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► **To cite this version:**

Mario Bravetti. Axiomatizing Maximal Progress and Discrete Time. Logical Methods in Computer Science, 2021, 10.23638/LMCS-17(1:1)2021 . hal-03340630

**HAL Id: hal-03340630**

**<https://inria.hal.science/hal-03340630>**

Submitted on 10 Sep 2021

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## AXIOMATIZING MAXIMAL PROGRESS AND DISCRETE TIME

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**ABSTRACT.** Milner’s complete proof system for observational congruence is crucially based on the possibility to equate  $\tau$  divergent expressions to non-divergent ones by means of the axiom  $recX.(\tau.X + E) = recX.\tau.E$ . In the presence of a notion of priority, where, e.g., actions of type  $\delta$  have a lower priority than silent  $\tau$  actions, this axiom is no longer sound. Such a form of priority is, however, common in timed process algebra, where, due to the interpretation of  $\delta$  as a time delay, it naturally arises from the maximal progress assumption. We here present our solution, based on introducing an auxiliary operator  $pri(E)$  defining a “priority scope”, to the long time open problem of axiomatizing priority using standard observational congruence: we provide a complete axiomatization for a basic process algebra with priority and (unguarded) recursion. We also show that, when the setting is extended by considering static operators of a discrete time calculus, an axiomatization that is complete over (a characterization of) finite-state terms can be developed by re-using techniques devised in the context of a cooperation with Prof. Jos Baeten.

### 1. INTRODUCTION

The necessity of extending the expressiveness of classical process algebras, so to make them suitable for the specification and analysis of real case studies, led to the definition of several timed calculi (see, e.g., [34, 31, 21, 25, 22, 23, 7, 12, 13, 11, 8, 6, 3, 9, 10]). Even if expressing different “kind” of times, e.g. discrete time or stochastic time, often such extensions led to the necessity of introducing a notion of priority among actions. A quite common technical design choice is to exploit priority for enacting the so-called *maximal progress assumption* [34, 31, 22]: the possibility of executing internal transitions prevents the execution of timed transitions, thus expressing that the system cannot wait if it has something internal to do.

Technical problems arising from expressing maximal progress were studied in the context of prioritized process algebras, see, e.g., [19, 30, 17] and [18] for a survey. One of the open questions in this context (see [18]) was finding a complete axiomatization for observational congruence in the presence of (unguarded) recursion.

In this paper we present our solution to such a problem. We proceed as follows. First, in Section 2, we do it in the context of a *basic calculus*, presenting the results in [14] and their full technical machinery. In particular, we consider the algebra of finite-state agents

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Research partly supported by the H2020-MSCA-RISE project ID 778233 “Behavioural Application Program Interfaces (BEHAPI)”.

(made up of choice, prefix and recursion only), used by Milner in [29] for axiomatizing observational congruence in presence of recursion, and we extend it with  $\delta$  prefixing, where  $\delta$  actions have lower *priority* than internal  $\tau$  actions. Such a calculus can be interpreted in this way:  $\delta$  actions represent “generic” time delays, standard actions are executed in zero time, and the priority of  $\tau$  actions over  $\delta$  actions derives from the maximal progress assumption. As in [19, 30, 17, 18, 24] we assume that visible actions never have pre-emptive power over lower priority  $\delta$  actions, because we see visible actions as indicating only the potential for execution. The presence of such a priority mechanism makes the standard Milner’s complete proof system for observational congruence (with  $\delta$  actions being treated as visible actions) no longer sound. In particular this happens for the fundamental *Ung* axiom  $recX.(\tau.X + E) = recX.\tau.E$  (a  $\delta$  action performable by  $E$  is pre-empted in the left-hand term but not in the right-hand term), which makes it possible to equate  $\tau$  divergent expressions to non-divergent ones, so to remove unguarded recursion. Such a problem was previously faced in [24], where an alternative axiom was found in the context of a  $\tau$ -divergent sensitive equivalence that is finer than observational congruence/weak bisimulation. Our solution, which does not require a modification of the equivalence, is based on the idea of introducing an auxiliary operator  $pri(E)$  that cuts behaviours in  $E$  starting with an unprioritized  $\delta$  action: it allows us to express the result of applying priority within a certain *scope*. By suitably modifying the axiom above and by introducing some new axioms, our technique provides a complete axiomatization for Milner’s standard observational congruence over a basic calculus with this simple kind of priority and (unguarded) recursion.

Then, in Section 3, we consider a full timed calculus, including also static operators like parallel composition. In particular, we interpret unprioritized  $\delta$  actions of the basic calculus as representing time delays in the context of discrete time (see [22] and the references therein), where a  $\delta$  action is called a “tick”. Ticks take a fixed (unspecified) amount of time, which is the same for all processes, and are assumed to synchronize over all system processes.

We first show that, under this “timed” interpretation for  $\delta$  actions, we can extend the basic calculus of Section 2 with *static* operators, like CSP [26] parallel composition and hiding, preserving the congruence property of standard observational congruence. We then consider a full discrete time calculus, which, besides including recursion and the above “timed” CSP parallel composition and hiding operators, is endowed with a timed variant, taken from [22], of the prefix and choice operators: a “timed prefix”  $a^t.P$ , which allows time to pass via  $\delta$  actions, while waiting for the action  $a$  to be performed; and a “timed choice”  $P +^t Q$ , which, similarly, allows time to pass via  $\delta$  actions performed (in synchronization) by  $P$  and  $Q$ , while waiting for the choice to be resolved by a standard action. Considering such a timed variant of the operators of the basic calculus is convenient from a modeling viewpoint and leads to *time-determinism*: any process of the discrete time calculus can perform at most one  $\delta$  transition. Such a property is natural in discrete time (see [22]) in that the mere passage of time is expected to modify system behavior in a deterministic way.

