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Discontinuous Galerkin Method based on Riemann fluxes for the time domain Maxwell System

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Abstract: In this report, we devise a discontinuous Galerkin method for the propagation of electromagnetic waves in a homogeneous medium. The numerical fluxes are deduced from the solution of a 1D electromagnetic problem. Numerical simulations illustrate the accuracy of the method.

Key-words: Maxwell equation, electromagnetic waves, Finite Element, Discontinuous Galerkin, Riemann solver

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Méthode de Gakerkine discontinue basée sur des flux de Riemann pour la propagation des ondes électromagnétiques harmoniques

Résumé : Dans ce rapport, nous construisons une méthode de Galerkine discontinue pour la propagation des ondes électromagnétiques dans un milieu homogène. Les flux numériques sont déduits de la résolution de problème électromagnétique 1D. Des simulations numériques illustrent la précision de la méthode.

Mots-clés : Équations de Maxwell, ondes électromagnétiques, Élements Finis, Galerkine Discontinue, solveur de Riemann

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Introduction

Discretization of time harmonic problems with classical methods [19] relies on the solution of a very large linear system. In the context of three dimensional modeling, this is really problematic due to the increase of memory consumption for LU decomposition.

An alternative consists in resorting to a domain decomposition based on a Trefftz method. Instead of one big problem, we have to solve numerous small problems. This strategy allows us to strongly reduce the memory usage and to make the computations less expensive.

One particularity of Trefftz methods [18], which generalizes the Ultra-Weak Variational Formulation [5], is to use solutions of the time harmonic Maxwell equations as basis functions [4]. Plane wave approximations are taken as basis functions in most of Trefftz solvers [20]. This leads to a major drawback since the involved linear system that has to be inverted is really poorly conditioned [1, 10]. A possible solution is to compute the plane waves approximations thank to an auxiliary solver. We call this type of methods quasi Trefftz methods [14].

In this report, we present one possible auxiliary solver : Discontinuous Galerkin Method based on numerical fluxes computed thanks to a Riemann solver. This method has the main advantage to be based on the time harmonic version of the first order hyperbolic system [15, 3, 12, 9] which involves the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$, both needed for Trefftz solver.

This report aims at presenting a Discontinuous Galerkin method, see [7, 11] for a general presentation. The method has been taken out from former works about Discontinuous Galerkin with general fluxes. These methods are associated to numerical fluxes which are defined thanks to four parameters which can be tricky to determine. In this work, we propose to resort to a Riemann solver [22, 2] to determine these coefficients. This avoids the optimization process to determine these parameters which can be a crucial step for the efficiency of the method.

The report is organized as follows. In the first section, we recall some well known results for the solution of time harmonic Maxwell system. In the second section, we present the variational formulation associated to the Discontinuous Galerkin method based on a time harmonic hyperbolic system of PDE. In the third section, we compute the numerical fluxes thanks to a Riemann solver. In the fourth section, we devise the final variational formula-

tion. The final section is devoted to the implementation and some numerical illustrations.

1 The functional framework for time harmonic Maxwell equations

For this subsection, the reader can refer to [21].

1.1 The normalized Maxwell equations

Let us consider a propagation domain Ω with boundary $\partial\Omega$. We suppose that the domain is sufficiently regular (it will be clarified in hypothesis 3 of subsection 1.5). We recall that the homogeneous time domain Maxwell system [24] takes the form

$$\left\{\begin{array}{rcll} \mathrm{rot}(\widetilde{\mathbf{E}}) & = & -\mu_0 \dfrac{\partial \widetilde{\mathbf{H}}}{\partial t} & \text{in } \Omega, \\ \mathrm{rot}(\widetilde{\mathbf{H}}) & = & \varepsilon_0 \dfrac{\partial \widetilde{\mathbf{E}}}{\partial t} & \text{in } \Omega. \end{array}\right. \tag{1}$$

It can be normalized by considering the following change of variables

$$\widetilde{\mathbf{H}} \;=\; H_0\,\mathbf{H} \text{ and } \widetilde{\mathbf{E}} \;=\; E_0\,\mathbf{E} \tag{2}$$

with $c = \dfrac{1}{\sqrt{\varepsilon_0\mu_0}}$ the speed of light, E_0 the characteristic electric magnitude and H_0 the characteristic magnetic magnitude which are related by

$$\sqrt{\mu_0}\,H_0 \;=\; \sqrt{\varepsilon_0}\,E_0. \tag{3}$$

The time domain normalized Maxwell equations are given by

$$\left\{\begin{array}{rcl} \mathrm{rot}(\mathbf{E}) & = & -\dfrac{1}{c}\dfrac{\partial \mathbf{H}}{\partial t}, \\ \mathrm{rot}(\mathbf{H}) & = & \dfrac{1}{c}\dfrac{\partial \mathbf{E}}{\partial t}, \end{array}\right.$$

where $\mathbf{E}$ is the normalized electric field and $\mathbf{H}$ is the normalized magnetic field.

Let ω be a wave angular frequency. We suppose that the solution is time harmonic, which mathematically means

$$\begin{array}{rcl} \mathbf{E}(\mathbf{x},t) & = & \Re\big(\exp(i\omega t)\mathbf{E}(\mathbf{x})\big), \\ \mathbf{H}(\mathbf{x},t) & = & \Re\big(\exp(i\omega t)\mathbf{H}(\mathbf{x})\big). \end{array}$$

This leads to the time harmonic Maxwell equation ($k = \dfrac{\omega}{c}$)

$$\begin{cases} \mathrm{rot}(\mathbf{E}) &= -ik\mathbf{H}, \\ \mathrm{rot}(\mathbf{H}) &= \quad ik\mathbf{E}. \end{cases} \tag{4}$$

This system of equations is often equipped with an impedance boundary condition imposed on the boundary $\partial\Omega$ of Ω

$$Y\gamma_t\mathbf{E} \;+\; \gamma_\times\mathbf{H} \;=\; \gamma_t\overrightarrow{\varphi}, \qquad \text{with } 0 \leq Y \leq 1, \tag{5}$$

where $\overrightarrow{\varphi}$ is a source term and the two tangential trace operators γ_t and $\gamma_\times$ are defined by

$$\gamma_t\mathbf{E} = \mathbf{E} - (\mathbf{E}\cdot\mathbf{n})\mathbf{n} \quad \text{and} \quad \gamma_\times\mathbf{E} = \mathbf{n}\times\mathbf{E} \tag{6}$$

with $\mathbf{n}$ the unit outward normal to $\partial\Omega$.
The first one is called the tangential component and the second one is also simply called the tangential trace.

1.2 Functional framework

Definition 1.2.1. The nabla operator $\nabla_{\partial\Omega}$ on the boundary $\partial\Omega$ of a function $\psi : \Omega \longrightarrow \mathbb{C}$ is defined by

$$\nabla_{\partial\Omega}\psi \;=\; \gamma_t\nabla\psi.$$

Definition 1.2.2. The divergence operator $\mathrm{div}_{\partial\Omega}$ on $\partial\Omega$ is defined by

$$\int_{\partial\Omega}(\mathrm{div}_{\partial\Omega}\overrightarrow{\varphi})\psi \;=\; -\int_{\partial\Omega}\overrightarrow{\varphi}\cdot\nabla_{\partial\Omega}\psi.$$

This definition is given in the sense of distributions for $\psi \in C^\infty(\partial\Omega)$ and for $\overrightarrow{\varphi}$ in the following functional space

$$C_t^\infty(\partial\Omega) \;=\; \Big\{\overrightarrow{\varphi} \;=\; \gamma_t\overrightarrow{\zeta} \text{ such that } \overrightarrow{\zeta} \in (C^\infty(\Omega))^3\Big\}.$$

Definition 1.2.3. We define a surface rotational operator $\mathrm{rot}_{\partial\Omega}$ which acts on tangential vector fields of $\overrightarrow{\varphi} : \Omega \longrightarrow \mathbb{C}^3$

$$\mathrm{rot}_{\partial\Omega}(\gamma_t\overrightarrow{\varphi}) \;=\; \gamma_n(\mathrm{rot}(\overrightarrow{\varphi})) \quad \text{with } \gamma_n\overrightarrow{\varphi} \;=\; \overrightarrow{\varphi}\cdot\mathbf{n},$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$.

We also have the following formula

$$\mathrm{rot}_{\partial\Omega}(\gamma_t \overrightarrow{\varphi}) \;=\; \mathrm{div}_{\partial\Omega}(\gamma_t \overrightarrow{\varphi} \times \mathbf{n}).$$

Its adjoint is denoted by $\overline{\mathrm{rot}_{\partial\Omega}}$. In the sense of distributions, it satisfies

$$\int_{\partial\Omega} (\mathrm{rot}_{\partial\Omega}(\gamma_t \overrightarrow{\varphi}))\psi \;=\; \int_{\partial\Omega} \gamma_t \overrightarrow{\varphi} \cdot \overline{\mathrm{rot}_{\partial\Omega}}\psi \text{ and } \overline{\mathrm{rot}_{\partial\Omega}}(\psi) = \mathbf{n} \times \nabla_{\partial\Omega}\psi. \quad (7)$$

We introduce the following functional spaces on Ω and on $\partial\Omega$:

- $L^2_t(\partial\Omega) = \Big\{ \overrightarrow{\varphi} \in \Big(L^2(\partial\Omega)\Big)^3 \;\Big|\; \overrightarrow{\varphi} \cdot \mathbf{n} = 0 \Big\}$,
- $H(\mathrm{rot}, \Omega) = \Big\{ \mathbf{E} \in \Big(L^2(\Omega)\Big)^3 \;\Big|\; \mathrm{rot}(\mathbf{E}) \in \Big(L^2(\Omega)\Big)^3 \Big\}$,
- $H^{\frac{1}{2}}(\partial\Omega) = \Big\{ \gamma\varphi \Big|\; \varphi \in H^1(\Omega) \Big\}$,
- $H^{\frac{1}{2}}_t(\partial\Omega) = \Big\{ \overrightarrow{\varphi} \in \Big(H^{\frac{1}{2}}(\partial\Omega)\Big)^3, \;\Big|\; \overrightarrow{\varphi} \cdot \mathbf{n} = 0 \Big\}$,
- $H^{-\frac{1}{2}}_t(\partial\Omega) = \Big(H^{\frac{1}{2}}_t(\partial\Omega)\Big)^*$,
- $H^{-\frac{1}{2}}_t(\mathrm{rot}_{\partial\Omega}, \partial\Omega) = \Big\{ \overrightarrow{\varphi} \in \Big(H^{-\frac{1}{2}}_t(\partial\Omega)\Big)^3, \mathrm{rot}_{\partial\Omega}\overrightarrow{\varphi} \in H^{-\frac{1}{2}}_t(\partial\Omega) \Big\}$,
- $H^{-\frac{1}{2}}_t(\mathrm{div}_{\partial\Omega}, \partial\Omega) \;=\; \Big\{ \overrightarrow{\varphi} \in \Big(H^{-\frac{1}{2}}_t(\partial\Omega)\Big)^3, \mathrm{div}_{\partial\Omega}\overrightarrow{\varphi} \in H^{-\frac{1}{2}}_t(\partial\Omega) \Big\}$.

Remark 1. The Hilbert space $H^{-\frac{1}{2}}_t(\mathrm{div}_{\partial\Omega}, \partial\Omega)$ is the dual space of $H^{-\frac{1}{2}}_t(\mathrm{rot}_{\partial\Omega}, \partial\Omega)$. Moreover $\times\mathbf{n}$ defines an isomorphism between $H^{-\frac{1}{2}}_t(\mathrm{rot}_{\partial\Omega}, \partial\Omega)$ and $H^{-\frac{1}{2}}_t(\mathrm{div}_{\partial\Omega}, \partial\Omega)$. The operator γ_t is continuous when equipped with the following functional framework

$$H(\mathrm{rot}, \Omega) \longrightarrow H^{-\frac{1}{2}}_t(\mathrm{rot}_{\partial\Omega}, \partial\Omega).$$

Therefore, the trace operator $\gamma_\times$ can be defined as a continuous mapping from $H(\mathrm{rot}, \Omega)$ to $H^{-\frac{1}{2}}_t(\mathrm{div}_{\partial\Omega}, \partial\Omega)$.
This allows to extend the bilinear form

$$\int_{\partial\Omega} \gamma_t \mathbf{E} \cdot (\mathbf{n} \times \mathbf{E}')$$

to $H(\mathrm{rot}, \Omega) \times H(\mathrm{rot}, \Omega)$ even if it is not anymore an integral but a duality pairing.

Remark 2. In the context of finite element methods, $\mathbf{E}$ and $\mathbf{H}$ are regular and the duality pairing is simply an integral.

1.3 The Stokes formula theorem

The Stokes formula is the main ingredient to derive variational formulation for the Maxwell system. It is analogue to the divergence theorem for the Laplace equation.

Lemma 1. Let $\mathbf{E} \in H(\mathrm{rot}, \Omega)$ and $\mathbf{H} \in H(\mathrm{rot}, \Omega)$. We have

$$\int_\Omega \mathrm{rot}\mathbf{E} \cdot \mathbf{H} - \int_\Omega \mathbf{E} \cdot \mathrm{rot}\mathbf{H} = -\int_{\partial\Omega} \gamma_t \mathbf{E} \cdot \gamma_\times \mathbf{H},$$

or equivalently

$$\int_\Omega \mathrm{rot}\mathbf{E} \cdot \mathbf{H} - \int_\Omega \mathbf{E} \cdot \mathrm{rot}\mathbf{H} = \int_{\partial\Omega} \gamma_\times \mathbf{E} \cdot \gamma_t \mathbf{H}, \tag{8}$$

where the last integral should be interpreted as a duality paring between $H_t^{-\frac{1}{2}}(\mathrm{rot}_{\partial\Omega}, \partial\Omega)$ and $H_t^{-\frac{1}{2}}(\mathrm{div}_{\partial\Omega}, \partial\Omega)$.

Remark 3. If the functions $\mathbf{E}$ and $\mathbf{H}$ are in $H_{imp}(\Omega)$ (we define $H_{imp}(\Omega)$ in the next subsection) then the duality pairing becomes simply an integral.

Proof. The Stokes theorem is a corollary of the divergence theorem

$$\int_\Omega \mathrm{rot}\mathbf{E} \cdot \mathbf{H} - \int_\Omega \mathbf{E} \cdot \mathrm{rot}\mathbf{H} = \int_{\partial\Omega} (\mathbf{E} \times \mathbf{H}) \cdot \mathbf{n}, \tag{9}$$

since

$$\mathrm{div}(\mathbf{E} \times \mathbf{H}) \;=\; \mathrm{rot}(\mathbf{E}) \cdot \mathbf{H} - \mathrm{rot}(\mathbf{H}) \cdot \mathbf{E}. \tag{10}$$

The conclusion follows from

$$(\mathbf{E} \times \mathbf{H}) \cdot \mathbf{n} \;=\; (\mathbf{n} \times \mathbf{E}) \cdot \mathbf{H}. \tag{11}$$

□

1.4 Existence and uniqueness of the solution

We consider the Maxwell equations on the domain Ω with the impedance boundary condition [24]

$$Y \gamma_t \mathbf{E} \;+\; \gamma_\times \mathbf{H} \;=\; \gamma_t \overrightarrow{\varphi}$$

with $\gamma_t \overrightarrow{\varphi}$ in $L_t^2(\partial\Omega)$.

We are looking for $\mathbb{E} := (\mathbf{E}, \mathbf{H})$ in the functional space

$$H_{imp}(\Omega) \;=\; \Big\{\mathbb{E} = (\mathbf{E}, \mathbf{H}) \in H(\mathrm{rot}, \Omega) \times H(\mathrm{rot}, \Omega)$$

$$\text{such that } \gamma_t \mathbf{E} \in L^2_t(\partial\Omega) \text{ and } \gamma_\times \mathbf{H} \in L^2_t(\partial\Omega)\Big\}. \quad (12)$$

The extra condition that the two tangential traces are $L^2_t(\partial\Omega)$ is an important hypothesis for the existence of the solution. Indeed, this condition is more restrictive than the ones imposed by $H(\mathrm{rot}, \Omega)$, that are $\gamma_t \mathbf{E} \in H_t^{-1/2}(\mathrm{rot}_{\partial\Omega}, \partial\Omega)$ and $\gamma_\times \mathbf{H} \in H_t^{-1/2}(\mathrm{div}_{\partial\Omega}, \partial\Omega)$.

Theorem 2 (see [21]). For every $\gamma_t \overrightarrow{\varphi} \in L^2_t(\partial\Omega)$, there exists a unique $\mathbb{E} = (\mathbf{E}, \mathbf{H}) \in H_{imp}$ such that

$$\begin{cases} \mathrm{rot}(\mathbf{E}) \;=\; -ik\mathbf{H} & \text{in } \Omega, \\ \mathrm{rot}(\mathbf{H}) \;=\; ik\mathbf{E} & \text{in } \Omega, \\ Y\gamma_t \mathbf{E} \;+\; \gamma_\times \mathbf{H} \;=\; \gamma_t \overrightarrow{\varphi} & \text{on } \partial\Omega. \end{cases} \quad (13)$$

1.5 The unique continuation theorem for the Maxwell system

In this subsection we suppose that Ω is a bounded domain. We suppose moreover that Ω fulfills the following regularity property.

Hypothesis 3. Every vectorial function $\mathbf{E} \in H(\mathrm{rot}, \Omega)$ such that $\gamma_t \mathbf{E} = 0$ on $\partial\Omega$ can be extended to all $\mathbb{R}^3$ as $\widetilde{\mathbf{E}} \in H(\mathrm{rot}, \mathbb{R}^3)$ defined by

$$\begin{cases} \widetilde{\mathbf{E}} = \mathbf{E} \text{ in } \Omega, \\ \widetilde{\mathbf{E}} = 0 \text{ outside of } \Omega. \end{cases} \quad (14)$$

Remark 4. A polyhedral domain defined thanks to a non degenerated mesh satisfies this hypothesis.

Theorem 4. If $\mathbf{E} \in H(\mathrm{rot}, \Omega)$ and $\mathbf{H} \in H(\mathrm{rot}, \Omega)$ are solution of

$$\begin{cases} \mathrm{rot}(\mathbf{E}) \;=\; -ik\mathbf{H}, \\ \mathrm{rot}(\mathbf{H}) \;=\; ik\mathbf{E}, \end{cases}$$

and satisfies on the boundary

$$\gamma_t \mathbf{E} = 0 \text{ and } \gamma_t \mathbf{H} = 0 \text{ on } \partial\Omega, \quad (15)$$

then $\mathbf{E} \equiv 0$ and $\mathbf{H} \equiv 0$ in Ω.

