star}\mathcal{Z}_W(t^\star) + \mathcal{Z}_S(t^\star) + o(1) \\ &= n^{1/2}f_S^{(n)}(\widehat{z}_n) - \frac{f_S'(z^\star)}{\alpha}\mathcal{Z}_W(t^\star) + \mathcal{Z}_S(t^\star) + o(1), \end{aligned} \tag{23}$$

where the last equality follows from the fact that $(\widehat{f}_S)'(t) = -(f_S)(e^{-t})e^{-t}$ and $e^{-t^\star} = z^\star$.

Using the similar arguments, we have the following analogues:

$$n^{-1/2}H_n^+(\tau_n^\star) = n^{1/2}f_{H^+}^{(n)}(\widehat{z}_n) - \frac{f_{H^+}'(z^\star)}{\alpha}\mathcal{Z}_W(t^\star) + \mathcal{Z}_{H^+}(t^\star) + o(1),$$

$$n^{-1/2}I_n^+(\tau_n^\star) = n^{1/2}f_{I^+}^{(n)}(\widehat{z}_n) - \frac{f_{I^+}'(z^\star)}{\alpha}\mathcal{Z}_W(t^\star) + \mathcal{Z}_{I^+}(t^\star) + o(1),$$

and

$$n^{-1/2}S_{x,\theta,\ell}^{(n)}(\tau_n^\star) = n^{1/2}s_{x,\theta,\ell}^{(n)}(\widehat{z}_n) - \frac{s_{x,\theta,\ell}'(z^\star)}{\alpha}\mathcal{Z}_W(t^\star) + \mathcal{Z}_{x,\theta,l}^*(t^\star) + o(1),$$

for all $x \in \mathcal{X}$ and $0 \leqslant \ell < \theta \leqslant d_x^+$. This completes the proof of Theorem 3.7.

6.8 Proof of Theorem 4.1

Using the notations as in the proof of Theorem 3.1 (see Section 6.3), we have $S_{x,\theta,\ell}^{(n)}(t) = U_{x,\theta,d_x^+-\ell}^{(n)}(t)$. Then we have

$$\sum_{x\in\mathcal{X}} \bar{L}_x^\Diamond \sum_{\theta=1}^{d_x^+}\sum_{\ell=1}^{\theta-1} \ell S_{x,\theta,\ell}^{(n)}(t) = \sum_{x\in\mathcal{X}} \bar{L}_x^\Diamond \sum_{\theta=1}^{d_x^+}\sum_{\ell=1}^{\theta-1} \ell U_{x,\theta,d_x^+-\ell}^{(n)}(t),$$

and,

$$\sum_{x\in\mathcal{X}} \bar{L}_x^\odot D_x^{(n)}(t) = \sum_{x\in\mathcal{X}} \bar{L}_x^\odot (n\mu_x^{(n)} - \sum_{\theta=1}^{d_x^+}\sum_{s=d_x^+-\theta+1}^{d_x^+} U_{x,\theta,s}^{(n)}(t)).$$

Since for all $x \in \mathcal{X}$, $\bar{L}_x^\odot$ and $\bar{L}_x^\Diamond$ are bounded, there exists some constant C such that the two above expressions are both dominated by

$$nC\sum_{x\in\mathcal{X}}(d_x^+ + d_x^-)\sum_{\theta=1}^{d_x^+}\sum_{s=d_x^+-\theta+1}^{d_x^+} U_{x,\theta,s}^{(n)}(t).$$

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Thus again by Lemma 6.4 and basic calculations, it follows that

$$\sup_{t\leqslant\tau_n}\Big|\frac{\Gamma_n^{\Diamond}(t)}{n}-f_{\Diamond}(e^{-t})\Big|\xrightarrow{p}0.$$

For $z^\star=0$, by Lemma 3.2, $\tau_n^\star\xrightarrow{p}\infty$. Then we have $e^{-\tau_n^\star}\xrightarrow{p}0$ and $f_\Diamond(0)=\bar\Gamma^\Diamond-\sum_{x\in\mathcal{X}}\bar L_x^{\odot}\mu_x$. Thus, by the continuity of $f_\Diamond$, it follows that

$$f_\Diamond(e^{-\tau_n^\star})=\bar\Gamma^\Diamond-\sum_{x\in\mathcal{X}}\bar L_x^{\odot}\mu_x+o_p(1).$$

We therefore have

$$\frac{\Gamma_n^{\Diamond}(\tau_n^\star)}{n}\xrightarrow{p}\bar\Gamma^\Diamond-\sum_{x\in\mathcal{X}}\bar L_x^{\odot}\mu_x.$$

For $z^\star\in(0,1]$ and $f_W'(z^\star)>0$, by Lemma 3.2, $\tau_n^\star\xrightarrow{p}-\ln z^\star$. Then a similar argument as above implies that

$$\frac{\Gamma_n^{\Diamond}(\tau_n^\star)}{n}\xrightarrow{p}f_\Diamond(z^\star).$$

This completes the proof of Theorem 4.1.

6.9 Proof of Theorem 4.2

Recall that we have defined $V_{x,\theta,s}^{(n)}(t)$ as the number of bins (institutions) with type $x\in\mathcal{X}$, threshold θ and at least s alive balls at time t. We notice that

$$\sum_{x\in\mathcal{X}}\bar L_x^{\odot}D_x^{(n)}(t)=\sum_{x\in\mathcal{X}}\bar L_x^{\odot}\big(n\mu_x^{(n)}-\sum_{\theta=1}^{d_x^+}V_{x,\theta,d_x^+-\theta+1}^{(n)}(t)\big),$$

and,

$$\sum_{\ell=1}^{\theta-1}\ell S_{x,\theta,\ell}^{(n)}(t)=(\theta-1)V_{x,\theta,d_x^+-\theta+1}^{(n)}(t)-\sum_{s=d_x^+-\theta+2}^{d_x^+}V_{x,\theta,s}^{(n)}(t).$$

Since for all $x\in\mathcal{X}$, $\bar L_x^{\odot}$ and $\bar L_x^{\Diamond}$ are bounded, there exists a constant C such that $\bar L_x^{\odot}+\bar L_x^{\Diamond}\leqslant C$ for all $x\in\mathcal{X}$. Hence, by using similar arguments as in Appendix 6.6 which has been used to prove the convergence of H_n^+, S_n and so on, Lemma 6.7 leads to the convergence of the following (infinite) sums

$$\sum_{x\in\mathcal{X}}\bar L_x^{\odot}\sum_{\theta=1}^{d_x^+}V_{x,\theta,d_x^+-\theta+1}^{*(n)}(t)\xrightarrow{d}\sum_{x\in\mathcal{X}}\bar L_x^{\odot}\sum_{\theta=1}^{d_x^+}\mathcal{Z}_{x,\theta,d_x^+-\theta+1}^{*}(t),$$

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and

$$\sum_{x\in\mathcal{X}} \bar{L}_x^{\Diamond} \sum_{\theta=1}^{d_x^+} [(\theta-1)V^{*(n)}_{x,\theta,d_x^+-\theta+1}(t) - \sum_{s=d_x^+-\theta+2}^{d_x^+} V^{*(n)}_{x,\theta,s}(t)]$$
$$\xrightarrow{d} \sum_{x\in\mathcal{X}} \bar{L}_x^{\Diamond} \sum_{\theta=1}^{d_x^+} [(\theta-1)\mathcal{Z}^*_{x,\theta,d_x^+-\theta+1}(t) - \sum_{s=d_x^+-\theta+2}^{d_x^+} \mathcal{Z}^*_{x,\theta,s}(t)].$$

Hence we have in $\mathcal{D}[0,\infty)$, as $n\to\infty$,

$$n^{-1/2}\left(\Gamma_n^{\Diamond}(t\wedge\tau_n) - n\widehat{f}_{\Diamond}^{(n)}(t\wedge\tau_n)\right) \xrightarrow{d} \mathcal{Z}_{\Diamond}(t\wedge t_0), \tag{24}$$

where $\mathcal{Z}_{\Diamond}$ is a continuous Gaussian process with,

$$\begin{aligned}\mathcal{Z}_{\Diamond} := &-\sum_{x\in\mathcal{X}} \bar{L}_x^{\Diamond} \sum_{\theta=1}^{d_x^+} \Big[(\theta-1)\mathcal{Z}^*_{x,\theta,d_x^+-\theta+1}(t) - \sum_{s=d_x^+-\theta+2}^{d_x^+} \mathcal{Z}^*_{x,\theta,s}(t)\Big] \\ &- \sum_{x\in\mathcal{X}} \bar{L}_x^{\odot} \sum_{\theta=1}^{d_x^+} \mathcal{Z}^*_{x,\theta,d_x^+-\theta+1}(t).\end{aligned} \tag{25}$$

For the final system-wide wealth $\Gamma_n^{\Diamond}(\tau_n^\star)$, the case where $z^\star = 0$ has been proved in Theorem 4.1. We consider now the case where $z^\star \in (0,1]$ and $f'_W(z^\star) > 0$. Note that the convergence result (24) hold jointly with the convergence of other processes H_n^+, S_n and so on. Hence, by a similar argument as in Section 6.7, we have that

$$n^{-1/2}\Gamma_n^{\Diamond}(\tau_n^\star) = n^{1/2} f_{\Diamond}^{(n)}(\widehat{z}_n) - \frac{f'_{\Diamond}(z^\star)}{\alpha}\mathcal{Z}_W(t^\star) + \mathcal{Z}_{\Diamond}(t^\star) + o(1).$$

This gives

$$\mathcal{Z}_{\Diamond}^\star := \mathcal{Z}_{\Diamond}(t^\star) - \alpha^{-1} f'_{\Diamond}(z^\star)\mathcal{Z}_W(t^\star) = \mathcal{Z}_{\Diamond}(t^\star) - \Delta(z^\star)\mathcal{Z}_W(t^\star).$$

Hence $\mathcal{Z}_{\Diamond}^\star$ is a centered Gaussian random variable, which completes the proof.

6.10 Proof of Theorem 5.1

Our proof of Theorem 5.1 is based on using Theorem 3.1 together with the ideas applied in [40] which studies the conditions for existence of giant component in the percolated random (non directed) graph with given vertex degrees.

We first remove all potential saved links by the planner from the network. Consider the type-dependent bond percolation model where we remove each incoming link to any institution of type $x \in \mathcal{X}$ with probability α_x. Note that this also includes extra removed links between solvent institutions that will not play any role in the default contagion process. Next we run the death process as described in Section 2.4 and Appendix 6.3. We denote by $\widetilde{U}^{(n)}_{x,\theta,\ell}(t)$ the number of bins (institutions) with type $x \in \mathcal{X}$, threshold θ and ℓ alive (in-)

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balls at time t in the percolated random graph. It is then not hard to show that for the percolated random graph, Lemma 6.4 changes to (as $n \to \infty$)

$$\sup_{t \leqslant \tau_n} \Big| \frac{1}{n} \sum_{s=r+1}^{d_x^+} \widetilde{U}^{(n)}_{x,\theta,s}(t) - \mu_x q_x(\theta) \sum_{s=r+1}^{d_x^+} b(d_x^+, \alpha_x + (1-\alpha_x)e^{-t}, s) \Big| \xrightarrow{p} 0.$$

Consider now $\widetilde{S}^{(n)}_{x,\theta,\ell}$, the number of solvent institutions with type x, threshold θ and $\ell = 0, \ldots, \theta - 1$ defaulted neighbors at time t. Hence, by writing above equation for $r_1 = d_x^+ - \ell$ and $r_2 = d_x^+ - \ell - 1$, then taking the difference, we obtain

$$\sup_{t \leqslant \tau_n} \Big| \frac{\widetilde{S}^{(n)}_{x,\theta,\ell}(t)}{n} - \mu_x q_x(\theta) b\left(d_x^+, (1-\alpha_x)(1-e^{-t}), \ell\right) \Big| \xrightarrow{p} 0, \quad \text{as} \quad n \to \infty.$$

Let $\widetilde{W}_n(t)$ and $\widetilde{D}_n(t)$ denote respectively the number of white balls and the total number of defaults at time t in the percolated random graph. Then, by following the same steps as for the proof of Theorem 3.1 in Section 6.3, we obtain

$$\sup_{t \leqslant \tau_n} \Big| \frac{\widetilde{W}_n(t)}{n} - f_W^{(\boldsymbol{\alpha})}(e^{-t}) \Big| \xrightarrow{p} 0, \quad \sup_{t \leqslant \tau_n} \Big| \frac{\widetilde{D}_n(t)}{n} - f_D^{(\boldsymbol{\alpha})}(e^{-t}) \Big| \xrightarrow{p} 0.$$