Even if we consider the discrete time calculus as a “specification level calculus” that does not include the standard prefix and choice operators of the basic calculus of Section 2, the latter, also due to presence of recursion, plays a fundamental role in its axiomatization. The idea is that, since all the operators of the full discrete time calculus have a sort of static behaviour (even  $a^t.P$  and  $P +^t Q$ ), we can use terms of the basic calculus of Section 2 (which essentially can be seen as a process algebraic representation of finite-state labeled transition systems) to express “normal forms” of processes. The complete axiomatization, developed

in Section 2, enforced over normal forms, would then yield completeness for finite-state processes of the discrete time calculus.

As a matter of fact, the introduction of the “timed choice”  $P +^t Q$  operator, makes standard observational congruence, with  $\delta$  actions being treated as standard visible actions, no longer a congruence. Similarly as in [16] we need to extend the root condition of the equivalence (the first step, leading to weak bisimulation), so that the (matched) execution of a  $\delta$  action does not lead to leaving the root condition, i.e. to weak bisimulation. So the modified notion of observational congruence still coincides with the standard one as far as standard actions are concerned (it is a conservative extension of it). Moreover, for any action, it still leads to standard weak bisimulation once the (strengthened) root condition is left. As a consequence, we have to slightly modify the axiomatization of the basic calculus considered in Section 2, in order to account for the modified way in which the discrete time observational congruence deals with  $\delta$  actions: this just causes an additional  $\delta$ -specific axiom to be added to standard *Tau* axioms and does not affect the *Ung* axioms (the ones we dealt with in Section 2 due to priority of  $\tau$  over  $\delta$ ).

Apart from axiomatizing the basic calculus, one of the technical difficulties concerning the (inductive) procedure of turning discrete time calculus processes into normal form is dealing with unguardedness generated by static operators, i.e., in our case by the hiding operator. During a two months visit of Prof. Jos Baeten (who was on sabbatical leave) at University of Bologna, a procedure of this kind was devised in the context of a standard (untimed) process algebra, based on the introduction of a specific axiom. Such a procedure, that is here reapplied in a much more complex setting, is contained in [4, 5]. The completeness result of the axiomatization for the discrete time calculus is then obtained by adopting a syntactical characterization that guarantees finite-stateness, which is similar, apart from treatment of the  $P +^t Q$  operator, to that of [15, 4, 5]. Notice that such a characterization includes the possibility of expressing unguarded recursion, which is dealt with by resorting to the basic calculus axiomatization in Section 2.

Finally, in Section 4 we discuss related work and in Section 5 we report remarks about future work.

Section 2 is an extended and revised version of [14] that presents, for the first time, complete proofs. Presenting such a technical machinery in its entirety is needed for Section 3 that builds its results and proofs on it.

## 2. AXIOMATIZING PRIORITIZED FINITE STATE BEHAVIORS

In this section we present the complete axiomatization of observational congruence for the basic calculus: the algebra of finite-state agents used by Milner in [29] extended with generic  $\delta$  prefixing, where  $\delta$  actions have lower priority than internal  $\tau$  actions.

**2.1. A Basic Calculus.** Prioritized observable actions are denoted by  $a, b, c, \dots$ . The denumerable set of all prioritized actions, which includes the silent action  $\tau$  denoting an internal move, is denoted by  $PAct$ , ranged over by  $\alpha, \alpha', \dots$ . The set of all actions is defined by  $Act = PAct \cup \{\delta\}$ , ranged over by  $\gamma, \gamma', \dots$ . The denumerable set of term variables is  $Var$ , ranged over by  $X, Y, \dots$ . The set  $\mathcal{E}$  of behavior expressions, ranged over by  $E, F, G, \dots$  is defined by the following syntax.

$$E ::= \underline{0} \mid X \mid \gamma.E \mid E + E \mid \text{rec}X.E$$

|  |   |
|--|---|
| $\gamma.P \xrightarrow{\gamma} P$  |   |
| $\frac{P \xrightarrow{\alpha} P'}{P + Q \xrightarrow{\alpha} P'}$              | $\frac{Q \xrightarrow{\alpha} Q'}{P + Q \xrightarrow{\alpha} Q'}$ |
| $\frac{E\{recX.E/X\} \xrightarrow{\gamma} P'}{recX.E \xrightarrow{\gamma} P'}$ |   |

Table 1: Standard Rules

The meaning of the operators is the standard one of [28, 29], where  $recX$  denotes recursion in the usual way. We adopt the standard definitions of [28, 29] for *free* variable, and *open* and *closed* term. The set of processes, i.e. closed terms, is denoted by  $\mathcal{P}$ , ranged over by  $P, Q, R, \dots$ .

As in [29] we take the liberty of identifying expressions which differ only by a change of bound variables (hence we do not need to deal with  $\alpha$ -conversion explicitly). We will write  $E\{F/X\}$  for the result of syntactically substituting  $F$  for each free occurrence of  $X$  in  $E$ , renaming bound variables as necessary. This is also generalized to sets of variables  $\tilde{X} = \{X_1, \dots, X_n\}$ :  $E\{\tilde{F}/\tilde{X}\}$ , where  $\tilde{F} = \{F_1, \dots, F_n\}$  stands for the result of substituting  $F_i$  for each free occurrence of  $X_i$  in  $E$  for all  $i \in \{1, \dots, n\}$ , renaming bound variables as necessary.

We adopt the following standard (see [29]) definitions concerning guardedness of variables.

**Definition 2.1.** A free occurrence of  $X$  in  $E$  is *weakly guarded* if it occurs within some subexpression of  $E$  of the form  $\gamma.F$ . It is *(strongly) guarded* if we additionally have that  $\gamma \neq \tau$ . It is *unguarded* if it is not (strongly) guarded. It is *fully unguarded* if it is not weakly guarded.

