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# UNIFORMLY ACCURATE SCHEMES FOR OSCILLATORY STOCHASTIC DIFFERENTIAL EQUATIONS

IBRAHIM ALMUSLIMANI<sup>1</sup>, PHILIPPE CHARTIER<sup>2</sup>, MOHAMMED LEMOU<sup>3</sup> AND FLORIAN MĚHATS<sup>4</sup>

ABSTRACT. In this work, we adapt the *micro-macro* methodology to stochastic differential equations for the purpose of numerically solving oscillatory evolution equations. The models we consider are addressed in a wide spectrum of regimes where oscillations may be slow or fast. We show that through an ad-hoc transformation (the micro-macro decomposition), it is possible to retain the usual orders of convergence of Euler-Maruyama method, that is to say, uniform weak order one and uniform strong order one half.

*Keywords:* highly-oscillatory, stochastic differential equations, micro-macro decomposition, uniform accuracy.

*AMS subject classification (2010):* 65L20, 74Q10, 35K15.

## 1. Introduction

In this paper, we aim at constructing *uniformly accurate* numerical schemes for solving Itô stochastic differential equations (SDEs) with a (possibly highly) oscillatory drift term of the form

$$dX(t) = f_{t/\epsilon}(X(t))dt + \sigma(X(t))dW(t), \quad X(0) = X_0 \in \mathbb{R}^d, \quad (1.1)$$

where  $X(t)$  is a stochastic process with values in  $\mathbb{R}^d$ , the drift function  $(\theta, x) \in \mathbb{T} \times \mathbb{R}^d \mapsto f_\theta(x)$  is assumed to be 1-periodic with respect to  $\theta$  (we shall denote accordingly the torus by  $\mathbb{T} = \mathbb{R}/\mathbb{Z} \equiv [0, 1)$ ), and the diffusion function  $\sigma$  is defined as a smooth function  $x \in \mathbb{R}^d \mapsto \sigma(x) \in \mathbb{R}^{d \times m}$ . Finally,  $W(t) = (W_1(t), \dots, W_m(t))^T$  is an array of  $m$  independent Wiener processes. Precise regularity assumptions on  $f$  and  $\sigma$  are made in Assumption 1.2. Let us emphasize that  $\epsilon$  is here a parameter whose value can freely vary in the interval  $]0, 1]$  and that equation (1.1) is not restricted to its asymptotic regime where  $\epsilon$  tends to zero.

Analogously to the case of deterministic differential equations, in the highly-oscillatory (stiff) regime ( $\epsilon \ll 1$ ), standard numerical methods for SDEs, such as Euler-Maruyama method, face a severe time step restriction ( $\Delta t = \mathcal{O}(\epsilon)$ ) when applied directly to (1.1). This issue is well documented in the literature for deterministic ODEs [16], and several classes of methods were introduced in order to deal with stiffness [5, 7, 9, 10, 11]. However, none of the methods introduced therein qualify as uniformly accurate methods as they do not produce numerical approximations with an accuracy and at a cost both independent of the value of  $\epsilon \in (0, 1]$ . This motivated the introduction of a new methodology based on averaging techniques and exposed in [8]: there, the authors elaborate a new technique enabling standard

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numerical methods to retain their non-stiff order with uniform accuracy for all  $\epsilon \in (0, 1]$ . In the case of stiff SDEs, as for ODEs, we can differentiate between two kinds of stiffness: stiff dissipative SDEs and highly-oscillatory SDEs. While for stiff dissipative SDEs, many interesting integrators were introduced in the last two decades with nice stability and convergence properties [1, 2, 4, 3], the numerical solution of highly-oscillatory SDEs has not received so much attention [18]. This contribution is, up to our knowledge, the first attempt to adapt the technique of micro-macro decomposition to the SDE context.

More precisely, we focus, in the present paper, on constructing uniformly accurate methods for highly-oscillatory SDEs in the spirit of the methodology explained in [8]. We derive a micro-macro system by introducing a change of variable that leads us to treat the average decay and the fast oscillations separately. We show that applying Euler-Maruyama method to the micro-macro system gives an approximation of uniform weak order 1 and strong order  $1/2$  for any value of  $\epsilon \in (0, 1]$  and with no restriction on the time step. In more mathematical terms, we prove that, under appropriate assumptions, the Euler-Maruyama scheme for solving the *micro-macro* system derived from (1.1), provides approximations  $X_n$  on a uniform grid  $\{t_n\}_{n=0}^N$  such that

$$\forall n = 0, \dots, N, \quad |\mathbb{E}(\phi(X(t_n))) - \mathbb{E}(\phi(X_n))| \leq Ch \text{ and } \mathbb{E}(|X(t_n) - X_n|^2)^{\frac{1}{2}} \leq Ch^{\frac{1}{2}},$$

where  $h$  is the time step and the constant  $C$  is independent of  $h$  and  $\epsilon$ . We prove as well that the same result can be obtained using the integral scheme introduced in Section 2. Interestingly, deriving the micro-macro method gives insight about possible generalizations to higher-order schemes.

Noteworthy, our results are obtained under quite standard assumptions that we recall below. In particular, we shall constantly suppose that Assumption 1.2 below holds true in the sequel.

**Definition 1.1.** *We define the following two sets of functions:*

$$C_{poly}^q := \left\{ \phi \in C^q(\mathbb{R}^d, \mathbb{R}) : \exists \kappa \in \mathbb{N}^*, \forall \mathbf{k} \in \mathbb{N}^d, |\mathbf{k}| \leq q, \|(1 + |x|^\kappa)^{-1} \partial_x^{\mathbf{k}} \phi\|_{L^\infty} < +\infty \right\},$$

$$C_{Lip}^q := \left\{ \phi \in C^q(\mathbb{R}^d, \mathbb{R}^d) : \forall \mathbf{k} \in \mathbb{N}^d, |\mathbf{k}| \leq q, \partial_x^{\mathbf{k}} \phi \text{ is Lipschitz continuous} \right\}.$$

**Assumption 1.2.** *The functions  $x \mapsto \sigma_{i,j}(x)$ ,  $i = 1, \dots, d$ ,  $j = 1, \dots, m$  are Lipschitz functions of the set  $C_{poly}^2$ . The function  $(\theta, x) \mapsto f_\theta(x) \in \mathbb{R}^d$  is defined on  $\mathbb{T} \times \mathbb{R}^d$  and its components  $(f_\theta(\cdot))_i$ ,  $i = 1, \dots, d$ , are functions of  $C_{Lip}^2$ . Furthermore,  $f$  is continuously differentiable with respect to  $\theta$  and  $x \mapsto \partial_\theta f_\theta(x)$  is Lipschitz continuous.*

**Remark 1.3.** *The Lipschitz continuity of  $f_\theta$  and  $\sigma$  ensures the existence and uniqueness of the solution of (1.1) in  $C([0, T]; L^2(\Omega))$ . It ensures in addition a linear growth, that is to say*

$$\forall (\theta, x) \in \mathbb{T} \times \mathbb{R}^d, \quad |f_\theta(x)| + |\sigma(x)| \leq K(1 + |x|).$$

The rest of the paper is organized as follows. In Section 2 we introduce the integral scheme and we prove its uniform convergence (weak and strong) with respect to the parameter  $\epsilon$ . In Section 3 we derive the micro-macro scheme, and we show its uniform convergence properties. Finally, in Section 4 we present some numerical experiments that illustrate the efficiency of the schemes.

**Notations.** *The derivative with respect to the space variable  $x$  will be denoted by a prime. For example,  $f'_\theta(x) = \partial_x f_\theta(x)$  and  $f''_\theta(x) = \partial_x^2 f_\theta(x)$ . The letter  $L$  will be used throughout the paper as the maximum Lipschitz constant of the concerned functions. The constants represented by*

the capital letter  $C$  (indexed or not) are generic constants. We denote by  $|\cdot|$  the Euclidean norm of  $\mathbb{R}^d$ .

