



**HAL**  
open science

## Bridging the multiscale hybrid-mixed and multiscale hybrid high-order methods

Théophile Chaumont-Frelet, Alexandre Ern, Simon Lemaire, Frédéric Valentin

► **To cite this version:**

Théophile Chaumont-Frelet, Alexandre Ern, Simon Lemaire, Frédéric Valentin. Bridging the multiscale hybrid-mixed and multiscale hybrid high-order methods. ICOSAHOM 2020 - International Conference on Spectral and High-Order Methods, Jul 2021, Vienna / Virtual, Austria. hal-03403920

**HAL Id: hal-03403920**

**<https://inria.hal.science/hal-03403920>**

Submitted on 26 Oct 2021

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Bridging the multiscale hybrid-mixed and multiscale hybrid high-order methods

---

T. Chaumont-Frelet<sup>\*,†</sup>, A. Ern<sup>‡,\*</sup>, S. Lemaire<sup>\*,§</sup>, F. Valentin<sup>#</sup>

ICOSAHOM, July 2021

\*Inria †Laboratoire J.A. Dieudonné, ‡CERMICS, §Laboratoire Paul Painlevé, #LNCC

# Outline

- 1 What are multiscale methods?
- 2 The multiscale hybrid-mixed (MHM) method
- 3 The multiscale hybrid high-order (MsHHO) method
- 4  $\text{MHM} = \text{HHO}$ ?

# **What are multiscale methods?**

---

# Multiscale problems

Consider  $\Omega \subset \mathbb{R}^d$ . Given  $f : \Omega \rightarrow \mathbb{R}$ , find  $u : \Omega \rightarrow \mathbb{R}$  such that

$$\begin{cases} -\nabla \cdot (\mathbf{A}_\varepsilon \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the diffusion tensor  $\mathbf{A}_\varepsilon$  exhibits “multiscale” features.

A key application is Darcy equations, modeling flows in porous rocks.

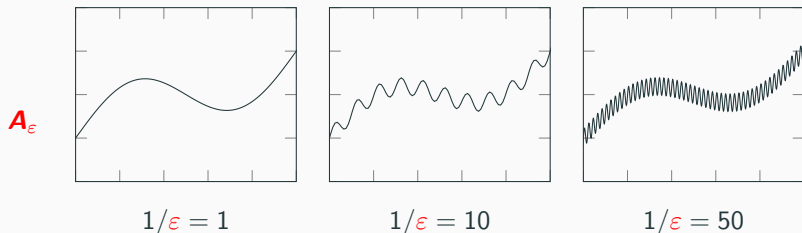
# What are these multiscale features?

For the sake, of simplicity, we will consider here the case where

$$\mathbf{A}_\varepsilon(\mathbf{x}) := \widehat{\mathbf{A}}\left(\mathbf{x}, \left\{\frac{\mathbf{x}}{\varepsilon}\right\}\right),$$

where  $\widehat{\mathbf{A}} : \Omega \times (0, 1)^d \rightarrow \mathbb{R}^{d \times d}$  is smooth, and

$\widehat{\mathbf{A}}(\mathbf{x}, \mathbf{y})$  is  $\mathbf{y}$ -periodic for all  $\mathbf{x} \in \Omega$ .



# Goal

Resolving the fine scale with a mesh size  $h \simeq \varepsilon$  is too expensive.

We want to capture the “macroscopic behavior” of  $u$ , with a mesh size  $H$ .

We would like error estimates in  $\mathcal{O}(H)$ , with  $H \gg \varepsilon$ .

# Limitations of standard FEM

Standard FEM fails. Indeed, we have

$$-f = \nabla \cdot (\mathbf{A}_\varepsilon \nabla u) = (\nabla \cdot \mathbf{A}_\varepsilon) \cdot \nabla u + \mathbf{A}_\varepsilon : \nabla^2 u$$

with  $|\nabla \cdot \mathbf{A}_\varepsilon| \simeq 1/\varepsilon$ , leading to

$$\|\nabla^2 u\|_\Omega \lesssim \frac{1}{\varepsilon} \|f\|_\Omega,$$

and

$$\|\nabla(u - u_H)\|_\Omega \lesssim \frac{H}{\varepsilon} \|f\|_\Omega,$$

with  $\mathcal{P}_1$  elements.

