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Semismooth and smoothing Newton methods for nonlinear systems with complementarity constraints: adaptivity and inexact resolution*

Ibtihel Ben Gharbia[†] Joëlle Ferzly^{†‡§} Martin Vohralík^{‡§} Soleiman Yousef[†]

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Abstract

We consider nonlinear algebraic systems with complementarity constraints stemming from numerical discretizations of nonlinear complementarity problems. The particularity is that they are non-differentiable, so that classical linearization schemes like the Newton method cannot be applied directly. To approximate the solution of such nonlinear systems, an iterative linearization algorithm like the semismooth Newton-min or an interior-point algorithm can be used. Alternatively, the non-differentiable nonlinearity can be smoothed, which allows a direct application of the Newton method. Corresponding linear systems can be solved only approximately using an iterative linear algebraic solver, leading to inexact approaches. In this work, we design a general framework to systematically steer these different ingredients. We first derive an a posteriori error estimate given by the norm of the considered system's residual. We then, relying on smoothing, design a simple strategy of tightening the smoothing parameter. We finally distinguish the smoothing, linearization, and algebraic error components, which enables us to formulate an adaptive algorithm where the linear and nonlinear solvers are stopped when the corresponding error components do not affect significantly the overall error. Numerical experiments indicate that the proposed algorithm, possibly in combination with the GMRES algebraic solver, ensures important savings in terms of the number of iterations and execution time. It appears rather promising in comparison with the other methods, namely since its performance seems remarkably stable over a range of academic and industrial problems.

Key words: nonlinear complementarity constraints, semismooth smoothing Newton methods, interior-point method, a posteriori error estimate, adaptivity, stopping criteria

1 Introduction

2 Consider a system of algebraic equations with complementarity constraints written in the following form: Find a
3 vector $\mathbf{X} \in \mathbb{R}^n$ such that

$$4 \quad \mathbb{E}\mathbf{X} = \mathbf{F}, \tag{1.1a}$$

$$5 \quad \mathbf{K}(\mathbf{X}) \geq \mathbf{0}, \mathbf{G}(\mathbf{X}) \geq \mathbf{0}, \mathbf{K}(\mathbf{X}) \cdot \mathbf{G}(\mathbf{X}) = \mathbf{0}, \tag{1.1b}$$

7 where for two integers $n > 1$ and $0 < m < n$, $\mathbb{E} \in \mathbb{R}^{n-m,n}$ is a matrix, $\mathbf{K} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $\mathbf{G} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ are (linear)
8 operators, and $\mathbf{F} \in \mathbb{R}^{n-m}$ is a given vector. The first line (1.1a) typically represents the discretization of a linear
9 partial differential equation. The second line (1.1b) then represents the complementarity constraints. It states that
10 the vectors $\mathbf{K}(\mathbf{X})$ and $\mathbf{G}(\mathbf{X})$ have nonnegative components and are orthogonal.

11 Complementarity problems have important applications in many fields: economics, engineering, operations
12 research, nonlinear analysis... In the literature, many theoretical results and numerical methods have been proposed
13 to solve problem (1.1), see for example the books of Facchinei and Pang [23, 24], Ito and Kunisch [30], Ulbrich [43],
14 Bonnans et al. [14], and the study of Aganagić [1].

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15 By means of so-called C-functions (C for complementarity), see [23, 24], the complementarity constraints (1.1b)
 16 can be rewritten as a system of equations $\mathbf{C}(\mathbf{X}) = \mathbf{0}$, where $\mathbf{C} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is nonlinear and semismooth. We then
 17 obtain the following equivalent formulation of problem (1.1): Find $\mathbf{X} \in \mathbb{R}^n$ such that

$$18 \quad \begin{aligned} \mathbb{E}\mathbf{X} &= \mathbf{F}, \\ \mathbf{C}(\mathbf{X}) &= \mathbf{0}. \end{aligned} \quad (1.2)$$

20 A direct application of the standard Newton method to (1.2) is, however, impeded by the fact that $\mathbf{C}(\mathbf{X})$ is not
 21 differentiable. An introduction of the Clarke differential [16] allows to give a weaker differentiability meaning and
 22 leads to the class of semismooth Newton methods, with reputedly good convergence properties [33, 24, 10, 11,
 23 12, 19, 20]. These methods are in certain cases equivalent to primal–dual active set strategies, see Hintermüller
 24 et al. [27]. Moreover, in [47], a regularized semismooth Newton method combined with a hyperplane projection
 25 technique was proposed.

26 Augmented Lagrangian method is one of the commonly used algorithms for constrained optimization, see, e.g.,
 27 [29] and the references therein. It seeks a solution by replacing the original constrained problem by a series of
 28 unconstrained problems and add to the objective function a penalty term, and another term designed to mimic a
 29 Lagrange multiplier.

30 An additional technique, often used in a function space setting, consists in introducing a proper regularization,
 31 motivated by the augmented Lagrangian method. It allows to apply an infinite-dimensional semismooth Newton
 32 method for the solution of the regularized problem, see, e.g., [43]. In the present context, this leads to replacing
 33 the complementarity conditions (1.1b) by

$$34 \quad \mathbf{K}(\mathbf{X}) + \min\{\mathbf{0}, -\mathbf{K}(\mathbf{X}) + \gamma\mathbf{G}(\mathbf{X})\} = \mathbf{0},$$

35 for a parameter $\gamma > 0$. This method can be combined with a path-following strategy to update the regularization
 36 parameter γ , see for instance [41, 28, 40].

37 Another important class of methods for constrained optimization problems of the form (1.1) is formed by interior-
 38 point methods. These methods consist in generating a sequence in the feasible region $\mathbf{K}(\mathbf{X}) \geq \mathbf{0}$ and $\mathbf{G}(\mathbf{X}) \geq \mathbf{0}$,
 39 under the assumption of knowing a feasible initial point. We refer to the work of Wright [46], Bellavia et al. [4],
 40 and the references therein for a review.

41 Lastly, an additional notable method is the smoothing Newton method. The main idea of this approach is to
 42 approximate the semismooth (non-differentiable) function \mathbf{C} from (1.2) by a smooth (differentiable) function that
 43 depends on a smoothing parameter. The problem is reformulated as a sequence of regularized smooth equations
 44 that can be solved by applying the standard Newton method, and where one drives the smoothing parameter down
 45 to zero, cf. [39, 36, 35] and the references therein.

46 In this work, we design a general framework to systematically steer the above different ingredients. Our main
 47 philosophy is adaptive smoothing (regularization). For $\mu^j > 0$, let a smoothed function $\mathbf{C}_{\mu^j}(\cdot)$, satisfy $\|\mathbf{C}_{\mu^j}(\mathbf{X}) -$
 48 $\mathbf{C}(\mathbf{X})\| \rightarrow 0$ as $\mu^j \rightarrow 0$, for $\mathbf{X} \in \mathbb{R}^n$. The smoothing parameter μ^j is reduced at each smoothing iteration $j \geq 1$.
 49 Thus, problem (1.1), or equivalently (1.2), can be reformulated as a system of smooth (differentiable) equations
 50 written in the form: Find $\mathbf{X}^j \in \mathbb{R}^n$ such that

$$51 \quad \begin{aligned} \mathbb{E}\mathbf{X}^j &= \mathbf{F}, \\ \mathbf{C}_{\mu^j}(\mathbf{X}^j) &= \mathbf{0}. \end{aligned} \quad (1.3)$$

53 Hence, Newton-type methods can be applied to solve system (1.3), yielding, at each linearization step $k \geq 1$, a
 54 linear system

$$55 \quad \mathbb{A}_{\mu^j}^{j,k-1} \mathbf{X}^{j,k} = \mathbf{B}_{\mu^j}^{j,k-1}, \quad (1.4)$$

56 where $\mathbb{A}_{\mu^j}^{j,k-1} \in \mathbb{R}^{n,n}$ is a matrix and $\mathbf{B}_{\mu^j}^{j,k-1} \in \mathbb{R}^n$ is a vector.

57 Solving (1.4) with a direct method may be very expensive. A popular approach is to solve it approximately
 58 by applying only a few steps of an iterative algebraic solver. Such inexact approaches can be found in [22, 32]
 59 for semismooth Newton methods, in [39, 26] for smoothing Newton methods, in [49] for augmented Lagrangian
 60 methods, and in [3] for interior-point methods. In the algorithms introduced therein, the iterations of different
 61 solvers are stopped according to a fixed maximal number of iterations, the Euclidean norm of the residual vector,
 62 or other parameters-dependant stopping criteria. In this work, the a posteriori estimate constitute a distinctive
 63 element at the heart of the proposed smoothing method. Importantly, it ensures the desired balance between each
 64 source of error at any resolution step, unlike existing approaches based on classical stopping criteria.

65 At each linear algebraic step $i \geq 1$ for (1.4), one in particular obtains $\mathbf{X}^{j,k,i} \in \mathbb{R}^n$ such that

$$66 \quad \mathbb{A}_{\mu^j}^{j,k-1} \mathbf{X}^{j,k,i} = \mathbf{B}_{\mu^j}^{j,k-1} - \mathbf{R}_{\text{alg}}^{j,k,i},$$

67 where $\mathbf{R}_{\text{alg}}^{j,k,i} \in \mathbb{R}^n$ is the algebraic residual vector of (1.4).

68 Our principal aim is to reduce the computational cost of the numerical resolution of (1.1) by employing an
 69 adaptive strategy based on a posteriori error estimates. There is a well-developed literature on a posteriori error
 70 estimates and *mesh adaptivity* for partial differential equations, see for instance the books of Ainsworth and Oden
 71 [2], Repin [37], and Nochetto et al. [34]. For variational inequalities, we can mention the contributions of Repin [38],
 72 Ben Belgacem et al. [5], Bürg and Schröder [15], and Dabaghi et al. [17]. Although smoothing Newton approaches
 73 have been widely studied, to the best of our knowledge, almost no work has been done to this day on a posteriori
 74 error estimates and adaptivity for *solvers* applied to discrete problems of the form (1.1).

75 We first derive an upper bound on the norm of the residual of system (1.2), given by

$$76 \quad \mathbf{R}(\mathbf{X}^{j,k,i}) := \begin{bmatrix} \mathbf{F} - \mathbb{E}\mathbf{X}^{j,k,i} \\ -\mathbf{C}(\mathbf{X}^{j,k,i}) \end{bmatrix}.$$

77 Then, decomposing $\mathbf{R}(\mathbf{X}^{j,k,i})$, we distinguish the different error components. This leads to an a posteriori control
 78 of the form

$$79 \quad \|\mathbf{R}(\mathbf{X}^{j,k,i})\|_r \leq \eta^{j,k,i} = \eta_{\text{sm}}^{j,k,i} + \eta_{\text{lin}}^{j,k,i} + \eta_{\text{alg}}^{j,k,i}. \quad (1.5)$$

80 Here, $\eta^{j,k,i}$ is a fully computable upper bound that holds true at any smoothing (regularization) step j , linearization
 81 step k , and algebraic solver step i , whereas the role of the estimators $\eta_{\text{sm}}^{j,k,i}$, $\eta_{\text{lin}}^{j,k,i}$, and $\eta_{\text{alg}}^{j,k,i}$ is to identify the
 82 smoothing, linearization, and algebraic components of the error. This error bound allows to define adaptive stopping
 83 criteria for the nonlinear and linear algebraic solvers, in the spirit of [21, 17], and the references therein. These
 84 criteria, as well as a simple way to tighten the smoothing parameter μ^j , are incorporated in a three-level adaptive
 85 algorithm. In contrast to common approaches, where the termination requires reaching a fixed threshold, the
 86 particularity of this adaptive algorithm is that the iterations are stopped when the error component of the concerned
 87 solver is smaller than the total error, up to a desired fraction. The efficiency of the proposed adaptive algorithm for
 88 (inexact) smoothing Newton methods and (inexact) interior-point methods is showcased numerically on practical
 89 problems.

90 It is relevant to mention that this work is extended in [8], where the present approach is applied to a system
 91 of PDEs with complementarity constraints in infinite-dimensional space. In particular, taking into account the
 92 discretization error allows to adaptively steer the smoothing in system (1.3). Although we do not address mesh
 93 adaptivity in our work, we underline that a posteriori estimators are an important tool for adaptive mesh refinement
 94 strategies, see, e.g., [18] and the references therein. Consequently, algorithms based on the previous criteria ensure
 95 significant computational gains in terms of total number of iterations and mesh cells.

96 Our manuscript is organized as follows. In Section 2, we recall a semismooth Newton method based on an
 97 equivalent reformulation of the complementarity constraints in the form (1.2). Section 3 is devoted to introduce
 98 our adaptive inexact smoothing Newton method based on the reformulation as a system of smooth equations as in
 99 (1.3). We establish here the a posteriori error estimates (1.5) and propose an adaptive algorithm with a posteriori
 100 stopping criteria. We survey a nonparametric interior-point method in Section 4, and introduce its adaptive version
 101 in Section 5. Finally, a detailed numerical study is presented in Sections 6 and 7.