We say that  $X$  is weakly guarded/(strongly) guarded in  $E$  if each free occurrence of  $X$  in  $E$  is weakly guarded/(strongly) guarded, respectively. Correspondingly, we say that  $X$  is unguarded/fully unguarded in  $E$  if some free occurrence of  $X$  in  $E$  is unguarded/fully unguarded, respectively. Moreover we say that a recursion  $recX.E$  is weakly guarded/(strongly) guarded/unguarded/ fully unguarded, if  $X$  is weakly guarded/(strongly) guarded/unguarded/fully unguarded in  $E$ , respectively. Finally, we say that an expression  $E$  is weakly guarded/(strongly) guarded if every subexpression of  $E$  which is a recursion is weakly guarded/(strongly) guarded, respectively. Correspondingly, we say that an expression  $E$  is unguarded/fully unguarded if some subexpression of  $E$  which is a recursion is unguarded/fully unguarded, respectively.

The operational semantics of the algebra terms is given as a relation  $\longrightarrow \subseteq \mathcal{P} \times Act \times \mathcal{P}$ .

We write  $P \xrightarrow{\gamma} Q$  for  $(P, \gamma, Q) \in \longrightarrow$ ,  $P \xrightarrow{\exists} Q$  for  $\exists Q : (P, \gamma, Q) \in \longrightarrow$  and  $P \xrightarrow{\exists} Q$  for  $\exists Q : (P, \gamma, Q) \in \longrightarrow$ .  $\longrightarrow$  is defined as the least relation satisfying the operational rules in Tables 1 and 2. Notice that, even if the rules in Table 2 include a negative premise, the operational semantics is well-defined in that the inference of transitions can be stratified (see, e.g., [20]).

$$\boxed{\begin{array}{c} \frac{P \xrightarrow{\delta} P' \quad Q \not\xrightarrow{\tau}}{P + Q \xrightarrow{\delta} P'} \quad \frac{Q \xrightarrow{\delta} Q' \quad P \not\xrightarrow{\tau}}{P + Q \xrightarrow{\delta} Q'} \end{array}}$$

Table 2: Special Rules Expressing Priority.

As in [30] we capture the priority of  $\tau$  actions over  $\delta$  actions by cutting transitions which cannot be performed directly in semantic models (and not by discarding them at the level of bisimulation definition as done in [24]) so that we can just apply the ordinary notion of observational congruence [28].

**Proposition 2.2** (maximal progress). *Relation  $\longrightarrow$  is such that:*

$$P \xrightarrow{\tau} \text{ implies } P \not\xrightarrow{\delta}$$

*Proof.* Easily shown by induction on the height of the inference tree of  $\tau$  transitions of terms  $P \in \mathcal{P}$  (the base case being a  $\tau$  prefix).  $\square$

**2.2. Weak Bisimulation Equivalence.** The equivalence notion we consider over the terms of our prioritized process algebra is the standard notion of observational congruence extended to open terms [28, 29].

As in [29] we use  $\xrightarrow{\gamma^+}$  to denote computations composed of all  $\xrightarrow{\gamma}$  transitions whose length is at least one and  $\xrightarrow{\gamma^*}$  to denote computations composed of all  $\xrightarrow{\gamma}$  transitions whose length is possibly zero. Let  $\xrightarrow{\hat{\gamma}}$  denote  $\xrightarrow{\tau^*} \xrightarrow{\gamma} \xrightarrow{\tau^*}$ . Moreover we define  $\xrightarrow{\hat{\gamma}} = \xrightarrow{\gamma}$  if  $\gamma \neq \tau$  and  $\xrightarrow{\hat{\tau}} = \xrightarrow{\tau^*}$ .

**Definition 2.3.** A relation  $\beta \subseteq \mathcal{B} \times \mathcal{B}$  is a weak bisimulation if, whenever  $(P, Q) \in \beta$ :

- If  $P \xrightarrow{\gamma} P'$  then, for some  $Q', Q \xrightarrow{\hat{\gamma}} Q'$  and  $(P', Q') \in \beta$ .
- If  $Q \xrightarrow{\gamma} Q'$  then, for some  $P', P \xrightarrow{\hat{\gamma}} P'$  and  $(P', Q') \in \beta$ .

Two processes  $P, Q$  are weakly bisimilar, written  $P \approx Q$ , iff  $(P, Q)$  is included in some weak bisimulation.

**Definition 2.4.** Two processes  $P, Q$  are observationally congruent, written  $P \simeq Q$ , iff:

- If  $P \xrightarrow{\gamma} P'$  then, for some  $Q', Q \xrightarrow{\hat{\gamma}} Q'$  and  $P' \approx Q'$ .
- If  $Q \xrightarrow{\gamma} Q'$  then, for some  $P', P \xrightarrow{\hat{\gamma}} P'$  and  $P' \approx Q'$ .

Weak bisimulation  $\approx$  and observational congruence  $\simeq$  are indeed equivalence relations [28].

Open terms are dealt with as follows [28].

**Definition 2.5.** Two open terms  $E, F$  are observationally congruent, written  $E \simeq F$ , if, assumed the set of variables  $\tilde{X}$  to include all free variables occurring in  $E$  and  $F$ , the following holds. For all sets  $\tilde{P}$  of closed terms we have  $E\{\tilde{P}/\tilde{X}\} \simeq F\{\tilde{P}/\tilde{X}\}$ .

**Corollary 2.6.** *If  $P \simeq Q$  then:*

$$P \xrightarrow{\tau} \quad \Leftrightarrow \quad Q \xrightarrow{\tau}$$

The following theorem shows that the presence of priority preserves the congruence property of observational congruence w.r.t. the operators of the algebra.

**Theorem 2.7.**  *$\simeq$  is a congruence w.r.t. prefix, choice and recursion operators.*

*Proof.* As far as the prefix operator is concerned, from  $P \simeq Q$  we immediately derive  $\gamma.P \simeq \gamma.Q$

Concerning the choice operator, from  $P \simeq Q$  we derive  $P + R \simeq Q + R$  as follows. Suppose  $P + R \xrightarrow{\gamma} P'$ , we have two cases.

