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# Polynomial-degree-robust a posteriori error estimation for Maxwell's equations

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# Outline

- 1 A posteriori error estimation 101
- 2 Equilibration and quasi-equilibration for Poisson's problem
- 3 Quasi-equilibration for Maxwell's equations
- 4 Numerical examples

# A posteriori error estimation 101

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# A posteriori error estimation

Consider a PDE problem set in a domain  $\Omega$ , with solution  $u$ .

The FEM discretizes the problem using a mesh  $\mathcal{T}_h$  of  $\Omega$ .

This process leads to an approximation  $u_h$  of  $u$ .

The goal of *a posteriori* estimates is to assess discretization error

$$\|u - u_h\|$$

measured in an “appropriate” norm  $\|\cdot\|$ .

## A posteriori error estimators

Typically, we attach with each element  $K \in \mathcal{T}_h$  a computable number  $\eta_K$ .

We call  $\eta_K$  the “error estimator” associated with  $K$ .

In this talk we are especially interested with two properties of  $\eta_K$ :

“Reliability”      and      “Efficiency”

# Reliability and efficiency

A posteriori error estimators provide a global error upper bound

## Reliability

$$\|u - u_h\|_{\Omega}^2 \leq C_{\text{rel}}^2 \sum_{K \in \mathcal{T}_h} \eta_K^2,$$

and local lower bounds

## Efficiency

$$\|u - u_h\|_{\tilde{K}} \geq C_{\text{eff}} \eta_K.$$

These two properties allow for efficient adaptive mesh refinements.

# Polynomial-degree-robustness

The constants  $C_{\text{rel}}$  and  $C_{\text{eff}}$  are required to be independent of  $h$ .

However, they “usually” depend on the polynomial degree  $p$  of the FEM.

Residual- and recovery-based estimator suffer from this for instance.

Here, the goal is to derive  $p$ -robust estimates for Maxwell's equations.

This is especially important for  $hp$ -adaptive algorithm.

We propose a novel idea called “quasi-equilibration”, to achieve this goal.



## Two simplifying assumptions

Throughout the talk, the right-hand sides are piecewise polynomials.

This is only for the sake of simplicity.

General right-hand sides are dealt with using usual “oscillation terms”.

Interior and exterior vertices (or edges) require distinct treatments.

When this happens, I will only treat the “interior case”.

This is only to save time, and the exterior treatment is always similar.

# Poisson's problem

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# Poisson's problem

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The model problem

# The Poisson problem

Consider a Lipschitz polyhedral domain  $\Omega \subset \mathbb{R}^3$ .

Given  $f : \Omega \rightarrow \mathbb{R}$  we search  $u : \Omega \rightarrow \mathbb{R}$  such that

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

More formally, if  $f \in L^2(\Omega)$ , there exists a unique  $u \in H_0^1(\Omega)$  s.t.

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega).$$

# FEM discretization

Fix  $p \geq 0$ , and consider the (Lagrange) discrete space

$$V_h := \mathcal{P}_{p+1}(\mathcal{T}_h) \cap H_0^1(\Omega),$$

for piecewise polynomials of degree  $p + 1$ .

There exists a unique  $u_h \in V_h$  such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall v_h \in V_h.$$

We say that  $u_h$  is the FEM approximation to  $u$ .

# Poisson's problem

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The concept of equilibrated flux

## The key idea behind flux-equilibration

Consider a vector field  $\sigma \in \mathbf{H}(\text{div}, \Omega)$  with  $\nabla \cdot \sigma = f$ . We have

$$(f, v) = (\nabla \cdot \sigma, v) = -(\sigma, \nabla v) \quad \forall v \in H_0^1(\Omega).$$

In particular, one sees that for  $v \in H_0^1(\Omega)$

$$\begin{aligned}(\nabla(u - u_h), \nabla v) &= (f, v) - (\nabla u_h, \nabla v) \\ &= (\nabla \cdot \sigma, v) - (\nabla u_h, \nabla v) \\ &= -(\sigma + \nabla u_h, \nabla v).\end{aligned}$$

Selecting  $v := u - u_h$ , it follows that

$$\|\nabla(u - u_h)\|_{\Omega} \leq \|\sigma + \nabla u_h\|_{\Omega}.$$

# Equilibrated flux and Prager-Synge theorem

## Equilibrated flux

$$\sigma \in \mathbf{H}(\operatorname{div}, \Omega) \quad \nabla \cdot \sigma = f$$

## Prager-Synge theorem

$$\|\nabla(u - u_h)\|_{\Omega} \leq \|\sigma + \nabla u_h\|_{\Omega}$$

Equilibrated fluxes provide fully computable (global) upper bounds!



