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► **To cite this version:**

Laurent Bourgeois, Sonia Fliss, Jean-François Fritsch, Christophe Hazard, Arnaud Recoquillay. Scattering in a partially open waveguide: the forward problem. 2021. hal-03407434v1

**HAL Id: hal-03407434**

**<https://inria.hal.science/hal-03407434v1>**

Preprint submitted on 28 Oct 2021 (v1), last revised 20 Dec 2022 (v2)

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# Scattering in a partially open waveguide: the forward problem

Laurent Bourgeois, Sonia Fliss, Jean-François Fritsch,  
Christophe Hazard and Arnaud Recoquillay

October 28, 2021

## Abstract

This paper is dedicated to an acoustic scattering problem in a two-dimensional partially open waveguide, in the sense that the left part of the waveguide is closed, that is with a *bounded* cross-section, while the right part is bounded in the transverse direction by some Perfectly Matched Layers that mimic the situation of an open waveguide, that is with an *unbounded* cross-section. We prove well-posedness of such scattering problem and exhibit the asymptotic behaviour of the solution in the longitudinal direction with the help of the Kondratiev approach. Having in mind the numerical computation of the solution, we also propose some transparent boundary conditions in such longitudinal direction, based on Dirichlet-to-Neumann operators. After proving that such artificial conditions actually enable us to approximate the exact solution, some numerical experiments illustrate the quality of such approximation.

## 1 Introduction

Elongated structures such as cables, pipes and bars are widely used in the field of civil engineering. A typical configuration is the case of a cable which is partially embedded into another elastic medium, the cable being for example made of steel while the surrounding medium is made of concrete. It may happen that some defects appear in the inaccessible part of the cable, namely the embedded part of the cable or at the interface between the cable and the surrounding medium. Non Destructive Testing (NDT) is commonly used to image defects in such structures. NDT consists in solving an inverse problem: one produces several incident waves coming from the accessible part of the cable, measures the resulting scattered waves due to the presence of the defect, then try to find the defect by using those multistatic data. Before addressing the inverse scattering problem, which will be the objective of a forthcoming article, it is important to analyze the forward problem, which consists for example in finding the scattering response of the partially embedded cable to an incident field coming from the part of the cable which is free. Such configuration can be seen as a junction between a closed waveguide and an open waveguide (i.e., with a *bounded* and an *unbounded* cross-section, respectively), this open waveguide being itself composed of a core and a sheath. We assume that the defect lies within the core of the open waveguide to simplify the presentation, having in mind that the case of a defect within the sheath would be treated the same way.

In order to simplify the analysis, we restrict ourselves to a scalar two-dimensional problem which models an isotropic antiplane shear situation, in other words we consider the sole SH waves. In addition, we restrict to the time-harmonic waves at a given frequency  $\omega$  assuming a  $e^{-i\omega t}$ -dependence with respect to time  $t$ . The configuration that we consider is the following. The closed part of the waveguide occupies the domain  $(-\infty, 0) \times (-h, h)$ , with  $h > 0$ , while the open part of the waveguide occupies the domain  $(0, +\infty) \times \mathbb{R}$ . The shear modulus and the density are denoted  $\mu$  and  $\rho$ , respectively, while the speed  $c$  and the wave number  $k$  are defined by

$$c := \sqrt{\frac{\mu}{\rho}} \quad \text{and} \quad k := \frac{\omega}{c}.$$

In the closed part of the waveguide, the shear modulus and the density are given by  $(\mu, \rho) = (\mu_0, \rho_0)$ , where  $\mu_0$  and  $\rho_0$  are given positive constants. In the open part of the waveguide, the shear modulus and the density are piecewise constants, more precisely  $(\mu, \rho) = (\mu_0, \rho_0)$  in the domain  $(0, +\infty) \times (-h, h)$  (the core) and  $(\mu, \rho) = (\mu_\infty, \rho_\infty)$  in the domain  $(0, +\infty) \times ((-\infty, -h) \cup (h, +\infty))$  (the sheath), where

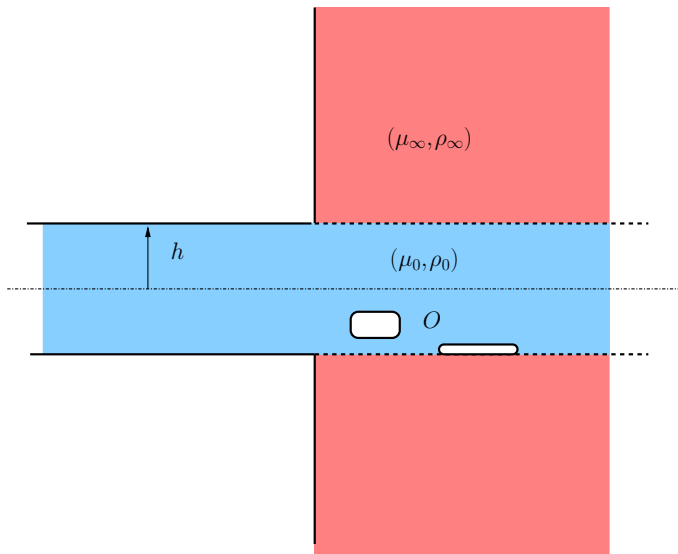


Figure 1: Original configuration

$\mu_\infty$  and  $\rho_\infty$  are given positive constants. Denoting the defect  $O$ , we have represented such original configuration in Figure 1.

Studying well-posedness of a scattering problem in this configuration is a challenging task due to the definition of radiation conditions in the open part of the waveguide. In this vein, well-posedness for a similar case, that is a junction between two open waveguides, is established in [Bonnet-Ben Dhia et al., 2011]. Such result requires some complicated tools and lots of technicalities. More precisely, the behaviour at infinity is specified by means of modal radiation conditions (based on generalized Fourier transforms adapted to both half waveguides) which extend the classical conditions used for closed waveguides. The main difference is that they involve a continuum of modes. The main originality of [Bonnet-Ben Dhia et al., 2011] concerns the proof of uniqueness which relies on an argument of analyticity with respect to the generalized Fourier variable. Another angle of attack is chosen in the present paper. More precisely, the surrounding unbounded medium will be modeled by transverse Perfectly Matched Layers of finite depth, which is a classical way of replacing the true open waveguide by an equivalent closed one made of an artificial material (see [Berenger, 1994]). The justification of such transverse PMLs is not addressed in the present paper, but some arguments given in [Chandler-Wilde & Monk, 2009] for a similar problem could probably be adapted to our configuration. The physical medium will occupy the domain  $(0, +\infty) \times (-h_{\text{in}}, h_{\text{in}})$ , with  $h_{\text{in}} \geq h$ , while the PMLs will occupy the domain  $(0, +\infty) \times ((-h_{\text{out}}, -h_{\text{in}}) \cup (h_{\text{in}}, h_{\text{out}}))$ , with  $h_{\text{out}} > h_{\text{in}}$ . The PMLs involve a complex-valued function  $\alpha$  such that  $\alpha = 1$  in the physical domain and  $\alpha$  having well chosen real and imaginary parts in the PMLs in order to absorb waves (see section 2). Let us introduce the domains  $\Omega^- := (-\infty, 0) \times (-h, h)$ ,  $\Omega^+ := (0, +\infty) \times (-h_{\text{out}}, h_{\text{out}})$  and  $\Omega := \Omega^- \cup \Sigma_0 \cup \Omega^+$ , with  $\Sigma_0 := \{0\} \times (-h, h)$ . On Figure 2, we have represented the configuration with PMLs, which will be the configuration of interest throughout the paper. The scattering problem we actually consider is the following: given an incident field  $u^i$ , which in practice is a mode coming from the left, find the total field  $u$  in the complete waveguide  $\Omega$  such that

$$\begin{cases} Pu = 0 & \text{in } \Omega \setminus \overline{O}, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \\ u = 0 & \text{on } \partial O, \\ u - u^i \text{ is outgoing,} \end{cases} \quad (1.1)$$

where the differential operator  $P$  is defined by

$$P := -\partial_y(\alpha\mu\partial_y\cdot) - \frac{\mu}{\alpha}\partial_{xx}\cdot - \frac{\mu}{\alpha}k^2 \quad (1.2)$$

and  $\nu$  is the outward unit normal vector to  $\partial\Omega$ . The last line of the system (1.1) is a radiation condition applied to the scattered field  $u^s := u - u^i$  that will be made precise in the following.

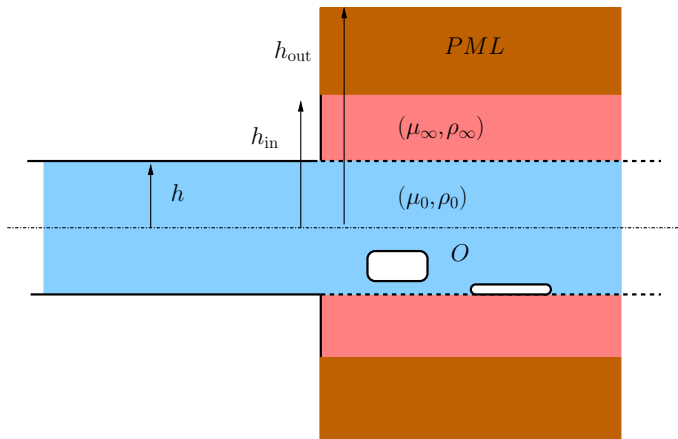


Figure 2: Configuration with PMLs

Our paper has mainly three objectives: the first one is to establish the well-posedness of the scattering problem (1.1) in an appropriate functional space, the second one is to study the asymptotic behaviour of the solution at infinity, and the third one is to introduce some transparent boundary conditions along the axis of the waveguide and corresponding approximations in order to compute numerical solutions. In the case of a homogeneous closed waveguide, all these questions can be addressed using the fact that the underlying transverse operator is self-adjoint, and a complete orthonormal basis of eigenvectors can be used to derive a modal decomposition of all fields. In the presence of PMLs, the corresponding transverse operator is not self-adjoint any more, which prevents us from exploiting the same technique. As a result, our forthcoming analysis of the scattering problem relies on the theory exposed in [Kondratiev, 1967] (see also [Maz'ya & Plamenevskii, 1977, Nazarov & Plamenevskii, 1994, Kozlov et al., 1997, Kozlov et al., 2001]). In the first part of the analysis, we consider a straight closed waveguide with PMLs which is unperturbed, that is without any defect. The idea is to introduce weighted Sobolev spaces and to apply the Fourier transform in the unbounded direction in order to handle one dimensional problems in a bounded interval, those problems being naturally parametrized by the Fourier variable. It should be noted that in the waveguide with PMLs, the analysis is considerably simplified by the absence of propagating modes, which results from an assumption on  $k_0$  and  $k_\infty$  (see section 2). In the second part, well-posedness of the scattering problem (1.1) in the junction of the two half-waveguides in the presence of the obstacle is obtained by using Dirichlet-to-Neumann operators in order to reduce the problem to an equivalent problem set in a bounded domain. The proper definition of these DtN operators relies on the first part of the analysis. With the help of the residue theorem, we additionally obtain a precise asymptotic behaviour of the solution at infinity. The Kondratiev approach is used in an abstract framework in [Nazarov & Plamenevskii, 1991], in [Nazarov, 2013] for an elastic waveguide, in [Bourgeois et al., 2019] for a 2D waveguide governed by a Kirchhoff-Love model, and in other different situations in [Nazarov, 1982, Nazarov & Taskinen, 2011, Bonnet-Ben Dhia et al., 2013, Bonnet-Ben Dhia & Chesnel, 2013].

Our article is organized as follows. In section 2, we give a brief review of the spectral theory of closed and open waveguides. Section 3 is dedicated to an analysis of the eigenvalues and eigenfunctions of the non-selfadjoint transverse operator in the presence of the PMLs which naturally comes into play in the Kondratiev approach. In section 4, we prove existence and uniqueness for the forward problem in a uniform unperturbed waveguide and show how the Kondratiev approach enables us to specify the behaviour at infinity of the solution. We prove well-posedness of the forward problem (1.1) of interest, that is the partially embedded closed waveguide containing a defect, in section 5. The behaviour at infinity of the solution is specified as well. In section 6, we introduce some transparent boundary conditions in the direction of the waveguide and prove they enable us to approximate the true solution in the unbounded domain. The proof relies on the previous result giving the behaviour at infinity of such true solution. Lastly, some numerical experiments are presented in section 7.

## 2 A short review of spectral theory of closed and open waveguides

This section essentially summarizes some results borrowed from [Goursaud, 2010]. Such section aims at describing the modes of closed and open waveguides, in particular to recall the effect of the infinite and truncated PMLs on the modes of open waveguides. It will help us to classify the computed modes in the numerical section.

### 2.1 The closed waveguide

Let us denote  $\tilde{\Omega} := \mathbb{R} \times I$ ,  $I := (-h, h)$ , a closed waveguide which is characterized by the constant material properties  $(\mu_0, \rho_0)$ , the celerity being  $c_0 := \sqrt{\mu_0/\rho_0}$  and the wave number being  $k_0 := \omega/c_0$ . A generic point in the waveguide has coordinates  $(x, y)$ , where  $x$  is the coordinate of the unbounded direction of the waveguide, while  $y$  is the coordinate in the transverse bounded section  $I$ . We consider the solutions  $u$  of the form  $u(x, y) = e^{\lambda x} \varphi(y)$  for some  $\lambda \in \mathbb{C}$  to the problem

$$\begin{cases} -\Delta u - k_0^2 u = 0 & \text{in } \tilde{\Omega} \\ \partial_\nu u = 0 & \text{on } \partial\tilde{\Omega}, \end{cases} \quad (2.3)$$

where  $\nu$  is the outward unit normal vector to  $\partial\tilde{\Omega}$ . We are then naturally led to investigate the numbers  $\gamma := \lambda^2$  and the corresponding functions  $\varphi$  which are respectively eigenvalues and eigenvectors of the transverse unbounded operator  $\tilde{L}: D(\tilde{L}) \subset L^2(I) \rightarrow L^2(I)$  defined by

$$\begin{cases} \tilde{L}\varphi := -d_{yy}\varphi - k_0^2\varphi \\ D(\tilde{L}) := \{\varphi \in H^2(I), d_y\varphi(-h) = d_y\varphi(h) = 0\}, \end{cases}$$

which is self-adjoint and has a compact resolvent. If we choose to normalize the eigenfunctions in  $L^2(I)$ , it is readily seen that pairs  $(\lambda, \varphi)$  are given by

$$\pm \tilde{\lambda}_n, \quad \tilde{\lambda}_n := i\tilde{\beta}_n, \quad \tilde{\beta}_n := -i\mathbb{R}^+ \sqrt{k_0^2 - \frac{n^2\pi^2}{4h^2}}, \quad n \in \mathbb{N}, \quad (2.4)$$

$$\tilde{\varphi}_0(y) := \frac{1}{\sqrt{2h}}, \quad \tilde{\varphi}_n(y) := \frac{1}{\sqrt{h}} \cos\left(\frac{n\pi}{2h}(y+h)\right), \quad n \in \mathbb{N}^*. \quad (2.5)$$

Given  $\alpha \in \mathbb{C}^*$ , the complex square root  $\alpha^{\mathbb{R}^+} \sqrt{\cdot}$  is defined for  $z \in \mathbb{C} \setminus \alpha\mathbb{R}^+$  by

$$\alpha^{\mathbb{R}^+} \sqrt{z} := \sqrt{|z|} e^{i \arg_{\alpha\mathbb{R}^+}(z)/2}, \quad \theta - 2\pi < \arg_{\alpha\mathbb{R}^+}(z) < \theta,$$

where  $\theta$  is the argument of  $\alpha$  in  $(0, 2\pi]$ .

It is important to note that the  $\tilde{\varphi}_n$ 's form a complete orthonormal basis of  $L^2(I)$ . We assume from now on a restriction on the wave number  $k_0$ .

**Assumption 2.1.** For all  $n \in \mathbb{N}$ ,  $n\pi/2h \neq k_0$ .

Assumption 2.1 amounts to saying that the  $\tilde{\lambda}_n$ 's never vanish. This implies in particular that there exists some  $\tilde{N} \in \mathbb{N}^*$  such that for  $n = 0, \dots, \tilde{N} - 1$ ,  $\tilde{\lambda}_n = i\tilde{\beta}_n$ , with  $\tilde{\beta}_n > 0$  while for  $n = \tilde{N}, \dots, +\infty$ ,  $\tilde{\lambda}_n < 0$ . The corresponding solutions  $u(x, y) = e^{\lambda x} \varphi(y)$  to the problem (2.3) are given by

$$\tilde{w}_n^\pm(x, y) := e^{\pm \tilde{\lambda}_n x} \tilde{\varphi}_n(y) \quad (2.6)$$

and are called the *modes*. For  $n = 0, \dots, \tilde{N} - 1$ , the  $\tilde{w}_n^+$  propagate from the left to the right, the  $\tilde{w}_n^-$  propagate from the right to the left. They are called the *guided modes*. For  $n = \tilde{N}, \dots, +\infty$ , the  $\tilde{w}_n^+$  are exponentially decaying from the left to the right, the  $\tilde{w}_n^-$  are exponentially decaying from the right to the left. They are called the *evanescent modes*.

The top-left part of Figure 3 shows the location of the spectrum of  $\tilde{L}$  not in the natural spectral variable  $\gamma$ , but rather using its complex square root  $\lambda$ , which acts as a complex wavenumber in the  $x$ -direction. This square root, which yields two complex numbers opposite to each other, leads to a symmetry of the spectrum with respect to  $\lambda = 0$ , which reflects the symmetry of the waveguide with respect to  $x = 0$ . For us, the regions of interest in this complex  $\lambda$ -plane are the top-left and bottom-right quarter-planes, which correspond respectively to right-going and left-going modes in our conventions.

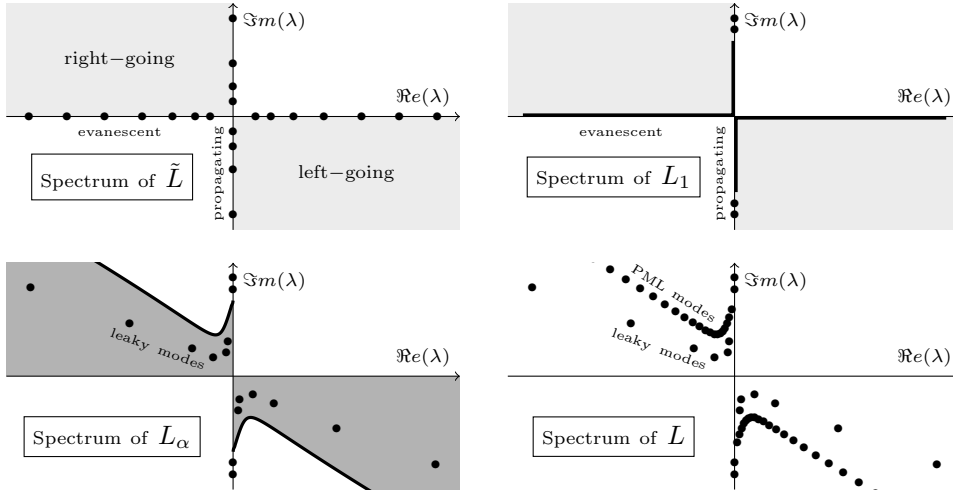


Figure 3: Location of the spectrum of  $\tilde{L}$ ,  $L_1$ ,  $L_\alpha$  and  $L$  in the complex  $\lambda$ -plane. Dots represent the discrete spectrum, whereas thick lines stands for essential spectrum.

## 2.2 The open waveguide

Let us consider a stratified medium also called an open waveguide which occupies the whole space  $\mathbb{R}^2$  and which is characterized by the piecewise constant material properties  $(\mu, \rho)$ , with  $(\mu, \rho) = (\mu_0, \rho_0)$  in  $\mathbb{R} \times (-h, h)$  (called the core of the waveguide) and  $(\mu, \rho) = (\mu_\infty, \rho_\infty)$  in  $\mathbb{R} \times ((-\infty, -h) \cup (h, +\infty))$  (called the sheath of the waveguide). Let us denote the associated celerities  $c_0$  and  $c_\infty$  as well as the associated wave numbers  $k_0$  and  $k_\infty$ . In some sense, we have embedded the previous closed waveguide in an infinite surrounding medium. Again, we consider the solutions  $u$  of the form  $u(x, y) = e^{\lambda x} \varphi(y)$  for some  $\lambda \in \mathbb{C}$  to the following problem in  $\mathbb{R}^2$ :

$$-\partial_y(\mu \partial_y u) - \mu \partial_{xx} u - \omega^2 \rho u = 0. \quad (2.7)$$

We then search the numbers  $\gamma := \lambda^2$  and the functions  $\varphi$  as the eigenvalues and eigenvectors of the unbounded self-adjoint transverse operator  $L_1 : D(L_1) \subset L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  defined by

$$\begin{cases} L_1 \varphi := -\frac{1}{\mu} d_y(\mu d_y \varphi) - k^2 \varphi \\ D(L_1) := \{\varphi \in H^1(\mathbb{R}), \mu d_y \varphi \in H^1(\mathbb{R})\}, \end{cases}$$

where  $k^2 = \omega^2 \mu^{-1} \rho$ . It can be shown (see [Goursaud, 2010]) that the spectrum of the operator  $L_1$  is divided into a discrete spectrum composed of a finite number of eigenvalues  $\gamma = \lambda^2$  in  $(-k_{\max}^2, -k_\infty^2)$ , where  $k_{\max} := \max(k_0, k_\infty)$  and an essential spectrum, which is equal to  $[-k_\infty^2, +\infty)$ . For some  $\gamma = \lambda^2$  in the discrete spectrum, by setting  $\gamma = (i\beta)^2$  with  $\beta \in (k_\infty, k_{\max})$ , the corresponding function  $w^\pm(x, y) = e^{\pm i\beta x} \varphi(y)$ , where  $\varphi \in D(L_1)$  is an eigenvector of  $L_1$  associated with the eigenvalue  $\gamma$ , is called a *guided mode* as in the case of a closed waveguide. Indeed, it is localized in the core of the open waveguide. For some  $\gamma = \lambda^2$  in the essential spectrum, the modes are given by  $w^\pm(x, y) = e^{\pm i\beta x} \Phi(y)$  (propagating) or  $w^\pm(x, y) = e^{\mp \lambda x} \Phi(y)$  (evanescent), with  $\beta, \lambda > 0$ , where  $\Phi$  is no more an eigenvector (it does not belong to  $L^2(\mathbb{R})$ ), but a so-called generalized eigenfunction. Such modes are called *radiation modes*, because they are oscillating at infinity in the transverse direction.