This is sharp: standard FEM needs to resolve the small scales ( $H \leq \varepsilon$ ).



## Key ideas in multiscale methods

We would like to keep the same number of dofs as the standard  $\mathcal{P}_1$  FEM.

Hence, we preserve the mesh size  $H$ , but we modify the shape functions.

These modified basis functions will “upscale” fine scale information.

# New shape functions via local problems

Consider a simplex  $K \in \mathcal{T}_H$ , one of its vertices  $\mathbf{a}$  and

$$\psi^{\mathbf{a}} \in \mathcal{P}_1(K), \quad \psi^{\mathbf{a}}(\mathbf{a}') = \delta_{\mathbf{a}\mathbf{a}'} \quad \forall \mathbf{a}' \in \mathcal{V}_K$$

the “hat function” associated with  $\mathbf{a}$ .

An alternate definition of  $\psi^{\mathbf{a}}$  is that it satisfies

$$\begin{cases} -\Delta \psi^{\mathbf{a}} = 0 & \text{in } K, \\ \psi^{\mathbf{a}} = \psi^{\mathbf{a}} & \text{on } \partial K. \end{cases}$$

This motivates the definition

$$\begin{cases} -\nabla \cdot (\mathbf{A}_\varepsilon \nabla \psi_\varepsilon^{\mathbf{a}}) = 0 & \text{in } K, \\ \psi_\varepsilon^{\mathbf{a}} = \psi^{\mathbf{a}} & \text{on } \partial K, \end{cases}$$

for the multiscale shape functions.

# Multiscale shape functions

Recall that:

$$\begin{cases} -\nabla \cdot (\mathbf{A}_\varepsilon \nabla \psi_\varepsilon^{\mathbf{a}}) = 0 & \text{in } K, \\ \psi_\varepsilon^{\mathbf{a}} = \psi^{\mathbf{a}} & \text{on } \partial K. \end{cases} \quad (1)$$

Since we preserved the boundary conditions at each element, we automatically construct an  $H_0^1(\Omega)$ -conforming basis.

The  $\psi_\varepsilon^{\mathbf{a}}$  will incorporate some  $\varepsilon$ -scale oscillations.

In practice, (1) is approximated numerically on a fine mesh, which is possible because the problem is local to the element  $K$ .

Step 1: compute for each vertex the hat function  $\psi_\epsilon^{\mathbf{a}}$ .

This is done by solving uncoupled fine scale problems on small domains.

Step 2: compute the multiscale approximation  $u_{\epsilon,H} \in \text{span}_{\mathbf{a} \in \mathcal{V}_H} \{\psi_\epsilon^{\mathbf{a}}\}$

$$(\mathbf{A}_\epsilon \nabla u_{\epsilon,H}, \nabla \psi_\epsilon^{\mathbf{a}'})_\Omega = (\mathbf{f}, \psi_\epsilon^{\mathbf{a}'})_\Omega \quad \forall \mathbf{a}' \in \mathcal{V}_H.$$

# Multiscale error estimates

## Standard FEM

$$\|\nabla(u - u_H)\|_{\Omega} \lesssim \frac{H}{\varepsilon} \|f\|_{\Omega}$$

## Multiscale FEM

$$\|\nabla(u - u_{\varepsilon,H})\|_{\Omega} \lesssim \left( H + \sqrt{\frac{\varepsilon}{H}} \right) \|f\|_{\Omega}$$



T.Y. Hou, X.H Wu, Z. Cai, 1999.

## Summary on multiscale methods

Multiscale methods are robust when  $H \gg \epsilon$ .