W. Prager and J.L. Synge, 1947



# Practical construction of an equilibrated flux

We have assumed that  $f \in \mathcal{P}_p(\mathcal{T}_h)$  is piecewise polynomial.

As a result, we can construct *discrete* equilibrated fluxes  $\sigma_h$ .

The appropriate tool is then the Raviart-Thomas finite element space  $\mathbf{X}_h$ .

## Ideal flux

$$\sigma_h := \operatorname{argmin}_{\substack{\tau_h \in \mathbf{X}_h \\ \nabla \cdot \tau_h = f}} \|\tau_h + \nabla u_h\|_{\Omega}$$

Unfortunately, this leads to global (expensive) saddle-point problem.

# Poisson's problem

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Localization techniques

# Main ideas behind localization

The idea is then to compute the flux locally. Assume we can decompose

$$f = \sum_{\mathbf{a}} f_{\mathbf{a}},$$

where each  $f_{\mathbf{a}}$  has a “small” support  $\omega_{\mathbf{a}}$ .

We could then (try to) define locally  $\sigma_h^{\mathbf{a}}$  such that

$$\nabla \cdot \sigma_h^{\mathbf{a}} = f_{\mathbf{a}} \text{ in } \omega_{\mathbf{a}} \quad \sigma_h^{\mathbf{a}} \cdot \mathbf{n} = 0 \text{ on } \partial\omega_{\mathbf{a}},$$

and the sum

$$\sigma_h := \sum_{\mathbf{a}} \sigma_h^{\mathbf{a}} \in \mathbf{H}(\text{div}, \Omega)$$

would be an equilibrated flux.

# It is trickier than it looks!

But... There is a catch! Stoke's theorem says that

$$\int_{\omega_a} f_a = \int_{\omega_a} \nabla \cdot \sigma_h^a = \int_{\partial\omega_a} \sigma_h^a \cdot \mathbf{n} = 0.$$

Hence, we cannot use *any* decomposition.

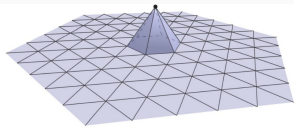
Actually, it is hard, but there is a solution:



P. Destuynder and B. Métivet, 1999

# Hat functions and vertex patches

We denote by  $\mathcal{V}_h$  the vertices of the mesh.  
 $\mathcal{T}_h^{\mathbf{a}}$  is the set of elements sharing  $\mathbf{a} \in \mathcal{V}_h$ .  
 $\omega_{\mathbf{a}}$  is the associated domain.



For each  $\mathbf{a} \in \mathcal{V}_h$ , there is a “hat function”  $\psi_{\mathbf{a}} \in \mathcal{P}_1(\mathcal{T}_h) \cap H^1(\Omega)$  s.t.

$$\psi_{\mathbf{a}}(\mathbf{a}') = \delta_{\mathbf{a}, \mathbf{a}'} \quad \forall \mathbf{a}' \in \mathcal{V}_h$$

They are locally supported,  $\text{supp } \psi_{\mathbf{a}} = \omega_{\mathbf{a}}$ , and

$$\sum_{\mathbf{a} \in \mathcal{V}_h} \psi_{\mathbf{a}} = 1.$$

## Compatible decomposition of the right-hand side

At this point, it is tempting to define  $f_a := \psi_a f$ , but it does not work!

The key idea is to observe that

$$0 = (f, \psi_a) - (\nabla u_h, \nabla \psi_a) = \int_{\omega_a} \psi_a f - \nabla \psi_a \cdot \nabla u_h$$

because  $\psi_a \in V_h$ , and then define

$$f_a := \psi_a f - \nabla \psi_a \cdot \nabla u_h.$$

# Poisson's problem

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Flux-equilibrated error estimator

# Flux-equilibrated error estimator

Let  $\mathbf{X}_h^a$  denote the Raviart-Thomas space on the vertex patch  $\mathcal{T}_h^a$ .

As we saw, the definition

$$\sigma_h^a := \underset{\substack{\tau_h^a \in \mathbf{X}_h^a \\ \nabla \cdot \tau_h^a = \psi_a f - \nabla \psi_a \cdot \nabla u_h \text{ in } \omega_a \\ \tau_h^a \cdot \mathbf{n} = 0 \text{ on } \partial\omega_a}}{\operatorname{argmin}} \|\tau_h^a + \psi_a \nabla u_h\|_{\omega_a}$$

makes sense. And correspond to local (cheap) saddle-point problems.