Let $\widetilde{\tau}_n^\star$ be the first time when $\widetilde{W}_n(\widetilde{\tau}_n^\star) = -1$. Then, similar to the proof of Lemma 3.2 in Section A.1, we find that $\widetilde{\tau}_n^\star \to -\ln z_{\boldsymbol{\alpha}}^\star$, where $z_{\boldsymbol{\alpha}}^\star := \sup\{z \in [0,1] : f_W^{(\boldsymbol{\alpha})}(z) = 0\}$ and

$$f_W^{(\boldsymbol{\alpha})}(z) := \lambda z - \sum_{x \in \mathcal{X}} \mu_x d_x^- \sum_{\theta=1}^{d_x^+} q_x(\theta) \beta\big(d_x^+, \alpha_x + (1-\alpha_x)z, d_x^+ - \theta + 1\big).$$

Next, by following the same steps as proof of Theorem 3.3 in Section 6.4, we obtain

$$\frac{S^{(n)}_{x,\theta,\ell}(\boldsymbol{\alpha}_n)}{n} \xrightarrow{p} s^{(\boldsymbol{\alpha})}_{x,\theta,\ell}(z_{\boldsymbol{\alpha}}^\star), \quad \frac{D_n(\boldsymbol{\alpha}_n)}{n} \xrightarrow{p} f_D^{(\boldsymbol{\alpha})}(z_{\boldsymbol{\alpha}}^\star),$$

which then implies (by definition) that system-wide wealth converges to

$$\frac{\Gamma_n^\Diamond(\boldsymbol{\alpha}_n)}{n} \xrightarrow{p} f_\Diamond^{(\boldsymbol{\alpha})}(z_{\boldsymbol{\alpha}}^\star) := \bar{\Gamma}^\Diamond - \sum_{x \in \mathcal{X}} \bar{L}_x^\odot f_D^{(\boldsymbol{\alpha})}(z_{\boldsymbol{\alpha}}^\star) - \sum_{x \in \mathcal{X}} \bar{L}_x^\Diamond \sum_{\theta=1}^{d_x^+} \sum_{\ell=1}^{\theta-1} \ell s^{(\boldsymbol{\alpha})}_{x,\theta,\ell}(z_{\boldsymbol{\alpha}}^\star).$$

The total cost of interventions $\boldsymbol{\alpha}_n$ for the planner converges to

$$\frac{\Phi_n(\boldsymbol{\alpha}_n)}{n} \xrightarrow{p} \phi(z_{\boldsymbol{\alpha}}^\star) := \sum_{x \in \mathcal{X}} \mu_x \alpha_x C_x \sum_{\ell=1}^{d_x^+} \ell b\left(d_x^+, 1 - z_{\boldsymbol{\alpha}}^\star, \ell\right).$$

This completes the proof of Theorem 5.1.

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7 Concluding Remarks

In this paper, we propose a general tractable framework to study default cascades and systemic risk in a heterogeneous financial network, subject to an exogenous macroeconomic shock. We state various limit theorems for the final state of default contagion and systemic risk depending on the network structure and institutions' (observable) characteristics.

We show that, under some regularity assumptions, the default cascade could be transferred to a death process problem represented by balls-and-bins model. Under suitable assumptions on the degree and threshold distributions, we show that the final size of default cascade and system-wide wealth aggregation functions have asymptotically Gaussian fluctuations. We also show how these results could be used by a social planner to optimally target interventions during a financial crisis, under partial information of the financial network and under some budget constraint.

The closed form interpretable limit theorems that we provide could also serve as a mandate for regulators to collect data on those specific network characteristics and assess systemic risk via more intensive computational methods.

Our results can also be used in a regulatory risk management framework. The regulator will impose capital requirements on each bank. In practice, the capital ratio constraint is the same for all banks. However, using our heterogeneous setup, we could allow the regulator to choose optimally this capital ratio according to the type of the banks. Fixing the time horizon T, the regulator's problem is how to choose optimally capital ratio for the institutions of type $x \in \mathcal{X}$ at time 0 to minimize $\rho\big(\Gamma_\diamond(T)\big)$. Then the regulator can update the type of each bank and capital ratios at time T according to the new data.

References

[1] Daron Acemoglu, Asuman Ozdaglar, and Alireza Tahbaz-Salehi. Systemic risk and stability in financial networks. *American Economic Review*, 105(2):564–608, 2015.

[2] Viral V Acharya, Lasse H Pedersen, Thomas Philippon, and Matthew Richardson. Measuring systemic risk. *The Review of Financial Studies*, 30(1):2–47, 2017.

[3] Mohammad Akbarpour, Suraj Malladi, and Amin Saberi. Diffusion, seeding, and the value of network information. In *Proceedings of the 2018 ACM Conference on Economics and Computation*, pages 641–641, 2018.

[4] Franklin Allen and Douglas Gale. Financial contagion. *Journal of political economy*, 108(1):1–33, 2000.

[5] H. Amini and N. Fountoulakis. Bootstrap percolation in power-law random graphs. *Journal of Statistical Physics*, 155:72–92, 2014.

[6] Hamed Amini. Bootstrap percolation and diffusion in random graphs with given vertex degrees. *Electronic Journal of Combinatorics, 17: R25*, 2010.

Electronic copy available at: https://ssrn.com/abstract=3811107

[7] Hamed Amini. Contagion risks and security investment in directed networks. *Available at SSRN 3654657*, 2020.

[8] Hamed Amini, Erhan Bayraktar, and Suman Chakraborty. A central limit theorem for diffusion in sparse random graphs, 2021.

[9] Hamed Amini, Rama Cont, and Andreea Minca. Resilience to contagion in financial networks. *Mathematical Finance*, 26(2):329–365, 2016.

[10] Hamed Amini and Zachary Feinstein. Optimal network compression. *arXiv preprint arXiv:2008.08733*, 2020.

[11] Hamed Amini and Andreea Minca. Inhomogeneous financial networks and contagious links. *Operations Research*, 64(5):1109–1120, 2016.

[12] Hamed Amini, Andreea Minca, and Agnes Sulem. Control of interbank contagion under partial information. *SIAM Journal on Financial Mathematics*, 6(1):1195–1219, 2015.

[13] Hamed Amini, Andreea Minca, and Agnès Sulem. A dynamic contagion risk model with recovery features. *Available at SSRN 3435257*, 2019.

[14] Kartik Anand, Ben Craig, and Goetz Von Peter. Filling in the blanks: Network structure and interbank contagion. *Quantitative Finance*, 15(4):625–636, 2015.

[15] Roc Armenter and Miklós Koren. A balls-and-bins model of trade. *American Economic Review*, 104(7):2127–51, 2014.

[16] P. Billingsley, Karreman Mathematics Research Collection, W.A. Shewhart, Wiley series in probability, mathematical statistics, and S.S. Wilks. *Convergence of Probability Measures*. Wiley Series in Probability and Mathematical Statistics. Wiley, 1968.

[17] Michael Boss, Helmut Elsinger, Martin Summer, and Stefan Thurner. Network topology of the interbank market. *Quantitative Finance*, 4(6):677–684, 2004.

[18] Antonio Cabrales, Piero Gottardi, and Fernando Vega-Redondo. Risk sharing and contagion in networks. *The Review of Financial Studies*, 30(9):3086–3127, 2017.

[19] Chen Chen, Garud Iyengar, and Ciamac C. Moallemi. An axiomatic approach to systemic risk. *Management Science*, 59(6):1373–1388, 2013.

[20] Matteo Chinazzi and Giorgio Fagiolo. Systemic risk, contagion, and financial networks: A survey. *Institute of Economics, Scuola Superiore Sant'Anna, Laboratory of Economics and Management (LEM) Working Paper Series*, (2013/08), 2015.

[21] Rama Cont, Amal Moussa, et al. Network structure and systemic risk in banking systems. *Edson Bastos e, Network Structure and Systemic Risk in Banking Systems (December 1, 2010)*, 2010.

Electronic copy available at: https://ssrn.com/abstract=3811107

[22] Emilie Coupechoux and Marc Lelarge. How clustering affects epidemics in random networks. *Advances in Applied Probability*, 46(4):985–1008, 2014.

[23] Ben Craig and Goetz Von Peter. Interbank tiering and money center banks. *Journal of Financial Intermediation*, 23(3):322–347, 2014.

[24] Marco D'Errico and Tarik Roukny. Compressing over-the-counter markets. *Available at SSRN 2962575*, 2019.

[25] Nils Detering, Thilo Meyer-Brandis, Konstantinos Panagiotou, and Daniel Ritter. Managing default contagion in inhomogeneous financial networks. *SIAM Journal on Financial Mathematics*, 10(2):578–614, 2019.

[26] Larry Eisenberg and Thomas H Noe. Systemic risk in financial systems. *Management Science*, 47(2):236–249, 2001.

[27] Matthew Elliott, Benjamin Golub, and Matthew O Jackson. Financial networks and contagion. *American Economic Review*, 104(10):3115–53, 2014.

[28] Selman Erol and Rakesh Vohra. Network formation and systemic risk. *Available at SSRN 2546310*, 2018.

[29] Zachary Feinstein, Birgit Rudloff, and Stefan Weber. Measures of systemic risk. *SIAM Journal on Financial Mathematics*, 8(1):672–708, 2017.

[30] Daniel Fricke and Thomas Lux. Core–periphery structure in the overnight money market: evidence from the e-mid trading platform. *Computational Economics*, 45(3):359–395, 2015.

[31] Prasanna Gai and Sujit Kapadia. Contagion in financial networks. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 466(2120):2401–2423, 2010.

[32] Andrea Galeotti, Benjamin Golub, and Sanjeev Goyal. Targeting interventions in networks. *Econometrica*, 88(6):2445–2471, 2020.

[33] Axel Gandy and Luitgard AM Veraart. A bayesian methodology for systemic risk assessment in financial networks. *Management Science*, 63(12):4428–4446, 2017.

[34] Paul Glasserman and H Peyton Young. How likely is contagion in financial networks? *Journal of Banking & Finance*, 50:383–399, 2015.

[35] Giulia Iori, Rosario N Mantegna, Luca Marotta, Salvatore Micciche, James Porter, and Michele Tumminello. Networked relationships in the e-mid interbank market: A trading model with memory. *Journal of Economic Dynamics and Control*, 50:98–116, 2015.

Electronic copy available at: https://ssrn.com/abstract=3811107

[36] Matthew O Jackson and Agathe Pernoud. Credit freezes, equilibrium multiplicity, and optimal bailouts in financial networks. *arXiv preprint arXiv:2012.12861*, 2020.

[37] Matthew O Jackson and Agathe Pernoud. Systemic risk in financial networks: A survey. *Available at SSRN*, 2020.

[38] Jean Jacod and Albert Shiryaev. *Limit theorems for stochastic processes*, volume 288. Springer Science & Business Media, 2013.

[39] S. Janson, T. Łuczak, T. Turova, and T. Vallier. Bootstrap percolation on the random graph $G_{n,p}$. *The Annals of Applied Probability*, 22(5):1989–2047, 2012.

[40] Svante Janson et al. On percolation in random graphs with given vertex degrees. *Electronic Journal of Probability*, 14:86–118, 2009.

[41] Svante Janson, Malwina J Luczak, et al. Asymptotic normality of the k-core in random graphs. *The annals of applied probability*, 18(3):1085–1137, 2008.

[42] Olav Kallenberg. *Foundations of modern probability*, volume 2. Springer, 1997.

[43] Jon Kleinberg. Cascading behavior in networks: Algorithmic and economic issues. *Algorithmic game theory*, 24:613–632, 2007.

[44] Eduard Kromer, Ludger Overbeck, and Katrin Zilch. Systemic risk measures on general probability spaces. *Mathematical Methods of Operations Research*, 84(2):323–357, 2016.

[45] Yaron Leitner. Financial networks: Contagion, commitment, and private sector bailouts. *The Journal of Finance*, 60(6):2925–2953, 2005.

[46] Marc Lelarge. Diffusion and cascading behavior in random networks. *Games and Economic Behavior*, 75(2):752–775, 2012.

[47] Dan Li and Norman Schürhoff. Dealer networks. *The Journal of Finance*, 74(1):91–144, 2019.

[48] Stephen Morris. Contagion. *The Review of Economic Studies*, 67(1):57–78, 2000.

[49] Stephen Morris and Hyun Song Shin. Illiquidity component of credit risk. Technical report, working paper, Princeton University, 2009.

[50] Jeannette Müller. Interbank credit lines as a channel of contagion. *Journal of Financial Services Research*, 29(1):37–60, 2006.

[51] Erlend Nier, Jing Yang, Tanju Yorulmazer, and Amadeo Alentorn. Network models and financial stability. *Journal of Economic Dynamics and Control*, 31(6):2033–2060, 2007.

Electronic copy available at: https://ssrn.com/abstract=3811107

[52] Leonard CG Rogers and Luitgard AM Veraart. Failure and rescue in an interbank network. *Management Science*, 59(4):882–898, 2013.