102 2 Semismooth Newton method

103 The purpose of this section is to briefly recall the semismooth Newton method to approximate the solution of the
 104 nonlinear system of equations (1.1), see, e.g., [33, 23, 17]. The complementarity constraints represented by (1.1b) as
 105 algebraic inequalities are here rewritten as non-differentiable algebraic equalities, using a complementarity function
 106 (C-function). A function $\tilde{\mathbf{C}} : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^m$, $m \geq 1$, is called a C-function if

$$107 \quad \tilde{\mathbf{C}}(\mathbf{x}, \mathbf{y}) = \mathbf{0} \iff \mathbf{x} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0}, \mathbf{x} \cdot \mathbf{y} = 0 \quad \forall (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^m \times \mathbb{R}^m.$$

108 A variety of C-functions can be found in the literature, see, e.g., [42, 25]. We give as examples the minimum (min)
 109 function and the Fischer–Burmeister (F–B) function: for $l = 1, \dots, m$,

$$110 \quad \left(\tilde{\mathbf{C}}_{\text{min}}(\mathbf{x}, \mathbf{y}) \right)_l := (\min(\mathbf{x}, \mathbf{y}))_l = (\mathbf{x}_l + \mathbf{y}_l)/2 - |\mathbf{x}_l - \mathbf{y}_l|/2, \quad (2.1)$$

$$111 \quad \left(\tilde{\mathbf{C}}_{\text{FB}}(\mathbf{x}, \mathbf{y}) \right)_l := \sqrt{\mathbf{x}_l^2 + \mathbf{y}_l^2} - (\mathbf{x}_l + \mathbf{y}_l). \quad (2.2)$$

112
 113 In general, the C-functions are not Fréchet differentiable. The min and the Fischer–Burmeister functions are,
 114 for example, differentiable everywhere except in $\mathbf{x} = \mathbf{y}$ and $(\mathbf{0}, \mathbf{0})$, respectively. Let us introduce a function

115 $C : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined as $C(\mathbf{X}) := \tilde{C}(\mathbf{K}(\mathbf{X}), \mathbf{G}(\mathbf{X}))$, where $\tilde{C} : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is any C-function. By using this
 116 reformulation in (1.1b), it is obvious that problem (1.1) can be equivalently rewritten as: Find a vector $\mathbf{X} \in \mathbb{R}^n$,
 117 such that

$$118 \quad \mathbb{E}\mathbf{X} = \mathbf{F}, \quad (2.3a)$$

$$119 \quad C(\mathbf{X}) = \mathbf{0}. \quad (2.3b)$$

121 Next, we detail the semismooth Newton linearization. Let an initial vector $\mathbf{X}^0 \in \mathbb{R}^n$ be given. At the step
 122 $k \geq 1$, one looks for $\mathbf{X}^k \in \mathbb{R}^n$ such that

$$123 \quad \mathbb{A}^{k-1} \mathbf{X}^k = \mathbf{B}^{k-1}, \quad (2.4)$$

124 where the square matrix $\mathbb{A}^{k-1} \in \mathbb{R}^{n,n}$ and the right-hand side vector $\mathbf{B}^{k-1} \in \mathbb{R}^n$ are given by

$$125 \quad \mathbb{A}^{k-1} := \begin{bmatrix} \mathbb{E} \\ \mathbf{J}_C(\mathbf{X}^{k-1}) \end{bmatrix}, \quad \mathbf{B}^{k-1} := \begin{bmatrix} \mathbf{F} \\ \mathbf{J}_C(\mathbf{X}^{k-1})\mathbf{X}^{k-1} - C(\mathbf{X}^{k-1}) \end{bmatrix}. \quad (2.5)$$

126 Note that the Jacobian corresponding to (2.3a) is constant and equal to \mathbb{E} since it is linear. The semismooth
 127 nonlinearity occurs in the second line (2.3b): the notation \mathbf{J}_C in (2.5) stands for the Jacobian matrix in the sense
 128 of Clarke of the function C , cf. [23, 24]. To give an example, consider the semismooth min function (2.1) and define
 129 the matrices \mathbb{K} and $\mathbb{G} \in \mathbb{R}^{m,n}$ respectively by $\mathbb{K} := [\nabla \mathbf{K}(\mathbf{X})]$ and $\mathbb{G} := [\nabla \mathbf{G}(\mathbf{X})]$. Then the l^{th} row of the Jacobian
 130 matrix in the sense of Clarke \mathbf{J}_C is either given by the l^{th} row of \mathbb{K} , if $(\mathbf{K}(\mathbf{X}^{k-1}))_l \leq (\mathbf{G}(\mathbf{X}^{k-1}))_l$, or by the l^{th}
 131 row of \mathbb{G} , if $(\mathbf{G}(\mathbf{X}^{k-1}))_l < (\mathbf{K}(\mathbf{X}^{k-1}))_l$.

132 We will need below the total residual vector of problem (2.3), defined by

$$133 \quad \mathbf{R}(\mathbf{V}) := \begin{bmatrix} \mathbf{F} - \mathbb{E}\mathbf{V} \\ -C(\mathbf{V}) \end{bmatrix}, \quad \mathbf{V} \in \mathbb{R}^n. \quad (2.6)$$

134 In this context, the relative norm of a vector $\mathbf{V} \in \mathbb{R}^n$ is given by $\|\mathbf{V}\|_r := \|\mathbf{V}\| / \|\mathbf{R}(\mathbf{X}^0)\|$, where $\|\cdot\|$ is the
 135 L_2 -norm.

136 3 Adaptive inexact smoothing Newton method

137 In this section we introduce our adaptive inexact smoothing Newton method. Based on a posteriori error estimators,
 138 adaptive stopping criteria are formulated to conceive an adaptive iterative algorithm.

139 3.1 Smoothing of the C-functions

140 The key of our developments is to smooth the non-differentiable equation formulation (2.3b) of the complementarity
 141 constraints (1.1b) with the help of a smooth (i.e. continuously differentiable) function. This smoothing allows us
 142 to approximately transform the nonsmooth nonlinear system (2.3) to a smooth system of nonlinear equations to be
 143 solved by using the standard Newton method.

144 Let $\mu > 0$ be a (small) smoothing parameter. We construct an approximation function $\tilde{C}_\mu : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ of
 145 a C-function \tilde{C} such that $\tilde{C}_\mu(\cdot, \cdot)$ is of class \mathcal{C}^1 on $\mathbb{R}^m \times \mathbb{R}^m$ and satisfies

$$146 \quad \|\tilde{C}(\mathbf{x}, \mathbf{y}) - \tilde{C}_\mu(\mathbf{x}, \mathbf{y})\| \rightarrow 0 \text{ as } \mu \rightarrow 0 \quad \text{for all } (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^m \times \mathbb{R}^m.$$

147 For example, for $l = 1, \dots, m$, a possible smoothing of the min and the Fischer–Burmeister functions (2.1) and (2.2)
 148 can be

$$149 \quad \left(\tilde{C}_{\min_\mu}(\mathbf{x}, \mathbf{y})\right)_l = \frac{\mathbf{x}_l + \mathbf{y}_l}{2} - \frac{\left(|\mathbf{x} - \mathbf{y}|_\mu\right)_l}{2}, \quad \text{with } (|\mathbf{z}|_\mu)_l = \sqrt{\mathbf{z}_l^2 + \mu^2}, \quad (3.1)$$

$$150 \quad \left(\tilde{C}_{\text{FB}_\mu}(\mathbf{x}, \mathbf{y})\right)_l = \sqrt{\mu^2 + \mathbf{x}_l^2 + \mathbf{y}_l^2} - (\mathbf{x}_l + \mathbf{y}_l), \quad (3.2)$$

152 where the μ -smoothed absolute value function $|\cdot|_\mu : \mathbb{R}^m \rightarrow \mathbb{R}_+^m$, $m \geq 0$, replaces the absolute value function (not
 153 differentiable at $\mathbf{0}$), see Figure 1. Note that both functions $|\cdot|_\mu$ and $\tilde{C}_{\text{FB},\mu}$ are of class \mathcal{C}^∞ .

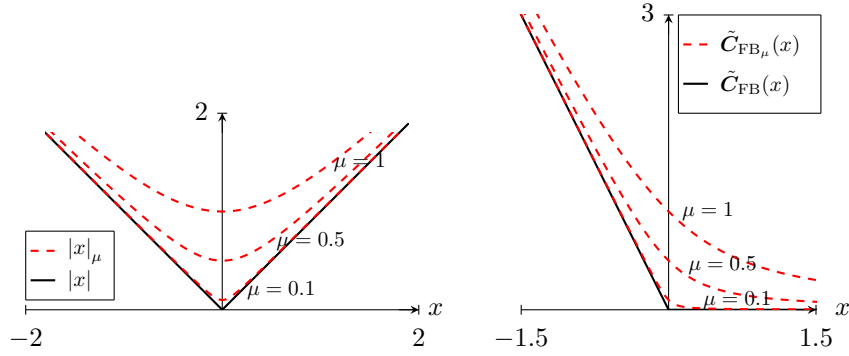


Figure 1: Left: Absolute value function $|\cdot|$ and smoothed absolute value function $|\cdot|_\mu$. Right: Fischer–Burmeister function $\tilde{C}_{\text{FB}}(\cdot)$ and smoothed Fischer–Burmeister function $\tilde{C}_{\text{FB}_\mu}(\cdot)$, for different values of the smoothing parameter μ .

154 We define the function $C_\mu : \mathbb{R}^n \rightarrow \mathbb{R}^m$ as $C_\mu(\mathbf{X}) := \tilde{C}_\mu(\mathbf{K}(\mathbf{X}), \mathbf{G}(\mathbf{X}))$, where $\tilde{C}_\mu : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is any
 155 smoothed C-function of at least class \mathcal{C}^1 . This allows to approximate problem (1.1) or (1.2) by a system of smooth
 156 equations: Find a vector $\mathbf{X} \in \mathbb{R}^n$, such that

$$\begin{aligned} \mathbb{E}\mathbf{X} &= \mathbf{F}, \\ \mathbf{C}_\mu(\mathbf{X}) &= \mathbf{0}. \end{aligned} \quad (3.3)$$

159 Thus, Newton-type methods can be applied to solve the system of nonlinear algebraic equations (3.3).

160 Fixing $\mu^1 > 0$, we now describe an iterative method for solving problem (2.3). At the beginning of each smoothing
 161 iteration (outer iteration) denoted hereafter by $j \geq 1$, an initial guess $\mathbf{X}^j \in \mathbb{R}^n$ is given, and a smoothing parameter
 162 μ^j is determined; μ^j will be driven down to zero. Then some iterative nonlinear solver like the Newton method is
 163 employed to solve the smoothed problem written in the form: Find $\mathbf{X}^j \in \mathbb{R}^n$ such that

$$\begin{aligned} \mathbb{E}\mathbf{X}^j &= \mathbf{F}, \\ \mathbf{C}_{\mu^j}(\mathbf{X}^j) &= \mathbf{0}. \end{aligned} \quad (3.4)$$

166 3.2 Newton linearization of the nonlinear algebraic system

167 In what follows, we detail the Newton method employed to solve problem (3.4) at a fixed outer smoothing step
 168 $j \geq 1$. Given an initial vector $\mathbf{X}^{j,0}$ (typically $\mathbf{X}^{j,0} = \mathbf{X}^{j-1}$), Newton's algorithm generates a sequence $(\mathbf{X}^{j,k})_{k \geq 1}$
 169 with $\mathbf{X}^{j,k} \in \mathbb{R}^n$ given by the following system of linear algebraic equations

$$\mathbb{A}_{\mu^j}^{j,k-1} \mathbf{X}^{j,k} = \mathbf{B}_{\mu^j}^{j,k-1}, \quad (3.5)$$

171 where the Jacobian matrix $\mathbb{A}_{\mu^j}^{j,k-1} \in \mathbb{R}^{n,n}$ and the right-hand side vector $\mathbf{B}_{\mu^j}^{j,k-1} \in \mathbb{R}^n$ are defined by

$$\mathbb{A}_{\mu^j}^{j,k-1} := \begin{bmatrix} \mathbb{E} \\ \mathbf{J}_{\mathbf{C}_{\mu^j}}(\mathbf{X}^{j,k-1}) \end{bmatrix}, \quad \mathbf{B}_{\mu^j}^{j,k-1} := \begin{bmatrix} \mathbf{F} \\ \mathbf{J}_{\mathbf{C}_{\mu^j}}(\mathbf{X}^{j,k-1})\mathbf{X}^{j,k-1} - \mathbf{C}_{\mu^j}(\mathbf{X}^{j,k-1}) \end{bmatrix}, \quad (3.6)$$

173 with $\mathbf{J}_{\mathbf{C}_{\mu^j}}(\mathbf{X}^{j,k-1})$ the Jacobian matrix of the smooth function \mathbf{C}_{μ^j} at $\mathbf{X}^{j,k-1}$.

174 3.3 Inexact solution of the linear algebraic system

175 The linearized system (3.5) may not be solved exactly, since the use of a direct method may be expensive. For this
 176 reason, we consider in this work also an inexact resolution. For a fixed smoothing step $j \geq 1$, a fixed Newton step
 177 $k \geq 1$, and an initial guess $\mathbf{X}^{j,k,0}$ (typically $\mathbf{X}^{j,k,0} = \mathbf{X}^{j,k-1}$), only a few steps of an iterative linear algebraic solver
 178 can be applied to find an approximate solution to (3.5), yielding, on step $i \geq 1$, an approximation $\mathbf{X}^{j,k,i}$ to $\mathbf{X}^{j,k}$.
 179 This satisfies (3.5) up to the residual vector given by

$$\mathbf{B}_{\mu^j}^{j,k-1} - \mathbb{A}_{\mu^j}^{j,k-1} \mathbf{X}^{j,k,i}. \quad (3.7)$$

181 Define now the linearization function $C_{\mu^j}^{j,k-1} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ of C_{μ^j} at smoothing step j and Newton step k as

$$182 \quad C_{\mu^j}^{j,k-1}(\mathbf{V}) := C_{\mu^j}(\mathbf{X}^{j,k-1}) + \mathbf{J}_{C_{\mu^j}}(\mathbf{X}^{j,k-1})(\mathbf{V} - \mathbf{X}^{j,k-1}) \quad \forall \mathbf{V} \in \mathbb{R}^n. \quad (3.8)$$

183 This allows us to write the algebraic residual vector for $\mathbf{V} \in \mathbb{R}^n$ as

$$184 \quad \mathbf{R}_{\text{alg}}^{\text{AISN}}(\mathbf{V}) := \mathbf{B}_{\mu^j}^{j,k-1} - \mathbb{A}_{\mu^j}^{j,k-1} \mathbf{V} = \begin{bmatrix} \mathbf{F} - \mathbb{E}\mathbf{V} \\ -C_{\mu^j}^{j,k-1}(\mathbf{V}) \end{bmatrix}. \quad (3.9)$$

185 3.4 An upper bound for the norm of the residual

186 We consider the total residual vector of problem (2.3) given in (2.6). By adding and subtracting $C_{\mu^j}(\mathbf{X}^{j,k,i})$ and
187 its linearization $C_{\mu^j}^{j,k-1}(\mathbf{X}^{j,k,i})$ given by (3.8), the total residual vector can be decomposed as follows:

$$188 \quad \mathbf{R}(\mathbf{X}^{j,k,i}) = \begin{bmatrix} \mathbf{F} - \mathbb{E}\mathbf{X}^{j,k,i} \\ -C(\mathbf{X}^{j,k,i}) \pm C_{\mu^j}(\mathbf{X}^{j,k,i}) \pm C_{\mu^j}^{j,k-1}(\mathbf{X}^{j,k,i}) \end{bmatrix}$$

$$189 \quad = \underbrace{\begin{bmatrix} \mathbf{0} \\ C_{\mu^j}(\mathbf{X}^{j,k,i}) - C(\mathbf{X}^{j,k,i}) \end{bmatrix}}_{\text{smoothing}} + \underbrace{\begin{bmatrix} \mathbf{0} \\ C_{\mu^j}^{j,k-1}(\mathbf{X}^{j,k,i}) - C_{\mu^j}(\mathbf{X}^{j,k,i}) \end{bmatrix}}_{\text{linearization}} + \underbrace{\begin{bmatrix} \mathbf{F} - \mathbb{E}\mathbf{X}^{j,k,i} \\ -C_{\mu^j}^{j,k-1}(\mathbf{X}^{j,k,i}) \end{bmatrix}}_{\text{algebraic}}.$$