In the complex  $\lambda$ -plane, the top-right part of Figure 3 shows the location of the spectrum of  $L_1$ . As for the closed waveguide, the part of the spectrum located on the imaginary axis corresponds to propagating modes in the  $x$ -direction. But now, it includes a discrete part (which may be empty) and a continuous one (which is never empty). On the other hand, on the real axis associated to evanescent modes, there is only essential spectrum.

**Assumption 2.2.** *In our paper we will assume that  $c_0 > c_\infty$  or in other words  $k_0 < k_\infty$ , which implies that  $k_{\max} = k_\infty$ . As a consequence, the discrete spectrum is empty: there are no guided modes in the open waveguide, only radiation modes.*

### 2.3 Adding infinite PMLs

Let us consider the previous open waveguide and let us introduce transverse infinite Perfectly Matched Layers. The PMLs play the role of a fictitious absorbing medium which replaces the physical medium outside a strip of finite thickness including the core of the waveguide. It exactly mimics the behaviour of the infinite non-dissipative sheath, in the sense that the interface between the physical medium and the PMLs does not reflect the outgoing waves. As a consequence, compared with the previous open waveguide (without PMLs), the acoustic field in this new artificial open waveguide is exactly the same in the physical medium (the bounded region between both PMLs), whereas *in the PMLs, the radiating field is transformed into an evanescent one*. The latter property makes PMLs a very powerful tool for both theory and numerics. Let us first consider infinite PMLs which occupy the region  $|y| > h_{\text{in}}$  with  $h_{\text{in}} \geq h$ . It consists in transforming the partial derivative  $\partial_y \cdot$  in equation (2.7) into the weighted partial derivative  $\alpha \partial_y \cdot$ , where  $\alpha : \mathbb{R} \rightarrow \mathbb{C}$  is such that

$$\alpha(y) = \begin{cases} 1 & \text{if } |y| < h_{\text{in}}, \\ \alpha_\infty & \text{otherwise,} \end{cases} \quad (2.8)$$

and  $\alpha_\infty$  is a complex constant such that

$$-\frac{\pi}{2} < \arg(\alpha_\infty) < 0. \quad (2.9)$$

As a consequence, let us now consider the solutions  $u$  of the form  $u(x, y) = e^{\lambda x} \varphi(y)$  to the following problem in  $\mathbb{R}^2$ :

$$-\alpha \partial_y (\alpha \mu \partial_y u) - \mu \partial_{xx} u - \omega^2 \rho u = 0,$$

which amounts to searching the numbers  $\gamma := \lambda^2$  and the functions  $\varphi$  respectively as eigenvalues and eigenvectors of the unbounded non self-adjoint transverse operator  $L_\alpha : D(L_\alpha) \subset L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  defined by

$$\begin{cases} L_\alpha \varphi := -\frac{\alpha}{\mu} d_y (\alpha \mu d_y \varphi) - k^2 \varphi \\ D(L_\alpha) := \{\varphi \in H^1(\mathbb{R}), \alpha \mu d_y \varphi \in H^1(\mathbb{R})\}. \end{cases}$$

**Remark 2.1.** *Note that if  $\alpha$  is identically equal to 1, then the operator  $L_\alpha$  coincides with the operator  $L_1$ .*

It can be shown that the essential spectrum of  $L_\alpha$  is  $-k_\infty^2 + \alpha_\infty^2 \mathbb{R}^+$ , that is the PMLs have the effect to rotate the essential spectrum of  $L_1$  of an angle  $2\arg(\alpha_\infty)$  in the complex plane with respect to the point  $-k_\infty^2$ . The effect of this rotation in the  $\lambda$ -plane is illustrated by Figure 4.

As concerns the discrete spectrum, it is composed of two parts. On the one hand, every  $\lambda$  in the discrete spectrum of  $L_1$  also belongs to the discrete spectrum of  $L_\alpha$  (recall that Assumption 2.2 implies that for us, there is no such  $\lambda$ ). On the other hand, the discrete spectrum may contain a countable set of eigenvalues in the domain  $\{\gamma \in \mathbb{C}, 2\arg(\alpha_\infty) < \arg(k_\infty^2 + \gamma) \leq 0\}$ . The function  $w(x, y) = e^{\lambda x} \varphi(y)$ , where  $\gamma = \lambda^2$  belongs to that set and  $\varphi$  is a corresponding eigenvector of  $L_\alpha$ , is called a *leaky mode*. A coarse interpretation of those leaky modes is that they mimic the role of the radiation modes in the absence of the PMLs. They actually teach us about the preferred ways for the energy to radiate in the sheath of the waveguide: they are kinds of “harmonic states of leakage”.

Coming back to Figure 4, we can understand that the motion of the essential spectrum of  $L_\alpha$  as  $\arg(\alpha_\infty)$  decreases from 0 *unveils* a “hidden” region of the complex plane (Riemann sheet) where the leaky modes live. This region is represented in gray in the bottom-left part of Figure 3, which also shows the unveiled leaky modes.

### 2.4 Truncating PMLs

For numerical purpose, we need to consider PMLs of finite thickness, which will from now on occupy the domain  $\mathbb{R} \times ((-h_{\text{out}}, -h_{\text{in}}) \cup (h_{\text{in}}, h_{\text{out}}))$  for  $h_{\text{out}} > h_{\text{in}}$ . We have then obtained a closed waveguide  $\Omega_{\text{out}} := \mathbb{R} \times I_{\text{out}}$ , with  $I_{\text{out}} := (-h_{\text{out}}, h_{\text{out}})$ , on the boundary of which we arbitrarily impose a Neumann boundary condition (the leaky modes within the PMLs are exponentially decaying in the transverse direction). The shear modulus and the density are given in the truncated domain  $\Omega_{\text{out}}$  by

$$(\mu, \rho)(y) := \begin{cases} (\mu_0, \rho_0) & \text{if } |y| < h, \\ (\mu_\infty, \rho_\infty) & \text{if } h < |y| < h_{\text{out}}, \end{cases} \quad (2.10)$$

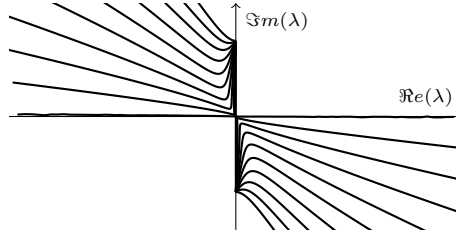


Figure 4: Successive locations of the essential spectrum of  $L_\alpha$  as  $\arg(\alpha_\infty)$  varies from 0 (two symmetric infinite “L”s, located on the real and imaginary axes) to  $-\pi/2$ .

where the PML complex function  $\alpha$  is defined in  $\Omega_{\text{out}}$  by

$$\alpha(y) := \begin{cases} 1 & \text{if } |y| < h_{\text{in}}, \\ \alpha_\infty & \text{if } h_{\text{in}} < |y| < h_{\text{out}}, \end{cases} \quad (2.11)$$

where  $\alpha_\infty$  is a complex constant satisfying (2.9). We hence consider the solutions  $u$  of the form  $u(x, y) = e^{\lambda x} \varphi(y)$  to the problem

$$\begin{cases} -\alpha \partial_y (\alpha \mu \partial_y u) - \mu \partial_{xx} u - \omega^2 \rho u = 0 & \text{in } \Omega_{\text{out}} \\ \partial_\nu u = 0 & \text{on } \partial \Omega_{\text{out}}, \end{cases} \quad (2.12)$$

where  $\nu$  is the outward unit normal vector to  $\partial \Omega_{\text{out}}$ . This amounts to searching the numbers  $\gamma := \lambda^2$  and the functions  $\varphi$  respectively as eigenvalues and eigenvectors of the unbounded transverse operator  $L : D(L) \subset L^2(I_{\text{out}}) \rightarrow L^2(I_{\text{out}})$  defined by

$$\begin{cases} L\varphi := -\frac{\alpha}{\mu} d_y (\alpha \mu d_y \varphi) - k^2 \varphi \\ D(L) := \{\varphi \in H^1(I_{\text{out}}), \alpha \mu d_y \varphi \in H^1(I_{\text{out}}), d_y \varphi(-h_{\text{out}}) = d_y \varphi(h_{\text{out}}) = 0\}. \end{cases} \quad (2.13)$$

We easily check that  $L$  is a non self-adjoint operator if  $\alpha_\infty \in \mathbb{C} \setminus \mathbb{R}$  and has a compact resolvent. Therefore, it has complex discrete eigenvalues. We have to distinguish two kinds of modes: those which correspond to the approximation of the leaky modes due to the truncation of the PMLs, which will still be called the *leaky modes*, and those which correspond to the discretization of the continuous spectrum, which will be called the *PML modes*. The leaky modes are localized in the physical part of the waveguide and have an intrinsic meaning, while the PML modes are localized in the PMLs and depend on the parameters of such PMLs.

The bottom-right part of Figure 3 illustrates the two kinds of modes. The leaky modes are closed to those of the infinite PMLs (the larger the PMLs, the better the approximation), whereas the PML modes compose a discrete approximation of the continuous spectrum of the infinite PMLs (the larger the PMLs, the denser the approximation).

**Remark 2.2.** *The function  $\alpha(y)$  in the PMLs given by (2.8) and (2.9) has a jump at  $y = \pm h_{\text{in}}$ . It hence defines the so-called abrupt PMLs. An alternative choice for the function  $\alpha$ , already used in [Nguyen et al., 2015], is given for  $|y| \in (h_{\text{in}}, h_{\text{out}})$ , by*

$$\alpha(y) = \frac{1}{1 + b(|y| - h_{\text{in}})^2 / h_{\text{out}}^2}, \quad (2.14)$$

with  $b \in \mathbb{C}$  satisfying  $\Re(b) > 0$  and  $\Im(b) > 0$ . It is important to note that all the theoretical results of the paper still hold if we choose the smooth function  $\alpha$  given by (2.14) instead of the one given by (2.11).

### 3 The case of truncated PMLs: description of the modes

In this section, we specify some important properties of the pairs  $(\lambda, \varphi)$ ,  $\varphi \neq 0$ , which satisfy (2.12). Rather than working with the unbounded operator  $L$  defined by (2.13), we now introduce the symbol



$\mathcal{L}(\lambda): \mathbb{H}^1(I_{\text{out}}) \rightarrow \mathbb{H}^1(I_{\text{out}})^*$ , where  $\mathbb{H}^1(I_{\text{out}})^*$  is the topological dual of  $\mathbb{H}^1(I_{\text{out}})$ , which is defined by

$$\langle \mathcal{L}(\lambda)\varphi, \psi \rangle_{I_{\text{out}}} := \int_{I_{\text{out}}} \left( \alpha \mu d_y \varphi d_y \psi - \frac{\mu}{\alpha} (\lambda^2 + k^2) \varphi \psi \right) dy, \quad \forall \varphi, \psi \in \mathbb{H}^1(I_{\text{out}}), \quad (3.15)$$

where the bracket  $\langle \cdot, \cdot \rangle_{I_{\text{out}}}$  means duality between  $\mathbb{H}^1(I_{\text{out}})$  and  $\mathbb{H}^1(I_{\text{out}})^*$ . This choice is dictated by the Kondratiev approach we adopt in the next section. However, the link between the operators  $L$  and  $\mathcal{L}$  is straightforward: finding the pairs  $(\lambda, \varphi) \in \mathbb{C} \times \mathbb{H}^1(I_{\text{out}}) \setminus \{0\}$  such that

$$\mathcal{L}(\lambda)\varphi = 0 \quad (3.16)$$

is equivalent to finding the eigenvalues  $\gamma = \lambda^2$  and eigenfunctions  $\varphi$  of the operator  $L$  defined by (2.13). We denote by  $\Lambda$  the discrete set of  $\lambda \in \mathbb{C}$  for which there exists a non zero  $\varphi \in \mathbb{H}^1(I_{\text{out}})$  such that  $(\lambda, \varphi)$  satisfies problem (3.16).

First of all, we establish that the spectrum  $\Lambda$  does not intersect the real and imaginary axes.

**Lemma 1.** *We have*

$$\Lambda \cap \{\lambda \in \mathbb{C}, \Re e(\lambda) \Im m(\lambda) = 0\} = \emptyset.$$

*Proof.* Let us assume that  $\lambda \in \Lambda$ . There exists  $\varphi \in \mathbb{H}^1(I_{\text{out}})$ ,  $\varphi \neq 0$ , such that for all  $\psi \in \mathbb{H}^1(I_{\text{out}})$ ,

$$\langle \mathcal{L}(\lambda)\varphi, \psi \rangle_{I_{\text{out}}} = 0.$$

Choosing  $\psi = \bar{\varphi}$ , we have in particular

$$\int_{I_{\text{out}}} \left( \alpha \mu |d_y \varphi|^2 - \frac{\mu}{\alpha} (\lambda^2 + k^2) |\varphi|^2 \right) dy = 0. \quad (3.17)$$

Let us show the result by contradiction. Assume first that  $\lambda \in \mathbb{R}$ . This implies that  $\lambda^2 + k^2 > 0$  in  $I_{\text{out}}$ . Since  $\arg(\alpha) \in (-\pi/2, 0]$  in  $I_{\text{out}}$  from (2.8) and (2.9), we have both  $\Im m(\alpha) \leq 0$  and  $\Im m(-1/\alpha) \leq 0$ , so that by taking the imaginary part in (3.17), we get

$$\Im m(\alpha) \mu |d_y \varphi|^2 + \mu (\lambda^2 + k^2) \Im m\left(-\frac{1}{\alpha}\right) |\varphi|^2 = 0 \quad \text{in } I_{\text{out}}.$$

For  $h_{\text{in}} < |y| < h_{\text{out}}$  (see Figure 2 for notations), we have  $\Im m(\alpha) < 0$  and  $\Im m(-1/\alpha) < 0$ , so that  $\varphi = 0$  in the domain  $h_{\text{in}} < |y| < h_{\text{out}}$ , and  $\varphi = 0$  in  $I_{\text{out}}$  by a unique continuation argument. We have obtained a contradiction.

Now let us assume that  $\lambda \in i\mathbb{R}$ , so that  $\lambda^2 = -|\lambda|^2$ . We have to discuss several cases. We recall that  $k_\infty > k_0$  (see Assumption 2.2).

- Case  $|\lambda| < k_0$ : we have  $\lambda^2 + k^2 > 0$  in  $I_{\text{out}}$ , so that we can conclude the same way as before.
- Case  $k_0 \leq |\lambda| < k_\infty$ : we have from (3.17)

$$\begin{aligned} & \int_{-h_{\text{in}}}^{h_{\text{in}}} (\mu |d_y \varphi|^2 - \mu (-|\lambda|^2 + k^2) |\varphi|^2) dy \\ & + \int_{I_{\text{out}} \cap \{|y| > h_{\text{in}}\}} \left( \alpha \mu_\infty |d_y \varphi|^2 - \frac{\mu_\infty}{\alpha} (-|\lambda|^2 + k_\infty^2) |\varphi|^2 \right) dy = 0. \end{aligned}$$

The first integral is purely real, so that by taking the imaginary part of the above identity and using the fact that  $-|\lambda|^2 + k_\infty^2 > 0$ , we conclude by the same reasoning as before.

- Case  $|\lambda| \geq k_\infty$ . Starting from (3.17) and taking the real part, we obtain

$$\begin{aligned} & \int_{-h}^h (\mu_0 |d_y \varphi|^2 - \mu_0 (-|\lambda|^2 + k_0^2) |\varphi|^2) dy \\ & + \int_{I_{\text{out}} \cap \{|y| > h\}} \left( \Re e(\alpha) \mu_\infty |d_y \varphi|^2 - \mu_\infty \Re e\left(\frac{1}{\alpha}\right) (-|\lambda|^2 + k_\infty^2) |\varphi|^2 \right) dy = 0. \end{aligned}$$

The two functions inside the two integrals are non negative since  $|\lambda| \geq k_\infty$ ,  $\Re e(\alpha) \geq 0$  and  $\Re e(1/\alpha) \geq 0$ . We hence obtain that  $\varphi = 0$  in  $(-h, h)$ , and then  $\varphi = 0$  in  $I_{\text{out}}$  by a unique continuation argument, which is a contradiction.

The proof is complete.  $\square$

We have the following result concerning the geometric multiplicity of the eigenvalues  $\lambda \in \Lambda$ , which is defined following [Kozlov et al., 1997] (see §5.1.1) by  $\dim \text{Ker}(\mathcal{L}(\lambda))$ , where  $\text{Ker}(\mathcal{L}(\lambda)) = \{\varphi \in H^1(I_{\text{out}}), \mathcal{L}(\lambda)(\varphi) = 0\}$ .

**Lemma 2.** *For all  $\lambda \in \Lambda$ , we have  $\dim \text{Ker}(\mathcal{L}(\lambda)) = 1$ .*

*Proof.* We observe that there is a unique function  $\Phi \in \text{Ker}(\mathcal{L}(\lambda))$  such that  $\Phi(-h_{\text{out}}) = 1$ . Indeed, denoting  $\Phi_1 := \Phi|_{(-h_{\text{out}}, -h_{\text{in}})}$ ,  $\Phi_1$  satisfies a second-order ordinary differential equation in the interval  $(-h_{\text{out}}, -h_{\text{in}})$  with constant coefficients and initial conditions  $\Phi_1(-h_{\text{out}}) = 1$  and  $d_y \Phi_1(-h_{\text{out}}) = 0$ . Hence  $\Phi_1$  is uniquely defined in  $(-h_{\text{out}}, -h_{\text{in}})$ . Next, denoting  $\Phi_2 := \Phi|_{(-h_{\text{in}}, -h)}$ ,  $\Phi_2$  also satisfies a second-order ODE in the interval  $(-h_{\text{in}}, -h)$  with initial conditions which are given by the transmission conditions at  $y = -h_{\text{in}}$  between  $\Phi_1$  and  $\Phi_2$  as well as  $d_y \Phi_1$  and  $d_y \Phi_2$ . Hence  $\Phi_2$  is uniquely defined in  $(-h_{\text{in}}, -h)$ . By reproducing this reasoning, we conclude that  $\Phi$  is also uniquely defined in  $(-h, h)$ ,  $(h, h_{\text{in}})$  and  $(h_{\text{in}}, h_{\text{out}})$ . That  $d_y \Phi(h_{\text{out}}) = 0$  is a consequence of the fact that  $\Phi \in \text{Ker}(\mathcal{L}(\lambda))$ . Then for any  $\varphi \in \text{Ker}(\mathcal{L}(\lambda))$ , by linearity we have  $\varphi = \varphi(-h_{\text{out}})\Phi$ , which implies that  $\dim \text{Ker}(\mathcal{L}(\lambda)) = 1$ .  $\square$

In what follows, we refer to the notion of *Jordan chain* of the operator pencil  $\mathcal{L}(\lambda)$  corresponding to a particular eigenvalue  $\lambda$  as well as the notion of *algebraic multiplicity* of  $\lambda$ . All these definitions are given in [Kozlov et al., 1997] (see §5.1.1). We have the following result concerning the algebraic multiplicity of the eigenvalues  $\lambda \in \Lambda$ .

**Lemma 3.** *Let us consider  $\lambda \in \Lambda$  and  $\varphi \in \text{Ker}(\mathcal{L}(\lambda))$  such that*

$$\int_{I_{\text{out}}} \frac{\mu}{\alpha} \varphi^2 dy \neq 0.$$

*Then the algebraic multiplicity of  $\lambda$  is 1.*

*Proof.* Let us consider  $\lambda \in \Lambda$ . Proving that the algebraic multiplicity of  $\lambda$  is 1 amounts to proving that the algebraic and geometric multiplicities of  $\lambda$  coincide, in other words that the length of any Jordan chain associated with  $\lambda$  is 1. Assume that such a Jordan chain is longer than 1. This implies that  $\mathcal{L}(\lambda)\varphi = 0$  and there exists some  $\hat{\varphi} \in H^1(I_{\text{out}})$ ,  $\hat{\varphi} \neq 0$ , such that

$$\mathcal{L}(\lambda)\hat{\varphi} + \frac{d\mathcal{L}}{d\lambda}(\lambda)\varphi = 0,$$

that is, for all  $\psi \in H^1(I_{\text{out}})$ ,

$$\int_{I_{\text{out}}} \left( \alpha \mu d_y \hat{\varphi} d_y \psi - \frac{\mu}{\alpha} (\lambda^2 + k^2) \hat{\varphi} \psi \right) dy - 2\lambda \int_{I_{\text{out}}} \frac{\mu}{\alpha} \varphi \psi dy = 0.$$

By choosing  $\psi = \varphi$ , we notice that the first integral vanishes because it coincides with  $\langle \mathcal{L}(\lambda)\varphi, \hat{\varphi} \rangle_{I_{\text{out}}} = 0$ . Since  $\lambda \neq 0$  (see Lemma 1), we obtain

$$\int_{I_{\text{out}}} \frac{\mu}{\alpha} \varphi^2 dy = 0,$$

which contradicts the assumption on  $\varphi$ .  $\square$

Since the spectrum  $\Lambda$  is discrete and in virtue of Lemma 1, we number the elements of  $\Lambda$  such that

$$\Lambda = \cup_{n=0}^{+\infty} \{-\lambda_n, \lambda_n\}, \quad \dots \leq \Re e(\lambda_{n+1}) \leq \Re e(\lambda_n) \leq \dots \leq \Re e(\lambda_0) < 0. \quad (3.18)$$

From Lemma 2, the geometric multiplicity of each element  $\pm\lambda_n$  in  $\Lambda$  is equal to 1: the eigenfunction corresponding to both  $\lambda_n$  and  $(-\lambda_n)$  is denoted  $\varphi_n$  and is defined up to a multiplicative constant. The modes are then given by

$$w_n^\pm(x, y) = e^{\pm\lambda_n x} \varphi_n(y). \quad (3.19)$$

Let us introduce the following fundamental assumption on the eigenfunctions  $\varphi_n$ .