Gathering the patch contributions, we construct the equilibrated flux

$$\sigma_h := \sum_{a \in \mathcal{V}_h} \sigma_h^a.$$



A. Ern and M. Vohralík, 2015



# Efficiency of the estimator

The reliability of the estimator follows from Prager-Synge theorem.

What about efficiency?

Unfortunately, I do not have time to show proofs.

Still, we can discuss the main results.

## Efficiency

$$\|\nabla u_h + \sigma_h^a\|_{\omega_a} \leq C_{\text{cont},a} C_{\text{st},a} \|\nabla(u - u_h)\|_{\omega_a}$$

In the efficiency estimate:

$C_{\text{cont},a}$  only depends on the geometry of the patch  $\omega_a$ ,

$C_{\text{st},a}$  depends on the geometry, but also involve the discrete space  $V_h$ .

It is a lot of work, but one may show that  $C_{\text{st},a}$  is independent of  $p$ :



D. Braess, V. Pillwein, J. Schöberl, 2009



A. Ern and M. Vohralík, 2019

I will get back to these constants when discussing Maxwell's equations...

# Summary on flux-equilibrated estimators

## Definition

$$\sigma_h^a := \underset{\substack{\tau_h^a \in X_h^a \\ \nabla \cdot \tau_h^a = \psi_a f - \nabla \psi_a \cdot \nabla u_h \text{ in } \omega_a \\ \tau_h^a \cdot n = 0 \text{ on } \partial \omega_a}}{\operatorname{argmin}} \|\tau_h^a + \psi_a \nabla u_h\|_{\omega_a}$$

## Reliability

$$\|\nabla(u - u_h)\|_{\Omega} \leq \|\sigma_h + \nabla u_h\|_{\Omega}$$

## Efficiency

$$\|\sigma_h^a + \nabla u_h\|_{\omega_a} \leq C_{\text{cont},a} C_{\text{st},a} \|\nabla(u - u_h)\|_{\omega_a}$$

# Poisson's problem

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Quasi-equilibrated error estimators

# Definition of the estimator

For the equilibrated error estimator, we considered

$$\sigma_h^a := \underset{\substack{\tau_h^a \in X_h^a \\ \nabla \cdot \tau_h^a = \psi_a f - \nabla \psi_a \cdot \nabla u_h \text{ in } \omega_a \\ \tau_h^a \cdot \mathbf{n} = 0 \text{ on } \partial\omega_a}}{\operatorname{argmin}} \|\tau_h^a + \psi_a \nabla u_h\|_{\omega_a},$$

where the BC and decomposition of  $f$  are important to build  $\sigma_h$ .

For quasi-equilibration, we instead set

$$\tilde{\sigma}_h^a := \underset{\substack{\tau_h^a \in X_h^a \\ \nabla \cdot \tau_h^a = f \text{ in } \omega_a}}{\operatorname{argmin}} \|\tau_h^a + \nabla u_h\|_{\omega_a},$$

where the boundary conditions, and the  $\psi_a$  have disappeared.

# Quasi-equilibration does not provide an equilibrated flux

The resulting field  $\tilde{\sigma}_h^a$  are “locally” equilibrated,  
but cannot be assembled into a globally equilibrated object.

Yet, we may associate with each  $a \in \mathcal{V}_h$  the quantity

$$\eta_a := \min_{\substack{\tau_h^a \in X_h^a \\ \nabla \cdot \tau_h^a = f \text{ in } \omega_a}} \|\tau_h^a + \nabla u_h\|_{\omega_a}.$$

It turns out that  $\eta_a$  makes a good estimator,  
even if we cannot use Prager-Theorem anymore.

# Summary of quasi-equilibrated estimator

## Definition

$$\eta_{\mathbf{a}} := \min_{\substack{\boldsymbol{\tau}_h^{\mathbf{a}} \in \mathbf{X}_h^{\mathbf{a}} \\ \nabla \cdot \boldsymbol{\tau}_h^{\mathbf{a}} = f \text{ in } \omega_{\mathbf{a}}}} \|\boldsymbol{\tau}_h^{\mathbf{a}} + \nabla u_h\|_{\omega_{\mathbf{a}}}.$$

## Reliability

$$\|\nabla(u - u_h)\|_{\Omega} \leq 2 \left( \sum_{\mathbf{a} \in \mathcal{T}_h} C_{\text{cont}, \mathbf{a}}^2 \eta_{\mathbf{a}}^2 \right)^{1/2}$$

## Efficiency

$$\eta_{\mathbf{a}} \leq C_{\text{st}, \mathbf{a}} \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}}$$

The estimates involve the same constants, but placed differently!