190 It is reasonable to get these three terms. Indeed, the first one reflects the error due to the approximation of
191 the semismooth function C by the smoothed function C_{μ^j} . The second term is related to the linearization of the
192 nonlinear smooth problem (3.4). Taking into account that the resolution of the smooth linearized problem (3.5)
193 is possibly done “inexactly”, the remaining term represents the error of the inexact algebraic resolution. By the
194 triangle inequality, the relative norm of $\mathbf{R}(\mathbf{X}^{j,k,i})$ is thus bounded by the smoothing, linearization, and algebraic
195 estimators respectively defined as

$$197 \quad \eta_{\text{sm,AISN}}^{j,k,i} := \left\| C_{\mu^j}(\mathbf{X}^{j,k,i}) - C(\mathbf{X}^{j,k,i}) \right\|_{\text{r}}, \quad (3.10a)$$

$$198 \quad \eta_{\text{lin,AISN}}^{j,k,i} := \left\| C_{\mu^j}^{j,k-1}(\mathbf{X}^{j,k,i}) - C_{\mu^j}(\mathbf{X}^{j,k,i}) \right\|_{\text{r}}, \quad (3.10b)$$

$$199 \quad \eta_{\text{alg,AISN}}^{j,k,i} := \left(\left\| \mathbf{F} - \mathbb{E}\mathbf{X}^{j,k,i} \right\|_{\text{r}}^2 + \left\| C_{\mu^j}^{j,k-1}(\mathbf{X}^{j,k,i}) \right\|_{\text{r}}^2 \right)^{\frac{1}{2}}. \quad (3.10c)$$

201 Note that $\eta_{\text{alg,AISN}}^{j,k,i}$ is exactly equal to the relative norm of $\mathbf{R}_{\text{alg}}^{\text{AISN}}(\mathbf{X}^{j,k,i})$ given by (3.9). From these developments
202 we conclude:

203 **Theorem 3.1** *Let $\mathbf{X}^{j,k,i} \in \mathbb{R}^n$ arise from an inexact solve of (3.5). We have*

$$204 \quad \left\| \mathbf{R}(\mathbf{X}^{j,k,i}) \right\|_{\text{r}} \leq \eta_{\text{AISN}}^{j,k,i} := \eta_{\text{sm,AISN}}^{j,k,i} + \eta_{\text{lin,AISN}}^{j,k,i} + \eta_{\text{alg,AISN}}^{j,k,i}.$$

205 3.5 Adaptive inexact smoothing Newton algorithm

206 Theorem 3.1 motivates the following. Let two real parameters α_{lin} and α_{alg} be given in $]0, 1]$, representing the
207 desired relative size of the algebraic and linearization errors, and let $\varepsilon > 0$ be a given desired tolerance for the
208 total error. The stopping criteria for the linearization, algebraic, and smoothing steps, with the bars denoting the
209 stopping indices, are respectively set as

$$210 \quad \eta_{\text{alg,AISN}}^{j,k,\bar{i}} < \alpha_{\text{alg}} \eta_{\text{lin,AISN}}^{j,k,\bar{i}}, \quad (3.11a)$$

$$211 \quad \eta_{\text{lin,AISN}}^{j,k,\bar{i}} < \alpha_{\text{lin}} \eta_{\text{sm,AISN}}^{j,k,\bar{i}}, \quad (3.11b)$$

$$212 \quad \left\| \mathbf{R}(\mathbf{X}^{\bar{j},\bar{k},\bar{i}}) \right\|_{\text{r}} < \varepsilon. \quad (3.11c)$$

214 The first criterion (3.11a) for the algebraic iterative solver expresses that there is no need to continue with the
215 algebraic steps when the linearization error becomes dominant. Similarly, the second one (3.11b) aims at stopping
216 the linearization iterations when the linearization error does not substantially contribute to the smoothing error.
217 Finally, the termination criterion for the smoothing steps (3.11c) is of the standard type, that is when we stop the
218 entire procedure, when the relative norm of the total residual vector lies below the desired tolerance ε .

219 The entire method is described by the following adaptive algorithm, which drives the smoothing parameter μ^j
220 to zero as $\mu^j := \alpha \mu^{j-1}$ at each smoothing iteration. Other common empirical ways to progressively reduce μ^j can
221 be found, e.g., in [45]. The adaptive inexact smoothing Newton algorithm is the following:

Algorithm 1 Adaptive inexact smoothing Newton algorithm

1. Initialization

Choose a tolerance $\varepsilon > 0$ and parameters $\alpha \in]0, 1[$ and $\alpha_{\text{lin}}, \alpha_{\text{alg}} \in]0, 1]$.
 Fix $\mu^1 > 0$ and an initial approximation $\mathbf{X}^0 \in \mathbb{R}^n$. Set $j := 1$.

2. Smoothing loop

2.1 Set $\mathbf{X}^{j,0} := \mathbf{X}^0$ as an initial guess for the nonlinear solver. Set $k := 1$.

2.2 Newton linearization loop

2.2.1 From $\mathbf{X}^{j,k-1}$ define $\mathbb{A}_{\mu^j}^{j,k-1} \in \mathbb{R}^{n,n}$ and $\mathbf{B}_{\mu^j}^{j,k-1} \in \mathbb{R}^n$ by (3.6).

2.2.2 Consider the problem of finding a solution $\mathbf{X}^{j,k}$ to

$$\mathbb{A}_{\mu^j}^{j,k-1} \mathbf{X}^{j,k} = \mathbf{B}_{\mu^j}^{j,k-1}. \quad (3.12)$$

2.2.3 Set $\mathbf{X}^{j,k,0} := \mathbf{X}^{j,k-1}$ as initial guess for the iterative algebraic solver. Set $i := 1$.

2.2.4 Algebraic solver loop

i) Starting from $\mathbf{X}^{j,k-1}$, perform a step of the iterative algebraic solver for the solution of (3.12), yielding, on step i an approximation $\mathbf{X}^{j,k,i}$ to $\mathbf{X}^{j,k}$ satisfying

$$\mathbb{A}_{\mu^j}^{j,k-1} \mathbf{X}^{j,k,i} = \mathbf{B}_{\mu^j}^{j,k-1} - \mathbf{R}_{\text{alg}}^{\text{AISN}}(\mathbf{X}^{j,k,i}).$$

ii) Compute the estimators given in (3.10).

iii) If $\eta_{\text{alg,AISN}}^{j,k,i} < \alpha_{\text{alg}} \eta_{\text{lin,AISN}}^{j,k,i}$, set $\bar{i} := i$ and stop. If not, set $i := i + 1$ and go to i).

2.2.5 If $\eta_{\text{lin,AISN}}^{j,k,\bar{i}} < \alpha_{\text{lin}} \eta_{\text{sm,AISN}}^{j,k,\bar{i}}$, set $\bar{k} := k$ and stop. If not, set $k := k + 1$ and go to **2.2.1**.

2.3 If $\|\mathbf{R}(\mathbf{X}^{j,\bar{k},\bar{i}})\|_{\text{r}} < \varepsilon$, set $\bar{j} := j$ and stop.

If not, set $j := j + 1$, $\mathbf{X}^{j,0} := \mathbf{X}^{j-1,\bar{k},\bar{i}}$, and $\mu^j := \alpha \mu^{j-1}$. Then set $k := 1$ and go to **2.2.1**.

222 4 Nonparametric interior-point method

223 Now we employ a nonparametric interior-point method to problem (1.1). More precisely, we consider the method
 224 introduced in [44] where a systematic strategy is used to steer the sequence of smoothing parameters towards zero.

225 We introduce a vector $\boldsymbol{\mu} = \mu \mathbf{1} \in \mathbb{R}^m$, where $\mu > 0$ is the smoothing parameter and $\mathbf{1} \in \mathbb{R}^m$ is the vector with
 226 all components equal to 1. The original nonsmooth problem (1.1) is replaced by a smoothed problem written in the
 227 form: Find $\mathbf{X} \in \mathbb{R}^n$ such that

$$228 \quad \mathbb{E}\mathbf{X} = \mathbf{F}, \tag{4.1a}$$

$$229 \quad \mathbf{K}(\mathbf{X}) \geq \mathbf{0}, \mathbf{G}(\mathbf{X}) \geq \mathbf{0}, \mathbf{K}(\mathbf{X})\mathbf{G}(\mathbf{X}) = \boldsymbol{\mu}, \tag{4.1b}$$

231 where $[(\mathbf{K}(\mathbf{X})\mathbf{G}(\mathbf{X}))]_m = [\mathbf{K}(\mathbf{X})]_m[\mathbf{G}(\mathbf{X})]_m$. In order to properly adjust the sequence of smoothing parameters,
 232 the smoothing parameter μ is treated as an unknown, by introducing the following new equation into system (4.1)

$$233 \quad \theta\mu + \mu^2 = 0, \tag{4.2}$$

234 where θ is a small positive real parameter, chosen once and for all. This equation prevents μ from rushing to zero
 235 in just one iteration, and ensures quadratic convergence, see [44]. The unknown of system (4.1) is now the enlarged
 236 vector $\boldsymbol{\mathcal{X}} = (\mathbf{X}, \mu)^T \in \mathbb{R}^{n+1}$. We are thus brought back to applying the standard Newton method to a smooth
 237 problem.

238 Let $\mathbf{X}^0 \in \mathbb{R}^n$ such that $\mathbf{K}(\mathbf{X}^0) \geq \mathbf{0}$ and $\mathbf{G}(\mathbf{X}^0) \geq \mathbf{0}$ be given. To update the iterate $\boldsymbol{\mathcal{X}}^{k-1}$, we compute a
 239 search direction denoted by $\mathbf{d}^k = [\mathbf{d}_{\mathbf{X}}^k, d_{\mu}^k] \in \mathbb{R}^{n+1}$, where $\mathbf{d}_{\mathbf{X}}^k \in \mathbb{R}^n$ and $d_{\mu}^k \in \mathbb{R}$. Then, to preserve positivity of
 240 $\mathbf{K}(\mathbf{X}^k)$ and $\mathbf{G}(\mathbf{X}^k)$ at each step of the nonlinear solver, a truncation of the Newton direction \mathbf{d}^k is performed so
 241 that the corresponding update satisfies $\mathbf{K}(\mathbf{X}^{k-1} + \kappa^k \mathbf{d}_{\mathbf{X}}^k) \geq \mathbf{0}$ and $\mathbf{G}(\mathbf{X}^{k-1} + \kappa^k \mathbf{d}_{\mathbf{X}}^k) \geq \mathbf{0}$ for some $\kappa^k \in]0, 1]$, as
 242 close to 1 as possible. After this, we can set

$$243 \quad \boldsymbol{\mathcal{X}}^k := \boldsymbol{\mathcal{X}}^{k-1} + \kappa^k \mathbf{d}^k.$$

244 Recall that our goal is to make μ equal to 0 in the limit while ensuring the positivity of the updated iterate. Another
 245 choice for the additional equation (4.2) added to system (4.1) was developed and introduced in a recent work, see
 246 [45, Section 3]. The proposed equation does not require to truncate the Newton direction, and couples μ and \mathbf{X} in
 247 a tighter way.

248 We rewrite system (4.1) as an enlarged system of $n + 1$ equations

$$249 \quad \begin{aligned} \mathbb{E}\mathbf{X} &= \mathbf{F}, \\ \mathbf{K}(\mathbf{X})\mathbf{G}(\mathbf{X}) - \boldsymbol{\mu} &= \mathbf{0}, \\ \theta\mu + \mu^2 &= 0. \end{aligned} \tag{4.3}$$

251 5 Adaptive inexact interior-point method

252 We present in this section our adaptive inexact version of the nonparametric interior point method of Section
 253 4. In contrast to Section 4, we consider, however, $\mu > 0$ as a parameter, and not as an unknown. At each
 254 smoothing step $j \geq 1$, we may solve the system of smoothing equations written as: Find $\mathbf{X}^j \in \mathbb{R}^n$ such that
 255 $\mathbf{K}(\mathbf{X}^j) \geq \mathbf{0}$, $\mathbf{G}(\mathbf{X}^j) \geq \mathbf{0}$, and

$$256 \quad \mathbb{E}\mathbf{X}^j = \mathbf{F}, \tag{5.1a}$$

$$257 \quad \mathbf{H}_{\mu^j}(\mathbf{X}^j) := \mathbf{K}(\mathbf{X}^j)\mathbf{G}(\mathbf{X}^j) - \boldsymbol{\mu}^j = \mathbf{0}. \tag{5.1b}$$

259 The values of μ^j are gradually decreased at each smoothing iteration, creating a sequence of suitable μ^j converging
 260 to zero.

261 5.1 Newton linearization of the nonlinear algebraic system

262 Let $\mathbf{X}^0 \in \mathbb{R}^n$ such that $\mathbf{K}(\mathbf{X}^0) \geq \mathbf{0}$ and $\mathbf{G}(\mathbf{X}^0) \geq \mathbf{0}$ be given. At each smoothing iteration $j \geq 1$ and each
 263 linearization step $k \geq 1$, starting with an initial approximation $\mathbf{X}^{j,0}$ such that $\mathbf{K}(\mathbf{X}^{j,0}) \geq \mathbf{0}$ and $\mathbf{G}(\mathbf{X}^{j,0}) \geq \mathbf{0}$
 264 (typically $\mathbf{X}^{j,0} = \mathbf{X}^{j-1}$), we try to approach the solution of problem (5.1) by finding $\mathbf{X}^{j,k} \in \mathbb{R}^n$ such that

$$265 \quad \mathbb{A}_{\mu^j}^{j,k-1} \mathbf{X}^{j,k} = \mathbf{B}_{\mu^j}^{j,k-1}, \tag{5.2}$$

266 where the Jacobian matrix $\mathbb{A}_{\mu^j}^{j,k-1} \in \mathbb{R}^{n,n}$ and the right-hand side vector $\mathbf{B}_{\mu^j}^{j,k-1} \in \mathbb{R}^n$ are defined by

$$267 \quad \mathbb{A}_{\mu^j}^{j,k-1} := \begin{bmatrix} \mathbb{E} \\ \mathbf{J}_{\mathbf{H}_{\mu^j}}(\mathbf{X}^{j,k-1}) \end{bmatrix}, \quad \mathbf{B}_{\mu^j}^{j,k-1} := \begin{bmatrix} \mathbf{F} \\ \mathbf{J}_{\mathbf{H}_{\mu^j}}(\mathbf{X}^{j,k-1})\mathbf{X}^{j,k-1} - \mathbf{H}_{\mu^j}(\mathbf{X}^{j,k-1}) \end{bmatrix}, \quad (5.3)$$

268 with $\mathbf{J}_{\mathbf{H}_{\mu^j}}$ the Jacobian matrix of \mathbf{H}_{μ^j} . To ensure the positivity of the complementarity constraints, we then define
269 the direction $\mathbf{d}^{j,k} := \mathbf{X}^{j,k} - \mathbf{X}^{j,k-1} \in \mathbb{R}^n$ and find $\kappa^{j,k} \in]0, 1]$ such that

$$270 \quad \mathbf{K}(\mathbf{X}^{j,k-1} + \kappa^{j,k}\mathbf{d}^{j,k}) \geq \mathbf{0} \quad \text{and} \quad \mathbf{G}(\mathbf{X}^{j,k-1} + \kappa^{j,k}\mathbf{d}^{j,k}) \geq \mathbf{0}.$$