**Assumption 3.1.** *The eigenvectors  $\varphi_n$  are such that for all  $n \in \mathbb{N}$ , we have*

$$\int_{I_{\text{out}}} \frac{\mu}{\alpha} \varphi_n^2 dy \neq 0.$$

We now establish the biorthogonality relationship.

**Proposition 4.** *With Assumption 3.1, the eigenvectors  $\varphi_n$  may be rescaled such that, for  $n, m \in \mathbb{N}$ , we have*

$$\int_{I_{\text{out}}} \frac{\mu}{\alpha} \varphi_n \varphi_m dy = \delta_{mn}, \quad (3.20)$$

where  $\delta_{mn} = 1$  if  $m = n$  and  $\delta_{mn} = 0$  otherwise.

*Proof.* Given Assumption 3.1, it is easy to rescale the functions  $\varphi_n$  such that, for all  $n \in \mathbb{N}$ ,  $\int_{I_{\text{out}}} \frac{\mu}{\alpha} \varphi_n^2 dy = 1$ . Now let us consider  $\varphi_n$  and  $\varphi_m$  for  $n \neq m$ . The corresponding eigenvalues are  $\pm \lambda_n$  and  $\pm \lambda_m$ , respectively, with  $\lambda_n^2 \neq \lambda_m^2$ . By using  $\mathcal{L}(\lambda_n)\varphi_n = 0$  and choosing  $\varphi = \varphi_n$  and  $\psi = \varphi_m$  in (3.15), we have

$$\int_{I_{\text{out}}} \left( \alpha \mu d_y \varphi_n d_y \varphi_m - \frac{\mu}{\alpha} (\lambda_n^2 + k^2) \varphi_n \varphi_m \right) dy = 0.$$

By using  $\mathcal{L}(\lambda_m)\varphi_m = 0$  and choosing  $\varphi = \varphi_m$  and  $\psi = \varphi_n$  in (3.15), we have

$$\int_{I_{\text{out}}} \left( \alpha \mu d_y \varphi_m d_y \varphi_n - \frac{\mu}{\alpha} (\lambda_m^2 + k^2) \varphi_m \varphi_n \right) dy = 0.$$

Subtracting the two previous identities implies that

$$(\lambda_n^2 - \lambda_m^2) \int_{I_{\text{out}}} \frac{\mu}{\alpha} \varphi_n \varphi_m dy = 0,$$

and since  $\lambda_n^2 \neq \lambda_m^2$ , we get

$$\int_{I_{\text{out}}} \frac{\mu}{\alpha} \varphi_n \varphi_m dy = 0,$$

which completes the proof.  $\square$

**Remark 3.1.** *It can be noted that for any profile function  $\alpha$ , a dispersion relationship can easily be derived, since in each interval  $(-h_{\text{out}}, -h_{\text{in}})$ ,  $(-h_{\text{in}}, -h)$ ,  $(-h, h)$ ,  $(h, h_{\text{in}})$ ,  $(h_{\text{in}}, h_{\text{out}})$ , the eigenvector  $\varphi$  satisfies a second order ordinary differential equation. Denoting*

$$\beta_0 := i\mathbb{R}^+ \sqrt{k_0^2 + \lambda} \quad \text{and} \quad \beta_\infty := i\mathbb{R}^+ \sqrt{k_\infty^2 + \lambda},$$

the eigenvalues  $\lambda$  are solutions to the equation

$$\left( \mu_\infty \beta_\infty \tan(\beta_\infty \gamma) + \mu_0 \beta_0 \tan(\beta_0 h) \right) \left( \mu_\infty \beta_\infty \tan(\beta_\infty \gamma) - \mu_0 \beta_0 \frac{1}{\tan(\beta_0 h)} \right) = 0, \quad (3.21)$$

where the first factor above corresponds to the symmetric modes and the second one to the antisymmetric ones, with

$$\gamma := \int_h^{h_{\text{out}}} \frac{1}{\alpha(y)} dy. \quad (3.22)$$

In particular, the dispersion relationship (3.21) holds when the function  $\alpha$  is given either by (2.11) for abrupt PMLs or by (2.14) for smooth PMLs.

## 4 Analysis of a straight unperturbed waveguide

In this section, we consider the following problem in the straight waveguide  $\Omega_{\text{out}}$ : for a source term  $f$  given in the domain  $\Omega_{\text{out}}$ , find a solution  $u$  in such domain  $\Omega_{\text{out}}$  to the problem

$$\begin{cases} -\partial_y(\alpha\mu\partial_y u) - \frac{\mu}{\alpha}\partial_{xx}u - \frac{\mu}{\alpha}k^2u = f & \text{in } \Omega_{\text{out}} \\ \partial_\nu u = 0 & \text{on } \partial\Omega_{\text{out}} \\ u \text{ is outgoing.} \end{cases} \quad (4.23)$$

Thanks to Assumption 2.2, which prevents the existence of propagating modes in  $\Omega_{\text{out}}$ , the radiation condition which characterizes *outgoing* waves reduces here to a simple decay condition at infinity, which will be imposed by searching for  $u$  in the space  $H^1(\Omega_{\text{out}})$ . In a view to study well-posedness of problem (4.23) using the Kondratiev theory (see [Kondratiev, 1967]), we introduce the weighted Sobolev spaces.

### 4.1 The weighted Sobolev spaces

For  $\beta \in \mathbb{R}$ , let us define the space

$$W_\beta^1(\Omega_{\text{out}}) := \{v \in \mathcal{D}'(\Omega_{\text{out}}), e^{\beta x}v, e^{\beta x}\partial_x v, e^{\beta x}\partial_y v \in L^2(\Omega_{\text{out}})\}, \quad (4.24)$$

where  $\mathcal{D}'(\Omega_{\text{out}})$  is the usual space of distributions, the space  $W_\beta^1(\Omega_{\text{out}})$  being endowed with its natural norm

$$\|v\|_{W_\beta^1(\Omega_{\text{out}})} := \left( \|e^{\beta x}v\|_{L^2(\Omega_{\text{out}})}^2 + \|e^{\beta x}\partial_x v\|_{L^2(\Omega_{\text{out}})}^2 + \|e^{\beta x}\partial_y v\|_{L^2(\Omega_{\text{out}})}^2 \right)^{1/2}.$$

Observe that for  $\beta=0$ , we have  $W_0^1(\Omega_{\text{out}}) = H^1(\Omega_{\text{out}})$ . We denote  $W_\beta^1(\Omega_{\text{out}})^*$  the topological dual space of  $W_\beta^1(\Omega_{\text{out}})$  endowed with the norm

$$\|f\|_{W_\beta^1(\Omega_{\text{out}})^*} := \sup_{v \in W_\beta^1(\Omega_{\text{out}}) \setminus \{0\}} \frac{|\langle f, \bar{v} \rangle_{\Omega_{\text{out}}}|}{\|v\|_{W_\beta^1(\Omega_{\text{out}})}}.$$

Here  $\langle \cdot, \cdot \rangle_{\Omega_{\text{out}}}$  refers to the bilinear duality pairing between  $W_\beta^1(\Omega_{\text{out}})^*$  and  $W_\beta^1(\Omega_{\text{out}})$ . For  $\beta \in \mathbb{R}$ , in relation with problem (4.23) we define the linear and bounded operator  $A_\beta : W_\beta^1(\Omega_{\text{out}}) \rightarrow W_{-\beta}^1(\Omega_{\text{out}})^*$  such that for all  $(u, v) \in W_\beta^1(\Omega_{\text{out}}) \times W_{-\beta}^1(\Omega_{\text{out}})$ ,

$$\langle A_\beta u, v \rangle_{\Omega_{\text{out}}} := \int_{\Omega_{\text{out}}} \left( \alpha\mu\partial_y u \partial_y v + \frac{\mu}{\alpha}\partial_x u \partial_x v - \frac{\mu}{\alpha}k^2 u v \right) dx dy. \quad (4.25)$$

We observe that the boundedness of the operator  $A_\beta$  stems from the fact that all the functions  $\alpha$ ,  $\alpha^{-1}$ ,  $\mu$  and  $k$ , which are piecewise constant in the transverse section, are bounded.

In what follows we will need the Fourier-Laplace transform  $\mathcal{F}_{x \rightarrow \lambda}$  which is given, for well chosen  $\lambda \in \mathbb{C}$ , by

$$\hat{v}(\lambda) = (\mathcal{F}_{x \rightarrow \lambda} v)(\lambda) := \int_{-\infty}^{+\infty} e^{-\lambda x} v(x) dx.$$

Let us recall some basic properties of the Fourier-Laplace transform, which inherit from the properties of the Fourier and the Laplace transforms. Denoting  $\mathcal{S}'(\mathbb{R})$  the usual space of tempered distributions, for any  $\beta \in \mathbb{R}$  and  $v \in \mathcal{S}'_\beta(\mathbb{R}) := \{v \in \mathcal{D}'(\mathbb{R}), e^{\beta x}v \in \mathcal{S}'(\mathbb{R})\}$ ,  $\hat{v}(\lambda)$  is well-defined for  $\lambda \in \ell_{-\beta}$ , where  $\ell_{-\beta} := \{\lambda = -\beta + is, s \in \mathbb{R}\}$ . In addition, the following formula applies to the derivative of  $v \in \mathcal{S}'_\beta(\mathbb{R})$  on  $\ell_{-\beta}$ :

$$\widehat{v'}(\lambda) = \lambda \hat{v}(\lambda). \quad (4.26)$$

Furthermore,  $\mathcal{F}_{x \rightarrow \lambda}$  is an isomorphism between  $L_\beta^2(\mathbb{R}) := \{v \in \mathcal{D}'(\mathbb{R}), e^{\beta x}v \in L^2(\mathbb{R})\}$  and  $L^2(\ell_{-\beta})$ , equipped with the norms

$$v \mapsto \|e^{\beta x}v\|_{L^2(\mathbb{R})} \quad \text{and} \quad \hat{v} \mapsto \sqrt{\frac{1}{i} \int_{\ell_{-\beta}} |\hat{v}(\lambda)|^2 d\lambda},$$

respectively, with the Plancherel identity

$$\|v\|_{L^2_\beta(\mathbb{R})}^2 = \frac{1}{2\pi i} \int_{\ell_{-\beta}} \|\hat{v}(\lambda)\|^2 d\lambda.$$

Besides, the inverse  $\mathcal{F}_{x \rightarrow \lambda}^{-1}$  is given by

$$\mathcal{F}_{x \rightarrow \lambda}^{-1} \hat{v}(x) = \frac{1}{2\pi i} \int_{\ell_{-\beta}} e^{\lambda x} \hat{v}(\lambda) d\lambda.$$

Lastly, for any  $\beta_1 < \beta_2 \in \mathbb{R}$ , for  $v \in \mathcal{S}'_{\beta_1}(\mathbb{R}) \cap \mathcal{S}'_{\beta_2}(\mathbb{R})$ , the function  $\hat{v}$  is holomorphic in the strip  $\{\lambda \in \mathbb{C}, -\beta_2 < \Re \lambda < \beta_1\}$ .

For functions of both variables  $x$  and  $y$ , we can define the partial Fourier-Laplace transform with respect to  $x$ , still denoted by  $\mathcal{F}_{x \rightarrow \lambda}$ , using the above lines with obvious changes. With the help of the above properties, we show that it is an isomorphism between

$$W_\beta^1(\Omega_{\text{out}}) \quad \text{and} \quad \widehat{W}_\beta^1 := \left\{ \hat{v} \in L^2(\ell_{-\beta}, H^1(I_{\text{out}})), \frac{1}{i} \int_{\ell_{-\beta}} \|\hat{v}(\lambda, \cdot)\|_{H^1(I_{\text{out}}, |\lambda|)}^2 d\lambda < +\infty \right\},$$

for all  $\beta \in \mathbb{R}$ , where

$$\|\varphi\|_{H^1(I_{\text{out}}, |\lambda|)} := \left( (1 + |\lambda|^2) \|\varphi\|_{L^2(I_{\text{out}})}^2 + \|d_y \varphi\|_{L^2(I_{\text{out}})}^2 \right)^{1/2}, \quad \forall \varphi \in H^1(I_{\text{out}}). \quad (4.27)$$

Note that for a fixed  $\lambda$ , the norms  $\|\cdot\|_{H^1(I_{\text{out}}, |\lambda|)}$  and  $\|\cdot\|_{H^1(I_{\text{out}})}$  are equivalent on  $H^1(I_{\text{out}})$ . However the constants which characterize this equivalence depend on  $|\lambda|$ . The choice of the above norm is justified by the fact that thanks to (4.26), we have the following Plancherel formula:

$$\|v\|_{W_\beta^1(\Omega_{\text{out}})}^2 = \frac{1}{2\pi i} \int_{\ell_{-\beta}} \|\hat{v}(\lambda, \cdot)\|_{H^1(I_{\text{out}}, |\lambda|)}^2 d\lambda =: \|\hat{v}\|_{\widehat{W}_\beta^1}^2. \quad (4.28)$$

Let us denote  $\widehat{W}_\beta^{1*}$  the topological dual space of  $\widehat{W}_\beta^1$  which can be characterized as

$$\widehat{W}_\beta^{1*} = \left\{ \hat{g} \in L^2(\ell_\beta, H^1(I_{\text{out}})^*), \int_{\ell_\beta} \|\hat{g}(\lambda, \cdot)\|_{H^1(I_{\text{out}}, |\lambda|)^*}^2 d\lambda < +\infty \right\}$$

where

$$\|g\|_{H^1(I_{\text{out}}, |\lambda|)^*} := \sup_{\varphi \in H^1(\Omega_{\text{out}}) \setminus \{0\}} \frac{|\langle g, \bar{\varphi} \rangle_{I_{\text{out}}}|}{\|\varphi\|_{H^1(I_{\text{out}}, |\lambda|)}}, \quad \forall g \in H^1(I_{\text{out}})^*. \quad (4.29)$$

The Fourier-Laplace Transform  $\mathcal{F}_{x \rightarrow \lambda}$  can be defined by duality for functions in  $W_\beta^1(\Omega_{\text{out}})^*$  as

$$\forall f \in W_\beta^1(\Omega_{\text{out}})^*, \forall \hat{v} \in \widehat{W}_\beta^1, \quad \langle \mathcal{F}_{x \rightarrow \lambda} f, \hat{v} \rangle_{\hat{\Omega}_{\text{out}}} := \langle f, \mathcal{F}_{x \rightarrow \lambda}^{-1} \hat{v} \rangle_{\Omega_{\text{out}}},$$

where  $\langle \cdot, \cdot \rangle_{\hat{\Omega}_{\text{out}}}$  refers to the duality pairing between  $\widehat{W}_\beta^{1*}$  and  $\widehat{W}_\beta^1$ . Finally, we have also a Plancherel formula

$$\|f\|_{W_\beta^1(\Omega_{\text{out}})^*}^2 = \frac{1}{2\pi i} \int_{\ell_\beta} \|\hat{f}(\lambda, \cdot)\|_{H^1(I_{\text{out}}, |\lambda|)^*}^2 d\lambda. \quad (4.30)$$

## 4.2 Reduction to problems in one dimension

By applying  $\mathcal{F}_{x \rightarrow \lambda}$  to the equation  $A_\beta u = f$  and with the help of the basic property (4.26), one obtains that  $\mathcal{L}(\lambda) \hat{u}(\lambda, \cdot) = \hat{f}(\lambda, \cdot)$  for  $\lambda \in \ell_{-\beta}$ , so that one is naturally led to study the symbol  $\mathcal{L}(\lambda)$  defined by (3.15). In the following lemma, we establish estimates in the  $\lambda$  dependent norms of  $H^1(I_{\text{out}}, |\lambda|)$  and  $H^1(I_{\text{out}}, |\lambda|)^*$ .

**Lemma 5.** *There is  $\tau_0 > 0$  such that for  $\lambda = i\tau$ ,  $\tau \in \mathbb{R}$  with  $|\tau| \geq \tau_0$ ,  $\mathcal{L}(\lambda) : H^1(I_{\text{out}}) \rightarrow H^1(I_{\text{out}})^*$  is an isomorphism. Moreover, if  $\varphi \in H^1(I_{\text{out}})$  satisfies  $\mathcal{L}(\lambda)\varphi = g \in H^1(I_{\text{out}})^*$ , then there holds*

$$\|\varphi\|_{H^1(I_{\text{out}}, |\lambda|)} \leq C \|g\|_{H^1(I_{\text{out}}, |\lambda|)^*}, \quad (4.31)$$

where  $C > 0$  is independent of  $g$  and  $\lambda$ .

*Proof.* For  $\lambda = i\tau$ , we have for all  $\varphi$  and  $\psi$  in  $H^1(I_{\text{out}})$ ,

$$\langle \mathcal{L}(i\tau)\varphi, \bar{\psi} \rangle_{I_{\text{out}}} = \int_{I_{\text{out}}} \alpha \mu dy \varphi dy \bar{\psi} + \frac{\mu}{\alpha} (\tau^2 - k^2) \varphi \bar{\psi} dy.$$

The proof relies on the Lax-Milgram Lemma. We know that in the waveguide  $\Omega_{\text{out}}$ , we have  $-\frac{\pi}{2} < \arg(\alpha) \leq 0$ , and that  $\alpha$  is bounded from above and bounded away from 0 in  $\Omega_{\text{out}}$ . Writing that

$$\alpha = |\alpha|e^{i\theta} \quad \text{and} \quad \frac{1}{\alpha} = \frac{1}{|\alpha|}e^{-i\theta},$$

with  $-\pi/2 < \theta \leq 0$ , there exists a constant  $C > 0$  such that

$$\Re e(\alpha) \geq C \quad \text{and} \quad \Re e(1/\alpha) \geq C.$$

If  $\tau \geq k_\infty$ , that  $k_0 < k_\infty$  implies that for all  $\varphi \in H^1(I_{\text{out}})$ ,

$$\Re e(\langle \mathcal{L}(i\tau)\varphi, \bar{\varphi} \rangle_{I_{\text{out}}}) \geq C \min(\mu_0, \mu_\infty) \int_{I_{\text{out}}} |d_y \varphi|^2 + (\tau^2 - k_\infty^2) |\varphi|^2 dy.$$

For some given  $\gamma \in (0, 1)$ , we have  $\tau^2 - k_\infty^2 \geq \gamma(1 + \tau^2)$  for  $\tau$  large enough, more precisely if  $\tau \geq \tau_0 := \sqrt{(k_\infty^2 + \gamma)/(1 - \gamma)}$ . We obtain that for all  $\varphi \in H^1(I_{\text{out}})$ ,

$$\Re e(\langle \mathcal{L}(i\tau)\varphi, \bar{\varphi} \rangle_{I_{\text{out}}}) \geq C\gamma \min(\mu_0, \mu_\infty) \int_{I_{\text{out}}} |d_y \varphi|^2 + (1 + \tau^2) |\varphi|^2 dy.$$

The estimate (4.31) follows from the above estimate, the fact that  $\tau^2 = |\lambda|^2$  and the very definitions of the norms of  $H^1(I_{\text{out}}, |\lambda|)$  and  $H^1(I_{\text{out}}, |\lambda|)^*$  given by (4.27) and (4.29), respectively.  $\square$

From Lemma 5, we deduce the following result with the help of the Fredholm alternative.

**Corollary 6.** *For all  $\lambda \in \mathbb{C}$ ,  $\mathcal{L}(\lambda): H^1(I_{\text{out}}) \rightarrow H^1(I_{\text{out}})^*$  is an isomorphism if and only if  $\lambda \notin \Lambda$ .*

**Remark 4.1.** *We also deduce from the analytic Fredholm theorem exposed in [Kozlov & Maz'ya, 1999] (see Proposition A.8.4) that the set  $\Lambda$  is discrete and does not have any accumulation point in  $\mathbb{C}$ .*

In order to apply the inverse Fourier-Laplace transform and use Plancherel formulas, we need estimates for  $\mathcal{L}(\lambda)^{-1}$  on lines  $\ell_\beta$ ,  $\beta \in \mathbb{R}$ , in the parameter dependent norms (4.27), (4.29).