Modified shape functions are computed via local fine scale problems. Here, we will assume these problems are exactly solved for simplicity.

The MHM and MsHHO are two multiscale methods recently introduced.

# The multiscale hybrid-mixed (MHM) method

---

# Some history



P.A. Raviart and J.M. Thomas, 1977: A source of inspiration.



R. Araya, C. Harder, D. Paredes and F. Valentin, 2013: Darcy equations.



D. Paredes, F. Valentin and H. Versieux, 2017: Multiscale error analysis.



R. Araya, C. Harder, A.H. Pozas and F. Valentin, 2017: Stoke's equations.



T. Chaumont-Frelet and F. Valentin, 2020: Helmholtz problems.



# **The multiscale hybrid-mixed (MHM) method**

---

**The primal hybrid formulation**

# The primal hybrid formulation

Consider the energy functional

$$J(v) := \frac{1}{2}(\mathbf{A}\nabla v, v) - (f, v)_\Omega, \quad v \in H_0^1(\Omega).$$

Then, we have the characterization

$$u = \arg \min_{v \in H_0^1(\Omega)} J(v).$$

The key idea of the primal hybrid formulation is to relax continuity through the use of a Lagrange multiplier.



P.A. Raviart and J.M. Thomas, 1977.

# Piecewise smooth functions

Consider a polytopal mesh  $\mathcal{T}_H$  of  $\Omega$  and the “broken” space

$$V := \{v \in L^2(\Omega) \mid v|_K \in H^1(K) \forall K \in \mathcal{T}_H\}.$$

We would like to characterize  $H_0^1(\Omega) \subset V$ .

Let  $v \in V$ . Then  $v \in H_0^1(\Omega)$  iff

$$\sum_{K \in \mathcal{T}_H} \int_{\partial K} \sigma \cdot \mathbf{n}_K v = 0 \quad \left( \simeq \sum_{F \in \mathcal{F}_H} \int_F \sigma \cdot \mathbf{n}_F \llbracket v \rrbracket \right)$$

for all  $\sigma \in \mathbf{H}(\text{div}, \Omega)$ .

# Lagrange multiplier

We see that traces of  $\mathbf{H}(\text{div}, \Omega)$  are involved. We thus set

$$\Lambda := \left\{ \lambda \in \prod_{K \in \mathcal{T}_H} H^{-1/2}(\partial K) \mid \exists \boldsymbol{\sigma} \in \mathbf{H}(\text{div}, \Omega); \lambda|_{\partial K} = \boldsymbol{\sigma} \cdot \mathbf{n}_K \quad \forall K \in \mathcal{T}_H \right\},$$

and

$$\langle \lambda, \mathbf{v} \rangle := \sum_{K \in \mathcal{T}_H} \int_{\partial K} \lambda \mathbf{v}.$$

Hence  $\mathbf{v} \in \mathbf{V}$  belongs to  $H_0^1(\Omega)$  iff

$$\langle \lambda, \mathbf{v} \rangle = 0 \quad \forall \lambda \in \Lambda.$$

# The primal hybrid formulation

We have

$$u = \arg \min_{v \in H_0^1(\Omega)} J(v) = \arg \min_{\substack{v \in V \\ \langle \mu, v \rangle = 0 \quad \forall \mu \in \Lambda}} J(v).$$

The primal hybrid formulation is the Euler-Lagrange equations:

Find  $(u, \lambda) \in V \times \Lambda$  such that

$$\begin{cases} (\mathbf{A} \nabla u, \nabla v)_{\mathcal{T}_H} + \langle \lambda, v \rangle = (f, v) & \forall v \in V, \\ \langle \mu, u \rangle = 0 & \forall \mu \in \Lambda. \end{cases}$$

$u$  is the original solution, and  $\lambda|_{\partial K} = \mathbf{A} \nabla u \cdot \mathbf{n}_K$ .



P.A. Raviart and J.M. Thomas, 1977.