Proof. Let us introduce the vectorial functions $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{H}}$ defined over all $\mathbb{R}^3$ by

$$\begin{cases} \widetilde{\mathbf{E}} = \mathbf{E} & \text{and} \quad \widetilde{\mathbf{H}} = \mathbf{H} \quad \text{in } \Omega, \\ \widetilde{\mathbf{E}} = 0 & \text{and} \quad \widetilde{\mathbf{H}} = 0 \quad \text{outside of } \Omega. \end{cases} \tag{16}$$

The computation of the curl operator of these functions in the sense of generalized functions leads to

$$\begin{cases} \mathrm{rot}(\widetilde{\mathbf{E}}) &= -ik\widetilde{\mathbf{H}}, \\ \mathrm{rot}(\widetilde{\mathbf{H}}) &= ik\widetilde{\mathbf{E}}. \end{cases} \tag{17}$$

Applying the divergence to this equation, we have $\mathrm{div}(\mathrm{rot}(\cdot)) = 0$

$$\begin{cases} \mathrm{div}(\widetilde{\mathbf{E}}) &= 0, \\ \mathrm{div}(\widetilde{\mathbf{H}}) &= 0. \end{cases} \tag{18}$$

We can therefore compute the vectorial laplacian $\Delta = \mathrm{grad}(\mathrm{div}(\cdot)) - \mathrm{rot}(\mathrm{rot}(\cdot))$. We obtain that both vectorial functions satisfy the vectorial Helmholtz equation

$$\Delta\widetilde{\mathbf{E}} + k^2\widetilde{\mathbf{E}} = 0 \text{ and } \Delta\widetilde{\mathbf{H}} + k^2\widetilde{\mathbf{H}} = 0 \text{ in } \mathbb{R}^3, \tag{19}$$

and

$$\widetilde{\mathbf{E}} = \widetilde{\mathbf{H}} = 0 \text{ outside of } \Omega. \tag{20}$$

The classical unique continuation theorem ensures that

$$\widetilde{\mathbf{E}} = \widetilde{\mathbf{H}} = 0 \text{ in } \Omega. \tag{21}$$

The result follows. □

2 The variational formulation of the numerical method

2.1 Geometrical definitions and numerical functional framework

We consider Maxwell equations on a three-dimensional domain Ω. We use a mesh which partitions Ω domain into elements. The set of all elements is denoted by $\mathcal{K}$, where each element $K \subset \overline{\Omega}$, with $\overline{\Omega}$ referring to the adherence of the domain [13].

Element boundaries are decomposed into faces. The set of all faces F in the mesh is denoted by $\mathcal{F}$. An element K has its own set of faces denoted by $\mathcal{F}_K$. Faces of an element K can either be on the boundary of the domain $\partial\Omega$ or be the interface with another element K' of the mesh. These two types of faces lead to two sets of faces in the mesh $\mathcal{F} := \mathcal{F}_{int} \cup \mathcal{F}_{ext}$

- internal faces denoted by $\mathcal{F}_{int}$: a face $F \in \mathcal{F}_{int}$ if and only if there exist two elements, denoted by K_F^+ and K_F^- such that F is one of their faces,
- exterior faces denoted by $\mathcal{F}_{ext}$: a face $F \in \mathcal{F}_{ext}$ if and only if F is a face of an element K and $F \subset \partial\Omega$.

The functional framework for a discontinuous Galerkin method [6] is defined as follows

$$\mathbb{X} = \prod_{K \in \mathcal{K}} \mathbb{X}_K$$

with the following Hilbert space defined for each element K

$$\mathbb{X}_K \;=\; H_{imp}(K).$$

2.2 First step, toward the numerical fluxes

Let us consider an element K whose boundary is ∂K. The solution $\mathbb{E} = (\mathbf{E}, \mathbf{H})$ is in $\mathbb{X}$. This means that the restriction of $\mathbb{E}$ to K can be written as following

$$\mathbb{E}_K := (\mathbf{E}_K, \mathbf{H}_K) \in \mathbb{X}_K. \tag{22}$$

It satisfies the Maxwell system

$$\left\{ \begin{array}{rcl} \mathrm{rot}(\mathbf{E}_K) + ik\mathbf{H}_K & = & 0, \\ -\mathrm{rot}(\mathbf{H}_K) + ik\mathbf{E}_K & = & 0. \end{array} \right. \tag{23}$$

We denote by $\mathbb{E}' = (\mathbf{E}', \mathbf{H}') \in \mathbb{X}$ the test function restricted to K

$$\mathbb{E}'_K \;=\; (\mathbf{E}'_K, \mathbf{H}'_K) \in \mathbb{X}_K. \tag{24}$$

We first multiply the Maxwell system written on K by a test function

$$\left\{ \begin{array}{rcl} \displaystyle\int_K \mathrm{rot}(\mathbf{E}_K) \cdot \overline{\mathbf{H}'_K} + ik\mathbf{H}_K \cdot \overline{\mathbf{H}'_K} & = & 0, \\ \displaystyle\int_K -\mathrm{rot}(\mathbf{H}_K) \cdot \overline{\mathbf{E}'_K} + ik\mathbf{E}_K \cdot \overline{\mathbf{E}'_K} & = & 0. \end{array} \right. \tag{25}$$

Then, summing over all the element $K \in \mathcal{K}$, we get

$$\left\{ \begin{array}{rcl} \displaystyle\sum_{K\in\mathcal{K}} \int_K \mathrm{rot}(\mathbf{E}_K) \cdot \overline{\mathbf{H}'_K} + ik\mathbf{H}_K \cdot \overline{\mathbf{H}'_K} & = & 0, \\ \displaystyle\sum_{K\in\mathcal{K}} \int_K -\mathrm{rot}(\mathbf{H}_K) \cdot \overline{\mathbf{E}'_K} + ik\mathbf{E}_K \cdot \overline{\mathbf{E}'_K} & = & 0. \end{array} \right. \tag{26}$$

We can then use the Stokes formula

$$\left\{ \begin{array}{rcl} \displaystyle\sum_{K\in\mathcal{K}} \int_K \mathbf{E}_K \cdot \overline{\mathrm{rot}(\mathbf{H}'_K)} + ik\mathbf{H}_K \cdot \overline{\mathbf{H}'_K} & = & \displaystyle\sum_{K\in\mathcal{K}} \int_{\partial K} \gamma_t \mathbf{E}_K \cdot \gamma_\times \overline{\mathbf{H}'_K}, \\ \displaystyle\sum_{K\in\mathcal{K}} \int_K -\mathbf{H}_K \cdot \overline{\mathrm{rot}(\mathbf{E}'_K)} + ik\mathbf{E}_K \cdot \overline{\mathbf{E}'_K} & = & \displaystyle -\sum_{K\in\mathcal{K}} \int_{\partial K} \gamma_t \mathbf{H}_K \cdot \gamma_\times \overline{\mathbf{E}'_K}. \end{array} \right. \tag{27}$$

It remains to take into account the boundary condition and the continuity of the solution.
We recall that the boundary ∂K of K can be decomposed into faces

$$F \in \mathcal{F}_K. \tag{28}$$

We will define the numerical tangential traces (also called numerical fluxes) of the solution on every face $F \subset \partial K$.
This face F can either be internal in $\mathcal{F}_{int}$ (shared by two elements K_F^+ and K_F^-) or be a boundary face in $\mathcal{F}_{ext}$ (a subset of ∂K and $\partial\Omega$).
These numerical fluxes will be denoted by

$$\gamma_t \widehat{\mathbf{E}}_F \text{ and } \gamma_t \widehat{\mathbf{H}}_F, \tag{29}$$

and have the same value for interior faces shared by two elements K_F^+ and K_F^-. This is reasonable since the solution is continuous. For the exact solution (but not for the numerical solution), we will have for internal faces

$$\gamma_t \widehat{\mathbf{E}}_F = \gamma_t \mathbf{E}_{K_F^+} = \gamma_t \mathbf{E}_{K_F^-} \text{ and } \gamma_t \widehat{\mathbf{H}}_F = \gamma_t \mathbf{H}_{K_F^+} = \gamma_t \mathbf{H}_{K_F^-},$$

and for boundary faces

$$\gamma_t \widehat{\mathbf{E}}_F = \gamma_t \mathbf{E}_{K_F} \text{ and } \gamma_t \widehat{\mathbf{H}}_F = \gamma_t \mathbf{H}_{K_F}.$$

In the right hand side of equation (27), we replace the tangential traces $\gamma_t \mathbf{E}_K$ and $\gamma_t \mathbf{H}_K$ by numerical fluxes

$$\left\{ \begin{array}{rcl} \displaystyle\sum_{K\in\mathcal{K}} \int_K \mathbf{E}_K \cdot \overline{\mathrm{rot}(\mathbf{H}'_K)} + ik\mathbf{H}_K \cdot \overline{\mathbf{H}'_K} & = & \displaystyle\sum_{K\in\mathcal{K}} \sum_{F\in\mathcal{F}^K} \int_F \gamma_t \widehat{\mathbf{E}}_F \cdot \gamma_\times \overline{\mathbf{H}'_K}, \\ \displaystyle\sum_{K\in\mathcal{K}} \int_K -\mathbf{H}_K \cdot \overline{\mathrm{rot}(\mathbf{E}'_K)} + ik\mathbf{E}_K \cdot \overline{\mathbf{E}'_K} & = & \displaystyle -\sum_{K\in\mathcal{K}} \sum_{F\in\mathcal{F}^K} \int_F \gamma_t \widehat{\mathbf{H}}_F \cdot \gamma_\times \overline{\mathbf{E}'_K}. \end{array} \right. \tag{30}$$

The last step of this subsection consists in doing a reverse Stokes formula

$$\left\{\begin{array}{rcl} \displaystyle\sum_{K\in\mathcal{K}}\int_K \mathrm{rot}(\mathbf{E}_K)\cdot\overline{\mathbf{H}'_K}+ik\mathbf{H}_K\cdot\overline{\mathbf{H}'_K} & = & \displaystyle\sum_{K\in\mathcal{K}}\int_{\partial K}\Big(\gamma_t\widehat{\mathbf{E}}_F-\gamma_t\mathbf{E}_K\Big)\cdot\overline{\gamma_\times\mathbf{H}'_K}, \\ \displaystyle\sum_{K\in\mathcal{K}}\int_K -\mathrm{rot}(\mathbf{H}_K)\cdot\overline{\mathbf{E}'_K}+ik\mathbf{E}_K\cdot\overline{\mathbf{E}'_K} & = & \displaystyle-\sum_{K\in\mathcal{K}}\int_{\partial K}\Big(\gamma_t\widehat{\mathbf{H}}_F-\gamma_t\mathbf{H}_K\Big)\cdot\overline{\gamma_\times\mathbf{E}'_K}. \end{array}\right. \tag{31}$$

Separating the interior and the exterior faces, we have

$$\left\{\begin{array}{rcl} \displaystyle\sum_{K\in\mathcal{K}}\int_K \mathrm{rot}(\mathbf{E}_K)\cdot\overline{\mathbf{H}'_K}+ik\mathbf{H}_K\cdot\overline{\mathbf{H}'_K} & = & \displaystyle\sum_{K\in\mathcal{K}}\sum_{F\in\mathcal{F}^K_{int}}\int_F\Big(\gamma_t\widehat{\mathbf{E}}_F-\gamma_t\mathbf{E}_K\Big)\cdot\overline{\gamma_\times\mathbf{H}'_K} \\ & & \displaystyle+\sum_{K\in\mathcal{K}}\sum_{F\in\mathcal{F}^K_{ext}}\int_F\Big(\gamma_t\widehat{\mathbf{E}}_F-\gamma_t\mathbf{E}_K\Big)\cdot\overline{\gamma_\times\mathbf{H}'_K}, \\ \displaystyle\sum_{K\in\mathcal{K}}\int_K -\mathrm{rot}(\mathbf{H}_K)\cdot\overline{\mathbf{E}'_K}+ik\mathbf{E}_K\cdot\overline{\mathbf{E}'_K} & = & \displaystyle-\sum_{K\in\mathcal{K}}\sum_{F\in\mathcal{F}^K_{int}}\int_F\Big(\gamma_t\widehat{\mathbf{H}}_F-\gamma_t\mathbf{H}_K\Big)\cdot\overline{\gamma_\times\mathbf{E}'_K} \\ & & \displaystyle+\sum_{K\in\mathcal{K}}\sum_{F\in\mathcal{F}^K_{ext}}\int_F\Big(\gamma_\times\widehat{\mathbf{H}}_F-\gamma_\times\mathbf{H}_K\Big)\cdot\overline{\gamma_t\mathbf{E}'_K}. \end{array}\right. \tag{32}$$

In the next subsection, the numerical fluxes will be determined thanks to a Riemann solver. These numerical fluxes will not have the same value if F is an interior face or a boundary face.

3 Riemann solver

Since meshing the domain causes discontinuities, we have to ensure the continuity of the solution across interfaces $F \in \mathcal{F}$. A Riemann solver can be used to find the solution of a problem with a discontinuous initial condition. In our case, the starting point of our problem is to know that the solution satisfies the Maxwell problem in each element K of the mesh. We then want to construct the solution in the whole domain. That is the reason why we will use the Riemann numerical fluxes to ensure the continuity between elements. Firstly, 1D case of Maxwell hyperbolic system will be considered. Then we deal with the case of a plane separating two 1D media. Finally, we consider the Riemann solver associated to a boundary condition.

3.1 1D Maxwell system as an hyperbolic system

Any wave propagation problem can be seen as an hyperbolic system. In particular, the solution $\mathbb{E}$ of the Maxwell system (4) defined on all $\mathbb{R}^3$ is also

satisfying the following general hyperbolic system

$$\frac{\partial \mathbb{E}}{\partial t}(\mathbf{x},t) = A_1 \frac{\partial \mathbb{E}}{\partial x_1}(\mathbf{x},t) + A_2 \frac{\partial \mathbb{E}}{\partial x_2}(\mathbf{x},t) + A_3 \frac{\partial \mathbb{E}}{\partial x_3}(\mathbf{x},t), \tag{33}$$

where $\mathbf{x} \in \mathbb{R}^3$, $t \in \mathbb{R}$, c is the speed velocity in vacuum, and where we adopt the notation

$$\mathbb{E} = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ H_1 \\ H_2 \\ H_3 \end{bmatrix}. \tag{34}$$

To determine numerical fluxes at interfaces, we have to get a continuity relation between elements. Before formulating this relation in three-dimensional coordinates, let us first consider the one related to 1D Maxwell equations. To get rid of two dimensions, let us suppose that initial conditions are independent of x_2 and x_3. That means

$$\mathbf{E}(x_1,x_2,x_3,0) = \mathbf{E}(x_1,0) \text{ and } \mathbf{H}(x_1,x_2,x_3,0) = \mathbf{H}(x_1,0).$$

Due to isotropy, the solution of the system is also independent of x_2 and x_3

$$\mathbf{E}(x_1,x_2,x_3,t) = \mathbf{E}(x_1,t) \text{ and } \mathbf{H}(x_1,x_2,x_3,t) = \mathbf{H}(x_1,t).$$

Assuming these independences, the hyperbolic system becomes

$$\frac{\partial \mathbb{E}}{\partial t}(x_1,t) = A_1 \frac{\partial \mathbb{E}}{\partial x_1}(x_1,t), \tag{35}$$

whose solution is also solution of the 1D Maxwell equation

$$\begin{cases} \mathrm{rot}(\mathbf{E}(x_1,t)) &= -ik\mathbf{H}(x_1,t), \\ \mathrm{rot}(\mathbf{H}(x_1,t)) &= \quad ik\mathbf{E}(x_1,t). \end{cases} \tag{36}$$

Theorem 5. The solution of the 1D Maxwell equation is given by

$$\left\{\begin{array}{lllll}
\mathbb{E}(x_1,t) & = & E_1(x_1,0) & \begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix} & + \; H_1(x_1,0) \begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix} \\
 & + & \dfrac{E_2+H_3}{2}(x_1-ct,0) & \begin{bmatrix}0\\1\\0\\0\\0\\1\end{bmatrix} & + \; \dfrac{E_3-H_2}{2}(x_1-ct,0) \begin{bmatrix}0\\0\\1\\0\\-1\\0\end{bmatrix} \\
 & + & \dfrac{E_2-H_3}{2}(x_1+ct,0) & \begin{bmatrix}0\\1\\0\\0\\0\\-1\end{bmatrix} & + \; \dfrac{E_3+H_2}{2}(x_1+ct,0) \begin{bmatrix}0\\0\\1\\0\\1\\0\end{bmatrix}
\end{array}\right. .$$

Proof. The Maxwell system reads

$$\left\{\begin{array}{l}
\dfrac{\partial E_3}{\partial x_2} - \dfrac{\partial E_2}{\partial x_3} = -\dfrac{1}{c}\dfrac{\partial H_1}{\partial t} \\
\dfrac{\partial E_1}{\partial x_3} - \dfrac{\partial E_3}{\partial x_1} = -\dfrac{1}{c}\dfrac{\partial H_2}{\partial t} \\
\dfrac{\partial E_2}{\partial x_1} - \dfrac{\partial E_1}{\partial x_2} = -\dfrac{1}{c}\dfrac{\partial H_3}{\partial t}
\end{array}\right. \qquad \left\{\begin{array}{l}
\dfrac{\partial H_3}{\partial x_2} - \dfrac{\partial H_2}{\partial x_3} = \dfrac{1}{c}\dfrac{\partial E_1}{\partial t} \\
\dfrac{\partial H_1}{\partial x_3} - \dfrac{\partial H_3}{\partial x_1} = \dfrac{1}{c}\dfrac{\partial E_2}{\partial t} \\
\dfrac{\partial H_2}{\partial x_1} - \dfrac{\partial H_1}{\partial x_2} = \dfrac{1}{c}\dfrac{\partial E_3}{\partial t}
\end{array}\right. .$$

Taking into account that the solution is independent of x_2 and x_3, we can simplify, thus leading to :

$$\left\{\begin{array}{l}
0 = -\dfrac{1}{c}\dfrac{\partial H_1}{\partial t} \\
-\dfrac{\partial E_3}{\partial x_1} = -\dfrac{1}{c}\dfrac{\partial H_2}{\partial t} \\
\dfrac{\partial E_2}{\partial x_1} = -\dfrac{1}{c}\dfrac{\partial H_3}{\partial t}
\end{array}\right. \qquad \left\{\begin{array}{l}
0 = \dfrac{1}{c}\dfrac{\partial E_1}{\partial t} \\
-\dfrac{\partial H_3}{\partial x_1} = \dfrac{1}{c}\dfrac{\partial E_2}{\partial t} \\
\dfrac{\partial H_2}{\partial x_1} = \dfrac{1}{c}\dfrac{\partial E_3}{\partial t}
\end{array}\right. .$$