[53] Tarik Roukny, Stefano Battiston, and Joseph E Stiglitz. Interconnectedness as a source of uncertainty in systemic risk. *Journal of Financial Stability*, 35:93–106, 2018.

[54] Adrian Tobias and Markus K Brunnermeier. Covar. *The American Economic Review*, 106(7):1705, 2016.

[55] Christian Upper and Andreas Worms. Estimating bilateral exposures in the german interbank market: Is there a danger of contagion? *European economic review*, 48(4):827–849, 2004.

[56] Remco van der Hofstad. *Random Graphs and Complex Networks*. Cambridge University Press, 2016.

[57] Remco van der Hofstad, Johan SH van Leeuwaarden, and Clara Stegehuis. Hierarchical configuration model. *arXiv preprint arXiv:1512.08397*, 2015.

[58] Luitgard Anna Maria Veraart. When does portfolio compression reduce systemic risk? *Available at SSRN 3688495*, 2020.

[59] Duncan J Watts. A simple model of global cascades on random networks. *Proceedings of the National Academy of Sciences*, 99(9):5766–5771, 2002.

[60] H Peyton Young. Innovation diffusion in heterogeneous populations: Contagion, social influence, and social learning. *American economic review*, 99(5):1899–1924, 2009.

A Proof of Lemmas

This appendix contains the proofs of all the lemmas in the main text.

A.1 Proof of Lemma 3.2

Recall that

$$f_W(z) := \lambda z - \sum_{x\in\mathcal{X}} \mu_x d_x^- \sum_{\theta=1}^{d_x^+} q_x(\theta)\beta\big(d_x^+, z, d_x^+ - \theta + 1\big),$$

and, $z^\star := \sup\big\{z \in [0,1] : f_W(z) = 0\big\}$. By the initial condition $q_x(0) > 0$ for some $x \in \mathcal{X}$. Hence, we have $f_W(1) > 0$ and $z^\star < 1$. Let us take a constant $t_1 > 0$ such that $t_1 < -\ln z^\star$. By continuity of $f_W(z)$ on $[0,1]$, it follows that $f_W(z) > 0$ on $(z^\star, 1]$. Hence, there exists some constant $C_1 > 0$ such that $f_W(e^{-t}) \geqslant C_1$ for $t \leqslant t_1$.

Since $W_n(\tau_n^\star) = -1$, if $\tau_n^\star \leqslant t_1$ then $W_n(\tau_n^\star)/n - f_W(e^{-\tau_n^\star}) \leqslant -C_1$. But on the other hand, we have

$$\sup_{t\leqslant\tau_n^\star} |\frac{W_n(t)}{n} - f_W(e^{-t}))| \xrightarrow{p} 0.$$

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This is a contradiction. Therefore, we must have $\mathbb{P}(\tau_n^\star \leqslant t_1) \longrightarrow 0$, as $n \to \infty$. In the case $z^\star = 0$, we can take any finite t_1, which implies that $\tau_n^\star \xrightarrow{p} \infty$.

We now consider the case $z^\star \in (0,1]$. Let $\varepsilon > 0$ small enough and fix the constant $t_2 \in (-\ln z^\star, -\ln(z^\star - \varepsilon))$. By using a similar argument and given the assumption $f'_W(z^\star) > 0$, we can show there exists some constant $C_2 > 0$, such that $W_n(\tau_n^\star)/n - f_W(e^{-\tau_n^\star}) \geqslant C_2$ if $\tau_n^\star \geqslant t_2$. Thus $\mathbb{P}(\tau_n \geqslant t_2) \longrightarrow 0$ as $n \to \infty$. Since t_1 and t_2 are arbitrary, tending both t_1 and t_2 to $-\ln z^\star$, implies that $\tau_n^\star \xrightarrow{p} -\ln z^\star$. This completes the proof of lemma.

A.2 Proof of Lemma 6.4

First note that $U^{(n)}_{x,\theta,0}(0) = N^{(n)}_{x,\theta}$ and $U^{(n)}_{x,\theta,s}(0) = 0$ for $\ell \geqslant 1$. Further, from Assumption 2, $N^{(n)}_x/n \to \mu_x$ and $q^{(n)}_x(\theta) \to q_x(\theta)$ as $n \to \infty$, for all $x \in \mathcal{X}$ and $\theta = 0, 1, \ldots, d_x^+$. Moreover, by the strong law of large numbers, $N^{(n)}_{x,\theta}/N^{(n)}_x \to q^{(n)}_x(\theta)$ a.s. as $n \to \infty$.

Consider now the death process as described in Section 2.4. Let us fix $x \in \mathcal{X}$ and integers θ, r with $0 \leqslant r \leqslant \theta \leqslant d_x^+$. Consider the $N^{(n)}_{x,\theta}$ bins which starts with d_x^+ alive in balls. For $k = 1, \ldots, N^{(n)}_{x,\theta}$, let T_k be the time that the $(d_x^+ - r)$-th ball is removed (killed) in the k-th such bin. Then we have $\#\{k : T_k \leqslant t\} = \sum_{s=0}^{r} U^{(n)}_{x,\theta,s}(t)$. Since the number of remaining balls in any of such bins at time t are i.i.d. random variables with distribution $\mathsf{Bin}(d_x^+, e^{-t})$, then we have $\mathbb{P}(T_k \leqslant t) = \sum_{s=0}^{r} b(d_x^+, e^{-t}, s)$. Hence, by using Glivenko-Cantelli theorem,

$$\sup_{t \leqslant \tau_n} \Big| \frac{1}{N^{(n)}_{x,\theta}} \sum_{s=r+1}^{d_x^+} U^{(n)}_{x,\theta,s}(t) - \sum_{s=r+1}^{d_x^+} b(d_x^+, e^{-t}, s) \Big| \xrightarrow{p} 0, \qquad \text{as} \quad N^{(n)}_{x,\theta} \to \infty.$$

Multiply the above equation by $N^{(n)}_{x,\theta}/N^{(n)}_x$, and by the law of large numbers, we have

$$\sup_{t \leqslant \tau_n} \Big| \frac{1}{N^{(n)}_x} \sum_{s=r+1}^{d_x^+} U^{(n)}_{x,\theta,s}(t) - q^{(n)}_x(\theta) \sum_{s=r+1}^{d_x^+} b(d_x^+, e^{-t}, s) \Big| \xrightarrow{p} 0, \qquad \text{as} \quad N^{(n)}_x \to \infty. \tag{26}$$

Moreover, by using Assumption 2,

$$\sup_{t \geqslant 0} \Big| \frac{N^{(n)}_x}{n} q^{(n)}_x(\theta) \sum_{s=r+1}^{d_x^+} b(d_x^+, e^{-t}, s) - \mu_x q_x(\theta) \sum_{s=r+1}^{d_x^+} b(d_x^+, e^{-t}, s) \Big| \longrightarrow 0, \quad \text{as} \quad n \to \infty.$$

By combining the two formulas above and multiplying (26) by $N^{(n)}_x/n$, we obtain

$$\sup_{t \leqslant \tau_n} \Big| \frac{1}{n} \sum_{s=r+1}^{d_x^+} U^{(n)}_{x,\theta,s}(t) - \mu_x q_x(\theta) \sum_{s=r+1}^{d_x^+} b(d_x^+, e^{-t}, s) \Big| \xrightarrow{p} 0, \quad \text{as} \quad n \to \infty. \tag{27}$$

Hence, by using (27) for $r_1 = \ell$ and $r_2 = \ell - 1$, and then taking the difference, we obtain

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that

$$\sup_{t\leqslant\tau_n}\Big|\frac{U^{(n)}_{x,\theta,\ell}(t)}{n}-\mu_x q_x(\theta)b\left(d^+_x,e^{-t},\ell\right)\Big|\xrightarrow{p}0,\quad\text{as}\quad n\to\infty.$$

Note that the above equation holds for all $x\in\mathcal{X}$ and $\theta=1,\dots,d^+_x$. Hence, the same convergence also holds for any partial sum over x and θ. In particular,

$$\sup_{t\leqslant\tau_n}\sum_{\theta=1}^{d^+_x}\sum_{s=d^+_x-\theta+1}^{d^+_x}\Big|\frac{1}{n}U^{(n)}_{x,\theta,s}(t)-\mu_x q_x(\theta)b(d^+_x,e^{-t},s)\Big|\xrightarrow{p}0. \tag{28}$$

Let $\mathcal{X}_K$ be the set of all characteristic $x\in\mathcal{X}$ such that $d^+_x+d^-_x\leqslant K$. Since (by Assumption 3a) $\lambda\in(0,\infty)$ then for arbitrary small $\varepsilon>0$, there exists K_ε such that $\sum_{x\in\mathcal{X}\setminus\mathcal{X}_{K_\varepsilon}}\mu_x(d^+_x+d^-_x)<\varepsilon$. Further, by Assumption 3a and dominated convergence,

$$\sum_{x\in\mathcal{X}\setminus\mathcal{X}_{K_\varepsilon}}(d^+_x+d^-_x)N^{(n)}_x/n\to\sum_{x\in\mathcal{X}\setminus\mathcal{X}_{K_\varepsilon}}(d^+_x+d^-_x)\mu_x<\varepsilon.$$

Hence for n large enough, we have $\sum_{x\in\mathcal{X}\setminus\mathcal{X}_{K_\varepsilon}}(d^+_x+d^-_x)N^{(n)}_x/n<2\varepsilon$. By (28), we obtain

$$\begin{aligned}
&\sup_{t\leqslant\tau_n}\sum_{x\in\mathcal{X}}(d^+_x+d^-_x)\sum_{\theta=1}^{d^+_x}\sum_{s=d^+_x-\theta+1}^{d^+_x}\left|U^{(n)}_{x,\theta,s}(t)/n-\mu_x q_x(\theta)b(d^+_x,e^{-t},s)\right|\\
\leqslant&\sup_{t\leqslant\tau_n}\sum_{x\in\mathcal{X}_{K_\varepsilon}}(d^+_x+d^-_x)\sum_{\theta=1}^{d^+_x}\sum_{s=d^+_x-\theta+1}^{d^+_x}\left|U^{(n)}_{x,\theta,s}(t)/n-\mu_x q_x(\theta)b(d^+_x,e^{-t},s)\right|\\
&+\sup_{t\leqslant\tau_n}\sum_{x\in\mathcal{X}\setminus\mathcal{X}_{K_\varepsilon}}(d^+_x+d^-_x)\sum_{\theta=1}^{d^+_x}\sum_{s=d^+_x-\theta+1}^{d^+_x}\left|U^{(n)}_{x,\theta,s}(t)/n-\mu_x q_x(\theta)b(d^+_x,e^{-t},s)\right|\\
\leqslant&o_p(1)+\sum_{x\in\mathcal{X}\setminus\mathcal{X}_{K_\varepsilon}}(d^+_x+d^-_x)(N^{(n)}_x/n+\mu_x)\leqslant o_p(1)+3\varepsilon.
\end{aligned}$$

Then let $\varepsilon\to0$, it follows that (12) holds, which completes the proof of Lemma 6.4

A.3 Proof of Lemma 6.5

Since $V^{(n)}_{x,\theta,s}$ changes by -1 when one of the alive (in) balls in $U^{(n)}_{x,\theta,s}$ bins dies, and there are $sU^{(n)}_{x,\theta,s}(t)$ such balls at time t, we obtain

$$dV^{(n)}_{x,\theta,s}(t)=-sU^{(n)}_{x,\theta,s}(t)dt+d\mathcal{M}'_t,$$

where $\mathcal{M}'$ is a martingale.