# The multiscale hybrid-mixed (MHM) method

---

The MHM formulation

## Local problems

We can rewrite the first equation of the primal hybrid formulation as

$$(\mathbf{A}\nabla u, \nabla v)_{\mathcal{T}_H} = (f, v)_{\Omega} - \langle \lambda, v \rangle \quad \forall v \in V,$$

which we may rewrite element-wise as

$$\begin{cases} -\nabla \cdot (\mathbf{A}\nabla u) = f & \text{in } K, \\ \mathbf{A}\nabla u \cdot \mathbf{n}_K = \lambda & \text{on } \partial K, \end{cases}$$

for all  $K \in \mathcal{T}_H$ .

If  $\lambda$  is known, we obtain  $u$  via Neumann problems.

This idea is behind the definition of multiscale shape functions.

# Kernel extraction

Neumann problems have a kernel: we need to extract constants.

Let  $V = \mathcal{P}_0(\mathcal{T}_H) \oplus V^\perp$  and  $u = u_0 + u^\perp$ . We have

$$(\mathbf{A}\nabla u^\perp, \nabla v^\perp)_{\mathcal{T}_H} + \langle \lambda, v^\perp \rangle = (\nabla u, \nabla v^\perp)_{\mathcal{T}_H} + \langle \lambda, v^\perp \rangle = \langle f, v^\perp \rangle$$

for all  $v^\perp \in V^\perp$ .

Hence,  $u^\perp = T\lambda + \hat{T}f$ , where the operators

$$(\mathbf{A}\nabla(T\mu), \nabla v^\perp) = -\langle \mu, v^\perp \rangle \quad (\mathbf{A}\nabla(\hat{T}k), \nabla v^\perp) = (k, v^\perp)_\Omega \quad \forall v^\perp \in V^\perp$$

for  $\mu \in \Lambda$  and  $k \in L^2(\Omega)$  are defined via local Neumann problems.



# The MHM formulation

Recall the primal hybrid formulaion

$$\begin{cases} (\mathbf{A}\nabla u, \nabla v)_{\mathcal{T}_H} + \langle \lambda, v \rangle = (f, v) & \forall v \in V, \\ \langle \mu, u \rangle = 0 & \forall \mu \in \Lambda. \end{cases}$$

the decomposition  $u = u_0 + u^\perp$ , and  $u^\perp = T\lambda + \widehat{T}f$ .

It gives: Find  $(u_0, \lambda) \in \mathcal{P}_0(\mathcal{T}_H) \times \Lambda$  such that

$$\begin{cases} \langle \mu, u_0 \rangle + \langle \mu, T\lambda \rangle = -\langle \mu, \widehat{T}f \rangle & \forall \mu \in \Lambda, \\ \langle \lambda, v_0 \rangle = (f, v_0)_\Omega & \forall v_0 \in \mathcal{P}_0(\mathcal{T}_H). \end{cases}$$

This global saddle point problem is the MHM formulation.

# The multiscale hybrid-mixed (MHM) method

---

The MHM method

# Discrete spaces

Recall that

$$\begin{cases} \langle \mu, u_0 \rangle + \langle \mu, T\lambda \rangle = -\langle \mu, \hat{T}f \rangle & \forall \mu \in \Lambda, \\ \langle \lambda, v_0 \rangle = (f, v_0)_\Omega & \forall v_0 \in \mathcal{P}_0(\mathcal{T}_H). \end{cases}$$

The space  $\mathcal{P}_0(\mathcal{T}_H)$  is already discrete. We thus need  $\Lambda_H \subset \Lambda$ .