It follows that H_1 and E_1 are time-independent. Then they are constant and equal to the initial condition for all t

$$H_1(\overrightarrow{x},t) = H_1(\overrightarrow{x},0) \text{ and } E_1(\overrightarrow{x},t) = E_1(\overrightarrow{x},0).$$

The other coordinates verify the 1D hyperbolic system

$$\frac{\partial \mathbf{U}}{\partial t}(x_1,t) = cA_1\frac{\partial \mathbf{U}}{\partial x_1}(x_1,t),$$

with

$$\mathbf{U} = \begin{bmatrix} E_2 \\ E_3 \\ H_2 \\ H_3 \end{bmatrix} \text{ and } A_1 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}. \tag{37}$$

We diagonalize the matrix A_1

$$A_1P = P\Lambda \text{ with } P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{38}$$

We introduce the orthogonal basis of eigenvectors $\overrightarrow{w}_i$

$$\overrightarrow{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \overrightarrow{w}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \quad \overrightarrow{w}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad \overrightarrow{w}_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}. \tag{39}$$

We have $A_1w_i = \lambda_i w_i$. More precisely, we have

$$A_1\overrightarrow{w}_1 = -\overrightarrow{w}_1, \quad A_1\overrightarrow{w}_2 = -\overrightarrow{w}_2, \quad A_1\overrightarrow{w}_3 = +\overrightarrow{w}_3, \quad A_1\overrightarrow{w}_4 = +\overrightarrow{w}_4.$$

The solution $\mathbf{U}$ can be decomposed in the introduced basis thanks to coefficients $(\varepsilon_i)_{i=1,4}$

$$\mathbf{U}(x_1,t) \;=\; \sum_{i=1}^{4} \varepsilon_i(x_1,t)\overrightarrow{w}_i\ .$$

Inserting this formula into the 1D hyperbolic system it follows

$$\sum_{i=1}^{4} \frac{\partial \varepsilon_i}{\partial t}(x_1,t)\overrightarrow{w}_i = c\sum_{i=1}^{4} \lambda_i \frac{\partial \varepsilon_i}{\partial x_1}(x_1,t)\overrightarrow{w}_i\ .$$

Therefore, the coordinates $\varepsilon_i(x_1,t)$ solves

$$\frac{\partial \varepsilon_i}{\partial t}(x_1,t) = c\lambda_i\frac{\partial \varepsilon_i}{\partial x_1}(x_1,t),$$

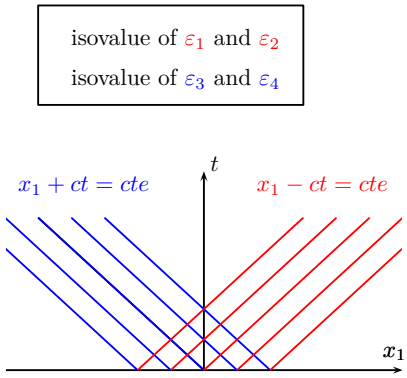

Figure 1: Isovalue of functions ε_i, $i = 1, 4$.

which is a transport equation whose solution is $\varepsilon_i(x_1, t) = \varepsilon_i(x_1 + \lambda_i ct, 0)$. It leads to

$$\begin{cases} \varepsilon_1(x_1,t) = \varepsilon_1(x_1 - ct, 0), & \varepsilon_2(x_1,t) = \varepsilon_2(x_1 - ct, 0), \\ \varepsilon_3(x_1,t) = \varepsilon_3(x_1 + ct, 0), & \varepsilon_4(x_1,t) = \varepsilon_4(x_1 + ct, 0). \end{cases}$$

Functions ε_1 and ε_2 are propagating in x_1 direction with velocity c. In a similar way, functions ε_3 and ε_4 are propagating in $-x_1$ direction with velocity c (see Fig.1).
Since $\|\overrightarrow{w}_i\|^2 = 2$, we can also normalize the basis thus leading to

$$\varepsilon_i(x_1,t) = \frac{\mathbf{U}(x_1,t) \cdot \overrightarrow{w}_i}{2}.$$

Therefore, we have

$$\begin{cases} \varepsilon_1(x_1,t) = \dfrac{\mathbf{U}(x_1,t) \cdot \overrightarrow{w}_1}{2} = \dfrac{E_2 + H_3}{2}(x_1,t) = \dfrac{E_2 + H_3}{2}(x_1 - ct, 0), \\ \varepsilon_2(x_1,t) = \dfrac{\mathbf{U}(x_1,t) \cdot \overrightarrow{w}_2}{2} = \dfrac{E_3 - H_2}{2}(x_1,t) = \dfrac{E_3 - H_2}{2}(x_1 - ct, 0), \\ \varepsilon_3(x_1,t) = \dfrac{\mathbf{U}(x_1,t) \cdot \overrightarrow{w}_3}{2} = \dfrac{E_2 - H_3}{2}(x_1,t) = \dfrac{E_2 - H_3}{2}(x_1 + ct, 0), \\ \varepsilon_4(x_1,t) = \dfrac{\mathbf{U}(x_1,t) \cdot \overrightarrow{w}_4}{2} = \dfrac{E_3 + H_2}{2}(x_1,t) = \dfrac{E_3 + H_2}{2}(x_1 + ct, 0). \end{cases} \tag{40}$$

The expression

$$\mathbf{U}(x_1,t) = \varepsilon_1(x_1 - ct, 0)\overrightarrow{w}_1 + \varepsilon_2(x_1 - ct, 0)\overrightarrow{w}_2 + \varepsilon_3(x_1 + ct, 0)\overrightarrow{w}_3 + \varepsilon_4(x_1 + ct, 0)\overrightarrow{w}_4, \tag{41}$$

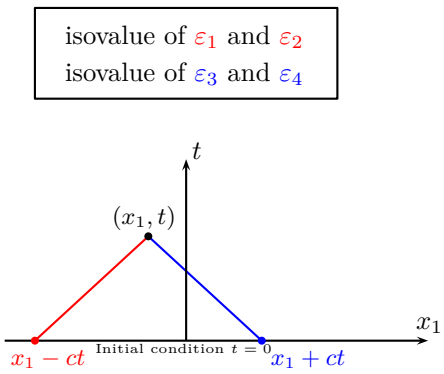

Figure 2: Characteristic lines involved in the computation of $\mathbf{U}(x_1, t)$ for an interior interface.

can then be evaluated thanks to (40). The formula (41) can then be graphically interpreted. The solution at point (x_1, t) is given thanks to the values of ε_i functions which are evaluated on the characteristic lines

1. $x - ct = cte$ for ε_1 and ϵ_2 (red color in Fig.2),
2. $x + ct = cte$ for ε_3 and ϵ_4 (blue color in Fig.2).

The theorem follows.

□

3.2 Riemann solver for interior interfaces

The interior fluxes are the information going through an interior face $F \in \mathcal{F}_{int}$. If we consider a face F interfacing two elements K_F^+ and K_F^-, and take $\alpha_{int} \in \mathbb{R}$ and $\beta_{int} \in \mathbb{R}$, interior fluxes on F can be written as following

$$\left\{ \begin{array}{lcl} \gamma_t \widehat{\mathbf{E}}_F & = & \{\gamma_t \mathbf{E}\}_F + \alpha_{int} [\![\gamma_\times \mathbf{H}]\!]_F, \\ \gamma_t \widehat{\mathbf{H}}_F & = & \{\gamma_t \mathbf{H}\}_F + \beta_{int} [\![\gamma_\times \mathbf{E}]\!]_F, \end{array} \right. \tag{42}$$

where we adopt the following notations

$$\left\{ \begin{array}{lcl} \{\gamma_t \mathbf{E}\}_F & = & \dfrac{\gamma_t \mathbf{E}^+ + \gamma_t \mathbf{E}^-}{2}, \\ [\![\gamma_\times \mathbf{E}]\!]_F & = & \mathbf{n}^- \times \mathbf{E}^- + \mathbf{n}^+ \times \mathbf{E}^+, \end{array} \right.$$

with $\mathbf{n}^+$ (resp. $\mathbf{n}^-$) the outward unit normal to K_F^+ (resp. K_F^-).

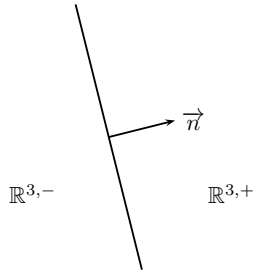

Figure 3: Two half spaces in $\mathbb{R}^3$.

The goals of the present part are to explain why interior fluxes take this form and to find the constants α_{int} and β_{int}. To do so, we consider the assumptions made on the independence of the initial conditions on x_2 and x_3. We can thus use the latter theorem to determine interior Riemann fluxes on interior faces. In other words, we have to find the values of functions ε_i, for $i = 1, 4$. The initial three-dimensional problem governed by the Maxwell equations and the initial data takes the following form

$$\mathbf{E}(\overrightarrow{x}, 0) = \mathbf{E}^- \text{ for } \overrightarrow{x} \in \mathbb{R}^{3,-} \text{ and } \mathbf{E}(\overrightarrow{x}, 0) = \mathbf{E}^+ \text{ for } \overrightarrow{x} \in \mathbb{R}^{3,+},$$

where $\mathbb{R}^3$ is decomposed into two half spaces (see Fig. 3)

$$\mathbb{R}^{3,+} = \left\{ \overrightarrow{x} \in \mathbb{R}^3 \text{ such that } \overrightarrow{x} \cdot \mathbf{n} \geq 0 \right\},$$

$$\mathbb{R}^{3,-} = \left\{ \overrightarrow{x} \in \mathbb{R}^3 \text{ such that } \overrightarrow{x} \cdot \mathbf{n} \leq 0 \right\}.$$

This decomposition allows us to see the interface between these two-half spaces as an interior face in the mesh. That is the reason why we choose our set of coordinates with respect to the normal $\mathbf{n}$

$$x_1 = \overrightarrow{x} \cdot \mathbf{n}.$$

We denote by x_2 and x_3 coordinates associated to an orthonormal basis $(\overrightarrow{e}_1 = \mathbf{n}, \overrightarrow{e}_2, \overrightarrow{e}_3)$, and we adopt the following notations

$$E_i = \mathbf{E} \cdot \overrightarrow{e}_i \quad \text{and} \quad H_i = \mathbf{H} \cdot \overrightarrow{e}_i.$$

The theorem 5 is the main ingredient to solve this transmission problem. It follows that the solution takes three values depending on the values of x_1 (see Fig. 4) :

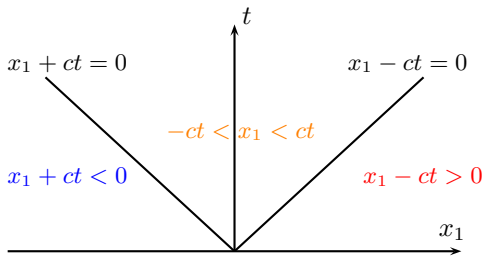

Figure 4: Variables dependant spaces.

- For $x_1 < -ct$ since $x_1 + ct < 0$

$$E_i(x_1, t) = E_i^- \text{ and } H_i(x_1, t) = H_i^-,$$

- For $-ct < x_1 < ct$, we have since $x_1 - ct < 0$ and $x_1 + ct > 0$

$$\begin{cases} E_2(x_1, t) = \dfrac{E_2^- + E_2^+}{2} - \dfrac{H_3^+ - H_3^-}{2}, \\ E_3(x_1, t) = \dfrac{E_3^- + E_3^+}{2} + \dfrac{H_2^+ - H_2^-}{2}, \\ H_2(x_1, t) = \dfrac{H_2^- + H_2^+}{2} + \dfrac{E_3^+ - E_3^-}{2}, \\ H_3(x_1, t) = \dfrac{H_3^- + H_3^+}{2} - \dfrac{E_2^+ - E_2^-}{2}, \end{cases}$$

  in that case, the value of $\mathbf{U}(x_1, t)$ depends on the four functions propagating in x_1 and $-x_1$ directions, see Fig. 2,

- For $x_1 > ct$, we have since $0 < x_1 - ct$

$$E_i(x_1, t) = E_i^+ \text{ and } H_i(x_1, t) = H_i^+.$$

The numerical fluxes used in our numerical method are determined by considering the value of the solution at $x_1 = 0$ which is independent of t

$$\begin{cases} E_2(0, t) = \dfrac{E_2^- + E_2^+}{2} - \dfrac{H_3^+ - H_3^-}{2}, \\ E_3(0, t) = \dfrac{E_3^- + E_3^+}{2} + \dfrac{H_2^+ - H_2^-}{2}, \\ H_2(0, t) = \dfrac{H_2^- + H_2^+}{2} + \dfrac{E_3^+ - E_3^-}{2}, \\ H_3(0, t) = \dfrac{H_3^- + H_3^+}{2} - \dfrac{E_2^+ - E_2^-}{2}. \end{cases}$$

Remark 5. Particularly, we take into account that

$$\begin{cases} \gamma_t \mathbf{E} = (0, E_2, E_3)^T, \\ \gamma_\times^+ \mathbf{E} = \mathbf{n}^+ \times \gamma_t \mathbf{E} = (0, E_3, -E_2)^T, \\ \gamma_\times^- \mathbf{E} = \mathbf{n}^- \times \gamma_t \mathbf{E} = (0, -E_3, E_2)^T, \\ \mathbf{n}^+ := -\mathbf{n} \text{ and } \mathbf{n}^- := \mathbf{n}, \end{cases}$$

which lead to the interior Riemann fluxes

$$\begin{cases} \gamma_t \widehat{\mathbf{E}}_F = \dfrac{\gamma_t \mathbf{E}^+ + \gamma_t \mathbf{E}^-}{2} - \dfrac{\gamma_\times^+ \mathbf{H}^+ + \gamma_\times^- \mathbf{H}^-}{2}, \\ \gamma_t \widehat{\mathbf{H}}_F = \dfrac{\gamma_t \mathbf{H}^+ + \gamma_t \mathbf{H}^-}{2} + \dfrac{\gamma_\times^+ \mathbf{E}^+ + \gamma_\times^- \mathbf{E}^-}{2}. \end{cases} \tag{43}$$

The latter can be condensed into

$$\begin{cases} \gamma_t \widehat{\mathbf{E}}_F &= \{\gamma_t \mathbf{E}\}_F - \dfrac{[\![\gamma_\times \mathbf{H}]\!]_F}{2}, \\ \gamma_t \widehat{\mathbf{H}}_F &= \{\gamma_t \mathbf{H}\}_F + \dfrac{[\![\gamma_\times \mathbf{E}]\!]_F}{2}. \end{cases} \tag{44}$$

The values of constants α_{int} and β_{int} follow

$$\begin{cases} \alpha_{int} &= -\dfrac{1}{2}, \\ \beta_{int} &= \dfrac{1}{2}. \end{cases} \tag{45}$$

We point out here that (47) are unoriented formulas. In other words, they are the same if we switch $\mathbb{R}^{3,+}$ and $\mathbb{R}^{3,-}$ domains.

Remark 6. The value of the numerical flux on an interior face F of an element K^+ is

$$\begin{cases} \gamma_t \widehat{\mathbf{E}}_F = \gamma_t \mathbf{E}^+ - \dfrac{\gamma_t \mathbf{E}^+ - \gamma_t \mathbf{E}^-}{2} - \dfrac{\gamma_\times^+ \mathbf{H}^+ + \gamma_\times^- \mathbf{H}^-}{2}, \\ \gamma_t \widehat{\mathbf{H}}_F = \gamma_t \mathbf{H}^+ - \dfrac{\gamma_t \mathbf{H}^+ - \gamma_t \mathbf{H}^-}{2} + \dfrac{\gamma_\times^+ \mathbf{E}^+ + \gamma_\times^- \mathbf{E}^-}{2}. \end{cases} \tag{46}$$

This leads to

$$\begin{cases} \gamma_t \widehat{\mathbf{E}}_F = \gamma_t \mathbf{E}^+ + \dfrac{\mathbf{n}^+ \times [\![\gamma_\times \mathbf{E}]\!]_F}{2} - \dfrac{[\![\gamma_\times \mathbf{H}]\!]_F}{2}, \\ \gamma_t \widehat{\mathbf{H}}_F = \gamma_t \mathbf{H}^+ + \dfrac{\mathbf{n}^+ \times [\![\gamma_\times \mathbf{H}]\!]_F}{2} + \dfrac{[\![\gamma_\times \mathbf{E}]\!]_F}{2}. \end{cases} \tag{47}$$

Remarks 5 and 6 lead to the following theorem.

Theorem 6. Riemann fluxes for an interior face F takes the form

$$\gamma_t \widehat{\mathbf{E}}_F = \{\gamma_t \mathbf{E}\}_F - \frac{[\![\gamma_\times \mathbf{H}]\!]_F}{2}, \gamma_t \widehat{\mathbf{H}}_F = \{\gamma_t \mathbf{H}\}_F + \frac{[\![\gamma_\times \mathbf{E}]\!]_F}{2}, \tag{48}$$

or equivalently

$$\left\{ \begin{array}{l} \gamma_t \widehat{\mathbf{E}}_F = \gamma_t \mathbf{E}^+ + \dfrac{\mathbf{n}^+ \times [\![\gamma_\times \mathbf{E}]\!]_F}{2} - \dfrac{[\![\gamma_\times \mathbf{H}]\!]_F}{2}, \\ \gamma_t \widehat{\mathbf{H}}_F = \gamma_t \mathbf{H}^+ + \dfrac{\mathbf{n}^+ \times [\![\gamma_\times \mathbf{H}]\!]_F}{2} + \dfrac{[\![\gamma_\times \mathbf{E}]\!]_F}{2}. \end{array} \right. \tag{49}$$

Remark 7. Since every regular solution of the Maxwell problem has zero-jumps

$$[\![\gamma_\times \mathbf{E}]\!]_F = 0 \text{ and } [\![\gamma_\times \mathbf{H}]\!]_F = 0,$$

these numerical fluxes applied to the exact solution simply give

$$\gamma_t \widehat{\mathbf{E}}_F = \gamma_t \mathbf{E}|_F \quad \text{and} \quad \gamma_t \widehat{\mathbf{H}}_F = \gamma_t \mathbf{H}|_F.$$

3.3 Riemann solver for the boundary

Let us now determine boundary Riemann fluxes. Boundary fluxes traduce the information going through the domain boundary.
For boundary faces we solve the Maxwell problem on the half space $\overrightarrow{x} \cdot \mathbf{n} \leq 0$. The initial three-dimensional problem is

$$\left\{ \begin{array}{ll} \mathrm{rot}(\mathbf{E}) = -\dfrac{1}{c}\dfrac{\partial \mathbf{H}}{\partial t}, & \text{for } \overrightarrow{x} \cdot \mathbf{n} \leq 0, \\ \mathrm{rot}(\mathbf{H}) = \dfrac{1}{c}\dfrac{\partial \mathbf{E}}{\partial t}, & \text{for } \overrightarrow{x} \cdot \mathbf{n} \leq 0. \end{array} \right.$$

These Maxwell equations are equipped with an impedance boundary condition

$$Y \gamma_t \mathbf{E} + \gamma_\times \mathbf{H} \ = \ \gamma_t \overrightarrow{\varphi}, \quad \text{for } \overrightarrow{x} \cdot \mathbf{n} = 0,$$

and the initial value problem

$$\mathbf{E}(\overrightarrow{x}, 0) = \mathbf{E}^- \text{ and } \mathbf{H}(\overrightarrow{x}, 0) = \mathbf{H}^-.$$

Let us show that taking $\alpha_{ext} \in \mathbb{R}$ and $\beta_{ext} \in \mathbb{R}$, boundary Riemann fluxes can be written as

$$\left\{ \begin{array}{l} \gamma_t \widehat{\mathbf{E}} = \gamma_t \mathbf{E}^- + \alpha_{ext}\Big(Y \gamma_t \mathbf{E}^- + \gamma_\times \mathbf{H}^- - \gamma_t \overrightarrow{\varphi}\Big) \\ \gamma_\times \widehat{\mathbf{H}} = \gamma_\times \mathbf{H}^- + \beta_{ext}\Big(Y \gamma_t \mathbf{E}^- + \gamma_\times \mathbf{H}^- - \gamma_t \overrightarrow{\varphi}\Big). \end{array} \right. \tag{50}$$

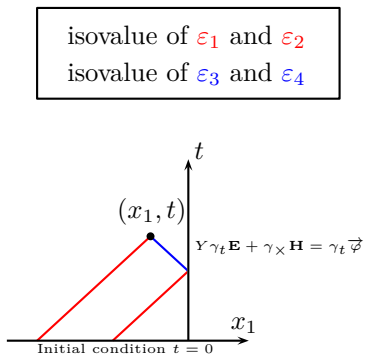

Figure 5: Characteristic lines involved in the computation of $\mathbf{U}(x_1, t)$ for a boundary interface.