- If  $P \xrightarrow{\gamma} P'$  (and  $R \not\xrightarrow{\tau}$ ) then (from  $P \simeq Q$ )  $Q \xrightarrow{\gamma} Q'$  for some  $P' \approx Q'$ . Hence we have  $Q + R \xrightarrow{\gamma} Q'$ . In particular if  $\gamma = \delta$  this derives from the fact that  $R \not\xrightarrow{\tau}$ .
- If  $R \xrightarrow{\gamma} P'$  (and  $P \not\xrightarrow{\tau}$ ) then  $Q + R \xrightarrow{\gamma} P'$ . In particular if  $\gamma = \delta$  this derives from the following consideration. From  $P \not\xrightarrow{\tau}$  we have also that  $Q \not\xrightarrow{\tau}$  (by Corollary 2.6).

Concerning the recursion operator, from  $E \simeq F$  we derive  $\text{rec}X.E \simeq \text{rec}X.F$  as follows. We show that

$$\beta = \{(G\{\text{rec}X.E/X\}, G\{\text{rec}X.F/X\}) \mid G \text{ contains at most } X \text{ free}\}$$

satisfies the condition:

$$\begin{aligned} &\text{if } G\{\text{rec}X.E/X\} \xrightarrow{\gamma} H \text{ then, for some } H', H'', \\ &\quad G\{\text{rec}X.F/X\} \xrightarrow{\gamma} H'' \text{ with } H'' \approx H' \text{ such that } (H, H') \in \beta, \\ &\text{and symmetrically for a move of } G\{\text{rec}X.F/X\}. \end{aligned}$$

This implies that  $\beta$  is a weak bisimulation up to  $\approx$ , see [33]. Moreover by taking  $G \equiv X$  we may conclude that  $\text{rec}X.E \simeq \text{rec}X.F$ .

In the following we begin the proof that  $(G\{\text{rec}X.E/X\}, G\{\text{rec}X.F/X\}) \in \beta$  satisfies the condition above by inducing on the height of the inference tree by which transitions of  $G\{\text{rec}X.E/X\} \xrightarrow{\alpha} H$  are inferred. That is, first we consider standard  $\alpha$  transitions (which are inferred from standard transitions only) and then  $\delta$  transitions.

We have the following cases depending on the structure of  $G$ .

- If  $G \equiv \mathbf{0}$  then the condition above trivially holds.
- If  $G \equiv X$  then  $G\{\text{rec}X.E/X\} \equiv \text{rec}X.E$  and  $G\{\text{rec}X.F/X\} \equiv \text{rec}X.F$ .  
 $\text{rec}X.E \xrightarrow{\alpha} H$  implies  $E\{\text{rec}X.E/X\} \xrightarrow{\alpha} H$ . Hence, by induction, for some  $H', H''$ ,  
 $E\{\text{rec}X.F/X\} \xrightarrow{\alpha} H''$  with  $H'' \approx H'$  such that  $(H, H') \in \beta$ . Since  $E \simeq F$ , we have also  
that  $F\{\text{rec}X.F/X\} \xrightarrow{\alpha} H'''$  with  $H''' \approx H''$ . Therefore  $\text{rec}X.F \xrightarrow{\alpha} H'''$  with  $H''' \approx H'$   
such that  $(H, H') \in \beta$ .
- If  $G \equiv \alpha.G'$  then  $G\{\text{rec}X.E/X\} \equiv \alpha.(G'\{\text{rec}X.E/X\})$  and  $G\{\text{rec}X.F/X\} \equiv \alpha.(G'\{\text{rec}X.F/X\})$ . The result trivially follows.
- If  $G \equiv G' + G''$  then  $G\{\text{rec}X.E/X\} \equiv G'\{\text{rec}X.E/X\} + G''\{\text{rec}X.E/X\}$  and  $G\{\text{rec}X.F/X\} \equiv G'\{\text{rec}X.F/X\} + G''\{\text{rec}X.F/X\}$ .

Suppose  $G'\{\text{rec}X.E/X\} + G''\{\text{rec}X.E/X\} \xrightarrow{\alpha} H$ , we have two cases.

- If  $G'\{\text{rec}X.E/X\} \xrightarrow{\alpha} H$  then (by induction)  $G'\{\text{rec}X.F/X\} \xrightarrow{\alpha} H''$  with  $H'' \approx H'$   
and  $(H, H') \in \beta$ . Therefore  $G'\{\text{rec}X.F/X\} + G''\{\text{rec}X.F/X\} \xrightarrow{\alpha} H''$  with  $H'' \approx H'$   
and  $(H, H') \in \beta$ .

- If  $G''\{recX.E/X\} \xrightarrow{\alpha} H$  then the result is derived in a similar way.
- If  $G \equiv recY.G'$ , with  $Y \neq X$ , then  $G\{recX.E/X\} \equiv recY.G'\{recX.E/X\}$  and  $G\{recX.F/X\} \equiv recY.G'\{recX.F/X\}$ .  
 $recY.G'\{recX.E/X\} \xrightarrow{\alpha} H$  implies  $(G'\{recX.E/X\})\{recY.G'\{recX.E/X\}/Y\} \equiv$   
 $(G'\{recY.G'/Y\})\{recX.E/X\} \xrightarrow{\alpha} H$ . By induction we have  $(G'\{recY.G'/Y\})\{recX.F/X\} \equiv$   
 $(G'\{recX.F/X\})\{recY.G'\{recX.F/X\}/Y\} \xrightarrow{\alpha} H''$  with  $H'' \approx H'$  and  $(H, H') \in \beta$ .  
Therefore  $recY.G'\{recX.F/X\} \xrightarrow{\alpha} H''$  with  $H'' \approx H'$  and  $(H, H') \in \beta$ .

A completely symmetric inductive proof is performed when we start from a transition of  $G\{recX.F/X\} \xrightarrow{\alpha} H$  in the condition above.