Fixing  $k \geq 0$ , we consider

$$\Lambda_H := \{ \mu_H \in \Lambda \mid \mu|_F \in \mathcal{P}_k(F) \forall F \in \mathcal{F}_H \}.$$

# The MHM method

The MHM method consists in finding  $(u_{0,H}, \lambda_H) \in \mathcal{P}_0(\mathcal{T}_H) \times \Lambda_H$  s.t.

$$\begin{cases} \langle \mu_H, u_{0,H} \rangle + \langle \mu_H, T\lambda_H \rangle = -\langle \mu_H, \hat{T}f \rangle & \forall \mu_H \in \Lambda_H, \\ \langle \lambda_H, v_0 \rangle = (f, v_0)_\Omega & \forall v_0 \in \mathcal{P}_0(\mathcal{T}_H). \end{cases}$$

and then setting

$$u_H := u_{0,H} + T\lambda_H + \hat{T}f.$$

# Implementation

Step 1: consider a basis  $(\mu^\ell)_{\ell=1}^N$  of  $\Lambda_H$ .

Step 2: solve the local Neumann problems: find  $\eta^\ell|_K \in H_*^1(K)$

$$(\mathbf{A}\nabla\eta^\ell, \nabla\mathbf{v}_*)_K = - \int_{\partial K} \mu^\ell \mathbf{v}_* \quad \forall \mathbf{v}_* \in H_*^1(K)$$

for all  $K \in \mathcal{T}_H$ . Compute  $\hat{T}\mathbf{f}$  similarly.

Step 3:  $T\lambda = \eta = \sum_{\ell=1}^N R_\ell \eta^\ell$ ,  $\mathbf{u}_{0,H} = \sum_{K \in \mathcal{T}_H} V_K \mathbf{1}_K$ , and





$$\begin{cases} \langle \mu_H, \mathbf{u}_{0,H} \rangle + \langle \mu_H, \eta \rangle = \langle \mu_H, \hat{T}\mathbf{f} \rangle & \forall \mu_H \in \Lambda_H \\ \langle \lambda_H, \mathbf{v}_{0,H} \rangle = (\mathbf{f}, \mathbf{v}_{0,H}) & \forall \mathbf{v}_{0,H} \in \mathcal{P}_0(\mathcal{T}_H) \end{cases}$$

Step 4: Assemble the solution

$$\mathbf{u}_H = \mathbf{u}_{0,H} + T\lambda_H + \hat{T}\mathbf{f} = \sum_{K \in \mathcal{T}_H} V_K \mathbf{1}_K + \sum_{\ell=1}^N R_\ell \eta^\ell + \hat{T}\mathbf{f}.$$

# The multiscale hybrid high-order (MsHHO) method

---

-  D. Di Pietro, A. Ern and S. Lemaire, 2014: original HHO article.
-  D. Di Pietro and A. Ern, 2015: Elasticity.
-  D. Di Pietro, and J. Droniou, 2017: Leray-Lions equations.
-  M. Cicuttin, A. Ern and S. Lemaire, 2019: multiscale extension.

# Degrees of freedom and reconstruction operator

The HHO method relies on the following set of discrete unknowns

$$\widehat{U}_H := \mathcal{P}_k(\mathcal{T}_H) \times \mathcal{P}_k(\mathcal{F}_H)$$

These unknowns are respectively meant to represent  $u|_K$  and  $u|_F$ .

The method “normally” hinges on a discrete reconstruction operator.

$\mathcal{G} : \widehat{U}_H \rightarrow \nabla \mathcal{P}_{k+1}(\mathcal{T}_H)$ , defined for  $u := (u_{\mathcal{T}}, u_{\mathcal{F}}) \in \widehat{U}_H$  by

$$(\mathcal{G}u, \nabla v)_K = (\nabla u_K, \nabla v)_K + (u_F - u_K, \nabla v \cdot \mathbf{n}_K)_{\partial K}$$

for all  $v \in \mathcal{P}_{k+1}(K)$  and all  $K \in \mathcal{T}_H$ .



# The multiscale variant

In the MsHHO, the “discrete” operator is replaced by local PDEs.  
This enables to upscale fine scale details onto the coarse mesh.

