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# Input-to-state stability of time-delay Persidskii systems

Wenjie Mei, Denis Efimov, Rosane Ushirobira

**Abstract**—We consider in this work a class of generalized Persidskii systems with time delays. The input-to-state stability and stabilization conditions for these nonlinear systems are introduced and checked through linear matrix inequalities. A numerical example demonstrates the efficacy of the proposed results.

## I. INTRODUCTION

Stability analysis for nonlinear dynamical systems is a complicated problem, especially in the presence of external perturbations [1], [2], [3]. One of the most general analysis methods is based on the input-to-state stability (ISS) concept [4], [5], which quantifies the boundedness and the convergence of the state of a nonlinear dynamical system with essentially bounded inputs. This stability property can be verified by finding the corresponding Lyapunov functions [5]. However, the main drawback of applying the ISS framework is the lack of a technique for assigning Lyapunov functions to a generic nonlinear system. The existence of time delays in nonlinear systems may require more complex stability analysis approaches, *e.g.*, using Lyapunov–Krasovskii functional or Lyapunov–Razumikhin function [6]. Moreover, a stabilization input design is usually necessary since time delays can degrade the control performance of dynamic systems.

Most existing approaches for synthesizing Lyapunov functions for nonlinear dynamics involve various canonical forms, such as Lur’e systems [7], Lipschitz dynamics, Persidskii systems [8], and homogeneous models. In this paper, we focus on Persidskii systems that have been extensively studied in the context of neural networks [9] and power systems [10], for instance. Such a choice is reasonable since there is a known canonical form of Lyapunov function for a Persidskii system [11], [8], [12], and also by recent advancements in [13], [14] where both stability problems, in the ISS and input-to-output stability (IOS) sense, are constructively analyzed using linear matrix inequalities (LMIs). However, there is still a lack of study on stability analysis of time-delay generalized Persidskii systems. Our primary goal in this work is to extend these results to the ISS conditions and stabilization for generalized Persidskii systems with time delays based on Lyapunov–Krasovskii functional approach,

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whose form has been proposed specifically for the considered systems. A numerical experiment is presented to illustrate the designing method for obtaining feedback gains.

The organization of this article is as follows. In Section II, the preliminaries are presented. The system under consideration and the problem statement are described in Section III. Section IV presents the ISS conditions of the considered systems, followed by a stabilization feedback design in Section V. In Section VI, we give an example to illustrate the proposed results.

## Notation

- The sets of natural, real and nonnegative real numbers are denoted by  $\mathbb{N}$ ,  $\mathbb{R}$  and  $\mathbb{R}_+$ , respectively. The symbol  $|\cdot|$  denotes the Euclidean norm on  $\mathbb{R}^n$  (and the induced matrix norm  $|A|$  for a matrix  $A \in \mathbb{R}^{m \times n}$ ).
- The set of  $n \times n$  diagonal matrices with nonnegative elements on the diagonal is denoted by  $\mathbb{D}_+^n$ . Also,  $I_n$  and  $\mathbb{O}_{p \times n}$  stand for the  $n \times n$  identity matrix and the  $p \times n$  zero matrix, respectively.
- For  $p, n \in \mathbb{N}$  with  $p \leq n$ , the notation  $\overline{p, n}$  is used to represent the set  $\{p, \dots, n\}$ .
- Let  $\mathbf{1}_{\mathcal{A}} : \mathcal{X} \rightarrow \overline{0, 1}$  denote the indicator function of a set  $\mathcal{A} \subset \mathcal{X}$ .
- For  $t_1, t_2 \in \mathbb{R}$ , with  $t_1 < t_2$ , we denote by  $C_{[t_1, t_2]}^n$  the Banach space of continuous functions  $\psi : [t_1, t_2] \rightarrow \mathbb{R}^n$  with the norm  $\|\psi\|_{[t_1, t_2]} = \sup_{t_1 \leq r \leq t_2} |\psi(r)|$ . For a Lebesgue measurable function  $d : [t_1, t_2] \rightarrow \mathbb{R}^m$ , define the norm  $\|d\|_{[t_1, t_2]} = \text{ess sup}_{r \in [t_1, t_2]} |d(r)|$  for  $[t_1, t_2] \subset \mathbb{R}_+$ . We denote by  $\mathcal{L}_{[t_1, t_2]}^m$  (or  $\mathcal{L}_{\infty}^m$ ) the space of functions  $d$  with  $\|d\|_{[t_1, t_2]} < +\infty$  (or  $\|d\|_{\infty} := \|d\|_{[0, \infty)} < +\infty$ ).
- A continuous function  $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to class  $\mathcal{K}$  if it is strictly increasing and  $\sigma(0) = 0$ ; it belongs to class  $\mathcal{K}_{\infty}$  if it is also unbounded. A continuous function  $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to class  $\mathcal{KL}$  if  $\beta(\cdot, r) \in \mathcal{K}$  and  $\beta(r, \cdot)$  is a decreasing function going to zero for any fixed  $r > 0$ .