271 5.2 Inexact solution of the linear algebraic system

272 For a fixed smoothing iteration $j \geq 1$, a fixed Newton step $k \geq 1$, and an initial guess $\mathbf{X}^{j,k,0}$ (typically $\mathbf{X}^{j,k,0} =$
273 $\mathbf{X}^{j,k-1}$), an iterative algebraic solver can be applied to approach the solution of (5.2), yielding, on step $i \geq 1$, an
274 approximation $\mathbf{X}^{j,k,i}$ to $\mathbf{X}^{j,k}$. This satisfies (5.2) up to a residual vector defined by

$$275 \quad \mathbf{B}_{\mu^j}^{j,k-1} - \mathbb{A}_{\mu^j}^{j,k-1}\mathbf{X}^{j,k,i}. \quad (5.4)$$

276 Introduce the linearization $\mathbf{H}_{\mu^j}^{j,k-1} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ of $\mathbf{H}_{\mu^j}(\cdot)$ such that for $\mathbf{V} \in \mathbb{R}^n$,

$$277 \quad \mathbf{H}_{\mu^j}^{j,k-1}(\mathbf{V}) := \mathbf{H}_{\mu^j}(\mathbf{X}^{j,k-1}) + \mathbf{J}_{\mathbf{H}_{\mu^j}}(\mathbf{X}^{j,k-1})(\mathbf{V} - \mathbf{X}^{j,k-1}). \quad (5.5)$$

278 Using (5.5), the algebraic residual vector can be written as follows

$$279 \quad \mathbf{R}_{\text{alg}}^{\text{AIP}}(\mathbf{V}) := \mathbf{B}_{\mu^j}^{j,k-1} - \mathbb{A}_{\mu^j}^{j,k-1}\mathbf{V} = \begin{bmatrix} \mathbf{F} - \mathbb{E}\mathbf{V} \\ -\mathbf{H}_{\mu^j}^{j,k-1}(\mathbf{V}) \end{bmatrix}, \quad \mathbf{V} \in \mathbb{R}^n. \quad (5.6)$$

280 We now define the function $\mathbf{H} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ by

$$\mathbf{H}(\mathbf{V}) := \mathbf{K}(\mathbf{V})\mathbf{G}(\mathbf{V}), \quad \mathbf{V} \in \mathbb{R}^n. \quad (5.7)$$

281 and the total residual vector associated to the adaptive inexact interior-point method by

$$282 \quad \mathbf{R}^{\text{AIP}}(\mathbf{V}) := \begin{bmatrix} \mathbf{F} - \mathbb{E}\mathbf{V} \\ -\mathbf{H}(\mathbf{V}) \end{bmatrix}, \quad \mathbf{V} \in \mathbb{R}^n. \quad (5.8)$$

283 Here again, the relative norm of a given vector $\mathbf{V} \in \mathbb{R}^n$ is given by $\|\mathbf{V}\|_r := \|\mathbf{V}\| / \|\mathbf{R}^{\text{AIP}}(\mathbf{X}^0)\|$.

284 5.3 An upper bound for the norm of the residual

285 In the same spirit as in Section 3.4, we decompose at each smoothing step $j \geq 1$, each linearization step $k \geq 1$, and
286 each algebraic step $i \geq 1$ the total residual vector given by (5.8)

$$287 \quad \mathbf{R}^{\text{AIP}}(\mathbf{X}^{j,k,i}) = \underbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{H}_{\mu^j}(\mathbf{X}^{j,k,i}) - \mathbf{H}(\mathbf{X}^{j,k,i}) \end{bmatrix}}_{\text{smoothing}} + \underbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{H}_{\mu^j}^{j,k-1}(\mathbf{X}^{j,k,i}) - \mathbf{H}_{\mu^j}(\mathbf{X}^{j,k,i}) \end{bmatrix}}_{\text{linearization}} \\ 288 \quad + \underbrace{\begin{bmatrix} \mathbf{F} - \mathbb{E}\mathbf{X}^{j,k,i} \\ -\mathbf{H}_{\mu^j}^{j,k-1}(\mathbf{X}^{j,k,i}) \end{bmatrix}}_{\text{algebraic}}. \quad (5.9a)$$

289 We then define the smoothing, linearization, and algebraic estimators by

$$290 \quad \eta_{\text{sm,AIP}}^{j,k,i} := \|\mathbf{H}_{\mu^j}(\mathbf{X}^{j,k,i}) - \mathbf{H}(\mathbf{X}^{j,k,i})\|_r = \|\mu^j\|_r, \quad (5.9a)$$

$$291 \quad \eta_{\text{lin,AIP}}^{j,k,i} := \|\mathbf{H}_{\mu^j}^{j,k-1}(\mathbf{X}^{j,k,i}) - \mathbf{H}_{\mu^j}(\mathbf{X}^{j,k,i})\|_r, \quad (5.9b)$$

$$292 \quad \eta_{\text{alg,AIP}}^{j,k,i} := \left(\|\mathbf{F} - \mathbb{E}\mathbf{X}^{j,k,i}\|_r^2 + \|\mathbf{H}_{\mu^j}^{j,k-1}(\mathbf{X}^{j,k,i})\|_r^2 \right)^{\frac{1}{2}}. \quad (5.9c)$$

293 Then we have an upper bound for the norm $\|\mathbf{R}^{\text{AIP}}(\mathbf{X}^{j,k,i})\|_r$:

294 **Theorem 5.1** *Let $\mathbf{X}^{j,k,i} \in \mathbb{R}^n$ be the approximation of \mathbf{X} given by an iterative algebraic solver. Then we have*

$$295 \quad \|\mathbf{R}^{\text{AIP}}(\mathbf{X}^{j,k,i})\|_r \leq \eta_{\text{AIP}}^{j,k,i} := \eta_{\text{sm,AIP}}^{j,k,i} + \eta_{\text{lin,AIP}}^{j,k,i} + \eta_{\text{alg,AIP}}^{j,k,i}.$$

5.4 Adaptive inexact interior-point algorithm

Our proposed adaptive inexact interior-point algorithm implements adaptive stopping criteria formulated using the error component estimators given by (5.9) is as follows:

Algorithm 2 Adaptive inexact interior-point algorithm

1. Initialization

Choose a tolerance $\varepsilon > 0$ and parameters $\alpha \in]0, 1[$ and $\alpha_{\text{lin}}, \alpha_{\text{alg}} \in]0, 1]$.

Fix $\mu^1 > 0$ and an initial vector $\mathbf{X}^0 \in \mathbb{R}^n$ such that $\mathbf{K}(\mathbf{X}^0) \geq \mathbf{0}$ and $\mathbf{G}(\mathbf{X}^0) \geq \mathbf{0}$. Set $j := 1$.

2. Smoothing loop

2.1 Set $\mathbf{X}^{j,0} := \mathbf{X}^0$ as an initial guess for the linearization loop and $k := 1$.

2.2 Interior-point linearization loop

2.2.1 From $\mathbf{X}^{j,k-1}$ define $\mathbb{A}_{\mu^j}^{j,k-1} \in \mathbb{R}^{n,n}$ and $\mathbf{B}_{\mu^j}^{j,k-1} \in \mathbb{R}^n$ by (5.3).

2.2.2 Consider the problem of finding $\mathbf{X}^{j,k} \in \mathbb{R}^n$ such that

$$\mathbb{A}_{\mu^j}^{j,k-1} \mathbf{X}^{j,k} = \mathbf{B}_{\mu^j}^{j,k-1}. \quad (5.10)$$

2.2.3 Set $\mathbf{X}^{j,k,0} := \mathbf{X}^{j,k-1}$ as initial guess for the iterative algebraic solver. Set $i := 1$.

2.2.4 Algebraic solver loop

i) Starting from $\mathbf{X}^{j,k-1}$ perform a step of the iterative algebraic solver for (5.10), yielding, at step $i \geq 1$, a vector $\mathbf{X}^{j,k,i} \in \mathbb{R}^n$ such that

$$\mathbb{A}_{\mu^j}^{j,k-1} \mathbf{X}^{j,k} = \mathbf{B}_{\mu^j}^{j,k-1} - \mathbf{R}_{\text{alg}}^{\text{AIP}}(\mathbf{X}^{j,k,i}).$$

ii) Set $\mathbf{d}^{j,k,i} := \mathbf{X}^{j,k} - \mathbf{X}^{j,k-1}$ and compute $\kappa^{j,k,i} \in]0, 1]$ such that

$$\mathbf{K}(\mathbf{X}^{j,k-1} + \kappa^{j,k,i} \mathbf{d}^{j,k,i}) \geq \mathbf{0} \text{ and } \mathbf{G}(\mathbf{X}^{j,k-1} + \kappa^{j,k,i} \mathbf{d}^{j,k,i}) \geq \mathbf{0}.$$

Then set $\mathbf{X}^{j,k,i} := \mathbf{X}^{j,k-1} + \kappa^{j,k,i} \mathbf{d}^{j,k,i}$.

iii) Compute the estimators given by (5.9).

iv) If $\eta_{\text{alg,AIP}}^{j,k,i} < \alpha_{\text{alg}} \eta_{\text{lin,AIP}}^{j,k,i}$, set $\bar{i} := i$ and stop. If not, set $i := i + 1$ and go to i).

2.2.5 If $\eta_{\text{lin,AIP}}^{j,k,\bar{i}} < \alpha_{\text{lin}} \eta_{\text{sm,AIP}}^{j,k,\bar{i}}$, set $\bar{k} := k$ and stop. If not, set $k := k + 1$ and go to **2.2.1**.

2.3 If $\left\| \mathbf{R}^{\text{AIP}}(\mathbf{X}^{j,\bar{k},\bar{i}}) \right\|_r < \varepsilon$, set $\bar{j} := j$ and stop. If not, set $j := j + 1$, $\mathbf{X}^{j,0} := \mathbf{X}^{j-1,\bar{k},\bar{i}}$, and $\mu^j := \alpha \mu^{j-1}$. Then set $k := 1$ and go to **2.2.1**.

6 Numerical experiments: Problem of contact between two membranes

This section reports some numerical illustrations obtained using the algorithms previously presented. We consider here the model problem of contact between two membranes.

6.1 Problem statement

Let $\Omega = (a, b)$ be a one-dimensional domain. The problem reads: Find u_1, u_2 , and λ such that

$$\begin{cases} -\mu_1 \Delta u_1 - \lambda = f_1 & \text{in } \Omega, \\ -\mu_2 \Delta u_2 + \lambda = f_2 & \text{in } \Omega, \\ (u_1 - u_2) \lambda = 0, \quad u_1 - u_2 \geq 0, \quad \lambda \geq 0 & \text{in } \Omega, \\ u_1 = g & \text{on } \partial\Omega, \\ u_2 = 0 & \text{on } \partial\Omega, \end{cases} \quad (6.1)$$

308 where u_1 and u_2 represent the vertical displacements of the two membranes and λ is a Lagrange multiplier char-
309 acterizing the action of the second membrane on the first one, $-\lambda$ being the reaction. The constant parameters
310 $\mu_1, \mu_2 > 0$ correspond to the tension of each membrane, whereas $f_1, f_2 \in L^2(\Omega)$ are given external forces. The
311 boundary condition prescribed by a constant $g > 0$ ensures that, on the boundary $\partial\Omega$, the first membrane is above
312 the second one. The third line of (6.1) represents the linear complementarity conditions which serve to distinguish
313 two different physical situations: either the membranes are separated ($u_1 > u_2$ and $\lambda = 0$), or they are in contact
314 ($u_1 = u_2$ and $\lambda > 0$). We discretize this problem by the finite volume method. The corresponding discretization
315 can be written under the form of problem (1.1).

316 6.2 Test problem setting

317 Following [5], we set $\Omega = (-1, 1)$ and consider the following analytical solution for $x \in \Omega$

$$318 \quad u_1(x) := g(2x^2 - 1), \quad u_2(x) := \begin{cases} 2g(1 - x^2)(2x^2 - 1) & \text{if } x < \frac{-1}{\sqrt{2}} \text{ or } x > \frac{1}{\sqrt{2}}, \\ g(2x^2 - 1) & \text{otherwise,} \end{cases}$$

$$319 \quad \lambda(x) := \begin{cases} 0 & \text{if } x < \frac{-1}{\sqrt{2}} \text{ or } x > \frac{1}{\sqrt{2}}, \\ 2g & \text{otherwise.} \end{cases}$$

321 This triple is the solution of (6.1) for the data f_1 and f_2 given by

$$322 \quad f_1(x) := \begin{cases} -4g & \text{if } x < \frac{-1}{\sqrt{2}} \text{ or } x > \frac{1}{\sqrt{2}}, \\ -6g & \text{otherwise,} \end{cases} \quad \text{and} \quad f_2(x) := \begin{cases} -12g(1 - 4x^2) & \text{if } x < \frac{-1}{\sqrt{2}} \text{ or } x > \frac{1}{\sqrt{2}}, \\ -2g & \text{otherwise.} \end{cases}$$

323 Throughout the computational experiments, the parameters μ_1 and μ_2 are set to 1 and the boundary condition
324 g for the first membrane is taken equal to 0.1. Let N be the number of mesh elements. The initial guess $\mathbf{X}^0 \in \mathbb{R}^{3N}$
325 has its first N components equal to g and its other components equal to zero for the semismooth and smoothing
326 Newton methods. For the nonparametric interior-point method (resp. the adaptive interior-point method), the
327 initialization is given by $\mathcal{X}^0 = [\mathbf{0.1} \ \mathbf{0} \ \mathbf{0.5} \ \mathbf{0.05}]^T \in \mathbb{R}^{3N+1}$ (resp. $\mathbf{X}^0 = [\mathbf{0.1} \ \mathbf{0} \ \mathbf{0.5}]^T \in \mathbb{R}^{3N}$). All the simulations
328 are performed in MATLAB. We consider $N = 25000$ elements, leading to the matrix \mathbb{A} of size $n = 75000$.

329 6.3 Semismooth Newton method

330 We start by presenting the numerical results of the semismooth Newton method described in Section 2, using the
331 F–B function (2.2). The stopping criterion is on the total residual vector (2.6)

$$332 \quad \|\mathbf{R}(\mathbf{X}^k)\|_r < 10^{-8}. \tag{6.2}$$

333 To achieve this stopping criterion, 527 semismooth Newton-F–B iterations (CPU time: 68.9s) and 2232 Newton-
334 min iterations (CPU time: 338.9s) are needed. Figure 2 represents the evolution of $\|\mathbf{R}(\mathbf{X}^k)\|_r$ as a function of the
335 semismooth Newton-F–B iterations. We can see that it decreases slowly during iterations, then the convergence
336 gets extremely fast at the end.

