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Frédéric Havet, Dorian Mazauric, Viet-Ha Nguyen. On the complexity of overlaying a hypergraph with a graph with bounded maximum degree. [Research Report] Inria; CNRS; I3S; Université Côte d'Azur. 2021. hal-03368214v2

**HAL Id: hal-03368214**

**<https://inria.hal.science/hal-03368214v2>**

Submitted on 11 Jan 2022

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# On the complexity of overlaying a hypergraph with a graph with bounded maximum degree.

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January 11, 2022

## Abstract

Let  $G$  and  $H$  be respectively a graph and a hypergraph defined on a same set of vertices, and let  $F$  be a graph. We say that  $G$   $F$ -overlays a hyperedge  $S$  of  $H$  if the subgraph of  $G$  induced by  $S$  contains  $F$  as a spanning subgraph, and that  $G$   $F$ -overlays  $H$  if it  $F$ -overlays every hyperedge of  $H$ . For a fixed graph  $F$  and a fixed integer  $k$ , the problem  $(\Delta \leq k)$ - $F$ -OVERLAY consists in deciding whether there exists a graph with maximum degree at most  $k$  that  $F$ -overlays a given hypergraph  $H$ . In this paper, we prove that for any graph  $F$  which is neither complete nor anticomplete, there exists an integer  $\text{np}(F)$  such that  $(\Delta \leq k)$ - $F$ -OVERLAY is NP-complete for all  $k \geq \text{np}(F)$ .

## 1 Introduction

In order to obtain the low resolution structure of molecule-macro assemblies the following problem arises : given a list of complexes, determine the plausible contacts between subunits of an assembly. A convenient way of modelling this uses graphs and hypergraphs: we are given a hypergraph  $H$  whose vertices represent the subunits and whose hyperedges represents complexes; the aim is then to find a graph  $G$  on the same vertex set whose edges represent contacts between subunits and satisfying some properties.

One of the properties is that the subgraph of  $G$  induced by each hyperedge must belong to a family  $\mathcal{F}$  of admissible graphs. Precisely, a graph  $G$   $\mathcal{F}$ -overlays a hyperedge  $S$  if there exists  $F \in \mathcal{F}$  such that  $F$  is a spanning subgraph of  $G[S]$ , and  $G$   $\mathcal{F}$ -overlays  $H$  if  $G$   $\mathcal{F}$ -overlays every hyperedge of  $H$ . In a typical example, the family  $\mathcal{F}$  is the set of trees (or equivalently connected graphs) and the goal is to minimize the number of edges in  $G$ . This was studied by Agarwal et al. [1] in the aforementioned context of structural biology, but also by several authors for various applications like the design of vacuum systems [6, 7], scalable overlay networks [4, 13], and reconfigurable interconnection networks [8, 9]. Some variants have also been considered in the contexts of inferring a most likely social network [2], determining winners of combinatorial auctions [5], as well as drawing hypergraphs [3, 12].

Motivated by the fact that a subunit (e.g. a protein) cannot be connected to many other subunits, Havet et al. [10] studied the problem in which the sought graph  $G$  must have bounded maximum degree. Therefore they introduced the following problem where  $\mathcal{F}$  is a fixed family of graphs,  $k$  a fixed integer and  $\Delta(G)$  denotes the maximum degree of  $G$ .

$(\Delta \leq k)$ - $\mathcal{F}$ -OVERLAY

Input: A hypergraph  $H$ .

Question: Does there exist a graph  $G$   $\mathcal{F}$ -overlying  $H$  such that  $\Delta(G) \leq k$  ?

They studied the complexity of this problem and the associated maximization problem  $(\Delta \leq k)$ - $\mathcal{F}$ -OVERLAY and MAX  $(\Delta \leq k)$ - $\mathcal{F}$ -OVERLAY which, given a hypergraph  $H$  and an integer  $p$ , consists in deciding whether or not there exists a graph  $G$  of maximum degree at most  $k$  and  $F$ -overlays at least  $p$  hyperedges of  $H$ . Special attention was paid to the particular case when the family  $\mathcal{F}$  contains a unique

graph  $F$  and  $H$  is then an  $|F|$ -uniform hypergraph (*i.e.* every hyperedge has  $|F|$  vertices). In this case, we abbreviate  $\{F\}$ -overlay into  $F$ -overlay, and  $(\Delta \leq k)$ - $\{F\}$ -OVERLAY into  $(\Delta \leq k)$ - $F$ -OVERLAY. For convenience, a graph  $F$ -overlaying  $H$  and with maximum degree at most  $k$  is called an  $(F, H, k)$ -**graph**. Examples of  $(F, H, k)$ -graphs are given in Figure 1 when  $F$  is  $O_3$  or  $P_3$ , the graphs with three vertices and respectively one edge and two edges.

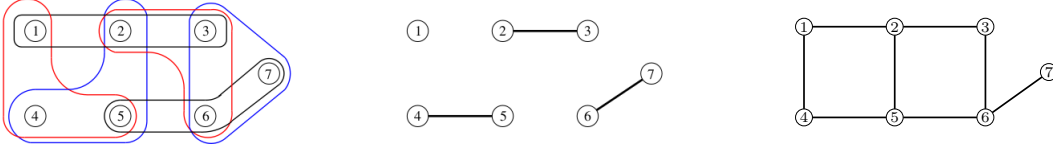


Figure 1: A hypergraph  $H$  (left), an  $(O_3, H, 1)$ -graph (middle) and a  $(P_3, H, 3)$ -graph(right).

Observe that if  $F$  is a graph with maximum degree greater than  $k$ , then solving  $(\Delta \leq k)$ - $F$ -OVERLAY or MAX  $(\Delta \leq k)$ - $F$ -OVERLAY is trivial as the answer is always ‘No’. Havet et al. [10] proved a complete polynomial/NP-complete dichotomy for MAX  $(\Delta \leq k)$ - $\mathcal{F}$ -OVERLAY depending on the pairs  $(F, k)$ . They proved that, except in a few exceptions, MAX  $(\Delta \leq k)$ - $F$ -OVERLAY is NP-complete if and only if  $\Delta(F) \leq k$ . The exceptions are when  $F$  is either an anticomplete graph  $\overline{K}_p$  or the complete graph on two vertices  $K_2$  in which case MAX  $(\Delta \leq k)$ - $F$ -OVERLAY is always polynomial-time solvable, or when  $F$  is the graph  $O_3$  with three vertices and one edge and  $k = 1$  with MAX  $(\Delta \leq 1)$ - $O_3$ -OVERLAY being polynomial-time solvable.

Regarding  $(\Delta \leq k)$ - $F$ -OVERLAY, establishing such a dichotomy seems more complicated. Indeed, Havet et al. [10] showed several pairs  $(F, k)$  (with  $\Delta(F) \leq k$ ) such that  $(\Delta \leq k)$ - $F$ -OVERLAY is polynomial-time and some such that  $(\Delta \leq k)$ - $F$ -OVERLAY is NP-complete, and posed the following problem.

**Problem 1** (Havet et al. [10]). Characterize the pairs  $(F, k)$  for which  $(\Delta \leq k)$ - $F$ -OVERLAY is polynomial-time solvable and those for which it is NP-complete.

In order to attack this problem, they propose the following conjecture.

**Conjecture 2** (Havet et al. [10]). If  $(\Delta \leq k)$ - $F$ -OVERLAY is  $\mathcal{NP}$ -complete, then  $(\Delta \leq k + 1)$ - $F$ -OVERLAY is also  $\mathcal{NP}$ -complete.

In this paper, we give some partial answers to Problem 1 and some evidences for Conjecture 2. We prove that except when  $F$  is complete or anticomplete, if  $k$  is large enough (with respect to  $F$ ), then  $(\Delta \leq k)$ - $F$ -OVERLAY is NP-complete. Recall that a graph is **complete** (resp. **anticomplete**) if its vertices are pairwise adjacent (resp. non-adjacent). The complete (resp. anticomplete) graph on  $p$  vertices is denoted by  $K_p$  (resp.  $\overline{K}_p$ ).

We define  $\text{np}(F)$  as the minimum integer  $k_0$  such that  $(\Delta \leq k)$ - $F$ -OVERLAY is NP-complete for all  $k \geq k_0$  or  $+\infty$  if no such  $k_0$  exists. The aim of this article is to prove the following theorem.

**Theorem 3.**  $\text{np}(F) = +\infty$  if and only if  $F$  is complete or anticomplete.

Let  $H$  be a  $p$ -uniform hypergraph. The anticomplete graph on  $V(H)$  vertices  $\overline{K}_p$ -overlays  $H$ . Thus, for any non-negative integer  $k$ , the answer to  $(\Delta \leq k)$ - $\overline{K}_p$ -OVERLAY is always affirmative, and so this problem can be trivially solved in polynomial time. Thus  $\text{np}(\overline{K}_p) = +\infty$  for all positive integer  $p$ .

If  $K_p$  is complete, then let  $G$  be the graph with vertex  $V(H)$  in which two vertices are adjacent if and only if they belong to a same hyperedge of  $H$ . Obviously, a graph  $K_p$ -overlays  $H$  if and only if it contains  $G$  as a subgraph. Hence, to solve  $(\Delta \leq k)$ - $K_p$ -OVERLAY it suffices to build  $G$  and to check whether  $\Delta(G) \leq k$ , which can be done in polynomial time. Thus  $\text{np}(K_p) = +\infty$  for all positive integer  $p$ .

Therefore to prove Theorem 3, it only remains to prove its sufficiency part, which is the following theorem.

**Theorem 4.** If  $F$  is neither a complete graph nor an anticomplete graph, then  $\text{np}(F) < +\infty$ .

A possibility to prove this theorem would be to prove Conjecture 2 and that, for every graph  $F$  which is not complete, there exists  $k$  such that  $(\Delta \leq k)$ - $F$ -OVERLAY is NP-complete. Unfortunately, we do not prove Conjecture 2. However, first show in Corollary 10 that it is sufficient to prove Theorem 4 for  $F$  with no isolated vertices. We then establish a weaker statement than Conjecture 2 for such graphs: in Lemma 12 we show that, for a graph  $F$  with no isolated vertices, as soon as there exists  $k$  such that  $(\Delta \leq k)$ - $F$ -OVERLAY is NP-complete, then  $\text{np}(F) < +\infty$ . This lemma, together with the following theorem, directly implies Theorem 4.

**Theorem 5.** *Let  $F$  be a graph with no isolated vertex and which is not complete. There exists  $k$  such that  $(\Delta \leq k)$ - $F$ -OVERLAY is NP-complete.*

In Section 4, we first prove Theorem 5 when  $F$  belongs to some particular classes of graphs :  $F$  is **regular** (Theorem 13), and  $F$  is a **complete graph minus an edge**, denoted by  $K^-$  (Theorem 14) and  $F$  is a **disjoint union of the complete bipartite graph**  $K_{a,a+1}$  (Theorem 15). Then, in Section 5, we prove Theorem 5 in full. Its proof requires the previously established particular cases and uses the techniques introduced in proving them. Finally, in Section 6, we give some final remarks and present some open questions for further research.

**Remark 6.** In all the paper, our aim is to prove that  $(\Delta \leq k)$ - $F$ -OVERLAY is NP-complete under some assumptions on  $k$  and  $F$ . Observe that, given an graph  $G$ , we can easily check whether  $G$   $F$ -overlays  $H$  or not in polynomial-time solvable, thus the problem is clearly in NP. Therefore, we only need to prove that the problem is NP-hard.

All our NP-hardness proofs are reductions from either (3,4)-SAT or 3-COLORABILITY of 4-regular graphs.

In (3,4)-SAT, an instance is a set of clauses, each of which being a conjunction of three literals on variables, such that every variable appears in at most 4 clauses; the problem consists in deciding whether there is a truth assignment to the variables such that every clause is satisfied. (3,4)-SAT has been proved NP-complete by Tovey [14].

3-COLORABILITY consists in deciding whether a given graph admits a proper 3-coloring. It has been proved NP-complete for 4-regular graphs by Holyer [11].

## 2 Notations and definitions

For a positive integer  $p$ , let  $[p] = \{1, \dots, p\}$ .

### 2.1 Graphs

Let  $G$  be graph. We denote by  $V(G)$  and  $E(G)$  its sets of vertices and edges, respectively. The **neighborhood** of a vertex  $v$ , denoted by  $N_G(v)$ , or simply  $N(v)$  when  $G$  is clear from the context, is the set of vertices adjacent to  $v$  and its **degree**, denoted by  $d_G(v)$  or simply  $d(v)$ , is the cardinality of  $N_G(v)$ . A vertex is **isolated** in  $G$  if it has degree 0. The minimum and maximum degree of  $G$  are respectively denoted by  $\delta(G)$  and  $\Delta(G)$ . Hence a graph  $F$  has no isolated vertices if and only if  $\delta(F) \geq 1$ . We denote by  $V_i$  (resp.  $V_{\leq i}$ ,  $V_{\geq i}$ ) the set of vertices of  $G$  that has degree exactly (resp. at most, at least)  $i$  in  $G$ .

For  $S \subseteq V(G)$ , the **subgraph induced by  $S$** , denoted by  $G[S]$ , is the graph with vertex set  $S$  and edge set  $\{uv \mid u \in S, v \in S \text{ and } uv \in E(G)\}$ .

The **degree sequence** of a graph  $F$  is the non-decreasing sequence  $\mathbf{d} = \{d_1, d_2, \dots, d_p\}$  such that there exists an ordering  $(v_1, \dots, v_p)$  of the vertices of  $F$  such that  $d(v_i) = d_i$  for all  $i \in [p]$ . We denote by  $\lambda_1 < \lambda_2 < \dots < \lambda_t$  the different values of  $\mathbf{d}$  (that are the integers  $\lambda$  in which there exists  $j$  such that  $d_j = \lambda$ ). We also denote by  $\alpha_i$  the **multiplicity** or number of occurrences of  $\lambda_i$  in  $\mathbf{d}$  :  $\alpha_i = |\{j \mid d_j = \lambda_i\}|$ . Observe that  $d_1 = \lambda_1 = \delta(F)$  and  $d_p = \lambda_t = \Delta(F)$ .

We denote by  $P_t$  the path on  $t$  vertices.

We denote by  $G_1 + G_2$  the disjoint union of the two graphs  $G_1$  and  $G_2$ .

## 2.2 Hypergraphs

Let  $H$  be a hypergraph. We denote by  $V(H)$  and  $E(H)$  its sets of vertices and hyperedges (a hyperedge is a subset of vertices of  $V(H)$ ), respectively.

A hypergraph is  $p$ -**uniform** for some  $p \in \mathbb{N}$ , if all its hyperedges have exactly  $p$  vertices. Observe that  $(\Delta \leq k)$ - $F$ -**OVERLAY** only makes sense for  $|V(F)|$ -uniform hypergraphs. Therefore in the paper, we only work with hypergraphs that are uniform, often without specifying it.

In a hypergraph  $H$ , a hyperedge  $S$  is **pendant** at a vertex  $x$ , if  $S$  is the unique hyperedge containing  $x$  for all  $v \in S \setminus \{x\}$ .

Let  $F$  be a graph,  $H$  a hypergraph, and  $G$  a graph  $F$ -overlying  $H$ . For each hyperedge  $S \in E(H)$ , one can choose a copy  $F_S$  of  $F$  which is a subgraph of  $G[S]$ . We then say that  $v$  is a  $\lambda$ -**vertex** in  $S$  if  $v$  has degree  $\lambda$  in  $F_S$ . If  $H'$  is a sub-hypergraph of  $H$  (typically a gadget in an NP-hardness proof), we denote by abbreviate  $G[V(H')]$  into  $G[H']$ . We also say that  $G$  has degree  $d$  in  $H'$  if it has degree  $d$  in  $G[H']$ .

## 3 Reduction to Theorem 5

### 3.1 Graphs with isolated vertices

**Lemma 7.** *Let  $F$  be a graph. If  $(\Delta \leq k)$ - $F$ -**OVERLAY** is NP-complete, then  $(\Delta \leq k)$ - $(F + \overline{K}_1)$ -**OVERLAY** is also NP-complete.*

*Proof.* We shall give a reduction from  $(\Delta \leq k)$ - $F$ -**OVERLAY** to  $(\Delta \leq k)$ - $(F + \overline{K}_1)$ -**OVERLAY**.

Let  $\mathbf{d}$  be the (non-decreasing) degree sequence of  $F$ , and let  $\lambda^+$  be the first non-zero value in this sequence. ( $\lambda^+ = \lambda_1$  if  $F$  has no isolated vertex, and  $\lambda^+ = \lambda_2$  otherwise.)

Let  $H$  be an  $|F|$ -uniform hypergraph. We construct an  $(|F| + 1)$ -uniform hypergraph  $H'$  as follows.

- Let  $H_1, \dots, H_t$  be  $t = \lfloor \frac{k}{\lambda^+} \rfloor |E(H)| + 1$  disjoint copies of  $H$ . We add  $V(H_i)$  to  $V(H')$  for all  $i \in [t]$ .
- For any  $S \in E(H)$ , we add a new vertex  $v_S$  to  $V(H')$ . For all  $i \in [t]$ , denoting by  $S_i$  the copy of  $S$  in  $H_i$ , we add the hyperedge  $S'_i = S_i \cup \{v_S\}$  to  $H'$ .

We shall prove that there is an  $(F, H, k)$ -graph  $G$  if and only if there exists an  $(F + \overline{K}_1, H', k)$ -graph  $G'$ .

Assume first that there is an  $(F, H, k)$ -graph. We build a graph  $G'$  by taking  $G'[H_i] = G$  for any  $i \in [t]$ . Observe that  $G'[S'_i]$  is  $(F + \overline{K}_1)$ -overlaid since  $G[S_i]$  is  $F$ -overlaid and  $v_S$  is an isolated vertex. Furthermore  $G'$  has at most degree  $k$ . Thus,  $G'$  is an  $(F + \overline{K}_1, H', k)$ -graph.

Conversely, assume that there exists an  $(F + \overline{K}_1, H', k)$ -graph  $G'$ . We will prove that there exists a copy  $H_i$  of  $H$  such that  $G'[H_i]$  is an  $(F, H, k)$ -graph. Observe that, for any  $S \in E(H)$ , the vertex  $v_S$  is either isolated or has degree at least  $\lambda^+$  in each  $G'[S'_i]$  for  $i \in [t]$ . Thus,  $v_S$  is not a 0-vertex in at most  $\lfloor \frac{k}{\lambda^+} \rfloor$  hyperedges. Since there are  $|E(H)|$  such vertices, there exists a copy  $H_i$  of  $H$  such that for any  $S \in E(H)$ ,  $v_S$  is a 0-vertex in all hyperedges  $G'[S'_i]$ . Thus  $G'[H_i]$  is an  $(F, H, k)$ -graph.  $\square$

Applying the lemma several times, we get the following.

**Corollary 8.** *Let  $F$  be a graph and  $q$  a positive integer. If  $(\Delta \leq k)$ - $F$ -**OVERLAY** is NP-complete, then  $(\Delta \leq k)$ - $(F + \overline{K}_q)$ -**OVERLAY** is also NP-complete. Hence  $\text{np}(F + \overline{K}_q) \leq \text{np}(F)$ .*

The family of graphs with isolated vertices to which this result does not apply is  $K_p + \overline{K}_q$  because  $(\Delta \leq k)$ - $K_p$ -**OVERLAY** is in P. We then need the following.

**Theorem 9.**  $\text{np}(K_p + \overline{K}_1) \leq 2p - 2$  for all  $p \geq 2$ .

*Proof.* Let  $p \geq 2$  and  $k \geq 2p - 2$ . Let  $q$  and  $r$  be the integers such that  $k = (p - 1)q + r$  with  $0 \leq r < p - 1$ . Note that  $q \geq 2$  since  $k \geq 2(p - 1)$ .

We shall prove that  $(\Delta \leq k)$ - $(K_p + \overline{K}_1)$ -**OVERLAY** is NP-complete with a reduction from 3-COLORABILITY on 4-regular graphs.

We need the following gadget. Let  $u$  be a vertex. A  $(p - 1)$ -**gadget at  $u$**  is the hypergraph  $H_u$  constructed as follows. The vertex set of  $H_u$  is the disjoint union of  $\{u, v\}$  and  $q + 1$  sets  $U_1, \dots, U_{q+1}$  of  $p - 1$  vertices, and its hyperedges are  $\{u, v\} \cup U_i$  for  $i \in [q + 1]$ .

**Claim 9.1.** Let  $H_u$  be a  $(p-1)$ -gadget at  $u$ .

- (i)  $u$  has degree at least  $p-1$  in every  $(K_p + \overline{K}_1, H_u, k)$ -graph.
- (ii) There is a  $(K_p + \overline{K}_1, H_u, k)$ -graph in which  $u$  has degree  $p-1$ .

*Proof of Claim.* (i) Let  $G_u$  be a  $(K_p + \overline{K}_1, H_u, k)$ -graph. Assume for a contradiction that  $u$  has degree less than  $p-1$  in  $G_u$ . Then  $u$  must be a 0-vertex in each  $S_i$ ,  $i \in [q+1]$ . Hence  $v$  must be adjacent to the  $p-1$  vertices of  $U_i$  in each  $S_i$ . Thus  $v$  has degree at least  $(p-1)(q+1) > k$  in  $G_u$ , a contradiction.

(ii) For  $i \in [q]$ , let  $F_i$  be a copy of  $K_p + \overline{K}_1$  in which  $u$  is isolated, and let  $F_{q+1}$  be a copy of  $K_p + \overline{K}_1$  in which  $v$  is isolated, and let  $G_u = \bigcup_{i \in [q+1]} F_i$ . Clearly,  $G_u$   $(K_p + \overline{K}_1)$ -overlays  $H_u$ ,  $v$  has degree  $q(p-1) \leq k$  in  $G_u$  and  $u$  has degree  $p-1$  in  $G_u$ . So  $G_u$  is the desired  $(K_p + \overline{K}_1, H_u, k)$ -graph.  $\diamond$

Given a 4-regular graph  $G$ , we build a  $p$ -uniform hypergraph  $H$  as follows.

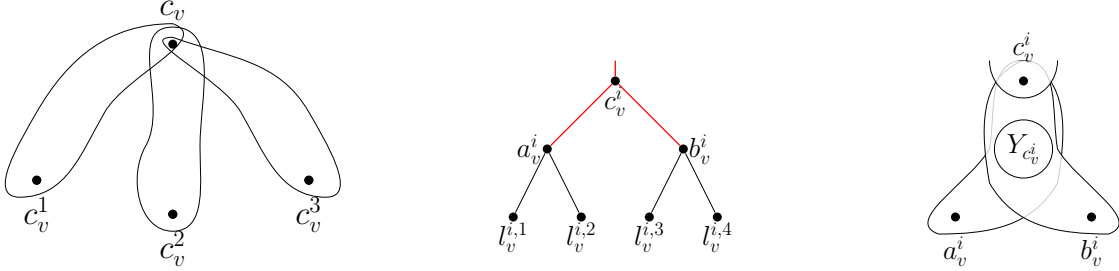


Figure 2: **Constructing the hypergraph  $H$ .** Left : the vertex gadget  $V_{G_v}$ . At each  $c_v^i$  of this gadget, we add a binary tree (center). Each pair of edges joining a vertex  $x$  to its two children in this tree is replaced by an  $x$ -edge-gadget. For example, the two red edges  $c_v^i a_v^i$  and  $c_v^i b_v^i$  are replaced by the  $c_v^i$ -edge-gadget to the right.