## II. PRELIMINARIES

Consider a functional differential equation [15]

$$\dot{x}(t) = f(x_t, d(t)), \quad t \geq 0, \quad (1)$$

where  $x(t) \in \mathbb{R}^n$ ;  $x_t \in C_{[-\tau, 0]}^n$  is the state function,  $x_t(s) = x(t+s)$  for  $s \in [-\tau, 0]$ ,  $\tau > 0$  is a constant delay;  $d(t) \in \mathbb{R}^m$  is the external input,  $d \in \mathcal{L}_{\infty}^m$ ;  $f : C_{[-\tau, 0]}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is a continuous functional,  $f(0, 0) = 0$ , and it ensures the existence and the uniqueness of solutions in forward time for the system (1). With the initial condition  $x_0 \in C_{[-\tau, 0]}^n$  and the input  $d \in \mathcal{L}_{\infty}^m$ , the unique solution is defined as  $x(t, x_0, d)$ ,

for which  $x_t(s, x_0, d) = x(t + s, x_0, d)$ ,  $s \in [-\tau, 0]$  denotes the corresponding state function.

For a continuous functional  $V : \mathbb{R} \times C_{[-\tau, 0]}^n \rightarrow \mathbb{R}_+$  the upper right Dini derivative along the solutions of (1) is defined as follows:

$$D^+V(t, \phi, d) = \limsup_{h \rightarrow 0^+} \frac{V(t+h, x_h(\phi, d)) - V(t, \phi)}{h},$$

where

$$x_h(\phi, d)(s) = \begin{cases} \phi(s+h), & s \in [-\tau, -h] \\ \phi(0) + f(\phi, d(h+s)), & s \in [-h, 0] \end{cases}$$

for any  $\phi \in C_{[-\tau, 0]}^n$  and  $d \in \mathbb{R}^m$ .

*Definition 1:* [16] The system (1) is called *input-to-state stable* (ISS), if there exist  $\beta \in \mathcal{KL}$  and  $\gamma \in \mathcal{K}$  such that

$$|x(t, x_0, d)| \leq \beta(\|x_0\|_{[-\tau, 0]}, t) + \gamma(\|d\|_\infty), \quad \forall t \in \mathbb{R}_+$$

for all  $x_0 \in C_{[-\tau, 0]}^n$  and  $d \in \mathcal{L}_\infty^m$ .

*Definition 2:* [16] The system (1) is said to possess the *asymptotic gain* (AG) property, if there exists  $\gamma \in \mathcal{K}$  such that

$$\lim_{t \rightarrow +\infty} |x(t, x_0, d)| \leq \gamma(\|d\|_\infty)$$

for all  $x_0 \in C_{[-\tau, 0]}^n$  and  $d \in \mathcal{L}_\infty^m$ .

*Definition 3:* [16] A continuous functional  $V : \mathbb{R} \times C_{[-\tau, 0]}^n \rightarrow \mathbb{R}_+$  is called an *ISS Lyapunov-Krasovskii functional* (LKF) if there exist some  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ ,  $\alpha_3, \chi \in \mathcal{K}$  such that

$$\alpha_1(|\phi(0)|) \leq V(t, \phi) \leq \alpha_2(\|\phi\|_{[-\tau, 0]}),$$

$$V(t, \phi) \geq \chi(|d|) \Rightarrow D^+V(t, \phi, d) \leq -\alpha_3(\|\phi\|_{[-\tau, 0]}),$$

for all  $t \in \mathbb{R}_+$ ,  $\phi \in C_{[-\tau, 0]}^n$  and  $d \in \mathbb{R}^m$ .

*Theorem 1:* [16] If the system (1) admits an ISS LKF, then it is ISS with the AG  $\gamma = \alpha_1^{-1} \circ \chi$ .

### III. PROBLEM STATEMENT

The main goal of this paper is to consider the input-to-state stability of a class of nonlinear systems in Persidskii form with constant bounded time delays

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{j=1}^M A_j F_j(x(t)) \\ &+ B_0 x(t - \tau_0) + \sum_{j=1}^M B_j F_j(x(t - \tau_j)) + d(t), \quad \forall t \in \mathbb{R}_+ \end{aligned} \quad (2)$$

with  $x(t) = [x_1(t) \dots x_n(t)]^\top \in \mathbb{R}^n$  the current value of the state and constant delays  $0 < \tau_s < +\infty$  for  $s \in \overline{0, M}$ ;  $A_s, B_s \in \mathbb{R}^{n \times n}$  for  $s \in \overline{0, M}$ ;  $F_j : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $F_j(x) = [f_j^1(x_1) \dots f_j^n(x_n)]^\top$  for  $j \in \overline{1, M}$  are the functions ensuring the existence of the solutions of the system (2) in the forward time at least locally;  $d(t) \in \mathbb{R}^n$  is the external perturbation,  $d \in \mathcal{L}_\infty^n$ ;  $\tau = \max_{s \in \overline{0, M}} \tau_s$ .

In this work, if the upper bound of an index is smaller than the lower one, then the corresponding term in a sum or a sequence must be omitted.

The sector restrictions on  $f_j^i$  ( $i \in \overline{1, n}$ ,  $j \in \overline{1, M}$ ) are imposed:

*Assumption 1:* Assume that for any  $i \in \overline{1, n}$  and  $j \in \overline{1, M}$ ,

$$v f_j^i(v) > 0, \quad \forall v \in \mathbb{R} \setminus \{0\}.$$

Under this assumption, with a reordering of nonlinearities and their decomposition, there exists an index  $m \in \overline{0, M}$  such that for all  $i \in \overline{1, n}$ ,  $a \in \overline{1, m}$

$$\lim_{v \rightarrow \pm\infty} f_a^i(v) = \pm\infty,$$

and that there exists  $\mu \in \overline{m, M}$  such that for all  $i \in \overline{1, n}$ ,  $b \in \overline{1, \mu}$

$$\lim_{v \rightarrow \pm\infty} \int_0^v f_b^i(s) ds = +\infty.$$

In such a case,  $m = 0$  implies that all nonlinearities are bounded.