# Poisson's problem

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Equilibration vs Quasi-equilibration



## Equilibration

$$\sigma_h^a := \underset{\substack{\tau_h^a \in X_h^a \\ \nabla \cdot \tau_h^a = \psi_a f - \nabla \psi_a \cdot \nabla u_h \text{ in } \omega_a \\ \tau_h^a \cdot n = 0 \text{ on } \partial \omega_a}}{\operatorname{argmin}} \|\tau_h^a + \psi_a \nabla u_h\|_{\omega_a}$$

## Quasi-equilibration

$$\eta_a := \underset{\substack{\tau_h^a \in X_h^a \\ \nabla \cdot \tau_h^a = f \text{ in } \omega_a}}{\min} \|\tau_h^a + \nabla u_h\|_{\omega_a}.$$

The constructions are similar. The quasi-equilibration is slightly simpler.

## Equilibration

$$\eta_{\mathbf{a}} \leq C_{\text{cont},\mathbf{a}} C_{\text{st},\mathbf{a}} \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}}$$

## Quasi-equilibration

$$\|\sigma_h^{\mathbf{a}} + \nabla u_h\|_{\omega_{\mathbf{a}}} \leq C_{\text{st},\mathbf{a}} \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}}$$

Both estimators are  $p$ -robust.

## Equilibration

$$\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{\Omega} \leq \|\boldsymbol{\sigma}_h + \nabla \mathbf{u}_h\|_{\Omega}$$

## Quasi-equilibration

$$\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{\Omega} \leq 2 \left( \sum_{\mathbf{a} \in \mathcal{T}_h} C_{\text{cont}, \mathbf{a}}^2 \eta_{\mathbf{a}}^2 \right)^{1/2}$$

Equilibrated-fluxes provide nicer upper bounds.

# Maxwell's equations

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# Maxwell's equations

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Model problem

# Model problem

Given  $\mathbf{J} : \Omega \rightarrow \mathbb{R}^3$ , with  $\nabla \cdot \mathbf{J} = 0$ , we search  $\mathbf{A} : \Omega \rightarrow \mathbb{R}^3$  such that

$$\left\{ \begin{array}{ll} \nabla \cdot \mathbf{A} = 0 & \text{in } \Omega, \\ \nabla \times \nabla \times \mathbf{A} = \mathbf{J} & \text{in } \Omega, \\ \mathbf{A} \times \mathbf{n} = \mathbf{o} & \text{on } \partial\Omega. \end{array} \right.$$

More formally, given  $\mathbf{J} \in \mathbf{H}(\text{div}^0, \Omega)$ , there exists a unique  $\mathbf{A} \in \mathbf{H}_0(\text{curl}, \Omega)$  such that

$$(\nabla \times \mathbf{A}, \nabla \times \mathbf{v}) = (\mathbf{J}, \mathbf{v}) \quad (\mathbf{A}, \nabla q) = 0$$

for all  $\mathbf{v} \in \mathbf{H}_0(\text{curl}, \Omega)$  and  $q \in H_0^1(\Omega)$ .



F. Brezzi, 1974

## Why is this problem interesting?

In magnetostatics, the magnetic field  $\mathbf{H}$  satisfies

$$\nabla \cdot \mathbf{H} = 0 \quad \nabla \times \mathbf{H} = \mathbf{J},$$

and then  $\mathbf{A}$  is the magnetic potential:  $\nabla \times \mathbf{A} = \mathbf{H}$ .

Also, our model problem is the “differential part” of:

$$-k^2 \mathbf{E} - \nabla \times \nabla \times \mathbf{E} = ik\mathbf{J},$$

known as time-harmonic Maxwell’s equations,  $\mathbf{E}$  being the electric field.



T. Chaumont-Frelet, A. Ern, M. Vohralík, 2019



S. Congreve, J. Gedicke, and I. Perugia, 2019

# Finite element discretization

We now consider the  $\mathbf{W}_h$ , the Nédélec finite-element space of degree  $p$ .