**Lemma 7.** *There are real positive constants  $\rho$  and  $\delta$  such that for all  $\lambda \in \mathbb{C}$  satisfying*

$$|\lambda| > \rho \quad \text{and} \quad |\Re \lambda| < \delta |\Im \lambda|,$$

*$\mathcal{L}(\lambda): H^1(I_{\text{out}}) \rightarrow H^1(I_{\text{out}})^*$  is an isomorphism. Moreover, if  $\varphi \in H^1(I_{\text{out}})$  satisfies  $\mathcal{L}(\lambda)\varphi = g \in H^1(I_{\text{out}})^*$ , then (4.31) holds (with a different constant  $C$ ).*

*Proof.* Lemma 5 ensures that  $\mathcal{L}(\lambda)$  is an isomorphism and that (4.31) holds for  $\lambda \in i\mathbb{R}$  with  $|\lambda| \geq \tau_0$ , where  $\tau_0$  is defined in the proof of Lemma 5. Now let us consider the case  $\lambda \notin i\mathbb{R}$ . We write  $\lambda$  as  $\lambda = \pm i|\lambda|e^{i\theta}$  with  $\theta \in (-\pi/2, \pi/2)$ . Set  $\tilde{\lambda} := \pm i|\lambda|$ . Since  $|\tilde{\lambda}| = |\lambda|$ , by definition of the parameter dependent norm (4.27), for  $\varphi \in H^1(I_{\text{out}})$ , we have  $\|\varphi\|_{H^1(I_{\text{out}}, |\lambda|)} = \|\varphi\|_{H^1(I_{\text{out}}, |\tilde{\lambda}|)}$ . Define  $\tilde{g} := \mathcal{L}(\tilde{\lambda})\varphi$ . Assume that  $|\lambda| \geq \tau_0$ . In that case, according to the first step of the proof, we have

$$\|\varphi\|_{H^1(I_{\text{out}}, |\lambda|)} = \|\varphi\|_{H^1(I_{\text{out}}, |\tilde{\lambda}|)} \leq C \|\tilde{g}\|_{H^1(I_{\text{out}}, |\tilde{\lambda}|)^*}. \quad (4.32)$$

Here and in what follows,  $C > 0$  is a constant which can change from one line to another but which is independent of  $\lambda, \varphi$ . Now we can write

$$\|\tilde{g}\|_{H^1(I_{\text{out}}, |\tilde{\lambda}|)^*} = \|\tilde{g}\|_{H^1(I_{\text{out}}, |\lambda|)^*} \leq \|g\|_{H^1(I_{\text{out}}, |\lambda|)^*} + \|\tilde{g} - g\|_{H^1(I_{\text{out}}, |\lambda|)^*}. \quad (4.33)$$

Besides, for all  $\psi \in H^1(I_{\text{out}})$ ,

$$\langle \tilde{g} - g, \bar{\psi} \rangle_{I_{\text{out}}} = \langle \mathcal{L}(\tilde{\lambda})\varphi - \mathcal{L}(\lambda)\varphi, \bar{\psi} \rangle_{I_{\text{out}}} = (\lambda^2 - \tilde{\lambda}^2) \int_{I_{\text{out}}} \frac{\mu}{\alpha} \varphi \bar{\psi} dy.$$

In view of (4.29) and using that  $|\lambda|\|\psi\|_{L^2(I_{\text{out}})} \leq \|\psi\|_{H^1(I_{\text{out}},|\lambda|)}$ , we deduce that

$$\|\tilde{g} - g\|_{H^1(I_{\text{out}},|\lambda|)^*} \leq C \frac{1}{|\lambda|} |\tilde{\lambda}^2 - \lambda^2| \|\varphi\|_{L^2(I_{\text{out}})} \leq C |e^{2i\theta} - 1| \|\varphi\|_{H^1(I_{\text{out}},|\lambda|)}.$$

Thus for all  $\varsigma > 0$ , there is  $\delta$  small enough so that one has  $\|\tilde{g} - g\|_{H^1(I_{\text{out}},|\lambda|)^*} \leq \varsigma \|\varphi\|_{H^1(I_{\text{out}},|\lambda|)}$  for all  $\lambda = \pm i|\lambda|e^{i\theta}$  such that  $|\theta| < \delta$ . Gathering the latter estimate, (4.32) and (4.33) leads to

$$\|\varphi\|_{H^1(I_{\text{out}},|\lambda|)} \leq C \|g\|_{H^1(I_{\text{out}},|\lambda|)^*} + C\varsigma \|\varphi\|_{H^1(I_{\text{out}},|\lambda|)}.$$

Taking  $\varsigma$  sufficiently small ( $\varsigma = 1/(2C)$  for example), finally we obtain (4.31).  $\square$

Lemma 7 gives in particular information on the location of elements of the spectrum  $\Lambda$ .

### 4.3 Well-posedness of problem (4.23)

From the previous lemma, we deduce the following result.

**Theorem 8.** *Let  $\beta \in \mathbb{R}$  be such that  $\Lambda$  has no intersection with the line  $-\beta + i\mathbb{R}$ . Then the operator  $A_\beta : W_\beta^1(\Omega_{\text{out}}) \rightarrow W_{-\beta}^1(\Omega_{\text{out}})^*$  defined in (4.25) is an isomorphism.*

*Proof.* Assume that  $\mathcal{L}$  has no eigenvalue on the line  $-\beta + i\mathbb{R}$ . Let us first suppose that  $u \in W_\beta^1(\Omega_{\text{out}})$  is such that  $A_\beta u = 0$ . Applying the partial Fourier-Laplace transform with respect to  $x$ , we obtain

$$\mathcal{L}(\lambda)\hat{u}(\lambda, \cdot) = 0, \quad \text{for a.e. } \lambda \in \ell_{-\beta}.$$

From Corollary 6, we deduce that for almost every  $\lambda \in \ell_{-\beta}$ ,  $\hat{u}(\lambda, \cdot) = 0$ . From the properties of the inverse Fourier-Laplace transform, we deduce that  $u \equiv 0$ . This shows that  $A_\beta$  is injective.

We prove now that  $A_\beta$  is onto. Let  $f \in W_{-\beta}^1(\Omega_{\text{out}})^*$ . First of all, the operator  $\mathcal{L}(\lambda)$  is invertible for all  $\lambda \in \ell_{-\beta}$  according to Corollary 6. In addition, Lemma 7 guarantees that there exist  $\nu_\beta > 0$  (which only depends on  $\beta$ ) and  $C > 0$  such that for  $\lambda \in \mathbb{C}$  such that  $\Re \lambda = -\beta$  and  $|\Im \lambda| \geq \nu_\beta$ , we have the estimate

$$\|\mathcal{L}(\lambda)^{-1}\hat{f}(\lambda, \cdot)\|_{H^1(I_{\text{out}},|\lambda|)} \leq C \|\hat{f}(\lambda, \cdot)\|_{H^1(I_{\text{out}},|\lambda|)^*}. \quad (4.34)$$

For  $\lambda \in -\beta + i[-\nu_\beta, \nu_\beta]$ , the continuity of  $\lambda \mapsto \mathcal{L}(\lambda)^{-1}$ , ensured by the analytic Fredholm theorem, guarantees that the estimate (4.34) also holds for  $\lambda$  in the compact set  $-\beta + i[-\nu_\beta, \nu_\beta]$ . Therefore (4.34) is valid for all  $\lambda \in \ell_{-\beta}$  with a constant  $C > 0$  independent of  $\lambda$ .

By definition

$$f \in W_{-\beta}^1(\Omega_{\text{out}})^* \quad \Rightarrow \quad \frac{1}{2\pi i} \int_{\ell_{-\beta}} \|\hat{f}(\lambda, \cdot)\|_{H^1(I_{\text{out}},|\lambda|)^*}^2 d\lambda < +\infty.$$

We deduce that

$$u(x, \cdot) = \frac{1}{2\pi i} \int_{\ell_{-\beta}} e^{\lambda x} \mathcal{L}(\lambda)^{-1} \hat{f}(\lambda, \cdot) d\lambda \in W_\beta^1(\Omega_{\text{out}}) \quad (4.35)$$

is solution of  $A_\beta u = f$  with, by the Plancherel formulas (4.28) and (4.30),

$$\|u\|_{W_\beta^1(\Omega_{\text{out}})} \leq C \|f\|_{W_{-\beta}^1(\Omega_{\text{out}})^*},$$

which completes the proof.  $\square$

We have proved that  $\Lambda \cap i\mathbb{R} = \emptyset$  (see Lemma 1). From Theorem 8, we deduce the following corollary.

**Corollary 9.** *The operator  $A_0$  is an isomorphism from  $H^1(\Omega_{\text{out}})$  to  $H^1(\Omega_{\text{out}})^*$ . In other words, for  $f \in H^1(\Omega_{\text{out}})^*$ , the problem (4.23) has a unique solution  $u$  in  $H^1(\Omega_{\text{out}})$ .*

#### 4.4 Behaviour at infinity of the solutions to problem (4.23)

In the following proposition, we compare two solutions  $u_1$  and  $u_2$  which are associated with two different operators  $A_{\beta_1}$  and  $A_{\beta_2}$ , with  $\beta_1 < \beta_2$ . From Lemma 7, the intersection of  $\Lambda$  with the strip  $\{\lambda \in \mathbb{C}, -\beta_2 < \Re \lambda < -\beta_1\}$  is a finite set.

**Proposition 10.** *Let us consider  $\beta_1 < \beta_2 \in \mathbb{R}$  such that  $\Lambda \cap (\ell_{-\beta_1} \cup \ell_{-\beta_2}) = \emptyset$ . The intersection of  $\Lambda$  with the strip  $\{\lambda \in \mathbb{C}, -\beta_2 < \Re \lambda < -\beta_1\}$  is supposed to be non empty and is denoted  $\{\mu_1, \mu_2, \dots, \mu_N\}$ .*

*Let  $f \in W_{-\beta_1}^1(\Omega_{\text{out}})^* \cap W_{-\beta_2}^1(\Omega_{\text{out}})^*$  and let us denote  $u_1 := A_{\beta_1}^{-1} f \in W_{\beta_1}^1(\Omega_{\text{out}})$  and  $u_2 := A_{\beta_2}^{-1} f \in W_{\beta_2}^1(\Omega_{\text{out}})$ . There exist complex numbers  $\{c_j\}_{j=1}^N$  such that*

$$u_2 = \sum_{j=1}^N c_j w_j + u_1,$$

where  $w_j(x, y) := e^{\mu_j x} \psi_j(y)$ , with  $\psi_j \in \text{Ker}(\mathcal{L}(\mu_j))$ .

*Proof.* As  $f \in W_{-\beta_1}^1(\Omega_{\text{out}})^* \cap W_{-\beta_2}^1(\Omega_{\text{out}})^*$ , one can show that  $\hat{f}(\lambda, \cdot)$  is holomorphic in the strip  $\{-\beta_2 < \Re \lambda < -\beta_1\}$ . The only singularities of the function  $e^{\lambda x} \mathcal{L}(\lambda)^{-1} \hat{f}(\lambda, \cdot)$  in  $\{-\beta_2 < \Re \lambda < -\beta_1\}$  are the poles of  $\mathcal{L}(\lambda)^{-1}$ , i.e. the elements of  $\Lambda$ .

Let  $\rho$  be sufficiently large so that

$$\Lambda \cap \{\lambda \in \mathbb{C}, -\beta_2 < \Re \lambda < -\beta_1\} \subset \{\lambda \in \mathbb{C}, -\beta_2 < \Re \lambda < -\beta_1, |\Im \lambda| < \rho\}.$$

Using the Residue Theorem, we get from (4.35) that

$$\begin{aligned} u_2(x, \cdot) &= \frac{1}{2i\pi} \lim_{\rho \rightarrow +\infty} \int_{-\beta_2 - i\rho}^{-\beta_2 + i\rho} e^{\lambda x} \mathcal{L}(\lambda)^{-1} \hat{f}(\lambda, \cdot) d\lambda \\ &= \frac{1}{2i\pi} \lim_{\rho \rightarrow +\infty} \left[ \int_{-\beta_1 - i\rho}^{-\beta_1 + i\rho} \dots d\lambda + \int_{-\beta_2 - i\rho}^{-\beta_1 - i\rho} \dots d\lambda - \int_{-\beta_2 + i\rho}^{-\beta_1 + i\rho} \dots d\lambda \right] \\ &\quad + \sum_{j=1}^N \text{Res}(e^{\lambda x} \mathcal{L}(\lambda)^{-1} \hat{f}(\lambda, \cdot), \mu_j). \end{aligned}$$

The first integral tends to  $u_1$ . For the second and the third ones, it suffices to extend [Kozlov et al., 1997] (see Lemma 5.4.1) to state that for all  $M$

$$\left\| \int_{-\beta_2 + i\rho}^{-\beta_1 + i\rho} e^{\lambda x} \mathcal{L}(\lambda)^{-1} \hat{f}(\lambda, \cdot) d\lambda \right\|_{L^2(\Omega_{\text{out}}^M)} \xrightarrow{\rho \rightarrow +\infty} 0,$$

where  $\Omega_{\text{out}}^M := \Omega_{\text{out}} \cap \{-M \leq x \leq M\}$ . To compute the residues, we apply [Kozlov et al., 1997] (see Theorem 5.1.1). Since the geometric and algebraic multiplicities of all eigenvalues  $\mu_j$  are equal to 1 (see Lemma 3), we get that in the neighborhood of  $\mu_j$ ,

$$\mathcal{L}(\lambda)^{-1} = \frac{P_j}{\lambda - \mu_j} + \mathcal{U}_j(\lambda)$$

where  $P_j$  is the projector on  $\psi_j$  and  $\mathcal{U}_j$  is an holomorphic operator function in a neighborhood of  $\mu_j$ .  $\square$

In order to quantify the exponentially growing or decaying behaviours as  $x \rightarrow \pm\infty$ , for  $\beta \in \mathbb{R}$ , let us introduce the weighted Sobolev space

$$\mathcal{W}_{\beta}^1(\Omega_{\text{out}}) := \{v \in \mathcal{D}'(\Omega_{\text{out}}), e^{-\beta|x|} v, e^{-\beta|x|} \partial_x v, e^{-\beta|x|} \partial_y v \in L^2(\Omega_{\text{out}})\}, \quad (4.36)$$

endowed with its natural norm

$$\|v\|_{\mathcal{W}_{\beta}^1(\Omega_{\text{out}})} := \left( \|e^{-\beta|x|} v\|_{L^2(\Omega_{\text{out}})}^2 + \|e^{-\beta|x|} \partial_x v\|_{L^2(\Omega_{\text{out}})}^2 + \|e^{-\beta|x|} \partial_y v\|_{L^2(\Omega_{\text{out}})}^2 \right)^{1/2}.$$



Remark the absolute value in the weight  $e^{-\beta|x|}$ , which implies that

$$\beta_1 \leq \beta_2 \quad \Rightarrow \quad \mathcal{W}_{\beta_1}^1(\Omega_{\text{out}}) \subset \mathcal{W}_{\beta_2}^1(\Omega_{\text{out}}). \quad (4.37)$$

Note that this property is not true for the spaces  $\mathcal{W}_{\beta}^1(\Omega_{\text{out}})$  introduced in (4.24). Observe also that we have  $\mathcal{W}_0^1(\Omega_{\text{out}}) = \mathcal{H}^1(\Omega_{\text{out}})$ . Let  $\langle \cdot, \cdot \rangle_{\Omega_{\text{out}}}$  stand for the bilinear duality pairing between  $\mathcal{W}_{\beta}^1(\Omega_{\text{out}})^*$  and  $\mathcal{W}_{\beta}^1(\Omega_{\text{out}})$ , where  $\mathcal{W}_{\beta}^1(\Omega_{\text{out}})^*$  is the topological dual space of  $\mathcal{W}_{\beta}^1(\Omega_{\text{out}})$  endowed with the norm

$$\|f\|_{\mathcal{W}_{\beta}^1(\Omega_{\text{out}})^*} = \sup_{v \in \mathcal{W}_{\beta}^1(\Omega_{\text{out}}) \setminus \{0\}} \frac{|\langle f, \bar{v} \rangle_{\Omega_{\text{out}}}|}{\|v\|_{\mathcal{W}_{\beta}^1(\Omega_{\text{out}})}}. \quad (4.38)$$

Due to (4.37), we have

$$\beta_1 \leq \beta_2 \quad \Rightarrow \quad \mathcal{W}_{\beta_2}^1(\Omega_{\text{out}})^* \subset \mathcal{W}_{\beta_1}^1(\Omega_{\text{out}})^*. \quad (4.39)$$

In what follows, we introduce the cut-off functions  $\chi^{\pm} \in \mathcal{C}^{\infty}(\mathbb{R}^2)$  which are equal to one for  $\pm x \geq 2L$  and to zero for  $\pm x \leq L$ , for a given  $L > 0$ . We have the following theorem.

**Theorem 11.** *Let us consider  $\beta > 0$  such that  $\Lambda \cap \ell_{-\beta} = \emptyset$  and let us assume that the set  $\Lambda \cap \{\lambda \in \mathbb{C}, -\beta < \Re \lambda < 0\}$  is non empty and denoted  $\{\lambda_0, \lambda_1, \dots, \lambda_{N_{\beta}-1}\}$ , where the  $\lambda_n$ 's are defined by (3.18).*

*For  $f \in \mathcal{W}_{\beta}^1(\Omega_{\text{out}})^* \subset \mathcal{H}^1(\Omega_{\text{out}})^*$ , let us denote  $u \in \mathcal{H}^1(\Omega_{\text{out}})$  the unique solution to the problem  $A_0 u = f$  given by Corollary 9. There exist some complex numbers  $a_n^+$ ,  $a_n^-$  and a function  $\tilde{u} \in \mathcal{W}_{-\beta}^1(\Omega_{\text{out}})$  such that*

$$u = \chi^+ \sum_{n=0}^{N_{\beta}-1} a_n^+ w_n^+ + \chi^- \sum_{n=0}^{N_{\beta}-1} a_n^- w_n^- + \tilde{u},$$

where we recall that  $w_n^{\pm}(x, y) = e^{\pm \lambda_n x} \varphi_n(y)$  with  $\varphi_n \in \text{Ker}(\mathcal{L}(\pm \lambda_n))$ .

*Proof.* Since  $f \in \mathcal{W}_{\beta}^1(\Omega_{\text{out}})^* \subset \mathcal{W}_{-\beta}^1(\Omega_{\text{out}})^*$ , by using Theorem 8, there exists a unique  $u_{\beta} \in \mathcal{W}_{\beta}^1(\Omega_{\text{out}})$  such that  $A_{\beta} u_{\beta} = f$ . By Proposition 10, we have that there exist some complex numbers  $a_n^+$  such that

$$u = \sum_{n=0}^{N_{\beta}-1} a_n^+ w_n^+ + u_{\beta}.$$

Similarly, since  $f \in \mathcal{W}_{\beta}^1(\Omega_{\text{out}})^* \subset \mathcal{W}_{\beta}^1(\Omega_{\text{out}})^*$ , there exists a unique  $u_{-\beta} \in \mathcal{W}_{-\beta}^1(\Omega_{\text{out}})$  such that  $A_{-\beta} u_{-\beta} = f$ . The set  $\Lambda$  is symmetrical with respect to 0, we have  $\Lambda \cap \{\lambda \in \mathbb{C}, 0 < \Re \lambda < \beta\} = \{-\lambda_0, -\lambda_1, \dots, -\lambda_{N_{\beta}-1}\}$ , hence there exist some complex numbers  $a_n^-$  such that

$$u = \sum_{n=0}^{N_{\beta}-1} a_n^- w_n^- + u_{-\beta}.$$

It remains to write

$$\begin{aligned} u &= \chi^+ u + \chi^- u + (1 - \chi^+ - \chi^-) u \\ &= \chi^+ \sum_{n=0}^{N_{\beta}-1} a_n^+ w_n^+ + \chi^- \sum_{n=0}^{N_{\beta}-1} a_n^- w_n^- + \tilde{u}, \end{aligned}$$

where

$$\tilde{u} := \chi^+ u_{\beta} + \chi^- u_{-\beta} + (1 - \chi^+ - \chi^-) u.$$

We easily check that the three terms of the above sum belong to  $\mathcal{W}_{-\beta}^1(\Omega_{\text{out}})$ .  $\square$

**Remark 4.2.** *In particular, if  $\lambda_0$  has the strictly largest real part among  $\{\lambda_0, \lambda_1, \dots, \lambda_{N_{\beta}-1}\}$  (which have negative real parts), the solution  $u$  behaves asymptotically as  $a_0^{\pm} e^{\pm \lambda_0 x} \varphi_0(y)$  for  $x \rightarrow \pm \infty$ .*

## 5 Well-posedness of the forward problem (1.1)

Let us come back to the problem (1.1) set in the introduction and illustrated by Figure 2, that is a junction between a half-closed waveguide and a half-open waveguide which is closed by finite PMLs. We recall that  $\Omega^- := (-\infty, 0) \times (-h, h)$ ,  $\Omega^+ := (0, +\infty) \times (-h_{\text{out}}, h_{\text{out}})$ ,  $\Sigma_0 := \{0\} \times (-h, h)$  and  $\Omega := \Omega^- \cup \Sigma_0 \cup \Omega^+$ . We also denote  $\Sigma_L := \{L\} \times (-h_{\text{out}}, h_{\text{out}})$ . We assume now that the obstacle  $O$  is a smooth bounded domain. It lies within the core of the open half-waveguide with PMLs, between the transverse sections  $\Sigma_0$  and  $\Sigma_L$ , so that  $O \subset (0, L) \times (-h, h)$ . Let us denote  $D := \Omega \setminus \overline{O}$  and  $D^+ := \Omega^+ \setminus \overline{O}$ . We hereafter study the scattering response of the structure  $D$  to an incident wave which is sent from the left. Let us hence define the incident wave  $u^i = \tilde{w}_{n,0}^+$  in  $\Omega$  as

$$\tilde{w}_{n,0}^+ := \begin{cases} \tilde{w}_n^+ & \text{in } \Omega^-, \\ 0 & \text{in } \Omega^+, \end{cases} \quad (5.40)$$

where  $\tilde{w}_n^+$  for  $n \in \mathbb{N}$  is a mode coming from the left closed waveguide and defined by (2.6). Let us introduce the space  $\tilde{\mathbf{H}}_{\text{loc}}^1(D)$  as the set of distributions  $v$  in  $D$  such that  $\chi v \in \mathbf{H}^1(D)$ , for all  $\chi \in \mathcal{C}^\infty(\mathbb{R}^2)$  vanishing for sufficiently large  $(-x)$  i.e., as  $x \rightarrow -\infty$ . Problem (1.1) can be rewritten as: find  $u \in \tilde{\mathbf{H}}_{\text{loc}}^1(D)$  such that

$$\left\{ \begin{array}{ll} -\Delta u - k_0^2 u = 0 & \text{in } \Omega^-, \\ -\partial_y(\alpha \mu \partial_y u) - \frac{\mu}{\alpha} \partial_{xx} u - \frac{\mu}{\alpha} k^2 u = 0 & \text{in } D^+, \\ \llbracket u \rrbracket = 0 & \text{on } \Sigma_0, \\ \llbracket \partial_x u \rrbracket = 0 & \text{on } \Sigma_0, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \\ u = 0 & \text{on } \partial O, \\ u - u^i \text{ is outgoing,} & \end{array} \right. \quad (5.41)$$

where  $u^i = \tilde{w}_{n,0}^+$  and  $\llbracket \cdot \rrbracket$  denotes the jump of the solution at the interface  $\Sigma_0$ . As for problem (4.23), the radiation condition on the right side of  $D$  consists in the choice of a  $\mathbf{H}^1$ -type functional space in view of the absence of propagating modes. On the left-side of  $D$ , such radiation condition will consist in selecting the scattered waves which either propagate or exponentially decrease from the right to the left. In order to analyze well-posedness of problem (5.41), let us find an equivalent problem set in a bounded domain. In this view we introduce Dirichlet-to-Neumann maps on the two sections  $\Sigma_0$  and  $\Sigma_L$ .