*Assumption 2:* Assume that for any  $i \in \overline{1, n}$  and  $j, j' \in \overline{1, M}$ ,  $z \in \overline{j+1, M}$ , there exist  $\eta_{0,j}^i, \eta_{1,jj'}^i, \eta_{2,jj'}^i, \eta_{3,jj'z}^i \geq 0$  such that

$$\begin{aligned} 2 \int_0^{x_i} f_j^i(s) ds &\leq \eta_{0,j}^i x_i^2 + \sum_{j'=1}^M f_{j'}^i(x_i) \left( \eta_{1,jj'}^i f_{j'}^i(x_i) \right. \\ &\left. + 2\eta_{2,jj'}^i x_i + 2 \sum_{z=j+1}^M \eta_{3,jj'z}^i f_z^i(x_i) \right) \end{aligned}$$

for all  $x \in \mathbb{R}^n$ .

This hypotheses is satisfied by many nonlinear functions: for polynomial ones, for example, it is sufficient to select  $\eta_{2,jj'}^i \neq 0$ . In the sequel, we denote the diagonal matrices:

$$\begin{aligned} \eta_{0,j} &= \text{diag}(\eta_{0,j}^1, \dots, \eta_{0,j}^n), \quad \eta_{1,jj'} = \text{diag}(\eta_{1,jj'}^1, \dots, \eta_{1,jj'}^n), \\ \eta_{2,jj'} &= \text{diag}(\eta_{2,jj'}^1, \dots, \eta_{2,jj'}^n), \quad \eta_{3,jj'z} = \text{diag}(\eta_{3,jj'z}^1, \dots, \eta_{3,jj'z}^n). \end{aligned}$$

### IV. ISS ANALYSIS

The following theorem is the main result of this paper.

*Theorem 2:* Let assumptions 1 and 2 be satisfied and let  $w_0 > 0$  be a given constant. If there exist  $\rho \in \mathbb{R}$ ;  $0 \leq P = P^\top$ ,  $S_0 = S_0^\top \in \mathbb{R}^{n \times n}$ ;  $0 < \Phi = \Phi^\top \in \mathbb{R}^{n \times n}$ ;  $\{\Lambda_j = \text{diag}(\Lambda_j^1, \dots, \Lambda_j^n)\}_{j=1}^M$ ,  $\{S_k\}_{k=1}^M$ ,  $\{\Xi^k\}_{k=0}^M$ ,  $\{\Upsilon_{s,z}\}_{0 \leq s < z \leq M} \subset \mathbb{D}_+^n$  such that

$$P + \rho \sum_{j=1}^{\mu} \Lambda_j > 0, \quad (3)$$

$$Q = Q^\top \leq 0, \quad (4)$$

where

$$Q = (Q_{a,b})_{a,b=1}^5$$

with

$$\begin{aligned}
Q_{1,1} &= A_0^\top P + PA_0 + S_0 + \Xi^0; \quad Q_{1,2} = PB_0, \\
Q_{1,3} &= PA + A_0^\top \Lambda + [\Upsilon_{0,1} \quad \dots \quad \Upsilon_{0,M}], \\
Q_{1,4} &= PB; \quad Q_{1,5} = P; \quad Q_{2,2} = -e^{-w_0 \tau_0} S_0, \\
Q_{2,3} &= B_0^\top \Lambda; \quad Q_{2,4} = \mathbb{O}_{n \times nM}; \quad Q_{2,5} = \mathbb{O}_{n \times n}, \\
Q_{3,3} &= Q_{3,3}^\top = (\widehat{Q}_{a,b})_{a,b=1}^M, \\
\widehat{Q}_{j,j} &= A_j^\top \Lambda_j + \Lambda_j A_j + \Xi^j + S_j, \quad j \in \overline{1, M}, \\
\widehat{Q}_{s,z} &= A_s^\top \Lambda_z + \Lambda_s A_z + \Upsilon_{s,z}, \quad s \in \overline{1, M-1}, z \in \overline{s+1, M}, \\
Q_{3,4} &= \Lambda^\top B; \quad Q_{3,5} = \Lambda^\top, \\
Q_{4,4} &= \text{diag}(-e^{-w_0 \tau_1} S_1, \dots, -e^{-w_0 \tau_M} S_M), \\
Q_{4,5} &= \mathbb{O}_{nM \times n}; \quad Q_{5,5} = -\Phi, \\
A &= [A_1 \quad \dots \quad A_M], \quad B = [B_1 \quad \dots \quad B_M], \\
\Lambda &= [\Lambda_1 \quad \dots \quad \Lambda_M],
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{1}_{\{0\}}(s) \cdot \Xi^0 + \mathbf{1}_{\overline{1, M}}(s) \cdot \Xi^s &\geq \xi \left[ \mathbf{1}_{\{0\}}(s) \left( P + \sum_{j=1}^M \Lambda_j \eta_{0,j} \right) \right. \\
&\quad \left. + \mathbf{1}_{\overline{1, M}}(s) \cdot \Lambda_s \sum_{j'=1}^M \eta_{1,sj'} \right], \quad (5) \\
\mathbf{1}_{\{0\}}(s) \cdot \Upsilon_{0,j} + \mathbf{1}_{\overline{1, M}}(s) \cdot \Upsilon_{s,z} &\geq \xi \left[ \mathbf{1}_{\{0\}}(s) \cdot \Lambda_j \sum_{j'=1}^M \eta_{2,jj'} \right. \\
&\quad \left. + \mathbf{1}_{\overline{1, M}}(s) \cdot \Lambda_s \sum_{j'=1}^M \eta_{3,sj'z} \right], \\
\Xi^0 - \xi \left( P + \sum_{j=1}^M \Lambda_j \eta_{0,j} \right) + \sum_{j=1}^M \left[ \Xi^j + \Upsilon_{0,j} \right. \\
&\quad \left. - \xi \Lambda_j \sum_{j'=1}^M (\eta_{1,jj'} + \eta_{2,jj'}) + \sum_{z'=j+1}^M \left( \Upsilon_{j,z'} - \xi \sum_{j'=1}^M \Lambda_j \eta_{3,jj'z'} \right) \right] > 0, \\
&\quad s \in \overline{0, M}, \quad j \in \overline{1, M}, \quad z \in \overline{s+1, M}
\end{aligned}$$