There exists a unique  $\mathbf{A}_h \in \mathbf{W}_h$  such that

$$(\nabla \times \mathbf{A}_h, \nabla \times \mathbf{v}_h) = (\mathbf{J}, \mathbf{v}_h) \quad (\mathbf{A}_h, \nabla q_h) = 0$$

for all  $\mathbf{v}_h \in \mathbf{W}_h$  and  $q_h \in V_h$ .

Notice in particular that

$$(\nabla \times (\mathbf{A} - \mathbf{A}_h), \nabla \times \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in \mathbf{W}_h.$$



F. Brezzi, 1974



# Maxwell's equations

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Flux equilibration?

# Prager-Synge theorem

## Equilibrated flux

$$\mathbf{B} \in H(\text{curl}, \Omega) \quad \nabla \times \mathbf{B} = \mathbf{J}$$

## Prager-Synge theorem

$$\|\nabla \times (\mathbf{A} - \mathbf{A}_h)\|_{\Omega} \leq \|\mathbf{B} - \nabla \times \mathbf{A}_h\|_{\Omega}$$

## Global construction

$$\mathbf{B}_h := \underset{\substack{\mathbf{M}_h \in \mathbf{W}_h \\ \nabla \times \mathbf{M}_h = \mathbf{J}}}{\text{argmin}} \|\mathbf{M}_h - \nabla \times \mathbf{A}_h\|_{\Omega}$$

# The issue with localization

We can (in principle) follow the same steps than for Poisson's problem:

Decompose the right-hand side

$$\mathbf{J} = \sum_a \mathbf{J}_a,$$

and find locally  $\mathbf{B}_h^a$  such that

$$\nabla \times \mathbf{B}_h^a = \mathbf{J}_a \text{ in } \omega_a \quad \mathbf{B}_h^a \times \mathbf{n} = \mathbf{0} \text{ on } \partial\omega_a.$$

The problem is that the compatibility conditions are “hard”.

The “natural choice”  $\mathbf{J}_a := \psi_a \mathbf{J} + \nabla \psi_a \times \nabla \times \mathbf{A}_h$  fails:  $\nabla \cdot \mathbf{J}_a \neq 0$ .

# Possible solutions

An initial idea for lowest-order elements:



D. Braess and J. Schöberl, 2008

An extension developed in the same time than the present work:



J. Gedicke, S. Geever and I. Perugia, 2019



J. Gedicke, S. Geever, I. Perugia and J. Schöberl, 2020

The construction is complex and requires 4 mesh traversals.

An alternative, quasi-equilibration:



T. Chaumont-Frelet, A. Ern and M. Vohralík, 2020

# Maxwell's equations

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Quasi-equilibration!

Quasi-equilibration does not require any compatible decomposition.

Actually, we can readily apply it to Maxwell's equations.

The resulting estimators are attached to the edges  $\ell \in \mathcal{E}_h$  of the mesh

$$\eta_\ell := \min_{\substack{\mathbf{B}_h^\ell \in \mathbf{W}_h^\ell \\ \nabla \times \mathbf{B}_h^\ell = \mathbf{J}}} \|\mathbf{B}_h^\ell - \nabla \times \mathbf{A}_h\|_{\omega_\ell},$$

where  $\omega_\ell$  is the set of tetrahedra  $K \in \mathcal{T}_h$  sharing the edge  $\ell$ .

# Key properties

## Reliability

$$\|\nabla \times (\mathbf{A} - \mathbf{A}_h)\|_{\Omega} \leq \sqrt{6} C_{\text{lift},\Omega} \left( \sum_{\ell \in \mathcal{E}_h} C_{\text{cont},\ell}^2 \eta_{\ell}^2 \right)^{1/2}$$

## Efficiency

$$\eta_{\ell} \leq C_{\text{st},\ell} \|\nabla \times (\mathbf{A} - \mathbf{A}_h)\|_{\omega_{\ell}}$$

The estimator is  $p$ -robust, all the constants are independent of  $p$ .

We do not have time for the proofs, but I can tell you where the constants come from, and what are the key ideas.

# Constants in the reliability estimate

## Reliability

$$\|\nabla \times (\mathbf{A} - \mathbf{A}_h)\|_{\Omega} \leq \sqrt{6} C_{\text{lift}, \Omega} \left( \sum_{\ell \in \mathcal{E}_h} C_{\text{cont}, \ell}^2 \eta_{\ell}^2 \right)^{1/2}$$



# Constants in the reliability estimate

## Reliability

$$\|\nabla \times (\mathbf{A} - \mathbf{A}_h)\|_{\Omega} \leq \sqrt{6} C_{\text{lift},\Omega} \left( \sum_{\ell \in \mathcal{E}_h} C_{\text{cont},\ell}^2 \eta_{\ell}^2 \right)^{1/2}$$

There are 6 edges in each tetrahedron!