## 2. Integral scheme

We consider the integral scheme

$$X_{n+1} = X_n + h \langle f \rangle (X_n) + \epsilon \int_{t_n/\epsilon}^{t_{n+1}/\epsilon} (f_\theta(X_n) - \langle f \rangle (X_n)) d\theta + \sigma(X_n) \Delta W_n, \quad (2.1)$$

where  $\langle f \rangle (x)$  is the value at  $x$  of the average of  $f$  defined as

$$\langle f \rangle (x) = \int_{\mathbb{T}} f_\theta(x) d\theta,$$

and  $\Delta W_n = W(t_{n+1}) - W(t_n) \sim \mathcal{N}(0, h)$  which can be replaced by  $\sqrt{h}\xi$  where  $\xi \sim \mathcal{N}(0, 1)$ . Note that the integral term in (2.1) can be rewritten as  $\epsilon(F_{t_{n+1}/\epsilon}(X_n) - F_{t_n/\epsilon}(X_n))$ , where

$$F_\theta(x) = \int_0^\theta (f_\tau(x) - \langle f \rangle (x)) d\tau.$$

**Theorem 2.1.** *Consider the integral scheme (2.1), for  $n = 0, \dots, N-1$  where  $T = Nh > 0$  is a given final time. Let  $X(t)$  be the solution of (1.1). Then, for all  $\phi \in C_{poly}^4$ , there exists  $C > 0$  independent of  $h$  and  $\epsilon \in (0, 1]$  such that,*

$$\forall n \in \{0, \dots, N\}, \quad |\mathbb{E}(\phi(X(t_n))) - \mathbb{E}(\phi(X_n))| \leq Ch, \quad (2.2)$$

$$\forall n \in \{0, \dots, N\}, \quad \mathbb{E}(|X(t_n) - X_n|^2)^{\frac{1}{2}} \leq Ch^{\frac{1}{2}}. \quad (2.3)$$

**Lemma 2.2.** *For  $0 \leq n \leq N-1$ ,  $h = t_{n+1} - t_n$  small enough, and  $t \in [t_n, t_{n+1}]$ , one has*

$$\left| \mathbb{E} \left( \int_{t_n}^{t_{n+1}} [f_{t/\epsilon}(X(t)) - f_{t/\epsilon}(X(t_n))] dt \right) \right| \leq Ch^2. \quad (2.4)$$

*Proof.* We first rewrite the integrand  $f_{t/\epsilon}(X(t)) - f_{t/\epsilon}(X(t_n))$  of (2.4) as

$$\begin{aligned} \int_{t_n}^t d(f_{t/\epsilon}(X(s))) &= \int_{t_n}^t [f'_{t/\epsilon}(X(s)) f_{t/\epsilon}(X(s)) \\ &\quad + \frac{1}{2} \sum_{r=1}^m f''_{t/\epsilon}(X(s)) (\sigma(X(s)) e_r, \sigma(X(s)) e_r)] ds \\ &\quad + \int_{t_n}^t f'_{t/\epsilon}(X(s)) \sigma(X(s)) dW(s), \end{aligned}$$

so that

$$\begin{aligned} \left| \mathbb{E} \left( \int_{t_n}^{t_{n+1}} [f_{t/\epsilon}(X(t)) - f_{t/\epsilon}(X(t_n))] dt \right) \right| &\leq \left| \mathbb{E} \left( \int_{t_n}^{t_{n+1}} \int_{t_n}^t [f'_{t/\epsilon}(X(s)) f_{t/\epsilon}(X(s)) \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \sum_{r=1}^m f''_{t/\epsilon}(X(s)) (\sigma(X(s)) e_r, \sigma(X(s)) e_r) \right] ds dt \right) \right| \\ &\quad + \left| \mathbb{E} \left( \int_{t_n}^{t_{n+1}} \int_{t_n}^t f'_{t/\epsilon}(X(s)) \sigma(X(s)) dW(s) dt \right) \right|, \end{aligned}$$

and using Fubini's theorem, we get the following upper-bound of the left-hand side of (2.4)

$$\left| \int_{t_n}^{t_{n+1}} \mathbb{E} \left( \int_{t_n}^t f'_{t/\epsilon}(X(s)) \sigma(X(s)) dW(s) \right) dt \right| + \mathcal{O}(h^2).$$

The expectation in previous integral is an expectation of a stochastic integral, hence it is null since  $X(s)$  is independent of the increments of  $W$  for times above  $s$ .  $\square$

**Lemma 2.3.** *For  $0 \leq n \leq N - 1$ ,  $h$  small enough, and  $t \in [t_n, t_{n+1}]$ , we have*

$$\mathbb{E}(|X(t) - X(t_n)|^2 ; X(t_n) = x) \leq C(x)(t - t_n). \quad (2.5)$$

*Proof.* From the integral form of equation (1.1)

$$X(t) = X(t_n) + \int_{t_n}^t f_{s/\epsilon}(X(s)) ds + \int_{t_n}^t \sigma(X(s)) dW(s).$$

we may write

$$\begin{aligned} \mathbb{E}(|X(t) - X(t_n)|^2 ; X(t_n) = x) &\leq \mathbb{E} \left( \left| \int_{t_n}^t f_{s/\epsilon}(X(s)) ds - \int_{t_n}^t \sigma(X(s)) dW(s) \right|^2 \right) \\ &\leq 2(t - t_n) \int_{t_n}^t \mathbb{E}(f_{s/\epsilon}(X(s))^2) ds + 2 \int_{t_n}^t \mathbb{E}(\sigma(X(s))^2) ds \\ &\leq 2(t - t_n) \int_{t_n}^t K(1 + \mathbb{E}(|X(t_n)|^2)) ds + 2 \int_{t_n}^t K(1 + \mathbb{E}(|X(t_n)|^2)) ds \\ &\leq C(x)(t - t_n), \end{aligned}$$

where we have used, between the first and second lines, Young's inequality, then Cauchy-Schwarz inequality for the first integral, and Itô isometry for the second integral. Finally, we have used the bound  $\mathbb{E}(|X(s)|^2) \leq K(1 + |x|^2)$ , borrowed from [12], between the third and fourth lines.  $\square$

*Proof of Theorem 2.1.* We hereby follow the methodology introduced in [17], which consists in bounding the moments (first step), then proving the weak convergence of the scheme (second step) and finally establishing its strong convergence (third step).

**Step 1.** In order to bound the moments of  $X_n$  of arbitrary order, we shall resort to [15, Lemma 2.2, p. 102], which requires the following estimates

$$\begin{aligned} |\mathbb{E}(X_{n+1} - X_n | X_n = x)| &= \left| \mathbb{E} \left( \epsilon \int_{t_n/\epsilon}^{t_{n+1}/\epsilon} f_\theta(x) d\theta \right) \right| \\ &\leq \epsilon \int_{t_n/\epsilon}^{t_{n+1}/\epsilon} K(1 + |x|) d\theta = K(1 + |x|)h, \end{aligned}$$

and

$$\begin{aligned} |X_{n+1} - X_n| &= \left| \epsilon \int_{t_n/\epsilon}^{t_{n+1}/\epsilon} f_\theta(X_n) d\theta + \sigma(X_n) \Delta W_n \right| \\ &\leq K(1 + |X_n|)h + K(1 + |X_n|) \frac{|\Delta W_n|}{\sqrt{h}} \sqrt{h} \leq M_n(1 + |X_n|)\sqrt{h}, \end{aligned}$$

where  $M_n = K \left( \sqrt{h} + \frac{|\Delta W_n|}{\sqrt{h}} \right)$  which is of bounded moments since  $\frac{|\Delta W_n|}{\sqrt{h}} \sim \mathcal{N}(0, 1)$ .