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We define further $\widehat{V}^{(n)}_{x,\theta,s}(t) := e^{st}V^{(n)}_{x,\theta,s}(t)$. Then by Ito's lemma,

$$\begin{aligned} d\widehat{V}^{(n)}_{x,\theta,s}(t) &= se^{st}V^{(n)}_{x,\theta,s}(t)dt + e^{st}dV^{(n)}_{x,\theta,s}(t) \\ &= se^{st}V^{(n)}_{x,\theta,s}(t)dt - se^{st}U^{(n)}_{x,\theta,s}(t)dt + e^{st}d\mathcal{M}'_t \\ &= se^{-t}\widehat{V}^{(n)}_{x,\theta,(s+1)}(t)dt + d\mathcal{M}_t, \end{aligned}$$

where $d\mathcal{M}_t = e^{st}d\mathcal{M}'_t$ is also a martingale differential. Thus

$$M^{(n)}_{x,\theta,s}(t) := \widehat{V}^{(n)}_{x,\theta,s}(t) - s\int_0^t e^{-r}\widehat{V}^{(n)}_{x,\theta,(s+1)}(r)dr \tag{29}$$

is also a martingale for every $0 \leqslant s \leqslant d_x^+$. We can calculate its quadratic variation by

$$\left[M^{(n)}_{x,\theta,s}, M^{(n)}_{x,\theta,s}\right]_t = \sum_{0<r\leqslant t} |\Delta M^{(n)}_{x,\theta,s}(r)|^2 = \sum_{0<r\leqslant t} |\Delta \widehat{V}^{(n)}_{x,\theta,s}(r)|^2 = \int_0^t e^{2sr}d(-V^{(n)}_{x,\theta,s}(r)). \tag{30}$$

Then,

$$\widetilde{M}^{(n)}_{x,\theta,s}(t) := n^{-1/2}\big(M^{(n)}_{x,\theta,s}(t) - M^{(n)}_{x,\theta,s}(0)\big),$$

is a martingale with initial value at 0. Further, from Lemma 6.4, as $n \to \infty$,

$$\sup_{t\leqslant\tau_n^\star} |V^{(n)}_{x,\theta,s}(t)/n - \mu_x q_x(\theta)\beta(d_x^+, e^{-t}, s)| \xrightarrow{p} 0. \tag{31}$$

For notational convenience, let us denote by

$$\varphi_{x,\theta,s}(y) := \mu_x q_x(\theta)\beta(d_x^+, y, s).$$

Using integration by parts, we obtain for all $t \leqslant \tau_n^\star$,

$$\begin{aligned} \left[\widetilde{M}^{(n)}_{x,\theta,s}, \widetilde{M}^{(n)}_{x,\theta,s}\right]_t &= n^{-1}\left[M^{(n)}_{x,\theta,s}, M^{(n)}_{x,\theta,s}\right]_t \\ &= n^{-1}\big(V^{(n)}_{x,\theta,s}(0) - e^{2st}V^{(n)}_{x,\theta,s}(t) + \int_0^t 2sV^{(n)}_{x,\theta,s}(r)e^{2sr}dr\big) \\ &= \varphi_{x,\theta,s}(1) - e^{2st}\varphi_{x,\theta,s}(e^{-t}) + \int_0^t 2s\varphi_{x,\theta,s}(e^{-r})e^{2sr}dr + o_p(1) \\ &= \int_0^t e^{2sr}d(-\varphi_{x,\theta,s}(e^{-r})) + o_p(1). \end{aligned}$$

Moreover, by (30), there exists a constant C such that

$$\left[\widetilde{M}^{(n)}_{x,\theta,s}, \widetilde{M}^{(n)}_{x,\theta,s}\right]_t = n^{-1}\left[M^{(n)}_{x,\theta,s}, M^{(n)}_{x,\theta,s}\right]_t \leqslant n^{-1}e^{2st}V^{(n)}_{x,\theta,s}(0) \leqslant Ce^{2st}, \tag{32}$$

which holds uniformly for n. Consequently, by Theorem 6.3, for the stopped process

$$\widetilde{M}^{(n)}_{x,\theta,s}(t\wedge\tau_n) \xrightarrow{d} Y_{x,\theta,s}(t\wedge t_0) \qquad \text{in} \quad \mathcal{D}\left[0,\infty\right), \tag{33}$$

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where $Y_{x,\theta,s}$ is a continuous Gaussian process with $\mathbb{E}[Y_{x,\theta,s}(t)] = 0$ and covariance

$$\mathbb{E}\left[Y_{x,\theta,s}(t)Y_{x,\theta,s}(u)\right] = \int_0^t e^{2sr} d(-\varphi_{x,\theta,s}(e^{-r})), \qquad 0 \leqslant t \leqslant u < \infty.$$

Furthermore, for $s \neq r$, we can show that $V^{(n)}_{x,\theta,r}$ and $V^{(n)}_{x,\theta,s}$ never change simultaneously, almost surely. Thus, $[\widetilde{M}^{(n)}_{x,\theta,r}, \widetilde{M}^{(n)}_{x,\theta,s}]_t = 0$.

Hence, by Theorem 6.3 applied to the vector-valued martingale $\left(\widetilde{M}^{(n)}_{x,\theta,s}\right)_{s=0,\ldots,d_x^+}$, the convergence holds jointly with a diagonal covariance matrix for $\left(Y_{x,\theta,s}\right)_{s=0,\ldots,d_x^+}$, which implies that the processes $Y_{x,\theta,0}, \ldots, Y_{x,\theta,d_x^+}$ are all independent.

As for two different types-thresholds (x,θ) and (x',θ'), the independence follows since for any $s = 0,\ldots,d_x^+$ and $s' = 0,\ldots,d_{x'}^+$, $V^{(n)}_{x,\theta,s}$ and $V^{(n)}_{x'\theta',s'}$ also a.s. never change simultaneously. This could also be viewed from the nature of our balls and bins representation: the balls die independently and a.s. never die at the same moment. Hence, the death processes in bins with different types are independent.

We now compute $\widehat{V}^{(n)}_{x,\theta,s}$, using the Definition-Equation (29) for $M^{(n)}_{x,\theta,s}(t)$, repeatedly. We find that for $s = d_x^+$,

$$\widehat{V}^{(n)}_{x,\theta,d_x^+}(t) = M^{(n)}_{x,\theta,d_x^+}(t),$$

and for $s = d_x^+ - 1$,

$$\widehat{V}^{(n)}_{x,\theta,s}(t) = M^{(n)}_{x,\theta,s}(t) + s\int_0^t e^{-r} M^{(n)}_{x,\theta,(s+1)}(r)dr.$$

Then for $s = d_x^+ - 2$, we find out

$$\begin{aligned}
\widehat{V}^{(n)}_{x,\theta,s}(t) =& M^{(n)}_{x,\theta,s}(t) + s\int_0^t e^{-r} M^{(n)}_{x,\theta,(s+1)}(r)dr \\
&+ \int_{r_2<r_1<t} s(s+1)e^{-(r_1+r_2)} M^{(n)}_{x,\theta,(s+2)}(r_2)dr_2dr_1 \\
=& M^{(n)}_{x,\theta,s}(t) + s\int_0^t e^{-r} M^{(n)}_{x,\theta,(s+1)}(r)dr \\
&+ \int_0^t s(s+1)(e^{-r} - e^{-t})e^{-r} M^{(n)}_{x,\theta,(s+2)}(r)dr \\
=& M^{(n)}_{x,\theta,s}(t) + \sum_{j=s+1}^{d_x^+} s\binom{j-1}{s}\int_0^t (e^{-r} - e^{-t})^{j-s-1} e^{-r} M^{(n)}_{x,\theta,j}(r)dr.
\end{aligned}$$

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Assume that the above formula holds for $s \leqslant k \leqslant d_x^+ - 1$. Then for $s-1$, we deduce

$$\begin{aligned}
\widehat{V}^{(n)}_{x,\theta,s-1}(t) =& M^{(n)}_{x,\theta,s-1}(t) + s\int_0^t e^{-r}\widehat{V}^{(n)}_{x,\theta,s}(r)dr \\
=& M^{(n)}_{x,\theta,s-1}(t) + (s-1)\int_0^t e^{-r}M^{(n)}_{x,\theta,s}(r)dr \\
&+ \sum_{j=s+1}^{d_x^+} \frac{s(s-1)}{j-s}\binom{j-1}{s}\int_0^t (e^{-r}-e^{-t})^{j-s}e^{-r}M^{(n)}_{x,\theta,j}(r)dr \\
=& M^{(n)}_{x,\theta,s-1}(t) + \sum_{j=s}^{d_x^+}(s-1)\binom{j-1}{s-1}\int_0^t (e^{-r}-e^{-t})^{j-s}e^{-r}M^{(n)}_{x,\theta,j}(r)dr.
\end{aligned}$$

By induction, we obtain that

$$\widehat{V}^{(n)}_{x,\theta,s}(t) = M^{(n)}_{x,\theta,s}(t) + \sum_{j=s+1}^{d_x^+} s\binom{j-1}{s}\int_0^t (e^{-r}-e^{-t})^{j-s-1}e^{-r}M^{(n)}_{x,\theta,j}(r)dr.$$

We next define $\widehat{v}^{(n)}_{x,\theta,s}(t)$ for all $t \geqslant 0$, as the conditional expectation of $\widehat{V}^{(n)}_{x,\theta,s}(t)$ given its initial value $V^{(n)}_{x,\theta,s}(0)$. Namely, we have

$$\begin{aligned}
\widehat{v}^{(n)}_{x,\theta,s}(t) :=& \mathbb{E}\big[\widehat{V}^{(n)}_{x,\theta,s}(t)|V^{(n)}_{x,\theta,s}(0)\big] \\
=& M^{(n)}_{x,\theta,s}(0) + \sum_{j=s+1}^{d_x^+} s\binom{j-1}{s}\int_0^t (e^{-r}-e^{-t})^{j-s-1}e^{-r}M^{(n)}_{x,\theta,j}(0)dr.
\end{aligned}$$

Note that by definition, $\mathbb{E}\big[\widehat{V}^{(n)}_{x,\theta,s}(t)\big] = e^{st}\mathbb{E}[V^{(n)}_{x,\theta,s}(t)]$. Further, $V^{(n)}_{x,\theta,s}(t)$ is the number of bins with type x, threshold θ and with at least s balls at time t in the death process where balls die independently with rate 1 (without stopping). Then at time t, each of such bins has the binomial probability $\beta(d_x^+, e^{-t}, s)$ to have at least s alive balls remaining. The initial number is $V^{(n)}_{x,\theta,s}(0) = N^{(n)}_{x,\theta}$. Consequently for all $s = 0, \ldots, d_x^+$, we have

$$\widehat{v}^{(n)}_{x,\theta,s} = e^{st}N^{(n)}_{x,\theta}\beta(d_x^+, e^{-t}, s). \tag{34}$$

We further define for $t \leqslant \tau_n^\star$,

$$\widetilde{V}^{(n)}_{x,\theta,s}(t) := n^{-1/2}\big(\widehat{V}^{(n)}_{x,\theta,s}(t) - \widehat{v}^{(n)}_{x,\theta,s}\big). \tag{35}$$

It is clear that,

$$\widetilde{V}^{(n)}_{x,\theta,s}(t) = \widetilde{M}^{(n)}_{x,\theta,s}(t) + \sum_{j=s+1}^{d_x^+} s\binom{j-1}{s}\int_0^t (e^{-r}-e^{-t})^{j-s-1}e^{-r}\widetilde{M}^{(n)}_{x,\theta,j}(r)dr.$$

We next apply Theorem 6.3 to the above finite sum and take the limit (in distribution)

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under the summation sign.

It follows that

$$\widetilde{V}^{(n)}_{x,\theta,s}(t \wedge \tau_n) \xrightarrow{d} \widetilde{\mathcal{Z}}_{x,\theta,s}(t \wedge t_0), \tag{36}$$

in $\mathcal{D}[0,\infty)$, where

$$\widetilde{\mathcal{Z}}_{x,\theta,s}(t) := Y_{x,\theta,s}(t) + \sum_{j=s+1}^{d_x^+} s\binom{j-1}{s} \int_0^t (e^{-r} - e^{-t})^{j-s-1} e^{-r} Y_{x,\theta,j}(r)dr.$$

Note that, although the initial $V^{(n)}_{x,\theta,s}(0)$ is random, by the standard (multidimensional) central limit theorem applied to $N^{(n)}_x = n\mu^{(n)}_x$ i.i.d. random variables $\mathbb{1}\{\Theta^{(n)}_i = \theta\}$, we have as $n \to \infty$,

$$n^{-1/2}\Big(N^{(n)}_{x,\theta} - n\mu^{(n)}_x q^{(n)}_x(\theta)\Big) \xrightarrow{d} \mathcal{Y}^*_{x,\theta}, \tag{37}$$

where $\mathcal{Y}^*_{x,\theta} \sim \mathcal{N}(0, \mu_x q_x(\theta)(1-q_x(\theta)))$ and $\mathrm{Cov}(\mathcal{Y}^*_{x,\theta_1}, \mathcal{Y}^*_{x,\theta_2}) = -\mu_x q_x(\theta_1) q_x(\theta_2)$ for $\theta_1 \neq \theta_2$. We denote by (for all θ_1, θ_2)

$$\psi_{x,\theta_1,\theta_2} := \mathrm{Cov}(\mathcal{Y}^*_{x,\theta_1}, \mathcal{Y}^*_{x,\theta_2}).$$

Notice that for all $x_1 \neq x_2$, $\mathcal{Y}^*_{x_1,\theta_1}, \mathcal{Y}^*_{x_2,\theta_2}$ are independent and $\mathrm{Cov}(\mathcal{Y}^*_{x_1,\theta_1}, \mathcal{Y}^*_{x_2,\theta_2}) = 0$. Then we have by (34) and (37) that jointly for all triple (x,θ,s), in $\mathcal{D}[0,\infty)$ as $n \to \infty$,

$$n^{-1/2}\Big(\widehat{v}^{(n)}_{x,\theta,s} - ne^{st}\mu^{(n)}_x q^{(n)}_x(\theta)\beta(d^+_x, e^{-t}, s)\Big) \xrightarrow{d} \widehat{\mathcal{Z}}_{x,\theta,s}(t), \tag{38}$$

where $\widehat{\mathcal{Z}}_{x,\theta,s}(t)$ is also a Gaussian process with mean 0 and covariance

$$\mathbb{E}\big[\widehat{\mathcal{Z}}_{x,\theta,s}(t)\widehat{\mathcal{Z}}_{x,\theta,s}(u)\big] = e^{(t+u)s}\beta(d^+_x, e^{-t}, s)\beta(d^+_x, e^{-u}, s)\psi_{x,\theta,\theta}.$$

We now analyse the independence between $\widetilde{\mathcal{Z}}_{x,\theta,s}$ and $\widehat{\mathcal{Z}}_{x,\theta,s}$. Let combine x and Θ_x as a new type $\varrho := (x,\theta)$ in the set of all possible combination for $x \in \mathcal{X}$ and $\theta \in \mathbb{N}$. Notice that we have fixed proportion for the type distribution $\mu^{(n)}_x$, $x \in \mathcal{X}$ in the network. But the threshold Θ_x for all $x \in \mathcal{X}$ is random. Thus the proportion (denoted by $\mu^{(n)}_\varrho$) for the new type ϱ is random. It does not satisfy Assumption 3b, but satisfies in probability, i.e. replacing $O(n)$ by $O_p(n)$ in Assumption 3b. In addition, $\mu^{(n)}_\varrho \xrightarrow{p} \mu_\varrho$ with the limit distribution $\mu_\varrho := \mu_x q_x(\theta)$ for $\varrho = (x,\theta)$. Further, by using the Skorokhod coupling theorem [42, Theorem 3.30] as also stated in [41, Lemma 8.2], we can define all the processes on a new common probability space such that, for the type distribution, $\mu^{(n)}_\varrho \to \mu_\varrho$ and Assumption 3b hold almost surely.