This factor comes from “covering” arguments.

# Constants in the reliability estimate

## Reliability

$$\|\nabla \times (\mathbf{A} - \mathbf{A}_h)\|_{\Omega} \leq \sqrt{6} C_{\text{lift},\Omega} \left( \sum_{\ell \in \mathcal{E}_h} C_{\text{cont},\ell}^2 \eta_{\ell}^2 \right)^{1/2}$$

For all  $\mathbf{v} \in \mathbf{H}_0(\text{curl}, \Omega)$ , there exists  $\mathbf{w} \in \mathbf{H}^1(\Omega) \cap \mathbf{H}_0(\text{curl}, \Omega)$  such that

$$\nabla \times \mathbf{w} = \nabla \times \mathbf{v} \quad \|\nabla \mathbf{w}\|_{\Omega} \leq C_{\text{lift},\Omega} \|\nabla \times \mathbf{v}\|_{\Omega}.$$

This is a usually trick, also employed in residual-based estimator:



R. Beck, R. Hiptmair, H.W. Hoppe and B. Wohlmuth, 2000



S. Nicaise and E. Creusé, 2003

$C_{\text{lift},\Omega} = 1$  for convex domains, but it is “hard” to compute in general.

# Constants in the reliability estimate

## Reliability

$$\|\nabla \times (\mathbf{A} - \mathbf{A}_h)\|_{\Omega} \leq \sqrt{6} C_{\text{lift},\Omega} \left( \sum_{\ell \in \mathcal{E}_h} C_{\text{cont},\ell}^2 \eta_{\ell}^2 \right)^{1/2}$$

We have

$$C_{\text{cont},\ell} := \|\psi_{\ell}\|_{L^{\infty}(\omega_{\ell})} + C_{P,\ell} h_{\ell} \|\nabla \times \psi_{\ell}\|_{L^{\infty}(\omega_{\ell})} \simeq 1$$

where

$C_{P,\ell}$  is the (computable) “Poincaré constant” of the patch  $\omega_{\ell}$

$h_{\ell}$  is the diameter of the patch  $\omega_{\ell}$

$\psi_{\ell}$  is the “edge function” associated with  $\ell$ .

Under the hood, we use a vectorial edge-based partition of unity.

# Constant in the efficiency estimate

## Efficiency

$$\eta_\ell := \min_{\substack{\mathbf{B}_h^\ell \in \mathbf{W}_h^\ell \\ \nabla \times \mathbf{B}_h^\ell = \mathbf{J}}} \|\mathbf{B}_h^\ell - \nabla \times \mathbf{A}_h\|_{\omega_\ell} \leq C_{\text{st},\ell} \|\nabla \times (\mathbf{A} - \mathbf{A}_h)\|_{\omega_\ell}$$

## Stable discrete minimization

$$\min_{\substack{\mathbf{B}_h^\ell \in \mathbf{W}_h^\ell \\ \nabla \times \mathbf{B}_h^\ell = \mathbf{j}_h^\ell}} \|\mathbf{B}_h^\ell - \boldsymbol{\theta}_h\|_{\omega_\ell} \leq C_{\text{st},\ell} \min_{\substack{\mathbf{B}^\ell \in \mathbf{H}(\text{curl}, \omega_\ell) \\ \nabla \times \mathbf{B}^\ell = \mathbf{j}_h^\ell}} \|\mathbf{B}^\ell - \boldsymbol{\theta}_h\|_{\omega_\ell}$$

The tough part is to show that  $C_{\text{st},\ell}$  does not depend on  $\rho$ !



A. Ern and M. Vohralík, 2019



T. Chaumont-Frelet, A. Ern and M. Vohralík, 2020

# Summary

We design an edge-based  $p$ -robust estimator

$$\eta_\ell := \min_{\substack{\mathbf{B}_h^\ell \in \mathbf{W}_h^\ell \\ \nabla \times \mathbf{B}_h^\ell = \mathbf{J}}} \|\mathbf{B}_h^\ell + \nabla \times \mathbf{A}_h\|_{\omega_\ell}.$$

It is cheap to compute, and only uses standard FEM matrices for its implementation.