for some  $\xi \in (0, w_0]$ , then the system (2) is ISS.

*Proof:* Our goal is to check the conditions in Definition 3 for a LKF taken as follows:

$$\begin{aligned}
V(t, x_t) &= x(t)^\top P x(t) + \int_{t-\tau_0}^t e^{-w_0(t-s)} x(s)^\top S_0 x(s) ds \\
&\quad + 2 \sum_{j=1}^M \sum_{i=1}^M \Lambda_j^i \int_0^{x_i(t)} f_j^i(s) ds \\
&\quad + \sum_{j=1}^M \int_{t-\tau_j}^t e^{-w_0(t-s)} F_j(x(s))^\top S_j F_j(x(s)) ds,
\end{aligned}$$

which verifies the required lower (due to (3) and Finsler's lemma [17]) and upper (since all matrices are nonnegative definite) bounds given in Definition 3, and whose time

derivative for (2) admits the following representation:

$$\begin{aligned}
\dot{V}(t, x_t) &= \dot{x}(t)^\top P x(t) + x(t)^\top P \dot{x}(t) \\
&\quad - w_0 \int_{t-\tau_0}^t e^{-w_0(t-s)} x(s)^\top S_0 x(s) ds \\
&\quad + x(t)^\top S_0 x(t) - e^{-w_0 \tau_0} x(t-\tau_0)^\top S_0 x(t-\tau_0) \\
&\quad + 2 \dot{x}(t)^\top \sum_{j=1}^M \Lambda_j F_j(x(t)) \\
&\quad - w_0 \sum_{j=1}^M \int_{t-\tau_j}^t e^{-w_0(t-s)} F_j(x(s))^\top S_j F_j(x(s)) ds \\
&\quad + \sum_{j=1}^M F_j(x(t))^\top S_j F_j(x(t)) \\
&\quad - \sum_{j=1}^M e^{-w_0 \tau_j} F_j(x(t-\tau_j))^\top S_j F_j(x(t-\tau_j)) \\
&= \begin{bmatrix} x(t) \\ x(t-\tau_0) \\ F_1(x(t)) \\ \vdots \\ F_M(x(t)) \\ F_1(x(t-\tau_1)) \\ \vdots \\ F_M(x(t-\tau_M)) \\ d(t) \end{bmatrix}^\top Q \begin{bmatrix} x(t) \\ x(t-\tau_0) \\ F_1(x(t)) \\ \vdots \\ F_M(x(t)) \\ F_1(x(t-\tau_1)) \\ \vdots \\ F_M(x(t-\tau_M)) \\ d(t) \end{bmatrix} \\
&\quad - x(t)^\top \Xi^0 x(t) - \sum_{j=1}^M F_j(x(t))^\top \Xi^j F_j(x(t)) \\
&\quad - 2 \sum_{j=1}^M x(t)^\top \Upsilon_{0,j} F_j(x(t)) \\
&\quad - 2 \sum_{s=1}^{M-1} \sum_{z=s+1}^M F_s(x(t))^\top \Upsilon_{s,z} F_z(x(t)) \\
&\quad - w_0 \int_{t+\tau_0}^t e^{-w_0(t-s)} x(s)^\top S_0 x(s) ds \\
&\quad - w_0 \sum_j \int_{t+\tau_j}^t e^{-w_0(t-s)} F_j(x(s))^\top S_j F_j(x(s)) ds \\
&\quad + d(t)^\top \Phi d(t).
\end{aligned}$$

Under the restriction  $\alpha(V) \geq d^\top \Phi d$  with  $\alpha(s) = \xi s$ , for  $\xi \in (0, w_0]$  and the condition (4) it follows that

$$\begin{aligned}
\dot{V}(t, x_t) &\leq -x(t)^\top \Xi^0 x(t) - \sum_{j=1}^M F_j(x(t))^\top \Xi^j F_j(x(t)) \\
&\quad - 2 \sum_{j=1}^M x(t)^\top \Upsilon_{0,j} F_j(x(t)) \\
&\quad - 2 \sum_{s=1}^{M-1} \sum_{z=s+1}^M F_s(x(t))^\top \Upsilon_{s,z} F_z(x(t)) \\
&\quad + \xi \left( x(t)^\top P x(t) + 2 \sum_{j=1}^M \sum_{i=1}^n \Lambda_j^i \int_0^{x_i(t)} f_j^i(s) ds \right).
\end{aligned}$$