We use the same set of coordinates than before. According to theorem 5, $E_1(\overrightarrow{x}, t) = E_1^-$ and $H_1(\overrightarrow{x}, t) = H_1^-$ are constant. We introduce the notation $\mathbf{U} = (E_2,\ E_3,\ H_2,\ H_3)^T$ and we mention the solution we got before

$$\mathbf{U}(x_1, t) = \varepsilon_1(x_1 - ct, 0)\overrightarrow{w}_1 + \varepsilon_2(x_1 - ct, 0)\overrightarrow{w}_2 + \varepsilon_3(x_1 + ct, 0)\overrightarrow{w}_3 + \varepsilon_4(x_1 + ct, 0)\overrightarrow{w}_4. \quad (51)$$

To obtain the values of $\mathbf{U}(x_1, t)$ for every $x_1 < 0$ and $t > 0$, we need to determine

$$\begin{cases} \varepsilon_i^-(x_1) &= \varepsilon_i(x_1) \text{ for } x_1 < 0 \text{ and } 1 \leq i \leq 4, \text{see Fig 6,} \\ \varepsilon_i^+(x_1) &= \varepsilon_i(x_1) \text{ for } x_1 > 0 \text{ and } 3 \leq i \leq 4, \text{see Fig 5.} \end{cases} \quad (52)$$

These functions will be determined thanks to

- the initial condition for ε_i^-, $1 \leq i \leq 4$,
- the impedance boundary condition for ε_3^+ and ε_4^+.

Writing (51) at $t = 0$, we have for $x_1 < 0$

$$\begin{pmatrix} E_2^-(x_1, 0) \\ E_3^-(x_1, 0) \\ H_2^-(x_1, 0) \\ H_3^-(x_1, 0) \end{pmatrix} = \varepsilon_1^-(x_1)\overrightarrow{w}_1 + \varepsilon_2^-(x_1)\overrightarrow{w}_2 + \varepsilon_3^-(x_1)\overrightarrow{w}_3 + \varepsilon_4^-(x_1)\overrightarrow{w}_4. \quad (53)$$

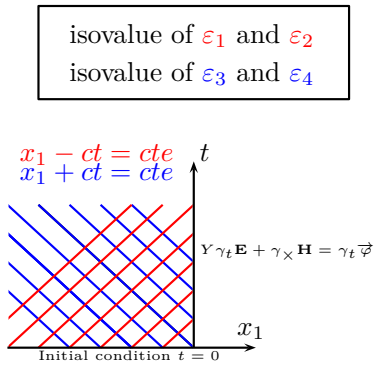

Figure 6: Characteristic lines encountering the initial condition and the boundary condition.

We have by acting similarly to (40)

$$\begin{cases} \varepsilon_1^- &= (E_2^- + H_3^-)/2, \\ \varepsilon_2^- &= (E_3^- - H_2^-)/2, \\ \varepsilon_3^- &= (E_2^- - H_3^-)/2, \\ \varepsilon_4^- &= (E_3^- + H_2^-)/2. \end{cases} \tag{54}$$

Consequently, $\mathbf{U}$ has the following form for $x_1 < -ct$

$$\begin{cases} E_2(\overrightarrow{x}, t) = E_2^-, \\ E_3(\overrightarrow{x}, t) = E_3^-, \\ H_2(\overrightarrow{x}, t) = H_2^-, \\ H_3(\overrightarrow{x}, t) = H_3^-. \end{cases} \tag{55}$$

Let us determine functions ε_3^+ and ε_4^+. The impedance boundary condition at $x_1 = 0$ and $t > 0$ is

$$Y\gamma_t\mathbf{E} + \gamma_\times\mathbf{H} \;=\; \gamma_t\overrightarrow{\varphi} \iff \begin{bmatrix} 0 \\ YE_2(0,t) \\ YE_3(0,t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ H_2(0,t) \\ H_3(0,t) \end{bmatrix} = \begin{bmatrix} 0 \\ \varphi_2(t) \\ \varphi_3(t) \end{bmatrix}. \tag{56}$$

This means that

$$\begin{cases} Y\ E_2(0,t) - H_3(0,t) = \varphi_2(t), \\ Y\ E_3(0,t) + H_2(0,t) = \varphi_3(t). \end{cases} \tag{57}$$

We have due to (51)

$$\mathbf{U}(0,t) = \varepsilon_1^-(-ct)\overrightarrow{w}_1 + \varepsilon_2^-(-ct)\overrightarrow{w}_2 + \varepsilon_3^+(ct)\overrightarrow{w}_3 + \varepsilon_4^+(ct)\overrightarrow{w}_4. \tag{58}$$

Taking into account (39), it means

$$\begin{pmatrix} E_2(0,t) \\ E_3(0,t) \\ H_2(0,t) \\ H_3(0,t) \end{pmatrix} = \begin{pmatrix} \varepsilon_1^-(-ct) + \varepsilon_3^+(ct) \\ \varepsilon_2^-(-ct) + \varepsilon_4^+(ct) \\ -\varepsilon_2^-(-ct) + \varepsilon_4^+(ct) \\ \varepsilon_1^-(-ct) - \varepsilon_3^+(ct) \end{pmatrix}. \tag{59}$$

This leads to the linear system

$$\begin{cases} Y\left(\varepsilon_1^- + \varepsilon_3^+\right) - \varepsilon_1^- + \varepsilon_3^+ = \varphi_2 \\ Y\left(\varepsilon_2^- + \varepsilon_4^+\right) - \varepsilon_2^- + \varepsilon_4^+ = \varphi_3. \end{cases} \tag{60}$$

We deduce ε_i^+ coefficients which are constant

$$\begin{cases} \varepsilon_3^+ = \dfrac{1}{1+Y}\varphi_2 + \dfrac{1-Y}{1+Y}\varepsilon_1^-, \\ \varepsilon_4^+ = \dfrac{1}{1+Y}\varphi_3 + \dfrac{1-Y}{1+Y}\varepsilon_2^-. \end{cases} \tag{61}$$

We denote by $R = \dfrac{1-Y}{1+Y}$ the reflexion coefficient. Since $\dfrac{1}{1+Y} = \dfrac{1+R}{2}$, we have

$$\begin{cases} \varepsilon_3^+ = \dfrac{1+R}{2}\varphi_2 + R\varepsilon_1^- = \dfrac{1+R}{2}\varphi_2 + R\dfrac{E_2^- + H_3^-}{2} \\ \varepsilon_4^+ = \dfrac{1+R}{2}\varphi_3 + R\varepsilon_2^- = \dfrac{1+R}{2}\varphi_3 + R\dfrac{E_3^- - H_2^-}{2}. \end{cases} \tag{62}$$

We recall that for $-ct < x_1 < 0$, $\mathbf{U}$ can be evaluated as

$$\mathbf{U}(x_1,t) = \varepsilon_1^-\overrightarrow{w}_1 + \varepsilon_2^-\overrightarrow{w}_2 + \varepsilon_3^+\overrightarrow{w}_3 + \varepsilon_4^+\overrightarrow{w}_4.$$

Therefore, we have thanks to (39) and (40)

$$\begin{bmatrix} E_2(x_1,t) \\ E_3(x_1,t) \\ H_2(x_1,t) \\ H_3(x_1,t) \end{bmatrix} = \begin{bmatrix} \varepsilon_1^- + \varepsilon_3^+ \\ \varepsilon_2^- + \varepsilon_4^+ \\ -\varepsilon_2^- + \varepsilon_4^+ \\ \varepsilon_1^- - \varepsilon_3^+ \end{bmatrix} = \begin{bmatrix} \dfrac{E_2^- + H_3^-}{2} + R\dfrac{E_2^- + H_3^-}{2} + \dfrac{R+1}{2}\varphi_2 \\ \dfrac{E_3^- - H_2^-}{2} + R\dfrac{E_3^- - H_2^-}{2} + \dfrac{R+1}{2}\varphi_3 \\ \dfrac{H_2^- - E_3^-}{2} - R\dfrac{H_2^- - E_3^-}{2} + \dfrac{R+1}{2}\varphi_3 \\ \dfrac{H_3^- + E_2^-}{2} - R\dfrac{H_3^- + E_2^-}{2} - \dfrac{R+1}{2}\varphi_2 \end{bmatrix}.$$

It follows for $-ct < x_1 < 0$

$$\begin{cases} E_2(x_1,t) = \dfrac{1+R}{2}\Big(E_2^- + H_3^-\Big) + \dfrac{R+1}{2}\varphi_2, \\ E_3(x_1,t) = \dfrac{1+R}{2}\Big(E_3^- - H_2^-\Big) + \dfrac{R+1}{2}\varphi_3, \\ H_2(x_1,t) = \dfrac{1-R}{2}\Big(H_2^- - E_3^-\Big) + \dfrac{R+1}{2}\varphi_3, \\ H_3(x_1,t) = \dfrac{1-R}{2}\Big(H_3^- + E_2^-\Big) - \dfrac{R+1}{2}\varphi_2. \end{cases}$$

Remark 8. We can now deduce the value of the numerical fluxes $\widehat{\mathbf{E}}$ and $\widehat{\mathbf{H}}$ thanks to the value of the electromagnetic fields at $x_1 = 0$

$$\begin{cases} \widehat{\mathbf{E}}_2 = \dfrac{1+R}{2}\Big(E_2^- + H_3^-\Big) + \dfrac{R+1}{2}\varphi_2, \\ \widehat{\mathbf{E}}_3 = \dfrac{1+R}{2}\Big(E_3^- - H_2^-\Big) + \dfrac{R+1}{2}\varphi_3, \\ \widehat{\mathbf{H}}_2 = \dfrac{1-R}{2}\Big(H_2^- - E_3^-\Big) + \dfrac{R+1}{2}\varphi_3, \\ \widehat{\mathbf{H}}_3 = \dfrac{1-R}{2}\Big(H_3^- + E_2^-\Big) - \dfrac{R+1}{2}\varphi_2. \end{cases}$$

Taking into account that

$$\begin{cases} \gamma_t\mathbf{E} = (0, E_2, E_3)^T, \\ \gamma_\times\mathbf{E} = \mathbf{n}^- \times \gamma_t\mathbf{E} = (0, -E_3, E_2)^T, \end{cases}$$

Riemann fluxes for an interface F take the following form

$$\begin{cases} \gamma_t\widehat{\mathbf{E}} = \dfrac{1+R}{2}\Big(\gamma_t\mathbf{E}^- - \gamma_\times\mathbf{H}^-\Big) + \dfrac{1+R}{2}\gamma_t\overrightarrow{\varphi} \\ \gamma_t\widehat{\mathbf{H}} = \dfrac{1-R}{2}\Big(\gamma_t\mathbf{H}^- + \gamma_\times\mathbf{E}^-\Big) - \dfrac{1+R}{2}\gamma_\times\overrightarrow{\varphi}. \end{cases} \tag{63}$$

Remark 9. If the impedance boundary condition

$$Y\gamma_t\mathbf{E}^- + \gamma_\times\mathbf{H}^- = \gamma_t\overrightarrow{\varphi}$$

is satisfied, then we have $\gamma_t\widehat{\mathbf{E}} = \gamma_t\mathbf{E}$ and $\gamma_t\widehat{\mathbf{H}} = \gamma_t\mathbf{H}$.
Indeed, by adding and subtracting fields traces, (63) is equivalent to

$$\begin{cases} \gamma_t\widehat{\mathbf{E}} = \gamma_t\mathbf{E}^- + \Big(\dfrac{1+R}{2} - 1\Big)\gamma_t\mathbf{E}^- - \dfrac{1+R}{2}\gamma_\times\mathbf{H}^- + \dfrac{R+1}{2}\gamma_t\overrightarrow{\varphi} \\ \gamma_t\widehat{\mathbf{H}} = \gamma_t\mathbf{H}^- + \Big(\dfrac{1-R}{2} - 1\Big)\gamma_t\mathbf{H}^- + \dfrac{1-R}{2}\gamma_\times\mathbf{E}^- - \dfrac{R+1}{2}\gamma_\times\overrightarrow{\varphi}. \end{cases} \tag{64}$$

Since $R = \dfrac{1-Y}{1+Y}$, we have

$$\frac{1+R}{2} = \frac{1}{1+Y}, \qquad \frac{1-R}{2} = \frac{Y}{1+Y},$$
$$\frac{1-R}{2} - 1 = -\frac{1}{1+Y}, \quad \frac{1+R}{2} - 1 = -\frac{Y}{1+Y},$$

then

$$\left\{ \begin{array}{l} \gamma_t \widehat{\mathbf{E}} = \gamma_t \mathbf{E}^- - \dfrac{Y}{1+Y}\gamma_t \mathbf{E}^- - \dfrac{1}{1+Y}\gamma_\times \mathbf{H}^- + \dfrac{\gamma_t \overrightarrow{\varphi}}{1+Y}, \\ \gamma_t \widehat{\mathbf{H}} = \gamma_t \mathbf{H} - \dfrac{1}{1+Y}\gamma_t \mathbf{H}^- + \dfrac{Y}{1+Y}\gamma_\times \mathbf{E}^- - \dfrac{\gamma_\times \overrightarrow{\varphi}}{1+Y}. \end{array} \right. \tag{65}$$

We can factorize this expression and it leads to

$$\left\{ \begin{array}{l} \gamma_t \widehat{\mathbf{E}} = \gamma_t \mathbf{E}^- - \dfrac{1}{1+Y}\Big(Y\gamma_t \mathbf{E}^- + \gamma_\times \mathbf{H}^- - \gamma_t \overrightarrow{\varphi}\Big), \\ \gamma_t \widehat{\mathbf{H}} = \gamma_t \mathbf{H}^- - \dfrac{1}{1+Y}\Big(\gamma_t \mathbf{H}^- - Y\gamma_\times \mathbf{E}^- + \gamma_\times \overrightarrow{\varphi}\Big). \end{array} \right. \tag{66}$$

The second line can be multiplied by $\mathbf{n}\times$:

$$\gamma_\times \widehat{\mathbf{H}} = \gamma_\times \mathbf{H}^- - \frac{1}{1+Y}\Big(Y\gamma_t \mathbf{E}^- + \gamma_\times \mathbf{H}^- - \gamma_t \overrightarrow{\varphi}\Big).$$

Finally, we have determined the values of constants α_{ext} and β_{ext}

$$\left\{ \begin{array}{lcl} \alpha_{ext} & = & -\dfrac{1}{1+Y}, \\ \beta_{ext} & = & -\dfrac{1}{1+Y}. \end{array} \right. \tag{67}$$

With remarks 8 and 9 we have proved the following result.