**Step 2.** For  $X(t_n) = X_n = x$ , it can be shown that

$$\begin{aligned}
 |\mathbb{E}(\phi(X(t_{n+1})) - \phi(X_{n+1}))| &\leq \left| \mathbb{E} \left[ \phi'(x) \left( \int_{t_n}^{t_{n+1}} (f_{t/\epsilon}(X(t)) - f_{t/\epsilon}(X(t_n))) dt \right) \right] \right| \\
 &+ \left| \mathbb{E} \left[ \phi''(x) \left( \int_{t_n}^{t_{n+1}} (\sigma(X(t)) - \sigma(X(t_n))) dW(t) \right) \left( \int_{t_n}^{t_{n+1}} (\sigma(X(t)) + \sigma(X(t_n))) dW(t) \right) \right] \right| \\
 &+ \frac{1}{6} \left| \mathbb{E} \left[ \int_0^1 (1-\tau)^3 \phi^{(4)}(x + \tau(X(t_{n+1}) - x)) d\tau (X(t_{n+1}) - x)^4 \right] \right| \\
 &+ \frac{1}{6} \left| \mathbb{E} \left[ \int_0^1 (1-\tau)^3 \phi^{(4)}(x + \tau(X_{n+1} - x)) d\tau (X_{n+1} - x)^4 \right] \right| + Ch^2 \\
 &:= \text{I} + \text{II} + \text{III} + \text{IV} + Ch^2,
 \end{aligned}$$

where  $C$  is independent of  $\epsilon$  and the term  $Ch^2$  comes from the remaining expectations of the second and third derivatives of  $\phi$  applied to the integrals, which are zero for odd number of stochastic integrals, and bounded by  $Ch^2$  otherwise. By Lemma 2.2, Lemma 2.3 and the boundedness of the moments, the terms I, III and IV are  $\mathcal{O}(h^2)$ . As for the integrand of the second term, we have, for  $t_n \leq t \leq t_{n+1}$ ,

$$\begin{aligned}
 \sigma(X(t)) - \sigma(X(t_n)) &= \int_{t_n}^t \left( \sigma'(X_s) f_{s/\epsilon}(X_s) + \frac{1}{2} \sum_{r=1}^m \sigma''(X_s) (\sigma(X_s) e_r, \sigma(X_s) e_r) \right) ds \\
 &+ \int_{t_n}^t \sigma'(X_s) \sigma(X_s) dW(s).
 \end{aligned}$$

Hence, using the Lipschitz-continuity of  $\sigma$ , the polynomial growth of  $\phi''$ , Lemma 2.3, and the following consequence of Itô isometry

$$\forall g, h \in L_{ad}^2([a, b] \times \Omega), \quad \mathbb{E} \left[ \left( \int_a^b g(t) dW(t) \right) \left( \int_a^b h(t) dW(t) \right) \right] = \mathbb{E} \left[ \int_a^b g(t) h(t) dt \right] \quad (2.6)$$

we have

$$\begin{aligned}
 \text{II} &\leq C(x) \left| \mathbb{E} \left[ \left( \int_{t_n}^{t_{n+1}} (\sigma(X(t)) - \sigma(X(t_n))) dW(t) \right) \left( \int_{t_n}^{t_{n+1}} (\sigma(X(t)) - \sigma(X(t_n))) dW(t) \right) \right. \right. \\
 &\quad \left. \left. + 2 \int_{t_n}^{t_{n+1}} \sigma(X(t_n)) dW(t) \right) \right] \right| \\
 &= C(x) \left| \mathbb{E} \left[ \left( \int_{t_n}^{t_{n+1}} (\sigma(X(t)) - \sigma(X(t_n)))^2 dt \right) \right] \right| \\
 &+ 2C(x) \left| \mathbb{E} \left[ \left( \int_{t_n}^{t_{n+1}} \sigma(X(t_n)) (\sigma(X(t)) - \sigma(X(t_n))) dt \right) \right] \right| \\
 &\leq C(x) L \int_{t_n}^{t_{n+1}} C(t - t_n) dt + 2 \left| \mathbb{E} \left[ \sigma(X(t_n)) \int_{t_n}^{t_{n+1}} \int_{t_n}^t d\sigma(X(s)) dt \right] \right| \\
 &= 2 \left| \mathbb{E} \left[ \sigma(X(t_n)) \int_{t_n}^{t_{n+1}} \int_{t_n}^t \left( \sigma'(X_s) f_{s/\epsilon}(X_s) + \frac{1}{2} \sum_{r=1}^m \sigma''(X_s) (\sigma(X_s) e_r, \sigma(X_s) e_r) \right) ds dt \right] \right| \\
 &+ 2 \left| \mathbb{E} \left[ \sigma(X(t_n)) \int_{t_n}^{t_{n+1}} \int_{t_n}^t \sigma'(X_s) \sigma(X_s) dW(s) dt \right] \right| + C_1 h^2.
 \end{aligned}$$

The first expectation after the last equal sign is clearly  $\mathcal{O}(h^2)$ . The second one is equal to zero since  $X(t_n)$  is independent of the increments  $W(r) - W(s)$  for  $t_n \leq s \leq r$ . The boundedness

of the moments and the local weak order 2 imply the global weak convergence of order 1 by a theorem from [14] (see also [15, Chapter 2.2]).

**Third step.** We first derive an upper-bound of  $\mathbb{E}(|X(t_{n+1}) - X_{n+1}|^2; X(t_n) = x)$  as follows

$$\begin{aligned} & \mathbb{E} \left( \left| \int_{t_n}^{t_{n+1}} (f_{t/\epsilon}(X(t)) - f_{t/\epsilon}(x)) dt + \int_{t_n}^{t_{n+1}} (\sigma(X(t)) - \sigma(x)) dW(t) \right|^2 \right) \\ & \leq 2\mathbb{E} \left( \left| \int_{t_n}^{t_{n+1}} (f_{t/\epsilon}(X(t)) - f_{t/\epsilon}(x)) dt \right|^2 \right) + 2\mathbb{E} \left( \left| \int_{t_n}^{t_{n+1}} (\sigma(X(t)) - \sigma(x)) dW(t) \right|^2 \right) \\ & \leq 2h\mathbb{E} \left( \int_{t_n}^{t_{n+1}} \left| f_{t/\epsilon}(X(t)) - f_{t/\epsilon}(X(t_n)) \right|^2 dt \right) + 2\mathbb{E} \left( \int_{t_n}^{t_{n+1}} \left| \sigma(X(t)) - \sigma(X(t_n)) \right|^2 dt \right) \\ & \leq 2L^2(h+1) \int_{t_n}^{t_{n+1}} \mathbb{E}(|X(t) - X(t_n)|^2) dt \leq 2L^2(h+1) \int_{t_n}^{t_{n+1}} C(t-t_n) dt \leq Ch^2, \end{aligned}$$

where we have used Lemma (2.3). Finally, we conclude that  $\mathbb{E}(|X(t_{n+1}) - X_{n+1}|^2)^{\frac{1}{2}} \leq Ch$ . A well-known theorem by Milstein [13] then allows to establish inequality (2.3).  $\square$

### 3. Micro-Macro method

Let us introduce the change of variables  $\Phi$ , defined, for all  $0 < \epsilon \leq 1$ , by the formula

$$\Phi_\theta(x) = x + \epsilon F_\theta(x) = x + \epsilon \int_0^\theta (f_\tau(x) - \langle f \rangle(x)) d\tau. \quad (3.1)$$

Separating slow and fast scales as follows

$$X = \Phi_{t/\epsilon}(\underline{X}) + Y \quad (3.2)$$

leads to the micro-macro system of the form

$$d\underline{X} = \langle f \rangle(\underline{X}) dt + \sigma(\underline{X}) dW, \quad \underline{X}(0) = X_0, \quad (3.3)$$

$$dY = \left( f_{t/\epsilon}(\Phi_{t/\epsilon}(\underline{X}) + Y) - f_{t/\epsilon}(\underline{X}) - \epsilon F'_{t/\epsilon}(\underline{X}) \langle f \rangle(\underline{X}) \right. \quad (3.4)$$

$$\left. - \frac{\epsilon}{2} \sum_{r=1}^m F''_{t/\epsilon}(\underline{X}) (\sigma(\underline{X}) e_r, \sigma(\underline{X}) e_r) \right) dt$$

$$+ \left( \sigma(\Phi_{t/\epsilon}(\underline{X}) + Y) - \sigma(\underline{X}) - \epsilon F'_{t/\epsilon}(\underline{X}) \sigma(\underline{X}) \right) dW, \quad Y(0) = 0,$$

where  $e_r$  is  $r^{\text{th}}$  canonical vector of  $\mathbb{R}^m$ .