Under conditions (5) and Assumption 2, the following inequality holds

$$\begin{aligned}
& \xi \left( x(t)^\top P x(t) + 2 \sum_{j=1}^M \sum_{i=1}^n \Lambda_j^i \int_0^{x_i(t)} f_j^i(s) ds \right) \\
\leq & x(t)^\top \xi \left( P + \sum_{j=1}^M \Lambda_j \eta_{0,j} \right) x(t) \\
& + \sum_{j=1}^M F_j(x(t))^\top \xi \left( \Lambda_j \cdot \sum_{j'=1}^M \eta_{1,jj'} \right) F_j(x(t)) \\
& + 2 \sum_{j=1}^M x(t)^\top \xi \left( \Lambda_j \cdot \sum_{j'=1}^M \eta_{2,jj'} \right) F_j(x(t)) \\
& + 2 \sum_{s=1}^{M-1} \sum_{z=s+1}^M F_s(x(t))^\top \xi \left( \Lambda_j \cdot \sum_{j'=1}^M \eta_{3,sj'z} \right) F_z(x(t)) \\
< & x(t)^\top \Xi^0 x(t) + \sum_{j=1}^M F_j(x(t))^\top \Xi^j F_j(x(t)) \\
& + 2 \sum_{j=1}^M x(t)^\top \Upsilon_{0,j} F_j(x(t)) \\
& + 2 \sum_{s=1}^{M-1} \sum_{z=s+1}^M F_s(x(t))^\top \Upsilon_{s,z} F_z(x(t)).
\end{aligned}$$

Therefore, by Definition 3 and Theorem 1, we can substantiate that system (2) is ISS as desired.  $\blacksquare$

## V. STABILIZATION

In this section, we design a feedback control to stabilize a system as (2) and study the ISS property of the resulting closed-loop system.

Consider a variation of system (2):

$$\begin{aligned}
\dot{x}(t) = & A_0 x(t) + \sum_{j=1}^M A_j F_j(x(t)) + B_0 x(t - \tau_0) \\
& + \sum_{j=1}^M B_j F_j(x(t - \tau_j)) + Gu(t) + d(t), \quad (6)
\end{aligned}$$

where all variables are defined as in (2),  $G \in \mathbb{R}^{n \times q}$  and  $u(t) \in \mathbb{R}^q$  is the control. A control law is introduced for stabilization as follows:

$$\begin{aligned}
u(t) = & K_{A,0} x(t) + \sum_{j=1}^M K_{A,j} F_j(x(t)) \\
& + K_{B,0} x(t - \tau_0) + \sum_{j=1}^M K_{B,j} F_j(x(t - \tau_j)),
\end{aligned}$$

where  $K_{A,s}, K_{B,s} \in \mathbb{R}^{q \times n}$ ,  $s \in \overline{0, M}$ . Then the closed-loop system can be presented in the form:

$$\begin{aligned}
\dot{x}(t) = & \tilde{A}_0 x(t) + \sum_{j=1}^M \tilde{A}_j F_j(x(t)) + \tilde{B}_0 x(t - \tau_0) \\
& + \sum_{j=1}^M \tilde{B}_j F_j(x(t - \tau_j)) + d(t), \quad (7)
\end{aligned}$$

where  $\tilde{A}_s = A_s + GK_{A,s}$ ,  $\tilde{B}_s = B_s + GK_{B,s}$  for  $s \in \overline{0, M}$ .

In the case that  $K_{A,s}, K_{B,s}$ ,  $s \in \overline{0, M}$  are given, we can directly formulate the following corollary to analyze the input-to-state stability of the closed-loop system (7).

*Corollary 1:* Assume that all conditions of Theorem 2 are satisfied under the substitutions  $A_s \rightarrow \tilde{A}_s$ ,  $B_s \rightarrow \tilde{B}_s$  for  $s \in \overline{0, M}$ . Then the system (7) is ISS.

We now state a theorem for designing the feedback gains  $K_{A,s}, K_{B,s}$ ,  $s \in \overline{0, M}$  that guarantee the ISS property of the system (7):

*Theorem 3:* Let assumptions 1 and 2 be satisfied with  $\eta_{0,j} = 0$  for  $j \in \overline{1, M}$  and let  $w_0 > 0$  be a given constant.

If there exist  $0 < \bar{P} \in \mathbb{D}_+^n$ ;  $\{\bar{\Lambda}_j = \text{diag}(\bar{\Lambda}_j^1, \dots, \bar{\Lambda}_j^n)\}_{j=1}^M$ ,  $\{\bar{S}_k\}_{k=0}^M$ ,  $\{\bar{\Xi}^k\}_{k=0}^M$ ,  $\{\bar{\Upsilon}_{s,z}\}_{0 \leq s < z \leq M} \subset \mathbb{D}_+^n$ ;  $0 < \Phi = \Phi^\top \in \mathbb{R}^{n \times n}$  and  $\{U_k\}_{k=0}^M, \{L_k\}_{k=0}^M \subset \mathbb{R}^{q \times n}$  such that

$$\bar{Q} = \bar{Q}^\top \leq 0, \quad \text{where } \bar{Q} = (\bar{Q}_{a,b})_{a,b=1}^5$$

$$\begin{aligned}
\text{with } \bar{Q}_{1,1} = & \bar{P} A_0^\top + U_0^\top G^\top + A_0 \bar{P} + G U_0 + \bar{S}_0 + \bar{\Xi}^0, \\
& \bar{Q}_{1,2} = B_0 \bar{P} + G L_0,
\end{aligned}$$

$$\bar{Q}_{1,3} = \left[ A_1 \bar{\Lambda}_1 + G U_1 + \bar{P} A_0^\top + U_0^\top G^\top + \bar{\Upsilon}_{0,1} \quad \dots \right.$$