An identical inductive proof is performed when we start from a transition of  $G\{recX.E/X\} \xrightarrow{\delta} H$  in the condition above (we just have to consider  $\delta$  instead of a generic  $\alpha$ ), apart when  $G \equiv G' + G''$ . In this case, to derive the final statement  $G'\{recX.F/X\} + G''\{recX.F/X\} \xrightarrow{\delta} H''$  with  $H'' \approx H'$  and  $(H, H') \in \beta$  we have to additionally prove that the  $\delta$  transition is not pre-empted. In the first subcase this is obtained as follows. It must be that  $G''\{recX.E/X\} \not\xrightarrow{\tau}$ , hence, since we already showed the condition for relation  $\beta$  presented above (and its symmetric one) to hold for any  $\alpha$  transition, we have also that  $G''\{recX.F/X\} \not\xrightarrow{\tau}$ . The other subcase is analogous. Finally, a completely symmetric inductive proof is performed when we start from a transition of  $G\{recX.F/X\} \xrightarrow{\delta} H$  in the condition above.  $\square$

In the following we show that, by exploiting the priority constraint, we can rewrite weak bisimulation definition into a less generic form. In particular we present two reformulations of weak bisimulation (the second one exploiting guardedness) and a reformulation of observational congruence.

**Proposition 2.8.** *A relation  $\beta \subseteq \mathcal{P} \times \mathcal{P}$  is a weak bisimulation iff, whenever  $(P, Q) \in \beta$ :*

- If  $P \xrightarrow{\alpha} P'$  then, for some  $Q', Q \xrightarrow{\hat{\alpha}} Q'$  and  $(P', Q') \in \beta$ .
- There exists  $Q''$  such that  $Q \xrightarrow{\tau}^* Q''$  and:
  - if  $P \xrightarrow{\delta} P'$  then, for some  $Q', Q'' \xrightarrow{\delta} \xrightarrow{\tau}^* Q'$  and  $(P', Q') \in \beta$ .
- If  $Q \xrightarrow{\alpha} Q'$  then, for some  $P', P \xrightarrow{\hat{\alpha}} P'$  and  $(P', Q') \in \beta$ .
- There exists  $P''$  such that  $P \xrightarrow{\tau}^* P''$  and:
  - if  $Q \xrightarrow{\delta} Q'$  then, for some  $P', P'' \xrightarrow{\delta} \xrightarrow{\tau}^* P'$  and  $(P', Q') \in \beta$ .

*Proof.* If  $\beta$  satisfies such a stricter condition then obviously  $\beta$  is a weak bisimulation. Conversely, suppose that  $\beta$  is a weak bisimulation; we now prove that it satisfies the condition above. If  $P \xrightarrow{\delta} P'$  then it must be that  $Q \xrightarrow{\tau}^* Q'' \xrightarrow{\delta} \xrightarrow{\tau}^*$  for some  $Q''$ . We now prove that such a  $Q''$  satisfies the related constraint in the condition above. Since  $Q \xrightarrow{\tau}^* Q''$ , it must be  $(P, Q'') \in \beta$ : because of the priority constraints, since  $P \xrightarrow{\delta}$ , the sequence of  $\tau$  transitions of  $Q$  can only be matched by zero length moves of  $P$ . From  $(P, Q'') \in \beta$  the constraint above is directly derived by observing that, since  $Q'' \xrightarrow{\delta}$ ,  $Q''$  cannot perform  $\tau$  transitions. Symmetrically for  $Q \xrightarrow{\delta}$ .  $\square$

**Proposition 2.9.** *If a relation  $\beta \subseteq \mathcal{P} \times \mathcal{P}$  is a weak bisimulation then, whenever  $(P, Q) \in \beta$ :*

- If  $P \xrightarrow{\alpha} P'$  then, for some  $Q', Q \xrightarrow{\hat{\alpha}} Q'$  and  $(P', Q') \in \beta$ .



- If  $P \xrightarrow{\delta} P'$  then, either  $Q \xrightarrow{\tau}$ , or, for some  $Q'$ ,  $Q \xrightarrow{\delta} \xrightarrow{\tau}^* Q'$  and  $(P', Q') \in \beta$ .
- If  $Q \xrightarrow{\alpha} Q'$  then, for some  $P'$ ,  $P \xrightarrow{\hat{\alpha}} P'$  and  $(P', Q') \in \beta$ .
- If  $Q \xrightarrow{\delta} Q'$  then, either  $P \xrightarrow{\tau}$ , or, for some  $P'$ ,  $P \xrightarrow{\delta} \xrightarrow{\tau}^* P'$  and  $(P', Q') \in \beta$ .

If, in addition,  $\beta$  is a relation over guarded processes of  $\mathcal{P}$ , then  $\beta$  is a weak bisimulation iff it satisfies the condition above.

*Proof.* We start by proving the first statement of the proposition. Suppose  $P \xrightarrow{\delta} P'$ , it is just sufficient to observe that: if  $Q \not\xrightarrow{\tau}$  then it must be  $Q'' \equiv Q$  in the condition of Proposition 2.8. Symmetrically for  $Q \xrightarrow{\delta} Q'$ .

Concerning the second statement, if  $\beta$  is a weak bisimulation then the condition above is satisfied as for the first statement. Conversely, suppose that  $\beta$  satisfies the condition above; we now prove that it satisfies the condition of Proposition 2.8. If  $P \xrightarrow{\delta}$  then either  $Q \not\xrightarrow{\tau}$  and we are done (in this case  $Q'' \equiv Q$  in the condition of Proposition 2.8 hence the two conditions coincide), or  $Q \xrightarrow{\tau} Q'$  for some  $Q'$ . In the latter case it must be  $(P, Q') \in \beta$  (because of the priority constraints,  $P$  can only match the  $\tau$  move of  $Q'$  by a zero length move). Then the same argument is repeated for the pair  $(P, Q')$  and so on... until a  $\xrightarrow{\tau}^* Q^n$  derivative of  $Q$  is reached such that  $Q^n \not\xrightarrow{\tau}$  and we are done. The guardedness constraint guarantees that such a derivative is always reached: the labeled transition system generated by a guarded process  $\mathcal{P}$  is such that every  $\tau$  path eventually reaches a state with no outgoing  $\tau$  transitions (since it is finite-state and no  $\tau$  loops can be generated with just strongly guarded recursion). Symmetrically for  $Q \xrightarrow{\delta}$ .  $\square$

A simple counterexample that shows why the guardedness constraint for processes is needed to obtain the “iff” in Proposition 2.9 is the following:  $\beta = \{recX.\tau.X, \delta.0\}$  satisfies the condition of the theorem but obviously it is not a weak bisimulation.