Theorem 7. The Riemann boundary fluxes are given by

$$\left\{ \begin{array}{l} \gamma_t \widehat{\mathbf{E}} = \dfrac{1+R}{2}\Big(\gamma_t \mathbf{E}^- - \gamma_\times \mathbf{H}^-\Big) + \dfrac{1+R}{2}\gamma_t \overrightarrow{\varphi}, \\ \gamma_t \widehat{\mathbf{H}} = \dfrac{1-R}{2}\Big(\gamma_t \mathbf{H}^- + \gamma_\times \mathbf{E}^-\Big) - \dfrac{1+R}{2}\gamma_\times \overrightarrow{\varphi}, \end{array} \right. \tag{68}$$

or equivalently

$$\left\{ \begin{array}{l} \gamma_t \widehat{\mathbf{E}} = \gamma_t \mathbf{E}^- - \dfrac{1}{1+Y}\Big(Y\gamma_t \mathbf{E}^- + \gamma_\times \mathbf{H}^- - \gamma_t \overrightarrow{\varphi}\Big), \\ \gamma_\times \widehat{\mathbf{H}} = \gamma_\times \mathbf{H}^- - \dfrac{1}{1+Y}\Big(Y\gamma_t \mathbf{E}^- + \gamma_\times \mathbf{H}^- - \gamma_t \overrightarrow{\varphi}\Big), \end{array} \right. \tag{69}$$

where the admitance Y is related to the reflexion coefficient R by

$$R = \frac{1-Y}{1+Y} \text{ or equivalently } Y = \frac{1-R}{1+R}\ . \tag{70}$$

4 Final variational formulation

The numerical fluxes have been computed thanks to a Riemann solver and theorems 6 and 7 state their values in (47) and (63). As a conclusion, we have the following numerical fluxes, for $F \in \mathcal{F}$,

$$\left\{\begin{array}{l} \gamma_t \widehat{\mathbf{E}}_F = \gamma_t \mathbf{E}_K + \dfrac{\mathbf{n}_K \times [\![\gamma_\times \mathbf{E}]\!]_F}{2} - \dfrac{[\![\gamma_\times \mathbf{H}]\!]_F}{2}, \\ \gamma_t \widehat{\mathbf{H}}_F = \gamma_t \mathbf{H}_K + \dfrac{\mathbf{n}_K \times [\![\gamma_\times \mathbf{H}]\!]_F}{2} + \dfrac{[\![\gamma_\times \mathbf{E}]\!]_F}{2}, \text{ for } F \in \mathcal{F}_{int}, \end{array}\right. \tag{71}$$

and

$$\left\{\begin{array}{l} \gamma_t \widehat{\mathbf{E}}_F = \gamma_t \mathbf{E}_K - \dfrac{1-R}{2}\gamma_t \mathbf{E}_K - \dfrac{1+R}{2}\gamma_\times \mathbf{H}_K + \dfrac{1+R}{2}\gamma_t \overrightarrow{\varphi}, \\ \gamma_\times \widehat{\mathbf{H}}_F = \gamma_\times \mathbf{H}_K - \dfrac{1-R}{2}\gamma_t \mathbf{E}_K - \dfrac{1+R}{2}\gamma_\times \mathbf{H}_K + \dfrac{1+R}{2}\gamma_t \overrightarrow{\varphi}, \quad \text{for } F \in \mathcal{F}_{ext}, \end{array}\right. \tag{72}$$

where $\mathbf{n}_K$ is the outward unit normal to the face F of the element K. These numerical fluxes are then inserted in (32) to obtain

$$\left\{\begin{array}{l} \displaystyle\sum_{K\in\mathcal{K}} \int_K \mathrm{rot}(\mathbf{E}_K)\cdot\overline{\mathbf{H}'_K} - ik\mathbf{H}_K\cdot\overline{\mathbf{H}'_K} = \\ \qquad \displaystyle\sum_{K\in\mathcal{K}}\sum_{F\in\mathcal{F}^K_{int}} \int_F \Big(\frac{\mathbf{n}_K\times[\![\gamma_\times\mathbf{E}]\!]_F}{2} - \frac{[\![\gamma_\times\mathbf{H}]\!]_F}{2}\Big)\cdot\overline{\gamma_\times\mathbf{H}'_K} \\ \qquad \displaystyle -\sum_{K\in\mathcal{K}}\sum_{F\in\mathcal{F}^K_{ext}} \int_F \Big(\frac{1-R}{2}\gamma_t\mathbf{E}_K + \frac{1+R}{2}\gamma_\times\mathbf{H}_K - \frac{1+R}{2}\gamma_t\overrightarrow{\varphi}\Big)\cdot\overline{\gamma_\times\mathbf{H}'_K}, \\ \displaystyle\sum_{K\in\mathcal{K}} \int_K -\mathrm{rot}(\mathbf{H}_K)\cdot\overline{\mathbf{E}'_K} - ik\mathbf{E}_K\cdot\overline{\mathbf{E}'_K} = \\ \qquad \displaystyle -\sum_{K\in\mathcal{K}}\sum_{F\in\mathcal{F}^K_{int}} \int_F \Big(\frac{\mathbf{n}_K\times[\![\gamma_\times\mathbf{H}]\!]_F}{2} + \frac{[\![\gamma_\times\mathbf{E}]\!]_F}{2}\Big)\cdot\overline{\gamma_\times\mathbf{E}'_K} \\ \qquad \displaystyle -\sum_{K\in\mathcal{K}}\sum_{F\in\mathcal{F}^K_{ext}} \int_F \Big(\frac{1-R}{2}\gamma_t\mathbf{E}_K + \frac{1+R}{2}\gamma_\times\mathbf{H}_K - \frac{1+R}{2}\gamma_t\overrightarrow{\varphi}\Big)\cdot\overline{\gamma_t\mathbf{E}'_K}. \end{array}\right. \tag{73}$$

Summing the latter equations we get the final variational formulation, which is

$$\text{Find } \mathbb{E} \in \mathbb{X} \text{ such that } a(\mathbb{E}, \mathbb{E}') = \ell(\mathbb{E}') \quad \text{for all } \mathbb{E}' \in \mathbb{X}, \tag{74}$$

with

$$\left\{\begin{array}{rcl} a(\mathbb{E},\mathbb{E}') & = & \displaystyle\sum_{K\in\mathcal{K}} a_K(\mathbb{E},\mathbb{E}') \;+\; b^{int}(\mathbb{E},\mathbb{E}') \;+\; b^{ext}(\mathbb{E},\mathbb{E}') \\ \ell(\mathbb{E}') & = & \displaystyle\frac{1+R}{2}\int_{\partial\Omega}\Big(\gamma_t\overrightarrow{\varphi}\cdot\overline{\gamma_t\mathbf{E}'}+\gamma_t\overrightarrow{\varphi}\cdot\overline{\gamma_\times\mathbf{H}'}\Big). \end{array}\right. \tag{75}$$

The bilinear form a_K can easily be identified

$$\left\{\begin{array}{rcl} a_K(\mathbb{E},\mathbb{E}') & = & \displaystyle\int_K \mathrm{rot}(\mathbf{E}_K)\cdot\overline{\mathbf{H}'_K}-\mathrm{rot}(\mathbf{H}_K)\cdot\overline{\mathbf{E}'_K} \\ & - & \displaystyle ik\int_K \mathbf{H}_K\cdot\overline{\mathbf{H}'_K}+\mathbf{E}_K\cdot\overline{\mathbf{E}'_K}. \end{array}\right. \tag{76}$$

The second one takes the form

$$\left\{\begin{array}{rcl} b_F^{ext}(\mathbb{E},\mathbb{E}') & = & \displaystyle\sum_{K\in\mathcal{K}}\sum_{F\in\mathcal{F}_{ext}^K}\int_F\Big(\frac{1-R}{2}\gamma_t\mathbf{E}_K+\frac{1+R}{2}\gamma_\times\mathbf{H}_K\Big)\cdot\overline{\gamma_\times\mathbf{H}'_K} \\ & + & \displaystyle\sum_{K\in\mathcal{K}}\sum_{F\in\mathcal{F}_{ext}^K}\int_F\Big(\frac{1-R}{2}\gamma_t\mathbf{E}_K+\frac{1+R}{2}\gamma_\times\mathbf{H}_K\Big)\cdot\overline{\gamma_t\mathbf{E}'_K}, \end{array}\right. \tag{77}$$

and can be simplified over all faces of the boundary as

$$\left\{\begin{array}{rcl} b_F^{ext}(\mathbb{E},\mathbb{E}') & = & \displaystyle\sum_{F\in\mathcal{F}_{ext}}\int_F\frac{1-R}{2}\gamma_t\mathbf{E}_K\cdot\overline{\gamma_t\mathbf{E}'_K}+\frac{1+R}{2}\gamma_\times\mathbf{H}_K\cdot\overline{\gamma_\times\mathbf{H}'_K} \\ & + & \displaystyle\sum_{F\in\mathcal{F}_{ext}}\int_F\frac{1-R}{2}\gamma_t\mathbf{E}_K\cdot\overline{\gamma_\times\mathbf{H}'_K}+\frac{1+R}{2}\gamma_\times\mathbf{H}_K\cdot\overline{\gamma_t\mathbf{E}'_K}. \end{array}\right. \tag{78}$$

The form on interior faces is

$$\left\{\begin{array}{rcl} b_F^{int}(\mathbb{E},\mathbb{E}') & = & \displaystyle\sum_{K\in\mathcal{K}}\sum_{F\in\mathcal{F}_{int}^K}\int_F\frac{[\![\gamma_\times\mathbf{H}]\!]_F}{2}\cdot\overline{\gamma_\times\mathbf{H}'_K}+\frac{[\![\gamma_\times\mathbf{E}]\!]_F}{2}\cdot\overline{\gamma_\times\mathbf{E}'_K} \\ & + & \displaystyle\sum_{K\in\mathcal{K}}\sum_{F\in\mathcal{F}_{int}^K}\int_F\frac{\mathbf{n}_K\times[\![\gamma_\times\mathbf{H}]\!]_F}{2}\cdot\overline{\gamma_\times\mathbf{E}'_K}-\frac{\mathbf{n}_K\times[\![\gamma_\times\mathbf{E}]\!]_F}{2}\cdot\overline{\gamma_\times\mathbf{H}'_K} \end{array}\right. \tag{79}$$

and is simplified as

$$\left\{\begin{array}{rcl} b_F^{int}(\mathbb{E},\mathbb{E}') & = & \displaystyle\sum_{K\in\mathcal{K}}\sum_{F\in\mathcal{F}_{int}^K}\int_F\frac{[\![\gamma_\times\mathbf{H}]\!]_F}{2}\cdot\overline{\gamma_\times\mathbf{H}'_K}+\frac{[\![\gamma_\times\mathbf{E}]\!]_F}{2}\cdot\overline{\gamma_\times\mathbf{E}'_K} \\ & + & \displaystyle\sum_{K\in\mathcal{K}}\sum_{F\in\mathcal{F}_{int}^K}\int_F\frac{[\![\gamma_\times\mathbf{H}]\!]_F\cdot\overline{\gamma_t\mathbf{E}'_K}}{2}-\frac{[\![\gamma_\times\mathbf{E}]\!]_F\cdot\overline{\gamma_t\mathbf{H}'_K}}{2} \end{array}\right. \tag{80}$$

The interior flux goes twice through an interior face interfacing two elements. That is why we can duplicate this face, and consider it either into K_F^- or into K_F^+. It follows that

$$\left\{\begin{array}{rcl} b_F^{int}(\mathbb{E},\mathbb{E}') & = & \displaystyle\sum_{F\in\mathcal{F}_{int}}\int_F \frac{[\![\gamma_\times \mathbf{H}]\!]_F}{2}\cdot\left(\overline{\gamma_\times^- \mathbf{H}'_{K_F^-}}+\overline{\gamma_\times^+ \mathbf{H}'_{K_F^+}}\right) \\ & + & \displaystyle\sum_{F\in\mathcal{F}_{int}}\int_F \frac{[\![\gamma_\times \mathbf{E}]\!]_F}{2}\cdot\left(\overline{\gamma_\times^- \mathbf{E}'_{K_F^-}}+\overline{\gamma_\times^+ \mathbf{E}'_{K_F^+}}\right) \\ & + & \displaystyle\sum_{F\in\mathcal{F}_{int}}\int_F [\![\gamma_\times \mathbf{H}]\!]_F\cdot\frac{\overline{\gamma_t \mathbf{E}'_{K_F^-}}+\overline{\gamma_t \mathbf{E}'_{K_F^+}}}{2} \\ & - & \displaystyle\sum_{F\in\mathcal{F}_{int}}\int_F [\![\gamma_\times \mathbf{E}]\!]_F\cdot\frac{\overline{\gamma_t \mathbf{H}'_{K_F^-}}+\overline{\gamma_t \mathbf{H}'_{K_F^+}}}{2}, \end{array}\right. \tag{81}$$

which leads to

$$\left\{\begin{array}{rcl} b_F^{int}(\mathbb{E},\mathbb{E}') & = & \displaystyle\sum_{F\in\mathcal{F}_{int}}\int_F \frac{[\![\gamma_\times \mathbf{H}]\!]_F\cdot\overline{[\![\gamma_\times \mathbf{H}']\!]_F}}{2}+\frac{[\![\gamma_\times \mathbf{E}]\!]_F\cdot\overline{[\![\gamma_\times \mathbf{E}']\!]_F}}{2} \\ & + & \displaystyle\sum_{F\in\mathcal{F}_{int}}\int_F [\![\gamma_\times \mathbf{H}]\!]_F\cdot\overline{\{\gamma_t \mathbf{E}'_{K_F^-}\}_F}-[\![\gamma_\times \mathbf{E}]\!]_F\cdot\overline{\{\gamma_t \mathbf{H}'_{K_F^-}\}_F}. \end{array}\right. \tag{82}$$

Theorem 8. The variational formulation of the Discontinuous Galerkin method based on Riemann fluxes is

$$\text{Find } \mathbb{E}\in\mathbb{X} \text{ such that } \mathsf{a}(\mathbb{E},\mathbb{E}')=\ell(\mathbb{E}') \quad \text{for all } \mathbb{E}'\in\mathbb{X} \tag{83}$$

with

$$\left\{\begin{array}{rcl} \mathsf{a}(\mathbb{E},\mathbb{E}') & = & \displaystyle\sum_{K\in\mathcal{K}} a_K(\mathbb{E},\mathbb{E}') + \sum_{F\in\mathcal{F}_{int}} b_F^{int}(\mathbb{E},\mathbb{E}') + \sum_{F\in\mathcal{F}_{ext}} b_F^{ext}(\mathbb{E},\mathbb{E}') \\ \ell(\mathbb{E}') & = & \displaystyle\frac{1+R}{2}\int_{\partial\Omega}\left(\gamma_t\overrightarrow{\varphi}\cdot\overline{\gamma_t\mathbf{E}'}+\gamma_t\overrightarrow{\varphi}\cdot\overline{\gamma_\times\mathbf{H}'}\right), \end{array}\right. \tag{84}$$

and

$$\left\{\begin{array}{rcl} a_K(\mathbb{E},\mathbb{E}') & = & \displaystyle\int_K \mathrm{rot}(\mathbf{E}_K)\cdot\overline{\mathbf{H}'_K}-\mathrm{rot}(\mathbf{H}_K)\cdot\overline{\mathbf{E}'_K} \\ & - & \displaystyle ik\int_K \mathbf{H}_K\cdot\overline{\mathbf{H}'_K}+\mathbf{E}_K\cdot\overline{\mathbf{E}'_K}\,, \end{array}\right. \tag{85}$$

$$\left\{\begin{array}{rcl} b_F^{int}(\mathbb{E},\mathbb{E}') & = & \displaystyle\int_F [\![\gamma_\times \mathbf{H}]\!]_F\cdot\overline{\{\gamma_t \mathbf{E}'_{K_F^-}\}}-[\![\gamma_\times \mathbf{E}]\!]_F\cdot\overline{\{\gamma_t \mathbf{H}'_{K_F^-}\}} \\ & + & \displaystyle\int_F \frac{[\![\gamma_\times \mathbf{H}]\!]_F\cdot\overline{[\![\gamma_\times \mathbf{H}']\!]}+[\![\gamma_\times \mathbf{E}]\!]_F\cdot\overline{[\![\gamma_\times \mathbf{E}']\!]}}{2}\,, \end{array}\right. \tag{86}$$

$$\left\{ \begin{array}{rcl} b_F^{ext}(\mathbb{E},\mathbb{E}') & = & \displaystyle\int_F \frac{1-R}{2}\gamma_t \mathbf{E}_K \cdot \overline{\gamma_t \mathbf{E}'_K} + \frac{1+R}{2}\gamma_\times \mathbf{H}_K \cdot \overline{\gamma_\times \mathbf{H}'_K} \\ & + & \displaystyle\int_F \frac{1-R}{2}\gamma_t \mathbf{E}_K \cdot \overline{\gamma_\times \mathbf{H}'_K} + \frac{1+R}{2}\gamma_\times \mathbf{H}_K \cdot \overline{\gamma_t \mathbf{E}'_K}\,. \end{array} \right. \tag{87}$$

5 Discretization of the variational formulation

The variational formulation can be discretized and then written as

$$\text{Find } \mathbb{E}_h \in \mathbb{X}_h \text{ such that } a(\mathbb{E}_h, \mathbb{E}'_h) = \ell(\mathbb{E}'_h) \quad \text{for all } \mathbb{E}'_h \in \mathbb{X}_h, \tag{88}$$

where $\mathbb{X}_h$ is the discretized space of basis functions, that will be defined later. We will see that this discrete problem can then be associated to a matrix problem. The latter will allow us to solve the main problem and find the discrete solution $\mathbb{E}_h$ approximating $\mathbb{E}$.
In order to define the discrete problem, we first introduce a finite element basis.

5.1 Definition of the finite element basis

5.1.1 The finite element space

Let us take an element K of the mesh defined in part 2.1. Each element K is a tetrahedron. The set of degrees of freedom (dof) of this element has dimension $\#dof$. We associate to each degree of freedom $\mathbf{X}$ a polynomial function p. Let $P_N(K)$ be the set of polynomials of total degree less than N defined on K. A polynomial function $p : K \longrightarrow \mathbb{C}$ is in $P_N(K)$ if an only if

$$p(\mathbf{X}) \;=\; \sum_{n_x=0}^{N} \sum_{n_y=0}^{N-n_x} \sum_{n_z=0}^{N-n_y-n_x} a_{n_x,n_y,n_z}\; X^{n_x}\; Y^{n_y}\; Z^{n_z}. \tag{89}$$

Since we are dealing with a discontinuous Galerkin method [16], there is no continuity relation between two touching elements and the scalar finite element space is simply the set of functions which are $P_N(K)$ in every element K

$$\mathbb{P}_N \;=\; \prod_{K\in\mathcal{K}} P_N(K). \tag{90}$$

The problem involves the unknown $\mathbb{E} = (\mathbf{E}, \mathbf{H})$ which has 6 components

$$E_i, \quad H_i, \qquad \text{for } 1 \le i \le 3. \tag{91}$$

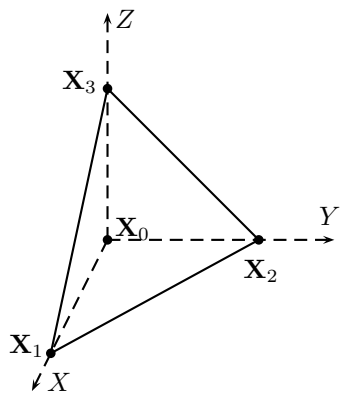

Figure 7: The reference tetrahedron.