Firstly, let us define the Dirichlet-to-Neumann operator  $T_0$  on  $\Sigma_0$ . Let us denote  $\mathbf{H}_{\text{loc}}^1(\Omega^-)$  the set of distributions  $v$  in  $\Omega^-$  such that  $\chi v \in \mathbf{H}^1(\Omega^-)$ , for all  $\chi \in \mathcal{C}^\infty(\mathbb{R}^2)$  vanishing for sufficiently large  $(-x)$ . For  $\varphi \in \mathbf{H}^{1/2}(\Sigma_0)$ , the problem set in the half-waveguide  $\Omega^-$ : find  $u^-(\varphi) \in \mathbf{H}_{\text{loc}}^1(\Omega^-)$  such that

$$\left\{ \begin{array}{ll} -\Delta u^- - k_0^2 u^- = 0 & \text{in } \Omega^-, \\ \partial_\nu u^- = 0 & \text{on } \partial\Omega^- \setminus \Sigma_0, \\ u^- = \varphi & \text{on } \Sigma_0, \\ u^- \text{ is outgoing,} & \end{array} \right. \quad (5.42)$$

is well-posed, so that we can define

$$\begin{aligned} T_0 : \mathbf{H}^{1/2}(\Sigma_0) &\rightarrow \tilde{\mathbf{H}}^{-1/2}(\Sigma_0) \\ \varphi &\mapsto T_0 \varphi = -\mu_0 \partial_x u^-(\varphi)|_{\Sigma_0}, \end{aligned}$$

where  $u^-(\varphi)$  is the solution to problem (5.42), or equivalently

$$\varphi \mapsto T_0 \varphi = \mu_0 \sum_{n \in \mathbb{N}} \tilde{\lambda}_n(\varphi, \tilde{\varphi}_n)_{L^2(\Sigma_0)} \tilde{\varphi}_n, \quad (5.43)$$

where the complex numbers  $\tilde{\lambda}_n$  and the functions  $\tilde{\varphi}_n$  are defined by (2.4) and (2.5), respectively. The space  $\tilde{\mathbf{H}}^{-1/2}(\Sigma_0)$  here denotes the topological dual space of  $\mathbf{H}^{1/2}(\Sigma_0)$  and coincides with the set of distributions in  $\mathbf{H}^{-1/2}(\{0\} \times \mathbb{R})$  the support of which belongs to the closure of  $\Sigma_0$ . Following [Lenoir & Tounsi, 1988], well-posedness of (5.42) is easily obtained by projecting  $\varphi = u|_{\Sigma_0}$  on the functions  $\tilde{\varphi}_n$ , which form a complete basis of  $L^2(\Sigma_0)$  (see also [Bourgeois & Lunéville, 2008]).

Secondly, we define the Dirichlet-to-Neumann operator  $T_L$  on  $\Sigma_L$ . Since it is not known that the functions  $\varphi_n$  form a complete basis of  $L^2(\Sigma_L)$ , the operator  $T_L$  cannot be defined as was  $T_0$ . In this view, we need the following Lemma, which is a consequence of section 4.

**Lemma 12.** *For  $\varphi \in H^{1/2}(\Sigma_L)$ , the problem set in the half-waveguide  $\Omega_L^+ := (L, +\infty) \times (-h_{\text{out}}, h_{\text{out}})$ : find  $u^+(\varphi) \in H^1(\Omega_L^+)$  such that*

$$\begin{cases} -\partial_y(\alpha\mu\partial_y u^+) - \frac{\mu}{\alpha}\partial_{xx}u^+ - \frac{\mu}{\alpha}k^2 u^+ = 0 & \text{in } \Omega_L^+, \\ \partial_\nu u^+ = 0 & \text{on } \partial\Omega_L^+ \setminus \Sigma_L, \\ u^+ = \varphi & \text{on } \Sigma_L \end{cases} \quad (5.44)$$

has a unique solution and there exists a constant  $C > 0$  independent on  $\varphi$  such that

$$\|u^+(\varphi)\|_{H^1(\Omega_L^+)} \leq C \|\varphi\|_{H^{1/2}(\Sigma_L)}, \quad \forall \varphi \in H^{1/2}(\Sigma_L).$$

*Proof.* The idea is to deduce the statement of the Lemma from the well-posedness of problem (4.23) using a symmetry argument. To do so, we first have to notice that for any  $\varphi \in H^{1/2}(\Sigma_L)$ , there exists a function  $U_\varphi^+ \in H^1(\Omega_L^+)$  such that  $U_\varphi^+|_{\Sigma_L} = \varphi$  and  $\partial_\nu U_\varphi^+|_{\partial\Omega_L^+ \setminus \Sigma_L} = 0$ . Hence, instead of solving (5.44), we can equivalently search for a function  $u_0^+ := u^+ - U_\varphi^+ \in H^1(\Omega_L^+)$  which satisfies

$$\begin{cases} Pu_0^+ = -PU_\varphi^+ & \text{in } \Omega_L^+, \\ \partial_\nu u_0^+ = 0 & \text{on } \partial\Omega_L^+ \setminus \Sigma_L, \\ u_0^+ = 0 & \text{on } \Sigma_L, \end{cases} \quad (5.45)$$

where we recall that  $P$  is the differential operator given by (1.2). We show below that thanks to the homogeneous boundary condition on  $\Sigma_L$ , this problem amounts to searching for solutions to (4.23) which are antisymmetric with respect to  $\Sigma_L$ .

Let us introduce some notations. We denote by  $\Omega_L^-$  the half-waveguide which is symmetric to  $\Omega_L^+$  with respect to section  $\Sigma_L$ . We thus have  $\Omega_L^- \cup \Sigma_L \cap \Omega_L^+ = \Omega_{\text{out}}$ . For any  $w \in H^1(\Omega_{\text{out}})$ , we denote respectively by  $\mathcal{S}w$  and  $\mathcal{A}w$  the symmetric and antisymmetric parts of  $w$ , namely,

$$\mathcal{S}w := \frac{1}{2}(w + \check{w}) \quad \text{and} \quad \mathcal{A}w := \frac{1}{2}(w - \check{w}) \quad \text{where} \quad \check{w}(x, y) := w(2L - x, y),$$

which both belong to  $H^1(\Omega_{\text{out}})$ . By transposition,  $\mathcal{S}$  and  $\mathcal{A}$  can be extended to any  $f \in H^1(\Omega_{\text{out}})^*$ , for example

$$\langle \mathcal{S}f, w \rangle_{\Omega_{\text{out}}} := \langle f, \mathcal{S}w \rangle_{\Omega_{\text{out}}}, \quad \forall w \in H^1(\Omega_{\text{out}}),$$

where  $\langle \cdot, \cdot \rangle_{\Omega_{\text{out}}}$  refers to the duality between  $H^1(\Omega_{\text{out}})^*$  and  $H^1(\Omega_{\text{out}})$ . Finally  $H_{0,L}^1(\Omega_L^+)$  stands for the subspace of functions in  $H^1(\Omega_L^+)$  which vanish on  $\Sigma_L$ .

Consider  $f_\varphi^+ := -PU_\varphi^+$  the right-hand side of the first equation of (5.45). As  $\partial_\nu U_\varphi^+|_{\partial\Omega_L^+ \setminus \Sigma_L} = 0$ , we remark that  $f_\varphi^+$  belongs to the dual space  $H_{0,L}^1(\Omega_L^+)^*$  of  $H_{0,L}^1(\Omega_L^+)$ . We can then define  $f_\varphi \in H^1(\Omega_{\text{out}})^*$  by setting

$$\langle f_\varphi, w \rangle_{\Omega_{\text{out}}} := 2 \left\langle f_\varphi^+, (\mathcal{A}w)|_{\Omega_L^+} \right\rangle_{\Omega_L^+}, \quad \forall w \in H^1(\Omega_{\text{out}}),$$

where  $\langle \cdot, \cdot \rangle_{\Omega_L^+}$  refers to the duality between  $H_{0,L}^1(\Omega_L^+)^*$  and  $H_{0,L}^1(\Omega_L^+)$  (note that this definition makes sense since  $(\mathcal{A}w)|_{\Omega_L^+} \in H_{0,L}^1(\Omega_L^+)$ ). The distribution  $f_\varphi$  is nothing but the antisymmetric extension of  $f_\varphi^+$ . Indeed, on the one hand, it is clear that  $f_\varphi$  is antisymmetric, since for any  $w \in H^1(\Omega_{\text{out}})$ ,

$$\langle \mathcal{S}f_\varphi, w \rangle_{\Omega_{\text{out}}} = \langle f_\varphi, \mathcal{S}w \rangle_{\Omega_{\text{out}}} = 2 \left\langle f_\varphi^+, (\mathcal{A}\mathcal{S}w)|_{\Omega_L^+} \right\rangle_{\Omega_L^+} = 0.$$

On the other hand, for any  $w^+ \in H_{0,L}^1(\Omega_L^+)$ , denoting by  $w \in H^1(\Omega_{\text{out}})$  its extension by 0 in  $\Omega_L^-$ , we have

$$\langle f_\varphi|_{\Omega_L^+}, w^+ \rangle_{\Omega_L^+} = \langle f_\varphi, w \rangle_{\Omega_{\text{out}}} = 2 \left\langle f_\varphi^+, (\mathcal{A}w)|_{\Omega_L^+} \right\rangle_{\Omega_L^+} = \langle f_\varphi^+, w^+ \rangle_{\Omega_L^+},$$

since  $(\mathcal{A}w)|_{\Omega_L^+} = w^+/2$ . This shows that  $f_\varphi|_{\Omega_L^+} = f_\varphi^+$ .

In virtue of Corollary 9, the unique solution to (4.23) for  $f = f_\varphi$  is  $u_0 := A_0^{-1} f_\varphi \in H^1(\Omega_{\text{out}})$ . Let us verify that  $u_0^+ := u_0|_{\Omega_L^+}$  is a solution to (5.45). As  $f_\varphi|_{\Omega_L^+} = f_\varphi^+$ , the first two equations of (5.45) are simply the restrictions of the first two equations of (4.23) to  $\Omega_L^+$  and  $\partial\Omega_L^+ \setminus \Sigma_L$ , respectively. For the third one, just notice that as  $f_\varphi$  is antisymmetric,  $u_0$  is antisymmetric as well (this follows from the injectivity of  $A_0$  and the fact that  $A_0\mathcal{S} = \mathcal{S}A_0$ ), so that  $u_0|_{\Sigma_L} = 0$ .

Finally, the uniqueness of the solution to (5.45) is straightforward by using again a symmetry argument and the injectivity of operator  $A_0$ .  $\square$

With the help of Lemma 12, we can define the operator

$$\begin{aligned} T_L : H^{1/2}(\Sigma_L) &\rightarrow \tilde{H}^{-1/2}(\Sigma_L) \\ \varphi &\mapsto T_L \varphi = \frac{\mu}{\alpha} \partial_x u^+(\varphi)|_{\Sigma_L}, \end{aligned} \quad (5.46)$$

where  $u^+(\varphi)$  is the solution to problem (5.44). Note that an explicit expression of  $T_L$  such as (5.43) is not available any longer.

That the scattered field  $u - \tilde{w}_{n,0}^+$  is outgoing in problem (5.41) is equivalent to the identities  $(\mu/\alpha)\partial_x u|_{\Sigma_L} = T_L u|_{\Sigma_L}$  and  $-\mu_0 \partial_x (u - \tilde{w}_n^+)|_{\Sigma_0} = T_0(u|_{\Sigma_0} - \tilde{w}_n^+|_{\Sigma_0})$ , which is itself equivalent to

$$-\mu_0 \partial_x u|_{\Sigma_0} = T_0 u|_{\Sigma_0} + g, \quad g := -2\mu_0 \tilde{\lambda}_n \tilde{w}_n^+|_{\Sigma_0}, \quad n \in \mathbb{N}. \quad (5.47)$$

As a result, the problem (5.41) set in  $D$  is equivalent to the following problem set in the bounded domain  $D_L$  of  $D$  which is delimited by the sections  $\Sigma_0$  and  $\Sigma_L$ : find  $u \in H^1(D_L)$  such that

$$\left\{ \begin{array}{ll} -\partial_y(\alpha\mu\partial_y u) - \frac{\mu}{\alpha}\partial_{xx}u - \frac{\mu}{\alpha}k^2u = 0 & \text{in } D_L, \\ \partial_\nu u = 0 & \text{on } \partial D_L \setminus (\Sigma_0 \cup \Sigma_L \cup \partial O), \\ u = 0 & \text{on } \partial O, \\ -\mu_0 \partial_x u = T_0 u + g & \text{on } \Sigma_0, \\ \frac{\mu}{\alpha} \partial_x u = T_L u & \text{on } \Sigma_L. \end{array} \right. \quad (5.48)$$

A weak formulation which is equivalent to problem (5.48) is the following, with  $H_{0,\partial O}^1(D_L)$  denoting the subspace of functions in  $H^1(D_L)$  which vanish on  $\partial O$ : find  $u \in H_{0,\partial O}^1(D_L)$  such that

$$a_L(u, v) = \ell(v), \quad \forall v \in H_{0,\partial O}^1(D_L), \quad (5.49)$$

where the sesquilinear form  $a_L$  is given by

$$a_L(u, v) = \int_{D_L} \left( \alpha\mu\partial_y u \partial_y \bar{v} + \frac{\mu}{\alpha}\partial_x u \partial_x \bar{v} - \frac{\mu}{\alpha}k^2 u \bar{v} \right) dx dy - \langle T_0 u, \bar{v} \rangle_{\Sigma_0} - \langle T_L u, \bar{v} \rangle_{\Sigma_L}, \quad (5.50)$$

$\langle \cdot, \cdot \rangle_{\Sigma_M}$  referring to the duality pairing between  $\tilde{H}^{-1/2}(\Sigma_M)$  and  $H^{1/2}(\Sigma_M)$  for  $M=0$  and  $M=L$ , and the antilinear form  $\ell$  is given by

$$\ell(v) := \int_{\Sigma_0} g \bar{v} ds, \quad (5.51)$$

where  $g$  is defined by (5.47). We are in a position to prove well-posedness of problem (5.41).

**Theorem 13.** *Assume that for  $u^i = 0$ , the problem (5.41) has only the trivial solution  $u = 0$ . Then the problem (5.41) is well-posed for  $u^i = \tilde{w}_{n,0}^+$ .*

*Proof.* Since we have uniqueness for problem (5.41), we also have uniqueness for problem (5.48). In order to prove existence for both problems, it suffices to prove that the problem (5.48) is of Fredholm type. Let us introduce a smooth real non-negative function  $\chi$  in  $\Omega_L^+$  which depends only on  $x$  and such that  $\chi$  vanishes for  $x \geq 2L$  and  $\chi = 1$  for  $L \leq x \leq 3L/2$ . We remark that for  $\varphi, \psi \in H^{1/2}(\Sigma_L)$ ,

$$\langle T_L \varphi, \bar{\psi} \rangle_{\Sigma_L} = \left\langle \frac{\mu}{\alpha} \partial_x u^+(\varphi), \overline{u^+(\psi)} \right\rangle_{\Sigma_L},$$

where  $u^+(\varphi)$  and  $u^+(\psi)$  are the solutions in  $\Omega_L^+$  to the problem (5.44) which correspond to data  $\varphi$  and  $\psi$ , respectively. Multiplying the equation satisfied by  $u^+(\varphi)$  by  $\chi \overline{u^+(\psi)}$  and using the Green Formula, we obtain

$$\begin{aligned} -\langle \frac{\mu}{\alpha} \partial_x u^+(\varphi), \overline{u^+(\psi)} \rangle_{\Sigma_L} &= \int_{\Omega_L^+} \left( \alpha \mu \chi \partial_y u^+(\varphi) \partial_y \overline{u^+(\psi)} + \frac{\mu}{\alpha} \chi \partial_x u^+(\varphi) \partial_x \overline{u^+(\psi)} \right) dx dy \\ &\quad + \int_{\Omega_L^+} \left( \frac{\mu}{\alpha} (\partial_x \chi) \partial_x u^+(\varphi) \overline{u^+(\psi)} - \frac{\mu}{\alpha} k^2 \chi u^+(\varphi) \overline{u^+(\psi)} \right) dx dy. \end{aligned}$$

Applying that identity to  $\varphi = u|_{\Sigma_L}$  and  $\psi = v|_{\Sigma_L}$ , it enables us to decompose the sesquilinear form  $a_L$  as  $a_L = c_L + k_L$ , with for  $u, v \in \mathbf{H}_{0,\partial O}^1(D_L)$ ,

$$\begin{aligned} c_L(u, v) &:= \int_{D_L} \left( \alpha \mu \partial_y u \partial_y \bar{v} + \frac{\mu}{\alpha} \partial_x u \partial_x \bar{v} \right) dx dy \\ &\quad + \int_{\Omega_L^+} \left( \alpha \mu \chi \partial_y u^+(u|_{\Sigma_L}) \partial_y \overline{u^+(v|_{\Sigma_L})} + \frac{\mu}{\alpha} \chi \partial_x u^+(u|_{\Sigma_L}) \partial_x \overline{u^+(v|_{\Sigma_L})} \right) dx dy \\ &\quad - \langle T_0 u, \bar{v} \rangle_{\Sigma_0} \end{aligned}$$

and

$$\begin{aligned} k_L(u, v) &:= - \int_{D_L} \frac{\mu}{\alpha} k^2 u \bar{v} dx dy \\ &\quad + \int_{\Omega_L^+} \left( \frac{\mu}{\alpha} (\partial_x \chi) \partial_x u^+(u|_{\Sigma_L}) \overline{u^+(v|_{\Sigma_L})} - \frac{\mu}{\alpha} k^2 \chi u^+(u|_{\Sigma_L}) \overline{u^+(v|_{\Sigma_L})} \right) dx dy. \end{aligned}$$

With the help of the Riesz theorem, we can define two operators  $C_L$  and  $K_L$  mapping  $\mathbf{H}_{0,\partial O}^1(D_L)$  to itself and which are associated with the sesquilinear forms  $c_L$  and  $k_L$ , respectively. It remains to prove that  $C_L$  is an isomorphism and  $K_L$  is a compact operator. Let us prove that  $c_L$  is coercive. Taking  $v = u$ , we get

$$\begin{aligned} c_L(u, u) &= \int_{D_L} \left( \alpha \mu |\partial_y u|^2 + \frac{\mu}{\alpha} |\partial_x u|^2 \right) dx dy \\ &\quad + \int_{\Omega_L^+} \left( \alpha \mu \chi |\partial_y u^+(u|_{\Sigma_L})|^2 + \frac{\mu}{\alpha} \chi |\partial_x u^+(u|_{\Sigma_L})|^2 \right) dx dy - \langle T_0 u, \bar{u} \rangle_{\Sigma_0}. \end{aligned}$$

Using (5.43), we have

$$\begin{aligned} -\langle T_0 u, \bar{u} \rangle_{\Sigma_0} &= -\mu_0 \sum_{n \in \mathbb{N}} \tilde{\lambda}_n |(u|_{\Sigma_0}, \tilde{\varphi}_n)_{L^2(\Sigma_0)}|^2 \\ &= -i\mu_0 \sum_{n=0}^{\tilde{N}-1} \tilde{\beta}_n |(u|_{\Sigma_0}, \tilde{\varphi}_n)_{L^2(\Sigma_0)}|^2 - \mu_0 \sum_{n=\tilde{N}}^{+\infty} \tilde{\lambda}_n |(u|_{\Sigma_0}, \tilde{\varphi}_n)_{L^2(\Sigma_0)}|^2, \end{aligned}$$

with  $\tilde{\beta}_n > 0$  for  $n=0, \dots, \tilde{N}-1$  and  $\tilde{\lambda}_n < 0$  for  $n \geq \tilde{N}$ . Hence

$$\Re(-\langle T_0 u, \bar{u} \rangle_{\Sigma_0}) = -\mu_0 \sum_{n=\tilde{N}}^{+\infty} \tilde{\lambda}_n |(u|_{\Sigma_0}, \tilde{\varphi}_n)_{L^2(\Sigma_0)}|^2 \geq 0.$$

Since there exists some  $C > 0$  such that  $\Re(\alpha) \geq C$  and  $\Re(1/\alpha) \geq C$ , in view of  $\chi \geq 0$  and using the Poincaré inequality, there exists some  $c > 0$  such that for all  $u \in \mathbf{H}_{0,\partial O}^1(D_L)$ ,

$$\Re(c_L(u, u)) \geq c \|u\|_{\mathbf{H}^1(D_L)}^2.$$

We have proved that  $c_L$  is coercive, thus  $C_L$  is an isomorphism. Now let us prove that  $K_L$  is compact. It suffices to prove that if  $(u_m, v_m)_{m \in \mathbb{N}}$  is a sequence which weakly converges to  $(u, v)$  in  $\mathbf{H}_{0,\partial O}^1(D_L) \times \mathbf{H}_{0,\partial O}^1(D_L)$ , then there exists a subsequence of  $(u_m, v_m)$ , still denoted  $(u_m, v_m)$ , such that  $k_L(u_m, v_m)$  converges to  $k_L(u, v)$ . Indeed, this result would have the consequence that if  $u_m \rightharpoonup u$  in  $\mathbf{H}_{0,\partial O}^1(D_L)$ , which implies that  $K_L u_m \rightharpoonup K_L u$  in  $\mathbf{H}_{0,\partial O}^1(D_L)$ , then

$$\|K_L(u_m - u)\|_{\mathbf{H}^1(D_L)}^2 = k_L(u_m - u, K_L(u_m - u)) \rightarrow 0.$$