We obtain efficiency estimate on “tight” edge patches.

But the constant in the upper bound is not always practically computable.

## Numerical examples

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# Numerical examples

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Smooth solution in a cube

## Smooth solution in a cube

Consider  $\Omega := (0, 1)^3$ , and

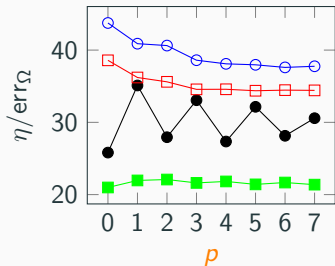
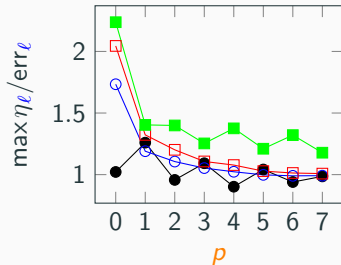
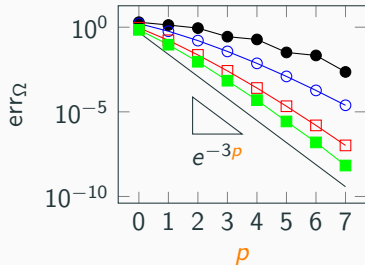
$$\mathbf{J}(\mathbf{x}) := 3\pi^3 \begin{pmatrix} \sin(\pi \mathbf{x}_1) \cos(\pi \mathbf{x}_2) \cos(\pi \mathbf{x}_3) \\ -\cos(\pi \mathbf{x}_1) \sin(\pi \mathbf{x}_2) \cos(\pi \mathbf{x}_3) \\ 0 \end{pmatrix}.$$

Then, the solution reads

$$\mathbf{A}(\mathbf{x}) := 3\pi^3 \begin{pmatrix} \sin(\pi \mathbf{x}_1) \cos(\pi \mathbf{x}_2) \cos(\pi \mathbf{x}_3) \\ -\cos(\pi \mathbf{x}_1) \sin(\pi \mathbf{x}_2) \cos(\pi \mathbf{x}_3) \\ 0 \end{pmatrix}.$$



# $p$ -robustness in the unit-cube experiment



## Numerical examples

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Singular solution in an “L brick”

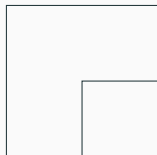
## L brick experiment

We now consider an “L brick”  $\Omega = L \times (0, 1)$ .

The right-hand side  $\mathbf{J} \in \mathbf{H}(\operatorname{div}^0, \Omega)$  is chosen such that

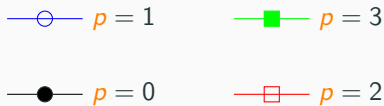
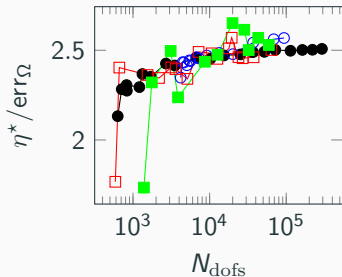
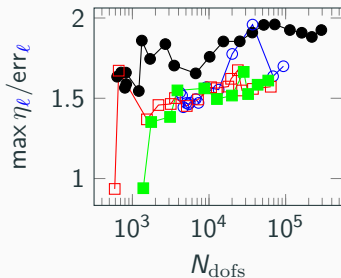
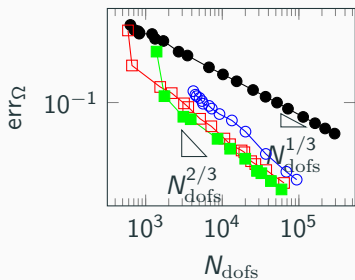
$$\mathbf{A}(\mathbf{x}) = \begin{pmatrix} 0 \\ 0 \\ \chi(r)r^\alpha \sin(\alpha\theta) \end{pmatrix},$$

where  $(r, \theta)$  the polar coordinates in  $L$ ,  $\chi$  is a cutoff and  $\alpha = 2/3$ .