We introduce the local (on an element K of the mesh) or global vectorial finite element spaces, which are respectively

$$X_N(K) = (P_N(K))^6 \quad \text{and} \quad \mathbb{X}_N = (\mathbb{P}_N)^6. \tag{92}$$

5.1.2 Geometric transformations

Let $\widehat{K}$ be the reference tetrahedron

$$\widehat{K} \;=\; \left\{ \mathbf{X} = (X, Y, Z) \in \mathbb{R}^3 \;\middle|\; X + Y + Z \leq 1,\; X \geq 0,\; Y \geq 0,\; Z \geq 0 \right\} \tag{93}$$

with vertices $\mathbf{X}_i = \mathbf{e}_i$ for $1 \leq i \leq 3$ and $\mathbf{X}_0 = (0, 0, 0)^T$. Any arbitrary tetrahedron K, with vertices $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, can be seen as the image of $\widehat{K}$ by an invertible affine mapping ψ_K defined by

$$\overrightarrow{\psi}_K(\mathbf{X}_i) = \mathbf{x}_i$$

with $\mathbf{x} = \overrightarrow{\psi}_K(\mathbf{X})$ and $K = \overrightarrow{\psi}_K(\widehat{K})$.
Any point in the tetrahedron can be written as

$$(x, y, z) = \mathbf{x} = \overrightarrow{\psi}_K(\mathbf{X}) = (1 - X - Y - Z)\mathbf{x}_0 + X\mathbf{x}_1 + Y\mathbf{x}_2 + Z\mathbf{x}_3. \tag{94}$$

Finally, we introduce the Jacobian matrix

$$J_{\overrightarrow{\psi}_K} = \begin{pmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{pmatrix} = \begin{pmatrix} x_1 - x_0 & x_2 - x_0 & x_3 - x_0 \\ y_1 - y_0 & y_2 - y_0 & y_3 - y_0 \\ z_1 - z_0 & z_2 - z_0 & z_3 - z_0 \end{pmatrix}.$$

5.1.3 The basis functions

First we define basis functions in the reference tetrahedron $\widehat{K}$, and after we will deal with an arbitrary tetrahedron K.
We use the classical Lagrange polynomial functions associated to the local Lagrange points $\mathbf{X}_{iloc_1,iloc_2,iloc_3}$ defined in the reference element

$$\mathbf{X}_{iloc_1,iloc_2,iloc_3} = \Big(\frac{iloc_1}{N}, \frac{iloc_2}{N}, \frac{iloc_3}{N}\Big). \tag{95}$$

We denote $iloc_0$ by

$$iloc_0 = N - iloc_1 - iloc_2 - iloc_3. \tag{96}$$

These reference interpolation points $\mathbf{X}_{iloc_1,iloc_2,iloc_3}$ are indexed by three integers $(iloc_k)_{k=1,3}$ which satisfy

$$0 \leq iloc_k \leq N, \qquad \text{for } 0 \leq k \leq 3. \tag{97}$$

In the reference tetrahedron, the Lagrange polynomial functions $W_{iloc_1,iloc_2,iloc_3}$ can be written as follows

$$W_{iloc_1,iloc_2,iloc_3}(\mathbf{X}_{iloc_1,iloc_2,iloc_3}) \;=\; Z_{iloc_0}(1-X-Y-Z)Z_{iloc_1}(X)Z_{iloc_2}(Y)Z_{iloc_3}(Z)$$

with 1D Lagrange polynomial functions

$$Z_i(X) = \prod_{k=0}^{i-1} \frac{X - \frac{k}{N}}{\frac{i}{N} - \frac{k}{N}}, \text{ for } i \geq 1, \text{ and } Z_0(X) = 1.$$

The Lagrange polynomial function $W_{iloc_1,iloc_2,iloc_3}$ is equal to 1 at interpolation point $X_{iloc_1,iloc_2,iloc_3}$ and is equal to 0 at other interpolation points.

On an arbitrary element K, the Lagrange basis is simply defined by

$$w_{K,iloc_1,iloc_2,iloc_3}(\mathbf{x}) \;=\; W_{iloc_1,iloc_2,iloc_3}(\mathbf{X}) \quad \text{with } \mathbf{x} = \overrightarrow{\psi}_K(\mathbf{X}),$$

N	1	2	3	4
$\#dof$	4	10	20	35
$\#vectorial\ dof$	24	60	120	210

Table 1: Number of degrees of freedom (dof) for the scalar and the vectorial finite element basis on one element.

where we recall that $\overrightarrow{\psi}_K$ is the invertible affine mapping that enables to go from the reference tetrahedron to any tetrahedron in the mesh. The triple index $_{iloc_1,iloc_2,iloc_3}$ is rather inconvenient to use. That is why we introduce a new numbering for the basis functions. We will use a unique integer number $1 \leq iloc \leq \#dof$ instead of three integer numbers, see table 1. Thus, the basis function $w_{K,iloc}$ in an arbitrary tetrahedron K is defined by

$$\begin{aligned} w_{K,iloc}(\mathbf{x}) &:= w_{K,iloc_1,iloc_2,iloc_3}(\mathbf{x}), \\ W_{iloc}(\mathbf{X}) &:= W_{iloc_1,iloc_2,iloc_3}(\mathbf{X}). \end{aligned} \tag{98}$$

To perform reordering we use the following algorithm

```
iloc=1
for iloc1 in [0,N]
    for iloc2 in [0,N-iloc1]
        for iloc3 in [0,N-iloc1-iloc2]
            iloc3_to_iloc(iloc1,iloc2,iloc3)=iloc
            iloc_to_iloc3(iloc)=(iloc1,iloc2,iloc3)
            iloc=iloc+1
        end
    end
end.
```

5.1.4 Solution approximation

To approximate the solution $\mathbb{E} = (\mathbf{E}, \mathbf{H})$ in K, we introduce the vector-basis $\overrightarrow{w}^j_{K,iloc}$ for $1 \leq j \leq 6$, where j is defined as the non-zero coordinate of $\mathbb{E}$. We recall that $\mathbb{E}$ takes the form (34). Thus, the vectors $\overrightarrow{w}^j_{K,iloc}$ are defined

as follows

$$\overrightarrow{w}^1_{K,iloc} = \begin{pmatrix} w_{K,iloc} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \overrightarrow{w}^2_{K,iloc} = \begin{pmatrix} 0 \\ w_{K,iloc} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \overrightarrow{w}^3_{K,iloc} = \begin{pmatrix} 0 \\ 0 \\ w_{K,iloc} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{99}$$

$$\overrightarrow{w}^4_{K,iloc} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ w_{K,iloc} \\ 0 \\ 0 \end{pmatrix}, \quad \overrightarrow{w}^5_{K,iloc} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ w_{K,iloc} \\ 0 \end{pmatrix}, \quad \overrightarrow{w}^6_{K,iloc} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ w_{K,iloc} \end{pmatrix}. \tag{100}$$

Remark 10. This is clearly not the only choice for the basis functions. However, this one is really easy to understand and can be quickly coded. A good idea would have been to use Nedelec finite element [17, 23] to benefit from their properties related to an easy evaluation of the tangential traces.

Consequently, the numerical approximation $\mathbb{E}_h$ of $\mathbb{E}$ on the element K is given by

$$\mathbb{E}_h|_K = \sum_{j=1}^{6} \sum_{iloc=1}^{\#dof} \mathbb{E}^j_{K,iloc} \overrightarrow{w}^j_{K,iloc} = \sum_{iloc=1}^{\#dof} \begin{pmatrix} E^1_{K,iloc} \\ E^2_{K,iloc} \\ E^3_{K,iloc} \\ H^1_{K,iloc} \\ H^2_{K,iloc} \\ H^3_{K,iloc} \end{pmatrix} w_{K,iloc} \tag{101}$$

with $E^i_{K,iloc}$ (respectively $H^j_{K,iloc}$) is the value of E^i_K (respectively H^j_K) on the point $\mathbf{x}_{K,iloc}$ for $1 \leq i, j \leq 3$ and $\#dof$ is number of degrees of freedom for

the element K

$$\begin{cases} \mathbb{E}^1_{K,iloc} &= E^1_{K,iloc}, \\ \mathbb{E}^2_{K,iloc} &= E^2_{K,iloc}, \\ \mathbb{E}^3_{K,iloc} &= E^3_{K,iloc}, \\ \mathbb{E}^4_{K,iloc} &= H^1_{K,iloc}, \\ \mathbb{E}^5_{K,iloc} &= H^2_{K,iloc}, \\ \mathbb{E}^6_{K,iloc} &= H^3_{K,iloc}. \end{cases} \tag{102}$$

We also introduce the following notation for the restrictions of the electric and the magnetic fields associated to $\mathbb{E}_h$

$$\mathbf{E}_h(\mathbf{x}) = \sum_{iloc=1}^{\#dof} \begin{pmatrix} E^1_{K,iloc} \\ E^2_{K,iloc} \\ E^3_{K,iloc} \end{pmatrix} w_{K,iloc}(\mathbf{x}) \quad \text{for } \mathbf{x} \in K, \tag{103}$$

$$\mathbf{H}_h(\mathbf{x}) = \sum_{iloc=1}^{\#dof} \begin{pmatrix} H^1_{K,iloc} \\ H^2_{K,iloc} \\ H^3_{K,iloc} \end{pmatrix} w_{K,iloc}(\mathbf{x}) \quad \text{for } \mathbf{x} \in K. \tag{104}$$

As mentioned before, Ω domain is split into a set of elements $\mathcal{K}$. Basis functions of an element K are defined on each of its vertices. They are then independent to each other. Thus, the basis functions defined on the whole degrees of freedom of the mesh are also independent to each other. Consequently, local functions $w^j_{K,iloc}$ can be considered with respect to the whole domain and they are called global basis functions $\overrightarrow{w}_{iglob}$. We define them as

$$\begin{aligned} &\overrightarrow{w}_{iglob}(\mathbf{x}) = \overrightarrow{w}^j_{K,iloc}(\mathbf{x}) && \text{if } \mathbf{x} \in K, \\ &\overrightarrow{w}_{iglob}(\mathbf{x}) = 0 && \text{if } \mathbf{x} \in K', \text{ and } K' \neq K, \end{aligned} \tag{105}$$

where $iglob = 1, \#global\ dof$ is a global index associated on the element K, $iloc = 1, \#dof$, and $j = 1, 6$.
This global numbering over full domain is constructed thanks to the following algorithm

```
iglob=1
for K=1 to #K
    for j=1 to 6
        for iloc= 1 to #dof
           Kjiloc_to_iglob(K,j,iloc)=iglob
```

```
                iglob_to_Kjiloc(iglob)=(K,j,iloc)
                iglob=iglob+1
            end
        end
    end .
```

Therefore, the associated degrees of freedom are defined by

$$\mathbb{E}_{iglob} \;=\; \mathbb{E}^{j}_{K,iloc}. \tag{106}$$

Remark 11. We can define the global approximation of $\mathbb{E}$ as

$$\mathbb{E}_h(\mathbf{x}) \;=\; \sum_{iglob=1}^{\#global\ dof} \mathbb{E}_{iglob}\,\overrightarrow{w}_{iglob}(\mathbf{x}). \tag{107}$$

The global number of degrees of freedom is given by

$$\#global\ dof = 6\ \#K\ \#dof, \tag{108}$$

where $\#K$ is the number of elements of the mesh.

We thus define the discretized space $\mathbb{X}_h$ of basis functions as

$$\mathbb{X}_h := \Big\{\mathbb{E}_h : \Omega \longrightarrow \mathbb{C}^6 \text{ such that } \mathbb{E}_h(\mathbf{x}) \;=\; \sum_{iglob=1}^{\#global\ dof} \mathbb{E}_{iglob}\,\overrightarrow{w}_{iglob}(\mathbf{x})\Big\}. \tag{109}$$

5.2 Definition of the involved matrices

The discrete variational formulation (88) enables us to solve the problem. Indeed, the discrete problem can be written as a matrix problem

$$\mathbb{A}\,\underline{\mathbb{E}} \;=\; \underline{\mathbb{F}}, \tag{110}$$

where $\mathbb{A}$ is the matrix associated to the sesquilinear form a, $\mathbb{F}$ is the vector associated to the linear form ℓ, and $\underline{\mathbb{E}}$ is the vector associated to the discrete approximation $\mathbb{E}_h$ of the main problem solution $\mathbb{E}$.
This matrix and theses vectors are defined in the present section.

5.2.1 The global matrix system

We represent a solution $\mathbb{E}_h$ of the discretized problem (88) according to (107).

The solution can be written as a vector $\underline{\mathbb{E}}$ and takes the form

$$\underline{\mathbb{E}} = \begin{pmatrix} \mathbb{E}_1 \\ \mathbb{E}_2 \\ \vdots \\ \mathbb{E}_{\#global\ dof} \end{pmatrix} = \begin{pmatrix} \mathbb{E}_{K_1} \\ \mathbb{E}_{K_2} \\ \vdots \\ \mathbb{E}_{K_{\#K}} \end{pmatrix} \tag{111}$$

with

$$\mathbb{E}_K = \begin{pmatrix} \mathbf{E}_K^1 \\ \mathbf{E}_K^2 \\ \mathbf{E}_K^3 \\ \mathbf{H}_K^1 \\ \mathbf{H}_K^2 \\ \mathbf{H}_K^3 \end{pmatrix}, \quad \mathbf{E}_K^j = \begin{pmatrix} E_{K,1}^j \\ E_{K,2}^j \\ \vdots \\ E_{K,\#dof}^j \end{pmatrix}, \quad \mathbf{H}_K^j = \begin{pmatrix} H_{K,1}^j \\ H_{K,2}^j \\ \vdots \\ H_{K,\#dof}^j \end{pmatrix}. \tag{112}$$

The exact solution is related to the vector $\underline{\mathbb{E}}$ by

$$\mathbb{E}_h = \sum_{jglob=1}^{\#globaldof} \mathbb{E}_{jglob}\ \overrightarrow{w}_{jglob}. \tag{113}$$

Taking as test function $\mathbb{E}' = \overrightarrow{w}_{jglob}$, we have for $1 \leq jglob \leq \#global\ dof$

$$\sum_{iglob,jglob=1}^{\#globaldof} \mathbb{E}_{jglob} a(\overrightarrow{w}_{jglob}, \overrightarrow{w}_{iglob}) = \ell(\overrightarrow{w}_{jglob}). \tag{114}$$

This system can be written as

$$\sum_{iglob,jglob=1}^{\#globaldof} \mathbb{A}_{iglob,jglob}\mathbb{E}_{jglob} = \mathbb{F}_{jglob} \tag{115}$$

with the matrix $\mathbb{A}$ and the vector $\mathbb{F}$ defined by

$$\mathbb{A}_{iglob,jglob} = a(\overrightarrow{w}_{jglob}, \overrightarrow{w}_{iglob}) \quad \text{and} \quad \mathbb{F}_{jglob} = \ell(\overrightarrow{w}_{jglob}) \tag{116}$$

with $iglob, jglob = 1, \#globaldof$.
We have finally defined matrices satisfying the matrix problem (110).

5.2.2 Decomposition of the matrix $\mathbb{A}$

Due to (84), we recall that the sesquilinear form a can be decomposed as following

$$\mathsf{a}(\mathbb{E},\mathbb{E}') \;=\; \sum_{K\in\mathcal{K}} a_K(\mathbb{E},\mathbb{E}') \;+\; \sum_{F\in\mathcal{F}_{int}} b_F^{int}(\mathbb{E},\mathbb{E}') \;+\; \sum_{F\in\mathcal{F}_{ext}} b_F^{ext}(\mathbb{E},\mathbb{E}'). \quad (117)$$

Therefore, we follow the same reasoning to decompose the matrix $\mathbb{A}$

$$\mathbb{A} \;=\; \sum_{K\in\mathcal{K}} \mathbb{A}_K \;+\; \sum_{F\in\mathcal{F}_{int}} \mathbb{B}_F^{int} \;+\; \sum_{F\in\mathcal{F}_{ext}} \mathbb{B}_F^{ext}, \quad (118)$$

where the different matrices are defined by

$$(\mathbb{A}_K\,)_{iglob,jglob} \;:=\; a_K(\overrightarrow{w}_{jglob},\overrightarrow{w}_{iglob}), \quad (119)$$

$$(\mathbb{B}_F^{int})_{iglob,jglob} \;:=\; b_F^{int}(\overrightarrow{w}_{jglob},\overrightarrow{w}_{iglob}), \quad (120)$$

$$(\mathbb{B}_F^{ext})_{iglob,jglob} \;:=\; b_F^{ext}(\overrightarrow{w}_{jglob},\overrightarrow{w}_{iglob}). \quad (121)$$

5.3 Implementation of the matrix $\mathbb{A}_K$

The matrix $\mathbb{A}_K$ is associated to the form a_K defined by (85), recalled here

$$\left\{\begin{array}{rcl} a_K(\mathbb{E},\mathbb{E}') & = & \displaystyle\int_K \mathrm{rot}(\mathbf{E}_K)\cdot\overline{\mathbf{H}'_K} - \mathrm{rot}(\mathbf{H}_K)\cdot\overline{\mathbf{E}'_K} \\ & - & \displaystyle ik\int_K \mathbf{H}_K\cdot\overline{\mathbf{H}'_K} + \mathbf{E}_K\cdot\overline{\mathbf{E}'_K}. \end{array}\right.$$

This sesquilinear form has to be evaluated for every pair of vectors $(\overrightarrow{w}_{iglob},\overrightarrow{w}_{jglob})$ in the finite element basis

$$(\mathbb{A}_K)_{iglob,jglob} \;=\; a_K(\overrightarrow{w}_{jglob},\overrightarrow{w}_{iglob}) \;=\; a_K(\overrightarrow{w}^{\ell_j}_{K_j,jloc},\overrightarrow{w}^{\ell_i}_{K_i,iloc}), \quad (122)$$

where $\overrightarrow{w}_{iglob} = \overrightarrow{w}^{\ell_i}_{K_i,iloc}$ and $\overrightarrow{w}_{jglob} = \overrightarrow{w}^{\ell_j}_{K_j,jloc}$, with K_i, K_j the elements, $iloc, jloc = 1, \#dof$, and $l_i, l_j = 1, 6$ the directions of the basis function, all respectively associated to $iglob, jglob$.