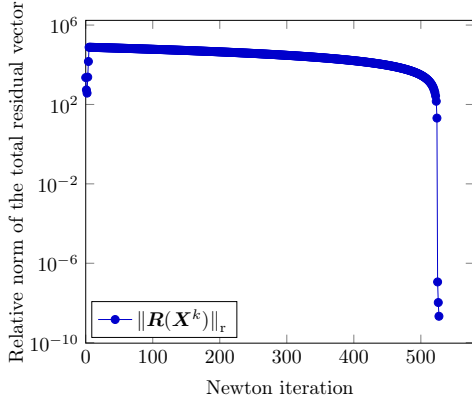


Figure 2: [Semismooth Newton method, F–B function (2.2), stopping criterion (6.2)] Relative norm of the total residual vector (2.6) as a function of semismooth Newton iterations.

6.4 Adaptive smoothing Newton method

We now test the adaptive smoothing Newton method, denoted by ASN, with the smoothed F–B function (3.2). This consists in employing the method presented in Section 3, summarized in Algorithm 1, but with an exact resolution of the nonlinear system (3.5). The linearization and smoothing estimators are respectively defined by

$$\eta_{\text{lin,ASN}}^{j,k} := \|\mathbf{C}_{\mu^j}(\mathbf{X}^{j,k})\|_{\text{r}}, \quad (6.3a)$$

$$\eta_{\text{sm,ASN}}^{j,k} := \|\mathbf{C}_{\mu^j}(\mathbf{X}^{j,k}) - \mathbf{C}(\mathbf{X}^{j,k})\|_{\text{r}}, \quad (6.3b)$$

and the total estimator by $\eta_{\text{ASN}}^{j,k} := \eta_{\text{sm,ASN}}^{j,k} + \eta_{\text{lin,ASN}}^{j,k}$.

First, we analyze the performance of the adaptive stopping criterion based on the estimators for stopping the linearization steps. We compare it with the classical approach in where the linearization is continued until the relative norm of the linearization estimator becomes smaller than a threshold taken as 10^{-8} , i.e.,

$$\text{Classical stopping criterion: } \eta_{\text{lin,ASN}}^{j,k} < 10^{-8}, \quad (6.4)$$

$$\text{Adaptive stopping criterion: } \eta_{\text{lin,ASN}}^{j,k} < \alpha_{\text{lin}} \eta_{\text{sm,ASN}}^{j,k}. \quad (6.5)$$

We set $\mu^1 = 1$, $\varepsilon = 10^{-8}$, $\alpha_{\text{lin}} = 1$, and $\alpha = 0.1$ in Algorithm 1. Figure 3 depicts the evolution of the estimators and the relative norm of the total residual vector $\mathbf{R}(\mathbf{X}^{j,k})$ given in (2.6) as a function of the smoothing Newton–F–B iterations, at a specific smoothing iteration $j = 1$ ($\mu^1 = 1$), left, and $j = 3$ ($\mu^3 = 10^{-2}$), right. We can observe from Figure 3, left, that, as expected, the smoothing estimator and $\|\mathbf{R}(\mathbf{X}^{j,k})\|_{\text{r}}$ stagnates after few steps, since here the smoothing parameter μ^1 is equal to 1, whereas the linearization estimator steadily decreases. If we consider the classical stopping criterion (6.4), the linearization will only be stopped at step $k = 8$. On the other hand, with our adaptive stopping criterion (6.5), only one iteration is necessary. Clearly after a few linearization steps, the linearization estimator no longer affects significantly the smoothing estimator, and we can economize many useless iterations.

Next, we provide in Table 1 the results obtained using the adaptive stopping criterion (6.5) to stop the nonlinear solver. We terminate the smoothing iterations using the relative norm of the total residual vector (2.6)

$$\|\mathbf{R}(\mathbf{X}^{j,\bar{k}})\|_{\text{r}} < 10^{-8}. \quad (6.6)$$

We present the cumulated number of Newton iterations Niter, the estimators (6.3), and the relative norm of the total residual vector (2.6) at each smoothing step j . In terms of numbers, 10 smoothing iterations and 36 cumulated Newton iterations (CPU time: 6.9s) are needed to achieve the stopping criterion (6.6). From Table 1, one can see that for each value of μ^j , the Newton iterations are stopped according to (6.5). $\|\mathbf{R}(\mathbf{X}^{j,\bar{k}})\|_{\text{r}}$ decreases until lying below 10^{-8} . Figure 4 displays the curve of the estimators as a function of cumulated Newton iterations and smoothing iterations, as well as the relative norm of the total residual vector as a function of smoothing iterations. The improvement of the performance with respect to the semismooth Newton–F–B method of Section 6.3 is spectacular.

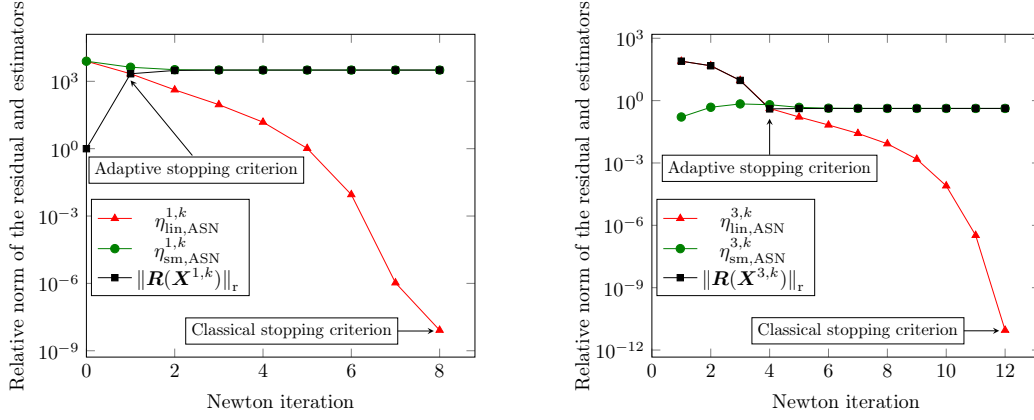


Figure 3: [Adaptive smoothing Newton method, smoothed F–B function (3.2), classical and adaptive stopping criteria (6.4) and (6.5)] Relative norm of the total residual vector (2.6) and estimators (6.3) as a function of Newton iterations k , at a specific smoothing iteration $j = 1$ ($\mu^1 = 1$), left, and at $j = 3$ ($\mu^3 = 10^{-2}$), right.

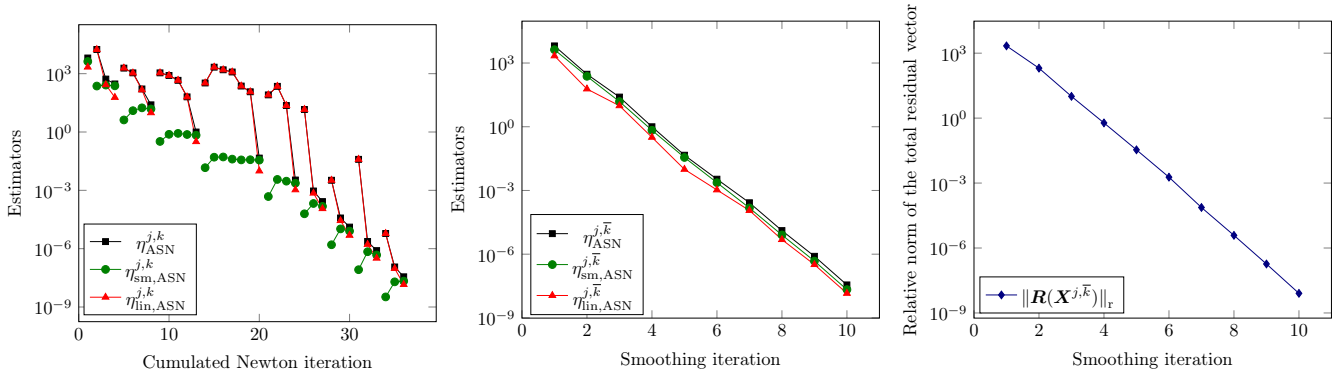


Figure 4: [Adaptive smoothing Newton method, smoothed F–B function (3.2), adaptive stopping criterion (6.5)] Estimators (6.3) as a function of cumulated Newton iterations (left). Estimators (6.3) (middle) and relative norm of the total residual vector (2.6) (right) as a function of smoothing iterations j at convergence of the linearization solver.

μ^j	Niter	$\eta_{\text{lin,ASN}}^{j,\bar{k}}$	$\eta_{\text{sm,ASN}}^{j,\bar{k}}$	$\ \mathbf{R}(\mathbf{X}^{j,\bar{k}})\ _r$
1e+00	1	2.17e+03	4.24e+03	2.17e+03
1e-01	3	6.00e+01	2.37e+02	2.03e+02
1e-02	4	9.73e+00	1.53e+01	1.01e+01
1e-03	5	3.18e-01	6.84e-01	6.00e-01
1e-04	7	9.87e-03	3.58e-02	3.43e-02
1e-05	4	1.06e-03	2.33e-03	1.87e-03
1e-06	3	1.14e-04	1.50e-04	7.45e-05
1e-07	3	4.85e-06	8.04e-06	3.84e-06
1e-08	3	3.23e-07	4.72e-07	1.83e-07
1e-09	3	1.43e-08	2.15e-08	8.04e-09

Table 1: [Adaptive smoothing Newton method, smoothed F–B function (3.2), adaptive stopping criterion (6.5)] Number of Newton iterations Niter, estimators (6.3), and relative norm of the total residual vector (2.6) at each smoothing iteration j , at convergence of the linearization solver.

371 With the intention to compare the proposed method to existing methods, we complete the semismooth Newton
372 method by a path-following strategy to solve problem (1.1), following [48]. For the sake of brevity, we shall not
373 detail this here. The following test compares the semismooth Newton method (SSN) and the semismooth Newton
374 method with path-following (SSN-pf) in which the linearization is stopped when the criterion (6.2) is satisfied,
375 to the adaptive smoothing Newton method, using the smoothed min and F–B functions (3.1) and (3.2) and the
376 stopping criteria (6.5) and (6.2) respectively for the linearization and smoothing iterations. We compare the number
377 of cumulated linearization iterations and the global CPU time of the simulation for the different strategies. The
378 results are displayed in Figure 5. They confirm the expected reduction of the computational cost of the numerical
379 resolution with our adaptive approaches. Actually, we notice that the semismooth Newton method with path-
380 following (red curve) and the adaptive smoothing Newton method (purple and dark blue curves) require significantly
381 fewer cumulated Newton iterations and time to converge, in comparison with the semismooth Newton method (green
382 and orange curves). Therefore, employing the path-following strategy or the adaptive strategy based on a posteriori
383 error estimates enables to save many unnecessary additional iterations, and yield much better results than the pure
384 semismooth Newton method. We note that, using the adaptive smoothing Newton method, one obtains similar
385 computational results using both the smoothed F–B or the smoothed min function.

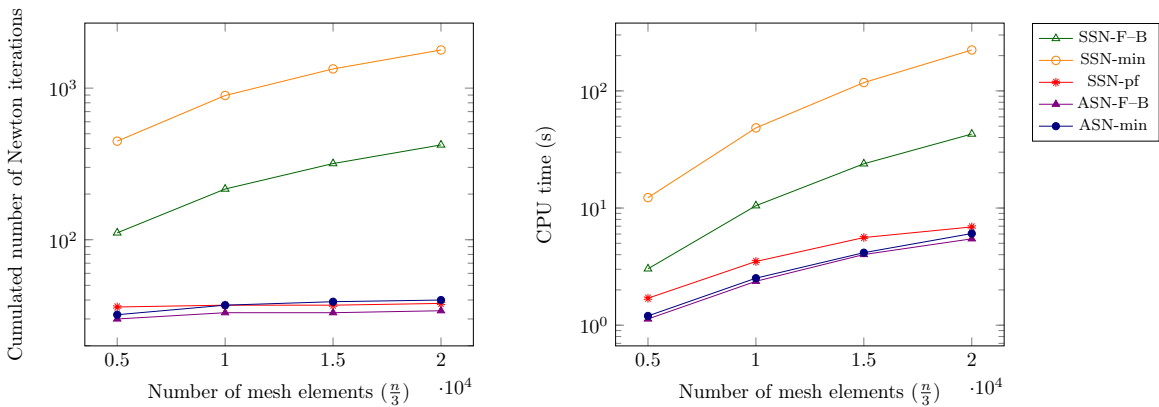


Figure 5: [Semismooth Newton method (with and without a path-following strategy) and adaptive smoothing method] Cumulated number of Newton iterations (left) and CPU time (right) as a function of the number of mesh elements.

386 6.5 Adaptive inexact smoothing Newton method

387 We focus in this section on the adaptive inexact Newton method introduced in Section 3 and investigate the
388 performance of Algorithm 1 using the smoothed F–B function (3.2) together with the restarted GMRES method.
389 Typically, we use a fixed restart parameter equal to 300. The behavior of the adaptive smoothing solvers can be
390 improved dramatically by using good preconditioners. Here, we merely use an ILU preconditioner to speed-up the
391 GMRES solver. For other possibilities for preconditioners, we refer to, e.g., [31] and the references therein. To
392 point out the efficiency of the adaptivity, we test two approaches. First, we stop the algebraic iterations using the
393 standard GMRES stopping criterion on the relative residual given by

$$394 \mathbf{R}_{\text{rel}} := \frac{\|\mathbb{M}_2 \setminus (\mathbb{M}_1 \setminus (\mathbf{B}_{\mu^j}^{j,k-1} - \mathbb{A}_{\mu^j}^{j,k-1} \mathbf{X}^{j,k,i}))\|}{\|\mathbb{M}_2 \setminus (\mathbb{M}_1 \setminus (\mathbf{B}_{\mu^j}^{j,k-1} - \mathbb{A}_{\mu^j}^{j,k-1} \mathbf{X}^{j,k-1}))\|} \leq 10^{-10}, \quad (6.7)$$

395 where \mathbb{M}_1 and \mathbb{M}_2 are the preconditioner matrices. Second, we incorporate the adaptive stopping criteria (3.11a)
396 for the algebraic solver in Algorithm 1. We set the parameters $\mu^1 = 1$, $\varepsilon = 10^{-5}$, $\alpha_{\text{alg}} = 10^{-3}$, $\alpha_{\text{lin}} = 1$, and $\alpha = 0.1$.
397 Figure 6 depicts the evolution of the algebraic and linearization estimators and the GMRES relative residual during
398 the algebraic resolution, for specific smoothing step j and linearization step k . For $j = 2$ and $k = 2$, we see that 22
399 GMRES iterations are needed to achieve the standard stopping criterion (6.7), whereas in the adaptive resolution
400 case, only 10 GMRES iterations are required to satisfy the adaptive stopping criterion (3.11a). In this case, we can
401 avoid many unnecessary iterations. One can also see from the right part of Figure 6, for $j = 3$ and $k = 1$, that the
402 overall gain in terms of algebraic iterations obtained using our stopping criteria is quite significant.