Further, note that the distribution of $\widetilde{\mathcal{Z}}_{x,\theta,s}$ do not depend on the random proportion $\mu^{(n)}_\varrho$, but only on the limit distribution μ_ϱ. Hence the arguments in the above paragraph guarantee that conditioning on the initial value $V^{(n)}_{x,\theta,s}(0)$ does not have any influence on the

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distribution of $\widetilde{\mathcal{Z}}_{x,\theta,s}$. Therefore $\widetilde{\mathcal{Z}}_{x,\theta,s}$ and $\widehat{\mathcal{Z}}_{x,\theta,s}$ are independent for all (x,θ,s).

The above argument shows that $\widetilde{V}^{(n)}_{x,\theta,s}$ converges to a Gaussian process. We next define

$$\widetilde{\widehat{V}}^{(n)}_{x,\theta,s}(t) := n^{-1/2}\Big(\widehat{V}^{(n)}_{x,\theta,s}(t) - ne^{st}\mu^{(n)}_x q^{(n)}_x(\theta)\beta(d^+_x, e^{-t}, s)\Big).$$

By (35), (36) and (38), we obtain that

$$\widetilde{\widehat{V}}^{(n)}_{x,\theta,s}(t\wedge\tau_n) \xrightarrow{d} \widetilde{\widehat{\mathcal{Z}}}_{x,\theta,s}(t\wedge t_0),$$

where $\widetilde{\widehat{\mathcal{Z}}}_{x,\theta,s} := \widehat{\mathcal{Z}}_{x,\theta,s} + \widetilde{\mathcal{Z}}_{x,\theta,s}$ is a Gaussian process with mean 0 and covariance

$$\mathbb{E}\big[\widetilde{\widehat{\mathcal{Z}}}_{x,\theta,s}(t)\widetilde{\widehat{\mathcal{Z}}}_{x,\theta,s}(u)\big] = \mathbb{E}\big[\widehat{\mathcal{Z}}_{x,\theta,s}(t)\widehat{\mathcal{Z}}_{x,\theta,s}(u)\big] + \mathbb{E}\big[\widetilde{\mathcal{Z}}_{x,\theta,s}(t)\widetilde{\mathcal{Z}}_{x,\theta,s}(u)\big],$$

for all $0 \leqslant t \leqslant u < \infty$.

Next, we define for all triple (x,θ,s),

$$V^{*(n)}_{x,\theta,s}(t) := e^{-st}\widetilde{\widehat{V}}^{(n)}_{x,\theta,s}(t), \quad \widetilde{\widetilde{\mathcal{Z}}}_{x,\theta,s}(t) := e^{-st}\widetilde{\mathcal{Z}}_{x,\theta,s}(t),$$

and

$$\widehat{\widetilde{\mathcal{Z}}}_{x,\theta,s}(t) := e^{-st}\widehat{\mathcal{Z}}_{x,\theta,s}(t), \quad \mathcal{Z}^*_{x,\theta,s}(t) := e^{-st}\widetilde{\widehat{\mathcal{Z}}}_{x,\theta,s}(t).$$

Then we have

$$V^{*(n)}_{x,\theta,s}(t\wedge\tau_n) \xrightarrow{d} \mathcal{Z}^*_{x,\theta,s}(t\wedge t_0). \tag{39}$$

We define further

$$\widehat{\sigma}_{x,\theta_1,\theta_2,r,s}(y) := \mathrm{Cov}\big((\widehat{\widetilde{\mathcal{Z}}}_{x,\theta_1,r}(-\ln y), \widehat{\widetilde{\mathcal{Z}}}_{x,\theta_2,s}(-\ln y))\big),$$

and,

$$\widetilde{\sigma}_{x,\theta,r,s}(y) := \mathrm{Cov}\big(\widetilde{\widetilde{\mathcal{Z}}}_{x,\theta,r}(-\ln y), \widetilde{\widetilde{\mathcal{Z}}}_{x,\theta,s}(-\ln y)\big).$$

By using all the independence and covariance formulas above, it follows that

$$\begin{aligned}
\mathrm{Cov}\big(\mathcal{Z}^*_{x_1,\theta_1,s_1}(t), \mathcal{Z}^*_{x_2,\theta_2,s_2}(t)\big) =& 0, \quad \text{for all} \quad x_1 \neq x_2,\\
\mathrm{Cov}\big(\mathcal{Z}^*_{x,\theta_1,s_1}(t), \mathcal{Z}^*_{x,\theta_2,s_2}(t)\big) =& \widehat{\sigma}_{x,\theta_1,\theta_2,s_1,s_2}(e^{-t}), \quad \text{for all} \quad \theta_1 \neq \theta_2,\\
\mathrm{Cov}\big(\mathcal{Z}^*_{x,\theta,s_1}(t), \mathcal{Z}^*_{x,\theta,s_2}(t)\big) =& \widehat{\sigma}_{x,\theta,\theta,s_1,s_2}(e^{-t}) + \widetilde{\sigma}_{x,\theta,s_1,s_2}(e^{-t}),
\end{aligned}$$

where

$$\begin{aligned}
&\widehat{\sigma}_{x,\theta_1,\theta_2,s_1,s_2}(e^{-t}) = \beta(d^+_x, e^{-t}, s_1)\beta(d^+_x, e^{-t}, s_2)\psi_{x,\theta_1,\theta_2},\\
&\psi_{x,\theta,\theta} = \mu_x q_x(\theta)(1-q_x(\theta)), \quad \psi_{x,\theta_1,\theta_2} = -\mu_x q_x(\theta_1)q_x(\theta_2) \quad \text{for all} \quad \theta_1 \neq \theta_2.
\end{aligned} \tag{40}$$

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Moreover, the covariance between $\mathcal{Z}^*_{x,\theta_1,s}(t)$ and $\mathcal{Y}^*_{x,\theta_2}$ is given by

$$\mathrm{Cov}\big(\mathcal{Z}^*_{x,\theta_1,s}(t), \mathcal{Y}^*_{x,\theta_2}\big) = \mathrm{Cov}\big(\beta(d_x^+, e^{-t}, s)\mathcal{Y}^*_{x,\theta_1}, \mathcal{Y}^*_{x,\theta_2}\big) = \beta(d_x^+, e^{-t}, s)\psi_{x,\theta_1,\theta_2},$$

and for $x_1 \neq x_2$, again by the independence the covariance is 0.

We now compute $\widetilde{\sigma}_{x,\theta,r,s}(y)$. Recall that

$$\mathrm{Cov}(Y_{x,\theta,s}(-\ln y), Y_{x,\theta,r}(-\ln y)) = \mathbb{1}\{r = s\} \int_y^1 u^{-2s} d\varphi_{x,\theta,s}(u).$$

Then we obtain

$$\begin{aligned}
\mathrm{Var}\big(\widetilde{\mathcal{Z}}_{x,\theta,s}(-\ln y)\big) &= \int_y^1 v^{-2s} d\varphi_{x,\theta,s}(v) \\
&+ \sum_{j=s+1}^{d_x^+} s^2 \binom{j-1}{s}^2 \iint_{y<u<z<1} (u-y)^{j-s-1}(z-y)^{j-s-1}\Big(\int_z^1 v^{-2j} d\varphi_{x,\theta,j}(v)\Big) du dz \\
&= \frac{1}{2}\sum_{j=s}^{d_x^+} \binom{j-1}{s-1}^2 \int_y^1 (v-y)^{2j-2s} v^{-2j} d\varphi_{x,\theta,j}(v).
\end{aligned}$$

For $r \geqslant s$, we can write $r = s + k$ for some $k \geqslant 1$, and deduce that

$$\begin{aligned}
&\mathrm{Cov}\big(\widetilde{\mathcal{Z}}_{x,\theta,s}(-\ln y), \widetilde{\mathcal{Z}}_{x,\theta,s+k}(-\ln y)\big) \\
&\quad = \frac{1}{2}\sum_{j=s+k}^{d_x^+} \binom{j-1}{s-1}\binom{j-1}{s+k-1} \int_y^1 (v-x)^{2j-2s-k} v^{-2j} d\varphi_{x,\theta,j}(v).
\end{aligned}$$

Hence we have

$$\begin{aligned}
\widetilde{\sigma}_{x,\theta,s,s+k}(y) :&= y^{2s+k}\mathrm{Cov}\big(\widetilde{\mathcal{Z}}_{x,\theta,s}(-\ln y), \widetilde{\mathcal{Z}}_{x,\theta,s+k}(-\ln y)\big) \\
&= \frac{1}{2} y^{2s+k} \sum_{j=s+k}^{d_x^+} \binom{j-1}{s-1}\binom{j-1}{s+k-1} \int_y^1 (v-y)^{2j-2s-k} v^{-2j} d\varphi_{x,\theta,j}(v).
\end{aligned} \tag{41}$$

This completes the proof of Lemma 6.5.

A.4 Proof of Lemma 6.6

We only provide the proof for f_{H^+}. The proof for the case $\clubsuit \in \{S, D, I^+, W\}$ follows in the same way. For all $d, y \in \mathbb{N}$ and $z \in [0,1]$, let us consider the following function

$$h((d,y);z) := \sum_{\ell=d-y+1}^{d} \ell b(d,z,\ell).$$

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We define a sequence of bi-dimensional nonnegative integer valued random variables $\{X_n\}$ and X with distributions

$$\mathbb{P}(X_n = (d, y)) = \sum_{x \in \mathcal{A}_d} \mu_x^{(n)} q_x^{(n)}(y),$$

and

$$\mathbb{P}(X = (d, y)) = \sum_{x \in \mathcal{A}_d} \mu_x q_x(y),$$

where $\mathcal{A}_d := \{x \in \mathcal{X} : d_x^+ = d\}$. Then it follows that

$$f_{H^+}^{(n)}(z) = \mathbb{E}h(X_n; z) \quad \text{and} \quad f_{H^+}(z) = \mathbb{E}h(X; z).$$

By Assumption 3a, we have $X_n \to X$ in distribution as $n \to \infty$. Moreover, for any $z \in [0, 1]$, $0 \leqslant h((d, y); z) \leqslant d$. Thus, $h(X_n; z) \leqslant X_n^{(1)}$, where $X_n^{(1)}$ is first dimensional component of X_n. Note also that by Assumption 3b, $X_n^{(1)}$ is uniformly integrable. Hence we have (as $n \to \infty$) for all $z \in [0, 1]$,

$$f_{H^+}^{(n)}(z) = \mathbb{E}h(X_n; z) \to \mathbb{E}h(X; z) = f_{H^+}(z).$$

Further, an elementary calculation gives that

$$\frac{\partial}{\partial z} b(d, z, \ell) = db(d-1, z, \ell-1) - db(d-1, z, \ell).$$

Combining now with the fact that $b(d, z, \ell) \in [0, 1]$, we have $|\frac{\partial}{\partial z} b(d, z, \ell)| \leqslant d$. In addition, using a simple induction gives that for every $j \geqslant 0$, $|\frac{\partial^j}{\partial z^j} b(v, z, \ell)| \leqslant d^j$. We therefore obtain that

$$\Big|\frac{\partial^j}{\partial z^j} h(X_n; z)\Big| \leqslant (X_n^{(1)})^j \sum_{\ell=1}^{X_n^{(1)}} \ell \leqslant (X_n^{(1)})^{j+2}. \tag{42}$$

This is again, by Assumption 3b, uniformly integrable. Hence, we also have (as $n \to \infty$)

$$\frac{\partial^j}{\partial z^j} f_{H^+}^{(n)}(z) = \mathbb{E}\frac{\partial^j}{\partial z^j} h(X_n; z) \to \mathbb{E}\frac{\partial^j}{\partial z^j} h(X; z) = \frac{\partial^j}{\partial z^j} f_{H^+}(z)$$

for all $z \in [0, 1]$. Moreover, (42) also implies all these derivatives are uniformly bounded. Thus by the Arzela-Ascoli theorem (see e.g., [42]), as $n \to \infty$, $f_{H^+}^{(n)}(z) \to f_{H^+}(z)$ together with all its derivatives uniformly on $[0, 1]$. This completes the proof for $\clubsuit = H^+$ and the proof for the case $\clubsuit \in \{S, D, I^+, W\}$ follows in the same way.