We first remark that $a_K(\overrightarrow{w}_{jglob},\overrightarrow{w}_{iglob}) = 0$ except if $K_i = K_j = K$, see Figure 8.
Let us recall (103) and (104). They lead to the following formulas, where we associate a basis function to an electric field and a magnetic field

$$\left\{\begin{array}{lcl}
\mathbf{E}_K[\overrightarrow{w}^1_{K,iloc}] & = & \begin{pmatrix} w_{K,iloc} \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{H}_K[\overrightarrow{w}^1_{K,iloc}] = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\mathbf{E}_K[\overrightarrow{w}^2_{K,iloc}] & = & \begin{pmatrix} 0 \\ w_{K,iloc} \\ 0 \end{pmatrix} \qquad \mathbf{H}_K[\overrightarrow{w}^2_{K,iloc}] = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\mathbf{E}_K[\overrightarrow{w}^3_{K,iloc}] & = & \begin{pmatrix} 0 \\ 0 \\ w_{K,iloc} \end{pmatrix} \qquad \mathbf{H}_K[\overrightarrow{w}^3_{K,iloc}] = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\mathbf{E}_K[\overrightarrow{w}^4_{K,iloc}] & = & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{H}_K[\overrightarrow{w}^4_{K,iloc}] = \begin{pmatrix} w_{K,iloc} \\ 0 \\ 0 \end{pmatrix} \\
\mathbf{E}_K[\overrightarrow{w}^5_{K,iloc}] & = & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{H}_K[\overrightarrow{w}^5_{K,iloc}] = \begin{pmatrix} 0 \\ w_{K,iloc} \\ 0 \end{pmatrix} \\
\mathbf{E}_K[\overrightarrow{w}^6_{K,iloc}] & = & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{H}_K[\overrightarrow{w}^6_{K,iloc}] = \begin{pmatrix} 0 \\ 0 \\ w_{K,iloc} \end{pmatrix}.
\end{array}\right. \tag{123}$$

Let us remark that we have $\overrightarrow{w}^{\ell}_{K,iloc} = w_{K,iloc}\overrightarrow{e}_{\ell}$, where $\overrightarrow{e}_{\ell}$ is the classical

basis vector. In the same way, we have

$$\left\{\begin{array}{lcl}
\mathrm{rot}(\mathbf{E}_K[\overrightarrow{w}^1_{K,iloc}]) & = & \begin{pmatrix} 0 \\ \partial_3 w_{K,iloc} \\ -\partial_2 w_{K,iloc} \end{pmatrix} \qquad \mathrm{rot}(\mathbf{H}_K[\overrightarrow{w}^1_{K,iloc}]) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\mathrm{rot}(\mathbf{E}_K[\overrightarrow{w}^2_{K,iloc}]) & = & \begin{pmatrix} -\partial_3 w_{K,iloc} \\ 0 \\ \partial_1 w_{K,iloc} \end{pmatrix} \qquad \mathrm{rot}(\mathbf{H}_K[\overrightarrow{w}^2_{K,iloc}]) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\mathrm{rot}(\mathbf{E}_K[\overrightarrow{w}^3_{K,iloc}]) & = & \begin{pmatrix} \partial_2 w_{K,iloc} \\ -\partial_1 w_{K,iloc} \\ 0 \end{pmatrix} \qquad \mathrm{rot}(\mathbf{H}_K[\overrightarrow{w}^3_{K,iloc}]) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\mathrm{rot}(\mathbf{E}_K[\overrightarrow{w}^4_{K,iloc}]) & = & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \mathrm{rot}(\mathbf{H}_K[\overrightarrow{w}^4_{K,iloc}]) = \begin{pmatrix} 0 \\ \partial_3 w_{K,iloc} \\ -\partial_2 w_{K,iloc} \end{pmatrix} \\
\mathrm{rot}(\mathbf{E}_K[\overrightarrow{w}^5_{K,iloc}]) & = & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \mathrm{rot}(\mathbf{H}_K[\overrightarrow{w}^5_{K,iloc}]) = \begin{pmatrix} -\partial_3 w_{K,iloc} \\ 0 \\ \partial_1 w_{K,iloc} \end{pmatrix} \\
\mathrm{rot}(\mathbf{E}_K[\overrightarrow{w}^6_{K,iloc}]) & = & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \mathrm{rot}(\mathbf{H}_K[\overrightarrow{w}^6_{K,iloc}]) = \begin{pmatrix} \partial_2 w_{K,iloc} \\ -\partial_1 w_{K,iloc} \\ 0 \end{pmatrix}.
\end{array}\right. \tag{124}$$

where $\partial_{x,y,z}$ are the partial derivatives associated to $\mathbf{E}_K$ and $\mathbf{H}_K$ components. We introduce the local mass matrix associated to an element K

$$\mathbb{M}_{K,iloc,jloc} = \int_K w_{K,iloc}\, w_{K,jloc}. \tag{125}$$

Also, we define the first order derivative matrices for $1 \leq iloc, jloc \leq \#dof$

$$\left\{\begin{array}{lcl}
\mathbb{R}^1_{K,iloc,jloc} & = & \displaystyle\int_K w_{K,iloc}\, \frac{\partial\, w_{K,jloc}}{\partial x}, \\
\mathbb{R}^2_{K,iloc,jloc} & = & \displaystyle\int_K w_{K,iloc}\, \frac{\partial\, w_{K,jloc}}{\partial y}, \\
\mathbb{R}^3_{K,iloc,jloc} & = & \displaystyle\int_K w_{K,iloc}\, \frac{\partial\, w_{K,jloc}}{\partial z}.
\end{array}\right. \tag{126}$$

The matrix $\mathbb{A}_K$ is given by

$$a_K(\overrightarrow{w}^{\ell_j}_{K_i,jloc}, \overrightarrow{w}^{\ell_i}_{K_j,iloc}) = 0 \tag{127}$$

except if $K_i = K_j = K$ where it takes the values

$$\begin{cases} a_K(\overrightarrow{w}^{\ell}_{K,jloc}, \overrightarrow{w}^{\ell}_{K,iloc}) = -ik\mathbb{M}_{K,iloc,jloc} \text{ for } 1 \leq \ell \leq 6, \\ a_K(\overrightarrow{w}^{1}_{K,jloc}, \overrightarrow{w}^{5}_{K,iloc}) = \quad \mathbb{R}^3_{K,iloc,jloc}, \\ a_K(\overrightarrow{w}^{1}_{K,jloc}, \overrightarrow{w}^{6}_{K,iloc}) = -\mathbb{R}^2_{K,iloc,jloc}, \\ a_K(\overrightarrow{w}^{2}_{K,jloc}, \overrightarrow{w}^{4}_{K,iloc}) = -\mathbb{R}^3_{K,iloc,jloc}, \\ a_K(\overrightarrow{w}^{2}_{K,jloc}, \overrightarrow{w}^{6}_{K,iloc}) = \quad \mathbb{R}^1_{K,iloc,jloc}, \\ a_K(\overrightarrow{w}^{3}_{K,jloc}, \overrightarrow{w}^{4}_{K,iloc}) = \quad \mathbb{R}^2_{K,iloc,jloc}, \\ a_K(\overrightarrow{w}^{3}_{K,jloc}, \overrightarrow{w}^{5}_{K,iloc}) = -\mathbb{R}^1_{K,iloc,jloc}, \\ a_K(\overrightarrow{w}^{4}_{K,jloc}, \overrightarrow{w}^{2}_{K,iloc}) = -\mathbb{R}^3_{K,iloc,jloc}, \\ a_K(\overrightarrow{w}^{4}_{K,jloc}, \overrightarrow{w}^{3}_{K,iloc}) = \quad \mathbb{R}^2_{K,iloc,jloc}, \\ a_K(\overrightarrow{w}^{5}_{K,jloc}, \overrightarrow{w}^{1}_{K,iloc}) = \quad \mathbb{R}^3_{K,iloc,jloc}, \\ a_K(\overrightarrow{w}^{5}_{K,jloc}, \overrightarrow{w}^{3}_{K,iloc}) = -\mathbb{R}^1_{K,iloc,jloc}, \\ a_K(\overrightarrow{w}^{6}_{K,jloc}, \overrightarrow{w}^{1}_{K,iloc}) = -\mathbb{R}^2_{K,iloc,jloc}, \\ a_K(\overrightarrow{w}^{6}_{K,jloc}, \overrightarrow{w}^{2}_{K,iloc}) = \quad \mathbb{R}^1_{K,iloc,jloc}. \end{cases} \tag{128}$$

For each element K, here are its non zero blocks of the matrix $\mathbb{A}_K$

$$\begin{bmatrix} -ik\mathbb{M}_{K,\cdot,\cdot} & 0 & 0 & 0 & \mathbb{R}^3_{K,\cdot,\cdot} & \mathbb{R}^2_{K,\cdot,\cdot} \\ 0 & -ik\mathbb{M}_{K,\cdot,\cdot} & 0 & -\mathbb{R}^3_{K,\cdot,\cdot} & 0 & \mathbb{R}^1_{K,\cdot,\cdot} \\ 0 & 0 & -ik\mathbb{M}_{K,\cdot,\cdot} & \mathbb{R}^2_{K,\cdot,\cdot} & -\mathbb{R}^1_{K,\cdot,\cdot} & 0 \\ 0 & -\mathbb{R}^3_{K,\cdot,\cdot} & \mathbb{R}^2_{K,\cdot,\cdot} & -ik\mathbb{M}_{K,\cdot,\cdot} & 0 & 0 \\ \mathbb{R}^3_{K,\cdot,\cdot} & 0 & -\mathbb{R}^1_{K,\cdot,\cdot} & 0 & -ik\mathbb{M}_{K,\cdot,\cdot} & 0 \\ -\mathbb{R}^2_{K,\cdot,\cdot} & \mathbb{R}^1_{K,\cdot,\cdot} & 0 & 0 & 0 & -ik\mathbb{M}_{K,\cdot,\cdot} \end{bmatrix}. \tag{129}$$

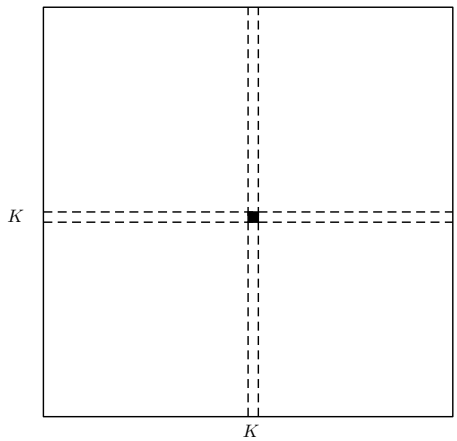

Figure 8: Localization of the non zero block of $\mathbb{A}_K$.

5.4 Implementation of the matrix $\mathbb{B}_F^{ext}$

The matrix $\mathbb{B}_F^{ext}$ is corresponding to the form b_F^{ext}. The latter is recalled here

$$\left\{ \begin{array}{rcl} b_F^{ext}(\mathbb{E},\mathbb{E}') & = & \displaystyle\int_F \frac{1-R}{2}\gamma_t\mathbf{E}_K\cdot\overline{\gamma_t\mathbf{E}'_K} + \frac{1+R}{2}\gamma_\times\mathbf{H}_K\cdot\overline{\gamma_\times\mathbf{H}'_K} \\ & + & \displaystyle\int_F \frac{1-R}{2}\gamma_t\mathbf{E}_K\cdot\overline{\gamma_\times\mathbf{H}'_K} + \frac{1+R}{2}\gamma_\times\mathbf{H}_K\cdot\overline{\gamma_t\mathbf{E}'_K}. \end{array} \right. \tag{130}$$

The matrix is defined by the formula

$$(\mathbb{B}_F^{ext})_{iglob,jglob} \;=\; b_F^{ext}(\overrightarrow{w}_{jglob},\overrightarrow{w}_{iglob}) \;=\; b_F^{ext}(\overrightarrow{w}^{\ell_j}_{K_j,jloc},\overrightarrow{w}^{\ell_i}_{K_i,iloc}), \tag{131}$$

where K_i, K_j the elements, $iloc, jloc = 1, \#dof$, and $l_i, l_j = 1, 6$ the directions of the basis function, all respectively associated to $iglob, jglob$.
We remark that $b_F^{ext}(\overrightarrow{w}_{jglob},\overrightarrow{w}_{iglob}) = 0$ except if $F \in \mathcal{F}_{ext}$ and $K_i = K_j$.
When $b_F^{ext}(\overrightarrow{w}_{jglob},\overrightarrow{w}_{iglob}) \neq 0$, *ie* $F \in \mathcal{F}_{ext}$ and $K_i = K_j = K$, we use tangential traces to compute its value. Let us introduce these traces

$$\left\{ \begin{array}{l} \gamma_t\mathbf{E} = \mathbf{E} - (\mathbf{E}\cdot\mathbf{n}_K)\mathbf{n}_K, \\ \gamma_\times^K\mathbf{E} = \mathbf{n}_K\times\gamma_t\mathbf{E}, \end{array} \right.$$

where $\mathbf{n}_K = (n_1, n_2, n_3)^T$ is the outward normal to element K on face F. To simplify notations in (131) and further in the document, we have taken into account that

$$\gamma_\times \mathbf{E}_K := \gamma_\times^K \mathbf{E} = \mathbf{n}_K \times \gamma_t \mathbf{E},$$

for all vector field $\mathbf{E}$ and $K \in \mathcal{K}$.
Using (123) and the above definitions, we define actions of the tangential trace operator

$$\left\{\begin{array}{llll}
\gamma_t \mathbf{E}_K[\overrightarrow{w}^1_{K,iloc}] & = w_{K,iloc}\begin{pmatrix}(1-n_1^2)\\ -n_1n_2\\ -n_1n_3\end{pmatrix} & \gamma_t \mathbf{H}_K[\overrightarrow{w}^1_{K,iloc}] & = w_{K,iloc}\begin{pmatrix}0\\0\\0\end{pmatrix}\\
\gamma_t \mathbf{E}_K[\overrightarrow{w}^2_{K,iloc}] & = w_{K,iloc}\begin{pmatrix}-n_1n_2\\ (1-n_2^2)\\ -n_2n_3\end{pmatrix} & \gamma_t \mathbf{H}_K[\overrightarrow{w}^2_{K,iloc}] & = w_{K,iloc}\begin{pmatrix}0\\0\\0\end{pmatrix}\\
\gamma_t \mathbf{E}_K[\overrightarrow{w}^3_{K,iloc}] & = w_{K,iloc}\begin{pmatrix}-n_1n_3\\ -n_2n_3\\ (1-n_3^2)\end{pmatrix} & \gamma_t \mathbf{H}_K[\overrightarrow{w}^3_{K,iloc}] & = w_{K,iloc}\begin{pmatrix}0\\0\\0\end{pmatrix}\\
\gamma_t \mathbf{E}_K[\overrightarrow{w}^4_{K,iloc}] & = w_{K,iloc}\begin{pmatrix}0\\0\\0\end{pmatrix} & \gamma_t \mathbf{H}_K[\overrightarrow{w}^4_{K,iloc}] & = w_{K,iloc}\begin{pmatrix}(1-n_1^2)\\ -n_1n_2\\ -n_1n_3\end{pmatrix}\\
\gamma_t \mathbf{E}_K[\overrightarrow{w}^5_{K,iloc}] & = w_{K,iloc}\begin{pmatrix}0\\0\\0\end{pmatrix} & \gamma_t \mathbf{H}_K[\overrightarrow{w}^5_{K,iloc}] & = w_{K,iloc}\begin{pmatrix}-n_1n_2\\ (1-n_2^2)\\ -n_2n_3\end{pmatrix}\\
\gamma_t \mathbf{E}_K[\overrightarrow{w}^6_{K,iloc}] & = w_{K,iloc}\begin{pmatrix}0\\0\\0\end{pmatrix} & \gamma_t \mathbf{H}_K[\overrightarrow{w}^6_{K,iloc}] & = w_{K,iloc}\begin{pmatrix}-n_1n_3\\ -n_2n_3\\ (1-n_3^2)\end{pmatrix}.
\end{array}\right. \tag{132}$$

In the same way, since $\gamma_\times \mathbf{E}_K = \mathbf{n}_K \times \gamma_t \mathbf{E}$, we define the cross operator on a

face F of element K

$$\left\{\begin{array}{llll}
\gamma_\times \mathbf{E}_K[\overrightarrow{w}^1_{K,iloc}] & = w_{K,iloc}\begin{pmatrix}0\\ n_3\\ -n_2\end{pmatrix} & \gamma_\times \mathbf{H}_K[\overrightarrow{w}^1_{K,iloc}] & = w_{K,iloc}\begin{pmatrix}0\\0\\0\end{pmatrix}\\
\gamma_\times \mathbf{E}_K[\overrightarrow{w}^2_{K,iloc}] & = w_{K,iloc}\begin{pmatrix}-n_3\\ 0\\ n_1\end{pmatrix} & \gamma_\times \mathbf{H}_K[\overrightarrow{w}^2_{K,iloc}] & = w_{K,iloc}\begin{pmatrix}0\\0\\0\end{pmatrix}\\
\gamma_\times \mathbf{E}_K[\overrightarrow{w}^3_{K,iloc}] & = w_{K,iloc}\begin{pmatrix}n_2\\ -n_1\\ 0\end{pmatrix} & \gamma_\times \mathbf{H}_K[\overrightarrow{w}^3_{K,iloc}] & = w_{K,iloc}\begin{pmatrix}0\\0\\0\end{pmatrix}\\
\gamma_\times \mathbf{E}_K[\overrightarrow{w}^4_{K,iloc}] & = w_{K,iloc}\begin{pmatrix}0\\0\\0\end{pmatrix} & \gamma_\times \mathbf{H}_K[\overrightarrow{w}^4_{K,iloc}] & = w_{K,iloc}\begin{pmatrix}0\\ n_3\\ -n_2\end{pmatrix}\\
\gamma_\times \mathbf{E}_K[\overrightarrow{w}^5_{K,iloc}] & = w_{K,iloc}\begin{pmatrix}0\\0\\0\end{pmatrix} & \gamma_\times \mathbf{H}_K[\overrightarrow{w}^5_{K,iloc}] & = w_{K,iloc}\begin{pmatrix}-n_3\\ 0\\ n_1\end{pmatrix}\\
\gamma_\times \mathbf{E}_K[\overrightarrow{w}^6_{K,iloc}] & = w_{K,iloc}\begin{pmatrix}0\\0\\0\end{pmatrix} & \gamma_\times \mathbf{H}_K[\overrightarrow{w}^6_{K,iloc}] & = w_{K,iloc}\begin{pmatrix}n_2\\ -n_1\\ 0\end{pmatrix}.
\end{array}\right. \tag{133}$$

Hereinafter, we adopt the notation $\mathbb{M}_F^{K,K'}$ for any face $F \in K$ (either interior face or boundary face)

$$\mathbb{M}^{K,K'}_{F,iloc,jloc} = \int_F w_{K',jloc} w_{K,iloc}. \tag{134}$$

For a boundary face $F \in \mathcal{F}_{ext}$, we use $K' = K$ in the above formula.
From the results above, we can then deduce the following formula. The matrix involving twice the tangential trace operator γ_t is given by

$$\int_F \gamma_t \mathbf{E}_K[\overrightarrow{w}^{\ell_j}_{K,jloc}] \cdot \gamma_t \mathbf{E}_K[\overrightarrow{w}^{\ell_i}_{K,iloc}] = C^{\ell_i,\ell_j}_{\mathbf{E},\mathbf{E}} \int_F w_{K,jloc} w_{K,iloc} = C^{\ell_i,\ell_j}_{K,\mathbf{E},\mathbf{E}} \mathbb{M}^{K,K}_{F,iloc,jloc},$$

with

$$C^{\cdot,\cdot}_{K,\mathbf{E},\mathbf{E}} = \begin{pmatrix}
1-n_1^2 & -n_1n_2 & -n_1n_3 & 0 & 0 & 0\\
-n_1n_2 & 1-n_2^2 & -n_2n_3 & 0 & 0 & 0\\
-n_1n_3 & -n_2n_3 & 1-n_3^2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

The matrix representing the form $\int_F \gamma_\times \mathbf{H}_K \cdot \gamma_\times \overline{\mathbf{H}'_K}$ is computed in the same way

$$\int_F \gamma_\times \mathbf{H}_K[\overrightarrow{w}^{\ell_i}_{K,jloc}] \cdot \gamma_\times \mathbf{H}_K[\overrightarrow{w}^{\ell_j}_{K,iloc}] \;=\; C^{\ell_i,\ell_j}_{K,\mathbf{H},\mathbf{H}} \mathbb{M}^{K,K}_{F,iloc,jloc}, \tag{135}$$

with

$$C^{\cdot,\cdot}_{K,\mathbf{H},\mathbf{H}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-n_1^2 & -n_1n_2 & -n_1n_3 \\ 0 & 0 & 0 & -n_1n_2 & 1-n_2^2 & -n_2n_3 \\ 0 & 0 & 0 & -n_1n_3 & -n_2n_3 & 1-n_3^2 \end{pmatrix}.$$