403 Figure 7, left, shows the evolution of the estimators during smoothing iterations, at convergence of the nonlinear
404 and linear solvers. As expected, the estimators decrease when μ decreases at each smoothing step. In the middle

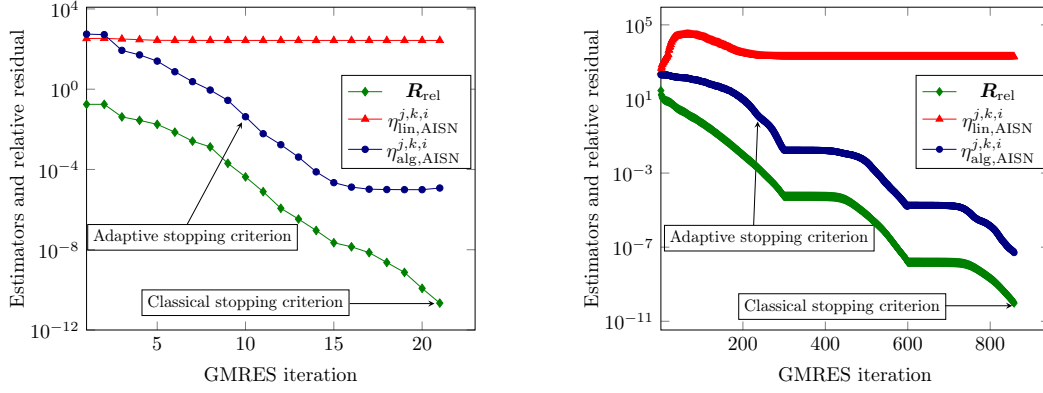


Figure 6: [Adaptive inexact smoothing Newton method, smoothed F–B function (3.2), Algorithm 1] Algebraic and linearization estimators (3.10) and GMRES relative residual as a function of the GMRES iterations i , for a fixed smoothing and linearization iterations, $j = 2, k = 2, i$ varies, left, and $j = 3, k = 1, i$ varies, right, using the classical stopping criterion (6.7) and the adaptive one (3.11a).

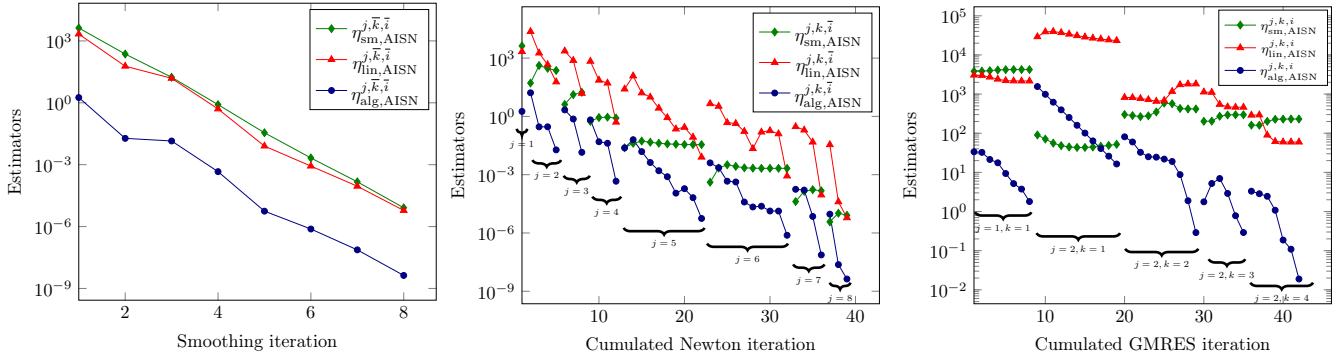


Figure 7: [Adaptive inexact smoothing Newton method, smoothed F–B function (3.2), Algorithm 1] Estimators (3.10) as a function of smoothing iterations j at convergence of the algebraic and linearization solvers, left. Estimators as a function of cumulated Newton iterations at convergence of the algebraic solver, middle. Estimators as a function of cumulated GMRES iterations during the first two smoothing iterations ($j = 1$ and $j = 2$), right.

405 part of Figure 7, we can observe the behavior of the estimators at the end of the algebraic iterations, during the
406 linearization iterations. We present 8 curves, each one corresponding to a specific value of μ^j . We can see that at
407 each smoothing iteration j , the smoothing estimator $\eta_{sm,AISN}^{j,k,i}$ stagnates after about two iterations. The linearization
408 estimator $\eta_{lin,AISN}^{j,k,i}$ decreases until becoming smaller than the smoothing estimator, satisfying the stopping criterion
409 (3.11b). Finally, the detected behavior in terms of all smoothing iterations j , linearization iterations k , and algebraic
410 solver iterations i is presented in Figure 7, right, for $j \leq 2$. The overall results are collected in Table 2. We present
411 in particular the number of linearization and cumulated algebraic iterations per smoothing step j , Niter and Giter
412 respectively, as well as the estimators (3.10) and the relative norm of the total residual vector (2.6) at the end of
413 each smoothing step j . Using the adaptive stopping criteria (3.11), 8 smoothing iterations, 39 cumulated Newton
414 iterations, and 5999 cumulated GMRES iterations are needed to ensure convergence. Figure 8 illustrates the
415 performance of the adaptive inexact smoothing Newton method. It represents the ratio between: 1) the number
416 of algebraic iterations (left) and the CPU time (right) using the classical GMRES stopping criterion (6.7) and 2)
417 the number of algebraic iterations and the CPU time using the adaptive stopping criterion (3.11a) for GMRES, as
418 a function of the number of elements. For larger systems, 20-times fewer iterations and 18-times faster execution
419 time are achieved.

μ^j	Niter	Giter	$\eta_{\text{lin,AISN}}^{j,\bar{k},\bar{i}}$	$\eta_{\text{sm,AISN}}^{j,\bar{k},\bar{i}}$	$\eta_{\text{alg,AISN}}^{j,\bar{k},\bar{i}}$	$\ \mathbf{R}(\mathbf{X}^{j,\bar{k},\bar{i}})\ _{\text{r}}$
1e+00	1	8	2.16e+03	4.24e+03	1.80e+00	2.19e+03
1e-01	4	34	5.95e+01	2.31e+02	1.89e-02	1.80e+02
1e-02	3	391	1.54e+01	1.73e+01	1.41e-02	6.75e+00
1e-03	4	198	5.04e-01	8.16e-01	4.60e-04	5.95e-01
1e-04	10	796	7.99e-03	3.53e-02	5.58e-06	3.43e-02
1e-05	10	684	8.54e-04	2.12e-03	7.61e-07	1.94e-03
1e-06	4	513	9.03e-05	1.48e-04	7.42e-08	1.05e-04
1e-07	3	3375	6.04e-06	8.14e-06	4.27e-09	4.26e-06

Table 2: [Adaptive inexact smoothing Newton method, smoothed F–B function (3.2), Algorithm 1] Number of Newton iterations and cumulated GMRES iterations, estimators (3.10), and relative norm of the total residual vector (2.6) at each smoothing iteration j , at convergence of the algebraic and linearization solvers.

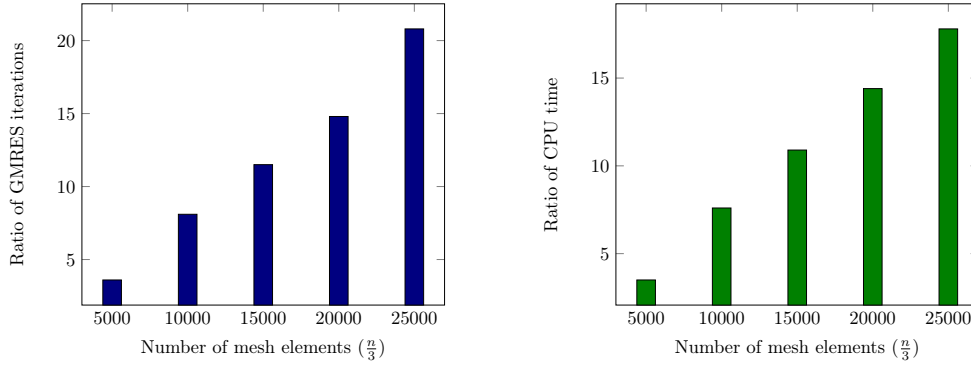


Figure 8: [Adaptive inexact smoothing Newton method, smoothed F–B function (3.2), Algorithm 1] Ratio between: the number of algebraic iterations (left) and CPU time (right) needed by the classical stopping criterion (6.7) to converge to the number and time needed by the adaptive stopping criterion (3.11a), as a function of the number of mesh elements.

6.6 Nonparametric interior-point method

We consider here the nonparametric interior-point approach of Section 4, where the dimension of the corresponding problem is $n = 3N + 1$. The value of the constant θ in the additional equation (4.2) is 10^{-1} . The stopping criterion is on the relative norm of the linearization residual vector

$$\|\mathbf{R}^{\text{IP}}(\mathcal{X}^k)\|_{\text{r}} := \|\mathbf{R}^{\text{IP}}(\mathcal{X}^k)\| / \|\mathbf{R}^{\text{IP}}(\mathcal{X}^0)\| < 10^{-8}, \quad (6.8)$$

with

$$\mathbf{R}^{\text{IP}}(\mathcal{X}^k) := \begin{bmatrix} \mathbf{F} - \mathbb{E}\mathbf{X}^k \\ \boldsymbol{\mu} - \mathbf{K}(\mathbf{X}^k)\mathbf{G}(\mathbf{X}^k) \\ -\theta\mu^k - (\mu^k)^2 \end{bmatrix}.$$

Using this method, 19 Newton iterations (CPU time: 6.7s) are needed to reach the end of the simulation. Figure 9 shows that $\|\mathbf{R}^{\text{IP}}(\mathcal{X}^k)\|_{\text{r}}$ decreases during the Newton interior-point iterations until satisfying the stopping criterion (6.8).

6.7 Adaptive interior-point method

Next, we consider the adaptive interior-point method, which is the method presented in Section 5, Algorithm 2 without applying an algebraic iterative solver to approximate the solution of the linear system (5.2). In this case, we can define the linearization and smoothing estimators respectively by

$$\eta_{\text{lin,AIP}}^{j,k} := \|\mathbf{H}_{\mu^j}(\mathbf{X}^{j,k})\|_{\text{r}}, \quad (6.9a)$$

$$\eta_{\text{sm,AIP}}^{j,k} := \|\boldsymbol{\mu}^j\|_{\text{r}}, \quad (6.9b)$$

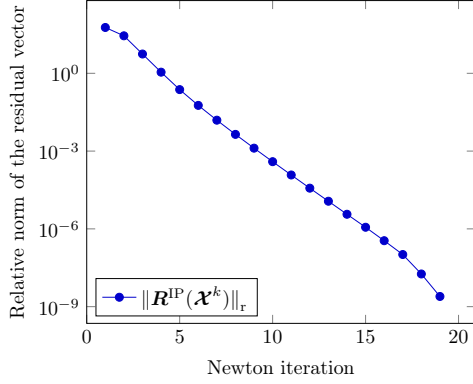


Figure 9: [Nonparametric interior-point method, stopping criterion (6.8)] Relative norm of the linearization residual vector (6.8) as a function of Newton iterations.

435 where $\mathbf{H}_{\mu^j}(\cdot)$ is defined in (5.1b), and the total estimator by $\eta_{\text{AIP}}^{j,k} := \eta_{\text{sm,AIP}}^{j,k} + \eta_{\text{lin,AIP}}^{j,k}$. Recall from (5.8) the
 436 definition of the total residual vector for $\mathbf{V} \in \mathbb{R}^n$ as

$$437 \quad \mathbf{R}^{\text{AIP}}(\mathbf{V}) := \begin{bmatrix} \mathbf{F} - \mathbb{E}\mathbf{V} \\ -\mathbf{H}(\mathbf{V}) \end{bmatrix}, \quad (6.10)$$

438 where $\mathbf{H}(\cdot)$ is defined in (5.7). The adaptive stopping criterion

$$439 \quad \eta_{\text{lin,AIP}}^{j,k} < \alpha_{\text{lin}} \eta_{\text{sm,AIP}}^{j,k} \quad (6.11)$$

440 is used to stop the nonlinear solver and a criterion on the relative norm of the total residual vector is applied to
 441 stop the smoothing iterations

$$442 \quad \|\mathbf{R}^{\text{AIP}}(\mathbf{X}^{j,\bar{k}})\|_{\text{r}} < 10^{-8}. \quad (6.12)$$

443 The initial smoothing vector is $\boldsymbol{\mu}^1 = [1, \dots, 1]^T \in \mathbb{R}^N$ and $\alpha_{\text{lin}} = 1$. Concerning the update of the smoothing
 444 parameter μ , we set $\alpha = 10^{-1}$. Table 3 summarizes the results. To achieve the stopping criterion (6.12), 11
 445 smoothing iterations and 20 cumulated Newton iterations are needed (CPU time: 5.0s). In Figure 10, we plot
 446 the estimators (6.9) as a function of the cumulated Newton iterations (left), the smoothing iterations (middle),
 447 and the relative norm of the residual vector as a function of the smoothing iterations (right). The behavior of
 448 $\|\mathbf{R}^{\text{AIP}}(\mathbf{X}^{j,\bar{k}})\|_{\text{r}}$ in Figure 10 appears a bit different from its behavior in Figure 4. This is related to the fact that
 449 the relative norm of the total residual given by (2.6) includes $\mathbf{C}(\mathbf{X})$ in the adaptive smoothing Newton method,
 450 whereas in this adaptive interior-point method, the relative norm of the total residual given by (6.10) includes
 451 $\mathbf{K}(\mathbf{X})\mathbf{G}(\mathbf{X})$.

μ^j	Niter	$\eta_{\text{lin,AIP}}^{j,\bar{k}}$	$\eta_{\text{sm,AIP}}^{j,\bar{k}}$	$\ \mathbf{R}^{\text{AIP}}(\mathbf{X}^{j,\bar{k}})\ _{\text{r}}$
1e+00	2	1.11e+01	2.00e+01	3.00e+01
1e-01	2	1.24e+00	2.00e+00	3.20e+00
1e-02	2	1.15e-01	2.00e-01	3.11e-01
1e-03	2	6.51e-03	2.00e-02	2.43e-02
1e-04	2	3.38e-04	2.00e-03	2.14e-03
1e-05	1	1.58e-04	2.00e-04	2.82e-04
1e-06	2	3.67e-06	2.00e-05	2.10e-05
1e-07	2	1.00e-07	2.00e-06	2.02e-06
1e-08	1	1.86e-07	2.00e-07	3.84e-07
1e-09	2	9.33e-10	2.00e-08	2.01e-08
1e-10	2	2.55e-11	2.00e-09	2.00e-09

Table 3: [Adaptive interior-point method] Number of Newton iterations, estimators (6.9), and relative norm of the total residual vector (6.10) at each smoothing step j , at convergence of the linearization solver.