- For each vertex  $v \in V(G)$ , we create a *vertex gadget*  $V_{G_v}$  with three hyperedges  $S_v^i = \{c_v, c_v^i\} \cup X_v^i$  for  $i \in [3]$  where  $|X_v^i| = p-2$ . We add  $q-2$   $(p-1)$ -gadgets at  $c_v$ . We say that  $S_v^i$  is the *parent hyperedge* of  $c_v^i$  for each  $i \in [3]$ .
- For each vertex  $v$  and each  $i \in [3]$ , we construct a *color gadget*  $CG_v^i$  for  $i \in [3]$  as follows.
  - We create a binary tree  $T_v^i$  with vertex set  $\{c_v^i, a_v^i, b_v^i, l_v^{i,1}, l_v^{i,2}, l_v^{i,3}, l_v^{i,4}\}$  and edge set  $\{c_v^i a_v^i, c_v^i b_v^i, a_v^i l_v^{i,1}, a_v^i l_v^{i,2}, b_v^i l_v^{i,3}, b_v^i l_v^{i,4}\}$ , rooted at  $c_v^i$ . In this tree,  $a_v^i$  and  $b_v^i$  are the children of  $c_v^i$ ,  $l_v^{i,1}$  and  $l_v^{i,2}$  are the children of  $a_v^i$ , and  $l_v^{i,3}$  and  $l_v^{i,4}$  are the children of  $b_v^i$ .
  - For any vertex  $x \in \{c_v^i, a_v^i, b_v^i\}$ , let  $y_1, y_2$  be its children in  $T_v^i$ , and let  $e_1 = xy_1, e_2 = xy_2$ . We construct an  *$x$ -edge-gadget* as follows: we add a set  $Y_x$  of  $p-2$  new vertices, the hyperedges  $S(e_1) = \{x, y_1\} \cup Y_x$  and  $S(e_2) = \{x, y_2\} \cup Y_x$ . For convenience, we say that  $S(xy_1)$  (resp.  $S(xy_2)$ ) is the *parent hyperedge* of  $y_1$  (resp.  $y_2$ ). Moreover, for any leaf  $l_v^{i,j}$ , we denote by  $S_v^{i,j}$  the hyperedge containing the vertex  $l_v^{i,j}$ . We then add  $q-1$   $(p-1)$ -gadgets at  $x$ .
- For every vertex  $v \in V(G)$ , let  $e_v^1, e_v^2, e_v^3, e_v^4$  be an ordering of the edges incident to  $v$ . For each edge  $uv \in E(G)$ , let  $j_u$  and  $j_v$  be the indices such that  $uv = e_u^{j_u} = e_v^{j_v}$ . Then, for all  $i \in [3]$ , we identify the vertices  $l_u^{i,j_u}$  and  $l_v^{i,j_v}$  and we add  $q-1$   $(p-1)$ -gadgets at this vertex.

Let us now prove that there is a proper 3-coloring of  $G$  if and only if there is a  $(K_p + \overline{K}_1, H, k)$ -graph  $G^*$ .

Assume first that there is a  $(K_p + \overline{K}_1, H, k)$ -graph  $G^*$ .

Let  $v \in V(G)$ . By Claim 9.1 (i), the vertex  $c_v$  has degree at least  $p-1$  in each of its  $(p-1)$ -gadgets. So it has at most  $2(p-1) + r$  neighbours in  $S_v^1 \cup S_v^2 \cup S_v^3$ . But those hyperedges pairwise intersect in  $\{c_v\}$ . Thus there is  $i \in [3]$  such that  $c_v$  is a 0-vertex in  $S_v^i$ . Since there is only one 0-vertex in  $S_v^i$ ,  $c_v^i$  must be a  $(p-1)$ -vertex in  $S_v^i$ . Therefore, we can define a 3-coloring  $\phi$  by  $\phi(v) = i$  where  $i$  is an index such that  $c_v^i$  is a  $(p-1)$ -vertex in  $S_v^i$ . Let us now prove that  $\phi$  is proper. We need the following claim.

**Claim 9.2.** Let  $v \in V(G)$  and  $i \in [3]$ . If  $c_v^i$  is a  $(p-1)$ -vertex in  $S_v^i$ , then so is the leaf  $l_v^{i,j}$  in  $S_v^{i,j}$  for all  $j \in [4]$ .

*Proof of Claim.* It suffices to prove that for any  $x \in \{c_v^i, b_v^i, a_v^i\}$ , if  $x$  is a  $(p-1)$ -vertex in its parent hyperedge, then so are both  $y_1, y_2$  in their parent hyperedges.

Assume that  $x$  is a  $(p-1)$ -vertex in its parent hyperedge. Since  $x$  has degree at least  $p-1$  in each of its  $(p-1)$ -gadgets by Claim 9.1 (i), it has at most  $r$  neighbors in  $S(xy_1) \cup S(xy_2)$ . It implies that  $x$  is a 0-vertex in both  $S(xy_1), S(xy_2)$ . Hence, the vertex  $y_1$  (resp.  $y_2$ ) must be a  $(p-1)$ -vertex in  $S(xy_1)$  (resp.  $S(xy_2)$ ).  $\diamond$

Consider an edge  $uv \in E(G)$ ,  $i \in [3]$ . By Claim 9.1 (i), the vertex  $\ell = \ell_u^{i,j_u} = \ell_v^{i,j_v}$  has degree at least  $p-1$  in each of its  $q-1$   $(p-1)$ -gadgets. Thus it has at most  $(p-1) + r$  neighbors in  $S_u^{i,j_u} \cup S_v^{i,j_v}$ . As  $\ell$  is the unique common vertex of  $S_u^{i,j_u}$  and  $S_v^{i,j_v}$ , it is a  $(p-1)$ -vertex in at most one of those. Hence, by Claim 9.2, at most one of  $c_u^i, c_v^i$  is a  $(p-1)$ -vertex in its parent hyperedge. Thus at most one of  $u, v$  is colored  $i$  by  $\phi$ . Therefore,  $\phi$  is a proper 3-coloring of  $G$ .

Conversely, let  $\phi$  be a proper 3-coloring of  $G$ . We construct a graph  $G^*$  as follows.

- For any vertex gadget  $VG_v$ ,  $i \in [3]$ , let  $G^*[S_v^i]$  be a copy of  $K_p + \bar{K}_1$  in which every vertex in  $X_v^i$  is a  $(p-1)$ -vertex, and  $c_v$  is a 0-vertex (resp.  $(p-1)$ -vertex) and  $c_v^i$  is a  $(p-1)$ -vertex (resp. 0-vertex) in  $S_v^i$  if  $\phi(v) = i$  (resp.  $\phi(v) \neq i$ ).
- In every color gadget  $CG_v^i$ , for  $x \in \{c_v^i, b_v^i, a_v^i\}$  with children  $y_1$  and  $y_2$ , let  $G^*[S(xy_1)]$  and  $G^*[S(xy_2)]$  be two similar copies of  $K_p + \bar{K}_1$  such that:
  - if  $i \neq \phi(v)$ , then  $x$  has degree  $p-1$  in  $G^*[S(xy_1)]$  and  $G^*[S(xy_2)]$ ;  $y_1$  and  $y_2$  are 0-vertices in  $S(xy_1)$  and  $S(xy_2)$  respectively (so  $x$  has degree  $p-1$  in  $G^*[S(xy_1) \cup S(xy_2)]$ ).
  - if  $i = \phi(v)$ , then  $x$  has degree 0 in  $G^*[S(xy_1)]$  and  $G^*[S(xy_2)]$ ;  $y_1$  and  $y_2$  are  $(p-1)$ -vertices in  $S(xy_1)$  and  $S(xy_2)$  respectively (so  $x$  has degree at most  $p-1$  in  $G^*[S(xy_1) \cup S(xy_2)]$ ).
  - every vertex in  $Y_x$  is a  $p-1$  vertex in both  $S(xy_1)$  and  $S(xy_2)$  and so has degree at most  $p$  in  $G^*[S(xy_1) \cup S(xy_2)]$ ;
- For any  $(p-1)$ -gadget  $H_x$  at vertex some  $x$ , we let  $G^*[H_x]$  be a  $(K_p + \bar{K}_1, H_x, k)$ -graph in which  $v$  has degree  $p-1$ . Such a copy exists by Claim 9.1 (ii).

By construction,  $G^*$   $(K_p + \bar{K}_1)$ -overlays  $H$ . Let us check that  $\Delta(G^*) \leq k$ . Let  $u$  be a vertex of  $G^*$ .

- If  $u$  is in at most two hyperedges (in particular, if  $u$  is in  $X_v^i$  or  $u$  is in  $Y_x$  for  $x$  internal vertex in some  $T_v^i$  or  $u$  is only in a  $(p-1)$ -gadget), then  $u$  has degree at most  $2(p-1)$ , and so at most  $k$ .
- Assume now that  $u \in \{c_v^i, a_v^i, b_v^i\}$  for  $i \in [3]$  with  $u$  parent of  $y_1, y_2$ . Then  $u$  has degree  $p-1$  in each of its  $q-2$   $(p-1)$ -gadgets. Moreover if  $i = \phi(v)$  (resp.  $i \neq \phi(v)$ ), then  $u$  has degree  $p-1$  (resp. 0) in its parent hyperedge and  $p-1$  (resp. 0) in  $G^*[S_{uy_1}^1 \cup S_{uy_2}^1]$ . Hence  $u$  has degree at most  $(q-1)(p-1) + (p-1) = q(p-1) \leq k$ .
- Assume that  $u$  is the identification of  $\ell_v^{i,j_v}$  and  $\ell_w^{i,j_w}$  for an edge  $vw \in E(G)$ . First,  $u$  has degree  $p-1$  in each of its  $q-1$   $(p-1)$ -gadgets. Moreover, since either  $\phi(v) \neq i$  or  $\phi(w) \neq i$ , then  $u$  has degree  $p-1$  in at most one of  $S_v^{i,j_v}, S_w^{i,j_w}$  and 0 in the other. Therefore,  $u$  has degree at most  $q(p-1) \leq k$  in  $G^*$ . Consequently,  $G^*$  is a  $(K_p + \bar{K}_1, H, k)$ -graph.  $\square$

Corollary 8 and Theorem 9 directly imply the following.

**Corollary 10.** *Theorem 4 holds if and only if it holds for graphs with no isolated vertices.*

### 3.2 Reduction to Theorem 5

By Corollary 10, one can restrict our study to graphs  $F$  with  $\delta(F) \geq 1$ . We shall now prove that for such an  $F$ , we have  $\text{np}(F) \leq +\infty$  as soon as there is some  $k$  for which  $(\Delta \leq k)$ - $F$ -OVERLAY is NP-complete. To prove this, we introduce the notion of degree-gadget that will be useful in almost all the following proofs.

Let  $F$  be graph with  $\delta(F) \geq 1$ . For any integer  $d \geq \lambda_1$ , a  **$d$ -degree-gadget** (with respect to  $F$ ) at vertex  $v$ , is the subgraph  $D(d, v)$  defined as follows. Let  $\alpha = \lfloor d/\lambda_1 \rfloor$  and  $\beta = d - \alpha\lambda_1$ . If  $\beta = 0$ , then  $D(d, v)$  is the union of  $\alpha$  pendant hyperedges at  $v$ . If  $\beta \geq 1$ , then  $D(d, v)$  is the union of  $\alpha - 1$  pendant hyperedges at  $v$  and two hyperedges which intersect in  $I \cup \{v\}$  where  $I$  is a set of  $\lambda_1 - \beta$  vertices. (See Figure 3).

Degree-gadgets are useful because of the following proposition whose easy proof is left to the reader.

**Proposition 11.** *Let  $F$  be graph with  $\delta(F) \geq 1$ . Then for any  $d \geq \lambda_1$ , we have the following.*

- (i) *In any graph  $G$  that  $F$ -overlays  $D(d, v)$ , vertex  $v$  has degree at least  $d$ .*
- (ii) *There is a graph  $G_v$  that  $F$ -overlays  $D(d, v)$  in which  $v$  has degree exactly  $d$ , and every other vertex has degree at most  $\Delta(F)$  if  $\delta(F)$  divides  $d$  (i.e.  $\beta = 0$ ) and at most  $2\Delta(F) - 1$  otherwise.*

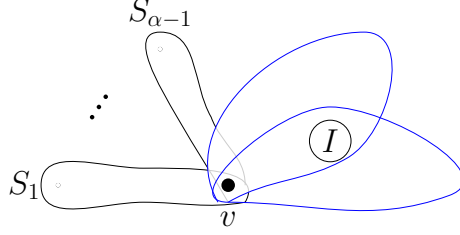


Figure 3: **A  $d$ -degree-gadget  $D$  at vertex  $v$ .** The set  $I \neq \emptyset$  is the intersection of the two blue hyperedges.  $\beta \neq 0$  when  $I$  is different from these two hyperedges; and  $\beta = 0$  when  $I$  and the two blue hyperedges are equal.

**Lemma 12.** *Let  $F$  be a graph with  $\delta(F) \geq 1$ . Assume that  $(\Delta \leq k_0)$ - $F$ -OVERLAY is NP-complete.*

- (i) *If  $\delta(F) = 1$ , then  $\text{np}(F) \leq k_0$ .*
- (ii)  *$\text{np}(F) \leq \max\{k_0 + \delta(F), 2\Delta(F) - 1\}$ .*

*Proof.* Observe that  $k_0 \geq \Delta(F)$ , because  $(\Delta \leq k)$ - $F$ -OVERLAY is trivially polynomial-time solvable for every  $k < \Delta(F)$ .

(i) Let  $k > k_0$ . We shall prove that  $(\Delta \leq k)$ - $F$ -OVERLAY is NP-complete. We give a reduction from  $(\Delta \leq k_0)$ - $F$ -OVERLAY. Let  $H_0$  be an  $|F|$ -uniform hypergraph. Let  $H$  be the hypergraph obtained from  $H_0$  by adding a  $(k - k_0)$ -degree-gadget  $DG_v$  on every vertex  $v$ . Such a degree-gadget exists because  $k - k_0 \geq 1 = \delta(F)$ . Let us prove that there is an  $(F, H_0, k_0)$ -graph  $G_0$  if and only if there exists an  $(F, H, k)$ -graph  $G$ .

Assume there is an  $(F, H_0, k_0)$ -graph  $G_0$ . By Proposition 11-(ii), for every  $v \in V(H_0)$ , there is a graph  $G_v$  that  $F$ -overlays  $DG_v$ , in which  $v$  has degree  $k - k_0$ , and every other vertex as degree at most  $\Delta(F) \leq k_0$ . Consider  $G = G_0 \cup \bigcup_{v \in V(H_0)} G_v$ . Clearly,  $G$  is an  $(F, H, k)$ -graph.

Conversely, assume that there is an  $(F, H, k)$ -graph  $G$ . By Proposition 11-(i), every vertex  $v$  of  $V(H_0)$  has degree at least  $k - k_0$  in  $DG_v$ . Thus it has degree at most  $k_0$  in  $G[H_0]$ . Therefore,  $G[H_0]$  is an  $(F, H_0, k_0)$ -graph.

(ii) The proof is identical to (i). Taking  $k \geq \max\{k_0 + \delta(F), 2\Delta(F) - 1\}$  and using the same reduction as above we get that  $(\Delta \leq k)$ - $F$ -OVERLAY is NP-complete. Note that  $(k - k_0)$ -degree-gadgets exist because  $k - k_0 \geq \delta(F)$ .  $\square$

By this lemma, in order to prove that  $\text{np}(F)$  is bounded, it suffices to prove that there exists  $k_0$  such that  $(\Delta \leq k_0)$ - $F$ -OVERLAY is NP-complete.

## 4 Particular cases

In this section, we prove the NP-completeness of  $(\Delta \leq k)$ - $F$ -OVERLAY for pairs  $(F, k)$  where  $F$  is either a **regular graph**, or a **complete graph minus an edge**  $K_p^-$  (*i.e.* it is obtained by removing an edge from  $K_p$ ) or a **disjoint union of the complete bipartite graph**  $K_{a,a+1}$ , and  $k$  is an integer (depending on  $F$ ).

### 4.1 Regular graphs

**Theorem 13.** *Let  $\lambda$  be a positive integer, and let  $F$  be a  $\lambda$ -regular graph which is not complete.*

*Then  $(\Delta \leq 6\lambda - 1)$ - $F$ -OVERLAY is NP-complete.*

*Proof.* Set  $p = |F|$ . Since  $F$  is not complete, we have  $p > \lambda + 1$ .

We give a reduction from (3,4)-SAT to  $(\Delta \leq 6\lambda - 1)$ - $F$ -OVERLAY.

Given a formula  $\Phi$  of (3,4)-SAT with  $n$  variables  $x_t, t \in [n]$ , and  $m$  clauses  $C_j, j \in [m]$ , we construct a  $p$ -uniform hypergraph  $H$  as follows.

- For each variable  $x_t$ , we construct a *variable gadget*  $H_t$  as follows. We first create a *center vertex*  $w_t$ , a set of  $4p - 4$  vertices  $U_t = \{u_t^1, \dots, u_t^{4p-4}\}$ , and  $4p - 4$  hyperedges  $S_t^j = \{w_t, u_t^j, \dots, u_t^{j+p-2}\}$  (superscripts are modulo 4) for  $j \in [4p - 4]$ . We then add a  $(2\lambda - 1)$ -degree-gadget at  $w_t$  and a  $4\lambda$ -degree-gadget on each  $u_t^{(p-1)i-j}$  for any  $i \in [4]$  and  $j \in [\lambda - 1]$ . For  $r \in [4]$  let  $x_t^r = u_t^{r(p-1)-p+2}$  and



$\bar{x}_t^r = u_t^{r(p-1)-p+3}$ . Set  $X_t = \{x_t^1, x_t^2, x_t^3, x_t^4\}$  and  $\bar{X}_t = \{\bar{x}_t^1, \bar{x}_t^2, \bar{x}_t^3, \bar{x}_t^4\}$ . The vertices of  $X_t$  (resp.  $\bar{X}_t$ ) are called the *non-negated* (resp. *negated*) literal vertices of  $H_t$ . See Figure 4.

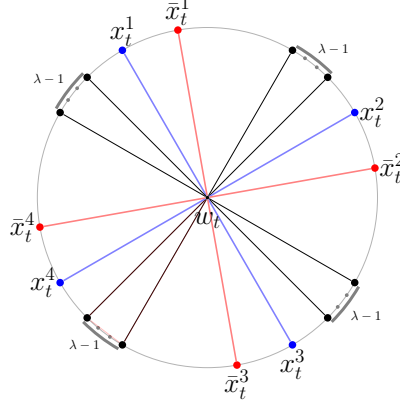


Figure 4: **The variable gadget  $H_t$ .** The center vertex  $w_t$  is in a  $(2\lambda - 1)$ -degree-gadget. There are four sets of  $\lambda - 1$  vertices (in black), each of which is adjacent to the center vertex  $w_t$  and in a  $4\lambda$ -degree-gadget. Blue and red vertices are respectively non-negated and negated literal vertices.

- For each clause  $C_j = (\ell_1 \vee \ell_2 \vee \ell_3)$  we identify  $y_1, y_2, y_3$  into a *clause vertex*  $c_j$ , where  $y_i = x_t^r$  if  $\ell_i = x_t$  and  $\ell_i$  is the  $r$ -th occurrence of  $x_t$ , and  $y_i = \bar{x}_t^r$  if  $\ell_i = \bar{x}_t$  and is the  $r$ -th occurrence of  $x_t$ .

We will prove that there exists an assignment  $\phi$  satisfying  $\Phi$  if and only if there is an  $(F, H, 6\lambda - 1)$ -graph  $G$ . The general idea is that a variable  $x_t = \text{true}$  (resp. *false*) if and only if the vertices of  $X_t$  (resp.  $\bar{X}_t$ ) have degree  $2\lambda - 1$  in  $G[H_t]$  and so they are adjacent to the center vertex while the ones of the other set are not.

Assume that there exists a truth assignment  $\phi$  satisfying  $\Phi$ . Let  $G$  be the graph obtained as follows.

For each  $t \in [n]$ , let  $(v_0, v_1, \dots, v_{p-1})$  be an ordering of  $V(F)$  such that  $N_F(v_0) = \{v_{p-\lambda+1}, \dots, v_{p-1}\} \cup \{v_1\}$  if  $\phi(x_t) = \text{true}$  and  $N_F(v_0) = \{v_{p-\lambda+1}, \dots, v_{p-1}\} \cup \{v_2\}$  if  $\phi(x_t) = \text{false}$ . For every  $j \in [4p - 4]$ , we let  $G[S_t^j]$  be the copy of  $F$  in which  $w_t$  corresponds to  $v_0$  and  $u_t^i$  for  $i \in \{j, \dots, j + p - 1\}$  corresponds to the vertex  $v_{i'}$  such that  $i \equiv i' \pmod{p-1}$ . Observe that each  $u_t^i$  corresponds to the same vertex of  $F$  in all the  $p - 1$  copies of  $F$  induced by the  $S_t^j$  to which it belongs. Therefore either  $u_t^i$  is not adjacent to  $w_t$  and it has  $2\lambda$  neighbors in  $G[H_t]$  or  $u_t^i$  is adjacent to  $w_t$  and it has  $2\lambda - 1$  neighbors in  $G[H_t]$ . In particular, if  $\phi(x_t) = \text{true}$  (resp.  $\phi(x_t) = \text{false}$ ), then all vertices of  $X_t$  (resp.  $\bar{X}_t$ ) have degree  $2\lambda - 1$  in  $G[H_t]$ . In addition, for every  $d$ -degree-gadget  $D$  at some vertex  $v$ , we let  $G[D]$  be an  $(F, D, 6\lambda - 1)$ -graph in which  $v$  has degree  $d$ .

Let us check that every vertex has degree at most  $6\lambda - 1$  in  $G$ .

- Each center vertex  $w_t$  has degree  $2\lambda - 1$  in its  $(2\lambda - 1)$ -degree-gadget and it is adjacent to  $4\lambda$  vertices in  $H_t$ , so  $6\lambda - 1$  in total.
- Every vertex in  $\{u_t^{(p-1)i-j} \mid i \in [4] \text{ and } j \in [\lambda - 1]\}$  has  $2\lambda - 1$  neighbors in  $H_t$  and  $4\lambda$  other in its  $4\lambda$ -degree-gadget. Hence its total degree is  $6\lambda - 1$ .
- Every vertex in  $U_t \setminus \{u_t^{(p-1)i-j} \mid i \in [4] \text{ and } j \in [\lambda - 1]\}$  which is not identified in a clause vertex has only neighbors in  $H_t$  and thus degree at most  $2\lambda < 6\lambda - 1$ .
- Each clause vertex is the identification of three literal vertices which have degree  $2\lambda$  or  $2\lambda - 1$  in their variable gadgets. Moreover, at least one of the literals is true, so at least one of those vertices has only  $2\lambda - 1$  neighbors in its variable gadget. Hence its degree in  $G$  is at most  $6\lambda - 1$ .