The matrix representing the form $\int_F \gamma_t \mathbf{E}_K \cdot \overline{\gamma_\times \mathbf{H}'_K}$ is given by

$$\int_F \gamma_t \mathbf{E}_K[\overrightarrow{w}^{\ell_j}_{K,jloc}] \cdot \gamma_\times \mathbf{H}_K[\overrightarrow{w}^{\ell_i}_{K,iloc}] = C^{\ell_i,\ell_j}_{\mathbf{E},\mathbf{H}} \int_F w_{K,iloc} w_{K,jloc} = C^{\ell_i,\ell_j}_{K,\mathbf{E},\mathbf{H}} \mathbb{M}^{K,K}_{F,iloc,jloc}, \tag{136}$$

$$C^{\cdot,\cdot}_{K,\mathbf{E},\mathbf{H}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & n_3 & -n_2 & 0 & 0 & 0 \\ -n_3 & 0 & n_1 & 0 & 0 & 0 \\ n_2 & -n_1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{137}$$

The matrix representing the form $\int_F \gamma_\times \mathbf{H}_K \cdot \overline{\gamma_t \mathbf{E}'_K}$ is given by

$$\int_F \gamma_\times \mathbf{H}_K[\overrightarrow{w}^{\ell_j}_{K,jloc}] \cdot \gamma_t \mathbf{E}_K[\overrightarrow{w}^{\ell_i}_{K,iloc}] = C^{\ell_i,\ell_j}_{\mathbf{H},\mathbf{E}} \int_F w_{K,iloc} w_{K,jloc} = C^{\ell_i,\ell_j}_{K,\mathbf{E},\mathbf{H}} \mathbb{M}^{K,K}_{F,iloc,jloc}, \tag{138}$$

with

$$C^{\cdot,\cdot}_{K,\mathbf{H},\mathbf{E}} = \begin{pmatrix} 0 & 0 & 0 & 0 & -n_3 & n_2 \\ 0 & 0 & 0 & n_3 & 0 & -n_1 \\ 0 & 0 & 0 & -n_2 & n_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{139}$$

Theorem 9. The matrix associated to the bilinear form b_F^{ext} is defined by

$$(\mathbb{B}_F^{ext})_{iglob,jglob} \;=\; b_F^{ext}(\overrightarrow{w}_{jglob}, \overrightarrow{w}_{iglob}) \;=\; 0, \tag{140}$$

and its coefficients are given by

$$\mathbb{B}^{ext}_{F,iglob,jglob} \;= \zeta^F_{K_i,K_j}\; D^{\ell_j}_{ext}\; C^{\ell_i,\ell_j}_K\; M^{K,K}_{F,iloc,jloc}, \tag{141}$$

with

$$C^{\ell_i,\ell_j}_K \;=\; C^{\ell_i,\ell_j}_{K,\mathbf{E},\mathbf{E}} + C^{\ell_i,\ell_j}_{K,\mathbf{E},\mathbf{H}} + C^{\ell_i,\ell_j}_{K,\mathbf{H},\mathbf{E}} + C^{\ell_i,\ell_j}_{K,\mathbf{H},\mathbf{H}}\;, \tag{142}$$

$$D^{\ell_j}_{ext} \;=\; \left(\frac{1-R}{2}\quad \frac{1-R}{2}\quad \frac{1-R}{2}\quad \frac{1+R}{2}\quad \frac{1+R}{2}\quad \frac{1+R}{2}\right), \tag{143}$$

and $\zeta^F_{K_i,K_j} = 0$ except if $F \in \mathcal{F}_{ext}$ and $K_i = K_j = K$ where we have

$$\zeta^F_{K_i,K_j} = 1. \tag{144}$$

Remark 12. The matrix C^{ℓ_i,ℓ_j}_K can be explicitly computed and is given by

$$C^{\ell_i,\ell_j}_K \;=\; \begin{pmatrix} 1-n_1^2 & -n_1n_2 & -n_1n_3 & 0 & -n_3 & n_2 \\ -n_1n_2 & 1-n_2^2 & -n_2n_3 & n_3 & 0 & -n_1 \\ -n_1n_3 & -n_2n_3 & 1-n_3^2 & -n_2 & n_1 & 0 \\ 0 & n_3 & -n_2 & 1-n_1^2 & -n_1n_2 & -n_1n_3 \\ -n_3 & 0 & n_1 & -n_1n_2 & 1-n_2^2 & -n_2n_3 \\ n_2 & -n_1 & 0 & -n_1n_3 & -n_2n_3 & 1-n_3^2 \end{pmatrix} \tag{145}$$

with $\mathbf{n}_F = (n_1, n_2, n_3)$ the outward normal to F, a face of element K.

Remark 13. The ζ factor ensures to consider only the faces situated on the domain boundary.

5.5 Implementation of the matrix $\mathbb{B}_F^{int}$

The matrix $\mathbb{B}_F^{int}$ is corresponding to the form b_F^{int}

$$\left\{\begin{array}{rcl} b_F^{int}(\mathbb{E},\mathbb{E}') & = & \displaystyle\int_F [\![\gamma_\times \mathbf{H}]\!]_F \cdot \overline{\{\gamma_t \mathbf{E}'_{K_F^-}\}} - [\![\gamma_\times \mathbf{E}]\!]_F \cdot \overline{\{\gamma_t \mathbf{H}'_{K_F^-}\}} \\ & + & \displaystyle\int_F \frac{[\![\gamma_\times \mathbf{H}]\!]_F \cdot \overline{[\![\gamma_\times \mathbf{H}']\!]} + [\![\gamma_\times \mathbf{E}]\!]_F \cdot \overline{[\![\gamma_\times \mathbf{E}']\!]}}{2}. \end{array}\right. \tag{146}$$

It is defined by the formula

$$(\mathbb{B}_F^{int})_{iglob,jglob} \;=\; b_F^{int}(\overrightarrow{w}_{jglob}, \overrightarrow{w}_{iglob}) \;=\; b_F^{int}(\overrightarrow{w}^{\ell_j}_{K_j,jloc}, \overrightarrow{w}^{\ell_i}_{K_i,iloc}) \tag{147}$$

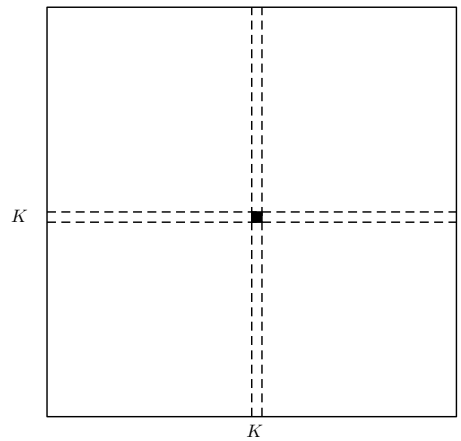

Figure 9: Localization of the non zero block of $\mathbb{B}_F^{ext}$.

where data is defined as in (131). To deal with interior faces, we consider a face F interfacing two elements K_F^+ and K_F^-. We define the outward normal $\mathbf{n}^+$ to element K_F^+

$$\mathbf{n}^+ := \mathbf{n}_{K_F^+} = (n_1, n_2, n_3)^T. \tag{148}$$

We intoduce the mean value and the jump with respect to this orientation

$$\left\{ \begin{array}{rcl} \{\gamma_t \mathbf{E}\}_F & = & \dfrac{\gamma_t \mathbf{E}_{K_F^+} + \gamma_t \mathbf{E}_{K_F^-}}{2} \\ [\![\gamma_\times \mathbf{E}]\!]_F & = & \mathbf{n}^+ \times \mathbf{E}_{K_F^+} + \mathbf{n}^- \times \mathbf{E}_{K_F^-} \ = \ \gamma_\times^{K_F^+} \mathbf{E}_{K_F^+} + \gamma_\times^{K_F^-} \mathbf{E}_{K_F^-} \end{array} \right. \tag{149}$$

with $\mathbf{n}^- = -\mathbf{n}^+$, F is a common face of K_F^+ and K_F^- and where we have

$$\left\{ \begin{array}{l} \gamma_t \mathbf{E} = \mathbf{E} - (\mathbf{E} \cdot \mathbf{n}^+)\mathbf{n}^+, \\ \gamma_\times^{K^+} \mathbf{E} = \mathbf{n}^+ \times \gamma_t \mathbf{E}. \end{array} \right.$$

We apply the same reasoning than for the boundary form and we get the following theorem.

Theorem 10. The matrix $\mathbb{B}_F^{int}$ representing the interior form is given by

$$\mathbb{B}_F^{int}(\overrightarrow{w}_{K_j,jloc}^{\ell_j}, \overrightarrow{w}_{K_i,iloc}^{\ell_i}) \ = \ \frac{1}{2}(\varepsilon_{K_i,K_j}^F \ C_{K_j}^{\ell_i,\ell_j} + \vartheta_{K_i,K_j}^F \ D_{K_j}^{\ell_i,\ell_j}) \ M_{F,iloc,jloc}^{K_i,K_j}, \tag{150}$$

with $\varepsilon_{K,K'}^F = 0$ except if $F \subset \{K, K'\}$, for which we have

$$\left\{ \begin{array}{l} \varepsilon_{K,K'}^F = 1 \text{ for } K = K', \\ \varepsilon_{K,K'}^F = -1 \text{ for } K \neq K', \end{array} \right. \qquad \left\{ \begin{array}{l} \vartheta_{K,K'}^F = 1 \text{ for } K = K', \\ \vartheta_{K,K'}^F = -1 \text{ for } K \neq K', \end{array} \right. \tag{151}$$

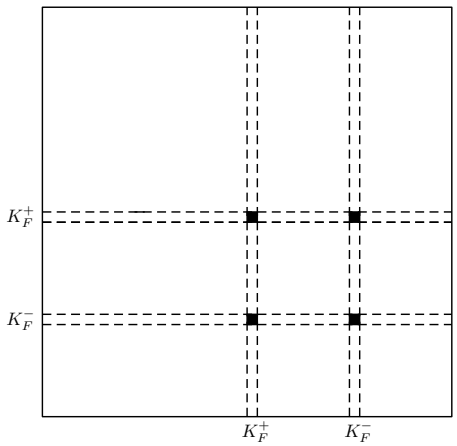

Figure 10: The non zero blocks $\mathbb{B}^{int}_{F,\cdot,\cdot}$.

and $C_K^{\ell,\ell'}$, $D_K^{\ell,\ell'}$ given by

$$\begin{aligned} C_K^{\ell,\ell'} &= C_{K,\mathbf{E},\mathbf{E}}^{\ell,\ell'} + C_{K,\mathbf{H},\mathbf{H}}^{\ell,\ell'}, \\ D_K^{\ell,\ell'} &= C_{K,\mathbf{E},\mathbf{H}}^{\ell,\ell'} + C_{K,\mathbf{H},\mathbf{E}}^{\ell,\ell'}. \end{aligned} \tag{152}$$

Remark 14. On one hand, conditions for ϑ values come from the change of sign that can have the forms $\int_F \gamma_\times \mathbf{H}_K \cdot \overline{\gamma_t \mathbf{E}'_{K'}}$ and $\int_F \gamma_t \mathbf{E}_K \cdot \overline{\gamma_\times \mathbf{H}'_{K'}}$. On the other hand, the normals involved in the scalar product in the forms $\int_F \gamma_t \mathbf{E}_K \cdot \overline{\gamma_t \mathbf{E}'_{K'}}$ and $\int_F \gamma_\times \mathbf{H}_K \cdot \overline{\gamma_\times \mathbf{H}'_{K'}}$ are each time multiplied to each other and thus give a positive sign. The only case where they will give a negative sign is when $K_F^+ \neq K'$. That is the reason why we have defined $\varepsilon^F_{K,K'}$.

Remark 15. Taking alternatively $K = K_F^+$, $K = K_F^-$, $K' = K_F^+$ and $K' = K_F^-$, and using (151), we remark that the non zero blocks of matrix $\mathbb{B}^{int}_F$ are

$$\left\{ \begin{array}{ll} \left[\frac{1}{2}(C^{\cdot,\cdot}_{K_F^+} + D^{\cdot,\cdot}_{K_F^+}) M^{K_F^+,K_F^+}_{F,\cdot,\cdot}\right]_{K_F^+,K_F^+} & \left[-\frac{1}{2}(C^{\cdot,\cdot}_{K_F^+} + D^{\cdot,\cdot}_{K_F^+}) M^{K_F^+,K_F^-}_{F,\cdot,\cdot}\right]_{K_F^+,K_F^-} \\ \left[\frac{1}{2}(D^{\cdot,\cdot}_{K_F^-} - C^{\cdot,\cdot}_{K_F^-}) M^{K_F^-,K_F^+}_{F,\cdot,\cdot}\right]_{K_F^-,K_F^+} & \left[\frac{1}{2}(C^{\cdot,\cdot}_{K_F^-} - D^{\cdot,\cdot}_{K_F^-}) M^{K_F^-,K_F^-}_{F,\cdot,\cdot}\right]_{K_F^-,K_F^-} \end{array} \right. . \tag{153}$$

5.6 Numerical results

The numerical results were done for different meshes of the cube domain which has also different sizes depending on the order of approximation. We impose that the wave number is $k = 2\pi$. That means that the wavelength is $\lambda = 1$.
On figure 5.6, we show the convergence of the method for different orders of approximation (from 1st order to 4th order). A convergence curve represents the error in function of the element size h. The error is computed thanks to the norm of the difference of the exact solution and the numerical solution. The exact solution can indeed be computed analytically. We can thus compare it to the solution we get with our numerical method. Coming from the energy fields definition on a face F, the L^2 norm can be defined as following

$$\|\mathbb{E}\|_0^2 = \int_F |\mathbf{E}|^2 + |\mathbf{H}|^2. \tag{154}$$

Here, we can point out that our method is converging. Indeed, when the size of meshes decreases then the $L2$-norm decreases as well.
The study of memory cost is also an important point. When we decrease the size of the mesh, we thus have to consider more elements, leading to a memory cost increase (see Figure 5.6). Therefore, to get a good approxmation of the solution, that means to get a small norm, we have to use a lot of memory (see Figure 5.6).
Unfortunately, we did not have time to perform the plane wave visualization on our cube mesh. However, we can ensure the correctness of the code thanks to those convergence curves.

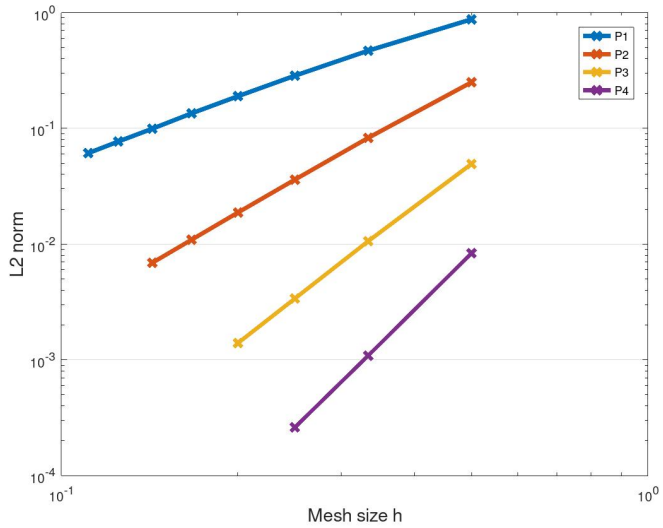

Figure 11: Convergence curves for different orders of approximation.

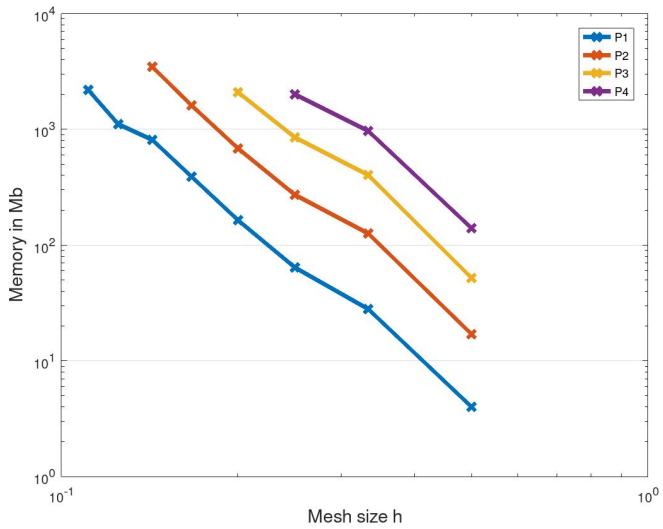

Figure 12: Memory burden for different orders of approximation.

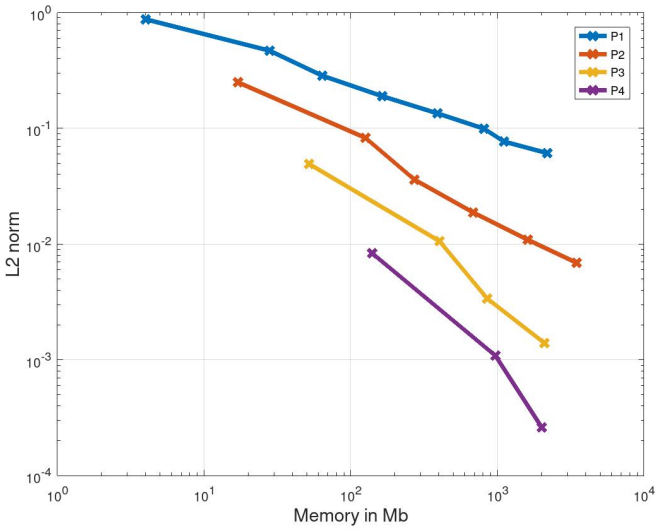

Figure 13: Convergence curves for different orders of approximation.

Conclusion

As for any near-field numerical method applied to indefinite systems of PDE (like Finite Differences, Finite Volumes, Finite Elements), the numerical inversion by LU direct solver becomes impossible even with HPC architectures when the size of the domain is larger than 20 wavelengths.
However, this study confirms the excellent convergence property of the Discontinuous Galerkin method. The latter enables to get both the electric and magnetic fields as output without post processing of the numerical method. In comparison with the method based on the quasi-elliptic formulation ($\mathrm{rot}(\mathrm{rot}(\mathbf{E})) - k^2\mathbf{E} = 0$), the size of the linear system is doubled.
Consequently, in the context of very large domains, we can benefit from these excellent properties of this DG method within the framework of a quasi-Trefftz solver [14].

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