Since the sequences  $(u_m)$  and  $(v_m)$  are bounded in  $H^1(D_L)$  and  $D_L$  is a bounded domain, by the Rellich theorem there exist subsequences  $(u_m)$  and  $(v_m)$ , still denoted  $(u_m)$  and  $(v_m)$ , such that  $u_m \rightarrow u$  and  $v_m \rightarrow v$  in  $L^2(D_L)$ , which implies that

$$\int_{D_L} \frac{\mu}{\alpha} k^2 u_m \overline{v_m} dx dy \rightarrow \int_{D_L} \frac{\mu}{\alpha} k^2 u \overline{v} dx dy.$$

Using the continuity of the trace, another consequence of  $u_m \rightarrow u$  in  $H_{0,\partial O}^1(D_L)$  is that  $u_m|_{\Sigma_L} \rightarrow u|_{\Sigma_L}$  in  $H^{1/2}(\Sigma_L)$ . Since the mapping  $h \rightarrow u^+(h)$  given by Lemma 12 is continuous from  $H^{1/2}(\Sigma_L)$  to  $H^1(\Omega_L^+)$ , we have that  $u^+(u_m|_{\Sigma_L}) \rightarrow u^+(u|_{\Sigma_L})$  and  $u^+(v_m|_{\Sigma_L}) \rightarrow u^+(v|_{\Sigma_L})$  in  $H^1(\Omega_L^+)$ , so that  $u^+(u_m|_{\Sigma_L}) \rightarrow u^+(u|_{\Sigma_L})$  and  $u^+(v_m|_{\Sigma_L}) \rightarrow u^+(v|_{\Sigma_L})$  in  $L^2(D_{L,2L})$ , where  $D_{L,2L} = (L, 2L) \times (-h_{\text{out}}, h_{\text{out}})$ . That the function  $\chi$  vanishes for  $x \geq 2L$  leads to

$$\int_{\Omega_L^+} \left( \frac{\mu}{\alpha} (\partial_x \chi) \partial_x u^+(u_m|_{\Sigma_L}) \overline{u^+(v_m|_{\Sigma_L})} \right) dx dy \rightarrow \int_{\Omega_L^+} \left( \frac{\mu}{\alpha} (\partial_x \chi) \partial_x u^+(u|_{\Sigma_L}) \overline{u^+(v|_{\Sigma_L})} \right) dx dy$$

and

$$\int_{\Omega_L^+} \left( \frac{\mu}{\alpha} k^2 \chi u^+(u_m|_{\Sigma_L}) \overline{u^+(v_m|_{\Sigma_L})} \right) dx dy \rightarrow \int_{\Omega_L^+} \left( \frac{\mu}{\alpha} k^2 \chi u^+(u|_{\Sigma_L}) \overline{u^+(v|_{\Sigma_L})} \right) dx dy.$$

As a conclusion we have  $k_L(u_m, v_m) \rightarrow k_L(u, v)$  and we have proved that  $K_L$  is compact. The problem (5.48) is thus of Fredholm type, which completes the proof.  $\square$

In order to specify the behaviour of solutions for  $x \rightarrow +\infty$ , let us introduce, for  $\beta \in \mathbb{R}$ , the space  $\dot{W}_\beta^1(D^+) = \{v \in W_\beta^1(D^+), v|_{\partial O} = 0\}$ , where  $W_\beta^1(D^+)$  is defined by (4.36) for  $\Omega_{\text{out}}$  replaced by  $D^+$ . Note that for  $\beta = 0$ , the space  $\dot{W}_0^1(D^+)$  coincides with the subspace of functions in  $H^1(D^+)$  which vanish on  $\partial O$ . Let us introduce a function  $\chi^+ \in \mathcal{C}^\infty(\mathbb{R}^2)$  such that  $\chi^+ = 1$  for  $x \geq 2L$  and  $\chi^+ = 0$  for  $x \leq L$ .

**Theorem 14.** *Assume that for  $u^i = 0$ , the problem (5.41) has only the trivial solution  $u = 0$ . Then the solution  $u$  to the problem (5.41) for  $u^i = \tilde{w}_{n,0}^+$  is such that for any  $\beta > 0$  with  $\Lambda \cap \ell_{-\beta} = \emptyset$ , if*

$$\Lambda \cap \{\lambda \in \mathbb{C}, -\beta < \Re \lambda < 0\} = \{\lambda_0, \lambda_1, \dots, \lambda_{N_\beta-1}\},$$

there exist some complex numbers  $a_n^+$  and a function  $\tilde{u} \in \dot{W}_{-\beta}^1(D^+)$  such that  $u$  satisfies

$$u = \chi^+ \sum_{n=0}^{N_\beta-1} a_n^+ w_n^+ + \tilde{u}.$$

*Proof.* We observe that  $\chi^+ u \in H^1(\Omega_{\text{out}})$  and  $f := A_0(\chi^+ u) \in H^1(\Omega_{\text{out}})^*$  (in fact, it could be noted that  $f \in L^2(\Omega_{\text{out}})$ ). Since  $f$  is compactly supported, we have  $f \in W_{-\beta}^1(\Omega_{\text{out}})^*$ . Then, let us denote  $u_\beta := A_\beta^{-1} f \in W_\beta^1(\Omega_{\text{out}})$ . By Proposition 10, there exist some complex numbers  $a_n^+$  such that

$$\chi^+ u = \sum_{n=0}^{N_\beta-1} a_n^+ w_n^+ + u_\beta.$$

We have

$$u = (\chi^+)^2 u + (1 - (\chi^+)^2) u = \chi^+ \left( \sum_{n=0}^{N_\beta-1} a_n^+ w_n^+ + u_\beta \right) + (1 - (\chi^+)^2) u.$$

We notice that the function

$$\tilde{u} := \chi^+ u_\beta + (1 - (\chi^+)^2) u,$$

belongs to  $\dot{W}_{-\beta}^1(D^+)$ , and the proof is complete.  $\square$

In order to provide a numerical approximation of problem (5.41), we introduce a DtN map with an overlap and an equivalent problem to problem (5.41) with the help of such map. We hence introduce, for  $M \geq L$ , the operator

$$\begin{aligned} T_{L,M} : H^{1/2}(\Sigma_L) &\rightarrow \tilde{H}^{-1/2}(\Sigma_M) \\ \varphi &\mapsto T_{L,M} \varphi = \frac{\mu}{\alpha} \partial_x u^+|_{\Sigma_M}, \end{aligned} \tag{5.52}$$

where  $u^+$  is the solution to problem (5.44). Then we introduce the problem in  $D_M$ : find  $u \in \mathbf{H}_{0,\partial O}^1(D_M)$  such that

$$\left\{ \begin{array}{ll} -\partial_y(\alpha\mu\partial_y u) - \frac{\mu}{\alpha}\partial_{xx}u - \frac{\mu}{\alpha}k^2u = 0 & \text{in } D_M, \\ \partial_\nu u = 0 & \text{on } \partial D_M \setminus (\Sigma_0 \cup \Sigma_M \cup \partial O), \\ u = 0 & \text{on } \partial O, \\ -\mu_0\partial_x u = T_0u + g & \text{on } \Sigma_0, \\ \frac{\mu}{\alpha}\partial_x u = T_{L,M}(u|_{\Sigma_L}) & \text{on } \Sigma_M. \end{array} \right. \quad (5.53)$$

Problem (5.53) is equivalent to the following weak formulation: find  $u \in \mathbf{H}_{0,\partial O}^1(D_M)$  such that

$$a_{L,M}(u, v) = \ell(v), \quad \forall v \in \mathbf{H}_{0,\partial O}^1(D_M), \quad (5.54)$$

where the sesquilinear form  $a_{L,M}$  and the antilinear form  $\ell$  are given by

$$a_{L,M}(u, v) := \int_{D_M} \left( \alpha\mu\partial_y u \partial_y \bar{v} + \frac{\mu}{\alpha}\partial_x u \partial_x \bar{v} - \frac{\mu}{\alpha}k^2 u \bar{v} \right) dx dy - \langle T_0 u, \bar{v} \rangle_{\Sigma_0} - \langle T_{L,M} u, \bar{v} \rangle_{\Sigma_M} \quad (5.55)$$

and (5.51), respectively.

We now prove that the initial problem (5.41) is equivalent to the problem (5.53) in the following sense.

**Proposition 15.** *If  $u$  satisfies the problem (5.41) in  $D$ , then  $u|_{D_M}$  satisfies the problem (5.53) in  $D_M$ . Conversely, assume that  $u_M$  satisfies the problem (5.53) in  $D_M$ . If  $u^-$  (resp.  $u^+$ ) denotes the solution to the problem (5.42) (resp. (5.44)) in  $\Omega^-$  (resp.  $\Omega_L^+$ ) for  $\varphi = u_M|_{\Sigma_0}$  (resp.  $\psi = u_M|_{\Sigma_L}$ ), then the functions  $u_M$  and  $u^+$  coincide in  $D_M \setminus D_L$  and the function  $u$  defined by  $u = u^-$  in  $\Omega^-$ ,  $u = u_M$  in  $D_M$  and  $u = u^+$  in  $\Omega_L^+$  satisfies problem (5.41).*

*Proof.* The first part of the proof is straightforward. As concerns the second part, we just have to check that the two functions  $u_M$  and  $u^+$  coincide in  $D_{L,M} := (L, M) \times (-h_{\text{out}}, h_{\text{out}})$ . Let us denote  $v := u_M - u^+$  in  $D_{L,M}$ . The very definition of  $u^+$  implies that  $v|_{\Sigma_L} = 0$ . Moreover, we have on  $\Sigma_M$ :

$$\frac{\mu}{\alpha}\partial_x u_M = T_{L,M}(u_M|_{\Sigma_L}) = T_{L,M}(\psi) = \frac{\mu}{\alpha}\partial_x u^+,$$

which implies that  $(\mu/\alpha)\partial_x v = 0$  on  $\Sigma_M$ . As a result, the function  $v \in \mathbf{H}^1(D_{L,M})$  satisfies in  $D_{L,M}$  the problem

$$\left\{ \begin{array}{l} Pv = 0 \text{ in } D_{L,M}, \\ \partial_\nu v = 0 \text{ on } \partial D_{L,M} \setminus (\Sigma_L \cup \Sigma_M), \\ v = 0 \text{ on } \Sigma_L, \\ \frac{\mu}{\alpha}\partial_x v = 0 \text{ on } \Sigma_M. \end{array} \right. \quad (5.56)$$

It remains to prove that any function  $v$  in  $\mathbf{H}^1(D_{L,M})$  which satisfies (5.56) actually vanishes. We observe that the space  $\mathbf{H}_{0,L}^1(D_{L,M})$ , which denotes the set of functions in  $\mathbf{H}^1(D_{L,M})$  which vanish on  $\Sigma_L$ , coincides with the set of functions

$$v(x, y) = \sum_{n=0}^{+\infty} v_n(y) \sin(\mu_n(x-L)), \quad \mu_n := (2n+1)\frac{\pi}{2} \frac{1}{M-L},$$

such that

$$\sum_{n=0}^{+\infty} (1+n^2) \|v_n\|_{L^2(I_{\text{out}})}^2 + \|d_y v_n\|_{L^2(I_{\text{out}})}^2 < +\infty.$$

Hence, any function  $v \in \mathbf{H}^1(D_{L,M})$  satisfying (5.56) has the previous decomposition, where the  $v_n \in \mathbf{H}^1(I_{\text{out}})$  satisfy in addition

$$\mathcal{L}(i\mu_n)(v_n) = 0, \quad \forall n \in \mathbb{N},$$

where  $\mathcal{L}(\lambda)$  is defined by (3.15). In view of Corollary 6 and Lemma 1, we observe that  $\mathcal{L}(i\mu_n)$  is invertible for all  $n \in \mathbb{N}$ . We conclude that  $v_n = 0$  for all  $n \in \mathbb{N}$ , hence  $v = 0$ . The proof is complete.  $\square$

## 6 Approximation of the DtN map

We first introduce the truncated DtN maps with or without an overlap and prove that they enable us to set problems of Fredholm type. For the DtN map with an overlap, we secondly prove a convergence result of the approximated solution obtained by a “truncation” of such DtN map to the true solution with respect to the size of the overlap.

### 6.1 The truncated DtN map with or without an overlap

For computational purpose, we need to approximate the DtN maps given by (5.46) and (5.52). Let us denote for  $M \in \mathbb{R}$  and  $\varphi, \psi \in L^2(\Sigma_M)$ ,

$$\langle \varphi, \psi \rangle_{\alpha, M} := \int_{\Sigma_M} \frac{\mu}{\alpha} \varphi \psi dy.$$

Drawing inspiration from the expression (5.43) for the closed waveguide, for a fixed  $\beta > 0$ , for  $M \geq L$ , let us introduce the operator  $T_{L, M}^\beta$  defined by

$$\begin{aligned} T_{L, M}^\beta : \mathbf{H}^{1/2}(\Sigma_L) &\rightarrow \tilde{\mathbf{H}}^{-1/2}(\Sigma_M) \\ \varphi &\mapsto T_{L, M}^\beta \varphi = \sum_{n=0}^{N_\beta-1} \frac{\mu}{\alpha} \lambda_n e^{\lambda_n(M-L)} \langle \varphi, \varphi_n \rangle_{\alpha, L} \varphi_n. \end{aligned} \quad (6.57)$$

Note that for  $M = L$ , the operator  $T_{L, L}^\beta$  is expected to be an approximation of the DtN map  $T_L$  given by (5.46). Making use of this approximate operator, we introduce a new problem set in the bounded domain  $D_M$ , which is the analogous of problem (5.53) with operator  $T_{L, M}$  replaced by operator  $T_{L, M}^\beta$ . This problem corresponds to the weak formulation: find  $u \in \mathbf{H}_{0, \partial O}^1(D_M)$  such that for all  $v \in \mathbf{H}_{0, \partial O}^1(D_M)$ ,

$$a_{L, M}^\beta(u, v) = \ell(v), \quad (6.58)$$

where the sesquilinear form  $a_{L, M}^\beta$  is given by

$$a_{L, M}^\beta(u, v) := b_M(u, v) - \int_{D_M} \frac{\mu}{\alpha} k^2 u \bar{v} dx dy - \langle T_{L, M}^\beta u, \bar{v} \rangle_{\Sigma_M}, \quad (6.59)$$

the sesquilinear form  $b_M$  is defined by

$$b_M(u, v) := \int_{D_M} \left( \alpha \mu \partial_y u \partial_y \bar{v} + \frac{\mu}{\alpha} \partial_x u \partial_x \bar{v} \right) dx dy - \langle T_0 u, \bar{v} \rangle_{\Sigma_0}, \quad (6.60)$$

while the antilinear form  $\ell$  is given by (5.51). We have the following proposition.

**Proposition 16.** *For  $M \geq L$ , assume that the problem (6.58) has at most one solution. Then the problem (6.58) is well-posed.*

*Proof.* By using the same arguments as in the proof of Theorem 13, the sesquilinear form  $b_M$  is associated by the Riesz theorem with an invertible operator, while the sesquilinear form  $a_{L, M}^\beta - b_M$  is associated with a compact operator. In particular, the sesquilinear form

$$(u, v) \in \mathbf{H}_{0, \partial O}^1(D_M) \times \mathbf{H}_{0, \partial O}^1(D_M) \mapsto \langle T_{L, M}^\beta u, \bar{v} \rangle_{\Sigma_M}$$

is associated with a finite rank operator in view of (6.57). This shows that the problem (6.58) is of Fredholm type, which completes the proof.  $\square$

### 6.2 An error estimate for the DtN map with an overlap

We wish now to prove that the solution  $u_{L, M}^\beta$  to the problem (6.58) is a good approximation of the solution  $u$  to the problem (5.41) in the domain  $D_L$  when those two problems are well-posed. In the simple case of a homogeneous closed waveguide, the convergence of the solution obtained with the



DtN operator without an overlap to the true solution can be proved by using that the eigenvectors of the transverse operator form a complete basis. More precisely, the convergence rate is of exponential type, both with respect to the distance between the artificial boundary and the perturbation and with respect to the number of terms in the truncated series. The proof is very similar to the one used in [Bécache et al., 2004] to justify the approximation of the true solution with Perfectly Matched Layers. In our case, since we do not know whether the eigenfunctions  $\varphi_n$  form a complete basis, the convergence estimate for the DtN operator without an overlap is an open question. By exploiting the description of the solution in terms of the modes up to an exponentially decaying remainder, we were able to derive an estimate for the DtN operator with an overlap. Moreover, the convergence rate is exponential with respect to the overlap  $(M - L)$ , but the convergence rate with respect to  $\beta$ , which determines the number of terms in the series (6.57), is not specified. Note that in the case of a periodic waveguide, an error estimate in the presence of an artificial boundary condition of the same type as our thick DtN boundary condition is derived in [Nazarov, 2018].

We begin with the following lemma, which specifies the properties of solutions to the half-waveguide problem (5.44). For  $\beta \in \mathbb{R}$ , we introduce the space  $\mathcal{W}_\beta^1(\Omega_L^\pm)$  defined by (4.36) for  $\Omega_{\text{out}}$  replaced by  $\Omega_L^\pm$ , equipped with the norm

$$\|v\|_{\mathcal{W}_\beta^1(\Omega_L^\pm)} := \left( \|e^{-\beta(x-L)}v\|_{L^2(\Omega_L^\pm)}^2 + \|e^{-\beta(x-L)}\partial_x v\|_{L^2(\Omega_L^\pm)}^2 + \|e^{-\beta(x-L)}\partial_y v\|_{L^2(\Omega_L^\pm)}^2 \right)^{1/2}.$$

**Lemma 17.** *For  $\varphi \in H^{1/2}(\Sigma_L)$ , let us denote by  $u^+(\varphi)$  the solution in  $H^1(\Omega_L^\pm)$  to the problem (5.44) in the half-waveguide  $\Omega_L^\pm$  given by Lemma 12. For any  $\beta > 0$  such that  $\Lambda \cap \ell_{-\beta} = \emptyset$ , if*

$$\Lambda \cap \{\lambda \in \mathbb{C}, -\beta < \Re \lambda < 0\} = \{\lambda_0, \lambda_1, \dots, \lambda_{N_\beta-1}\},$$

*there exist some unique complex numbers  $a_n^+(\varphi)$  and a unique function  $u_\beta(\varphi) \in \mathcal{W}_{-\beta}^1(\Omega_L^\pm)$  such that  $u^+(\varphi)$  satisfies*

$$u^+(\varphi) = \sum_{n=0}^{N_\beta-1} a_n^+(\varphi) w_n^+ + u_\beta(\varphi). \quad (6.61)$$

*Moreover, the coefficients  $a_n^+$  are given by*

$$a_n^+(\varphi) = e^{-\lambda_n L} \langle \varphi, \varphi_n \rangle_{\alpha, L} \quad (6.62)$$

*and there exists a constant  $C_\beta$  such that for all  $\varphi \in H^{1/2}(\Sigma_L)$ ,*

$$\sum_{n=0}^{N_\beta-1} |a_n^+(\varphi)| + \|u_\beta(\varphi)\|_{\mathcal{W}_{-\beta}^1(\Omega_L^\pm)} \leq C_\beta \|\varphi\|_{H^{1/2}(\Sigma_L)}. \quad (6.63)$$

*Proof.* Existence of coefficients  $a_n^+(\varphi)$  and the function  $u_\beta(\varphi) \in \mathcal{W}_{-\beta}^1(\Omega_L^\pm)$  in the decomposition (6.61) follows the same lines as the proof of Theorem 14. For  $M \geq L$ , let us introduce the bilinear form

$$q_M(u, v) := \left\langle \frac{\mu}{\alpha} \partial_x u, v \right\rangle_{\Sigma_M} - \left\langle \frac{\mu}{\alpha} \partial_x v, u \right\rangle_{\Sigma_M},$$

whenever the two duality brackets are well defined. It should be remarked that if  $u$  and  $v$  in  $H^1(\Omega_L^\pm)$  both satisfy the equations

$$\begin{cases} -\partial_y(\alpha \mu \partial_y w) - \frac{\mu}{\alpha} \partial_{xx} w - \frac{\mu}{\alpha} k^2 w = 0 & \text{in } \Omega_L^+ \\ \partial_\nu w = 0 & \text{on } \partial\Omega_L^+ \setminus \Sigma_L, \end{cases} \quad (6.64)$$

then by using the Green Formula in  $(L, M) \times I_{\text{out}}$ , the quantity  $q_M(u, v)$  is well-defined for  $M \geq L$  and does not depend on  $M$ . Let us prove that

$$a_n^+(\varphi) = \frac{q_L(u^+(\varphi), w_n^-)}{2\lambda_n}, \quad \text{and} \quad q_L(u^+(\varphi), w_n^+) = 0. \quad (6.65)$$

Using the decomposition of  $u^+(\varphi)$  given by (6.61), we get

$$q_L(u^+(\varphi), w_n^\pm) = \sum_{m=0}^{N_\beta-1} a_m^+(\varphi) q_L(w_m^+, w_n^\pm) + q_L(u_\beta(\varphi), w_n^\pm).$$