$$\left. A_M \bar{\Lambda}_M + G U_M + \bar{P} A_0^\top + U_0^\top G^\top + \bar{\Upsilon}_{0,M} \right],$$

$$\bar{Q}_{1,4} = [B_1 \bar{\Lambda}_1 + G L_1 \quad \dots \quad B_M \bar{\Lambda}_M + G L_M],$$

$$\bar{Q}_{1,5} = I_n; \bar{Q}_{2,2} = -e^{-w_0 \tau_0} \bar{S}_0,$$

$$\bar{Q}_{2,3} = [\bar{P} B_0^\top + L_0^\top G^\top \quad \dots \quad \bar{P} B_0^\top + L_0^\top G^\top],$$

$$\bar{Q}_{2,4} = \mathbb{O}_{n \times nM}; \bar{Q}_{2,5} = \mathbb{O}_{n \times n},$$

$$\bar{Q}_{3,3} = \bar{Q}_{3,3}^\top = (\hat{Q}'_{a,b})_{a,b=1}^M,$$

$$\hat{Q}'_{j,j} = \bar{\Lambda}_j A_j^\top + U_j^\top G^\top$$

$$+ A_j \bar{\Lambda}_j + G U_j + \bar{\Xi}^j + \bar{S}_j, \quad j \in \overline{1, M},$$

$$\hat{Q}'_{s,z} = \bar{\Lambda}_s A_s^\top + U_s^\top G^\top$$

$$+ A_z \bar{\Lambda}_z + G U_z + \bar{\Upsilon}_{s,z}, \quad s \in \overline{1, M-1}, z \in \overline{s+1, M},$$

$$\bar{Q}_{3,4} = \begin{bmatrix} B_1 \bar{\Lambda}_1 + G L_1 & \dots & B_M \bar{\Lambda}_M + G L_M \\ \vdots & \ddots & \vdots \\ B_1 \bar{\Lambda}_1 + G L_1 & \dots & B_M \bar{\Lambda}_M + G L_M \end{bmatrix},$$

$$\bar{Q}_{3,5} = [I_n \quad \dots \quad I_n]^\top,$$

$$\bar{Q}_{4,4} = \text{diag}(-e^{-w_0 \tau_1} \bar{S}_1, \dots, -e^{-w_0 \tau_M} \bar{S}_M),$$

$$\bar{Q}_{4,5} = \mathbb{O}_{nM \times n}; \bar{Q}_{5,5} = -\Phi,$$

and

$$\mathbf{1}_{\{0\}}(s) \cdot \bar{\Xi}^0 + \mathbf{1}_{\overline{1, M}}(s) \cdot \bar{\Xi}^s \geq \xi \left[ \mathbf{1}_{\{0\}}(s) \cdot \bar{P} \quad (8)$$

$$+ \mathbf{1}_{\overline{1, M}}(s) \cdot \bar{\Lambda}_s \sum_{j'=1}^M \eta_{1,sj'} \right],$$

$$\mathbf{1}_{\{0\}}(s) \cdot \bar{\Upsilon}_{0,j} + \mathbf{1}_{\overline{1, M}}(s) \cdot \bar{\Upsilon}_{s,z} \geq \xi \left[ \mathbf{1}_{\{0\}}(s) \cdot \bar{P} \sum_{j'=1}^M \eta_{2,jj'}$$

$$+ \mathbf{1}_{\overline{1, M}}(s) \cdot \bar{\Lambda}_z \sum_{j'=1}^M \eta_{3,sj'z} \right],$$

$$\begin{aligned} & \bar{\Xi}^0 - \xi \bar{P} + \sum_{j=1}^M \left[ \bar{\Xi}^j + \bar{Y}_{0,j} - \xi \bar{\Lambda}_j \sum_{j'=1}^M \eta_{1,jj'} \right. \\ & \left. - \xi \bar{P} \sum_{j'=1}^M \eta_{2,jj'} + \sum_{z'=j+1}^M \left( \bar{Y}_{j,z'} - \xi \bar{\Lambda}_{z'} \sum_{j'=1}^M \eta_{3,jj'z'} \right) \right] > 0, \\ & s \in \overline{0, M}, j \in \overline{1, M}, z \in \overline{s+1, M} \end{aligned}$$

for some  $\xi \in (0, w_0]$ , then (7) is ISS with feedback gains

$$K_{A,k} = \begin{cases} U_0 \bar{P}^{-1}, & k = 0 \\ U_k (\bar{\Lambda}_k)^{-1}, & k \in \overline{1, M}, \end{cases}$$

$$K_{B,k} = \begin{cases} L_0 \bar{P}^{-1}, & k = 0 \\ L_k (\bar{\Lambda}_k)^{-1}, & k \in \overline{1, M}. \end{cases}$$