Our aim is now to prove a uniform (in  $\epsilon$ ) convergence result for the following micro-macro scheme, which is nothing but the Euler-Maruyama method applied to (3.3, 3.4)

$$\underline{X}_{n+1} = \underline{X}_n + h \langle f \rangle(\underline{X}_n) + \sqrt{h} \sigma(\underline{X}_n) \xi, \quad \underline{X}_0 = X_0, \quad (3.5)$$

$$Y_{n+1} = Y_n + h \left( f_{t/\epsilon}(\Phi_{t_n/\epsilon}(\underline{X}_n) + Y) - f_{t_n/\epsilon}(\underline{X}_n) \right) \quad (3.6)$$

$$- \epsilon h \left( F'_{t_n/\epsilon}(\underline{X}_n) \langle f \rangle(\underline{X}_n) + \frac{1}{2} \sum_{r=1}^m F''_{t_n/\epsilon}(\underline{X}_n) (\sigma(\underline{X}_n) e_r, \sigma(\underline{X}_n) e_r) \right)$$

$$+ \sqrt{h} \left( \sigma(\Phi_{t_n/\epsilon}(\underline{X}_n) + Y_n) - \sigma(\underline{X}_n) - \epsilon F'_{t_n/\epsilon}(\underline{X}_n) \sigma(\underline{X}_n) \right) \xi, \quad Y_0 = 0,$$

where the increment  $\xi$  is a random quantity sampled from a normalized Gaussian centered at zero and with variance 1.

### 3.1. Main result.

**Theorem 3.1.** *Consider the Euler-Maruyama scheme (3.5, 3.6) for solving the micro-macro system (3.3, 3.4) and let*

$$X_n = \Phi_{t_n/\epsilon}(\underline{X}_n) + Y_n$$

for  $n = 0, \dots, N$  where  $T = Nh > 0$  is a given final time. Let  $X(t)$  be the solution of (1.1). Then, for all  $\phi \in C_{poly}^4$ , there exists  $C > 0$  such that, one has for all  $h$  and  $\epsilon \in (0, 1]$ ,

$$\forall n \in \{0, \dots, N\}, \quad |\mathbb{E}(\phi(X(t_n))) - \mathbb{E}(\phi(X_n))| \leq Ch, \quad (3.7)$$

$$\forall n \in \{0, \dots, N\}, \quad \mathbb{E}(|X(t_n) - X_n|^2)^{\frac{1}{2}} \leq Ch^{\frac{1}{2}}. \quad (3.8)$$

Again, the proof of the theorem follows the usual steps from [17]. Note that the main novelty of our result lies in the fact that estimate (3.7) is uniform w.r.t.  $\epsilon \in (0, 1]$ .

**Lemma 3.2.** *There exists  $c > 0$  such that for all  $\epsilon \in (0, 1]$  and all  $t \in [0, T]$ , one has*

$$\mathbb{E}(|Y(t)|) \leq c\epsilon.$$

*Proof.* First, note that we have for all  $t \in [0, T]$  and all positive integer  $m$  (see [12])

$$\mathbb{E}(|\underline{X}(t)|^{2m}) \leq K(1 + |X_0|^{2m}).$$

Moreover, we have

$$\begin{aligned} \mathbb{E}(|Y(t)|^2) &\leq 2t\mathbb{E} \left( \int_0^t \left| f_{s/\epsilon}(\Phi_{s/\epsilon}(\underline{X}(s)) + Y(s)) - f_{s/\epsilon}(\underline{X}(s)) \right. \right. \\ &\quad \left. \left. - \epsilon \left( F'_{s/\epsilon}(\underline{X}(s)) \langle f \rangle (\underline{X}(s)) + \frac{1}{2} \sum_{r=1}^m F''_{s/\epsilon}(\underline{X}(s)) (\sigma(\underline{X}(s)) e_r, \sigma(\underline{X}(s)) e_r) \right) \right|^2 ds \right) \\ &\quad + 2\mathbb{E} \left( \int_0^t \left| \sigma(\Phi_{s/\epsilon}(\underline{X}(s)) + Y(s)) - \sigma(\underline{X}(s)) - \epsilon F'_{s/\epsilon}(\underline{X}(s)) \sigma(\underline{X}(s)) \right|^2 ds \right) \\ &\leq 4t \int_0^t \mathbb{E} \left( \left| f_{s/\epsilon}(\Phi_{s/\epsilon}(\underline{X}(s)) + Y(s)) - f_{s/\epsilon}(\underline{X}(s)) \right|^2 \right) ds \\ &\quad + 4t\epsilon^2 \int_0^t \mathbb{E} \left( \left| F'_{s/\epsilon}(\underline{X}(s)) \langle f \rangle (\underline{X}(s)) + \frac{1}{2} \sum_{r=1}^m F''_{s/\epsilon}(\underline{X}(s)) (\sigma(\underline{X}(s)) e_r, \sigma(\underline{X}(s)) e_r) \right|^2 \right) ds \\ &\quad + 4 \int_0^t \mathbb{E} \left( \left| \sigma(\Phi_{s/\epsilon}(\underline{X}(s)) + Y(s)) - \sigma(\underline{X}(s)) \right|^2 \right) ds + 4\epsilon^2 \int_0^t \mathbb{E} \left( \left| F'_{s/\epsilon}(\underline{X}(s)) \sigma(\underline{X}(s)) \right|^2 \right) ds \\ &:= 4tA + 4t\epsilon^2B + 4C + 4\epsilon^2D. \end{aligned}$$

In the first inequality, we have used Cauchy-Schwarz inequality for the first term and Itô isometry for the second one. We have also used the triangular inequality as well as Young's inequality. Given that  $\mathbb{E}(|\underline{X}(s)|^{2m}) < +\infty$  and that the components of  $F_\theta$  are in  $C_{poly}^2$  by Assumption 1.2, we can conclude that the terms  $B$  and  $D$  are uniformly bounded. Now, by definition of the change of variables (3.1) and the uniform Lipschitz continuity of  $f_\theta$  and  $\sigma$ ,



we get

$$\begin{aligned} 4tA + 4C &\leq 4tL^2 \int_0^t \mathbb{E}(|\epsilon F'_{s/\epsilon}(\underline{X}(s)) + Y(s)|^2) + 4L^2 \int_0^t \mathbb{E}(|\epsilon F'_{s/\epsilon}(\underline{X}(s)) + Y(s)|^2) \\ &\leq 8L^2(T+1)\epsilon^2 \int_0^t \mathbb{E}(|F'_{s/\epsilon}(\underline{X}(s))|^2) ds + 8L^2(T+1) \int_0^t \mathbb{E}(|Y(s)|^2) ds \\ &\leq C_1\epsilon^2 + C_2 \int_0^t \mathbb{E}(|Y(s)|^2) ds. \end{aligned}$$

Therefore,  $\mathbb{E}(|Y(t)|^2)$  satisfies the following inequality

$$\mathbb{E}(|Y(t)|^2) \leq P\epsilon^2 + Q \int_0^t \mathbb{E}(|Y(s)|^2) ds.$$

Gronwall's lemma imply that  $\mathbb{E}(|Y(s)|^2) \leq c^2\epsilon^2$ . Finally,  $\mathbb{E}(|Y(s)|) \leq \mathbb{E}(|Y(s)|^2) = c^2\epsilon^2$ , thus  $\mathbb{E}(|Y(s)|) \leq c\epsilon$ .  $\square$

*Proof of Theorem 3.1.* We begin by the proof of the **weak convergence**: Collecting the variables  $\underline{X}$  and  $Y$  into  $Z = (\underline{X}, Y)$  and similarly  $z = (\underline{x}, y)$ , we may rewrite the micro-macro system (3.3, 3.4) as

$$dZ = g_{t/\epsilon}(Z)dt + \sigma_{t/\epsilon}(Z)dW$$

where

$$g_\theta(z) = \begin{pmatrix} \langle f \rangle(\underline{x}) \\ f_\theta(\Phi_\theta(\underline{x}) + y) - f_\theta(\underline{x}) - \epsilon F'_\theta(\underline{x}) \langle f \rangle(\underline{x}) - \frac{\epsilon}{2} \sum_{r=1}^m F''_\theta(\underline{x}) (\sigma(\underline{x})e_r, \sigma(\underline{x})e_r) \end{pmatrix}$$

and

$$\Sigma_\theta(z) = \begin{pmatrix} \sigma(\underline{x}) \\ \sigma(\Phi_\theta(\underline{x}) + y) - \sigma(\underline{x}) - \epsilon F'_\theta(\underline{x})\sigma(\underline{x}) \end{pmatrix}$$

and accordingly equations (3.5), (3.6) as

$$Z_{n+1} = Z_n + hg_{t_n/\epsilon}(Z_n) + \sqrt{h}\Sigma_{t_n/\epsilon}(Z_n)\xi_n$$

where the  $\xi_n \sim \mathcal{N}(0, I_m)$  are independent Gaussian random variables.