Hence,  $G$  is an  $(F, H, 6\lambda - 1)$ -graph.

Conversely, assume that  $G$  is an  $(F, H, 6\lambda - 1)$ -graph.

Consider a variable gadget  $H_t$ . The center vertex  $w_t$  has degree at least  $2\lambda - 1$  in its  $(2\lambda - 1)$ -degree-gadget, so it has at most  $4\lambda$  neighbors in  $V(H_t)$ . But  $w_t$  has degree at least  $\lambda$  in each of the  $S_t^j$ , and the hyperedges  $S_t^j, S_t^{j+p-1}, S_t^{j+2p-2}, S_t^{j+3p-3}$  pairwise intersect only in  $w_t$ . So this vertex has exactly  $\lambda$  neighbors in each of these sets, and so exactly  $\lambda$  neighbors in each  $S_t^j$ . Furthermore, if  $u_t^j$  is adjacent to  $w_t$ , then  $w_t$  has  $\lambda - 1$

neighbors in  $\{u_t^{j+1}, \dots, u_t^{j+p-2}\}$  and so  $u_t^{j+p-1}$  is adjacent to  $w_t$  because  $S_t^{j+1}$  contains  $\lambda$  neighbors of  $w_t$ . In particular, the vertices of  $X_t$  (resp.  $\bar{X}_t$ ) are either all adjacent to  $w_t$  or all non-adjacent to  $w_t$ .

Now each of the  $\lambda-1$  vertices in  $\{u_t^{p-r+2}, \dots, u_t^{p-1}\}$  is in a  $4\lambda$ -gadget in which it has degree  $4\lambda$ . Therefore, it has degree  $2\lambda-1$  in  $G[H_t]$  and must be adjacent to  $w_t$ . Hence at most one vertex in  $\{x_t^1, \bar{x}_t^1\}$  is adjacent to  $w_t$ . Thus the vertices of  $X_t$  and those of  $\bar{X}_t$  cannot be simultaneously adjacent to  $w_t$ .

Let  $\phi$  be the truth assignment defined by  $\phi(x_t) = \text{true}$  if  $w_t$  is adjacent to  $X_t$ , and  $\phi(x_t) = \text{false}$  otherwise. In any clause vertex  $c_j$ , we identified three literal vertices corresponding to the three literals. But  $c_j$  has degree at most  $6\lambda-1$ , so there is at least one literal vertex having degree  $2\lambda-1$  in its variable gadget. This implies that this literal is true. Therefore,  $\phi$  satisfies  $\Phi$ .  $\square$

## 4.2 Complete graph minus an edge

**Theorem 14.**  $(\Delta \leq 3p-1)$ - $K_p^-$ -OVERLAY is NP-complete for all  $p \geq 3$ .

*Proof.* Reduction from (3,4)-SAT. Given a formula  $\Phi$  of (3,4)-SAT with variables  $x_t, t \in [n]$  and clauses  $C_j, j \in [m]$ , we build a hypergraph  $H$  as follows.

- For each variable  $x_t$ , we add a *variable gadget*  $H_t$  containing a *center set*  $C_t$  of size  $p-2$ , a set  $U_t$  of 8 vertices  $U_t = \{u_t^1, \dots, u_t^8\}$ , and 8 hyperedges  $S_t^i = C_t \cup \{u_t^i, u_t^{i+1}\}$  (superscripts are modulo 8) for  $i \in [8]$ . Set  $X_t = \{u_t^{2i-1} \mid i \in [4]\}$  and  $\bar{X}_t = \{u_t^{2i} \mid i \in [4]\}$ . The vertices of  $X_t$  (resp.  $\bar{X}_t$ ) are called the *non-negated literal vertices* (resp. *negated literal vertices*).
- For each clause  $C_j = (\ell_1 \vee \ell_2 \vee \ell_3)$ , we add a *clause vertex*  $c_j$  in which, for each literal  $\ell_i$  which is the  $r$ -th occurrence of the variable  $x_t$ , we identify  $u_t^{2r-1}$  (resp.  $u_t^{2r}$ ) if  $\ell_i = x_t$  (resp.  $\ell_i = \bar{x}_t$ ).
- In any center set  $C_t$ , if  $p=3$ , in which case  $|C_t|=1$ , we add a  $(2p-2)$ -degree-gadget at the vertex of  $C_t$ ; if  $p \geq 4$ , we add a  $(2p-4)$ -degree-gadget at  $\max\{0, 6-p\}$  vertices of  $C_t$  and a  $(2p-5)$ -degree-gadget at  $\min\{4, 2p-8\}$  vertices among the other ones.

We will show that there is an assignment  $\phi$  satisfying  $\Phi$  if and only there is a  $(K_p^-, H, 3p-1)$ -graph  $G$ .

Assume that  $\phi$  satisfies  $\Phi$ , then we construct  $G$  as follows.

In a variable gadget  $H_t$ , for every  $i \in [8]$ , we let  $G[S_t^i]$  be a copy of  $K_p^-$  such that

- every vertex in  $C_t$  which is not in any degree-gadget is a  $(p-1)$ -vertex, and so is adjacent to all vertices of  $H_t$ ;
- if  $\phi(x_t) = \text{true}$  (resp.  $\phi(x_t) = \text{false}$ ), then each vertex in  $X_t$  (resp.  $\bar{X}_t$ ) is a  $(p-2)$ -vertex in every hyperedge containing it and each vertex in  $\bar{X}_t$  (resp.  $X_t$ ) is a  $(p-1)$ -vertex in every hyperedge containing it.
- any vertex in  $C_t$  which is in a  $d$ -degree-gadget is adjacent to all vertices in  $H_t$  except  $p+5-(3p-1-d)$  literal vertices in exactly one of the two sets  $X_t, \bar{X}_t$ .

For any  $d$ -degree-gadget  $D$  at a vertex  $v$ , let  $G[D]$  be a  $(K_p^-, D, 3p-1)$ -graph in which  $v$  has degree  $d$ .

Let us check that  $\Delta(G) \leq 3p-1$ .

- Each vertex in  $C_t$  which is not in any degree-gadget is adjacent to all vertices of  $H_t$ . So it has degree at most  $p+5 \leq 3p-1$  in  $G$ .
- Each vertex in  $C_t$  which is in a  $d$ -degree-gadget has  $d$  neighbors in its degree-gadget and is adjacent to  $3p-1-d$  vertices of  $H_t$ . So it has degree  $3p-1$  in  $G$ .
- Any literal vertex which is not identified in any clause vertex has either  $p-1$  or  $p$  neighbors in its variable gadget. Thus it has degree less than  $3p-1$ .
- Each clause vertex is the identification of three literal vertices. Each of those has degree either  $p-1$  or  $p$  in its variable gadget. Moreover at least one of the literals is true so its corresponding literal vertex has degree  $p-1$  in its variable gadget. Therefore the clause vertex has degree at most  $3p-1$ .
- Any vertex which is in a degree-gadget but in no variable gadget belongs to at most two hyperedges. Thus it has degree at most  $2(p-1) < 3p-1$ .

Hence,  $G$  is a  $(K_p^-, H, 3p-1)$ -graph.

Conversely, assume that  $G$  is a  $(K_p^-, H, 3p-1)$ -graph. For every hyperedge  $S$  of  $H$ , let  $F_S$  be a subgraph of  $G[S]$  isomorphic to  $K_p^-$ .

**Claim 14.1.** For every  $t \in [n]$ , we have the following:

- (i)  $G[C_t]$  is complete.

- (i) *There are exactly four non-edges between  $X_t \cup \overline{X}_t$  and  $C_t$ . Moreover, either each vertex of  $X_t$  is incident to one of those non-edges, or each vertex of  $\overline{X}_t$  is incident to one of those non-edges.*

*Proof of Claim.* Assume that  $G[C_t]$  is not complete, then there is an edge  $uv \notin G[C_t]$  for  $u, v \in V(C_t)$ . Since all hyperedges of  $H_t$  are  $K_p^-$ -overlaid the edge  $uv$  is the only one missing in each subgraph  $G[S_t^i]$ ,  $i \in [8]$ . Thus  $u_t^i$  has degree  $p$  in  $G[H_t]$ . Therefore every vertex of  $C_t$  is adjacent to all vertices of  $X_t \cup \overline{X}_t$ . A vertex  $z$  of  $C_t$  is in a  $d$ -degree-gadget with  $d \geq 2p - 5$  so it has at least  $2p - 5$  neighbors in this gadget. It is adjacent to at least  $p - 3$  vertices in  $C_t$  and the eight of  $X_t \cup \overline{X}_t$ . So it has degree at least  $3p$ , a contradiction. This proves (i)

(ii) Consider a vertex of  $C_t$  that is in a  $(2p - 6 + i)$ -degree-gadget. It has degree at least  $(2p - 6 + i)$  in its gadget and  $p - 1$  in  $C_t$  by (i). Hence it has at most  $8 - i$  neighbors in  $X_t \cup \overline{X}_t$  and thus is non-adjacent to  $i$  vertices in  $X_t \cup \overline{X}_t$ ; Hence if  $p = 3$  then the vertex of  $C_t$  is non-adjacent to four vertices in  $X_t \cup \overline{X}_t$ ; if  $p = 4$ , then two vertices of  $C_t$  are non-adjacent to two vertices in  $X_t \cup \overline{X}_t$  each; if  $p = 5$ , then the one vertex of  $C_t$  non-adjacent to two vertices in  $X_t \cup \overline{X}_t$ , and two other vertices are non-adjacent to one vertex in  $X_t \cup \overline{X}_t$  each; if  $p \geq 6$ , then four vertices of  $C_t$  are non-adjacent to one vertex in  $X_t \cup \overline{X}_t$  each. In all cases, there four non-edges between  $X_t \cup \overline{X}_t$  and  $C_t$ . Now since every  $G[S_t^i]$  has at most one non-edge, there are exactly four non-edges between  $X_t \cup \overline{X}_t$  and  $C_t$ , and each vertex of  $X_t$  is incident to one of those non-edges, or each vertex of  $\overline{X}_t$  is incident to one of those non-edges. This proves (ii).  $\diamond$

By Claim 14.1, we define a truth assignment  $\phi$  by  $\phi(x_t) = \text{true}$  (resp.  $\phi(x_t) = \text{false}$ ) if th four non-edges between  $X_t \cup \overline{X}_t$  and  $C_t$  are. incident to vertices of  $X_t$  (resp.  $\overline{X}_t$ ). Observe that a literal vertex has degree  $p - 1$  (resp.  $p$ ) in  $H_t$  if its corresponding literal is true (resp. false).

A clause vertex  $c_j$  is the identification of three literal vertices. Since it has degree at most  $3p - 1$ , then at least one of those literal vertices has degree at most  $p - 1$  in its variable gadget. Thus this vertex corresponds to a true literal in the clause  $C_j$ . Therefore,  $\phi$  satisfies  $\Phi$ .  $\square$

### 4.3 Disjoint union of the complete bipartite graph $K_{a,a+1}$

In this section, we study on the family of disjoint union of the graph  $K_{a,a+1}$ . We aim to prove the following.

**Theorem 15.** *Let  $rK_{a,a+1}$  be the disjoint union of  $r$  copies of  $K_{a,a+1}$ . Then  $\text{np}(rK_{a,a+1}) \leq 3a + 5$ .*

In order prove this theorem, we first prove Theorem 16 which show that  $\text{np}(K_{a,a+1}) \leq 3a + 5$ , and then deduce it using Lemma 17.

**Theorem 16.**  $(\Delta \leq 3a + 5)$ - $K_{a,a+1}$ -OVERLAY is NP-complete.

*Proof.* Reduction from (3,4)-SAT. Given a formula  $\Phi$  of (3,4)-SAT with variables  $x_t, t \in [n]$  and clauses  $C_j, j \in [m]$ , we build a hypergraph  $H$  as follows.

- For each variable  $x_t$ , we add a *variable gadget*  $H_t$  containing a set  $C_t^1$  of size  $a$ , a set  $C_t^2$  of size  $a - 1$  and a set  $U_t$  of eight vertices  $U_t = \{u_t^1, \dots, u_t^8\}$ , and eight hyperedges  $S_t^i = C_t^1 \cup C_t^2 \cup \{u_t^i, u_t^{i+1}\}$  (superscripts are modulo 8) for  $i \in [8]$ . Set  $X_t = \{u_t^1, u_t^3, u_t^5, u_t^7\}$  and  $\overline{X}_t = \{u_t^2, u_t^4, u_t^6, u_t^8\}$ . The vertices of  $X_t$  (resp.  $\overline{X}_t$ ) are called the *non-negated literal vertices* (resp. *negated literal vertices*).
- For each clause  $C_j = (\ell_1 \vee \ell_2 \vee \ell_3)$ , we add a *clause vertex*  $c_j$  in which, for each literal  $\ell_i$  which is the  $r$ th occurrence of the variable  $x_t$ , we identify  $u_t^{2r-1}$  (resp.  $u_t^{2r}$ ) if  $\ell_i = x_t$  (resp.  $\ell_i = \bar{x}_t$ ).
- We add degree-gadgets on some vertices:
  - we add a  $(2a + 2)$ -degree-gadget at each of vertices in  $C_t^1$ .
  - we add a  $(2a + 1)$ -degree-gadget at each of vertices in  $C_t^2$ .

We will show that there is an assignment  $\phi$  satisfying  $\Phi$  if and only there is a  $(K_{a,a+1}, H, 3a + 5)$ -graph  $G$ .

Assume that  $\phi$  satisfies  $\Phi$ , then we construct  $G$  as follows.

In a variable gadget  $H_t$ , for every  $i \in [8]$ , we let  $G[S_t^i]$  be a copy of  $K_{a,a+1}$  such that

- every vertex in  $C_t^1$  is an  $a$ -vertex and each vertex in  $C_t^2$  is an  $(a + 1)$ -vertex (so  $G[C_t]$  is  $K_{a,a-1}$  with partition  $(C_t^1, C_t^2)$ );

- if  $\phi(x_t) = true$  (resp.  $\phi(x_t) = false$ ), then each vertex in  $X_t$  (resp.  $\overline{X}_t$ ) is an  $a$ -vertex in every hyperedge containing it and each vertex in  $\overline{X}_t$  (resp.  $X_t$ ) is an  $(a + 1)$ -vertex in every hyperedge containing it.

For any  $d$ -degree-gadget  $D$  at a vertex  $v$ , let  $G[D]$  be a  $(K_{a,a+1}, D, 3a + 5)$ -graph in which  $v$  has degree  $d$ .

Let us check that  $\Delta(G) \leq 3a + 5$ .

- Each vertex in  $C_t^1$  has degree  $2a + 2$  in its  $(2a + 2)$ -degree-gadget. It is also adjacent to the  $a - 1$  vertices in  $C_t^2$ , and to the four vertices of exactly one of the two sets  $X_t, \overline{X}_t$ . Thus, this vertex has degree  $3a + 5$  in  $G$ .
- Each vertex in  $C_t^2$  has degree  $2a + 1$  in its  $(2a + 1)$ -degree-gadget. It is also adjacent to the  $a$  vertices in  $C_t^1$  and to the four vertices in exactly one of sets  $X_t, \overline{X}_t$ . Hence, it has degree  $3a + 5$ .
- Any literal vertex which is not identified in any clause vertex has degree at most  $a + 2$  in its variable gadget. So, it has degree  $a + 2 < 3a + 5$  in  $G$ .
- Each clause vertex is the identification of three literal vertices. Each of those has degree either  $a + 1$  or  $a + 2$  in its variable gadget. Moreover at least one of the literals is true so its corresponding literal vertex has degree  $a + 1$  in its variable gadget. Therefore the clause vertex has degree at most  $2(a + 2) + a + 1 = 3a + 5$ .
- Any vertex which is in a degree-gadget but in no variable gadget has degree at most  $2(a + 1) < 3a + 5$  since it belongs to at most two hyperedges.

Hence,  $G$  is a  $(K_{a,a+1}, H, 3a + 5)$ -graph.

Conversely, assume that  $G$  is a  $(K_{a,a+1}, H, 3a + 5)$ -graph. For every hyperedge  $S$  of  $H$ , let  $F_S$  be a subgraph of  $G[S]$  isomorphic to  $K_{a,a+1}$ . Free to remove some edges, we may assume that  $G$  is the union of the  $F_S$  over all hyperedges  $S$  of  $H$ . We have the following.

**Claim 16.1.** *For every  $t \in [n]$ , the following hold.*

- In a hyperedge of  $H_t$ , the two literal vertices cannot be both  $a$ -vertices or both  $(a + 1)$ -vertices.*
- In every hyperedge of  $H_t$ , the vertices in  $C_t^1$  are  $a$ -vertices and the vertices in  $C_t^2$  are  $(a + 1)$ -vertices.*
- The vertices of one of the two sets  $X_t, \overline{X}_t$  are  $a$ -vertices in all hyperedges of  $H_t$  to which they belong, and the vertices of the other of those sets are  $(a + 1)$ -vertices in all hyperedges of  $H_t$ .*

*Proof of Claim.* Observe that any vertex in  $C_t^1$  is in a  $(2a + 2)$ -degree-gadget, so it has degree at most  $a + 3$  in  $G[H_t]$ . Similarly, any vertex in  $C_t^2$  is in a  $(2a + 1)$ -degree-gadget, so it has degree at most  $a + 4$  in  $G[H_t]$ .

(i) Assume for a contradiction that there is  $i \in [8]$  such that  $u_t^i, u_t^{i+1}$  are both  $a$ -vertices in  $S_t^i$ . There are  $a - 1$  other  $a$ -vertices in  $S_t^i$ . Thus, at least one vertex  $v$  in  $C_t^1$  is an  $(a + 1)$ -vertex in  $S_t^i$ , and thus adjacent to  $u_t^i, u_t^{i+1}$  and the  $a - 1$  other  $a$ -vertices in  $S_t^i$  which are in  $C_t^1 \cup C_t^2$ .

Assume for a contradiction that  $v$  is adjacent to exactly  $a - 1$  vertices in  $C_t^1 \cup C_t^2$ . Then because  $v$  has degree at least  $a$  in every hyperedge, it must be adjacent to at least one literal vertex in each  $S_t^{i'}$  for all  $i' \in [8]$ . In particular  $v$  is adjacent to at least one literal vertex in  $S_t^{i+2}, S_t^{i+4}$ , and  $S_t^{i+6}$ . Hence  $v$  has degree  $a + 4$  in  $G[H_t]$ , a contradiction to the above observation.

Consequently,  $v$  is adjacent to at least  $a$  and at most  $a + 1$  vertices in  $C_t^1 \cup C_t^2$ .

- If  $v$  is adjacent to exactly  $a$  vertices in  $C_t^1 \cup C_t^2$ , then there is a vertex  $u$  in  $C_t^1 \setminus \{v\}$  which is adjacent to  $v$  since there are only  $a - 1$  vertices in  $C_t^2$ . Vertex  $u$  has degree at least  $a$  in  $S_t^i$ . Since  $v$  has degree  $a + 2$  in  $S_t^i \cup C_t^1 \cup C_t^2$ , it is adjacent to at most one vertex, among the six literal vertices  $u_t^{i+1+j}$ ,  $j \in [6]$ . Hence there are two hyperedges  $S, S'$  in  $\{S_t^{i+2}, S_t^{i+4}, S_t^{i+6}\}$  such that  $v$  is adjacent to no literal vertex of  $S$  and  $S'$ . Now in each of those two hyperedges,  $v$  has degree exactly  $a$ . Hence it must be an  $a$ -vertex, and each of its neighbors, including  $u$ , is an  $(a + 1)$ -vertex and thus is adjacent to the two literal vertices. Hence  $u$  is adjacent to at least  $a + 4$  vertices in  $G[H_t]$  (at least  $a$  in  $S_t^i$  plus the four literal vertices of  $S$  and  $S'$ ). This is a contradiction.
- If  $v$  is adjacent to  $a + 1$  vertices in  $C_t^1 \cup C_t^2$ , then there are two vertices  $u, u'$  in  $C_t^1 \setminus \{v\}$  which are adjacent to  $v$  and each of them has degree at least  $a$  in  $S_t^i$ . Since  $v$  has degree  $a + 2$  in  $S_t^i \cup C_t^1 \cup C_t^2$ , it is not adjacent to any of the six other literal vertices than  $u_t^i, u_t^{i+1}$ . Consider the three hyperedges  $S_t^{i+2}, S_t^{i+4}, S_t^{i+6}$ ;
  - if  $v$  is an  $(a + 1)$ -vertex in one of these hyperedges, then  $u, u'$  must be  $a$ -vertices and thus adjacent to the two literal vertices in this hyperedge.

- if  $v$  is an  $a$ -vertex in one of those hyperedges, then at least one of  $u, u'$  is an  $(a+1)$ -vertex in this hyperedge and is adjacent to its two literal vertex.

Thus at least one of  $u, u'$  is adjacent to at least four literal vertices in  $S_t^{i+2} \cup S_t^{i+4} \cup S_t^{i+6}$ , and so has degree at least  $a+4$ , a contradiction.

This proves that the two literal vertices of a hyperedge of  $H_t$  are not both  $a$ -vertices.

Let us now prove that the two literal vertices of a hyperedge of  $H_t$  cannot be both  $(a+1)$ -vertices. Assume for a contradiction that there is  $i \in [8]$  such that  $u_t^i, u_t^{i+1}$  are both  $(a+1)$ -vertices. Any  $a$ -vertex  $x$  in  $S_t^i$  is adjacent to  $u_t^i, u_t^{i+1}$  and at least  $a-2$  vertices in  $C_t^1 \cup C_t^2$ . If  $x$  is adjacent to exactly  $a-2$  (resp.  $a-1$ ) vertices in  $C_t^1 \cup C_t^2$ , then, since it has degree at least  $a$  in any hyperedge, it is adjacent to all six (resp. at least three) literal vertices in  $S_t^{i+2} \cup S_t^{i+4} \cup S_t^{i+6}$ . Thus  $x$  has degree  $a+6$  (resp.  $a+4$ ) in  $G[H_t]$ , a contradiction. Hence every  $a$ -vertex in  $S_t^i$  has at least  $a$  neighbors in  $C_t^1 \cup C_t^2$ .

There are  $a+1$   $a$ -vertices in  $S_t^i$ , so there must be one, say  $v$ , in  $C_t^1$ . It has degree at most  $a+3$  in  $G[H_t]$  and at least  $a+2$  in  $S_t^i$ . Thus it is adjacent to at most one literal vertex in  $S_t^{i+2} \cup S_t^{i+4} \cup S_t^{i+6}$ . Hence  $v$  is not adjacent to the literal vertices of two hyperedges  $S, S'$  in  $S_t^{i+2} \cup S_t^{i+4} \cup S_t^{i+6}$ . Thus the literal vertices of the hyperedge  $S$  (resp.  $S'$ ) are both in a same part of  $F_S$  (resp.  $F_{S'}$ ), and so they are  $(a+1)$ -vertices.