**Proposition 2.10.** *Two processes  $P, Q$  are observationally congruent ( $P \simeq Q$ ), iff:*

- If  $P \xrightarrow{\alpha} P'$  then, for some  $Q'$ ,  $Q \xrightarrow{\alpha} Q'$  and  $P' \approx Q'$ .
- If  $P \xrightarrow{\delta} P'$  then, for some  $Q'$ ,  $Q \xrightarrow{\delta} \xrightarrow{\tau}^* Q'$  and  $P' \approx Q'$ .
- If  $Q \xrightarrow{\alpha} Q'$  then, for some  $P'$ ,  $P \xrightarrow{\alpha} P'$  and  $P' \approx Q'$ .
- If  $Q \xrightarrow{\delta} Q'$  then, for some  $P'$ ,  $P \xrightarrow{\delta} \xrightarrow{\tau}^* P'$  and  $P' \approx Q'$ .

*Proof.* If  $P, Q$  satisfy such a stricter condition then obviously  $P \simeq Q$ . Conversely, suppose that  $P \simeq Q$ ; we now prove that they satisfy the condition above. Suppose  $P \xrightarrow{\delta} P'$ , due to the priority constraints, we must have that  $Q \not\xrightarrow{\tau}$  (otherwise, by contradiction, being  $P \simeq Q$ , also  $P$  should perform a  $\tau$  move), therefore the condition above directly derives from the definition of observational congruence. Symmetrically for  $Q \xrightarrow{\delta} Q'$ .  $\square$

**2.3. Axiomatization.** We now present an axiomatization of  $\simeq$  which is complete over processes  $P \in \mathcal{P}$  of our algebra.

As we already explained in the introduction, the law of ordinary CCS which makes it possible escape  $\tau$  divergence:

$$recX.(\tau.X + E) = recX.\tau.E$$

$$\boxed{\frac{P \xrightarrow{\alpha} P'}{pri(P) \xrightarrow{\alpha} P'}}$$

Table 3: Rule for Auxiliary Pri Operator.

is not sound in a calculus with this kind of priority. Consider for instance the divergent term  $F \equiv recX.(\tau.X + \delta.\underline{0})$ . Because of priority of “ $\tau$ ” actions over “ $\delta$ ” actions the operational semantics of  $F$  is isomorphic to that of  $recX.\tau.X$ . Hence  $F$  is an infinitely looping term which can never escape from divergence by executing the action “ $\delta$ ”. If the cited law were sound, we would obtain  $F = recX.\tau.\delta.\underline{0}$  and this is certainly not the case.

In general the behavior of  $E'$  such that  $recX.(\tau.X + E) = recX.\tau.E'$  is obtained from that of  $E$  by removing all “ $\delta$ ” actions (and subsequent behaviors) performable in  $E$ . We denote such  $E'$ , representing the “prioritized” behavior of  $E$ , with  $pri(E)$ . The operational semantics of the auxiliary operator  $pri$  is simply that in Table 3. As we discussed in the introduction, this auxiliary operator is crucial for being able to axiomatize the priority of  $\tau$  actions over  $\delta$  actions in that it allows us to represent a *scope* to which priority is applied.

The axiomatization of “ $\simeq$ ” we propose is made over the set of terms  $\mathcal{E}_{pri}$ , generated by extending the syntax to include the new operator  $pri(E)$ .

We start by noting that the congruence property of “ $\simeq$ ” trivially extends to the new operator.

**Theorem 2.11.**  $\simeq$  is a congruence w.r.t. the new operator  $pri(E)$ .

*Proof.* Let us suppose  $P \simeq Q$ . If  $pri(P) \xrightarrow{\gamma} P'$  then, according to the semantics of Table 3, it must be  $\gamma \neq \delta$  and  $P \xrightarrow{\gamma} P'$ . Hence  $Q \xrightarrow{\gamma} Q'$  with  $P' \approx Q'$ . Thus, being  $\gamma \neq \delta$ ,  $pri(Q) \xrightarrow{\gamma} Q'$ . Symmetrically for transitions of  $pri(Q)$ .  $\square$

We adopt the following notion of *serial* variable, which is used in the axiomatization.

**Definition 2.12.**  $X$  is *serial* in  $E \in \mathcal{E}_{pri}$  if every subexpression of  $E$  which contains a free occurrence of  $X$ , apart from  $X$  itself, is of the form  $\gamma.F$ ,  $F' + F''$  or  $recY.F$  for some variable  $Y$ .

The axiom system  $\mathcal{A}$  is formed by the axioms presented in Table 4. The axiom (*Pri6*) expresses the priority of  $\tau$  actions over  $\delta$  actions. Note that from (*Pri6*) we can derive  $\tau.E + \delta.E = \tau.E$  by applying (*Pri3*). The axioms (*Rec1*), (*Rec2*) handle strongly guarded recursion in the standard way [29]. The axioms (*Ung1*) and (*Ung2*) are used to turn unguarded terms into strongly guarded ones similarly as in [29]. The axiom (*Ung3*) and the new axiom (*Ung4*) are used to transform weakly guarded recursions into the form required by the axiom (*Ung2*), so that they can be turned into strongly guarded ones. In particular the role of axiom (*Ung4*) is to remove unnecessary occurrences of terms  $pri(X)$  in weakly guarded recursions. In the following, when transforming terms via axiom applications, we will often omit mentioning axioms (*A1*) – (*A4*), which are just used to rearrange arguments and eliminate duplicates and  $\underline{0}$  in sums.