A.5 Proof of Lemma 6.7

We only provide the proof of (15). Other convergences are simpler and could be proved by using similar arguments.

First note that, for $\widetilde{V}_{x,s}^{(n)}(t) = \sum_\theta \widetilde{V}_{x,\theta,s}^{(n)}(t)$, we have

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$$\widetilde{V}^{(n)}_{x,s}(t) = \sum_{\theta=1}^{d_x^+} \widetilde{M}^{(n)}_{x,\theta,s}(t) + \sum_{j=s+1}^{d_x^+} s\binom{j-1}{s} \int_0^t (e^{-r} - e^{-t})^{j-s-1} e^{-r} \sum_{\theta=1}^{d_x^+} \widetilde{M}^{(n)}_{x,\theta,j}(r)dr$$
$$= \widetilde{M}^{(n)}_{x,s}(t) + \sum_{j=s+1}^{d_x^+} s\binom{j-1}{s} \int_0^t (e^{-r} - e^{-t})^{j-s-1} e^{-r} \widetilde{M}^{(n)}_{x,j}(r)dr,$$

where $\widetilde{M}^{(n)}_{x,s}(t) := \sum_{\theta=1}^{d_x^+} \widetilde{M}^{(n)}_{x,\theta,s}(t)$. This is again a partial sum and Theorem 6.3 applies. We therefore have

$$\widetilde{V}^{(n)}_{x,s}(t \wedge \tau_n) \xrightarrow{d} \widetilde{\mathcal{Z}}_{x,s}(t \wedge t_0) := \sum_{\theta=1}^{d_x^+} \widetilde{\mathcal{Z}}_{x,\theta,s}(t \wedge t_0),$$

in $\mathcal{D}[0,\infty)$. More precisely,

$$\widetilde{M}^{(n)}_{x,s}(t \wedge \tau_n) \xrightarrow{d} Y_{x,s}(t \wedge t_0) \qquad \text{in} \quad \mathcal{D}[0,\infty),$$

where $Y_{x,s}$ is a continuous Gaussian process with $\mathbb{E}Y_{x,s}(t) = 0$ and covariance

$$\mathbb{E}\big[Y_{x,s}(t)Y_{x,s}(u)\big] = \sum_{\theta=1}^{d_x^+} \int_0^t e^{2sr} d(-\varphi_{x,\theta,s}(e^{-r})) = \int_0^t e^{2sr} d(-\varphi_{x,s}(e^{-r})),$$

for $0 \leqslant t \leqslant u < \infty$ and

$$\varphi_{x,s}(y) = \sum_{\theta=1}^{d_x^+} \varphi_{x,\theta,s}(y) = \mu_x \beta(d_x^+, y, s).$$

We now prove that the convergence also hold for an infinite sum which is used to prove our final result. Let

$$Q^{(n)}_{x,\theta,s}(t) := e^{-st} n^{-1/2} \big(\widehat{v}^{(n)}_{x,\theta,s} - n e^{st} \mu^{(n)}_x q^{(n)}_x(\theta) \beta(d_x^+, e^{-t}, s)\big).$$

Then we have for all (x, θ, s) and all $t > 0$,

$$\mathrm{Var}(Q^{(n)}_{x,\theta,s}(t)) \leqslant \mu^{(n)}_x q^{(n)}_x(\theta)(1 - q^{(n)}_x(\theta)).$$

Recall that $\mathcal{X}_s^+$ and $\mathcal{X}_s^-$ denote the set of characteristics which have in-degree $d_x^+ \geqslant s$ and out-degree $d_x^- \geqslant s$ respectively. Let Θ_x be an arbitrary subset of $\{1, \ldots, d_x^+\}$. By (38), the convergence holds for the finite sum $\sum_{x \in \mathcal{X} \setminus \mathcal{X}_\ell^+} \sum_{\theta \in \Theta_x} Q^{(n)}_{x,\theta,s}(t)$. We now consider the following infinite sum

$$\sum_{x \in \mathcal{X}_\ell^+} (d_x^+ + d_x^-) \sum_{\theta \in \Theta_x} Q^{(n)}_{x,\theta,s}(t).$$

Since power function can be controlled by exponential function, there exists a constant

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$C > 1$ such that for all $t > 0$,

$$\begin{aligned}\sum_{x\in\mathcal{X}_\ell^+}((d_x^+)^2+(d_x^-)^2)\mathrm{Var}(\sum_{\theta\in\Theta_x}Q^{(n)}_{x,\theta,s}(t)) &\leqslant \sum_{x\in\mathcal{X}_\ell^+}((d_x^+)^2+(d_x^-)^2)\sum_{\theta\in\Theta_x}\mathrm{Var}(Q^{(n)}_{x,\theta,s}(t))\\ &\leqslant \sum_{x\in\mathcal{X}_\ell^-}(d_x^-)^2\mu_x^{(n)}+2\sum_{x\in\mathcal{X}_\ell^+}(d_x^+)^2\mu_x^{(n)}\\ &\leqslant \sum_{x\in\mathcal{X}_\ell^-}C^{d_x^-}\mu_x^{(n)}+2\sum_{x\in\mathcal{X}_\ell^+}C^{d_x^+}\mu_x^{(n)}.\end{aligned}$$

Thus we have by Assumption 3b, for n large enough, as $\ell\to\infty$,

$$\mathbb{E}[\sup_{t>0}|\sum_{x\in\mathcal{X}_\ell^+}(d_x^++d_x^-)\sum_{\theta=1}^{d_x^+}Q^{(n)}_{x,\theta,s}(t)|]\to 0. \tag{43}$$

Then using the convergence of the partial sums of $Q^{(n)}_{x,\theta,s}$, we can extend the convergence to an infinite sum of $Q^{(n)}_{x,\theta,s}$, by using e.g. [16, Theorem 4.2]. Further, the limit is also continuous. We therefore have in $\mathcal{D}[0,\infty)$,

$$\sum_{x\in\mathcal{X}}(d_x^++d_x^-)\sum_{\theta\in\Theta_x}Q^{(n)}_{x,\theta,s}(t)\xrightarrow{d}\sum_{x\in\mathcal{X}}(d_x^++d_x^-)\sum_{\theta\in\Theta_x}\widehat{\mathcal{Z}}_{x,\theta,s}(t). \tag{44}$$

On the other hand, for the following infinite sum, we have

$$\begin{aligned}\bar{V}_s^{(n)}(t) :&= \sum_{x\in\mathcal{X}}(d_x^++d_x^-)\sum_{\theta\in\Theta_x}\widetilde{V}^{(n)}_{x,\theta,s}(t)\\ &= \sum_{x\in\mathcal{X}}(d_x^++d_x^-)\sum_{\theta\in\Theta_x}\widetilde{M}^{(n)}_{x,\theta,s}(t)\\ &\quad+\sum_{x\in\mathcal{X}}(d_x^++d_x^-)\sum_{\theta\in\Theta_x}\sum_{j=s+1}^{d_x^+}s\binom{j-1}{s}\int_0^t(e^{-r}-e^{-t})^{j-s-1}e^{-r}\widetilde{M}^{(n)}_{x,\theta,j}(r)dr\\ &= \overline{M}_s^{(n)}(t)+\sum_{j=s+1}^{\infty}s\binom{j-1}{s}\int_0^t(e^{-r}-e^{-t})^{j-s-1}e^{-r}\overline{M}_j^{(n)}(r)dr,\end{aligned}$$

where $\overline{M}_s^{(n)}(t) := \sum_{x\in\mathcal{X}}(d_x^++d_x^-)\sum_{\theta\in\Theta_x}\widetilde{M}^{(n)}_{x,\theta,s}(t)$ is a martingale with initial value 0.

We set the convention that $\widetilde{M}^{(n)}_{x,s}(t)=0$ for $d_x^+<s$. By a similar way as done for (32), we obtain that the quadratic variation of $\overline{M}_s^{(n)}$, which is

$$[\overline{M}_s^{(n)},\overline{M}_s^{(n)}]_t\leqslant 2e^{2st}\sum_{x\in\mathcal{X}_s^+}((d_x^+)^2+(d_x^-)^2)V^{(n)}_{x,s}(0)/n.$$

Using Assumption 3b, for all $A>1$, there exists constants C_s and $C_{s,A}$ such that

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$$\sum_{x\in\mathcal{X}_s^+}((d_x^+)^2+(d_x^-)^2)V_{x,s}^{(n)}(0)\leqslant\sum_{x\in\mathcal{X}_s^-}(d_x^-)^2V_{x,s}^{(n)}(0)+2\sum_{x\in\mathcal{X}_s^+}(d_x^+)^2V_{x,s}^{(n)}(0)$$
$$\leqslant A^{-s}\sum_{x\in\mathcal{X}}\mu_x^{(n)}(2(C_sA)^{d_x^+}+(C_sA)^{d_x^-})\leqslant C_{s,A}A^{-s}n.$$

Thus for any $t>0$ and a fixed T, by choosing $A=e^{2t+4T}$ we get

$$[\overline{M}_s^{(n)},\overline{M}_s^{(n)}]_t\leqslant 2\sum_{x\in\mathcal{X}_s^+}((d_x^+)^2+(d_x^-)^2)e^{2st}V_s^{(n)}(0)/n\leqslant C_{s,A}e^{-4Ts}.$$

By Doob's L^2 inequality, we have (for some constant $C'_{s,T}$)

$$\mathbb{E}[\sup_{t\leqslant T}(\overline{M}_s^{(n)}(t))^2]\leqslant 4\mathbb{E}[\overline{M}_s^{(n)},\overline{M}_s^{(n)}]_T\leqslant C'_{s,T}e^{-4Ts}.$$

Then combining the Cauchy-Schwarz inequality, we obtain (for some constant $C''_{s,T}$)

$$\mathbb{E}[\sup_{t\leqslant T}|\overline{M}_s^{(n)}(t)|]\leqslant C''_{s,T}e^{-2Ts}. \tag{45}$$

Let us define

$$\xi_{N,n}(t):=\sum_{j=N}^{\infty}s\binom{j-1}{s}\int_0^t(e^{-r}-e^{-t})^{j-s-1}e^{-r}\overline{M}_j^{(n)}(r)dr.$$

Then by (45) and some (simple) calculations we find that

$$\begin{aligned}\mathbb{E}\big(\sup_{t\leqslant T}|\xi_{N,n}(t)|\big)&\leqslant\sum_{j=N}^{\infty}s\binom{j-1}{s}\int_0^t(e^{-r}-e^{-t})^{j-s-1}e^{-r}\mathbb{E}\big[\sup_{t\leqslant T}|\overline{M}_j^{(n)}(t)|\big]dr\\&\leqslant C''_{s,T}T\sum_{j=N}^{\infty}s\binom{j-1}{s}(1-e^{-T})^{j-s-1}e^{-2Tj}\\&\leqslant C''_{s,T}Tse^{sT}\sum_{j=N}^{\infty}e^{-2Tj},\end{aligned}$$

which implies for any fixed s and T, $\mathbb{E}(\sup_{t\leqslant T}|\xi_{N,n}(t)|)\to 0$ as $N\to\infty$, uniformly in n.