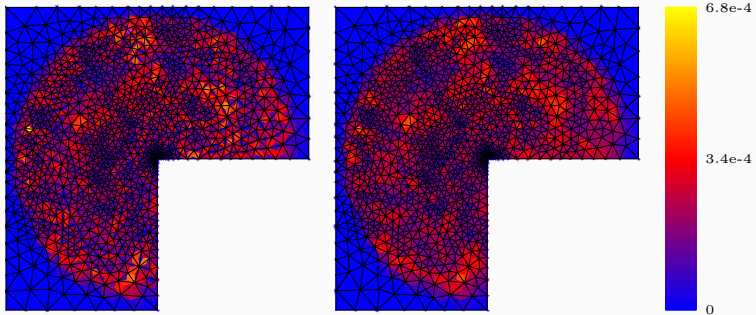


The solution feature an “edge singularity”:  $\nabla \times \mathbf{A} \in \mathbf{H}^{2/3}(\Omega)$ .

# $h$ -adaptivity in the L brick experiment



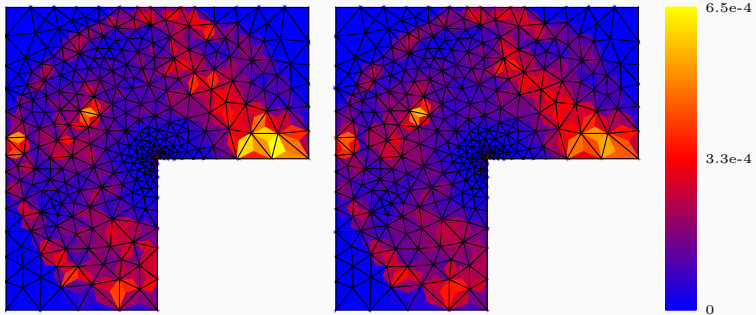
# Error distribution in the L brick experiment $\rho = 0$



$\eta_\ell$

$\|\nabla \times (\mathbf{A} - \mathbf{A}_h)\|_{\omega_\ell}$

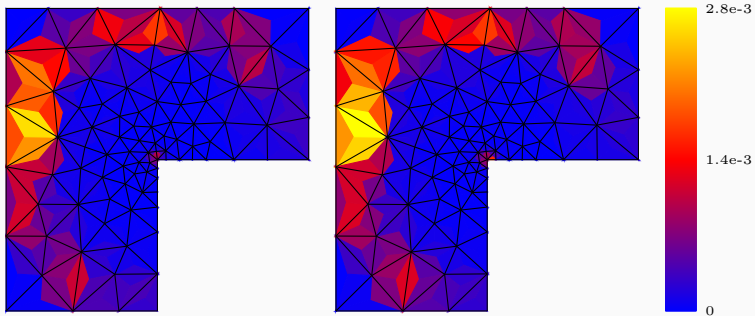
# Error distribution in the L brick experiment $\rho = 1$



$\eta_\ell$

$\|\nabla \times (\mathbf{A} - \mathbf{A}_h)\|_{\omega_\ell}$

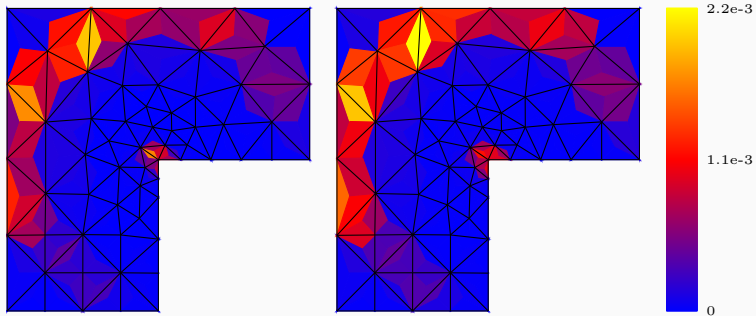
# Error distribution in the L brick experiment $\rho = 2$



$\eta_\ell$

$\|\nabla \times (\mathbf{A} - \mathbf{A}_h)\|_{\omega_\ell}$

# Error distribution in the L brick experiment $\rho = 3$



$\eta_\ell$

$\|\nabla \times (\mathbf{A} - \mathbf{A}_h)\|_{\omega_\ell}$



# Conclusion

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# Conclusion

Quasi-equilibration is an alternative to flux-equilibration.

Both methods provide polynomial-degree-robust estimates.

Flux-equilibration leads to constant-free upper bounds.

Quasi-equilibration relies on simpler local problems.

Importantly, quasi-equilibration bypass “compatible decompositions” of the right-hand side which is especially handy for Maxwell’s equations.

I believe that the present estimator is the “simplest”  $p$ -robust estimator currently available for Maxwell’s equations in the literature.



T. Chaumont-Frelet, A. Ern and M. Vohralík, 2020