**Step 1.** Since the vector fields  $g_{t/\epsilon}$  and  $\Sigma_{t/\epsilon}$  are uniformly Lipschitz continuous (this follows from the uniform Lipschitz continuity of  $f$ ,  $\sigma$ ,  $\Phi$ ,  $F$ ,  $F'$ , and  $F''$ ), the proof of the boundedness of the moments of the numerical solution can be obtained following the same arguments as in Milstein's lemma for the non-oscillating case ( $\epsilon = 1$ ). As a matter of fact, we have that

- (i)  $\mathbb{E}(|Z_0|^{2m}) < +\infty$  since  $Z_0 = (X_0, 0) \in \mathbb{R}^{2d}$  (deterministic);
- (ii)  $|\mathbb{E}(Z_{n+1} - Z_n \mid Z_n = z)| = h|g_{t_n/\epsilon}(z)| \leq K(1 + |z|)h$ ;
- (iii)  $|Z_{n+1} - Z_n| \leq h|g_{t_n/\epsilon}(Z_n)| + |\Sigma_{t_n/\epsilon}(Z_n)| |\Delta W_n| \leq M_n(1 + |Z_n|)\sqrt{h}$ , with  $\Delta W_n = W(t_{n+1}) - W(t_n) \sim \mathcal{N}(0, h)$  and  $M_n = K|\frac{\Delta W_n}{\sqrt{h}}|$  has clearly bounded moments uniformly in  $n$ .

We have used the fact that  $g$  and  $\Sigma$  grow linearly in  $Z$  as a consequence of their Lipschitz continuity. Under the above conditions, Lemma 2.2 from [15, p. 102] implies the boundedness of the moments of arbitrary order of  $Z_n$ .

**Step 2.** We have

$$\begin{aligned} \mathbb{E}(\phi(Z_{n+1})|Z_n = z) &\leq \phi(z) + h(\mathcal{L}_{t_n/\epsilon}\phi)(z) + Ch^2 \\ &+ \frac{1}{6}\mathbb{E}\left(\int_0^1 (1-\tau)^3 \phi^{(4)}(z + \tau(Z_{n+1} - z))d\tau (Z_{n+1} - z)^4 \Big| Z_n = z\right). \end{aligned}$$

where, as in the proof of Theorem 2.1, the constant  $C$  is independent of  $\epsilon$  and where the term  $Ch^2$  comes from the remaining expectations of the second and third derivatives of  $\phi$  applied repeatedly to  $hg + \sqrt{h}\Sigma\xi_n$ , which are zero for odd moments of  $\xi_n$ , and bounded by  $Ch^2$  otherwise. For the part of the remainder coming from the above first order Taylor expansion of the expectation of  $\phi(Z_{n+1})$ , it is clearly bounded by  $Ch^2$ , where the constant  $C$  is independent of  $\epsilon$ . This stems from the polynomial growth of the test function  $\phi$  and its derivatives up to order 4, as well as from the bound  $|Z_{n+1} - Z_n| \leq M_n(1 + |Z_n|)\sqrt{h}$ .

For the part of the remainder coming from the first order Taylor expansion of the expectation of the test function  $\phi$  applied to the exact solution, it is of the form (see [14])

$$\left| \mathbb{E}\left(\int_{t_n}^{t_{n+1}} \left(\int_{t_n}^s (\mathcal{L}_{\tau/\epsilon}^2 \phi)(Z(\tau))d\tau\right) ds \Big| Z(t_n) = z\right) \right|,$$

where, for a test function  $\psi$  that depend explicitly on  $\theta$  and  $z$ ,

$$(\mathcal{L}_\theta \psi)(z) = \partial_\theta \psi(\theta, z) + \partial_z \psi(\theta, z)g_\theta(z) + \frac{1}{2} \sum_{r=1}^m \partial_z^2 \psi(\theta, z) (\Sigma_\theta(z)e_r, \Sigma_\theta(z)e_r). \quad (3.9)$$

Let us notice that the time derivative will not appear in  $\mathcal{L}_{t_n/\epsilon}\phi$  because our test function  $\phi$  does not depend explicitly on time. However, it will appear in  $\mathcal{L}_{t_n/\epsilon}^2\phi$  owing to the explicit dependence of  $g_{t/\epsilon}$  and  $\Sigma_{t/\epsilon}$  on  $t$ .

Let  $g_\theta^1$  and  $g_\theta^2$  be the two components of  $g_\theta$  (both belong to  $\mathbb{R}^d$ ), then  $\partial_\theta g_\theta(z) = (\partial_\theta g_\theta^1(z), \partial_\theta g_\theta^2(z))^T = (0, \partial_\theta g_\theta^2(z))^T$ , with

$$\begin{aligned} \partial_\theta(g_\theta^2(z)) &= \partial_\theta(g_\theta^2(\underline{x}, y)) = \frac{1}{\epsilon} \partial_\theta f_{t/\epsilon}(\Phi_{t/\epsilon}(\underline{x}) + y) + \frac{1}{\epsilon} f'_{t/\epsilon}(\Phi_{t/\epsilon}(\underline{x}) + y) \partial_\theta \Phi_{t/\epsilon}(\underline{x}) \\ &- \frac{1}{\epsilon} \partial_\theta f_{t/\epsilon}(\underline{x}) - \partial_\theta F'_{t/\epsilon}(\underline{x}) \langle f \rangle(\underline{x}) \\ &- \frac{1}{2} \sum_{r=1}^m \partial_\theta F''_{t/\epsilon}(\underline{x})(\sigma(\underline{x})e_r, \sigma(\underline{x})e_r). \end{aligned}$$

Using the same notations for  $\Sigma_\theta(z)$ , we have

$$\partial_\theta(\Sigma_{t/\epsilon}^2(z)) = \frac{1}{\epsilon} \sigma'(\Phi_{t/\epsilon}(\underline{x}) + y) \partial_\theta \Phi_{t/\epsilon}(\underline{x}) - \partial_\theta F'_{t/\epsilon}(\underline{x}) \sigma(\underline{x}).$$

It remains to bound the difference

$$\frac{1}{\epsilon} \partial_\theta f_{\tau/\epsilon}(\Phi_{\tau/\epsilon}(\underline{X}(\tau)) + Y(\tau)) - \frac{1}{\epsilon} \partial_\theta f_{\tau/\epsilon}(\underline{X}(\tau)), \quad (3.10)$$

since all other terms are bounded independently of  $\epsilon$  thanks to the smoothnes of  $f_\theta$ ,  $\sigma$  and their derivatives with respect to the space variable, and using the fact that  $\partial_\theta \Phi_{\tau/\epsilon}(X) = \mathcal{O}(\epsilon)$ . Now, taking into account the uniform Lipschitz continuity of  $\partial_\theta f_\theta$ , the polynomial growth of the test function  $\phi$ , and the above lemma, we have

$$\begin{aligned}
\left| \mathbb{E} \left( \frac{1}{\epsilon} \partial_\theta f_{\tau/\epsilon}(\Phi_{\tau/\epsilon}(\underline{X}(\tau)) + Y(\tau)) - \frac{1}{\epsilon} \partial_\theta f_{\tau/\epsilon}(\underline{X}(\tau)) \right) \right| &\leq \frac{L}{\epsilon} \mathbb{E} |\Phi_{\tau/\epsilon}(\underline{X}(\tau)) + Y(\tau) - \underline{X}(\tau)| \\
&\leq \frac{L}{\epsilon} \mathbb{E} |\epsilon F_{\tau/\epsilon}(\underline{X}(\tau))| + \frac{L}{\epsilon} \mathbb{E} |Y(\tau)| \\
&= L \mathbb{E} |F_{\tau/\epsilon}(\underline{X}(\tau))| + Lc \\
&\leq L(K(1 + \mathbb{E} |\underline{X}(\tau)|) + c) < +\infty
\end{aligned}$$

independently of the value of  $\epsilon$ . Hence, we have

$$\left| \mathbb{E} \left( \int_{t_n}^{t_{n+1}} \left( \int_{t_n}^s (\mathcal{L}_{\tau/\epsilon}^2 \phi)(Z(\tau)) d\tau \right) ds \middle| Z(t_n) = z \right) \right| \leq C'h^2,$$

and the local order 2 is proved.