|        |  |
|--------|--|
| (A1)   | $E + F = F + E$  |
| (A2)   | $(E + F) + G = E + (F + G)$  |
| (A3)   | $E + E = E$  |
| (A4)   | $E + \underline{0} = E$  |
| (Tau1) | $\gamma.\tau.E = \gamma.E$   |
| (Tau2) | $E + \tau.E = \tau.E$  |
| (Tau3) | $\gamma.(E + \tau.F) + \gamma.F = \gamma.(E + \tau.F)$   |
| (Pri1) | $\text{pri}(\underline{0}) = \underline{0}$  |
| (Pri2) | $\text{pri}(\alpha.E) = \alpha.E$  |
| (Pri3) | $\text{pri}(\delta.E) = \underline{0}$   |
| (Pri4) | $\text{pri}(E + F) = \text{pri}(E) + \text{pri}(F)$  |
| (Pri5) | $\text{pri}(\text{pri}(E)) = \text{pri}(E)$  |
| (Pri6) | $\tau.E + F = \tau.E + \text{pri}(F)$  |
| (Rec1) | $\text{rec}X.E = E\{\text{rec}X.E/X\}$   |
| (Rec2) | $F = E\{F/X\} \Rightarrow F = \text{rec}X.E$ provided that X is serial and strongly guarded in E |
| (Ung1) | $\text{rec}X.(X + E) = \text{rec}X.E$  |
| (Ung2) | $\text{rec}X.(\tau.X + E) = \text{rec}X.(\tau.\text{pri}(E))$                                    |
| (Ung3) | $\text{rec}X.(\tau.(X + E) + F) = \text{rec}X.(\tau.X + E + F)$                                  |
| (Ung4) | $\text{rec}X.(\tau.(\text{pri}(X) + E) + F) = \text{rec}X.(\tau.X + E + F)$                      |

Table 4: Axiom system  $\mathcal{A}$ 

**2.4. Soundness.** We start by showing that the axiom system  $\mathcal{A}$  is sound.

**Theorem 2.13.** *Given  $E, F \in \mathcal{E}_{\text{pri}}$ , if  $\mathcal{A} \vdash E = F$  then  $E \simeq F$ .*

*Proof.* We prove the soundness of the new laws (Ung2) and (Ung4). For the other laws (Ung1) and (Ung3) the proof is a similar adaptation of the standard one. The soundness of the laws (Rec1) and (Rec2) is shown as in [28] (version corrected according to [33] concerning the structure of the weak bisimulation up to  $\approx$  to be considered) by assuming transition labels to also encompass the  $\delta$  action.

The soundness of the new equations (Pri1) – (Pri6) easily derives from the fact that the semantic models of left-hand and right-hand terms are isomorphic.

**Soundness of (Ung2)** In order to prove the soundness of (Ung2) we show that:

$$\beta = \{ (G\{\text{rec}X.(\tau.X + E)/X\}, G\{\text{rec}X.(\tau.\text{pri}(E))/X\}) \mid G \text{ contains at most } X \text{ free} \} \cup \{ (\text{rec}X.(\tau.X + E), \text{pri}(E)\{\text{rec}X.(\tau.\text{pri}(E))/X\}) \}$$

is a weak bisimulation. From this result it straightforwardly follows that  $\text{rec}X.(\tau.X + E) \simeq \text{rec}X.(\tau.\text{pri}(E))$ .

We start the proof that  $\beta$  is a weak bisimulation by considering all the pairs  $(G\{\text{rec}X.(\tau.X + E)/X\}, G\{\text{rec}X.(\tau.\text{pri}(E))/X\})$  such that  $G$  contains at most  $X$  free. In particular we prove that  $\beta$  satisfies the following stronger condition

$$\text{if } G\{\text{rec}X.(\tau.X + E)/X\} \xrightarrow{\gamma} H \text{ then, for some } H', \\ G\{\text{rec}X.(\tau.\text{pri}(E))/X\} \xrightarrow{\gamma} H' \text{ with } (H, H') \in \beta,$$

and symmetrically for a move of  $G\{recX.(\tau.pri(E))/X\}$

We begin proving it by induction on the depth of the inference by which transitions  $G\{recX.(\tau.X + E)/X\} \xrightarrow{\alpha} H$  are inferred. We have the following cases depending on the structure of  $G$ .