Now in each hyperedge of  $H_t$ , there are more  $a$ -vertices than  $(a+1)$ -vertices. Thus there is a vertex  $z$  which is an  $a$ -vertex in at least three hyperedges  $S_1, S_2, S_3$  in  $S_t^i \cup S_t^{i+2} \cup S_t^{i+4} \cup S_t^{i+6}$ . In any of these hyperedges at least one of the literal vertices is an  $(a+1)$ -vertex, and in at least two of them the two literal vertices are  $(a+1)$ -vertices. Hence  $z$  is adjacent to at least five literal vertices. Moreover, as above, we can show that  $z$  has at least  $a$  neighbours in  $C_t^1 \cup C_t^2$ . Thus  $z$  has degree at least  $a+5$  in  $G[H_t]$ , a contradiction. This completes the proof of (i).

(ii) Assume for a contradiction that a vertex  $w \in C_t^1$  is an  $(a+1)$ -vertex in  $S_t^i$ . By (i),  $w$  is adjacent to a literal vertex in  $S_t^i$ , and so it is adjacent to  $a$  other vertices in  $C_t^1 \cup C_t^2$ . Furthermore, by (i), in each hyperedge of  $H_t$ ,  $w$  is adjacent to a literal vertex (either to an  $a$ -vertex or an  $(a+1)$ -vertex). Thus  $w$  is adjacent to four literal vertices in  $H_t$ , and so has degree at least  $a+4$  in  $G[H_t]$ , a contradiction. Therefore the  $a$  vertices of  $C_t^1$  are  $a$ -vertices. Moreover, by (i), one of the literal vertex of each  $S_t^i$  is an  $a$ -vertex. Therefore all vertices of  $C_t^2$  must be  $(a+1)$ -vertices.

(iii) Let  $v$  be a vertex in  $C_t^1$ . It is an  $a$ -vertex in each  $S_t^i$ , so by (i) it is adjacent to one vertex in  $\{u_t^i, u_t^{i+1}\}$  for all  $i \in [8]$  and it is adjacent to the  $a-1$  vertices of  $C_t^2$ . But  $v$  has degree at most  $a+3$  in  $G[H_t]$ , so  $v$  is either adjacent to all vertices of  $X_t$  and non-adjacent to all vertices of  $\bar{X}_t$ , or non-adjacent to all vertices of  $X_t$  and adjacent to all vertices of  $\bar{X}_t$ .  $\diamond$

By Claim 16.1, we define a truth assignment  $\phi$  by  $\phi(x_t) = \text{true}$  (resp.  $\phi(x_t) = \text{false}$ ) if all vertices in  $X_t$  are  $a$ -vertices (resp.  $(a+1)$ -vertices) in the hyperedges of  $H_t$  to which they belong.

Observe that, by Claim 16.1, if a literal vertex  $u_t^i$  is an  $(a+1)$ -vertex in the hyperedges of  $H_t$  to which it belongs then it has degree at least  $a+2$  in  $G[H_t]$  because it is adjacent to the  $a$  vertices of  $C_t^1$  and the two literal vertices  $u_t^{i-1}, u_t^{i+1}$ .

A clause vertex  $c_j$  is the identification of three literal vertices. Since it has degree at most  $3a+5$ , then at least one of those literal vertices has degree at most  $a+1$  in its variable gadget. By the above observation, this vertex is an  $a$ -vertex in the hyperedges of  $H_t$  to which it belongs. Thus this vertex corresponds to a true literal in the clause  $C_j$ . Therefore,  $\phi$  satisfies  $\Phi$ .  $\square$

**Lemma 17.** *Let  $r$  be a positive integer. If  $(\Delta \leq k)$ - $K_{a,a+1}$ -OVERLAY is NP-complete, then  $(\Delta \leq k)$ - $rK_{a,a+1}$ -OVERLAY is NP-complete.*

*Proof.*  $K_{a,a+1}$  has  $a+1$  vertices of degree  $a$  and  $a$  vertices of degree  $a+1$ . Hence, in  $rK_{a,a+1}$ , there are  $r(a+1)$  vertices of degree  $a$  and  $ra$  vertices of degree  $a+1$ .

We shall give a reduction from  $(\Delta \leq k)$ - $K_{a,a+1}$ -OVERLAY to  $(\Delta \leq k)$ - $rK_{a,a+1}$ -OVERLAY.

Let  $H$  be a  $(2a+1)$ -uniform hypergraph. We construct an  $r(2a+1)$ -uniform hypergraph  $H'$  from  $H$  as follows. We create a set  $A$  of  $(r-1)(a+1)$  vertices, a set  $B$  of  $(r-1)a$  vertices, and a set  $C$  of  $2a+1$  vertices. We add the hyperedge  $S_C = A \cup B \cup C$  to  $E(H')$ , and for every hyperedge  $S$  of  $H$ , we add the hyperedge  $S' = S \cup A \cup B$  to  $E(H')$ . Finally, we add a  $(k-a)$ -degree-gadget at every vertex in  $A$  and a  $(k-a-1)$ -degree-gadget at every vertex in  $B$ .

Let us prove that there is a  $(K_{a,a+1}, H, k)$ -graph  $G$  if and only if there is an  $(rK_{a,a+1}, H', k)$ -graph  $G'$ .

Assume that  $G$  is a  $(K_{a,a+1}, H, k)$ -graph. We construct  $G'$  from  $G$  as follows. Let  $G'[H] = G[H]$ , so  $G'[S] = G[S]$  for each  $S \in E(H)$ ; let  $G'[C]$  be a copy of  $K_{a,a+1}$ ; let  $G'[A \cup B]$  be a copy of  $(r-1)K_{a,a+1}$  in which every vertex in  $A$  has degree  $a$  and every vertex in  $B$  has degree  $a+1$ ; for each  $d$ -degree-gadget  $D$  at a vertex  $v$ , let  $G'[D]$  be an  $(rK_{a,a+1}, D, k)$ -graph in which  $v$  has degree  $d$ . Clearly, for any  $S' \in E(H')$ ,  $G'[S']$  contains  $rK_{a,a+1}$  and so does  $G'[S_C]$ . Moreover, one easily checks that every vertex of  $G'$  has degree at most  $k$ . Therefore,  $G'$  is an  $(rK_{a,a+1}, H', k)$ -graph.

Assume now that there is an  $(rK_{a,a+1}, H', k)$ -graph  $G'$ . Every vertex  $v \in A$  is in a  $(k-a)$ -degree-gadget, so it has degree at most  $a$  in  $G'[V(H) \cup A \cup B \cup C]$ . Thus it must be an  $a$ -vertex in every hyperedge  $S'$  for all  $S \in E(H)$ .

Let  $v$  be a vertex in  $A$ . It is adjacent to  $a-i$  vertices in  $B$ . Then  $v$  must be adjacent to at least  $i$  vertices in  $C$  and  $i$  vertices in  $V(H)$ . Thus the degree of  $v$  is at least  $a+i$  in  $G'[V(H) \cup A \cup B \cup C]$ . Therefore  $i=0$ , so  $v$  is adjacent to  $a$  vertices in  $B$  and no vertex in  $V(H) \cup C$ .

This implies that there are  $(d-1)a(a+1)$  edges between  $A$  and  $B$ . But every vertex  $u \in B$  is in a  $(k-a-1)$ -degree-gadget, and so has degree at most  $a+1$  in  $G'[V(H) \cup A \cup B \cup C]$ . Thus, each vertex in  $B$  has  $a+1$  neighbors in  $A$ , and is adjacent to vertex in  $V(H) \cup C$ .

Consider now a hyperedge  $S' = S \cup A \cup B$ . The graph  $G'[S']$  contains  $rK_{a,a+1}$ . Since there is no edge between  $A \cup B$  and  $V(H)$ , necessarily  $G'[S]$  contains  $K_{a,a+1}$ . So  $S$  is  $K_{a,a+1}$ -overlaid by  $G'$ . Consequently,  $G = G'[V(H)]$  is a  $(K_{a,a+1}, H, k)$ -graph.  $\square$

## 5 Proof of Theorem 5

The aim of this section is to prove Theorem 5. The proof divides into four cases, Theorem 13, Theorem 18, Theorem 19 and Theorem 20 as follows.

*Proof of Theorem 5 (assuming Theorems 18, 19 and 20).* Let  $F$  be a graph with degree values  $1 \leq \delta(F) = \lambda_1 < \dots < \lambda_t = \Delta(F)$ .

If  $t=1$ , (i.e.  $F$  is regular), then we have the result by Theorem 13. Henceforth, we may assume that  $t \geq 2$ .

If there exists  $i \in [t-1]$  such that  $\lambda_{i+1} > \lambda_i + 1$ , then Theorem 18 yields the result. Henceforth, we may assume that  $\lambda_{i+1} = \lambda_i + 1$  for all  $i \in [t-1]$ .

If  $t \geq 3$ , then  $\lambda_t + \lambda_1 \geq 2\lambda_2$  and Theorem 19 yields the result. Henceforth, we may assume that  $t=2$  which we then have the result by Theorem 20.  $\square$

It thus remains to prove Theorems 18, 19 and 20.

The proofs of the first two are reductions from 3-COLORABILITY on 4-regular graphs which are similar to the one used to prove Theorem 9. Given a 4-regular graph  $G$ , we build a hypergraph  $H$  which includes, for each vertex  $v \in V(G)$ , a *vertex gadget* with three hyperedges which makes three choices of degrees on vertices  $c_v^1, c_v^2, c_v^3$  (as three colors labeled 1, 2, 3 of vertex  $v$ ) and a *color gadget* represented as a binary tree with 4 leaves which copies each choice to four (leaves) vertices in other hyperedges (with respect to four neighbors of  $v \in V(G)$ ). For any edge  $uv$ , we simply identifies the two leaves of  $u, v$ . The idea is that for a proper coloring  $c$  of  $G$ ,  $c(v)$  corresponds to a vertex  $c_v^i$  having a certain degree  $d$ ; then  $c(v) = i$  if and only if  $c_v^i$  as degree  $d$  in its vertex gadget (see Figure 5). However, the set of hyperedges which are in a color gadget of the two theorems are different, see Figure 6 in Theorem 18 and Figure 7 in Theorem 19.

**Theorem 18.** *Let  $F$  be a graph on  $p$  vertices with degree values  $1 \leq \lambda_1 < \dots < \lambda_t$ . If there exists  $i^* \in \{2, \dots, t\}$  such that  $\lambda_{i^*} \geq \lambda_{i^*-1} + 2$ , then there is  $k$  such that  $(\Delta \leq k)$ - $F$ -OVERLAY is NP-complete.*

*Proof.* Set  $k = \max\{2\lambda_t, 2\lambda_{i^*} + \lambda_{i^*-1} + \lambda_1\}$ . We give a reduction from 3-COLORABILITY on 4-regular graphs. Given a 4-regular graph  $G$ , we build a hypergraph  $H$  as follows.

- For each vertex  $v \in V(G)$ , we create a *vertex gadget*  $H_v$  with three hyperedges  $S_v^i = \{c_v, c_v^i\} \cup X_v^i \cup Y_v^i$  for  $i \in [3]$  where  $|X_v^i| = \sum_{j=1}^{i^*-1} \alpha_j - 1$ ,  $|Y_v^i| = p - |X_v^i| - 2$ . We add a  $(k - \lambda_{i^*} + 1)$ -degree-gadget at each vertex  $x \in X_v^i$  for  $i \in [3]$ , a  $(k - 2\lambda_{i^*} - \lambda_{i^*-1})$ -degree-gadget at  $c_v$ . We say that  $S_v^i$  is the *parent hyperedge* of  $c_v^i$  for each  $i \in [3]$ .

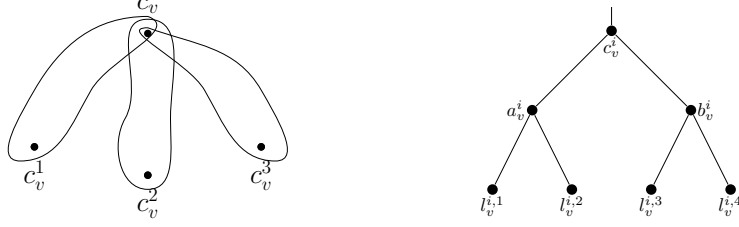


Figure 5: **The construction of the reduction.** The vertex gadget for vertex  $v$  (left) and the binary tree representing the color gadget (right). In the construction, each edge of this tree is replaced by hyperedges such that the degree of the root  $c_v^i$  is transmitted to all its descendants.

- For each vertex  $v$  and each  $i \in [3]$ , we construct a *color gadget*  $H_v^i$  for  $i \in [3]$  as follows.
  - We create a binary tree  $T_v^i$  with vertex set  $\{c_v^i, a_v^i, b_v^i, \ell_v^{i,1}, \ell_v^{i,2}, \ell_v^{i,3}, \ell_v^{i,4}\}$  and edge set  $\{c_v^i a_v^i, c_v^i b_v^i, a_v^i \ell_v^{i,1}, a_v^i \ell_v^{i,2}, b_v^i \ell_v^{i,3}, b_v^i \ell_v^{i,4}\}$ , rooted at  $c_v^i$ . In this tree,  $a_v^i$  and  $b_v^i$  are the children of  $c_v^i$ ,  $\ell_v^{i,1}$  and  $\ell_v^{i,2}$  are the children of  $a_v^i$ , and  $\ell_v^{i,3}$  and  $\ell_v^{i,4}$  are the children of  $b_v^i$ .
  - For any vertex  $x \in \{c_v^i, a_v^i, b_v^i\}$ , let  $y_1, y_2$  be its children in  $T_v^i$ , and let  $e_1 = xy_1, e_2 = xy_2$ . We first add a  $(k - 2\lambda_{i^*} + 1)$ -degree-gadget at  $x$ . Then we construct an  *$x$ -edge-gadget* as follows: we add a set  $A_x$  of  $\sum_{j=1}^{i^*-1} \alpha_j - 1$  new vertices and a set  $B_x$  of  $p - |A_x| - 2$  new vertices, the hyperedges  $S(e_1) = \{x, y_1\} \cup A_x \cup B_x$  and  $S(e_2) = \{x, y_2\} \cup A_x \cup B_x$ , and a  $(k - \lambda_{i^*} + 1)$ -degree-gadget at every vertex  $a \in A_x$ . For convenience, we say that  $S(xy_1)$  (resp.  $S(xy_2)$ ) is the parent hyperedge of  $y_1$  (resp.  $y_2$ ). Moreover, for any leaf  $\ell_v^{i,j}$ , we denote by  $S_v^{i,j}$  the hyperedge containing the vertex  $\ell_v^{i,j}$ . See Figure 6.
- For every vertex  $v \in V(G)$ , let  $e_v^1, e_v^2, e_v^3, e_v^4$ , be an ordering of the edges incident to  $v$ . For each edge  $uv \in E(G)$ , let  $j_u$  and  $j_v$  be the indices such that  $uv = e_u^{j_u} = e_v^{j_v}$ . Then, for all  $i \in [3]$ , we identify the vertices  $\ell_u^{i,j_u}$  and  $\ell_v^{i,j_v}$  and we add a  $(k - \lambda_{i^*} - \lambda_1)$ -degree-vertex at this vertex.

Note that each of the  $d$ -degree-gadgets exists because we have  $d \geq \lambda_1$  by our choice of  $k$ .

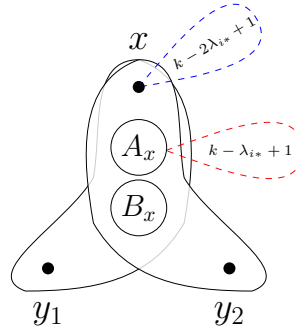


Figure 6: The  $x$ -edge-gadget with degree-gadgets at  $x$  and every vertex in  $A_x$ .

Let us now prove that there is a proper 3-coloring of  $G$  if and only if there is an  $(F, H, k)$ -graph  $G^*$ .

Assume first that there is an  $(F, H, k)$ -graph  $G^*$ .

Let  $v \in V(G)$ . The vertex  $c_v$  has degree at least  $(k - 2\lambda_{i^*} - \lambda_{i^*-1})$  in its  $(k - 2\lambda_{i^*} - \lambda_{i^*-1})$ -degree-gadget. Hence  $c_v$  has degree at most  $2\lambda_{i^*} + \lambda_{i^*-1}$  in  $S_v^1 \cup S_v^2 \cup S_v^3$ . But those hyperedges pairwise intersect in  $\{c_v\}$ . Thus there is  $i \in [3]$  such that  $c_v$  has degree less than  $\lambda_{i^*}$  in  $S_v^i$ . Moreover, since any vertex  $x \in X_v^i$  has degree at least  $k - \lambda_{i^*} + 1$  in its  $(k - \lambda_{i^*} + 1)$ -degree-gadget, so it has degree less than  $\lambda_{i^*}$  in  $S_v^i$ . Thus  $c_v^i$  must have degree at least  $\lambda_{i^*}$  in  $S_v^i$ . Therefore, we can define a 3-coloring  $\phi$  by  $\phi(v) = i$  where  $i$  is an index such that  $c_v^i$  has degree at least  $\lambda_{i^*}$  in  $S_v^i$ .

Let us now prove that  $\phi$  is proper. We need the following claim.

**Claim 18.1.** *Let  $v \in V(G)$  and  $i \in [3]$ . If  $c_v^i$  has degree at least  $\lambda_{i^*}$  in  $S_v^i$ , then so does the leaf  $\ell_v^{i,j}$  in  $S_v^{i,j}$  for all  $j \in [4]$ .*

*Proof of Claim.* It suffices to prove that for any  $x \in \{c_v^i, b_v^i, a_v^i\}$ , if  $x$  has degree at least  $\lambda_{i^*}$  in its parent hyperedge, then both  $y_1, y_2$  have degree at least  $\lambda_{i^*}$  in their parent hyperedges.

Assume that  $x$  has degree at least  $\lambda_{i^*}$  in its parent hyperedge. Since  $x$  has degree at least  $k - 2\lambda_{i^*} + 1$  in its  $(k - 2\lambda_{i^*} + 1)$ -degree-gadget, it has degree at most  $\lambda_{i^*} - 1$  in  $S(xy_1) \cup S(xy_2)$ . Moreover, any  $a \in A_x$  has degree at least  $k - \lambda_{i^*} + 1$  in its  $(k - \lambda_{i^*} + 1)$ -degree-gadget and so has degree less than  $\lambda_{i^*}$  in  $S(xy_1) \cup S(xy_2)$  and so in each of  $S(xy_1), S(xy_2)$ . Since  $A_x$  is of size  $\sum_{j=1}^{i^*-1} \alpha_1 - 1$ , the vertex  $y_1$  (resp.  $y_2$ ) must have degree at least  $\lambda_{i^*}$  in  $S(xy_1)$  (resp.  $S(xy_2)$ ).  $\diamond$

Consider an edge  $uv \in E(G)$ ,  $i \in [3]$ . The vertex  $\ell = \ell_u^{i,j_u} = \ell_v^{i,j_v}$  has degree at least  $k - \lambda_{i^*} - \lambda_1$  in its  $(k - \lambda_{i^*} - \lambda_1)$ -degree-gadget and is the unique common vertex of the hyperedges  $S_u^{i,j_u}$  and  $S_v^{i,j_v}$ . Therefore it has degree  $\lambda_{i^*}$  in at most one of  $S_u^{i,j_u}, S_v^{i,j_v}$ . Hence, by the Claim 18.1, at most one of  $c_u^i, c_v^i$  has degree  $\lambda_{i^*}$  in its parent hyperedge. Thus at most one of  $u, v$  is colored  $i$  by  $\phi$ . Therefore,  $\phi$  is a proper 3-coloring of  $G$ .

Assume now that  $\phi$  is a proper 3-coloring of  $G$ . We construct a graph  $G^*$  as follows.

- For any vertex gadget  $H_v$ ,  $i \in [3]$ , let  $G^*[S_v^i]$  be a copy of  $F$  in which every vertex in  $X_v^i$  has degree at most  $\lambda_{i^*-1}$ , every vertex in  $Y_v^i$  has degree at least  $\lambda_{i^*}$ , and  $c_v$  has degree  $\lambda_{i^*-1}$  (resp.  $\lambda_{i^*}$ ) and  $c_v^i$  has degree  $\lambda_1$  (resp.  $\lambda_{i^*}$ ) in  $S_v^i$  if  $\phi(v) = i$  (resp.  $\phi(v) \neq i$ ).
- In every color gadget  $H_v^i$ , for  $x \in \{c_v^i, b_v^i, a_v^i\}$  with children  $y_1$  and  $y_2$ , let  $G^*[S(xy_1)]$  and  $G^*[S(xy_2)]$  be two similar copies of  $F$  such that:
  - if  $i \neq \phi(v)$ , then  $x$  has degree  $\lambda_{i^*}$  in  $G^*[S(xy_1)]$  and  $G^*[S(xy_2)]$  (and so at most  $\lambda_{i^*} + 1$  in  $G^*[S(xy_1) \cup S(xy_2)]$ ) and  $y_1$  and  $y_2$  have degree  $\lambda_1$  in  $G^*[S(xy_1)]$  and  $G^*[S(xy_2)]$  respectively.
  - if  $i = \phi(v)$ , then  $x$  has degree  $\lambda_{i^*-1}$  in  $G^*[S(xy_1)]$  and  $G^*[S(xy_2)]$  (and so at most  $\lambda_{i^*-1} + 1$  in  $G^*[S(xy_1) \cup S(xy_2)]$ ) and  $y_1$  and  $y_2$  have degree  $\lambda_{i^*}$  in  $G^*[S(xy_1)]$  and  $G^*[S(xy_2)]$  respectively.
  - every vertex in  $A_x$  has degree at most  $\lambda_{i^*-1}$  in both  $G^*[S(xy_1)]$  and  $G^*[S(xy_2)]$  and so at most  $\lambda_{i^*} + 1$  in  $G^*[S(xy_1) \cup S(xy_2)]$ ;
  - every vertex in  $B_x$  is degree at least  $\lambda_{i^*}$  in both  $G^*[S(xy_1)]$  and  $G^*[S(xy_2)]$  and so at most  $\lambda_t + 1$  in  $G^*[S(xy_1) \cup S(xy_2)]$ ;
- For any  $d$ -degree-gadget  $D$  at vertex  $v$ , we let  $G^*[D]$  be an  $(F, D, k)$ -graph in which  $v$  has degree  $d$ .

By construction,  $G^*$   $F$ -overlays  $H$ . Let us check that  $\Delta(G^*) \leq k$ . Let  $u$  be a vertex of  $G^*$ .