*Proof:* Since  $\bar{P}$  and  $\bar{\Lambda}_j$  for  $j = \overline{1, M}$  are positive definite, select the LKF  $V$  given in the proof of Theorem 2:  $\bar{P} = P^{-1}$ ,  $\bar{\Lambda}_j = \Lambda_j^{-1}$ ,  $j \in \overline{1, M}$ , then  $V$  inherits the same property. Furthermore, consider the conditions and the proof of Theorem 2 and denote by  $\bar{Q}$  the block matrix  $\bar{Q}$  (in Theorem 2) under the substitutions  $A_s \rightarrow \bar{A}_s$ ,  $B_s \rightarrow \bar{B}_s$  for  $s \in \overline{0, M}$ , define

$$\bar{Q} = H^\top \bar{Q} H,$$

$$H = \text{diag}(P^{-1}, P^{-1}, \Lambda_1^{-1}, \dots, \Lambda_M^{-1}, \Lambda_1^{-1}, \dots, \Lambda_M^{-1}, I_n)$$

under the settings of

$$\bar{\Xi}^k = \begin{cases} \bar{P}^{-1} \bar{\Xi}^0 \bar{P}^{-1}, & k = 0 \\ \bar{\Lambda}_k^{-1} \bar{\Xi}^k \bar{\Lambda}_k^{-1}, & k \in \overline{1, M}, \end{cases}, S_k = \begin{cases} \bar{P}^{-1} \bar{S}_0 \bar{P}^{-1}, & k = 0 \\ \bar{\Lambda}_k^{-1} \bar{S}_k \bar{\Lambda}_k^{-1}, & k \in \overline{1, M}, \end{cases}$$

$$\Upsilon_{s,z} = \begin{cases} \bar{P}^{-1} \bar{Y}_{0,j} \bar{\Lambda}_j^{-1}, & s = 0, j \in \overline{1, M} \\ \bar{\Lambda}_s^{-1} \bar{Y}_{s,z} \bar{\Lambda}_z^{-1}, & s \in \overline{1, M-1}, z \in \overline{s+1, M}, \end{cases}$$

by which we see that

$$\bar{Q} \leq 0 \Leftrightarrow Q \leq 0$$

and the conditions (8) are equivalent to (5). This completes the proof.  $\blacksquare$

To find the control gains as solutions of LMIs in Theorem 3, more restrictive conditions are imposed than in Corollary 1: the matrices  $\bar{P}, \bar{\Lambda}_j$  are assumed to be diagonal and positive definite, with  $\eta_{0,j} = 0$  for  $j = \overline{1, M}$ .

## VI. EXAMPLE

In this section, an example of the controlled Persidskii system (7) and Theorem 3 are considered. Let  $n = 2$ ,  $M = 1$ ,  $\tau_0 = 1.5$ ,  $\tau_1 = 2.5$ ,  $w_0 = 0.3$  and  $d, F_1, G, A_0, B_0, A_1, B_1, K_{A,0}$  be chosen as follows:

$$A_0 = \begin{bmatrix} -0.75 & -4.29 \\ 1.07 & 4.25 \end{bmatrix}, A_1 = \begin{bmatrix} 1.42 & 2 \\ -3.95 & -1.04 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} -4.15 & -2.51 \\ -2.85 & -2.73 \end{bmatrix}, B_1 = \begin{bmatrix} 2.03 & 0.47 \\ 2.54 & 0.53 \end{bmatrix},$$

$$F_1(x) = \begin{bmatrix} e^{x_1} - 1 \\ e^{x_2} - 1 \end{bmatrix}, d(t) = \begin{bmatrix} 5 \sin(3t) \\ 1 \end{bmatrix},$$

$$G = \begin{bmatrix} 2.66 & -3.61 & -3.48 \\ 2.51 & -1.5 & -0.03 \end{bmatrix},$$

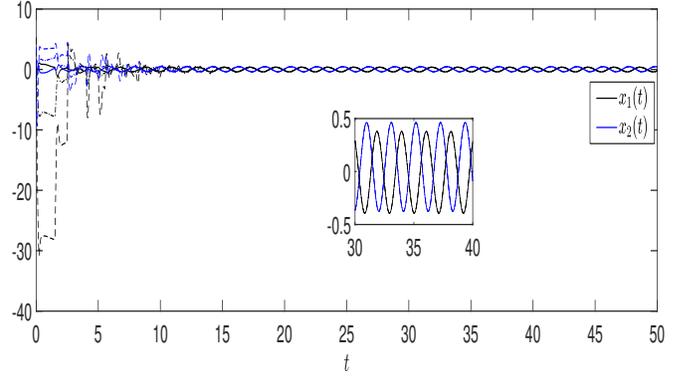


Fig. 1. The state trajectories

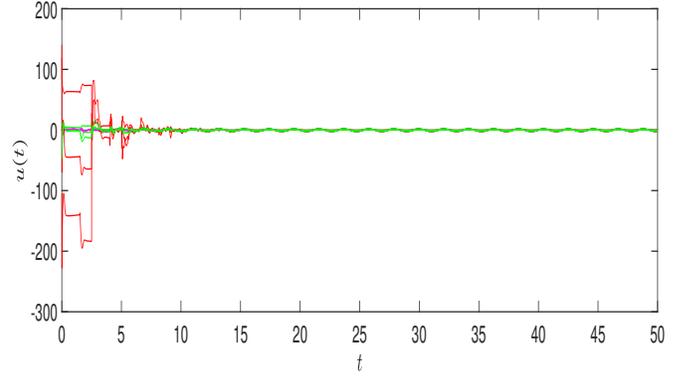


Fig. 2. The input trajectories

Assumption 2 is satisfied for

$$\eta_{0,1}^1 = \eta_{0,1}^2 = 0, \eta_{1,11}^1 = \eta_{1,11}^2 = 0.1,$$

$$\eta_{2,11}^1 = \eta_{2,11}^2 = 1.$$

By solving the LMIs proposed in Theorem 3, we conclude that they are verified by

$$\bar{P} = 10^4 \times \begin{bmatrix} 1.9060 & 0 \\ 0 & 1.3746 \end{bmatrix},$$