A straightforward computation shows that

$$q_L(w_m^+, w_n^+) = 0 \quad \text{and} \quad q_L(w_m^+, w_n^-) = (\lambda_m + \lambda_n) \delta_{m,n},$$

where we have used the biorthogonality relationship  $\langle \varphi_m, \varphi_n \rangle_{\alpha, L} = \delta_{m,n}$ . Next, let us prove that  $q_L(u_\beta(\varphi), w_n^\pm) = 0$ . The functions  $u_\beta(\varphi)$  and  $w_n^\pm$  both satisfy (6.64), so that  $q_M(u_\beta(\varphi), w_n^\pm)$  does not depend on  $M \geq L$ . We have for  $M > L$ ,

$$\begin{aligned} q_L(u_\beta(\varphi), w_n^-) &= q_M(u_\beta(\varphi), w_n^-) \\ &= e^{-\lambda_n M} \langle \frac{\mu}{\alpha} \partial_x u_\beta, \varphi_n \rangle_{\Sigma_M} + \lambda_n e^{-\lambda_n M} \langle \frac{\mu}{\alpha} \varphi_n, u_\beta \rangle_{\Sigma_M} \\ &= e^{-\lambda_n M} e^{-\beta(M-L)} \left( \langle e^{\beta(M-L)} \frac{\mu}{\alpha} \partial_x u_\beta, \varphi_n \rangle_{\Sigma_M} + \lambda_n \langle \frac{\mu}{\alpha} \varphi_n, e^{\beta(M-L)} u_\beta \rangle_{\Sigma_M} \right). \end{aligned}$$

The solution in  $H^1(\Omega_M^+)$  to the problem (5.44) corresponding to  $\varphi = \varphi_n$  is nothing but  $v_{M,n}(x, y) := w_n^+(x-M, y)$ , then by the Green formula in  $\Omega_M^+$ ,

$$\begin{aligned} & - \langle e^{\beta(M-L)} \frac{\mu}{\alpha} \partial_x u_\beta, \varphi_n \rangle_{\Sigma_M} \\ &= \int_{\Omega_M^+} \left( \alpha \mu \partial_y u_\beta \partial_y v_{M,n} e^{\beta(x-L)} + \frac{\mu}{\alpha} \partial_x u_\beta \partial_x (e^{\beta(x-L)} v_{M,n}) - \frac{\mu}{\alpha} k^2 u_\beta v_{M,n} e^{\beta(x-L)} \right) dx dy. \end{aligned}$$

We then obtain the estimate

$$| \langle e^{\beta(M-L)} \frac{\mu}{\alpha} \partial_x u_\beta, \varphi_n \rangle_{\Sigma_M} | \leq C \max(1, \beta) \|u_\beta(\varphi)\|_{\mathcal{W}_{-\beta}^1(\Omega_L^+)} \|v_{M,n}\|_{H^1(\Omega_M^+)}.$$

Similarly, we obtain

$$\begin{aligned} & - \lambda_n \langle \frac{\mu}{\alpha} \varphi_n, e^{\beta(M-L)} u_\beta \rangle_{\Sigma_M} = - \langle \frac{\mu}{\alpha} \partial_x v_{M,n}, e^{\beta(M-L)} u_\beta \rangle_{\Sigma_M} \\ &= \int_{\Omega_M^+} \left( \alpha \mu \partial_y v_{M,n} \partial_y u_\beta e^{\beta(x-L)} + \frac{\mu}{\alpha} \partial_x v_{M,n} \partial_x (e^{\beta(x-L)} u_\beta) - \frac{\mu}{\alpha} k^2 v_{M,n} u_\beta e^{\beta(x-L)} \right) dx dy, \end{aligned}$$

which implies the estimate

$$| \lambda_n \langle \frac{\mu}{\alpha} \varphi_n, e^{\beta(M-L)} u_\beta \rangle_{\Sigma_M} | \leq C \max(1, \beta) \|u_\beta(\varphi)\|_{\mathcal{W}_{-\beta}^1(\Omega_L^+)} \|v_{M,n}\|_{H^1(\Omega_M^+)}.$$

Gathering the previous estimates, and using that  $\|v_{M,n}\|_{H^1(\Omega_M^+)} = \|w_n^+\|_{H^1(\Omega^+)}$  does not depend on  $M$ , provides

$$|q_L(u_\beta(\varphi), w_n^-)| \leq C e^{\beta L} e^{-(\Re e(\lambda_n) + \beta)M} \max(1, \beta) \|u_\beta(\varphi)\|_{\mathcal{W}_{-\beta}^1(\Omega_L^+)}.$$

Since  $\Re e(\lambda_n) > -\beta$  for all  $n=0, \dots, N_\beta-1$ , passing to the limit  $M \rightarrow +\infty$ , we finally obtain  $q_L(u_\beta(\varphi), w_n^-) = 0$ . We would prove with the same arguments that  $q_L(u_\beta(\varphi), w_n^+) = 0$ . This implies the identities (6.65). From (6.65), we observe that by linearity

$$a_n^+ = \frac{q_L(u^+(\varphi), w_n^- - z w_n^+)}{2\lambda_n}, \quad \forall z \in \mathbb{C}.$$

Choosing  $z$  such that  $w_n^- - z w_n^+ = 0$  on  $\Sigma_L$ , that is  $z = e^{-2\lambda_n L}$ , implies that

$$a_n^+ = -\frac{1}{2\lambda_n} \langle \frac{\mu}{\alpha} \partial_x (w_n^- - e^{-2\lambda_n L} w_n^+), u^+(\varphi) \rangle_{\Sigma_L} = e^{-\lambda_n L} \langle \frac{\mu}{\alpha} \varphi_n, \varphi \rangle_{\Sigma_L},$$

which is (6.62). This in particular implies the uniqueness of the coefficients  $a_n^+$  and of the function  $u_\beta$  in the decomposition (6.61).

In order to prove the continuity estimate (6.63), let us consider, for a fixed  $\beta > 0$ , the Hilbert space  $V_{\text{aug}} := \mathbb{C}^{N_\beta} \times V$ , with

$$V := \{v \in \mathcal{W}_{-\beta}^1(\Omega_L^+), \quad Pv = 0 \quad \text{in} \quad \Omega_L^+, \quad \partial_\nu v = 0 \quad \text{on} \quad \partial\Omega_L^+ \setminus \Sigma_L\},$$

equipped with the norm  $\|\cdot\|_{V_{\text{aug}}}$  defined by

$$\|(b_0, b_1, \dots, b_{N_\beta-1}, v)\|_{V_{\text{aug}}}^2 := \sum_{n=0}^{N_\beta-1} |b_n|^2 + \|v\|_{\mathcal{W}_{-\beta}^1(\Omega_L^+)}^2$$

and the operator  $T_{\text{aug}}: V_{\text{aug}} \rightarrow \mathbf{H}^{1/2}(\Sigma_L)$  such that

$$(b_0, b_1, \dots, b_{N_\beta-1}, v) \mapsto w|_{\Sigma_L}, \quad w := \sum_{n=0}^{N_\beta-1} b_n w_n^+ + v.$$

From the first part of the proof, the operator  $T_{\text{aug}}$  is invertible. Its inverse is continuous from the Banach theorem, which implies (6.63).  $\square$

We are now in a position to state the main theorem of this section.

**Theorem 18.** *For a fixed  $L$  and  $M$  large enough, if problem (5.41) is well-posed, then problem (6.58) is well-posed as well. Moreover, if  $u$  is the solution to problem (5.41) and  $u_{L,M}^\beta$  is the solution to problem (6.58), there exists a constant  $C > 0$  which is independent of  $\beta$  and  $M$  such that*

$$\|u - u_{L,M}^\beta\|_{\mathbf{H}^1(D_L)} \leq C C_\beta \max(1, \beta) e^{-\beta(M-L)} \|u\|_{\mathbf{H}^1(D^+)}, \quad (6.66)$$

where  $C_\beta$  is the constant (independent of  $M$ ) given in (6.63).

*Proof.* Let us introduce the operators  $A_{L,M}$  and  $A_{L,M}^\beta$  which map the space  $\mathbf{H}_{0,\partial O}^1(D_M)$  to its dual space  $\mathbf{H}_{0,\partial O}^1(D_M)^*$  and such that for all  $(u, v) \in \mathbf{H}_{0,\partial O}^1(D_M)$ ,

$$\langle A_{L,M} u, \bar{v} \rangle_{D_M} := a_{L,M}(u, v), \quad \langle A_{L,M}^\beta u, \bar{v} \rangle_{D_M} := a_{L,M}^\beta(u, v),$$

where  $\langle \cdot, \cdot \rangle_{D_M}$  refers to the duality pairing between  $\mathbf{H}_{0,\partial O}^1(D_M)^*$  and  $\mathbf{H}_{0,\partial O}^1(D_M)$ . Let us define  $G \in \mathbf{H}_{0,\partial O}^1(D_M)^*$  by

$$\langle G, \bar{v} \rangle_{D_M} := \int_{\Sigma_0} g \bar{v} dy, \quad \forall v \in \mathbf{H}_{0,\partial O}^1(D_M). \quad (6.67)$$

The problems (5.54) and (6.58) are then respectively equivalent to

$$A_{L,M} u = G, \quad A_{L,M}^\beta u_{L,M}^\beta = G. \quad (6.68)$$

Let us take any  $w \in \mathbf{H}_{0,\partial O}^1(D_M)$ . For  $\varphi = w|_{\Sigma_L}$ , let us denote  $u^+(\varphi)$  the solution to problem (5.44) in  $\Omega_L^+$  with datum  $\varphi$  on  $\Sigma_L$ . Lemma 17 implies that  $u^+(\varphi)$  satisfies on  $\Sigma_M$  the identity

$$\frac{\mu}{\alpha} \partial_x u^+(\varphi)|_{\Sigma_M} = \sum_{n=0}^{N_\beta-1} a_n^+(\varphi) \frac{\mu}{\alpha} \lambda_n e^{\lambda_n M} \varphi_n + \frac{\mu}{\alpha} \partial_x u_\beta(\varphi)|_{\Sigma_M}.$$

Using (6.62), we obtain that

$$T_{L,M} \varphi = T_{L,M}^\beta \varphi + \frac{\mu}{\alpha} \partial_x u_\beta(\varphi)|_{\Sigma_M},$$

and given that for all  $w, v \in \mathbf{H}_{0,\partial O}^1(D_M)$ ,

$$\langle A_{L,M} w, v \rangle_{D_M} - \langle A_{L,M}^\beta w, v \rangle_{D_M} = \langle T_{L,M}^\beta w|_{\Sigma_L}, v|_{\Sigma_M} \rangle_{\Sigma_M} - \langle T_{L,M} w|_{\Sigma_L}, v|_{\Sigma_M} \rangle_{\Sigma_M},$$

we end up with, for all  $w, v \in \mathbf{H}_{0,\partial O}^1(D_M)$ ,

$$\langle A_{L,M}^\beta w, v \rangle_{D_M} - \langle A_{L,M} w, v \rangle_{D_M} = \langle R_{L,M}^\beta w|_{\Sigma_L}, v|_{\Sigma_M} \rangle_{\Sigma_M}, \quad (6.69)$$

where the operator  $R_{L,M}^\beta: \mathbf{H}^{1/2}(\Sigma_L) \rightarrow \tilde{\mathbf{H}}^{-1/2}(\Sigma_M)$  is defined, for all  $\varphi \in \mathbf{H}^{1/2}(\Sigma_L)$  by

$$R_{L,M}^\beta \varphi := \frac{\mu}{\alpha} \partial_x u_\beta(\varphi)|_{\Sigma_M}.$$

For any  $\varphi \in \mathbf{H}^{1/2}(\Sigma_L)$  and any  $\psi \in \mathbf{H}^{1/2}(\Sigma_M)$ , we have

$$\begin{aligned} & - \langle e^{\beta(M-L)} R_{L,M}^\beta \varphi, \psi \rangle_{\Sigma_M} \\ & = \int_{\Omega_M^+} \left( \alpha \mu \partial_y u_\beta(\varphi) \partial_y v e^{\beta(x-L)} + \frac{\mu}{\alpha} \partial_x u_\beta(\varphi) \partial_x (e^{\beta(x-L)} v) - \frac{\mu}{\alpha} k^2 u_\beta(\varphi) e^{\beta(x-L)} v \right) dx dy, \end{aligned}$$

where  $v \in \mathbf{H}^1(\Omega_M^+)$  is the unique solution to the problem (5.44) with  $L$  replaced by  $M$  and data  $\varphi$  replaced by  $\psi$ . In addition,  $\|v\|_{\mathbf{H}^1(\Omega_M^+)} \leq C \|\psi\|_{\mathbf{H}^{1/2}(\Sigma_M)}$ , for a constant  $C$  which does not depend on  $M$ . Since  $u_\beta(\varphi) \in \mathcal{W}_{-\beta}^1(\Omega_L^+)$ , we have that there exists a constant  $C$  (independent of  $\beta$  and  $M$ ) such that

$$\left| \langle e^{\beta(M-L)} R_{L,M}^\beta \varphi, \psi \rangle_{\Sigma_M} \right| \leq C \max(1, \beta) \|u_\beta(\varphi)\|_{\mathcal{W}_{-\beta}^1(\Omega_L^+)} \|\psi\|_{\mathbf{H}^{1/2}(\Sigma_M)}.$$

In view of (6.63), we conclude that

$$\|R_{L,M}^\beta\| \leq C C_\beta \max(1, \beta) e^{-\beta(M-L)}, \quad (6.70)$$

where  $\|\cdot\|$  denotes the norm of operator  $R_{L,M}^\beta$  from  $\mathbf{H}^{1/2}(\Sigma_L)$  to  $\tilde{\mathbf{H}}^{-1/2}(\Sigma_M)$  and  $C_\beta$  is the continuity constant in (6.63). From (6.69) we get that

$$\|A_{L,M}^\beta - A_{L,M}\| \leq C \|R_{L,M}^\beta\|, \quad (6.71)$$

where  $\|\cdot\|$  on the left hand side denotes the norm of operators  $A_{L,M}^\beta$  and  $A_{L,M}$  from  $\mathbf{H}_{0,\partial O}^1(D_M)$  to  $\mathbf{H}_{0,\partial O}^1(D_M)^*$ , and  $C$  is a constant which depends neither on  $\beta$  nor on  $M$ .

In order to consider source terms  $G$  supported in  $D_L$ , we need to introduce the restriction operator  $\mathcal{R}: \mathbf{H}_{0,\partial O}^1(D_M) \mapsto \mathbf{H}_{0,\partial O}^1(D_L)$  as well as its adjoint operator  $\mathcal{R}^*: \mathbf{H}_{0,\partial O}^1(D_L)^* \mapsto \mathbf{H}_{0,\partial O}^1(D_M)^*$ , which provides the continuation of a source in  $D_L$  by 0 in  $D_M \setminus D_L$ . Since the operator  $\mathcal{R}$  is surjective, it has a right inverse operator  $\mathcal{J}: \mathbf{H}_{0,\partial O}^1(D_L) \rightarrow \mathbf{H}_{0,\partial O}^1(D_M)$ , which implies that  $\mathcal{J}^*: \mathbf{H}_{0,\partial O}^1(D_M)^* \rightarrow \mathbf{H}_{0,\partial O}^1(D_L)^*$  is a left inverse operator for  $\mathcal{R}^*$ , in other words:

$$\mathcal{R}\mathcal{J} = I_{\mathbf{H}_{0,\partial O}^1(D_L)}, \quad \mathcal{J}^*\mathcal{R}^* = I_{\mathbf{H}_{0,\partial O}^1(D_L)^*}. \quad (6.72)$$

We introduce the space

$$V_M := \{v \in \mathbf{H}_{0,\partial O}^1(D_M), A_{L,M}v, A_{L,M}^\beta v \in \text{Range}(\mathcal{R}^*)\}.$$

Let us check that the space  $\text{Range}(\mathcal{R}^*)$  is closed, which in turn will imply that  $V_M$  is a Hilbert space. Indeed, we consider a sequence  $(F_m)_{m \in \mathbb{N}}$  in  $\mathbf{H}_{0,\partial O}^1(D_L)^*$  such that  $\mathcal{R}^*(F_m) \rightarrow G$  in  $\mathbf{H}_{0,\partial O}^1(D_M)^*$  when  $m \rightarrow +\infty$ . In view of (6.72), we have that  $F_m = \mathcal{J}^*\mathcal{R}^*F_m \rightarrow \mathcal{J}^*(G)$  in  $\mathbf{H}_{0,\partial O}^1(D_L)^*$ , then  $G = \mathcal{R}^*\mathcal{J}^*G$ , which completes the verification. We now introduce the operators  $\tilde{A}_{L,M} := A_{L,M}|_{V_M}: V_M \rightarrow \text{Range}(\mathcal{R}^*)$ ,  $\tilde{A}_{L,M}^\beta := A_{L,M}^\beta|_{V_M}: V_M \rightarrow \text{Range}(\mathcal{R}^*)$  and the identity operator  $\tilde{I}: V_M \rightarrow V_M$ . If  $A_{L,M}$  is invertible, then the operator  $\tilde{A}_{L,M}$  is invertible as well since it is injective and surjective. The operators  $\tilde{A}_{L,M}$  and  $\tilde{A}_{L,M}^\beta$  are then related to each other by

$$\tilde{A}_{L,M}^\beta = \tilde{A}_{L,M}(\tilde{I} + \tilde{A}_{L,M}^{-1}(\tilde{A}_{L,M}^\beta - \tilde{A}_{L,M})). \quad (6.73)$$

The objective is now to prove that for large  $M$ , the operator  $\tilde{A}_{L,M}^\beta$  is invertible, its inverse having a bound which is independent of  $\beta$  and  $M$ . In this view, we introduce the sesquilinear form in  $\mathbf{H}_{0,\partial O}^1(D^+) \times \mathbf{H}_{0,\partial O}^1(D^+)$  defined by

$$a(u, v) := \int_{D^+} \left( \alpha \mu \partial_y u \partial_y \bar{v} + \frac{\mu}{\alpha} \partial_x u \partial_x \bar{v} - \frac{\mu}{\alpha} k^2 u \bar{v} \right) dx dy - \langle T_0 u, \bar{v} \rangle_{\Sigma_0}, \quad (6.74)$$

as well as the operator  $A: \mathbf{H}_{0,\partial O}^1(D^+) \rightarrow \mathbf{H}_{0,\partial O}^1(D^+)^*$  such that for all  $(u, v) \in \mathbf{H}_{0,\partial O}^1(D^+) \times \mathbf{H}_{0,\partial O}^1(D^+)$ ,

$$\langle Au, \bar{v} \rangle_{D^+} := a(u, v).$$

We also introduce the restriction operator  $\mathcal{R}_{\infty \rightarrow a}: \mathbf{H}_{0,\partial O}^1(D^+) \mapsto \mathbf{H}_{0,\partial O}^1(D_a)$  as well as its adjoint operator  $\mathcal{R}_{\infty \rightarrow a}^*: \mathbf{H}_{0,\partial O}^1(D_a)^* \mapsto \mathbf{H}_{0,\partial O}^1(D^+)^*$ , for  $a = L, M$ . By an obvious extension of Proposition 15 to

any source term  $G \in H^1(D_L)^*$ , we have that  $A_{L,M}$  is invertible if and only if  $A$  is invertible, and in that case:

$$\tilde{A}_{L,M}^{-1} = \mathcal{R}_{\infty \rightarrow M} A^{-1} \mathcal{R}_{\infty \rightarrow L}^*.$$

In what follows, the norm of all operators are denoted  $\|\cdot\|$ , but their meaning of course depends on the spaces involved. Given that  $\|\mathcal{R}_{\infty \rightarrow a}\| = \|\mathcal{R}_{\infty \rightarrow a}^*\| \leq 1$  for  $a = L, M$ , we end up with

$$\|\tilde{A}_{L,M}^{-1}\| \leq \|A^{-1}\|. \quad (6.75)$$

In view of (6.70), (6.71), (6.73) and (6.75), we conclude that if  $A_{L,M}$  is invertible, then  $\tilde{A}_{L,M}^\beta$  is invertible as well for sufficiently large  $M$  and

$$\|(\tilde{A}_{L,M}^\beta)^{-1}\| \leq 2\|\tilde{A}_{L,M}^{-1}\| \leq 2\|A^{-1}\|, \quad (6.76)$$

which is the required result.

Besides, from (6.68) we obtain

$$A_{L,M}^\beta(u - u_{L,M}^\beta) = (A_{L,M}^\beta - A_{L,M})u,$$

which implies, since  $u - u_{L,M}^\beta \in V_M$  and  $(A_{L,M}^\beta - A_{L,M})u \in \text{Range}(\mathcal{R}^*)$ , that

$$\|u - u_{L,M}^\beta\|_{H_{0,\partial\Omega}^1(D_M)} \leq \|(\tilde{A}_{L,M}^\beta)^{-1}\| \|A_{L,M}^\beta - A_{L,M}\| \|u\|_{H^1(D_M)}.$$

This estimate, together with (6.70), (6.71) and (6.76), leads to

$$\|u - u_{L,M}^\beta\|_{H^1(D_L)} \leq C C_\beta \max(1, \beta) e^{-\beta(M-L)} \|u\|_{H^1(D^+)}.$$

The proof is complete.  $\square$

**Remark 6.1.** *The estimate (6.66) shows that for a fixed  $\beta$  the error  $\|u - u_{L,M}^\beta\|_{H^1(\Omega_L)}$  tends exponentially to 0 when  $M - L$  tends to  $+\infty$ , with an exponential rate equal to  $\beta$ . However, it would be desirable to specify how the constant  $C_\beta$  in (6.66), which coincides with the constant  $C_\beta$  in (6.63), depends on  $\beta$ . For example, if one could prove that  $C_\beta$  has a polynomial behaviour with respect to  $\beta$ , then for a fixed  $M > L$ , one could also deduce that the error  $\|u - u_{L,M}^\beta\|_{H^1(\Omega_L)}$  tends exponentially to 0 when  $\beta$  tends to  $+\infty$ . Once again, the major obstacle is the fact that we have no idea whether the eigenfunctions  $\varphi_n$  of the transverse operator  $L$  given by (2.13) form a complete basis or not. Such non-selfadjoint operator, due to the jump of the coefficients  $\alpha$  and  $\mu$  in the principal part, cannot be seen as a first order perturbation of the Laplace operator, therefore is not a particular case of the second-order Sturm-Liouville operators studied in [Naimark, 1967], for example.*

## 7 Numerical experiments

In the following numerical computations, we set  $h = 5 \times 10^{-2} m$ ,  $h_{\text{in}} = 7.5 \times 10^{-2} m$  and  $h_{\text{out}} = 12.5 \times 10^{-2} m$ . The core is made of steel, with  $\mu_0 = 84.298 \times 10^9 Pa$  and  $\rho_0 = 7932 kg.m^{-3}$ , while the sheath is made of concrete, with  $\mu_\infty = 15.908 \times 10^9 Pa$  and  $\rho_\infty = 2300 kg.m^{-3}$ . Using those values, it is easy to check that  $c_0 > c_\infty$ , which is consistent with Assumption 2.2. We also take  $\omega = 2\pi \times 10^5 Hz$ . All the numerical experiments have been implemented using the FEM-BEM library XLiFE++ (see [Kielbasiewicz & Lunéville, 2019]).