- If  $u$  is in at most two hyperedges (in particular, if  $u$  is in  $Y_v^i$  or  $u$  is in  $B_x$  for  $x$  internal vertex in some  $T_v^i$  or  $u$  is only in a  $d$ -degree-gadget), then  $u$  has degree at most  $2\lambda_t \leq k$ .
- If  $u \in X_v^i$  for  $v \in V(G)$ , then  $u$  has degree  $k - \lambda_{i^*} + 1$  in its  $(k - \lambda_{i^*} + 1)$ -degree-gadget and at most  $\lambda_{i^*-1}$  in  $S_v^i$ , thus  $u$  has degree at most  $k - \lambda_{i^*} + \lambda_{i^*-1} + 1 \leq k$ .
- If  $u \in A_x$  for  $v \in V(G)$  and  $x$  internal vertex in some tree  $T_v^i$ , then  $u$  has degree  $k - \lambda_{i^*} + 1$  in its  $(k - \lambda_{i^*} + 1)$ -degree-gadget and at most  $\lambda_{i^*-1} + 1$  in  $G^*[S(xy_1) \cup S(xy_2)]$ , thus  $u$  has degree at most  $k - \lambda_{i^*} + \lambda_{i^*-1} + 2 \leq k$ .
- For  $u \in \{c_v^i, a_v^i, b_v^i\}$  for  $i \in [3]$  with  $u$  parent of  $y_1, y_2$ , it has degree  $k - 2\lambda_{i^*} + 1$  in its  $(k - 2\lambda_{i^*} + 1)$ -degree-gadget. And if  $i = \phi(v)$  (resp.  $i \neq \phi(v)$ ), then  $u$  has degree  $\lambda_{i^*}$  (resp.  $\lambda_1$ ) in its parent hyperedge and  $\lambda_{i^*-1} + 1$  (resp.  $\lambda_{i^*} + 1$ ) in  $G^*[S_{uy_1}^1 \cup S_{uy_2}^1]$ . Hence  $u$  has degree at most  $k - \lambda_{i^*} + \lambda_{i^*-1} + 2 \leq k$ .
- Assume that  $u$  is the identification of  $\ell_v^{i,j_v}$  and  $\ell_w^{i,j_w}$  for an edge  $vw \in E(G)$ . First,  $u$  has degree  $k - \lambda_{i^*} - \lambda_1$  in its  $(k - \lambda_{i^*} - \lambda_1)$ -degree-gadget. Moreover, since either  $\phi(v) \neq i$  or  $\phi(w) \neq i$ , then  $u$  has degree  $\lambda_1$  in one of  $S_v^{i,j_v}, S_w^{i,j_w}$  and at most  $\lambda_{i^*}$  in the other. Therefore,  $u$  has degree at most  $k$  in  $G^*$ .

Consequently,  $G^*$  is an  $(F, H, k)$ -graph.  $\square$

**Theorem 19.** *Let a graph  $F$  on  $p$  vertices with degree sequence  $\mathbf{d} = (d_1, \dots, d_p)$  such that  $\lambda_t + \lambda_1 \geq 2\lambda_2$ . Then there exists  $k$  such that  $(\Delta \leq k)$ - $F$ -OVERLAY is NP-complete.*

*Proof.* Observe that the condition  $\lambda_t + \lambda_1 \geq 2\lambda_2$  implies  $t \geq 3$ . Set  $k = 2\lambda_t + \lambda_{t-1}$ .

We give a reduction from 3-COLORABILITY on 4-regular graphs.

Given a 4-regular graph  $G$ , we build a  $p$ -uniform hypergraph  $H$  as follows.



- For each vertex  $v \in V(G)$ , we create a *vertex gadget*  $H_v$  with three hyperedges  $S_v^i = \{c_v, c_v^i\} \cup X_v^i \cup Y_v^i$  for  $i \in [3]$  where  $|X_v^i| = \sum_{i=1}^{t-1} \alpha_i - 1$ ,  $|Y_v^i| = p - \alpha_t - 1$ . For  $i \in [3]$ , we add a  $(k - \lambda_{t-1})$ -degree-gadget at each vertex  $x \in X_v^i$ . We say that  $S_v^i$  is the parent hyperedge of each  $c_v^i$ ,  $i \in [3]$ .
- For each vertex  $v \in V(G)$  and each  $i \in [3]$ , we construct a *color gadget*  $H_v^i$  for  $i \in [3]$  as follows.
  - We create a binary tree  $T_v^i$  with vertex set  $\{c_v^i, a_v^i, b_v^i, \ell_v^{i,1}, \ell_v^{i,2}, \ell_v^{i,3}, \ell_v^{i,4}\}$  and edge set  $\{c_v^i a_v^i, c_v^i b_v^i, a_v^i \ell_v^{i,1}, a_v^i \ell_v^{i,2}, b_v^i \ell_v^{i,3}, b_v^i \ell_v^{i,4}\}$ , rooted at  $c_v^i$ . In this tree,  $a_v^i$  and  $b_v^i$  are the children of  $c_v^i$ ,  $\ell_v^{i,1}$  and  $\ell_v^{i,2}$  are the children of  $a_v^i$ , and  $\ell_v^{i,3}$  and  $\ell_v^{i,4}$  are the children of  $b_v^i$ .
  - For each edge  $e = xy$  of  $T_v^i$  with  $x$  the parent of  $y$  in  $T_v^i$ , we construct an *edge-gadget* containing  $x, y$ , a new vertex  $z_e$ , and four disjoint sets  $U_e^1 W_e^1, U_e^2, W_e^2$  of new vertices,  $U_e^1$  of size  $\alpha_i - 1$ ,  $W_e^1$  of size  $p - |U_e^1| - 1$ ,  $U_e^2$  of size  $p - \alpha_t - 1$ ,  $W_e^2$  of size  $\alpha_t - 1$ . We add the hyperedges  $S_e^1 = \{x, z_e\} \cup U_e^1 \cup W_e^1$  and  $S_e^2 = \{z_e, y\} \cup U_e^2 \cup W_e^2$ . See Figure 7. We finally add a  $(k - \lambda_t - 2\lambda_1)$ -degree-gadget at  $x$ , a  $(k - \lambda_1)$ -degree-gadget at each vertex of  $U_e^1$ , a  $(k - \lambda_t + 1)$ -degree-gadget at each of  $U_e^2$ , and a  $(k - \lambda_2 - \lambda_t + 1)$ -degree-gadget pendant at  $z_e$ .
- For every vertex  $v \in V(G)$ , let  $e_v^1, e_v^2, e_v^3, e_v^4$  be an ordering of the edges incident to  $v$ . For each edge  $uv \in E(G)$ , let  $j_u$  and  $j_v$  be the indices such that  $uv = e_u^{j_u} = e_v^{j_v}$ . Then, for all  $i \in [3]$ , we identify the vertices  $\ell_u^{i,j_u}$  and  $\ell_v^{i,j_v}$  and we add a  $(k - 2\lambda_t + 1)$ -degree-gadget at this vertex.

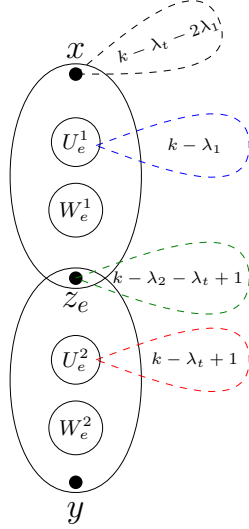


Figure 7: The edge-gadget for an edge  $e = xy$  with degree-gadgets at  $x, z_e$  and every vertex in  $U_e^1, U_e^2$ .

Let us now prove that there is a proper 3-coloring of  $G$  if and only if there is an  $(F, H, k)$ -graph  $G^*$ .

Assume first that there is an  $(F, H, k)$ -graph  $G^*$ .

Let  $v \in V(G)$ . The vertex  $c_v$  has degree at most  $2\lambda_t + \lambda_{t-1}$  in  $S_v^1 \cup S_v^2 \cup S_v^3$ . But those hyperedges pairwise intersect in  $\{c_v\}$ . Thus there is  $i \in [3]$  such that  $c_v$  has degree less than  $\lambda_t$  in  $S_v^i$ .

Moreover, each vertex  $x \in X_v^i$  has degree at least  $k - \lambda_{t-1}$  in its  $(k - \lambda_{t-1})$ -degree-gadget, and so at most  $\lambda_{t-1}$  in  $S_v^i$ . Together with  $c_v$ , there are  $\sum_{i=1}^{t-1} \alpha_t$  vertices of degree at most  $\lambda_{t-1}$  in  $S_v^i$ . Thus  $c_v^i$  have degree  $\lambda_t$  in its parent hyperedge  $S_v^i$ . Therefore, we can define a 3-coloring  $\phi$  by  $\phi(v) = i$  where  $i$  is an index such that  $c_v^i$  has degree  $\lambda_t$  in  $S_v^i$ .

Let us now prove that  $\phi$  is proper. We need the following claim.

**Claim 19.1.** *Let  $v \in V(G)$  and  $i \in [3]$ . If  $c_v^i$  has degree  $\lambda_t$  in  $S_v^i$ , then so does any leaf  $\ell_v^{i,j}$  in  $S_v^{i,j}$  for  $j \in [4]$ .*

*Proof of Claim.* It suffices to prove that for any  $x \in \{c_v^i, b_v^i, a_v^i\}$ , if  $x$  has degree  $\lambda_t$  in its parent hyperedge, then both  $y_1, y_2$  have degree  $\lambda_t$  in their parent hyperedges.

Assume that  $x$  is a  $\lambda_t$ -vertex in its parent hyperedge. Since  $x$  has degree at least  $k - \lambda_t - 2\lambda_1$  in its  $(k - \lambda_t - 2\lambda_1)$ -degree-gadget, and degree  $\lambda_t$  in its parent hyperedge, it has degree at most  $2\lambda_1$  in  $S_{xy_1}^1 \cup S_{xy_2}^1$ , and so  $\lambda_1$  in each of  $S_{xy_1}^1, S_{xy_2}^1$ . Let  $e = xy$  be one of the two edges  $xy_1, xy_2$ . Any vertex in  $U_e^1$  has degree at least  $k - \lambda_1$  in its  $(k - \lambda_1)$ -degree-gadget, and thus  $\lambda_1$  in  $S_e^1$ . It implies that  $z_e$  has degree at least  $\lambda_2$  in  $S_e^1$ . Since it is also in a  $(k - \lambda_2 - \lambda_t + 1)$ -degree-gadget,  $z_e$  has degree less than  $\lambda_t$  in  $S_e^2$ . Moreover, any vertex in  $U_e^2$  is in a  $(k - \lambda_t + 1)$ -degree-gadget, then none of them has degree  $\lambda_t$  in  $S_e^2$  except those in  $W_e^2$  which is of size  $\alpha_t - 1$ . Thus,  $y$  must have degree  $\lambda_t$  in  $S_e^2$ .  $\diamond$

Consider an edge  $uv \in E(G)$ ,  $i \in [3]$ . The vertex  $\ell = \ell_u^{i,j_u} = \ell_v^{i,j_v}$  has degree at least  $k - 2\lambda_t + 1$  in its  $(k - 2\lambda_t + 1)$ -degree-gadget and is the unique common vertex of the hyperedges  $S_u^{i,j_u}$  and  $S_v^{i,j_v}$ . Therefore it has degree  $\lambda_t$  in at most one of  $S_u^{i,j_u}$  and  $G^*S_v^{i,j_v}$ . Hence, by Claim 19.1, at most one of  $c_u^i, c_v^i$  has degree  $\lambda_t$  in its parent hyperedge. Thus at most one of  $u, v$  is colored  $i$  by  $\phi$ . Therefore,  $\phi$  is a proper 3-coloring of  $G$ .

Assume now that  $\phi$  is a proper 3-coloring of  $G$ . We construct a graph  $G^*$  as follows.

- For any vertex gadget  $H_v$ ,  $i \in [3]$ , let  $G^*[S_v^i]$  be a copy of  $F$  in which every vertex in  $X_v^i$  has degree at most  $\lambda_{t-1}$ , every vertex in  $Y_v^i$  has degree  $\lambda_t$ , and  $c_v$  has degree  $\lambda_{t-1}$  (resp.  $\lambda_t$ ) and  $c_v^i$  has degree  $\lambda_t$  (resp.  $\lambda_1$ ) if  $\phi(v) = i$  (resp.  $\phi(v) \neq i$ ).
- In every color gadget  $H_v^i$ , for each edge  $e = xy$  of  $T_v^i$  with  $x$  parent of  $y$ , let  $G^*[S_e^1], G^*[S_e^2]$  be copies of  $F$  such that:
  - every vertex in  $U_e^1$  has degree  $\lambda_1$ ;
  - every vertex in  $U_e^2$  has degree at most  $\lambda_{t-1}$ ;
  - every vertex in  $W_e^2$  has degree  $\lambda_t$ ;
  - if  $i = \phi(v)$ , then  $x$  has degree  $\lambda_1$  in  $S_e^1$ ,  $z_e$  has degree  $\lambda_2$  in  $S_e^1$  and  $\lambda_{t-1}$  in  $S_e^2$ , and  $y$  has degree  $\lambda_t$  in  $S_e^2$ ;
  - if  $i \neq \phi(v)$ , then  $x$  has degree  $\lambda_2$  in  $S_e^1$ ,  $z_e$  has degree  $\lambda_1$  in  $S_e^1$  and  $\lambda_t$  in  $S_e^2$ , and  $y$  has degree  $\lambda_1$  in  $S_e^2$ .
- For any  $d$ -degree-gadget  $D$  at vertex  $v$ , we let  $G^*[D]$  be an  $(F, D, k)$ -graph in which  $v$  has degree  $d$ .

By construction,  $G^*$   $F$ -overlays  $H$ . Let us check that  $\Delta(G^*) \leq k$ . Let  $u$  be a vertex of  $G^*$ .

- If  $u$  is in at most two hyperedges, (in particular if  $u$  is in  $Y_v^i$ , in  $W_e^1 \cup W_e^2$  in an edge-gadget or only in a degree-gadget), then  $u$  has degree at most  $2\lambda_t < k$  in  $G^*$ .
- If  $u = c_v$ , then it has degree  $\lambda_{t-1}$  in  $S_v^i$  for the index  $i = \phi(v)$ , and  $\lambda_t$  in  $S_v^i$  for the two indices  $i \neq \phi(v)$ . Hence  $c_v$  has degree  $2\lambda_1 + \lambda_{t-1} = k$ .
- If  $u \in X_v^i$  for  $v \in V(G)$ , then  $u$  has degree  $k - \lambda_{t-1}$  in its  $(k - \lambda_{t-1})$ -degree-gadget and at most  $\lambda_{t-1}$  in  $S_v^i$ , thus  $u$  has degree at most  $k$  in  $G^*$ .
- If  $u \in U_e^1$  for some edge  $e$  of  $T_v^i$ , then  $u$  has degree  $k - \lambda_1$  in its degree-gadget and  $\lambda_1$  in  $S_e^1$ , thus  $u$  has degree  $k$  in  $G^*$ .
- If  $u \in U_e^2$ , then  $u$  has degree  $k - \lambda_t + 1$  in its degree-gadget and at most  $\lambda_{t-1}$  in  $S_e^2$ , thus  $u$  has degree at most  $k$  in  $G^*$ .
- If  $u \in \{c_v^i, a_v^i, b_v^i\}$  for  $i \in [3]$  with children  $y_1, y_2$ , then  $u$  has degree  $k - \lambda_t - 2\lambda_1$  in its  $(k - \lambda_t - 2\lambda_1)$ -degree-gadget. Moreover, if  $i = \phi(v)$  (resp.  $i \neq \phi(v)$ ), then  $u$  has degree  $\lambda_t$  (resp.  $\lambda_1$ ) in its parent hyperedge and  $\lambda_1$  (resp.  $\lambda_2$ ) in both  $S_{uy_1}^1, S_{uy_2}^1$ . Hence  $u$  has degree at most  $k - \lambda_t - 2\lambda_1 + \lambda_t + 2\lambda_1 = k$  (resp.  $k - \lambda_t - 2\lambda_1 + \lambda_1 + 2\lambda_2 \leq k$  by the assumption  $\lambda_t + \lambda_1 \geq 2\lambda_2$ ) in  $G^*$ .
- If  $u = z_e$  for some edge  $e$  of  $T_v^i$ , then  $u$  has degree  $k - \lambda_2 - \lambda_t + 1$  in its  $(k - \lambda_2 - \lambda_t + 1)$ -degree-gadget. Moreover, if  $i = \phi(v)$  (resp.  $i \neq \phi(v)$ ), then  $u$  has degree  $\lambda_2$  (resp.  $\lambda_1$ ) in  $S_e^1$  and  $\lambda_{t-1}$  (resp.  $\lambda_t$ ) in  $S_e^2$ . Hence,  $u$  has degree at most  $k - \lambda_2 - \lambda_t + 1 + \lambda_2 + \lambda_{t-1} \leq k$  (resp.  $k - \lambda_2 - \lambda_t + 1 + \lambda_1 + \lambda_t \leq k$ ) in  $G^*$ .
- Assume that  $u$  is the identification of  $\ell_v^{i,j_v}$  and  $\ell_w^{i,j_w}$  for an edge  $vw \in E(G)$ . First,  $u$  has degree  $k - 2\lambda_t + 1$  in its  $(k - 2\lambda_t + 1)$ -degree-gadget. Moreover, since either  $\phi(v) \neq i$  or  $\phi(w) \neq i$ , then  $u$  has degree less than  $\lambda_t$  in one of  $S_v^{i,j_v}, S_w^{i,j_w}$ . Therefore,  $u$  has degree at most  $k$  in  $G^*$ .

Consequently,  $G^*$  is an  $(F, H, k)$ -graph.  $\square$

**Theorem 20.** *Let  $F$  be a graph with  $\alpha_1$  vertices of positive degree  $\lambda_1$  and  $\alpha_2 = p - \alpha_1$  vertices of degree  $\lambda_2 = \lambda_1 + 1$ . Then  $(\Delta \leq k)$ - $F$ -OVERLAY is NP-complete for some  $k$ .*

There are several cases in the proof, depending on the structure of graph  $F$ . In each case, we give a reduction from (3,4)-SAT problem, which follows the same general idea as the proof of Theorem 13 : we construct variable gadgets  $H_t$  containing some negated and non-negated *literal vertices* and identify some of them in such a way that for an assignment  $\phi$  satisfying  $\Phi$ ,  $\phi(x_t) = \text{true}$  (resp. *false*) if and only if non-negated (resp. negated) literal vertices in the variable gadget are adjacent to  $w_t$  in an  $(F, H, k)$ -graph.

**Lemma 21.** *Let  $F$  be a graph on  $p$  vertices with  $\alpha_1$  vertices of degree  $\lambda_1$  and  $\alpha_2 = p - \alpha_1$  vertices of degree  $\lambda_2 > \lambda_1$  such that  $F[V_{\lambda_2}]$  is  $\mu$ -regular but neither complete nor anticomplete. Then there exists  $k$  such that  $(\Delta \leq k)$ - $F$ -OVERLAY is NP-complete.*

*Proof.* Set  $\gamma = \lambda_2 - \mu$  and  $k = \max\{4\gamma(\alpha_2 - 1) + 4\mu + \lambda_1, 3\gamma(\alpha_2 - 1) + 6\mu - 1 + \lambda_1\}$ .

We give a reduction from (3,4)-SAT. Given a formula  $\Phi$  of (3,4)-SAT with variables  $x_t, t \in [n]$  and clauses  $C_j, j \in [m]$ , we construct a hypergraph  $H$  as follows.

1. For each variable  $x_t$ , we construct a *variable gadget*  $H_t$  in the following way.

We first create a *center vertex*  $w_t$ , a set of  $4(\alpha_2 - 1)$  vertices  $U_t = \{u_t^1, \dots, u_t^{4(\alpha_2-1)}\}$ , and for each  $i \in [4(\alpha_2 - 1)]$ , create a set of  $\alpha_1$  new vertices  $W_t^i$ , and a hyperedge  $S_t^i = W_t^i \cup \{w_t, u_t^i, \dots, u_t^{i+\alpha_2-2}\}$  (superscripts are modulo  $4(\alpha_2 - 1)$ ).

For  $r \in [4]$ , let  $x_t^r = u_t^{r(\alpha_2-1)-\alpha_2+2}$  and  $\bar{x}_t^r = u_t^{r(\alpha_2-1)-\alpha_2+3}$ . Set  $X_t = \{x_t^1, x_t^2, x_t^3, x_t^4\}$  and  $\bar{X}_t = \{\bar{x}_t^1, \bar{x}_t^2, \bar{x}_t^3, \bar{x}_t^4\}$ . The vertices of  $X_t$  (resp.  $\bar{X}_t$ ) are called the *non-negated* (resp. *negated*) *literal vertices* of  $H_t$ .

2. For each clause  $C_j = (\ell_1 \vee \ell_2 \vee \ell_3)$ , we identify  $y_1, y_2, y_3$  into a *clause vertex*  $c_j$ , where  $y_i = x_t^r$  if  $\ell_i = x_t$  and  $\ell_i$  is the  $r$ -th occurrence of  $x_t$ , and  $y_i = \bar{x}_t^r$  if  $\ell_i = \bar{x}_t$  and is the  $r$ -th occurrence of  $x_t$ .
3. Finally, we add degree-gadgets on some vertices.
  - We add a  $(k - 4\gamma(\alpha_2 - 1) - 4\mu)$ -degree gadget on vertex  $w_t$ .
  - We add a  $(k - \lambda_1)$ -degree gadget at every vertex in  $W_t^i$  for all  $i \in [4(\alpha_2 - 1)]$ .
  - For  $i \in [\mu - 1]$ , we add a  $(k - \gamma(\alpha_2 - 1) - 2\mu + 1)$ -degree-gadget on each  $u_t^{(\alpha_2-1)r-j}$  for  $r \in [4]$ .

Observe that every vertex in  $W_t^i$ , for  $i \in [4(\alpha_2 - 1)]$ , has degree at least  $k - \lambda_1$  in its  $(k - \lambda_1)$ -degree-gadget, and so degree at most  $\lambda_1$  in  $S_t^i$ . Thus, each of those vertices must have degree  $\lambda_1$  in  $S_t^i$ . It implies that all the other vertices must have degree at least  $\lambda_2$  in any hyperedge of  $H_t$ . In particular,  $w_t$  has degree  $\lambda_2$  in any hyperedge of  $H_t$ . Since  $w_t$  is in a  $(k - 4\gamma(\alpha_2 - 1) - 4\mu)$ -degree-gadget and is adjacent to  $\gamma$  vertices in every  $W_t^i$  for  $i \in [4(\alpha_2 - 1)]$ , it has degree at most  $4\mu$  in  $\bigcup_{i=1}^{4(\alpha_2-1)} S_t^i \setminus W_t^i$ .

Moreover, each vertex  $u_t^i \in U_t$  is a  $\lambda_2$ -vertex in every hyperedge  $S_t^{i'}$  of  $H_t$  containing it, and so adjacent to  $\gamma$  vertices in  $W_t^{i'}$ . Since  $u_t^i$  belongs to  $\alpha_2 - 1$  hyperedges of  $H_t$ , thus  $u_t^i$  is adjacent to  $\gamma(\alpha_2 - 1)$  vertices in  $\bigcup_{i'=1}^{4(\alpha_2-1)} W_t^{i'}$ . For  $i \in [\mu - 1]$ ,  $u_t^{(\alpha_2-1)r-i}$  is in a  $(k - \gamma(\alpha_2 - 1) - 2\mu + 1)$ -degree-gadget, then it has degree at most  $\gamma(\alpha_2 - 1) + 2\mu - 1$  in  $H_t$ , and so at most  $2\mu - 1$  in  $\bigcup_{i'=1}^{4(\alpha_2-1)} S_t^{i'}$ . Moreover,  $F[V_{\lambda_2}]$  is  $\mu$ -regular (but not complete or anticomplete). The following is then similar to the proof of Theorem 13 for  $F[V_{\lambda_2}]$ . So we just sketch it.