**Step 3.** The boundedness of the moments and the local weak order 2 imply the global weak convergence of order 1 by a theorem from [14] (see also [15, chap2.2]).

We have proved, for any test function  $\phi \in C_{poly}^4(\mathbb{R}^{2d}, \mathbb{R})$ , for all  $n = 0, 1, \dots, N$ , that

$$|\mathbb{E}(\phi(Z(t_n)) - \phi(Z_n))| \leq Ch,$$

where  $Z = (\underline{X}, Y)$  and  $C$  is independent of  $n$  and  $\epsilon$ . We need to prove that for any test function  $\phi \in C_{poly}^4(\mathbb{R}^d, \mathbb{R})$ , for all  $n = 0, 1, \dots, N$ ,

$$|\mathbb{E}(\phi(X(t_n)) - \phi(X_n))| = |\mathbb{E}(\phi(\Phi_{t_n/\epsilon}(\underline{X}(t_n)) + Y(t_n)) - \phi(\Phi_{t_n/\epsilon}(\underline{X}_n) + Y_n))| \leq C_1 h.$$

where  $C_1$  is independent of  $n$  and  $\epsilon$ . Let  $\phi \in C_{poly}^4(\mathbb{R}^d, \mathbb{R})$ , for each fixed parameter  $\theta \in \mathbb{T}$ , we consider the test function

$$\psi_\theta(Z) = \phi(\Phi_\theta(\underline{X}) + Y).$$

Note that  $\psi_\theta \in C_{poly}^4(\mathbb{R}^d, \mathbb{R})$ , since  $\Phi_\theta \in C_{poly}^4(\mathbb{R}^d, \mathbb{R}^d)$  (by assumption on  $f_\theta$ ). Hence, for each  $n = 0, 1, \dots, N$ , we have

$$\begin{aligned}
|\mathbb{E}(\psi_{t_n/\epsilon}(Z(t_n)) - \psi_{t_n/\epsilon}(Z_n))| &= |\mathbb{E}(\phi(\Phi_{t_n/\epsilon}(\underline{X}(t_n)) + Y(t_n)) - \phi(\Phi_{t_n/\epsilon}(\underline{X}_n) + Y_n))| \\
&= |\mathbb{E}(\phi(X(t_n)) - \phi(X_n))| \leq C_1 h,
\end{aligned}$$

where  $C_1 = C$ .

**Strong convergence.** For the strong convergence with order 1/2 (for  $Z$ ), in addition to the bounded moments and the local weak order 2, using [13], it is sufficient to show first order strong convergence after one step, i.e.,

$$\mathbb{E}(|Z(t_{n+1}) - Z_{n+1}|^2; Z(t_n) = z)^{\frac{1}{2}} \leq Ch, \quad (3.11)$$

where the generic constant  $C$  is independent of  $n, h$ , and  $\epsilon$ . By Wagner-Platen expansion [14, chap1.2.2], we have

$$\begin{aligned}
Z(t_{n+1}) - Z_{n+1} &= \int_{t_n}^{t_{n+1}} \int_{t_n}^s \Lambda_{\tau/\epsilon} \Sigma_{\tau/\epsilon}(Z(\tau)) dW(\tau) dW(s) + \int_{t_n}^{t_{n+1}} \int_{t_n}^s \mathcal{L}_{\tau/\epsilon} \Sigma_{\tau/\epsilon}(Z(\tau)) d\tau dW(s) \\
&\quad + \int_{t_n}^{t_{n+1}} \int_{t_n}^s \Lambda_{\tau/\epsilon} g_{\tau/\epsilon}(Z(\tau)) dW(\tau) d(s) + \int_{t_n}^{t_{n+1}} \int_{t_n}^s \mathcal{L}_{\tau/\epsilon} g_{\tau/\epsilon}(Z(\tau)) d\tau d(s)
\end{aligned}$$

where  $\Lambda_\theta \varphi(z) = \varphi'_\theta(z) \Sigma_\theta(z)$ ,  $\mathcal{L}_\theta$  is defined in (3.9) and

$$(\Lambda_\theta \psi)(z) = \psi'(z) \Sigma_\theta(z). \quad (3.12)$$

The only terms we need to bound are the derivatives with respect to  $\theta$  of  $g_\theta(Z)$  and  $\Sigma_\theta(Z)$  at  $\theta = \tau/\epsilon$  arising from the application of the differential operator  $\mathcal{L}_{\tau/\epsilon}$  to  $g_{\tau/\epsilon}(Z)$  and  $\Sigma_{\tau/\epsilon}(Z)$  respectively. It can be checked that the other terms in the integrals are again uniformly bounded by the regularity assumptions and (3.2). Once this is done, we can conclude that (3.11) is satisfied. The uniform boundedness of  $\partial_\theta g_{\tau/\epsilon}(Z)$  was already proved in the weak convergence case. Now, we have  $\partial_\theta \Sigma_{\tau/\epsilon}(Z) = (0, \partial_\theta \Sigma_\theta^2(Z))$ , and

$$\partial_\theta \Sigma_{\tau/\epsilon}^2(Z) = \frac{1}{\epsilon} \sigma'(\Phi_{\tau/\epsilon}(\underline{X}) + Y) \partial_\theta \Phi_{\tau/\epsilon}(\underline{X}) - \partial_\theta F'_{\tau/\epsilon}(\underline{X}) \sigma(\underline{X})$$

The right hand side is uniformly bounded since  $\partial_\theta \Phi_\theta(x) = \mathcal{O}(\epsilon)$ ,  $\sigma'(x)$  and  $\partial_\theta F'_\theta(x)$  have polynomial growth, and  $\sigma$  is Lipschitz. It follows that, after several applications of Itô isometry, Cauchy-Schwarz and Young inequalities and from the upper bound

$$\mathbb{E}(|Z(t_{n+1}) - Z_{n+1}|^2) \leq Ch^2$$

that  $\mathbb{E}(|Z(t_{n+1}) - Z_{n+1}|^2)^{\frac{1}{2}} \leq Ch$ .

The strong order 1/2 for  $X$  follows straightforwardly. Indeed, we have

$$\begin{aligned} \mathbb{E}|X(t_n) - X_n| &= \mathbb{E}|\Phi_{t_n/\epsilon}(X(t_n)) + Y(t_n) - \Phi_{t_n/\epsilon}(X_n) + Y_n| \\ &\leq L' \mathbb{E}|X(t_n) - X_n| + \mathbb{E}|Y(t_n) - Y_n| \\ &\leq L'Ch^{\frac{1}{2}} + Ch^{\frac{1}{2}} = C(1 + L')h^{\frac{1}{2}}. \end{aligned}$$

□

#### 4. Numerical experiments

Throughout this section (except for the last experiment) we will focus on the Hénon-Heiles model. We consider the Hamiltonian

$$H(p, q) = \frac{p_1^2}{2\epsilon} + \frac{p_2^2}{2} + \frac{q_1^2}{2\epsilon} + \frac{q_2^2}{2} + q_1^2 q_2 - \frac{1}{3} q_2^3.$$

Let

$$\begin{cases} X_1(t) = \cos\left(\frac{t}{\epsilon}\right) q_1(t) - \sin\left(\frac{t}{\epsilon}\right) p_1(t), \\ X_2(t) = q_2(t), \\ X_3(t) = \sin\left(\frac{t}{\epsilon}\right) q_1(t) + \cos\left(\frac{t}{\epsilon}\right) p_1(t), \\ X_4(t) = p_2(t). \end{cases}$$