### 7.1 Computation of the modes

First of all, we wish to compute the transverse eigenvalues  $\pm\lambda_n$  in  $\Lambda$  by using one-dimensional Lagrange  $P1$  finite elements and a regular mesh in the transverse section  $(-h_{\text{out}}, h_{\text{out}})$  of size  $10^{-3} cm$ . These approximate eigenvalues are plotted on the Figure 5 for the abrupt PMLs ( $\alpha$  is defined by (2.11) with  $\alpha_\infty = e^{-i\pi/3}$ ) and for the smooth PMLs ( $\alpha$  is defined by (2.14) with  $b = 3(3 + 4i)$ ). We observe three kinds of computed eigenvalues: those which correspond to leaky modes (the red circles), those which correspond to PML modes (the blue squares) and those which correspond to spurious modes (the green triangles), that is modes which are due to the discretization and not to the truncation of the PMLs

(see for example [Kim & Pasciak, 2009, Nannen & Wess, 2018] for discussions about spurious modes in the presence of PMLs). We now discuss how we distinguish the different modes. The spurious modes are identified by using the dispersion relationship (3.21), which is not satisfied by the spurious eigenvalues, but is by the other modes (up to the discretization error). We observe that within our computation window, spurious modes only appear in the case of the abrupt PMLs. Since the support of a leaky mode is mostly included in the physical part of the medium while the support of a PML mode is mostly included in the PMLs, we adopt the following criterium to separate the computed leaky modes from the computed PML modes. Following [Nguyen et al., 2015], we associate to sections  $S^{\text{tot}} := (-h_{\text{out}}, h_{\text{out}})$  and  $S^{\text{PML}} := (-h_{\text{out}}, -h_{\text{in}}) \cup (h_{\text{in}}, h_{\text{out}})$  the generalized kinetic energy of the mode  $\varphi_n$  defined by

$$E_n^{\text{tot}} := \frac{\omega^2}{4} \int_{S^{\text{tot}}} \frac{\rho}{\alpha} |\varphi_n|^2 dy, \quad E_n^{\text{PML}} := \frac{\omega^2}{4} \int_{S^{\text{PML}}} \frac{\rho}{\alpha} |\varphi_n|^2 dy.$$

For some prescribed  $\eta \in (0, 1)$ , if  $|E_n^{\text{PML}}|/|E_n^{\text{tot}}| < \eta$ , the mode  $\varphi_n$  is considered as a leaky mode. If not, it is considered as a PML mode. In practice, we choose  $\eta = 0.4$  for abrupt PMLs and  $\eta = 0.9$  for smooth PMLs. Indeed, the numerical experiments have shown that both PML and spurious modes are almost totally confined in the PML for the smooth profile, whereas it is not as sharp for the abrupt profile. As a consequence, the choice of  $\eta$  is easier for the smooth profile.

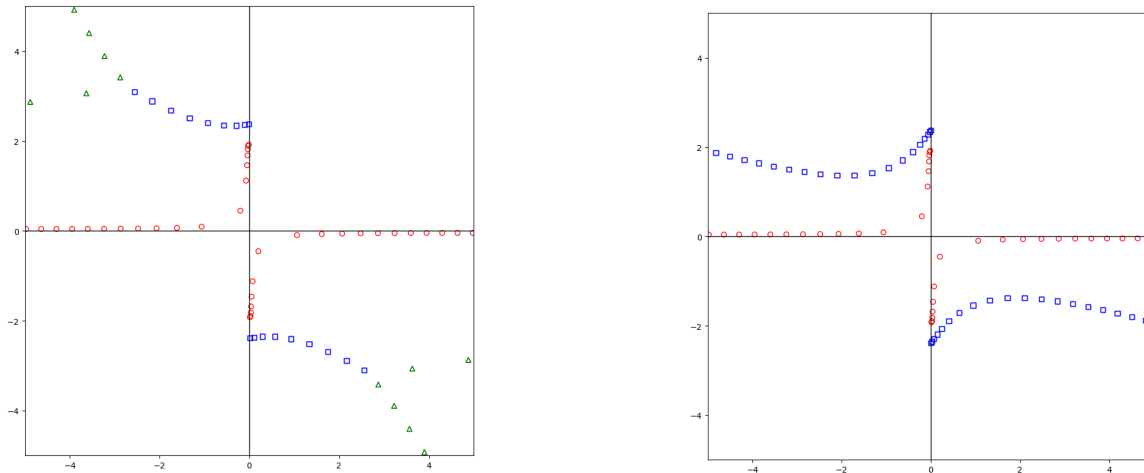


Figure 5: Discretized transverse eigenvalues related to the open waveguide closed by truncated PMLs (red circles correspond to leaky modes, blue squares correspond to PML modes, green triangles correspond to spurious modes). Left: abrupt PMLs. Right: smooth PMLs

## 7.2 The role of PMLs

In order to emphasize the role of PMLs, we consider the approximation of the total solution  $u$  of the problem (5.41) without any obstacle in the whole domain  $\Omega$ , for  $u^i = \tilde{w}_{0,0}^+$ , where  $\tilde{w}_{0,0}^+$  is defined by (5.40) and  $\tilde{w}_0^+ = e^{ik_0 x} / \sqrt{2h}$ . In other words, we send the first propagating mode of the closed waveguide coming from the left, which happens to be a plane wave. In Figure 6, where the real part of the solution  $u_{L,M}^\beta$  to the problem (6.58) is represented, we compare the abrupt PMLs given by (2.11) with  $\alpha_\infty = e^{-i\pi/3}$  (left picture) and the smooth PMLs given by (2.14) with  $b = 3(3+4i)$  (right picture). The computation is based on a two-dimensional Lagrange  $P2$  finite element method, the mesh size being  $0.15\text{cm}$ . The left Dirichlet-to-Neumann operator  $T_0$  given by (5.43) is discretized by truncating the series by the first 22 terms corresponding to 7 propagating modes and 15 evanescent modes. The Dirichlet-to-Neumann operator with an overlap  $T_{L,M}^\beta$  given by (6.57) is used on the right with  $\beta = 1.07$ ,  $L = 15\text{cm}$  and  $M = 25\text{cm}$ . For the particular functions  $\alpha$  that we chose, we remark that in the abrupt PMLs the solution  $u_{L,M}^\beta$  vanishes almost everywhere while in the smooth ones it vanishes at the outer boundary of the PMLs. The advantage of smooth PMLs is that they are continuous at the boundary between the physical and the artificial media (at  $y = \pm h_{\text{in}}$ ), which at the discrete level reduces spurious reflexions (see for example [Joly, 2012]).

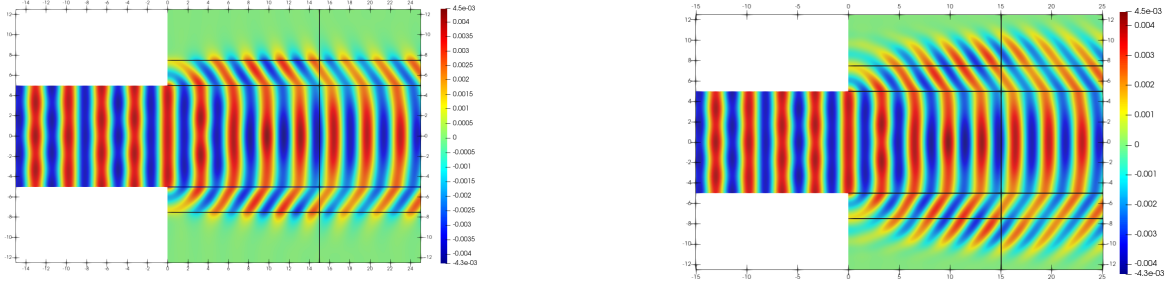


Figure 6: Left: abrupt PMLs. Right: smooth PMLs.

### 7.3 The role of the overlap in the right DtN operator

We now emphasize the role of the overlap in the right Dirichlet-to-Neumann operator  $T_{L,M}^\beta$  for the problem (5.53) in the presence of a Dirichlet obstacle formed by a disk centered at point  $(6\text{ cm}, 1.75\text{ cm})$  and of radius  $1.25\text{ cm}$ . The incident field  $u^i$  is the same as above, that is we send the first propagating mode of the left closed waveguide. We represent on Figure 7 the real part of the total solution  $u$  when either a thin DtN operator ( $M=L=10\text{ cm}$ ) or a thick DtN operator ( $L=10\text{ cm}$  and  $M=20\text{ cm}$ ) is used. Here we have chosen the smooth PMLs and  $\beta=0.04$ . The computation is again based on a two-dimensional Lagrange  $P2$  finite element method, the mesh size being  $0.15\text{ cm}$  and the approximation of the operator  $T_0$  is the same as previously. On Figure 7, small differences in the physical domain between the two solutions can be seen. To refine our study, we compare the solution  $u_{L,M}^\beta$  obtained

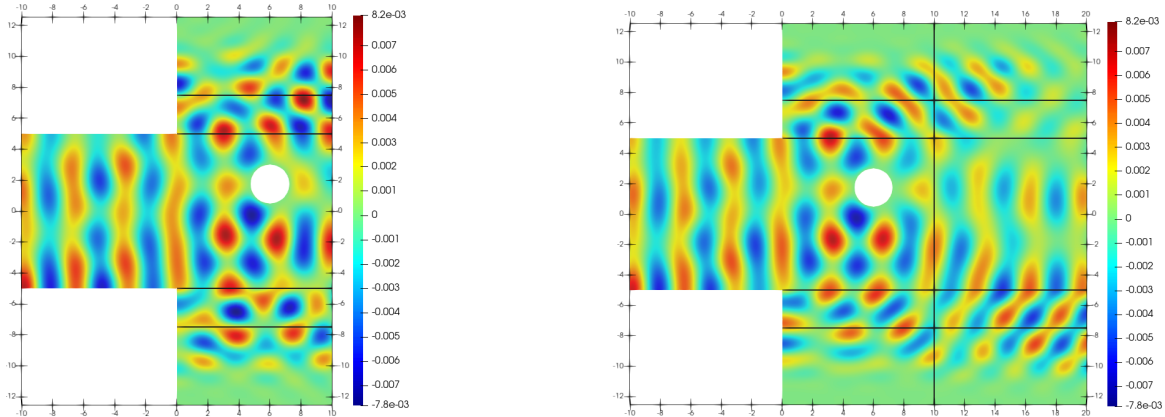


Figure 7: Left: thin DtN. Right: DtN with an overlap.

for different values of the overlap and  $\beta=0.21$ , precisely  $L=10$  and  $M=10$  (thin DtN),  $M=11$ ,  $M=12$ ,  $M=13$  and  $M=14$ , with a reference solution  $u_{\text{ref}} := u_{L,M_{\text{ref}}}^{\beta_{\text{ref}}}$  obtained with a large value of  $\beta$ , that is  $\beta_{\text{ref}}=1.62$ , and a large overlap, that is  $M_{\text{ref}}=20$ . The computation is again based on a two-dimensional Lagrange  $P2$  finite element method, but with a refined mesh of size  $0.05\text{ cm}$ , while the truncated series defining the left Dirichlet-to-Neumann operator  $T_0$  contains 37 terms, that is 7 propagating modes and 30 evanescent modes. We have plotted the function  $|u_{L,M}^\beta - u_{\text{ref}}| / \|u_{\text{ref}}\|_\infty$  on Figure 8. In view of the estimate (6.66), we expect that the error between the exact solution and the solution approached with the DtN operator improves with respect to the length of the overlap ( $M-L$ ), which is exactly what Figure 8 illustrates.

In order to be more quantitative, we complete this numerical section by testing the estimate (6.66) numerically, that is we plug the graph of the function  $f$  such that

$$\log \left( \frac{\|u_{L,M}^\beta - u_{\text{ref}}\|_{H^1(D_L)}}{\|u_{\text{ref}}\|_{H^1(D_L)}} \right) = f(M-L),$$

which is supposed to be at worst a line of slope  $-\beta$ . On the top left picture of Figure 9, we have drawn this function in the case of the abrupt PMLs for different increasing values of  $\beta$ . Increasing  $\beta$

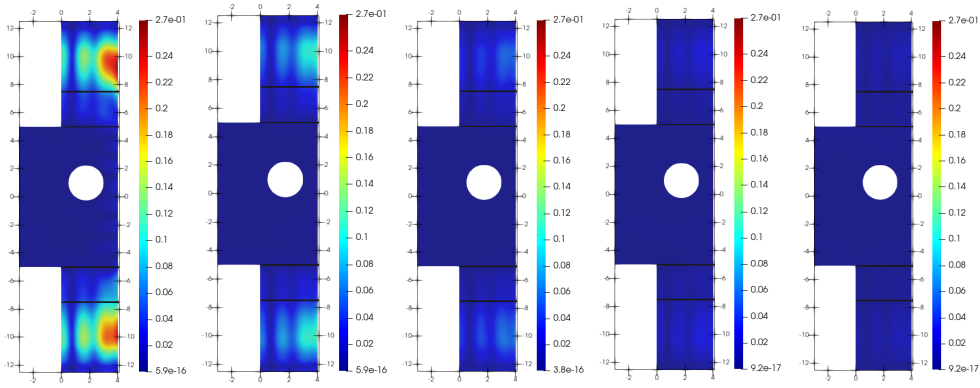


Figure 8: Impact of the overlap on the numerical error. From left to right:  $M=L=10$ ,  $M=11$ ,  $M=12$ ,  $M=13$ ,  $M=14$ .

amounts to increase the number of modes  $N_\beta$  taken in the series (6.57). In the Figure 10, we have specified, for each value of  $\beta$ , the number  $N_{\text{leaky}}$  of leaky modes and the number  $N_{\text{PML}}$  of PML modes. We observe that when  $\beta$  is not too large, the graph of  $f$  looks like a line, the slope of which increases with  $\beta$ , which is consistent with the estimate (6.66). However, the absolute value of these slopes are much larger than  $\beta$ , which is hence a pessimistic value of the slope. In addition, it seems that, except for  $\beta=1.07$  and  $\beta=1.62$ , the larger is  $\beta$ , the better is the approximation. The swap between the curves corresponding to  $\beta=1.07$  and  $\beta=1.62$  for small values of the overlap ( $M-L$ ) is due to a bad discrete approximation of the PML modes  $\varphi_n$  which correspond to large absolute values of the real parts  $\Re(\lambda_n)$ : the solver does not really separate those PML modes from the spurious modes. A precise explanation of that phenomenon for abrupt PMLs (based on the notion of pseudo-spectrum) can be found in [Goursaud, 2010], where it is shown that when the distance between the PMLs and the core increases, the approximation of both leaky and PML modes worsens and the spurious modes appear for smaller  $|\Re(\lambda)|$ . The consequence is that it deteriorates the approximation of the Dirichlet-to-Neumann operator, and therefore the approximation of the solution. The top right and bottom pictures of Figure 9 illustrate two different ways to fix this problem. The first one consists in using our smooth PMLs instead of the abrupt ones. The bad approximation of high order PML modes seems not to happen with the smooth PMLs that we chose, which is a consequence of the fact that, as already emphasized on Figure 5, spurious modes appear for larger values of  $|\Re(\lambda_n)|$  than for abrupt PMLs. The second way consists in sticking the abrupt PMLs to the core of the waveguide, that is  $h_{\text{in}}=h$ , the thickness ( $h_{\text{out}}-h_{\text{in}}$ ) of the PML being unchanged, which improves the computation of the PML modes. However, it should be noted that such technique is not satisfactory if one wants to compute the solution in the sheath and not only in the core of the waveguide.

Lastly, we note that the curves which correspond to  $\beta=1.07$  and  $\beta=1.62$  almost coincide for large ( $M-L$ ), whatever the PMLs, which is a saturation effect due to the fact that the mesh is too coarse for such a number of modes.

**Remark 7.1.** *The reader should not conclude from our numerical results that smooth PMLs give better results than abrupt PMLs in general. Our experience is that those results depend in a complex way on the properties of the function  $\alpha$  together with the mesh size. A comprehensive comparison between these two kinds of profiles was not the objective of the present paper. We only claim that taking into account spurious modes worsens the quality of the numerical approximation of the DtN operators.*

## Acknowledgment

The authors are indebted to Anne-Sophie Bonnet-Ben Dhia for helpful discussions.



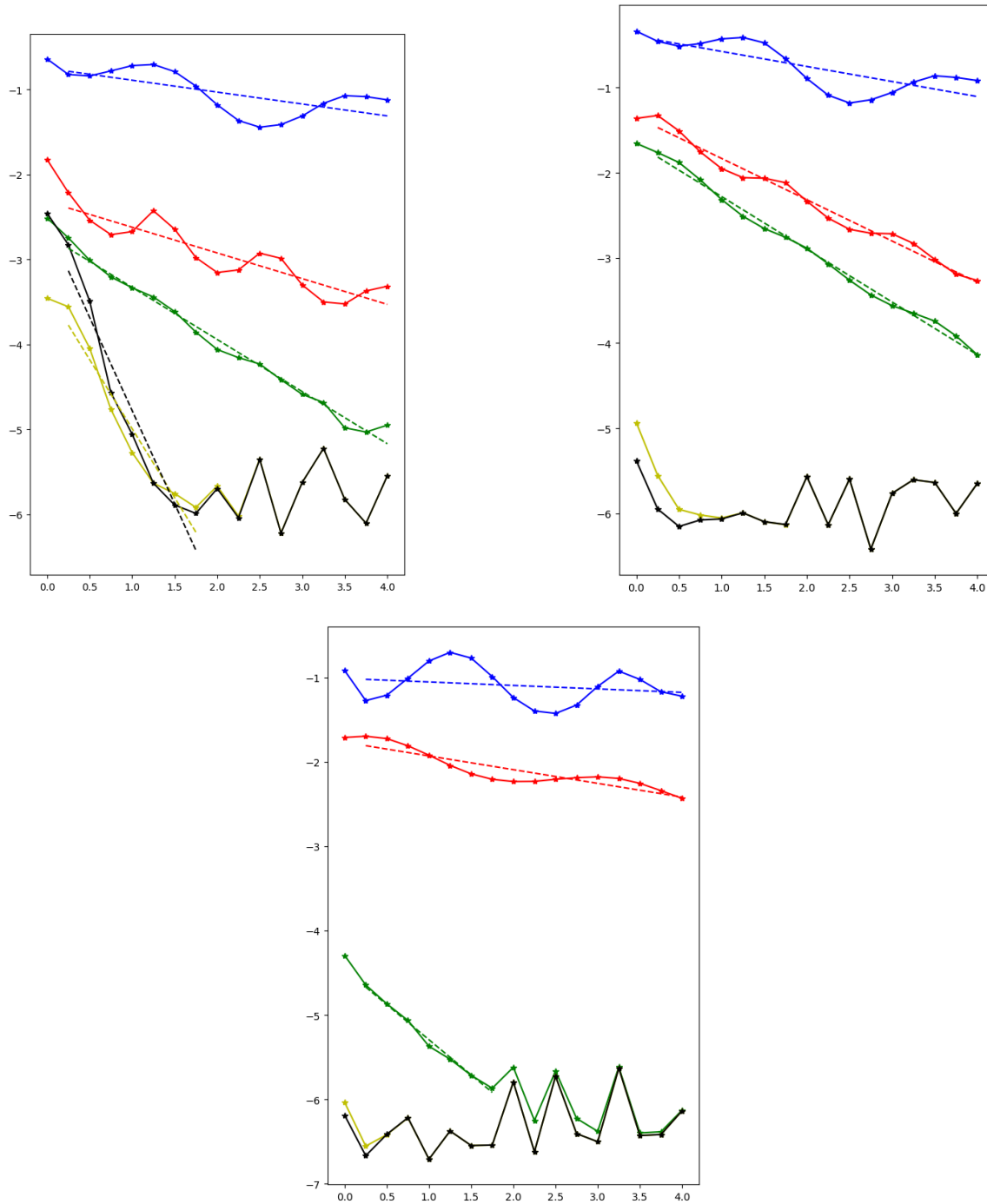


Figure 9: Error estimate with respect to  $(M-L)$ . Blue:  $\beta=0.038$ . Red:  $\beta=0.073$ . Green:  $\beta=0.21$ . Yellow:  $\beta=1.07$ . Black:  $\beta=1.62$ . Top left: abrupt PMLs. Top right: smooth PMLs. Bottom: abrupt PMLs stuck to the core

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	Abrupt PMLs		Smooth PMLs		Stuck abrupt PMLs	
$\beta$	$N_{\text{leaky}}$	$N_{\text{PML}}$	$N_{\text{leaky}}$	$N_{\text{PML}}$	$N_{\text{leaky}}$	$N_{\text{PML}}$
0.038	3	2	3	4	3	2
0.073	6	2	6	6	5	2
0.21	7	4	7	8	7	4
1.07	8	10	8	12	8	8
1.62	9	12	9	14	9	12

Figure 10: Number of leaky and PML modes for the different values of  $\beta$  in Figure 9

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