Assume that there exists a truth assignment  $\phi$  satisfying  $\Phi$ . Let  $G$  be the graph obtained as follows.

We let  $(v_0, v_1, \dots, v_{\alpha_2-1})$  be an ordering of  $V_{\lambda_2}$  such that  $N_F(v_0) = \{v_{\alpha_2-\mu+1}, \dots, v_{\alpha_2-1}\} \cup \{v_1\}$  if  $\phi(x_t) = \text{true}$  and such that  $N_F(v_0) = \{v_{\alpha_2-\mu+1}, \dots, v_{\alpha_2-1}\} \cup \{v_2\}$  if  $\phi(x_t) = \text{false}$ . For every  $i \in [4\alpha_2 - 4]$ , we let  $G[S_t^i]$  be the copy of  $F$  in which every vertex in  $W_t^i$  is a  $\lambda_1$ -vertex,  $w_t$  corresponds to  $v_0$  and  $u_t^{i'}$  for  $i' \in \{i, \dots, i + \alpha_2 - 1\}$  corresponds to the vertex  $v_{i''}$  such that  $i' \equiv i'' \pmod{\alpha_2 - 1}$ . In addition, for every  $d$ -degree-gadget  $D$  at some vertex  $v$ , we let  $G[D]$  be an  $(F, D, k)$ -graph in which  $v$  has degree  $d$ .

The graph  $G$   $F$ -overlays  $H$  and one can check that  $\Delta(F) \leq k$ .

Conversely, assume that  $G$  is an  $(F, H, k)$ -graph. One can prove the following claim.

**Claim 21.1.** *For every  $t \in [n]$  the following hold.*

- (a) *Every vertex in  $W_t^i$  for  $i \in [4\alpha_2 - 4]$  is a  $\lambda_1$ -vertex in  $S_t^i$ .*
- (b)  *$w_t$  is a  $\lambda_2$ -vertex in every hyperedges of  $H_t$ . Furthermore, it is adjacent to  $\gamma$  vertices in each  $W_t^i$  and the vertices  $u_t^{(\alpha_2-1)r-i}$  for  $r \in [4]$ ,  $i \in [\mu - 1]$ .*

Therefore the truth assignment  $\phi$  defined by  $\phi(x_t) = \text{true}$  (resp. *false*) if  $w_t$  is adjacent to  $X_t$  (resp.  $\bar{X}_t$ ), satisfies  $\Phi$ .  $\square$

*Proof of Theorem 20.* Let  $V_d$  be the set of vertices of degree  $d$  in  $F$  and  $\mathbf{d} = (d_1, \dots, d_p)$  be the non-decreasing degree sequence of  $F$ . Let  $N_s$  be the set of vertices of  $V_{\lambda_2}$  having exactly  $s$  neighbors in  $V_{\lambda_1}$ , and let  $N_{\geq s} = \bigcup_{s' \geq s} N_{s'}$ .

For technical reasons, we distinguish several cases as follows.

If  $F[V_{\lambda_1}]$  is not anticomplete, then see **Case A**. Otherwise,  $F[V_{\lambda_1}]$  is anticomplete.

Assume first that  $F[V_{\lambda_2}]$  is regular. If  $F[V_{\lambda_2}]$  is neither complete nor anticomplete, then we have the result by Lemma 21. If  $F[V_{\lambda_2}]$  is anticomplete, then  $F$  is a disjoint union of  $K_{\lambda_1, \lambda_1+1}$  and we have the result by Theorem 15.

Hence we may assume that  $F[V_{\lambda_2}]$  is complete. Observe  $\alpha_2 \geq \lambda_1$  because a vertex of  $V_{\lambda_1}$  has all its neighbors in  $V_{\lambda_2}$  and  $\alpha_2 \leq \lambda_1 + 1$  because every vertex of  $V_{\lambda_2}$  is adjacent to all other vertices of  $V_{\lambda_2}$  and at least one in  $V_{\lambda_1}$ . If  $\alpha_2 = \lambda_1 + 1$ , then every vertex of  $V_{\lambda_2}$  has exactly one neighbor in  $V_{\lambda_1}$ , and so  $\alpha_2 = \lambda_1 \times \alpha_1$ . Hence  $\alpha_2 = 2 = \alpha_1$  and  $\lambda_1 = 1$ . Thus  $F = K_3^-$  and we have the result by Theorem 14. If  $\alpha_2 = \lambda_1$ , then every vertex of  $V_{\lambda_1}$  is adjacent to all vertices of  $V_{\lambda_2}$ . Thus  $F$  is  $K_{\lambda_1+2}^-$  and we have the result by Theorem 14.

Assume now that  $F[V_{\lambda_2}]$  is not regular, that is  $F[V_{\lambda_2}]$  has at least two degree values. In particular,  $\alpha_2 \geq 2$ .

If  $N_{\geq 2}$  is empty, then  $V_{\lambda_2} = N_0 \cup N_1$  and both  $N_0, N_1$  are non-empty. See **Case B**-(i).

If there is a vertex in  $N_{\geq 2}$  which is not adjacent to a vertex in  $V_{\lambda_1}$ , see **Case C**-(i).

Otherwise, every vertex in  $N_{\geq 2}$  is adjacent to all vertices in  $V_{\lambda_1}$  (so here  $N_{\geq 2} = N_{\alpha_1}$  with  $\alpha_1 \geq 2$ ). If  $N_1 = \emptyset$ , then see **Case C**-(ii). Otherwise,  $N_1 \neq \emptyset$  and any vertex in  $V_{\lambda_1}$  is not adjacent to all vertices in  $N_1$ , see **Case B**.

**Case A:** We set  $k$  depending on the subgraph  $F[V_{\lambda_1}]$  of  $F$ .

(1) If  $F[V_{\lambda_1}]$  is not complete, then  $k = 6\lambda_1 - 1$ .

(2) If  $F[V_{\lambda_1}]$  is complete, then every vertex of  $V_{\lambda_1}$  is not adjacent to some vertex in  $V_{\lambda_2}$ . We set  $k = 6\lambda_1 + 3$ .

We give a reduction from (3,4)-SAT.

Given a formula  $\Phi$  of (3,4)-SAT with variables  $x_t, t \in [n]$  and clauses  $C_j, j \in [m]$ , we build a hypergraph  $H$  as follows.

1. For each variable  $x_t$ , we construct a *variable gadget*  $H_t$  in the following way.

We first create a *center vertex*  $w_t$ , a set of  $4p - 4$  vertices  $U_t = \{u_t^1, \dots, u_t^{4p-4}\}$ , and  $4p - 4$  hyperedges  $S_t^i = \{w_t, u_t^i, \dots, u_t^{i+p-2}\}$  (superscripts are modulo  $4(p-1)$ ) for  $i \in [4p-4]$ .

For  $r \in [4]$ , let  $x_t^r = u_t^{r(p-1)-p+2}$  and  $\bar{x}_t^r = u_t^{r(p-1)-p+3}$ . Set  $X_t = \{x_t^1, x_t^2, x_t^3, x_t^4\}$  and  $\bar{X}_t = \{\bar{x}_t^1, \bar{x}_t^2, \bar{x}_t^3, \bar{x}_t^4\}$ . The vertices of  $X_t$  (resp.  $\bar{X}_t$ ) are called the *non-negated* (resp. *negated*) *literal vertices* of  $H_t$ .

2. For each variable  $x_t$ , we add a set of  $p - \lambda_1$  vertices  $W_t$ , and a hyperedge  $S'_t = W_t \cup \{u_t^{p-1}, \dots, u_t^{p-\lambda_1+1}\}$  and we add a  $(k - 4\lambda_1 - 1)$ -degree-gadget at  $w_t$ .

3. For each clause  $C_j = (\ell_1 \vee \ell_2 \vee \ell_3)$ , we identify  $y_1, y_2, y_3$  into a *clause vertex*  $c_j$ , where for all  $i \in [3]$ ,  $y_i = x_t^i$  if  $\ell_i = x_t$  and  $\ell_i$  is the  $r$ -th occurrence of  $x_t$ , and  $y_i = \bar{x}_t^i$  if  $\ell_i = \bar{x}_t$  and is the  $r$ -th occurrence of  $x_t$ .

Let  $z$  be a vertex in  $V_{\lambda_1}$  which is adjacent to the minimum number  $a > 0$  of vertices in this set. Let  $(z, z_1, \dots, z_{p-1})$  be an ordering of  $F$  such that :

- $z_j$  has degree  $\lambda_1$  and is adjacent to  $z$  for all  $j \in [a]$ ,
- $z_j$  has degree  $\lambda_1$  and is not adjacent to  $z$  for all  $a + 1 \leq j \leq \alpha_1 - 1$ ,
- $z_j$  has degree  $\lambda_2$  and is adjacent to  $z$  for all  $\alpha_1 \leq j \leq \alpha_1 + \lambda_1 - a - 1$ .
- $z_j$  has degree  $\lambda_2$  and is not adjacent to  $z$  for all  $\alpha_1 + \lambda_1 - a \leq j \leq p - 1$ .

We will show that there is an assignment  $\phi$  satisfying  $\Phi$  if and only there is an  $(F, H, k)$ -graph  $G$ .

Assume that  $\phi$  satisfies  $\Phi$ , then we construct  $G$  as follows. For all  $i \in [4p-4]$ , let  $G[S_t^i]$  be copies of  $F$  such that  $w_t$  corresponds to the vertex  $z$  and the following hold.

In **Case A**-(1),

- if  $\phi(x_t) = \text{true}$  (resp.  $\phi(x_t) = \text{false}$ ), then each vertex in  $X_t$  (resp.  $\bar{X}_t$ ) corresponds to  $z_1$ , and each of  $\bar{X}_t$  (resp.  $X_t$ ) corresponds to  $z_{\alpha_1-1}$ .
- for all  $r \in [4]$  and  $2 \leq i \leq \alpha_1 - 2$ ,  $u_t^{(p-1)r+1-i}$  corresponds to  $z_i$ .

- for all  $r \in [4]$  and  $\alpha_1 \leq i \leq p-1$ ,  $u^{(p-1)r+2-i}$  corresponds to  $z_i$ .

In **Case A-(2)**,

- if  $\phi(x_t) = \text{true}$  (resp.  $\phi(x_t) = \text{false}$ ), then each vertex in  $X_t$  (resp.  $\overline{X}_t$ ) corresponds to  $z_1$ , and each of  $\overline{X}_t$  (resp.  $X_t$ ) corresponds to  $z_{p-1}$ .

- for all  $r \in [4]$  and  $2 \leq i \leq p-2$ ,  $u^{(p-1)r+1-i}$  corresponds to  $z_i$ .

For any  $d$ -degree-gadget  $D$  at a vertex  $v$ , let  $G[D]$  be an  $(F, D, k)$ -graph in which  $v$  has degree  $d$ .

Let us check that  $\Delta(G) \leq k$ .

-  $w_t$  is adjacent to  $4\lambda_1$  vertices in  $H_t$  and one more in  $W_t \subset V(S'_t)$ , and it has degree  $k - 4\lambda_1 - 1$  in its degree-gadget. Thus  $w_t$  has degree  $k$  in total.

- Any literal vertex which is not identified to any clause vertex and is not in  $S'_t$  has degree at most  $2\lambda_2$  in its variable gadget. So, it has degree less than  $k$ .

- Any literal vertex which is in  $S'_t$  has degree at most  $2\lambda_2$  in its variable gadget and it is adjacent to at most  $\lambda_1$  vertices in  $W_t$ . So, it has degree less than  $k$ .

- Each clause vertex  $c_j$  is in three literal variable gadgets. In **Case A-(1)** (resp. **Case A-(2)**),  $c_j$  has degree at most  $2\lambda_1$  (resp.  $2\lambda_2$ ) in each variable gadget. Moreover at least one of the literals is true so its corresponding literal vertex has degree  $2\lambda_1 - 1$  (resp.  $2\lambda_2 - 1$ ). Therefore  $c_j$  has degree at most  $6\lambda_1 - 1$  (resp.  $6\lambda_2 - 1$ ) in its variable gadget. Now it has degree  $k - 6\lambda_1 + 1$  (resp.  $k - 6\lambda_2 + 1$ ) in its degree-gadget, and so at most  $k$  in total.

- Any vertex which is in a degree-gadget but in no variable gadget has degree at most  $2\lambda_t \leq k$  since it belongs to at most two hyperedges.

- Any vertex in  $W_t$  has degree at most  $\lambda_2 < k$ .

Hence,  $G$  is an  $(F, H, k)$ -graph.

Conversely, let  $G$  be an  $(F, H, k)$ -graph.

**Claim 21.2.** *For every  $t \in [n]$  the following hold.*

(a)  $w_t$  has degree  $4\lambda_1$  in  $H_t$  and  $w_t$  has degree exactly  $\lambda_1$  in every hyperedge containing it.

(b) There is  $I \in [p-1]$  of size  $\lambda_1$  such that for all  $i \in I$  and  $r \in [4]$ ,  $u_t^{(p-1)r-i+1}$  is adjacent to  $w_t$ ; and  $[\lambda_1 - 1] \subset I$ .

*Proof of Claim.* Observe that  $w_t$  is in a  $(k - 4\lambda_1 - 1)$ -degree-gadget, so it has degree at most  $4\lambda_1 + 1$  in  $H_t \cup S'_t$ . Since  $w_t$  is in  $S'_t$  which intersects  $S_t^1$  in  $\lambda_1 - 1$  vertices and it is at least  $\lambda_1$  in  $S'_t$ , then  $w_t$  is adjacent to at least one vertex in  $W_t \subset V(S'_t)$ . Thus,  $w_t$  has degree at most  $4\lambda_1$  in  $H_t$ . Now for every  $i \in [p-1]$ ,  $w_t$  belongs to the four hyperedges  $S_t^{(p-1)r-i}$ ,  $r \in [4]$ , which pairwise intersect in  $\{w_t\}$ . Hence  $w_t$  has degree exactly  $\lambda_1$  in each  $S_t^{(p-1)r-i}$  and then  $4\lambda_1$  in  $H_t$ . This proves (a).

Now, if a vertex  $u_t^i$  is adjacent to  $w_t$ , then so is  $u_t^{i+p-1}$  because  $w_t$  has degree exactly  $\lambda_1$  in both  $S_t^i$  and  $S_t^{i+1}$ . Therefore there is  $I \in [p-1]$  of size  $\lambda_1$  such that  $w_t$  is adjacent to  $u_t^{(p-1)r-i+1}$  for all  $i \in I$  and  $r \in [4]$ . Since  $w_t$  has degree  $4\lambda_1$  in  $H_t$ , then  $w_t$  is adjacent to exactly one vertex in  $W_t$  and so must be adjacent to  $\lambda_1 - 1$  vertices in  $S'_t \setminus W_t$  which are  $u_t^{p-1}, \dots, u_t^{p-\lambda_1+1}$ . It implies that  $[\lambda_1 - 1] \subset I$ . This proves (b).  $\diamond$

Claim 21.2 implies that the vertices of  $X_t$  (resp.  $\overline{X}_t$ ) are either all adjacent to  $w_t$  or all non-adjacent to  $w_t$ . Moreover,  $w_t$  is adjacent to  $\lambda_1 - 1$  vertices not in  $X_t \cup \overline{X}_t$ . Hence if the vertices of  $X_t$  are adjacent to  $w_t$ , the vertices of  $\overline{X}_t$  are not (and vice-versa).

Let  $\phi$  be the truth assignment defined by  $\phi(x_t) = \text{true}$  if  $w_t$  is adjacent to  $H_t$ , and  $\phi(x_t) = \text{false}$  otherwise. In any clause vertex  $c_j$ , we identified three literal vertices corresponding to the three literals.

- In **Case A-(1)**,  $c_j$  has degree at most  $k = 6\lambda_1 - 1$ , so it has degree less than  $2\lambda_1$  in one of its three variable gadgets  $H_t$ . Since any vertex  $u_t^i$  for  $i \in [4p-4]$  belongs to two hyperedges  $S_t^i$  and  $S_t^{i-p+2}$  which intersect in  $\{u_t^i, w_t\}$  and has degree at least  $\lambda_1$  in each, then it has degree  $2\lambda_1 - 1$  in  $H_t$  only if it is adjacent to  $w_t$ . Hence,  $c_j$  is adjacent to  $w_t$ .

- In **Case A-(2)**,  $c_j$  has degree at most  $k = 6\lambda_1 + 3 < 6\lambda_2$ , so it has degree less than  $2\lambda_2$  in one of its three variable gadgets  $H_t$ .

Moreover,  $F[V_{\lambda_1}]$  is complete, then  $w_t$  is adjacent to all  $\lambda_1$ -vertices in every hyperedges of  $H_t$  (because it is a  $\lambda_1$ -vertex in every hyperedge of  $H_t$ ). If  $c_j$  is not adjacent to no center vertex of the three variable gadgets it belongs to, then it must be a  $\lambda_2$ -vertex in each hyperedge of those gadgets. Thus it has

degree at least  $2\lambda_2$  in each variable gadget and so at least  $6\lambda_2$  in total, a contradiction. Thus  $c_j$  is adjacent to the center of at least one variable gadget  $w_t$ .

Hence, the corresponding literal to the literal vertex adjacent to  $w_t$  for variable  $x_t$  is true and clause  $C_j$  is satisfied.

Consequently,  $\phi$  satisfies  $\Phi$ .

**Case B:** Recall that in that case  $N_1 \neq \emptyset$ . Let  $\gamma = \max\{|N(v) \cap V_{\lambda_1}| \mid v \in V_{\lambda_2}\}$ . We have  $V_{\lambda_2} = \bigcup_{s=0}^{\gamma} N_s$ . Let  $k$  as follows.

- (i) If  $N_0 \neq \emptyset$ , then set  $k = \max\{6\lambda_2 - 1 + \lambda_1, \gamma\alpha_2 + 2(\lambda_2 - \gamma) + \lambda_1\}$ .
- (ii) If  $N_0 = \emptyset$ ,  $N_{\geq 2} \neq \emptyset$  and every vertex of  $N_{\geq 2}$  is adjacent to all vertices of  $V_{\lambda_1}$ , then set  $k = \max\{6\lambda_1 + 3\alpha_2 - 1 + \lambda_1, \gamma\alpha_2 + 2(\lambda_2 - \gamma) + \lambda_1\}$ . Note that in that case every vertex in  $V_{\lambda_1}$  is adjacent to a vertex in  $N_1$  but not all.

We give a reduction from (3,4)-SAT.

Given a formula  $\Phi$  of (3,4)-SAT with variables  $x_t, t \in [n]$  and clauses  $C_j, j \in [m]$ , we build a hypergraph as follows.

1. For each variable  $x_t$ , we construct a *variable gadget*  $H_t$  in the following way.
  - We first create a *center vertex*  $w_t$ ,  $4\alpha_2$  sets of  $\alpha_1 - 1$  vertices  $A_t^i$  for  $i \in [4\alpha_2]$ , a set of  $4\alpha_2$  vertices  $U_i = \{u_t^1, \dots, u_t^{4\alpha_2}\}$ , and  $4\alpha_2$  hyperedges  $S_t^i = A_t^i \cup \{w_t, u_t^i, \dots, u_t^{i+\alpha_2-1}\}$  (superscripts are modulo  $4\alpha_2$ ) for  $i \in [4\alpha_2]$ .
  - For  $r \in [4]$ , let  $x_t^r = u_t^{\alpha_2(r-1)+1}$  and  $\bar{x}_t^r = u_t^{\alpha_2(r-1)+2}$ . Set  $X_t = \{x_t^1, x_t^2, x_t^3, x_t^4\}$  and  $\bar{X}_t = \{\bar{x}_t^1, \bar{x}_t^2, \bar{x}_t^3, \bar{x}_t^4\}$ . The vertices of  $X_t$  (resp.  $\bar{X}_t$ ) are called the *non-negated* (resp. *negated*) *literal vertices* of  $H_t$ .
2. For each variable  $x_t$ ,
  - we create a set of  $p - \lambda_1$  vertices  $B_t$  and a hyperedge  $S'_t = B_t \cup \{w_t, u_t^{\alpha_2}, \dots, u_t^{\alpha_2 - \lambda_1 + 2}\}$ .
  - add a  $(k - 4\lambda_1 - 1)$ -degree-gadget on  $w_t$ .
  - add a  $(k - \lambda_1)$ -degree-gadget on every vertex in  $A_t^i$  for  $i \in [4\alpha_2]$ .
3. For each clause  $C_j = (\ell_1 \vee \ell_2 \vee \ell_3)$ , we identify  $y_1, y_2, y_3$  into a *clause vertex*  $c_j$ , where  $y_i = x_t^r$  if  $\ell_i = x_t$  and  $\ell_i$  is the  $r$ -th occurrence of  $x_t$ , and  $y_i = \bar{x}_t^r$  if  $\ell_i = \bar{x}_t$  and is the  $r$ -th occurrence of  $x_t$ . We also add a  $(k - 6\lambda_2 - 1)$ -degree vertex at  $c_j$  in **Case B**-(i), and a  $(k - 6\lambda_1 - 3\alpha_2 + 1)$ -degree vertex at  $c_j$  in **Case B**-(ii).

We will show that there is an assignment  $\phi$  satisfying  $\Phi$  if and only there is an  $(F, H, k)$ -graph  $G$ .

Let  $z$  be a vertex in  $V_{\lambda_1}$  adjacent to a vertex  $y$  in  $N_1$  and let  $\bar{y}$  be a vertex in  $N_0$  in **Case B**-(i) or a vertex in  $N_1$  not adjacent to  $z$  in **Case B**-(ii). Note that  $\bar{y}$  and  $z$  are not adjacent. Let  $(y_1, \dots, y_{\alpha_2-2})$  be an ordering of  $V_{\lambda_2} \setminus \{y, \bar{y}\}$  such that  $y_1, \dots, y_{\lambda_1-1}$  are adjacent to  $z$  and  $y_{\lambda_1}, \dots, y_{\alpha_2-2}$  are not adjacent to  $z$ .

Assume that there is  $\phi$  satisfying  $\Phi$ , we construct a graph  $G$  as follows. Let  $G[S_t^i]$  be a copy of  $F$  such that  $w_t$  has degree  $\lambda_1$  and is adjacent to the  $\lambda_1 - 1$  vertices  $u_t^{\alpha_2}, \dots, u_t^{\alpha_2 - \lambda_1 + 2}$ .