It can be easily checked that the variable  $X(t)$  satisfies the following ODE

$$\frac{dX}{dt}(t) = f_{t/\epsilon}(X(t)),$$

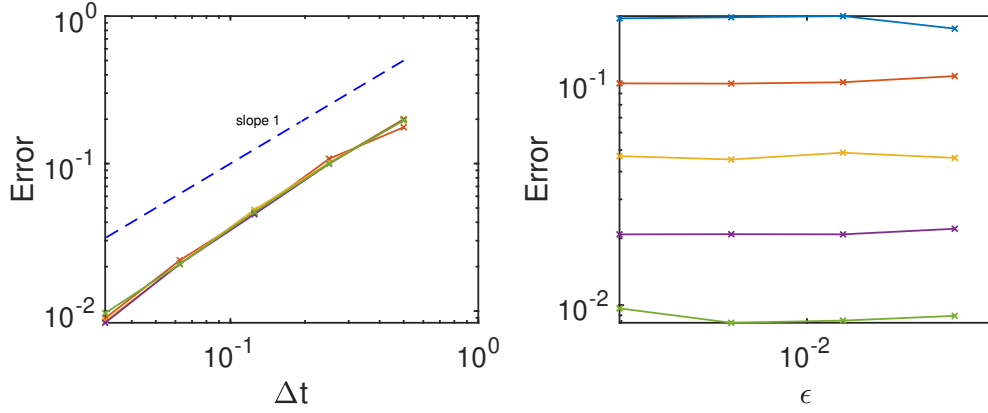
with,

$$\begin{cases} f_\theta^1(X) = 2 \sin \theta (X_1 \cos \theta + X_3 \sin \theta) X_2, \\ f_\theta^2(X) = X_4, \\ f_\theta^3(X) = -2 \cos \theta (X_1 \cos \theta + X_3 \sin \theta) X_2, \\ f_\theta^4(X) = -2(X_1 \cos \theta + X_3 \sin \theta)^2 + X_2^2 - X_2. \end{cases}$$

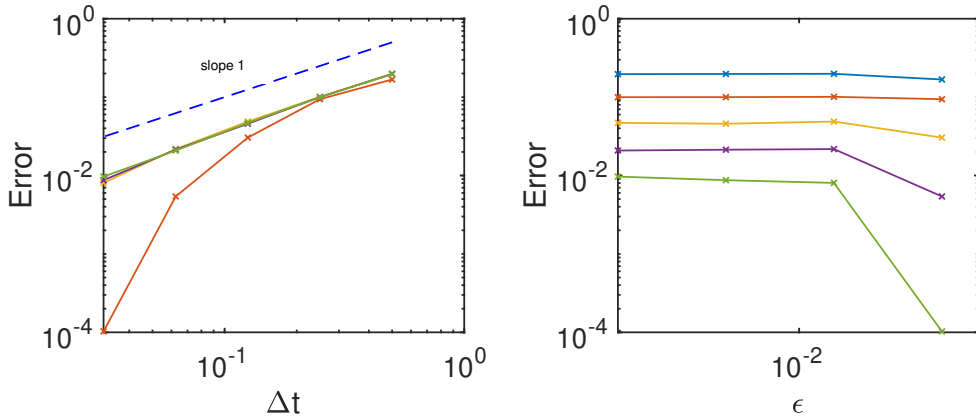
We consider the SDE

$$dX = f_{t/\epsilon}(X)dt + \sigma(X)dW(t). \quad (4.1)$$

In all our experiments  $\epsilon = 2^{-2i}$ ,  $i = 2, 3, 4, 5$ , final time  $T = 1$ ;



(A) Micro-Macro scheme.



(B) Integral scheme.

FIGURE 1. Weak convergence with multiplicative noise for the micro-macro method (3.5)-(3.6) and the integral scheme(2.1) .

4.1. **Weak convergence.** In this section we use  $X^0 = (0.7, 0.7, 0.7, 0.7)$ ,  $\Delta t = 2^{-i}$ ,  $i = 1, \dots, 5$ ,  $M = 10^4$ .

4.1.1. *Multiplicative noise.* We consider the above SDE (4.1) with multiplicative noise where  $\sigma(X) = 0.2(0, 0, X_1, X_2)^T$ . We use the test function  $\phi(X) = X_1$  to measure the weak convergence. In Figure 1A we plot the weak error with respect to the time step for different values of  $\epsilon$  (left figure) using the micro-macro method (3.5)-(3.6). We can see that the convergence behavior looks almost the same, with weak order one, for all the different values of  $\epsilon$ . The right picture of Figure 1A shows that for a fixed time step, the weak error remains almost constant when varying  $\epsilon$ . The above description applies also to the integral scheme (see Figure 1B).

4.1.2. *Additive noise.* The weak error of the micro-macro scheme (3.5)-(3.6) applied to the SDE (4.1) with additive noise is shown in Figure 2. We set  $\sigma(X) = (0, 0, 0.2, 0.2)^T$ , and we

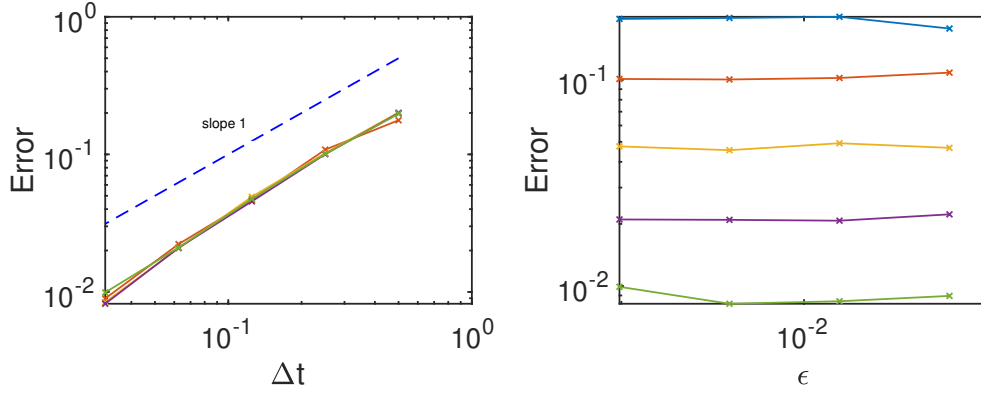
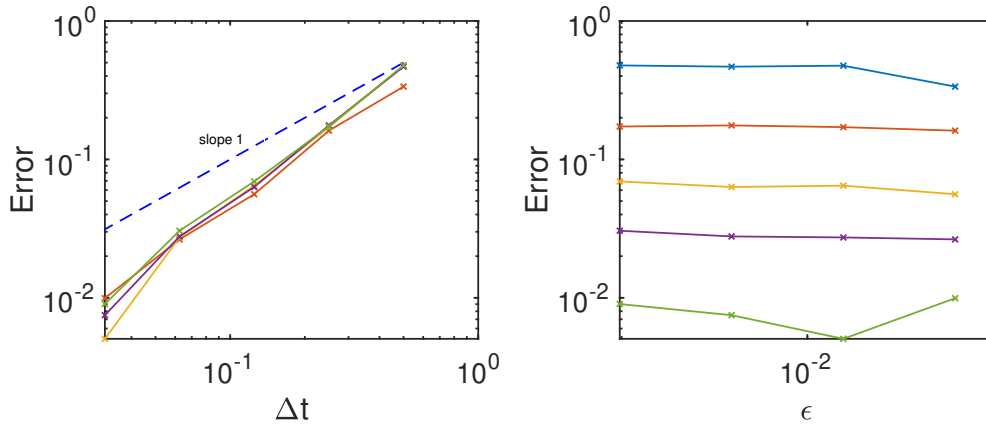

 (A)  $\phi(X) = X_1$ .

 (B)  $\phi(X) = \sum_{i=1}^4 X_i$ .