For all  $\hat{u} := (u_T, u_F) \in \hat{U}_H$  and  $K \in \mathcal{T}_H$ ,  $\exists! r_K(u) \in \mathcal{U}_k(K)$  such that

$$\begin{aligned}(\mathbf{A}\nabla r_K(\hat{u}), \nabla v)_K &= -(u_K, \nabla \cdot (\mathbf{A}\nabla v))_K + (u_F, \mathbf{A}\nabla v \cdot \mathbf{n}_K)_{\partial K} \\(r_K(\hat{u}), \mathbf{1}_K)_K &= (u_F, \mathbf{1}_F)_{\partial K}\end{aligned}$$

for all  $v \in \mathcal{U}_k(K)$ , with the finite-dimensional space

$$\mathcal{U}_k(K) := \{v \in H^1(K) \mid \nabla \cdot (\mathbf{A}\nabla v) \in \mathcal{P}_k(K), \mathbf{A}\nabla v \cdot \mathbf{n}_K|_F \in \mathcal{P}_k(F)\}.$$

# The MsHHO method

Step 1: Assemble the  $\mathcal{U}_k(K)$  spaces by solving local Neumann problems.

Step 2: Find dofs  $\hat{u} \in \hat{U}_H$  such that

$$\sum_{K \in \mathcal{T}_H} (\mathbf{A} \nabla r_K(\hat{u}), \nabla r_K(\hat{v}))_K = \sum_{K \in \mathcal{T}_H} (f, \hat{v}_K)_K$$

for all  $\hat{v} \in \hat{U}_H$ .

Step 3: Assemble the solution

$$u_H = \sum_{K \in \mathcal{T}_H} r_K(\hat{u}).$$

**MHM = HHO?**

---

# A first naive look

Cons:

The number of dofs does not match!

The dofs of MHM correspond  $u_{0,K} := \int_K u$  and  $\lambda|_{\partial K} = \mathbf{A} \nabla u \cdot \mathbf{n}_K$ .

The dofs of MsHHO correspond to  $u|_K$  and  $u|_{\partial K}$ .

The global linear systems have different features:

Saddle-point for MHM, SPD for MsHHO.

Pros:

Both methods employ Neumann problems to construct the multiscale basis functions.

# The central result

## Theorem

For all  $f \in L^2(\Omega)$ , there exists a unique  $u \in H^1(\mathcal{T}_H)$  s.t.

$$-\nabla \cdot (\mathbf{A} \nabla_H u) = f \text{ in } \Omega$$

and

$$\mathbf{A} \nabla_H u \cdot \mathbf{n}_F \in \mathcal{P}_k(F) \quad ([u], q)_F = 0 \quad \forall q \in \mathcal{P}_k(F)$$

for all  $F \in \mathcal{T}_H$ .

This  $u$  is linked to the solutions of MHM and MsHHO methods.

## The MHM solution (1/2)

Recall that for the MHM method

$$u_H := u_{0,H} + T\lambda_H + \hat{T}f.$$

Hence,

$$-\nabla \cdot (\mathbf{A}\nabla_H u_H) = -\nabla \cdot (\mathbf{A}\nabla_H T\lambda_H) - \nabla \cdot (\mathbf{A}\nabla_H \hat{T}f) = f + \mathcal{P}_0(\mathcal{T}_H)$$

and

$$(\mathbf{A}\nabla_H u_H) \cdot \mathbf{n}_F = (\mathbf{A}\nabla_H T\lambda_H) \cdot \mathbf{n}_F + (\mathbf{A}\nabla_H \hat{T}f) \cdot \mathbf{n}_F = \lambda_H|_F \in \mathcal{P}_k(F)$$

for all  $F \in \mathcal{F}_H$ .

By construction, the MHM solution automatically fulfills two conditions.