Using again the convergence of the partial sums of $\widetilde{V}_{x,\theta,s}^{*(n)}$, we can extend the convergence to some infinite sums of $\widetilde{V}_{x,\theta,s}^{(n)}$. It follows that in $\mathcal{D}[0,\infty)$,

$$\sum_{x\in\mathcal{X}}(d_x^++d_x^-)\sum_{\theta\in\Theta_x}e^{-st}\widetilde{V}_{x,\theta,s}^{(n)}(t\wedge\tau_n)\xrightarrow{d}\sum_{x\in\mathcal{X}}(d_x^++d_x^-)\sum_{\theta\in\Theta_x}\widetilde{\widetilde{\mathcal{Z}}}_{x,\theta,s}(t\wedge t_0). \tag{46}$$

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Combining now (44) and (46), it then follows that jointly for any $s > 0$,

$$\sum_{x\in\mathcal{X}}(d_x^+ + d_x^-)\sum_{\theta\in\Theta_x} V_{x,\theta,s}^{*(n)}(t\wedge\tau_n) \xrightarrow{d} \sum_{x\in\mathcal{X}}(d_x^+ + d_x^-)\sum_{\theta\in\Theta_x} \mathcal{Z}_{x,\theta,s}^{*}(t\wedge t_0),$$

and also the partial sum for any fixed r,

$$\sum_{s=1}^{r}\sum_{x\in\mathcal{X}}(d_x^+ + d_x^-)\sum_{\theta\in\Theta_x} V_{x,\theta,s}^{*(n)}(t\wedge\tau_n) \xrightarrow{d} \sum_{s=1}^{r}\sum_{x\in\mathcal{X}}(d_x^+ + d_x^-)\sum_{\theta\in\Theta_x} \mathcal{Z}_{x,\theta,s}^{*}(t\wedge t_0). \tag{47}$$

Then notice that for any $x\in\mathcal{X}$, $d_x^+ > 0$, thus we have that

$$\sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d_x^+}\sum_{s=d_x^+-\theta+2}^{d_x^+} V_{x,\theta,s}^{*(n)}(t) = \sum_{s=2}^{\infty}\sum_{x\in\mathcal{X}}\sum_{\theta\in\Theta_x} V_{x,\theta,s}^{*(n)}(t),$$

with taking $\Theta_x = \{1,\dots,d_x^+\}$. Similarly, we define the following infinite tail sum

$$\bar{\xi}_{N,n}(t) := \sum_{s=N}^{\infty}\sum_{x\in\mathcal{X}}\sum_{\theta\in\Theta_x} \widetilde{V}_{x,\theta,s}^{(n)}.$$

Note that when s is large, $C''_{s,T}$ can be bounded by another constant C_T only depending on T. Then by the same way as above and (45), we obtain

$$\begin{aligned}
\mathbb{E}\big(\sup_{t\leqslant T}|\bar{\xi}_{N,n}(t)|\big) &\leqslant \sum_{s=N}^{\infty}\sum_{j=s+1}^{\infty} s\binom{j-1}{s}\int_0^t (e^{-r}-e^{-t})^{j-s-1}e^{-r}\mathbb{E}\big[\sup_{t\leqslant T}|\overline{M}_j^{(n)}(t)|\big]dr \\
&\quad + \sum_{s=N}^{\infty}\overline{M}_s^{(n)}(t) \\
&\leqslant C_T\sum_{s=N}^{\infty} e^{-2Ts} + \leqslant C_T T\sum_{j=N+1}^{\infty}\sum_{s=N}^{j-1} s\binom{j-1}{s}(1-e^{-T})^{j-s-1}e^{-2Tj} \\
&\leqslant C_T\sum_{s=N}^{\infty} e^{-2Ts} + C_T\sum_{j=N+1}^{\infty} je^{jT}e^{-2Tj} \leqslant 2C_T\sum_{s=N}^{\infty} se^{-sT},
\end{aligned}$$

which implies that for any fixed $T > 0$, $\mathbb{E}(\sup_{t\leqslant T}|\xi_{N,n}(t)|) \to 0$ as $N\to\infty$, uniformly in n. Combine with (43), we therefore have that for any $T > 0$ fixed, as $\ell\to\infty$,

$$\mathbb{E}\big[\sup_{t\leqslant T}\big|\sum_{x\in\mathcal{X}_\ell^+}\sum_{\theta=1}^{d_x^+}\sum_{s=d_x^+-\theta+2}^{d_x^+} V_{x,\theta,s}^{*(n)}(t)\big|\big] \to 0.$$

Hence the same argument [16, Theorem 4.2] allows us to pass the limit under the infinite sum and with the limit being continuous. It then follows that, by using (47) and letting

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$r \to \infty$, to obtain that in $\mathcal{D}[0,\infty)$,

$$\sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d_x^+}\sum_{s=d_x^+-\theta+2}^{d_x^+} V_{x,\theta,s}^{*(n)}(t\wedge\tau_n) \xrightarrow{d} \sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d_x^+}\sum_{s=d_x^+-\theta+2}^{d_x^+} \mathcal{Z}_{x,\theta,s}^{*}(t\wedge t_0).$$

A.6 Covariances in Theorem 3.6

Using all the notations of Section 6.5 and 6.6, we now compute the covariances in Theorem 3.6 for the continuous Gaussian processes $\mathcal{Z}_S, \mathcal{Z}_{H^+}, \mathcal{Z}_{I^+}$ and $\mathcal{Z}_W$.

For convenience, we make a change of variable $y = e^{-t}$, which decreases from 1 to 0 as t varies from 0 to ∞. We use the notations

$$\sigma_L(y) := \mathrm{Var}(\mathcal{Z}_L(-\ln y)), \quad \sigma_{x,\theta,s}^L(y) := \mathrm{Cov}\big(\mathcal{Z}_L(-\ln y), \mathcal{Z}_{x,\theta,s}^*(-\ln y)\big).$$

We now compute $\sigma_L(y)$ and $\sigma_{x,\theta,s}^L(y)$. In order to compute $\sigma_{x,\theta,s}^L$, we apply Theorem 6.3 to $\widetilde{L}_n$ and $\widetilde{M}_{x,\theta,s}^{(n)}$ for all $s = d_x^+ - \theta + 1, \ldots, d_x^+$. Observe that each time $V_{x,\theta,s}^{(n)}$ decreases by 1, also an in ball dies and thus L_n decreases by 1. Hence, the quadratic covariation is

$$\begin{aligned}[\widetilde{M}_{x,\theta,s}^{(n)}, \widetilde{L}_n]_t &= n^{-1}\sum_{0<r\leqslant t}\Delta M_{x,\theta,s}^{(n)}(r)\Delta\widehat{L}_n(r) = n^{-1}\sum_{0<r\leqslant t}\Delta\widehat{V}_{x,\theta,s}^{(n)}(r)\Delta\widehat{L}_n(r)\\ &= n^{-1}\sum_{0<r\leqslant t} e^{(s+1)r}\Delta V_{x,\theta,s}^{(n)}(r)\Delta L_n(r) = n^{-1}\int_0^t e^{(s+1)r} d(-V_{x,\theta,s}^{(n)}(r)).\end{aligned}$$

Using integration by parts as before and formula (31), we can obtain that

$$\begin{aligned}[\widetilde{M}_{x,\theta,s}^{(n)}, \widetilde{L}_n]_t &= \int_0^t e^{(s+1)r} d(-\varphi_{x,\theta,s}(e^{-r})) + o_p(1)\\ &= \int_{e^{-t}}^1 u^{-(s+1)} d\varphi_{x,\theta,s}(u) + o_p(1).\end{aligned}$$

We can then compute all needed covariances. First, the above analysis together with Theorem 6.3, gives that for all $(x,\theta,s)s$,

$$\mathrm{Cov}\big(Y_{x,\theta,s}(-\ln y), \widetilde{\mathcal{Z}}_L(-\ln y)\big) = \int_y^1 u^{-(s+1)} d\varphi_{x,\theta,s}(u),$$

where $Y_{x,\theta,s}$ is defined in (33).

For $\sigma_L(y)$ we have

$$\sigma_L(y) := \mathrm{Var}(\mathcal{Z}_L(-\ln y)) = \mathrm{Var}(y\widetilde{\mathcal{Z}}_L(-\ln y)) = \lambda(y - y^2)/2. \tag{48}$$

Similarly, the above analysis together with Theorem 6.3, gives that for all x, θ,

$$\mathrm{Cov}\big(Y_{x,\theta,s}(-\ln y), \widetilde{\mathcal{Z}}_L(-\ln y)\big) = \int_y^1 u^{-(s+1)} d\varphi_{x,\theta,s}(u).$$

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On the other hand, for $v \leqslant t$, $\mathrm{Cov}(Y_{x,\theta,s}(v), \widetilde{\mathcal{Z}}_L(t)) = \mathrm{Cov}(Y_{x,\theta,s}(v), \widetilde{\mathcal{Z}}_L(v))$. Thus we have

$$\begin{aligned}
&\mathrm{Cov}(\widetilde{\mathcal{Z}}_{x,\theta,s}(-\ln y), \widetilde{\mathcal{Z}}_L(-\ln y)) \\
=&\mathrm{Cov}(Y_{x,\theta,s}(t), \widetilde{\mathcal{Z}}_L(t)) + \int_0^t (e^{-r} - e^{-t})^{j-s-1} e^{-r} \mathrm{Cov}(Y_{x,\theta,s}(r), \widetilde{\mathcal{Z}}_L(r)) dr \\
=&\int_y^1 u^{-(s+1)} d\varphi_{x,\theta,s}(u) + \sum_{j=s+1}^{d_x^+} s \binom{j-1}{s} f_{sj}(y),
\end{aligned}$$

where, with a change of variable $u = e^{-r}$, the function $f_{sj}(y)$ is defined as

$$\begin{aligned}
f_{sj}(y) :=& \int_y^1 (u-y)^{j-s-1} \int_u^1 v^{-(j+1)} d\varphi_{x,\theta,j}(v) du \\
=& \int_y^1 \int_y^v (u-y)^{j-s-1} v^{-(j+1)} du d\varphi_{x,\theta,j}(v) \\
=& \frac{1}{j-s} \int_y^1 (v-y)^{j-s} v^{-(j+1)} d\varphi_{x,\theta,j}(v).
\end{aligned}$$

We thus obtain

$$\mathrm{Cov}(\widetilde{\mathcal{Z}}_{x,\theta,s}(-\ln y), \widetilde{\mathcal{Z}}_L(-\ln y)) = \sum_{j=s}^{d_x^+} \binom{j-1}{s-1} \int_y^1 (v-y)^{j-s} v^{-(j+1)} d\varphi_{x,\theta,j}(v).$$

Also, notice that $\mathrm{Cov}(\widehat{\mathcal{Z}}_{x,\theta,s}(-\ln y), \widetilde{\mathcal{Z}}_L(-\ln y)) = 0$ since they are independent. Then we conclude that

$$\begin{aligned}
\sigma^L_{x,\theta,s}(y) :=& y^{s+1} \mathrm{Cov}(\widetilde{\mathcal{Z}}_{x,\theta,s}(-\ln y), \widetilde{\mathcal{Z}}_L(-\ln y)) \\
=& y^{s+1} \sum_{j=s}^{d_x^+} \binom{j-1}{s-1} \int_y^1 (v-y)^{j-s} v^{-(j+1)} d\varphi_{x,\theta,j}(v).
\end{aligned} \tag{49}$$

We can write now the covariances for the processes $\mathcal{Z}_S$, $\mathcal{Z}_{H^+}$, $\mathcal{Z}_{I^+}$ and $\mathcal{Z}_W$ by using $\sigma_L(y)$ and $\sigma^L_{x,\theta,s}(y)$ (calculated above), $\widetilde{\sigma}_{x,\theta,r,s}(y)$ and $\widehat{\sigma}_{x,\theta_1,\theta_2,s_1,s_2}$ (given in Lemma 6.5). We only write the covariances between $\mathcal{Z}_S$, $\mathcal{Z}_{H^+}$ and $\mathcal{Z}_W$; the covariances of $\mathcal{Z}_{I^+}$ could be easily deduced from those of $\mathcal{Z}_{H^+}$. For convenience, we set $\pi_x(\theta) := d_x^+ - \theta + 1$ in the following.