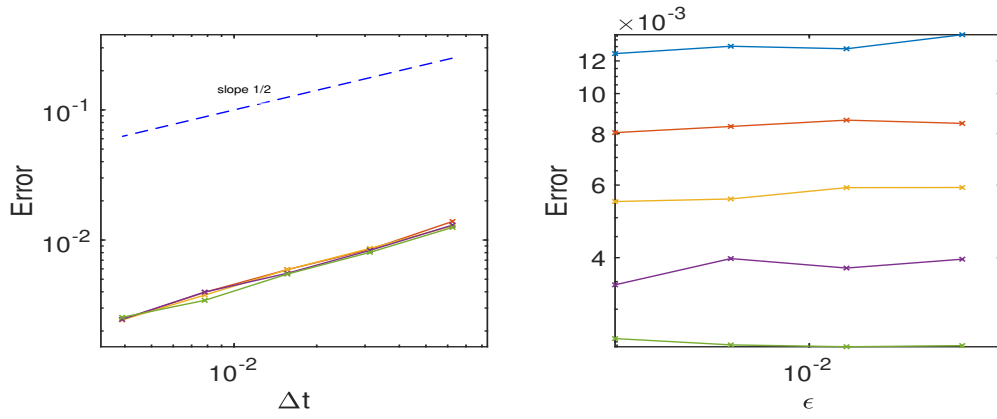
FIGURE 2. Weak convergence with additive noise for the micro-macro method (3.5)-(3.6).

perform the test with two different test functions. We see again the uniform weak order one.

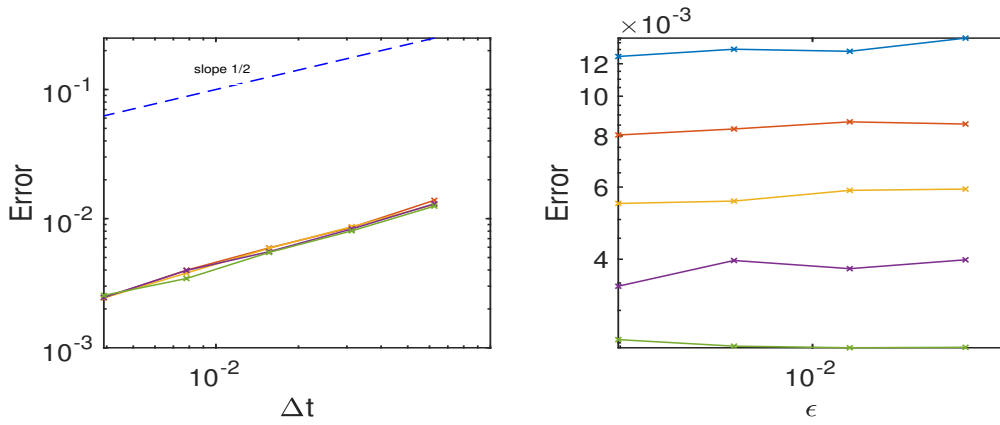
**4.2. Strong convergence.** In this section we use  $X^0 = (0.12, 0.12, 0.12, 0.12)$ ,  $\Delta t = 2^{-i}$ ,  $i = 4, \dots, 8$ ,  $M = 10^2$ .

**4.2.1. Multiplicative noise.** We consider the above SDE (4.1) with multiplicative noise where  $\sigma(X) = 0.5X$ . Figure 3 shows the uniform strong order  $\frac{1}{2}$  for both methods.

**4.2.2. Additive noise.** We consider the above SDE (4.1) with additive noise where  $\sigma(X) = (0, 0, 0.5, 0.5)^T$ . In addition to the uniform convergence, Figure 4 shows strong order one for the micro-macro method (3.5)-(3.6) since when the noise is additive, Euler-Maruyama method coincides with Milstein method of strong order one. This applies to uniformly accurate methods too.



(A) Micro-Macro.



(B) Integral scheme.

FIGURE 3. Strong convergence with multiplicative noise for methods (3.5)-(3.6) and (2.1).

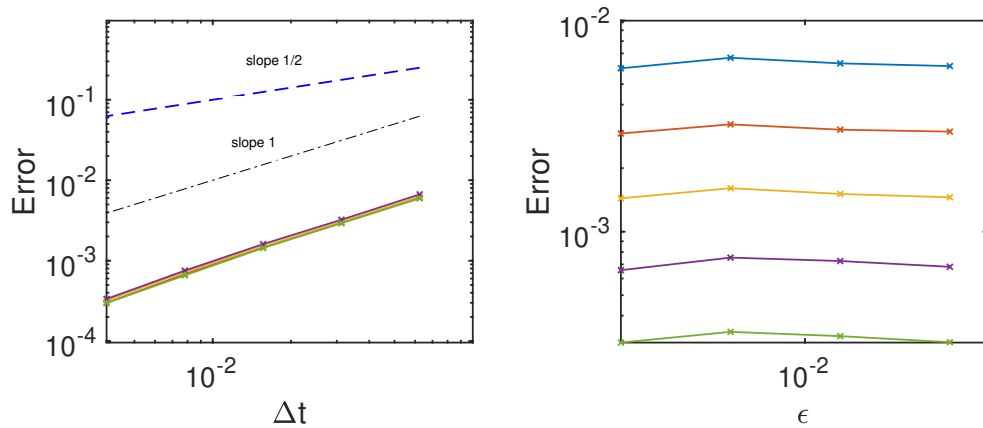


FIGURE 4. Strong convergence with additive noise for the micro-macro method (3.5)-(3.6).

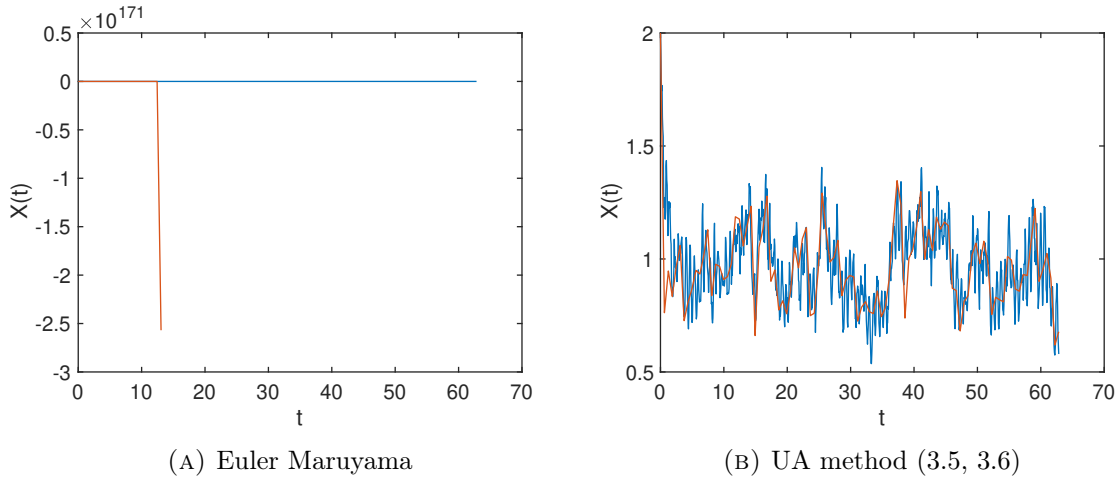


FIGURE 5. Failure of EM for particular time steps.

**4.3. Inefficiency of Euler-Maruyama method for particular time steps.** Although Euler-Maruyama method seems to work quite well, it still fails for some particular choices of time steps, while the Micro-Macro method (3.5)-(3.6) does not. We recall that the importance of UA methods appears more when using higher order schemes. See Figure 5. We consider the logistic SDE

$$dX = \left( X(1 - X) + \sin \frac{t}{\epsilon} \right) dt + 0.2X dW(t), \quad X(0) = 2. \quad (4.2)$$

We plot in Figure 5A the reference solution (in blue) calculated with very small time step using the integral scheme and the solution obtained using EM with time step  $h = 0.99(2\pi\epsilon)$ . In Figure 5B, we plot the reference solution (in blue) calculated with very small time step using the integral scheme and the solution obtained using UA (3.5)-(3.6) with time step  $h = 0.99(2\pi\epsilon)$ .

## 5. Conclusion

In this work, we have introduced two uniformly accurate methods for solving numerically oscillatory differential equations of stochastic nature. The first one is the so-called integral scheme (2.1) and can be derived quite straightforwardly, whereas the second one is obtained through a more elaborate transformation, namely a micro-macro decomposition (3.3, 3.4). Both schemes exhibit weak-order 1 and strong-order 1/2, as proved in the corresponding sections and confirmed numerically in Section 4. Given their comparable performance, the first scheme is arguably better for its simplicity. However, it is our belief that the micro-macro scheme exposed here could be generalized to higher order methods which would be the stochastic counterpart of existing deterministic UA methods [6, 8].

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