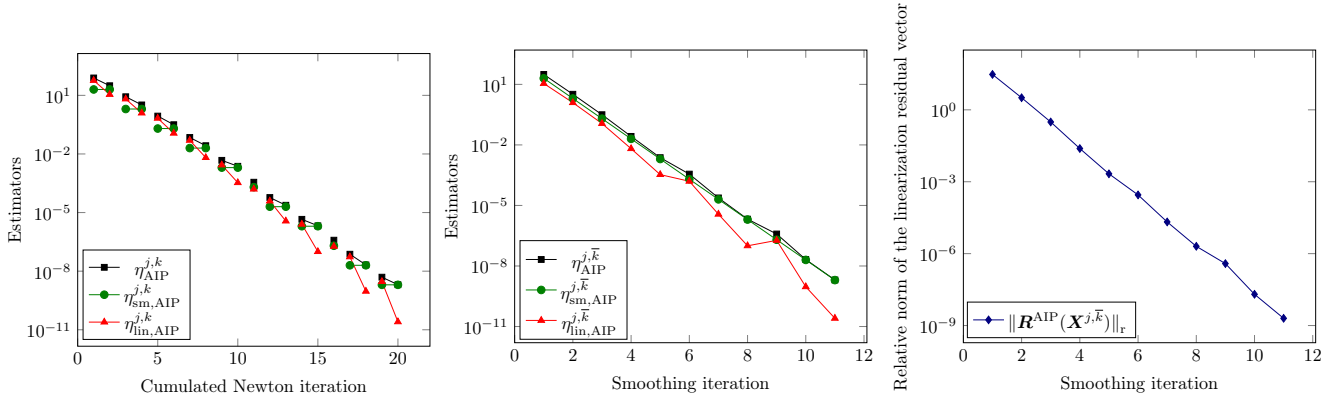


Figure 10: [Adaptive interior-point method] Estimators (6.9) as a function of cumulated Newton iterations (left). Estimators (6.9) (middle) and relative norm of the total residual vector (6.10) (right) as a function of smoothing iterations j at convergence of the linearization solver.

6.8 Adaptive inexact interior-point method

Let us now present the numerical results of the adaptive inexact interior-point method, detailed in Section 5. We employ Algorithm 2 with the GMRES algebraic solver and an ILU preconditioner. The parameters in Algorithm 2 are set as $\boldsymbol{\mu}^1 = [1, \dots, 1]^T \in \mathbb{R}^N$, $\varepsilon = 10^{-5}$, $\alpha_{\text{alg}} = 1$, $\alpha_{\text{lin}} = 1$, and $\alpha = 0.1$. The restart parameter of restarted GMRES is chosen equal to 300. From Table 4, we can see that the method converged after 8 smoothing iterations, 20 cumulated linearization iterations, and 760 cumulated GMRES iterations. Figure 11, left, displays the curves of the estimators (5.9) as a function of the smoothing iteration. One can see that the estimators satisfy the adaptive stopping criteria incorporated in Algorithm 2. In Figure 11, right, the estimators are shown as a function of cumulated Newton iterations, at convergence of the linear solver.

μ^j	Niter	Giter	$\eta_{\text{lin,AIP}}^{j,\bar{k},\bar{i}}$	$\eta_{\text{sm,AIP}}^{j,\bar{k},\bar{i}}$	$\eta_{\text{alg,AIP}}^{j,\bar{k},\bar{i}}$	$\ \mathbf{R}^{\text{AIP}}(\mathbf{X}^{j,\bar{k},\bar{i}})\ _r$
1e+00	3	7	1.15e+01	2.00e+01	3.36e+00	5.59e+00
1e-01	2	12	5.44e-01	2.00e+00	1.78e-02	2.00e+00
1e-02	3	20	9.75e-02	2.00e-01	2.80e-02	2.04e-01
1e-03	3	29	4.82e-03	2.00e-02	1.74e-03	2.01e-02
1e-04	3	56	2.52e-04	2.00e-03	2.19e-04	2.01e-03
1e-05	2	62	1.77e-04	2.00e-04	1.08e-04	2.29e-04
1e-06	2	110	1.49e-05	2.00e-05	1.42e-05	2.46e-05
1e-07	2	464	1.34e-06	2.00e-06	1.16e-06	2.31e-06

Table 4: [Adaptive inexact interior-point method, Algorithm 2] Number of cumulated Newton and GMRES iterations, estimators (5.9), and relative norm of the total residual vector (5.8) at each smoothing iteration j , at convergence of the algebraic and linearization solvers.

6.9 Comparison of the methods

This section is devoted to compare the semismooth Newton method (SSN), semismooth Newton method with path-following (SSN-pf), nonparametric interior-point method (IP), adaptive smoothing Newton method (ASN), and adaptive interior-point method (AIP). For this purpose, we introduce a unified residual vector, for $\mathbf{V} \in \mathbb{R}^n$

$$\mathbf{R}_{\text{unif}}(\mathbf{V}) := \begin{bmatrix} \mathbf{F} - \mathbf{E}\mathbf{V} \\ \min(\mathbf{0}, \mathbf{K}(\mathbf{V})) \\ \min(\mathbf{0}, \mathbf{G}(\mathbf{V})) \\ \mathbf{K}(\mathbf{V}) \cdot \mathbf{G}(\mathbf{V}) \end{bmatrix}, \quad (6.13)$$

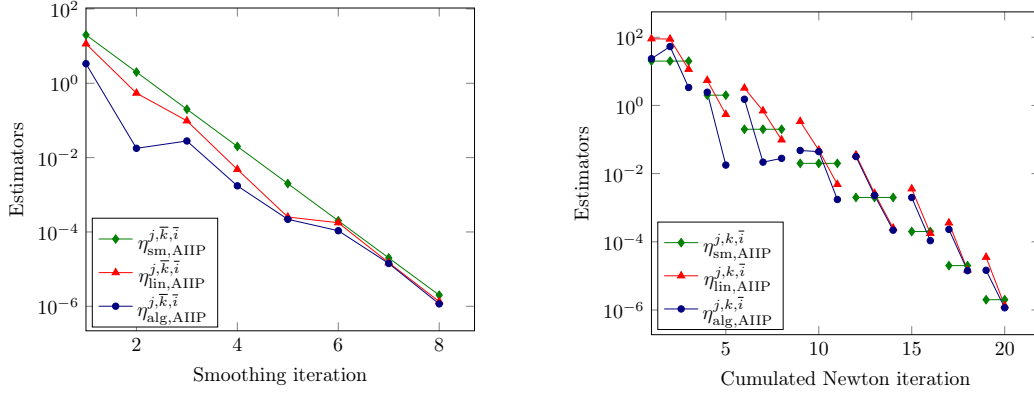


Figure 11: [Adaptive inexact interior-point method, Algorithm 2] Estimators (5.9) as a function of smoothing iterations j at convergence of the algebraic and linearization solvers (left). Estimators as a function of cumulated Newton iterations k at convergence of the algebraic solver (right).

466 independent of the way the nonlinear complementarity constraints are reformulated. The stopping criterion of
 467 the nonlinear solver for the classical methods (SSN, SSN-pf, IP) is on the relative unified residual $\|\mathbf{R}_{unif}(\mathbf{X}^k)\|_r$
 468 lying below 10^{-8} . Regarding the adaptive methods (ASN, AIP), to stop the nonlinear solver, we use the adaptive
 469 stopping criteria given respectively in (6.5) and (6.11). To stop the smoothing iterations, $\|\mathbf{R}_{unif}(\mathbf{X}^{j,k})\|_r$ is requested
 470 to become smaller than 10^{-8} .

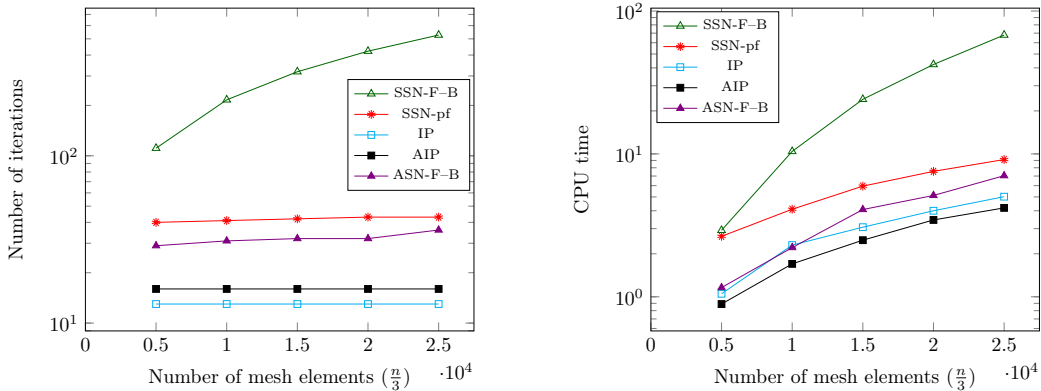


Figure 12: [Semismooth Newton method (F–B function (2.2)), semismooth Newton method with a path-following strategy, nonparametric interior-point method, adaptive interior-point method, and adaptive smoothing Newton method (smoothed F–B function (3.2))] Number of cumulated Newton iterations (left) and CPU time (right) as a function of the number of mesh elements, employing a stopping criterion on the relative norm of the unified residual vector (6.13).

471 In Figure 12, we plot the cumulated number of the Newton iterations (left) and the CPU time (right) required by
 472 each method, as a function of the number of mesh elements. It is clearly seen that the semismooth Newton method
 473 (green curve) is typically more costly, both in terms of the required number of iterations and the CPU time, in
 474 comparison with the other methods. Precisely, we can observe an important gain between the semismooth Newton
 475 method (green curve) and the adaptive smoothing Newton method (purple curve). Moreover, as we can remark
 476 from the red curve, the combination of a path-following strategy to the semismooth Newton method seems to be
 477 efficient. Finally, one does not see a remarkable difference between the results of the nonparametric interior-point
 478 method (cyan curve) and the adaptive interior-point method (black curve) in this test case.

7 Numerical experiments: Two-phase flow with phase transition

The second model problem that we consider in our numerical tests is a two-phase flow model (liquid–gas) with phase transition in porous media following [6, 13, 9]. Each of the liquid phase, denoted by l, and the gas phase, denoted by g, is composed of two components, water and hydrogen, denoted respectively by w and h.

7.1 Problem statement

The problem at hand can be formulated as a system of nonlinear partial differential equations with nonlinear complementarity constraints at each time step τ_ν . Let \mathcal{T}_h be the spatial mesh, we denote respectively by S_K^ν, P_K^ν , and χ_K^ν the discrete elementwise unknowns approximating the values of the saturation S^l , the pressure P^l , and the molar fraction of hydrogen in the liquid phase χ_h^l in the element $K \in \mathcal{T}_h$ and on time step $1 \leq \nu \leq N_t$. Let N be the number of elements in the mesh \mathcal{T}_h . If one introduces the appropriate nonlinear function $H_{c,K}^\nu : \mathbb{R}^{3N} \rightarrow \mathbb{R}$, $c \in \{w, h\}$, and suitable functions $F_K : \mathbb{R}^3 \rightarrow \mathbb{R}$ and $G_K : \mathbb{R}^3 \rightarrow \mathbb{R}$, the discrete problem written elementwise consists in finding $\mathbf{X}^\nu := (\mathbf{X}_K^\nu)_{K \in \mathcal{T}_h} \in \mathbb{R}^n$, where $n = 3N$, and $\mathbf{X}_K^\nu := [S_K^\nu, P_K^\nu, \chi_K^\nu] \in \mathbb{R}^3$, such that for all $K \in \mathcal{T}_h$

$$H_{c,K}^\nu(\mathbf{X}^\nu) = 0, \quad c \in \{w, h\}, \quad (7.1a)$$

$$F_K(\mathbf{X}_K^\nu) \geq 0, \quad G_K(\mathbf{X}_K^\nu) \geq 0, \quad F_K(\mathbf{X}_K^\nu)G_K(\mathbf{X}_K^\nu) = 0. \quad (7.1b)$$

The formulation (7.1) allows to model the transition from a single-phase flow to a two-phase flow during the appearance and disappearance of the gas phase and vice versa. As an example, a detailed finite volume discretization can be found in [7, Section 3.2]. The first $2N$ lines of system (7.1) can be written globally as

$$\mathcal{H}^\nu(\mathbf{X}^\nu) = 0,$$

where $\mathcal{H}^\nu : \mathbb{R}^{3N} \rightarrow \mathbb{R}^{2N}$ is defined elementwise by (7.1a).

Considering a C-function C^ν , for $1 \leq \nu \leq N_t$, we define a function $\mathcal{C}^\nu : \mathbb{R}^{3N} \rightarrow \mathbb{R}^N$ as $\mathcal{C}^\nu(\mathbf{X}^\nu) = C^\nu((F_K(\mathbf{X}_K^\nu))_{K \in \mathcal{T}_h}, (G_K(\mathbf{X}_K^\nu))_{K \in \mathcal{T}_h})$. This leads us to apply a semismooth Newton method to find a solution for problem (7.1) written as

$$\begin{aligned} \mathcal{H}^\nu(\mathbf{X}^\nu) &= 0, \\ \mathcal{C}^\nu(\mathbf{X}^\nu) &= 0. \end{aligned} \quad (7.2)$$

The total residual vector $\mathbf{R}(\mathbf{V})$ of problem (7.2) is thus given by

$$\mathbf{R}(\mathbf{V}) := \begin{bmatrix} -\mathcal{H}^\nu(\mathbf{V}) \\ -\mathcal{C}^\nu(\mathbf{V}) \end{bmatrix}, \quad \forall \mathbf{V} \in \mathbb{R}^n. \quad (7.3)$$

7.2 Adaptive smoothing Newton method

We introduce a function $\mathcal{C}_\mu^\nu : \mathbb{R}^{3N} \rightarrow \mathbb{R}^N$ defined as $\mathcal{C}_\mu^\nu(\mathbf{X}^\nu) = C_\mu^\nu((F_K(\mathbf{X}_K^\nu))_{K \in \mathcal{T}_h}, (G_K(\mathbf{X}_K^\nu))_{K \in \mathcal{T}_h})$, for $1 \leq \nu \leq N_t$, where C_μ^ν is a smoothed C-function. Line (7.1b) can be approximated as a smoothed nonlinear equation $\mathcal{C}_\mu^\nu(\mathbf{X}^\nu) = 0$, making it possible to apply the standard Newton method to solve the resulting nonlinear system in the form: Find $\mathbf{X}^{\nu,j} \in \mathbb{R}^{3N}$ at each time step ν , $1 \leq \nu \leq N_t$, satisfying

$$\begin{aligned} \mathcal{H}^\nu(\mathbf{X}^{\nu,j}) &= 0, \\ \mathcal{C}_{\mu^{j\nu}}^\nu(\mathbf{X}^{\nu,j}) &= 0. \end{aligned} \quad (7.4)$$