In a variable gadget  $H_t$ , for every  $i \in [4\alpha_2]$ , we let  $G[S_t^i]$  be a copy of  $F$  such that  $w_t$  corresponds to the vertex  $z$ , and

- $A_t^i$  corresponds to  $V_{\lambda_1} \setminus \{z\}$ .
- if  $\phi(x_t) = \text{true}$  (resp.  $\phi(x_t) = \text{false}$ ), then each vertex in  $X_t$  (resp.  $\bar{X}_t$ ) corresponds to  $y$ , and each vertex in  $\bar{X}_t$  (resp.  $X_t$ ) to  $\bar{y}$ .
- for  $i \in [\alpha_2 - 2]$ ,  $u_t^{(p-1)r-i+1}$  corresponds to  $y_i$ .

For any  $d$ -degree-gadget  $D$  at a vertex  $v$ , let  $G[D]$  be an  $(F, D, k)$ -graph in which  $v$  has degree  $d$ .

Let us check that  $\Delta(G) \leq k$ .

- $w_t$  is adjacent to  $4\lambda_1$  vertices in  $H_t$  and one more in  $W_t \subset V(S'_t)$ , and it has degree  $k - 4\lambda_1 - 1$  in its degree-gadget, then  $w_t$  has degree  $k$  in total.
- Any literal vertex which is not identified to any clause vertex and not in  $S'_t$  has degree at most  $\gamma\alpha_2 + 2(\lambda_2 - \gamma)$  in its variable gadget (it is adjacent to at most  $\gamma$  vertices in each  $A_t^i$  in a hyperedge to which it belongs and there are  $\alpha_2$  such hyperedges; and  $f(x) = x\alpha_2 + 2(\lambda_2 - x)$  is increasing). So, it has degree less than  $k$ .
- Any literal vertex which is not identified to any clause vertex and in  $S'_t$  has degree at most  $\gamma\alpha_2 + 2(\lambda_2 - \gamma)$  in its variable gadget and is adjacent to at most  $\lambda_1$  vertices in  $B_t$ . So it has degree at most  $k$ .

- Each clause vertex  $c_j$  is in three variable gadget. In **Case B-(i)**,  $c_j$  (resp. **Case B-(ii)**), in each of these gadgets,  $c_j$  has degree either  $2\lambda_2 - 1$  if it is adjacent to  $w_t$  or  $2\lambda_2$  (resp.  $\alpha_2 + 2(\lambda_2 - 1)$ ) otherwise. Moreover at least one of the literals is true, its corresponding literal vertex has degree  $2\lambda_2 - 1$  in its variable gadget. Therefore  $c_j$  at most  $6\lambda_2 - 1$  neighbors (resp.  $2\alpha_2 + 6(\lambda_2 - 1) + 1$ ) in variable gadgets. It also has  $k - 6\lambda_2 + 1$  (resp.  $k - 6\lambda_1 - 3\alpha_2 + 1$ ) neighbors in its degree-gadget. Hence, in  $G$ , it has degree at most  $k$ .
  - Any vertex which is in a degree-gadget but in no variable gadget has degree at most  $2\lambda_t \leq k$  since it belongs to at most two hyperedges.
  - Any vertex in  $B_t$  has degree at most  $\lambda_2 < k$ .
- Hence,  $G$  is an  $(F, H, k)$ -graph.

Conversely, let  $G$  be an  $(F, H, k)$ -graph.

Observe that any vertex in  $A_t^i$  for  $i \in [4\alpha_2]$  is in a  $(k - \lambda_1)$ -degree-gadget, then it has degree at most  $\lambda_1$  in  $S_t^i$ . Since any vertex has degree at least  $\lambda_1$  in a hyperedge, then every vertex in  $\bigcup_{i=1}^{4\alpha_2} A_t^i$  is a  $\lambda_1$ -vertex in any hyperedge to which it belongs.

**Claim 21.3.** *For every  $t \in [n]$  the following hold.*

- (a)  $w_t$  has degree  $4\lambda_1$  in  $H_t$  and  $w_t$  is a  $\lambda_1$ -vertex in every hyperedge of  $H_t$ .
- (b) There is  $I \in [\alpha_2]$  of size  $\lambda_1$  such that for all  $i \in I$  and  $r \in [4]$ ,  $u_t^{\alpha_2 r - i + 1}$  is adjacent to  $w_t$ ; and  $[\lambda_1 - 1] \subset I$ .

This claim can be proved in exactly the same way as Claim 21.2.

We have that every vertex in  $U_t$  is a  $\lambda_2$ -vertex in any hyperedge to which it belongs (since  $w_t$  and  $\alpha_1 - 1$  vertices of  $A_t^i$  for  $i \in [4\alpha_2]$  are  $\lambda_1$ -vertices).

Claim 21.3 implies that the vertices of  $X_t$  (resp.  $\overline{X}_t$ ) are either all adjacent to  $w_t$  or all non-adjacent to  $w_t$ . Moreover,  $w_t$  is adjacent to  $\lambda_1 - 1$  vertices in  $U_t$  but not in  $X_t \cup \overline{X}_t$ . Hence if the vertices of  $X_t$  are adjacent to  $w_t$ , the vertices of  $\overline{X}_t$  are not (and vice-versa).

Let  $\phi$  be the truth assignment defined by  $\phi(x_t) = \text{true}$  if  $w_t$  is adjacent to all vertices of  $X_t$  in  $H_t$ , and  $\phi(x_t) = \text{false}$  otherwise.

A clause vertex  $c_j$  has degree at most  $k$ . Because of its degree-gadget, in **Case B-(i)** (resp. **Case B-(ii)**), it has degree at most  $6\lambda_2 - 1$  (resp.  $6\lambda_1 + 3\alpha_2 - 1$ ) in  $H_t$ . Now, since it is the identification of three literal vertices,  $c_j$  has degree less than  $2\lambda_2$  (resp.  $2\lambda_1 + \alpha_2$ ) in one variable gadget  $H_t$ .

**Claim 21.4.** *Let  $i \in [4\alpha_2]$ . If  $u_t^i$  is not adjacent to  $w_t$ , then the following holds.*

- (i)  $u_t^i$  has degree at least  $2\lambda_2$  in  $G[H_t]$ ;
- (ii) If  $N_0 = \emptyset$ , then  $u_t^i$  has degree at least  $2\lambda_1 + \alpha_2$  in  $G[H_t]$ ;

*Proof of Claim.*  $u_t^i$  has at least  $\lambda_2$  neighbors in each of  $S_t^i$  and  $S_t^{i-\alpha_1+1}$ . But the intersection of those hyperedges is  $\{w_t, u_t^i\}$ . As it is not adjacent to  $w_t$ ,  $u_t^i$  has at least  $2\lambda_2$  neighbors in  $S_t^i \cup S_t^{i-\alpha_1+1}$ . This proves (i).

If  $N_0 = \emptyset$ , then for all  $i - \alpha_2 + 1 < i' < i$ ,  $u_t^i$  must be adjacent to at least one  $\lambda_1$ -vertex of  $S_t^{i'}$  which is in  $A_t^{i'}$ . Hence  $u_t^i$  has at least  $\alpha_2 - 2$  in  $\bigcup_{i-\alpha_2+1 < i' < i} A_t^{i'}$  which is disjoint from  $S_t^i \cup S_t^{i-\alpha_1+1}$ . Hence  $u_t^i$  has degree at least  $2\lambda_2 + \alpha_2 - 2 = 2\lambda_1 + \alpha_2$  in  $G[H_t]$ . This proves (ii).  $\diamond$

This claim implies that there is at least one variable gadget  $H_t$  in which  $c_j$  is adjacent to  $w_t$ . It implies that the corresponding literal of this vertex in  $C_j$  is true, and so  $C_j$  is satisfied. Consequently,  $\phi$  satisfies  $\Phi$ .

**Case C:** In this case,  $F[V_{\lambda_1}]$  is anticomplete,  $F[V_{\lambda_2}]$  is not regular, and  $V_{\lambda_2}$  satisfies one of the following.

- (i) there is a vertex of  $N_{\geq 2}$  that is not adjacent to all vertices in  $V_{\lambda_1}$  in  $F$ .
- (ii)  $V_{\lambda_2} = N_{\geq 2} \cup N_0$  and every vertex of  $N_{\geq 2}$  is adjacent to all vertices of  $V_{\lambda_1}$ . Since  $F[V_{\lambda_2}]$  is not complete, then there is a vertex in  $N_{\geq 2}$  which is not adjacent to a vertex in either  $N_0$  **Case C-(ii)-a** or  $N_{\geq 2}$  **Case C-(ii)-b**.

We set  $a = \max_{\substack{u \in V_{\lambda_1} \\ v \in N(u)}} |N(v) \cap N(u)|$ , and let  $k = 4(p - 2)(2\lambda_1 - a) + 4\lambda_1 + 1$ .

For conveniences, we denote some vertices of graph  $F$  as follows. Let  $z_0$  be a vertex in  $N_{\geq 2}$  such that there is  $z_1 \in V_{\lambda_1}$  adjacent to  $z_0$  with  $a = |N(z_0) \cap N(z_1)|$ . Let  $z \in N_{\geq 2}$  which is adjacent to the minimum number of vertices in  $N_0$ , and  $y, y' \in V_{\lambda_1}$  be vertices adjacent to  $z$  and

- in **Case C-(i)**, let  $\bar{y} \in V_{\lambda_1}$  be a vertex not adjacent to  $z$ .
- in **Case C-(ii)**, let  $\bar{y}$  be a vertex not adjacent to  $z$  such that  $\bar{y} \in N_0$  if  $z$  is not adjacent to all vertices in  $N_0$  and  $\bar{y} \in N_{\geq 2}$  otherwise.

We give a reduction from (3,4)-SAT.

Given a formula  $\Phi$  of (3,4)-SAT with variables  $x_t, t \in [n]$  and clauses  $C_j, j \in [m]$ , we build a hypergraph as follows.

1. For each variable  $x_t$ , we construct a *variable gadget*  $H_t$  in the following way.
 

We first create a *center vertex*  $w_t$ , a set of  $4(p-2)$  vertices  $D_t = \{d_t^1, \dots, d_t^{4(p-2)}\}$ , a set of  $4(p-2)$  vertices  $U_t = \{u_t^1, \dots, u_t^{4(p-2)}\}$ , and  $4(p-2)$  hyperedges  $S_t^i = \{w_t, d_t^i, u_t^i, \dots, u_t^{i+p-3}\}$  (superscripts are modulo  $4(p-2)$ ) for  $i \in [4(p-2)]$ .

For  $r \in [4]$ , let  $x_t^r = u_t^{r(p-2)-p+3}$  and  $\bar{x}_t^r = u_t^{r(p-2)-p+4}$ . Set  $X_t = \{x_t^1, x_t^2, x_t^3, x_t^4\}$  and  $\bar{X}_t = \{\bar{x}_t^1, \bar{x}_t^2, \bar{x}_t^3, \bar{x}_t^4\}$ . The vertices of  $X_t$  (resp.  $\bar{X}_t$ ) are called the *non-negated* (resp. *negated*) *literal vertices* of  $H_t$ .
2. For each variable  $x_t$ ,
  - We create a set  $Y_t$  of  $p - \lambda_1$  vertices and a hyperedge  $S'_t = Y_t \cup \{w_t\} \cup \{u_t^{p-2}, \dots, u_t^{p-\lambda_1}\}$ .
  - For any  $i \in [4(p-2)]$ , we add two sets of  $p - \lambda_1 - 1$  vertices  $A_t^i, B_t^i$  and a set of  $\lambda_1 - 1$  vertices  $C_t^i$ , and two hyperedges  $A_t^i \cup C_t^i \cup \{d_t^i, w_t\}$  and  $B_t^i \cup C_t^i \cup \{d_t^i, w_t\}$ . We call this a *fickle-gadget*  $F_t^i$ .
  - We add a  $(k - 2\lambda_1 + 1)$ -degree-gadget on every vertex in  $D_t$ .
3. For each clause  $C_j = (\ell_1 \vee \ell_2 \vee \ell_3)$ , we identify  $y_1, y_2, y_3$  into a *clause vertex*  $c_j$ , where  $y_i = x_t^r$  if  $\ell_i = x_t$  and  $\ell_i$  is the  $r$ -th occurrence of  $x_t$ , and  $y_i = \bar{x}_t^r$  if  $\ell_i = \bar{x}_t$  and is the  $r$ -th occurrence of  $x_t$ . We also add at  $c_j$  a  $(k - 6\lambda_1 + 1)$ -degree-gadget in **Case C-(i)**, a  $(k - 6\lambda_1 - 3)$ -degree-gadget in **Case C-(ii)-a**, and a  $(k - 6\lambda_1 - 3p + 6)$ -degree-gadget in **Case C-(ii)-b**.

We will show that there is an assignment  $\phi$  satisfying  $\Phi$  if and only if there is an  $(F, H, k)$ -graph  $G$ .

Assume that there is  $\phi$  satisfying  $\Phi$ , we construct a graph  $G$  as follows.

Let  $(y_1, \dots, y_{p-4})$  be an ordering of  $V(F) \setminus \{z, y, y', \bar{y}\}$  such that  $y_1, \dots, y_{\lambda_1-2}$  are adjacent to  $z$  and  $y_{\lambda_1-1}, \dots, y_{p-4}$  are not adjacent to  $z$ .

In both hyperedges of any fickle-gadget  $F_t^i$ ,  $w_t$  corresponds to  $z_0$  and  $d_t^i$  corresponds  $z_1$ .  $d_t^i$  is adjacent to  $w_t$  and all vertices in  $C_t^i$ , while  $w_t$  is adjacent to  $a$  vertices in  $C_t^i$  and  $\lambda_2 - a$  other ones in each of  $A_t^i, B_t^i$ . Let  $G[S'_t]$  be a copy of  $F$  such that  $w_t$  has degree  $\lambda_1$  and is adjacent to the  $\lambda_1 - 1$  vertices  $u_t^{p-2}, \dots, u_t^{p-\lambda_1}$ .

For each variable gadget  $H_t$ , for every  $i \in [4\alpha_2]$ , let  $G[S'_t]$  be a copy of  $F$  such that  $w_t$  corresponds to  $z$ , and

- if  $\phi(x_t) = \text{true}$  (resp. *false*), then each vertex of  $X_t$  (resp.  $\bar{X}_t$ ) corresponds to  $y$  and each vertex of  $\bar{X}_t$  (resp.  $X_t$ ) corresponds to  $\bar{y}$ .
- in any  $S'_t, d_t^i$  corresponds to  $y'$ .
- for  $i \in [p-4]$  and  $r \in [4]$ ,  $u_t^{(p-2)r+1-i}$  corresponds to  $y_i$ .

For any  $d$ -degree-gadget  $D$  at a vertex  $v$ , let  $G[D]$  be an  $(F, D, k)$ -graph in which  $v$  has degree  $d$ .

Let us check that  $G$  has degree at most  $k$ .

- Any vertex  $d_t^i \in D_t$  has degree  $(k - 2\lambda_1 + 1)$  in its degree-gadget and  $\lambda_1$  in the fickle-gadget  $F_t^i$  and  $\lambda_1 - 1$  other vertices in  $V(S'_t \setminus \{w_t\})$ , thus it has degree  $k$ .
- Any vertex in  $Y_t \subset V(S'_t)$  has degree at most  $\lambda_2$ .
- $w_t$  has  $(2\lambda_1 - a)$  neighbor in each of the  $4(p-2)$  fickle-gadgets and is adjacent to  $4\lambda_1$  vertices in  $U_t$  and one more in  $Y_t$ . Thus it has degree  $k$  in  $G$ .
- Any vertex in a degree-gadget which is not in  $H_t$  has degree at most  $2\lambda_2$ .
- Any vertex in  $U_t$  but not in  $X_t \cup \bar{X}_t \cup S'_t$  has degree at most  $2\lambda_2$  if it is not adjacent to any vertex in  $D_t$  or at most  $2\lambda_2 + p - 2$  if adjacent to a vertex in  $D_t$  for each hyperedge to which it belongs.
- Any vertex in  $U_t \cap S'_t$  has degree at most  $2\lambda_1 + p - 2$  in  $H_t$  and it is adjacent to at most  $\lambda_1$  vertices in  $Y_t$ , so it has degree less than  $k$ .
- Any clause vertex  $c_j$  has degree  $d$  in its  $d$ -degree-gadget. Moreover, in each of its variable gadget,  $c_j$  has degree either  $2\lambda_1 - 1$  if it adjacent to the center vertex or  $2\lambda_1$  in **Case C-(i)**,  $2\lambda_2$  in **Case C-(ii)-a**,



and  $2\lambda_1 + p - 2$  in **Case C**-(ii)-b otherwise. Since there at least one of three literals of the clause  $C_j$  is true,  $c_j$  has  $2\lambda_1 - 1$  in one of its variable gadget, and thus degree at most  $k$  in total.

Hence,  $G$  is an  $(F, H, k)$ -graph.

Conversely, let  $G$  be an  $(F, H, k)$ -graph.

**Claim 21.5.** *For every  $t \in [n]$  the following hold.*

- (a) *For all  $i \in [4(p-2)]$ ,  $d_t^i$  is adjacent to  $w_t$  and has degree  $\lambda_1$  in any hyperedge of  $F_t^i \cup S_t^i$ .*
- (b)  *$w_t$  is a  $\lambda_2$ -vertex in every hyperedge of  $H_t$ . Furthermore, there is  $I \in [p-2]$  of size  $\lambda_1$  such that for all  $i \in I$  and  $r \in [4]$ ,  $u_t^{(p-2)r-i+1}$  is adjacent to  $w_t$  and  $[\lambda_1 - 1] \subset I$ .*
- (c)  *$w_t$  has degree  $k$  in  $G$ .*

*Proof of Claim.* Observe that any vertex  $d_t^i \in X_t$  is in a  $(k - 2\lambda_1 + 1)$ -degree-gadget, then it has degree at most  $2\lambda_1 - 1$  in  $G[F_t^i \cup S_t^i]$ . Since  $S_t^i$  and  $F_t^i$  intersect only in  $\{d_t^i, w_t\}$  and  $d_t^i$  has degree at least  $\lambda_1$  in each, then  $d_t^i$  has degree at least  $2\lambda_1 - 1$  in  $G[F_t^i \cup S_t^i]$ . The equality holds when  $d_t^i$  is adjacent to  $w_t$  and  $\lambda_1 - 1$  vertices in  $C_t^i$ . Thus,  $d_t^i$  has degree  $\lambda_1$  in all hyperedges of  $F_t^i \cup S_t^i$  and is adjacent to  $w_t$ . This proves (a).

In any fickle-gadget  $F_t^i$ , from (a), every vertex in  $\{w_t\} \cup C_t^i$  is adjacent to  $d_t^i$  and must be a  $\lambda_2$ -vertex in the two hyperedges of  $F_t^i$ . Thus,  $w_t$  is adjacent to at most  $a$  vertices in  $C_t^i$ , and so has degree at least  $2\lambda_2 - a$  in  $G[F_t^i]$ . Since  $w_t$  is in  $4(p-2)$  fickle-gadgets, then it has degree at least  $4(p-2)(2\lambda_2 - a)$  in  $G[\bigcup_{i=1}^{4(p-2)} F_t^i]$ . Moreover, from (a), for  $i \in [4(p-2)]$ ,  $w_t$  is adjacent to  $d_t^i$  which has degree  $\lambda_1$  in  $S_t^i$ . Thus  $w_t$  must be a  $\lambda_2$ -vertex in  $S_t^i$  because  $F$  is anticomplete.

Since  $w_t$  is in  $S_t^i$  which intersects  $H_t$  in  $\lambda_1 - 1$  vertices and  $w_t$  must have degree at least  $\lambda_1$  in  $G[S_t^i]$ , then it is adjacent to at least one vertex in  $Y_t \subset V(S_t^i)$ . Therefore,  $w_t$  is adjacent to at most  $k - 4(p-2)(2\lambda_2 - a) - 1 = 4\lambda_1$  vertices in  $H_t$ .

Now for every  $i \in [p-2]$ ,  $w_t$  belongs to four hyperedges  $S_t^{(p-2)r-i}$ ,  $r \in [4]$ , which pairwise intersect in  $\{w_t\}$ . Hence  $w_t$  has degree exactly  $\lambda_1$  in each  $S_t^{(p-2)r-i} \setminus \{d_t^{(p-2)r-i}\}$  and then is adjacent to  $4\lambda_1$  vertices in  $U_t$ .

If a vertex  $u_t^i$  is adjacent to  $w_t$ , then so is  $u_t^{i+p-2}$  because  $w_t$  has degree exactly  $\lambda_1$  in both  $S_t^i \setminus \{d_t^i\}$  and  $S_t^{i+1} \setminus \{d_t^{i+1}\}$ . Therefore there is  $I \in [p-2]$  of size  $\lambda_1$  such that  $w_t$  is adjacent to  $u_t^{(p-2)r-i+1}$  for all  $i \in I$  and  $r \in [4]$ .

Since  $w_t$  has degree  $4\lambda_1$  in  $H_t$ , then  $w_t$  is adjacent to exactly one vertex in  $Y_t$  and so must be adjacent to  $\lambda_1 - 1$  vertices in  $S_t^i \setminus Y_t$  which are  $u_t^{p-2}, \dots, u_t^{p-\lambda_1}$ . It implies that  $[\lambda_1 - 1] \subset I$ . This completes the proof of (b).

From (a), (b) we have that  $w_t$  has degree  $4(p-2)(2\lambda_2 - a)$  in  $G[\bigcup_{i=1}^{4(p-2)} F_t^i]$ , it is adjacent to  $4\lambda_1$  vertices in  $U_t$  and one in  $Y_t$ . Thus,  $w_t$  has degree  $k$  in total. This proves (c).  $\diamond$

Claim 21.5(b) implies that the vertices of  $X_t$  (resp.  $\bar{X}_t$ ) are either all adjacent to  $w_t$  or all non-adjacent to  $w_t$ . Moreover,  $w_t$  is adjacent to  $4(\lambda_1 - 1)$  vertices in  $U_t \setminus (X_t \cup \bar{X}_t)$ . Hence if the vertices of  $X_t$  are adjacent to  $w_t$ , the vertices of  $\bar{X}_t$  are not (and vice-versa).

Let  $\phi$  be the truth assignment defined by  $\phi(x_t) = \text{true}$  if  $w_t$  is adjacent to all vertices of  $X_t$  in  $H_t$ , and  $\phi(x_t) = \text{false}$  otherwise. Observe the following.

- In **Case C**-(i), any clause vertex  $c_j$  is in a  $(k - 6\lambda_1 + 1)$ -degree-gadget, so it has degree at most  $6\lambda_1 - 1$  in the union of its three variable gadgets. Thus it has degree less than  $2\lambda_1$  in one of its variable gadgets  $H_t$ . Since any vertex in  $U_t$  has degree at least  $2\lambda_1 - 1$  in  $G[H_t]$ , with equality only if it is adjacent to  $w_t$ , the vertex  $c_j$  is adjacent to  $w_t$ . Hence, the corresponding literal to this literal vertex is true and so  $C_j$  is satisfied.
- In **Case C**-(ii), any clause vertex  $c_j$  is in a  $k - d$ -degree-gadget, then has degree at most  $d$  in the union of its three variable gadgets. Hence  $c_j$  has degree at most  $\lfloor d/3 \rfloor$  neighbors in one of those variable gadget, say  $H_t$ . Let  $i$  be the index such that  $c_j = u_t^i$ .