## The MHM solution (2/2)

Recall the global MHM saddle point problem

$$\begin{cases} \langle \mu_H, u_{0,H} \rangle + \langle \mu_H, T\lambda_H \rangle = -\langle \mu_H, \widehat{T}f \rangle & \forall \mu_H \in \Lambda_H, \\ \langle \lambda_H, v_0 \rangle = (f, v_0)_\Omega & \forall v_0 \in \mathcal{P}_0(\mathcal{T}_H). \end{cases}$$

We may rewrite the first line as

$$\langle \mu_H, u_H \rangle = 0 \quad \forall \mu_H \in \Lambda_H,$$

or after breaking the elementwise integrals, as

$$\sum_{F \in \mathcal{F}_H} (\mu_H, \llbracket u_H \rrbracket)_F = 0 \quad \forall \mu_H \in \Lambda_H.$$

The second line sets the element-wise constant of  $-\nabla \cdot (\mathbf{A}\nabla u_H)$ .

The MHM solution is the unique function of the main theorem!

## The MsHHO solution (1/2)

One easily shows that

$$(r_K(\hat{v}), q_K)_K = (\hat{v}_K, q_K)_K \quad \forall q_K \in \mathcal{P}_k(K)$$

$$(r_K(\hat{v}), q_F)_F = (\hat{v}_F, q_F)_F \quad \forall q_F \in \mathcal{P}_k(F)$$

for all  $\hat{v} \in \hat{U}_H$

Since the  $\hat{v}_F$  are univalued on faces, we see that

$$(\llbracket r_K(\hat{v}) \rrbracket, q)_F = 0 \quad \forall q \in \mathcal{P}_k(F)$$

for all  $F \in \mathcal{F}_H$ .

The MsHHO solution satisfies the third condition by construction.



## The MsHHO solution (2/2)

Recall the global MsHHO problem

$$\sum_{K \in \mathcal{T}_H} (\mathbf{A} \nabla r_K(\hat{u}), \nabla r_K(\hat{v}))_K = \sum_{K \in \mathcal{T}_H} (f, v_K)_K,$$

which we may rewrite as

$$- \sum_{K \in \mathcal{T}_H} (\nabla \cdot (\mathbf{A} \nabla r_K(\hat{u})), \hat{v}_K)_K + \sum_{F \in \mathcal{T}_H} ([\mathbf{A} \nabla r_K(\hat{u})] \cdot \mathbf{n}_F, \hat{v}_F)_F = \sum_{K \in \mathcal{T}_H} (\Pi_k f, v_K)_K,$$

since  $\nabla \cdot (\mathbf{A} \nabla r_K(\hat{u})) \in \mathcal{P}_k(K)$  and  $\mathbf{A} \nabla r_K(\hat{u}) \cdot \mathbf{n}_F \in \mathcal{P}_k(F)$ .

It follows that

$$-\nabla \cdot (\mathbf{A} \nabla_{H} u_H) = \Pi_k f \quad \mathbf{A} \nabla u_H \cdot \mathbf{n}_F \in \mathcal{P}_k(F).$$

**What have we learnt?**

---

# MHM = HHO?

Whenever  $f \in \mathcal{P}_k(\mathcal{T}_H)$ , the MHM and MsHHO methods produce the same solution  $u_H$ .

This solution can be clearly identified by conditions on the flux, the normal traces and the jumps.

## More insights on the two methods

The two methods may be seen as “dual” to each other.

The MHM method spans a space of functions such that

$$-\nabla \cdot (\mathbf{A} \nabla_{H^V} v_H) = f + \mathcal{P}_0(\mathcal{T}_H) \quad (\mathbf{A} \nabla_{H^V} v_H) \cdot \mathbf{n}_F \in \mathcal{P}_k(F)$$

for all  $F \in \mathcal{F}_H$ . The element-wise constants and weak continuity are enforced by the global system

$$([\![u_H]\!] , q)_F = 0 \quad \forall q \in \mathcal{P}_k(F).$$

The MsHHO method works exactly the other way around.

## Perspective for improvements

We hope this work will shed a new light on the design of skeletal based multiscale methods.

We believe that several opportunities of improvement will arise in the future. For example, saddle-point MHM problems can be replaced by MsHHO SPD problems.

Thanks for you attention!