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For the variances, by using (19)-(22), we have:

$$\sigma_{W,W}(y) = \sum_{x\in\mathcal{X}} \sum_{\theta=1}^{d_x^+} \Big[(d_x^-)^2 \widetilde{\sigma}_{x,\theta,\pi_x(\theta),\pi_x(\theta)}(y) - 2d_x^- \sigma^L_{x,\theta,\pi_x(\theta)}(y) \Big] + \sigma_L(y) \\ + \sum_{x\in\mathcal{X}} (d_x^-)^2 \sum_{\theta_1=1}^{d_x^+} \sum_{\theta_2=1}^{d_x^+} \widehat{\sigma}_{x,\theta_1,\theta_2,\pi_x(\theta_1),\pi_x(\theta_2)}(y). \tag{50}$$

$$\sigma_{S,S}(y) = \sum_{x\in\mathcal{X}} \sum_{\theta=1}^{d_x^+} \widetilde{\sigma}_{x,\theta,\pi_x(\theta),\pi_x(\theta)}(y) + \sum_{x\in\mathcal{X}} \sum_{\theta_1=1}^{d_x^+} \sum_{\theta_2=1}^{d_x^+} \widehat{\sigma}_{x,\theta_1,\theta_2,\pi_x(\theta_1),\pi_x(\theta_2)}(y), \tag{51}$$

and,

$$\sigma_{H^+,H^+}(y) = \sum_{x\in\mathcal{X}} \sum_{\theta=1}^{d_x^+} \Big[\pi_x^2(\theta) \widetilde{\sigma}_{x,\theta,\pi_x(\theta),\pi_x(\theta)}(y) + 2 \sum_{s=\pi_x(\theta)+1}^{d_x^+} \pi_x(\theta) \widetilde{\sigma}_{x,\theta,\pi_x(\theta),s}(y) \\ + \sum_{s=\pi_x(\theta)+1}^{d_x^+} \sum_{r=\pi_x(\theta)+1}^{d_x^+} \widetilde{\sigma}_{x,\theta,r,s}(y) \Big] \\ + \sum_{x\in\mathcal{X}} \sum_{\theta_1=1}^{d_x^+} \sum_{\theta_2=1}^{d_x^+} \Big[\sum_{s_1=\pi_x(\theta_1)+1}^{d_x^+} \sum_{s_2=\pi_x(\theta_2)+1}^{d_x^+} \pi_x(\theta_1)\pi_x(\theta_2) \widehat{\sigma}_{x,\theta_1,\theta_2,\pi_x(\theta_1),\pi_x(\theta_2)}(y) \\ + \widehat{\sigma}_{x,\theta_1,\theta_2,s_1,s_2}(y) + 2\pi_x(\theta_2) \sum_{s_2=\pi_x(\theta_2)+1}^{d_x^+} \widehat{\sigma}_{x,\theta_1,\theta_2,s,\pi_x(\theta_2)}(y) \Big], \tag{52}$$

For the covariances, by using again (19)-(22), we have:

$$\sigma_{S,H^+}(y) = \sum_{x\in\mathcal{X}} \sum_{\theta=1}^{d_x^+} \Big[\pi_x(\theta) \widetilde{\sigma}_{x,\theta,\pi_x(\theta),\pi_x(\theta)}(y) + \sum_{s=\pi_x(\theta)+1}^{d_x^+} \widetilde{\sigma}_{x,\theta,\pi_x(\theta),s}(y) \Big] \\ + \sum_{x\in\mathcal{X}} \sum_{\theta_1,\theta_2=1}^{d_x^+} \Big[\pi_x(\theta_1) \widehat{\sigma}_{x,\theta_1,\theta_2,\pi_x(\theta_1),\pi_x(\theta_2)}(y) + \sum_{s=\pi_x(\theta_1)+1}^{d_x^+} \widehat{\sigma}_{x,\theta_1,\theta_2,s,\pi_x(\theta_2)}(y) \Big], \tag{53}$$

$$\sigma_{S,W}(y) = \sum_{x\in\mathcal{X}} \sum_{\theta=1}^{d_x^+} \Big[\sigma^L_{x,\theta,\pi_x(\theta)}(y) - d_x^- \widetilde{\sigma}_{x,\theta,\pi_x(\theta),\pi_x(\theta)}(y) \Big] \\ - \sum_{x\in\mathcal{X}} d_x^- \sum_{\theta_1=1}^{d_x^+} \sum_{\theta_2=1}^{d_x^+} \widehat{\sigma}_{x,\theta_1,\theta_2,\pi_x(\theta_1),\pi_x(\theta_2)}(y), \tag{54}$$

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and,

$$\sigma_{H^+,W}(y) = \sum_{x\in\mathcal{X}} \sum_{\theta=1}^{d_x^+} \Big[-d_x^- \pi_x(\theta) \widetilde{\sigma}_{x,\theta,\pi_x(\theta),\pi_x(\theta)}(y) + \pi_x(\theta) \sigma^L_{x,\theta,\pi_x(\theta)}(y) + \sum_{s=\pi_x(\theta)+1}^{d_x^+} \big(\sigma^L_{x,\theta,s}(y) - d_x^- \widetilde{\sigma}_{x,\theta,\pi_x(\theta),s}(y) \big) \Big] \tag{55}$$

$$- \sum_{x\in\mathcal{X}} d_x^- \sum_{\theta_1,\theta_2=1}^{d_x^+} \Big[\sum_{s=\pi_x(\theta_1)+1}^{d_x^+} \widehat{\sigma}_{x,\theta_1,\theta_2,s,\pi_x(\theta_2)}(y) + \pi_x(\theta_1) \widehat{\sigma}_{x,\theta_1,\theta_2,\pi_x(\theta_1),\pi_x(\theta_2)}(y) \Big].$$

As for the covariances of $\mathcal{Z}_{I^+}$, we can deduce from the above six formulas by using the relation (21).

A.7 Covariances in Theorem 4.2

In this section, we provide the variances for Gaussian processes $\mathcal{Z}_\Diamond(t)$ and $\mathcal{Z}^\star_\Diamond$ in Theorem 4.2. First, let us define

$$\mathcal{Z}^{(1)}_\Diamond := \sum_{x\in\mathcal{X}} \bar{L}^{\odot}_x \sum_{\theta=1}^{d_x^+} \mathcal{Z}^*_{x,\theta,d_x^+-\theta+1}(t),$$

$$\mathcal{Z}^{(2)}_\Diamond := \sum_{x\in\mathcal{X}} \bar{L}^{\Diamond}_x \sum_{\theta=1}^{d_x^+} [(\theta-1)\mathcal{Z}^*_{x,\theta,d_x^+-\theta+1}(t) - \sum_{s=d_x^+-\theta+1+1}^{d_x^+} \mathcal{Z}^*_{x,\theta,s}(t)].$$

Let $\sigma_{i,j}(e^{-t}) := \mathrm{Cov}(\mathcal{Z}^{(i)}_\Diamond(t), \mathcal{Z}^{(j)}_\Diamond(t))$, $i,j = 1,2$. The variance $\sigma_\Diamond(t)$ is therefore

$$\sigma_\Diamond(t) = \sigma_{1,1}(e^{-t}) + 2\sigma_{1,2}(e^{-t}) + \sigma_{2,2}(e^{-t}), \tag{56}$$

where $\sigma_{i,j}(y)$ (for $i,j = 1,2$) are calculated in the following. By using results of Section A.6, we have (recall that $\pi_x(\theta) := d_x^+ - \theta + 1$):

$$\sigma_{1,1}(y) = \sum_{x\in\mathcal{X}} (\bar{L}^{\odot}_x)^2 \Big(\sum_{\theta=1}^{d_x^+} \widetilde{\sigma}_{x,\theta,\pi_x(\theta),\pi_x(\theta)}(y) + \sum_{\theta_1,\theta_2=1}^{d_x^+} \widehat{\sigma}_{x,\theta_1,\theta_2,\pi_x(\theta_1),\pi_x(\theta_2)}(y) \Big),$$

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$$\sigma_{2,2}(y) = \sum_{x\in\mathcal{X}} (\bar{L}_x^{\Diamond})^2 \sum_{\theta=1}^{d_x^+} \Big[(\theta-1)^2 \widetilde{\sigma}_{x,\theta,\pi_x(\theta),\pi_x(\theta)}(y) - 2(\theta-1) \sum_{s=\pi_x(\theta)+1}^{d_x^+} \widetilde{\sigma}_{x,\theta,\pi_x(\theta),s}(y)$$

$$+ \sum_{r,s=\pi_x(\theta)+1}^{d_x^+} \widetilde{\sigma}_{x,\theta,r,s}(y) \Big] + \sum_{x\in\mathcal{X}} (\bar{L}_x^{\Diamond})^2 \sum_{\theta_1,\theta_2=1}^{d_x^+} \Big[\sum_{s_1=\pi_x(\theta_1)+1}^{d_x^+} \sum_{s_2=\pi_x(\theta_2)+1}^{d_x^+} \widehat{\sigma}_{x,\theta_1,\theta_2,s_1,s_2}(y)$$

$$+ (\theta_1-1)(\theta_2-1)\widehat{\sigma}_{x,\theta_1,\theta_2,\pi_x(\theta_1),\pi_x(\theta_2)}(y) - 2(\theta_2-1) \sum_{s=\pi_x(\theta_1)+1}^{d_x^+} \widehat{\sigma}_{x,\theta_1,\theta_2,s,\pi_x(\theta_2)}(y) \Big],$$

and,

$$\sigma_{1,2}(y) = \sum_{x\in\mathcal{X}} \sum_{\theta=1}^{d_x^+} \Big[(\theta-1)\widetilde{\sigma}_{x,\theta,\pi_x(\theta),\pi_x(\theta)}(y) - \sum_{s=\pi_x(\theta)+1}^{d_x^+} \widetilde{\sigma}_{x,\theta,\pi_x(\theta),s}(y) \Big]$$

$$+ \sum_{x\in\mathcal{X}} \sum_{\theta_1,\theta_2=1}^{d_x^+} \Big[(\theta_1-1)\widehat{\sigma}_{x,\theta_1,\theta_2,\pi_x(\theta_1),\pi_x(\theta_2)}(y) - \sum_{s=\pi_x(\theta_1)+1}^{d_x^+} \widehat{\sigma}_{x,\theta_1,\theta_2,s,\pi_x(\theta_2)}(y) \Big].$$

Now we calculate the variance of $\mathcal{Z}_{\Diamond}^{\star}$. Since it has been shown that

$$\mathcal{Z}_{\Diamond}^{\star} := \mathcal{Z}_{\Diamond}(t^{\star}) - \alpha^{-1} f_{\Diamond}'(z^{\star}) \mathcal{Z}_W(t^{\star}) = \mathcal{Z}_{\Diamond}(t^{\star}) - \Delta(z^{\star}) \mathcal{Z}_W(t^{\star}),$$

where $\Delta(z^{\star}) := \alpha^{-1} f_{\Diamond}'(z^{\star})$. We have that

$$\sigma_{\Diamond}^{\star} := \sigma_{\Diamond}(t^{\star}) + \Delta(z^{\star})^2 \sigma_{W,W}(z^{\star}) - 2\Delta(z^{\star}) \sigma_{\Diamond,W}(z^{\star}). \tag{57}$$

We already calculated $\sigma_{\Diamond}$ and $\sigma_{W,W}$ (see Appendix A.6). The term $\sigma_{\Diamond,W}(e^{-t})$ can be calculated by using

$$\begin{aligned} \sigma_{\Diamond,W}(e^{-t}) &= -\operatorname{Cov}(Z_{\Diamond}^{(1)}(t), Z_W(t)) - \operatorname{Cov}(Z_{\Diamond}^{(2)}(t), Z_W(t)) \\ &= -\sigma_{\Diamond,W}^{(1)}(e^{-t}) - \sigma_{\Diamond,W}^{(2)}(e^{-t}), \end{aligned}$$

where

$$\begin{aligned} \sigma_{\Diamond,W}^{(1)}(y) = &\sum_{x\in\mathcal{X}} \bar{L}_x^{\odot} \sum_{\theta=1}^{d_x^+} \widehat{\sigma}_{x,\theta,\pi_x(\theta)}(y) - \sum_{x\in\mathcal{X}} \bar{L}_x^{\odot} d_x^- \sum_{\theta=1}^{d_x^+} \widetilde{\sigma}_{x,\theta,\pi_x(\theta),\pi_x(\theta)}(y) \\ &- \sum_{x\in\mathcal{X}} \bar{L}_x^{\odot} d_x^- \sum_{\theta_1,\theta_2=1}^{d_x^+} \widehat{\sigma}_{x,\theta_1,\theta_2,\pi_x(\theta_1),\pi_x(\theta_2)}(y), \end{aligned}$$

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and,

$$\sigma^{(2)}_{\Diamond,W}(y) = \sum_{x\in\mathcal{X}} \bar{L}^{\Diamond}_x \sum_{\theta=1}^{d^+_x} \Big[-d^-_x(\theta-1)\widetilde{\sigma}_{x,\theta,\pi_x(\theta),\pi_x(\theta)}(y) + (\theta-1)\widehat{\sigma}_{x,\theta,\pi_x(\theta)}(y)$$
$$- \sum_{s=\pi_x(\theta)+1}^{d^+_x} \Big(\sigma^L_{x,\theta,s}(y) - d^-_x \widetilde{\sigma}_{x,\theta,\pi_x(\theta),s}(y) \Big) \Big]$$
$$+ \sum_{x\in\mathcal{X}} \bar{L}^{\Diamond}_x d^-_x \sum_{\theta_1,\theta_2=1}^{d^+_x} \Big[\sum_{s=\pi_x(\theta_1)+1}^{d^+_x} \widehat{\sigma}_{x,\theta_1,\theta_2,s,\pi_x(\theta_2)}(y) - (\theta_1-1)\widehat{\sigma}_{x,\theta_1,\theta_2,\pi_x(\theta_1),\pi_x(\theta_2)}(y) \Big].$$

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