$$\bar{\Lambda}_1 = 10^4 \times \begin{bmatrix} 8.2154 & 0 \\ 0 & 8.2217 \end{bmatrix},$$

$$\bar{S}_0 = 10^4 \times \begin{bmatrix} 8.6301 & 0 \\ 0 & 8.6262 \end{bmatrix},$$

$$\bar{S}_1 = 10^4 \times \begin{bmatrix} 9.4469 & 0 \\ 0 & 9.4427 \end{bmatrix},$$

$$\bar{\Xi}^0 = 10^4 \times \begin{bmatrix} 7.4488 & 0 \\ 0 & 7.3931 \end{bmatrix},$$

$$\bar{\Xi}^1 = 10^4 \times \begin{bmatrix} 7.3367 & 0 \\ 0 & 7.3343 \end{bmatrix},$$

$$\begin{aligned}\bar{Y}_{0,1} &= 10^5 \times \begin{bmatrix} 1.1548 & 0 \\ 0 & 1.1496 \end{bmatrix}, \\ \Phi &= 10^4 \times \begin{bmatrix} 9.86 & 5.024 \\ 5.024 & 9.86 \end{bmatrix}, \\ U_0 &= 10^4 \times \begin{bmatrix} -8.1381 & -6.7348 \\ 1.4473 & 0.3149 \\ -4.8921 & -9.6405 \end{bmatrix}, \\ U_1 &= 10^5 \times \begin{bmatrix} 0.6605 & -0.1054 \\ 0.1309 & 0.0274 \\ 1.0384 & 0.0513 \end{bmatrix}, \\ L_0 &= 10^4 \times \begin{bmatrix} -1.0728 & -1.4498 \\ 0.0979 & 0.5767 \\ -0.8247 & -0.3281 \end{bmatrix}, \\ L_1 &= 10^4 \times \begin{bmatrix} -9.6272 & -4.069 \\ 3.3737 & 1.6205 \\ -3.6963 & -1.311 \end{bmatrix}, \quad \xi = 0.2.\end{aligned}$$

The state and input trajectories with three different initial conditions are shown in Fig. 1 and Fig. 2, respectively.

## VII. CONCLUSION

This paper has proposed ISS and stabilization conditions with given feedback control and the design of feedback gains for generalized Persidskii systems with constant time delays. The formulated conditions were obtained in the form of linear matrix inequalities. Therefore, they can be constructively verified. A simulation was presented for the examination of the proposed results.

## REFERENCES

- [1] M. Vidyasagar, *Input-output analysis of large-scale interconnected systems*, ser. Lecture Notes in Control and Information Sciences. Berlin: Springer-Verlag, 1981, vol. 29, decomposition, well-posedness and stability.
- [2] A. van der Schaft, *L<sub>2</sub>-gain and passivity techniques in nonlinear control*, ser. Lecture Notes in Control and Information Sciences. London: Springer-Verlag London Ltd., 1996, vol. 218.
- [3] H. Khalil, *Nonlinear systems*. Upper Saddle River, NJ: Prentice-Hall, 2002.
- [4] E. D. Sontag and Y. Wang, "Notions of input to output stability," *Systems & Control Letters*, vol. 38, no. 4-5, pp. 235–248, 1999.
- [5] E. Sontag and Y. Wang, "Lyapunov characterizations of input to output stability," *SIAM Journal on Control and Optimization*, vol. 39, no. 1, pp. 226–249, 2000.
- [6] E. Fridman, *Introduction to Time-Delay Systems*, ser. Systems & Control: Foundations & Applications. Cham: Springer International Publishing, 2014.
- [7] C. Guiver and H. Logemann, "A circle criterion for strong integral input-to-state stability," *Automatica*, vol. 111, p. 108641, 2020.
- [8] S. Persidskii, "Concerning problem of absolute stability," *Automation and Remote Control*, pp. 5–11, 1969.
- [9] L. Ferreira, E. Kaszkurewicz, and A. Bhaya, "Solving systems of linear equations via gradient systems with discontinuous righthand sides: Application to ls-svm," *IEEE Transactions on Neural Networks*, vol. 16, pp. 501–505, 2005.
- [10] H. Liu and L. Colvara, "The inclusion of automatic voltage regulators in power system transient stability analysis using Lyapunov functions," *IFAC Proceedings Volumes*, vol. 20, pp. 1–6, 1987.
- [11] E. Barbashin, "On construction of Lyapunov functions for nonlinear systems," in *Proc. 1st IFAC World Congress*, Moscow, 1961, pp. 742–751.
- [12] E. Kaszkurewicz and H. Liu, "Stability of nonlinear systems," *Automatica*, vol. 15, pp. 609–614, 1979.
- [13] D. Efimov and A. Aleksandrov, "Robust stability analysis and implementation of Persidskii systems," *58th IEEE Conference on Decision and Control (CDC)*, 2019.

- [14] W. Mei, D. Efimov, and R. Ushirobira, "Feedback synchronization in Persidskii systems," *Proc. 21th IFAC World Congress*, 2020.
- [15] V. Kolmanovskiy and V. Nosov, *Stability of functional differential equations*. San Diego: CA: Academic, 1986.
- [16] P. Pepe and Z.-P. Jiang, "A Lyapunov-Krasovskii methodology for ISS and iISS of time-delay systems," *Systems & Control Letters*, vol. 55, no. 12, pp. 1006–1014, 2006.
- [17] Y. Ebihara, D. Peaucelle, and D. Arzelier, *S-Variable Approach to LMI-Based Robust Control*, ser. Communications and Control Engineering. London: Springer London, 2015.