At each time step $1 \leq \nu \leq N_t$, each smoothing step $j \geq 1$, and each linearization step $k \geq 1$, fixing $\mathbf{X}^{\nu,j,0} \in \mathbb{R}^n$, we try to approach the solution of problem (7.4) by finding a solution $\mathbf{X}^{\nu,j,k} \in \mathbb{R}^n$ such that

$$\mathbb{A}_{\mu^{j\nu}}^{\nu,j,k-1} \mathbf{X}^{\nu,j,k} = \mathbf{B}_{\mu^{j\nu}}^{\nu,j,k-1}, \quad (7.5)$$

where the Jacobian matrix $\mathbb{A}_{\mu^{j\nu}}^{\nu,j,k-1} \in \mathbb{R}^{n,n}$ and the right-hand side vector $\mathbf{B}_{\mu^{j\nu}}^{\nu,j,k-1} \in \mathbb{R}^n$ are defined by

$$\mathbb{A}_{\mu^{j\nu}}^{\nu,j,k-1} := \begin{bmatrix} \mathbf{J}_{\mathcal{H}^\nu}(\mathbf{X}^{\nu,j,k-1}) \\ \mathbf{J}_{\mathcal{C}_{\mu^{j\nu}}^\nu}(\mathbf{X}^{\nu,j,k-1}) \end{bmatrix}, \quad (7.6a)$$

$$\mathbf{B}_{\mu^{j\nu}}^{\nu,j,k-1} := \begin{bmatrix} \mathbf{J}_{\mathcal{H}^\nu}(\mathbf{X}^{\nu,j,k-1})\mathbf{X}^{\nu,j,k-1} - \mathcal{H}^\nu(\mathbf{X}^{\nu,j,k-1}) \\ \mathbf{J}_{\mathcal{C}_{\mu^{j\nu}}^\nu}(\mathbf{X}^{\nu,j,k-1})\mathbf{X}^{\nu,j,k-1} - \mathcal{C}_{\mu^{j\nu}}^\nu(\mathbf{X}^{\nu,j,k-1}) \end{bmatrix}, \quad (7.6b)$$

with $\mathbf{J}_{\mathcal{H}^\nu}(\mathbf{X}^{\nu,j,k-1})$ and $\mathbf{J}_{\mathcal{C}_{\mu^{j\nu}}^\nu}(\mathbf{X}^{\nu,j,k-1})$ the Jacobian matrices of the function \mathcal{H}^ν and the smoothed function $\mathcal{C}_{\mu^{j\nu}}^\nu$, respectively, at the point $\mathbf{X}^{\nu,j,k-1}$ obtained by a Newton linearization.

522 7.3 Adaptive smoothing Newton algorithm

523 Let $\varepsilon > 0$ be the desired relative tolerance, $\alpha_{\text{lin}} \in]0, 1[$ be the desired relative size of the linearization error, and
 524 $\alpha \in]0, 1[$ the smoothing decrease parameter. The unsteady adaptive smoothing Newton algorithm reads as follows:

Algorithm 3 Unsteady adaptive smoothing Newton algorithm

Initialization: Fix $\varepsilon > 0$, $\alpha \in]0, 1[$, and $\alpha_{\text{lin}} \in]0, 1[$. Set $\nu := 1$ and $t_\nu := 0$. Choose $\mathbf{X}^{\nu,0} \in \mathbb{R}^n$.

Time loop

1. Fix $\mu^{j\nu} > 0$ and set $j := 1$.

2. **Smoothing loop**

2.1 Set $\mathbf{X}^{\nu,j,0} := \mathbf{X}^{\nu,0}$ and $k := 1$.

2.2 **Newton linearization loop**

2.2.1 From $\mathbf{X}^{\nu,j,k-1}$ define $\mathbb{A}_{\mu^{j\nu}}^{\nu,j,k-1} \in \mathbb{R}^{n,n}$ and $\mathbf{B}_{\mu^{j\nu}}^{\nu,j,k-1} \in \mathbb{R}^n$ given by (7.6).

2.2.2 Find $\mathbf{X}^{\nu,j,k}$ solution to the linear system

$$\mathbb{A}_{\mu^{j\nu}}^{\nu,j,k-1} \mathbf{X}^{\nu,j,k} = \mathbf{B}_{\mu^{j\nu}}^{\nu,j,k-1}.$$

2.2.3 Compute the estimators and check the stopping criterion for the nonlinear solver

$$\left(\eta_{\text{lin,ASN}}^{\nu,j,k} < \alpha_{\text{lin}} \eta_{\text{sm,ASN}}^{\nu,j,k} \right) \quad \text{or} \quad \left(\eta_{\text{lin,ASN}}^{\nu,j,k} < \varepsilon \right). \quad (7.7)$$

If satisfied, set $\bar{k} := k$ and stop. If not, set $k := k + 1$ and go to 2.2.1.

2.3 Check the stopping criterion for the smoothing iterations in the form:

$$\max \left\{ \eta_{\text{sm,ASN}}^{\nu,j,\bar{k}}, \left\| \mathbf{R}(\mathbf{X}^{\nu,j,\bar{k}}) \right\|_{\Gamma} \right\} < \varepsilon. \quad (7.8)$$

If satisfied, set $\bar{j} := j$ and stop. If not, set $j := j + 1$, $\mathbf{X}^{\nu,j,0} := \mathbf{X}^{\nu,j-1,\bar{k}}$, and $\mu^{j\nu} := \alpha \mu^{(j-1)\nu}$. Then set $k := 1$ and go to 2.2.1.

If $\nu = N_t$, stop. If not, set $\nu := \nu + 1$, $j = 1$, $\mathbf{X}^{\nu,j,0} := \mathbf{X}^{\nu-1,\bar{j}}$, and $t_\nu := \tau_\nu + t_{\nu-1}$. Then set $\mu^{j\nu} = \mu^{\bar{j}\nu-1}$, $k = 1$, and go to 2.2.1.

525 **Description of Algorithm 3** For the first time step $\nu = 1$, starting with an initial approximation $\mathbf{X}^{\nu,0} \in \mathbb{R}^n$ and
 526 an initial smoothing parameter $\mu^{\nu,1} > 0$, we solve the smoothed nonlinear system (7.5) by the Newton linearization
 527 solver, and decrease the smoothing parameter $\mu^{\nu,j}$ at each smoothing step j , until the stopping criterion (7.8) on
 528 the smoothing estimator or the relative norm of the total residual vector is satisfied at step \bar{j} . Then, we continue
 529 the time loop, for $2 \leq \nu \leq N_t$, starting for $j = 1$ with $\mathbf{X}^{\nu,j,0} := \mathbf{X}^{\nu-1,\bar{j}}$ and $\mu^{j\nu} := \mu^{\bar{j}\nu-1}$, until satisfying the
 530 stopping criterion (7.8).

531 7.4 Numerical results

532 We consider a homogeneous porous medium in one dimension, supposed to be horizontal with length 2m, and a
 533 uniform spatial mesh with $N = 1000$ elements. The final time of simulation is $t_F = 100s$, and the time step is
 534 constant $\tau_\nu = 10s$. We assume that the medium is initially saturated with liquid, $S^l = 1$, and containing no
 535 hydrogen, $\chi_h^l = 0$, on which we impose an injection of gas (hydrogen), constant in time, in the first cell of the mesh.
 536 The initial conditions are $S^{l,\nu=0} = 1$, $P^{l,\nu=0} = 10^6 \text{Pa}$, and $\chi_h^{l,\nu=0} = 0$.

537 **Semismooth Newton method.** We begin by employing the semismooth Newton method presented in Section
 538 2, with the min function (2.1) to solve the nonlinear system (7.2). On each time step $\nu \geq 1$, we request the relative
 539 norm of the total residual vector $\mathbf{R}(\mathbf{X}^{\nu,k})$ given by (7.3) to drop below 10^{-4} .

540 In Figure 13, the evolution of $\|\mathbf{R}(\mathbf{X}^{\nu,k})\|_r$ is shown at each time step. 31 cumulated Newton iterations are
 541 needed.

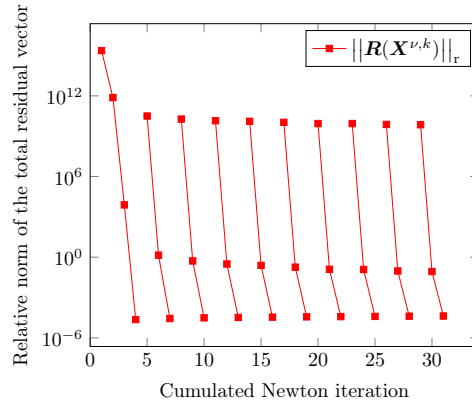


Figure 13: [Semismooth Newton method, min function (2.1)] Relative norm of the total residual vector (7.3) as a function of cumulated Newton iterations along the time steps ν .

542 **Adaptive smoothing Newton method.** Next, we present the results obtained using the adaptive smoothing
 543 Newton method, summarized in Algorithm 3, with the smoothed min function (3.1) to solve the smoothed nonlinear
 544 problem (7.4) at each time step τ_ν , $1 \leq \nu \leq N_t$. The parameters are set as $\mu^{j_1} = 10^{-1}$, $\varepsilon = 10^{-4}$, $\alpha_{\text{lin}} = 1$, and
 545 $\alpha = 0.1$.

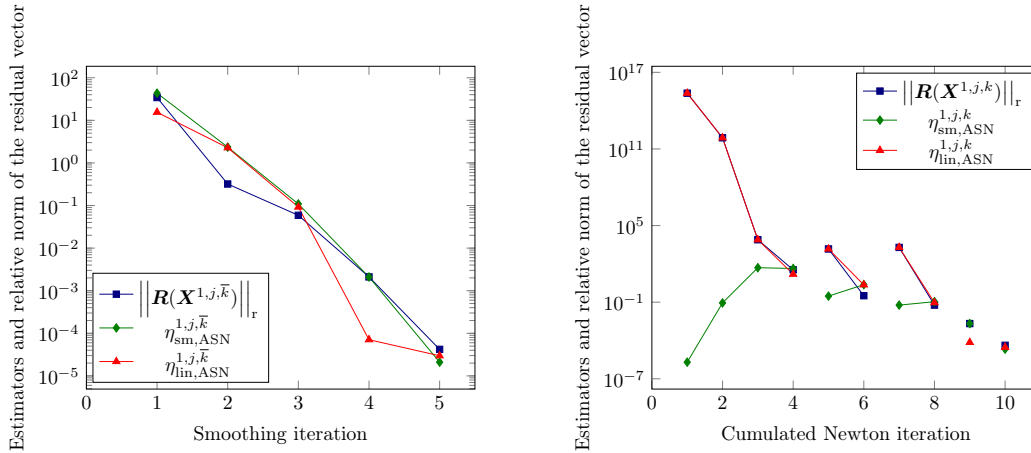


Figure 14: [Adaptive smoothing Newton method, smoothed min function (3.1), Algorithm 3] Estimators (6.3) and relative norm of the total residual vector (7.3) at the first time step $\nu = 1$ as a function of smoothing iterations j , at convergence of the linearization solver ($\nu = 1$ fixed, j varies, $k = \bar{k}$), left, and of cumulated Newton iterations, right, ($\nu = 1$ fixed, j and k vary).

ν	$\mu^{j\nu}$	Niter	$\eta_{\text{lin,ASN}}^{\nu,j,\bar{k}}$	$\eta_{\text{sm,ASN}}^{\nu,j,\bar{k}}$	$\ \mathbf{R}(\mathbf{X}^{\nu,j,\bar{k}})\ _{\text{r}}$
2	1e-05	3	2.15e-07	3.13e-07	2.86e-07
3	1e-05	3	3.13e-07	3.68e-07	4.24e-07
4	1e-05	3	3.93e-07	1.19e-07	3.47e-07
5	1e-05	3	4.62e-07	1.59e-07	4.01e-07
6	1e-05	3	5.04e-07	1.88e-06	1.87e-06
7	1e-05	3	5.58e-07	1.74e-07	4.94e-07
8	1e-05	3	5.94e-07	3.76e-07	7.08e-07
9	1e-05	3	6.64e-07	2.77e-07	7.50e-07
10	1e-05	3	7.01e-07	3.00e-07	7.89e-07

Table 5: [Adaptive smoothing Newton method, smoothed min function (3.1), Algorithm 3] Relative norm of the total residual vector (7.3) and estimators (6.3) at each time step ν , at convergence of the linearization solver.

From Figure 14, one can see that at the first time step $\nu = 1$ and at each smoothing step $j \leq 4$, the linearization estimator decreases until lying below the smoothing estimator. The smoothing iterations are thus stopped in the first possibility according to the stopping criterion (7.7). On the other hand, at the 5th smoothing step, $\eta_{\text{lin,ASN}}^{\nu,j,\bar{k}}$ is smaller than the fixed tolerance but not smaller than $\eta_{\text{sm,ASN}}^{\nu,j,\bar{k}}$. Even after additional Newton iterations at this smoothing step, we will have the same observation. This justifies the modification applied in the adaptive stopping criterion (7.7). In Figure 14, right, we report the estimators and $\|\mathbf{R}(\mathbf{X}^{1,j,\bar{k}})\|_{\text{r}}$ as a function of cumulated Newton iteration for $\nu = 1$. The stopping criterion (7.8) is satisfied after 5 cumulated smoothing iterations, and 10 cumulated Newton iterations. Then, as presented in Table 5, starting at the second time step ($\nu = 2$) with $\mu^{j\nu} = 10^{-5}$, the smoothing parameter does not decrease since the stopping criterion (7.8) is satisfied at each time step after one smoothing step only. To reach the end of the simulation, 9 cumulated smoothing steps and 31 cumulated linearization steps are needed.

As a conclusion, the results confirm the expected behavior of Algorithm 3 featuring an adaptive stopping criterion for the nonlinear solver. In this case, though, the stopping criteria in the adaptive smoothing Newton method do not bring the number of iterations down since the semismooth Newton method already behaves very well here.

8 Conclusion and outlook

In this work, we have considered nonlinear algebraic systems with inequalities in a form of complementarity constraints. We have considered some existing methods, like the semismooth Newton method, possibly combined with a path-following strategy, or a nonparametric interior-point method. Our goal was to propose a systematic way to drive such methods with adaptive stopping criteria and possibly inexact algebraic solvers. We have achieved this by a reformulation of the complementarity constraints using a smoothed function and a posteriori error estimate that enables to distinguish the different error components. Numerical experiments confirmed that the proposed adaptive approaches yield significant computational savings compared to some standard approaches from literature. Moreover, their numerical performance seems to be notably good across a range of test problems. In [8], we also take into account the discretization error of the considered problem, enabling to adaptively stop the outer smoothing loop in Algorithm 1, and employ the method to solve more involved problems.

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