Suppose for a contradiction that  $c_j$  is not adjacent to  $w_t$ . Then it is a  $\lambda_2$ -vertex in every hyperedge of  $H_t$ .

Vertex  $c_j$  has at least  $\lambda_2$  neighbours in each of  $S_t^i$  and  $S_t^{i-p-3}$  which intersect in  $\{c_j, w_t\}$ . Hence  $c_j$  has at least  $2\lambda_2$  neighbours in  $S_t^i \cup S_t^{i-p-3}$ . In **Case C**-(ii)-a,  $\lfloor d/3 \rfloor = 2\lambda_1 + 1 < 2\lambda_2$ , so we get a contradiction.

In **Case C**-(ii)-b, since  $w_t$  is adjacent to all  $d_t^i$  by Claim 21.5 (b),  $c_j$  corresponds to a vertex in  $N_{\geq 2}$  in every hyperedge of  $H_t$  to which it belongs. Therefore it is adjacent to all  $\lambda_1$ -vertices in these hyperedges and thus in particular to all  $d_t^{i'}$  for all  $i - p + 3 < i' < i$ . Hence  $c_j$  has degree at least  $2\lambda_2 + p - 4$  in  $V(H_t)$ . But  $\lfloor d/3 \rfloor = 2\lambda_1 + p - 3 = 2\lambda_2 + p - 5$ , a contradiction.

In both subcases, the vertex  $c_j$  is adjacent to  $w_t$ . Hence, the corresponding literal to this literal vertex is true and so  $C_j$  is satisfied.

Consequently,  $\phi$  satisfies  $\Phi$ . □

## 6 Further research

Problem 1 asks for a characterization of the pairs  $(F, k)$  for which  $(\Delta \leq k)$ - $F$ -OVERLAY is polynomial-time solvable and those for which it is NP-complete. As a partial answer, we proved that  $\text{np}(F) < +\infty$  if and only if  $F$  is **standard**, that is neither a complete graph nor an anticomplete graph. We believe that the following holds.

**Conjecture 22.** For every graph  $F$ ,  $(\Delta \leq k)$ - $F$ -OVERLAY is polynomial-time solvable when  $k < \text{np}(F)$  (and NP-complete otherwise).

Thus answering Problem 1 is equivalent to determining  $\text{np}(F)$ . However, it would already be interesting to prove that for any pair  $(F, k)$ ,  $(\Delta \leq k)$ - $F$ -OVERLAY is either polynomial-time solvable or NP-complete. A first step to prove this is to prove Conjecture 2.

Furthermore, we made no attempt to minimize the upper bound on  $\text{np}(F)$ , our goal was just to prove such a bound exists. In fact, our proof in Section 5 shows the general upper bound  $\text{np}(F) \leq 8|F|\delta(F)$  for every standard graph  $F$ . However, there are many graphs for which the proof shows  $\text{np}(F) \leq 6\Delta(F)$ . It motivates the following questions.

**Problem 23.** Does there exist a constant  $c$  such that  $\text{np}(F) \leq c \cdot \Delta(F)$  for every standard graph  $F$  ?

Moreover, better upper bounds can certainly be obtained for certain classes of graph  $F$ . For example, for every path  $P_p$  with  $p \geq 4$ , Theorem 20 **Case B** (i) yields  $\text{np}(P_p) \leq p + 1$ , and if  $F$  is a disjoint union of paths, then Theorem 20 yields  $\text{np}(F) \leq 8|F| - 11$  (the worst case is given by **Case C** (i) when  $F$  has a component which is a  $P_3$ ). In Appendix A, we show  $\text{np}(P_p) \leq 4$ . We also obtain a better upper bound for disjoint union of paths: we prove that  $\text{np}(F) \leq 5$  for such a graph  $F$ .

Getting lower bounds would also be interesting. The trivial lower bound is  $\text{np}(F) \geq \Delta(F)$ . There are graphs for which this lower bound is attained (the graphs with one edge of order at least 4 for example), and other for which it is not (the paths for example, see Havet et al. [10]). It would be nice to characterize the graphs such that  $\text{np}(F) = \Delta(F)$ . It would also be nice to find graphs  $F$  such that  $\text{np}(F) - \Delta(F)$  is large. The largest known difference is 2 for  $C_4$ , the cycle on four vertices. Indeed Havet et al. [10] proved  $\text{np}(C_4) \geq 4 = \Delta(C_4) + 2$ .

There are very few standard graphs  $F$  for which  $\text{np}(F)$  is known. The only ones are the graphs with one edge. Havet et al. [10] proved  $\text{np}(O_3) = 2$  and  $\text{np}(O_p) = 1$  for all  $p \geq 4$ . It would be nice to determine  $\text{np}(F)$  for other graphs. A first problem is to do it for paths. Havet et al. [10] proved that  $\text{np}(P_p) > 2$  for all  $p$ . In Appendix A of this paper, we prove  $\text{np}(P_p) \leq 4$  for  $p \geq 3$ . Hence it is open to answer whether  $\text{np}(P_p) = 3$ . Note that paths have minimum degree 1, so by Lemma 12, Conjecture 2 holds for such graphs. A second natural step is to determine  $\text{np}(F)$  when  $F$  is a cycle. Indeed, Conjecture 2 is not yet proved for such graphs and there are non-trivial polynomial-time algorithms when  $F$  is a cycle, as shown by the example of  $C_4$ .

In this paper, we only considered the case when the family  $\mathcal{F}$  of admissible graphs has size 1. It is natural and interesting to study the more general case when  $\mathcal{F}$  can have an arbitrary size, finite or infinite.

**Problem 24.** Characterize the pairs  $(\mathcal{F}, k)$  for which  $(\Delta \leq k)$ - $\mathcal{F}$ -OVERLAY is polynomial-time solvable and those for which it is NP-complete.

We believe that Conjectures 25 and 2 extends to any family  $\mathcal{F}$ .

**Conjecture 25.** For every family of graphs  $\mathcal{F}$ , there exists an integer  $k_0$  such that  $(\Delta \leq k)$ - $\mathcal{F}$ -OVERLAY is polynomial-time solvable when  $k < k_0$  and NP-complete otherwise.

**Conjecture 26.** If  $(\Delta \leq k)$ - $\mathcal{F}$ -OVERLAY is NP-complete, then  $(\Delta \leq k+1)$ - $\mathcal{F}$ -OVERLAY is also NP-complete.

We also strongly believe that Theorem 3 can be extended to any family  $\mathcal{F}$ . Defining  $\text{np}(\mathcal{F})$  as the minimum integer  $k_0$  such that  $(\Delta \leq k)$ - $\mathcal{F}$ -OVERLAY is NP-complete for all  $k \geq k_0$  or  $+\infty$  if no such  $k_0$  exists, we conjecture the following.

**Conjecture 27.**  $\text{np}(\mathcal{F}) = +\infty$  if and only if all elements of  $\mathcal{F}$  are complete or anticomplete.

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## A Disjoint union of paths

In this appendix, we study on the family of disjoint union of paths and aim to prove the following theorem.

**Theorem 28.** *Let  $F$  be a disjoint union of paths with  $\delta(F) \geq 1$ . If  $F \neq K_2$ , then  $\text{np}(F) \leq 5$ .*

A disjoint union of paths contains several paths with their lengths. The following result allows us to consider only a longest path in a disjoint of paths which is simpler.

**Lemma 29.** *Let  $F$  be a disjoint union of paths among which  $P$  is a shortest one. If  $(\Delta \leq k)$ - $(F - P)$ -OVERLAY is NP-complete, then  $(\Delta \leq k)$ - $F$ -OVERLAY is NP-complete. Hence  $\text{np}(F) \leq \text{np}(F - P)$ .*

*Proof.* Set  $|F| = p$ ,  $|P| = q$ , and  $F' = F - P$ . So  $|F'| = p - q$ .

We give a reduction from  $(\Delta \leq k)$ - $F'$ -OVERLAY to  $(\Delta \leq k)$ - $F$ -OVERLAY.

Let  $H'$  be a  $(p - q)$ -uniform hypergraph. Let us build a  $p$ -uniform hypergraph  $H$  from  $H'$ . For every hyperedge  $S$  of  $H'$ , we do the following: we create a set  $U_S = \{u_S^1, \dots, u_S^q\}$  of  $q$  vertices, and a set  $W_S$  of  $p - q$  vertices; we add the hyperedges  $S \cup U_S$  and  $U_S \cup W_S$  to  $E(H)$ ; we add  $k - 1$  pendant hyperedges at  $u_S^1$  and  $u_S^q$ , and  $k - 2$  pendant hyperedges at each  $u_S^i$  for  $2 \leq i \leq q - 1$ .

We shall prove that there is an  $(F, H, k)$ -graph  $G$  if and only if there is an  $(F', H', k)$ -graph  $G'$ .

Assume first that there is an  $(F, H, k)$ -graph  $G$ . Note that every vertex in a hyperedge  $S$  of  $H$  has degree at least 1 in  $G[S]$ . For each hyperedge  $S$  of  $H'$ , the vertex  $u_S^1$  is in  $k - 1$  pendant hyperedges, in each of which it has degree 1. Therefore  $u_S^1$  has degree at most 1 in  $G[S \cup U_S \cup W_S]$ . Now  $u_S^1$  has degree 1 in  $G[S \cup U_S]$  and  $G[U_S \cup W_S]$ , so necessarily  $u_S^1$  has a unique neighbor in  $U_S$  and no neighbor in  $S \cup W_S$ . Similarly,  $u_S^q$  has a unique neighbor in  $U_S$  and no neighbor in  $S \cup W_S$ . If  $q = 2$ , then  $G[U_S] = P$  and there is no edge between  $U_S$  and  $S$  in  $G$ . If  $q > 2$ , then the neighbor of  $u_S^1$  in  $U_S$  cannot be  $u_S^q$  for otherwise  $G[\{u_S^1, u_S^q\}]$  would be a connected component of  $G[S \cup U_S]$  on two vertices, a contradiction to the fact that  $P$  is the smallest component of  $F$ . Hence, without loss of generality, we may assume that the unique neighbor of  $u_S^1$  in  $U_S$  is  $u_S^2$ . But now since  $q > 2$ , the vertex  $u_S^2$  has at least two neighbors in both  $G[S \cup U_S]$  and  $G[U_S \cup W_S]$ . Moreover,  $u_S^2$  is in  $k - 2$  pendant hyperedges. Therefore  $u_S^2$  has exactly two neighbors in  $U_S$  and no neighbor in  $S \cup W_S$ . If  $q = 3$ , then  $G[U_S] = P$  and there is no edge between  $U_S$  and  $S$  in  $G$ . If  $q > 3$ , then the neighbor of  $u_S^2$  distinct from  $u_S^1$  in  $U_S$  cannot be  $u_S^q$  for otherwise  $G[\{u_S^1, u_S^2, u_S^q\}]$  would be a connected component of  $G[S \cup U_S]$  of order 3, a contradiction to the fact that  $P$  is the smallest component of  $F$ . Hence, without loss of generality, we may assume that the neighbor of  $u_S^2$  distinct from  $u_S^1$  in  $U_S$  is  $u_S^3$ . And so on, by induction on  $i \leq q - 1$ , one can show that the neighbors of  $u_S^i$  in  $G[S \cup U_S \cup W_S]$  are  $u_S^{i-1}$  and  $u_S^i$ . Therefore,  $G[U_S] = P$  and there is no edge between  $U_S$  and  $S$  in  $G$ , and so  $G[S] = F - P = F'$ . Consequently,  $G' = G[H']$   $F'$ -overlays  $H'$  and so is an  $(F', H', k)$ -graph  $G'$ .

Assume now that there is an  $(F', H', k)$ -graph  $G'$ . Let  $G$  be the graph built from  $G'$  as follows. For each hyperedge  $S$ , we let  $G[U_S \cup W_S]$  be a copy of  $F$  such that  $G[U_S] = P$ , and the subgraph of  $G$  induced by every pendant hyperedge at some vertex  $x$  is any copy of  $F$  in which  $x$  has degree 1. Observe that  $G[S \cup U_S] = G'[S] \cup P$ . Thus  $G[S \cup U_S]$  contains  $F$  because  $G'[S]$  contains  $F'$ . Therefore  $G$   $F$ -overlays  $H$ . One easily checks that  $\Delta(G) \leq k$ , so  $G$  is an  $(F, H, k)$ -graph.  $\square$

By Lemma 29, it is sufficient to prove Theorem 28 for paths, and  $2K_2$  – the 1-regular graph on four vertices. By Theorem 13, we have  $\text{np}(2K_2) \leq 5$ , so it suffices to prove the result for paths.

**Theorem 30.**  $\text{np}(P_p) \leq 4$  for all  $p \geq 4$ .

*Proof.* By Lemma 12, it suffices to prove that  $(\Delta \leq 4)$ - $P_p$ -OVERLAY is NP-complete for  $p \geq 4$ . We give a reduction from (3,4)-SAT.

Given a formula  $\Phi$  of (3,4)-SAT with  $n$  variables  $x_t, t \in [n]$ , and  $m$  clauses  $C_j, j \in [m]$ , we construct a hypergraph  $H$  as follows.

- For each variable  $x_t$ , we add a *variable gadget*  $H(x_t)$  containing a *center vertex*  $c_t$ , four pairs of *literal vertices*  $x_t^1, \bar{x}_t^1, \dots, x_t^4, \bar{x}_t^4$ , and eight sets of  $p - 3$  vertices  $U_t^1, \dots, U_t^4$  and  $T_t^1, \dots, T_t^4$ . Set  $X_t = \{x_t^i \mid i \in [4]\}$  and  $\bar{X}_t = \{\bar{x}_t^i \mid i \in [4]\}$ . We add the four hyperedges  $\{c_t, x_t^i, \bar{x}_t^i\} \cup U_t^i$  for  $i \in [4]$ , and the four hyperedges  $\{c_t, \bar{x}_t^i, x_t^{i+1}\} \cup T_t^i$  for  $i \in [4]$  (superscripts are modulo 4).
- For each clause  $C_j = (\ell_1 \vee \ell_2 \vee \ell_3)$ , we add a *clause gadget*  $H(C_j)$  containing two *clause vertices*  $y_j, z_j$ , one set of  $p - 2$  vertices  $V_j$ , three sets of  $p - 3$  vertices  $W_j^1, W_j^2, W_j^3$ , and we distinguish a vertex  $w_j^i \in W_j^i$  for each  $i \in [3]$ . We add one hyperedge  $S_j = \{y_j, z_j\} \cup V_j$ , and for each literal  $\ell_i$  which is the  $r_i$ -th occurrence of this variable,  $r_i \in [4]$ , we add one hyperedge  $S_j^{\ell_i} = \{y_j, z_j, \ell_i^{r_i}\} \cup W_j^i$ . Finally, for  $i \in [3]$  we add three pendant hyperedges at each  $w_j^i$  (with new vertices).

We will prove that there exists a truth assignment  $\phi$  satisfying  $\Phi$  if and only if there is a  $(P_p, H, 4)$ -graph  $G$ . The general idea is that a variable  $x_t = \text{true}$  (resp.  $\text{false}$ ) if and only if the center vertex  $c_t$  is adjacent to all vertices of  $X_t$  (resp.  $\bar{X}_t$ ) (so each  $x_t^i$  has degree 2 (resp. 3) in  $G[H(x_t)]$  for all  $i \in [4]$ ).

Assume that there exists a truth assignment  $\phi$  satisfying  $\Phi$ . Let  $G$  be the graph obtained as follows.

- We first consider each variable  $x_t$ ,  $t \in [n]$ . If  $\phi(x_t) = \text{true}$  (resp.  $\text{false}$ ), then any subgraph induced by a hyperedge  $S$  of  $H(x_t)$  is a copy of  $P_p$  whose two endpoints are  $c_t$  and the vertex of  $S \cap X_t$  (resp.  $S \cap \bar{X}_t$ ), and in which  $c_t$  is adjacent to the vertex of  $S \cap \bar{X}_t$  (resp.  $S \cap X_t$ ). Then  $G[H(x_t)]$  is called a *true variable subgraph* (resp. *false variable subgraph*) on  $x_t$ , if in this subgraph the literal vertices in  $X_t$  have degree 2 (resp. 3) and the literal vertices in  $\bar{X}_t$  have degree 3 (respectively 2).
- We then consider each clause  $C_j = (\ell_1 \vee \ell_2 \vee \ell_3)$ ,  $j \in [m]$ . The induced subgraph  $G[S_j]$  is a copy of  $P_p$  where  $y_j$  is an endpoint and  $y_j$  is adjacent to  $z_j$  (therefore  $y_j$  has degree 1 and  $z_j$  has degree 2 in  $G[S_j]$ ). Let  $i \in [3]$ . If  $\phi(\ell_i) = \text{true}$ , then let  $G[S_j^{\ell_i}]$  be a *true literal subgraph*, that is it is a copy of  $P_p$  starting with  $(z_j, y_j, \ell_i^{\alpha_i})$  and ending at  $w_j^i$ ; it increases the degree of  $y_j$  by 1 in  $G$  and does not increase the degree of  $z_j$  in  $G$ . If  $\phi(\ell_i) = \text{false}$ , then let  $G[S_j^{\ell_i}]$  be a *false literal subgraph*, that is a copy of  $P_p$  starting with  $(w_j^i, z_j, y_j)$  and ending at  $\ell_i^{\alpha_i}$ ; it increases the degree of both  $y_j$  and  $z_j$  by 1 in  $G$ .

Finally for any hyperedge pendant at  $w_j^i$ , the subgraph induced by this hyperedge is a copy of  $P_p$  in which  $w_j^i$  as degree 1.

Observe that all the vertices of  $G$  have degree at most 4. In particular, for each clause  $C_j$ , the vertex  $z_j$  has degree 4 because there is a literal such that  $\phi(\ell_i) = \text{true}$ . Hence,  $G$  is a  $(P_p, H, 4)$ -graph.

Conversely, assume that  $G$  is a  $(P_p, H, 4)$ -graph.

Observe that the subgraph induced by the hyperedges of each variable gadget is either a true or a false variable subgraph. Indeed,  $c_t$  has degree at most 4, and degree 1 in each of the eight hyperedges of the variable gadget. In order to have all hyperedges  $P_p$ -overlaid it must contain either the four edges  $c_t x_t^i$ ,  $i \in [4]$  (true vertex subgraph), or the four edges  $c_t \bar{x}_t^i$  for  $i \in [4]$  (false vertex subgraph). Thus, we define a truth assignment  $\phi$  by setting  $\phi(x_t) = \text{true}$  (resp.  $\text{false}$ ) if  $H(x_t)$  is a true (resp. false) variable subgraph. We shall prove that  $\phi$  satisfies  $\Phi$ . We need the following claim.

**Claim 30.1.** *For any clause gadget  $H(C_j)$ ,  $y_j z_j \in E(G)$  and at least one of three literal vertices in  $H(C_j)$  has at least two neighbors in  $V(H(C_j))$ .*

*Proof of Claim.* Observe that in each  $G[S_j^{\ell_i}]$ , the vertex  $w_j^i$  must have degree 1, because it has three neighbors in its three pendant hyperedges.

Assume for contradiction that  $y_j z_j \notin E(G)$ , then both  $y_j$  and  $z_j$  have a neighbor in  $V_j$ . Moreover, in each  $G[S_j^{\ell_i}]$  at least one of  $y_j, z_j$  has degree at least 2 because  $w_j^i$  has degree 1. Consequently, at least one of  $y_j, z_j$  has degree more than 4 in  $G$ , a contradiction. Therefore,  $y_j z_j \in E(G)$ .

Assume for a contradiction that the three literal vertices have only one neighbor in  $V(H(C_j))$ . Then both  $y_j$  and  $z_j$  have at least two neighbours in each  $S_j^{\ell_i}$ . Moreover, one of them, say  $z_j$ , has two neighbors in  $S_j$ . Thus  $z_j$  is adjacent to  $y_j$  and at least one vertex in each of the four disjoint sets  $S_j^{\ell_i} \setminus \{y_j, z_j\}$ ,  $i \in [3]$  and  $V_j$ . Hence  $z_j$  has degree 5 in  $G$ , a contradiction.  $\diamond$

From this claim, in any  $G[H(C_j)]$ , at least one of the three literal vertices in  $H(C_j)$ , say  $\ell_i^{\alpha_i}$ , has at least two neighbors in  $V(H(C_j))$ . But then  $\ell_i^{\alpha_i}$  must have degree 2 in its variable gadget. Therefore, by definition of  $\phi$ , we have  $\phi(\ell_i) = \text{true}$ . We conclude that  $\phi$  satisfies  $\Phi$ .  $\square$

**Theorem 31.**  $\text{np}(P_3) \leq 4$ .

*Proof.* By Lemma 12, it suffices to prove that  $(\Delta \leq 4)$ - $P_3$ -OVERLAY is NP-complete. We give a reduction from (3,4)-SAT problem. The proof is very similar to that of Theorem 30, the differences lying in the construction of the clause gadgets. Therefore, we just give a sketch of the proof and leave the details to the reader.

Given a formula  $\Phi$  of (3,4)-SAT with  $n$  variables  $x_t, t \in [n]$ , and  $m$  clauses  $C_j, j \in [m]$ , we construct a 3-uniform hypergraph  $H$  as follows.

- For each variable  $x_t$ , we add a variable gadget  $H(x_t)$  containing  $c_t$ , four couples  $(x_t^i, \bar{x}_t^i)$  and the hyperedges are of the form  $\{c_t, x_t^i, \bar{x}_t^i\}$  and  $\{c_t, \bar{x}_t^i, x_t^{i+1}\}$  for  $i \in [4]$  (superscript are modulo 4).
- For each clause  $C_j = (\ell_1 \vee \ell_2 \vee \ell_3)$ , we add a clause gadget  $H(C_j)$  six new vertices  $y_j^1, y_j^2, y_j^3, w_j^1, w_j^2, w_j^3$ . We add the hyperedge  $S_j = \{y_j^1, y_j^2, y_j^3\}$ . For each literal  $\ell_i$  for  $i \in [3]$  which is the  $r_i$ -th occurrence of its variable, we add  $S_j^{\ell_i} = \{y_j^i, \ell_i^{r_i}, w_j^i\}$ . For each  $i \in [3]$ , we add three pendant hyperedges at  $w_j^i$  and a pendant hyperedge at  $y_j^i$ .

One that then can easily prove the following analogue of Claim 30.1.

**Claim 31.1.** *For any clause gadget  $H(C_j)$ , in any  $(P_3, H, 4)$ -graph, at least one of three literal vertices in  $H(C_j)$  has at least two neighbors in  $V(H(C_j))$ .*

This allows to show that there exists a truth assignment  $\phi$  satisfying  $\Phi$  if and only if there is a  $(P_3, H, 4)$ -graph  $G$ .  $\square$

Lemma 29 and Theorems 30, 31, and 13 directly imply Theorem 28.