- If  $G \equiv 0$ , then the condition above obviously holds.
- If  $G \equiv X$ , then  $G\{recX.(\tau.X + E)/X\} \equiv recX.(\tau.X + E)$  and  $G\{recX.(\tau.pri(E))/X\} \equiv recX.(\tau.pri(E))$ .  
Suppose  $recX.(\tau.X + E) \xrightarrow{\alpha} H$ , we have two cases:
  - If  $recX.(\tau.X + E) \xrightarrow{\tau} H \equiv recX.(\tau.X + E)$ , then  $recX.(\tau.pri(E)) \xrightarrow{\tau} H' \equiv pri(E)\{recX.(\tau.pri(E))/X\}$  with  $(H, H') \in \beta$ .
  - If  $recX.(\tau.X + E) \xrightarrow{\alpha} H$  with  $E\{recX.(\tau.X + E)/X\} \xrightarrow{\alpha} H$ , then, by induction, we derive that, for some  $H'$ ,  $E\{recX.(\tau.pri(E))/X\} \xrightarrow{\alpha} H'$  with  $(H, H') \in \beta$ . Hence,  $recX.(\tau.pri(E)) \xrightarrow{\tau} \xrightarrow{\alpha} H'$ .
- If  $G \equiv \alpha.G'$ , then  $G\{recX.(\tau.X + E)/X\} \equiv \alpha.(G'\{recX.(\tau.X + E)/X\})$  and  $G\{recX.(\tau.pri(E))/X\} \equiv \alpha.(G'\{recX.(\tau.pri(E))/X\})$ . The result trivially follows.
- If  $G \equiv G' + G''$ , then  $G\{recX.(\tau.X + E)/X\} \equiv G'\{recX.(\tau.X + E)/X\} + G''\{recX.(\tau.X + E)/X\}$  and  $G\{recX.(\tau.pri(E))/X\} \equiv G'\{recX.(\tau.pri(E))/X\} + G''\{recX.(\tau.pri(E))/X\}$ .  
Suppose  $G'\{recX.(\tau.X + E)/X\} + G''\{recX.(\tau.X + E)/X\} \xrightarrow{\alpha} H$ , we have two cases.
  - If  $G'\{recX.(\tau.X + E)/X\} \xrightarrow{\alpha} H$ , then (by induction) for some  $H'$ ,  $G'\{recX.(\tau.pri(E))/X\} \xrightarrow{\alpha} H'$  and  $(H, H') \in \beta$ . Therefore  $G'\{recX.(\tau.pri(E))/X\} + G''\{recX.(\tau.pri(E))/X\} \xrightarrow{\alpha} H'$ .
  - If  $G''\{recX.(\tau.X + E)/X\} \xrightarrow{\alpha} H$ , then the result is derived in a similar way.
- If  $G \equiv recY.G'$  then  $G\{recX.(\tau.X + E)/X\} \equiv recY.G'\{recX.(\tau.X + E)/X\}$  and  $G\{recX.(\tau.pri(E))/X\} \equiv recY.G'\{recX.(\tau.pri(E))/X\}$ .  
 $recY.G'\{recX.(\tau.X + E)/X\} \xrightarrow{\alpha} H$  implies  $(G'\{recX.(\tau.X + E)/X\})\{recY.G'\{recX.(\tau.X + E)/X\}/Y\} \equiv (G'\{recY.G'/Y\})\{recX.(\tau.X + E)/X\} \xrightarrow{\alpha} H$ . By induction we have  $(G'\{recY.G'/Y\})\{recX.(\tau.pri(E))/X\} \equiv (G'\{recX.(\tau.pri(E))/X\})\{recY.G'\{recX.(\tau.pri(E))/X\}/Y\} \xrightarrow{\alpha} H'$  with  $(H, H') \in \beta$ . Therefore  $recY.G'\{recX.(\tau.pri(E))/X\} \xrightarrow{\alpha} H'$  with  $(H, H') \in \beta$ .

A symmetric inductive proof is performed when we start from a transition  $G\{recX.(\tau.X + E)/X\} \xrightarrow{\alpha} H$  in the condition above. The only exception to symmetry is the case of  $G \equiv X$ . The proof of this case follows.

If  $recX.(\tau.pri(E)) \xrightarrow{\alpha} H$ , we have  $\alpha = \tau$  and  $H \equiv pri(E)\{recX.(\tau.pri(E))/X\}$ . In this situation  $recX.(\tau.X + E) \xrightarrow{\tau} recX.(\tau.X + E)$ .

An identical inductive proof is performed when we start from a transition  $G\{recX.(\tau.X + E)/X\} \xrightarrow{\delta} H$  in the condition above (we just have to consider  $\delta$  instead of a generic  $\alpha$ ), apart from the cases  $G \equiv X$  and  $G \equiv G' + G''$ . In the case  $G \equiv X$ , due to the priority constraints, no  $\delta$  transitions can be performed, so the condition obviously holds. In the case  $G \equiv G' + G''$ , to derive the final statement  $G'\{recX.(\tau.pri(E))/X\} + G''\{recX.(\tau.pri(E))/X\} \xrightarrow{\delta} H'$  with  $(H, H') \in \beta$ , we have to additionally prove that the  $\delta$  transition is not pre-empted. In the first subcase this is obtained as follows. It must be that  $G''\{recX.(\tau.X + E)/X\} \not\xrightarrow{\tau}$ , hence, since we already showed the condition for relation  $\beta$  presented above (and its symmetric

one) to hold for any  $\alpha$  transition, we have also that  $G''\{recX.(\tau.pri(E))/X\} \not\rightarrow^{\tau}$ . The other subcase is analogous.

We conclude the proof that  $\beta$  is a weak bisimulation by considering the pair:  $(recX.(\tau.X + E), pri(E)\{recX.(\tau.pri(E))/X\})$ .

Suppose  $recX.(\tau.X + E) \xrightarrow{\gamma} H$ , first of all we note that  $\gamma \neq \delta$  for the priority constraints. We have the following two cases.

- If  $recX.(\tau.X + E) \xrightarrow{\tau} H \equiv recX.(\tau.X + E)$ , then  $pri(E)\{recX.(\tau.pri(E))/X\}$  just makes no move. Denoting the latter term with  $H'$ , we have  $(H, H') \in \beta$ .
- If  $recX.(\tau.X + E) \xrightarrow{\alpha} H$  with  $E\{recX.(\tau.X + E)/X\} \xrightarrow{\alpha} H$ , then, by the first part of the proof, we derive that, for some  $H'$ ,  $E\{recX.(\tau.pri(E))/X\} \xrightarrow{\alpha} H'$  with  $(H, H') \in \beta$ . Hence,  $pri(E)\{recX.(\tau.pri(E))/X\} \xrightarrow{\alpha} H'$ .

Vice-versa if  $pri(E)\{recX.(\tau.pri(E))/X\} \xrightarrow{\gamma} H$  (it must be  $\gamma \neq \delta$  for the definition of operator  $pri$ ) we have the following case.

- If  $pri(E)\{recX.(\tau.pri(E))/X\} \xrightarrow{\alpha} H$  with  $E\{recX.(\tau.pri(E))/X\} \xrightarrow{\alpha} H$ , then, by the first part of the proof, we derive that, for some  $H'$ ,  $E\{recX.(\tau.X + E)/X\} \xrightarrow{\alpha} H'$  with  $(H, H') \in \beta$ . Hence,  $recX.(\tau.X + E) \xrightarrow{\alpha} H'$ .

**Soundness of (Ung4)** The proof that (Ung4) is sound is similar. We show that:

$$\beta = \{(G\{recX.(\tau.(pri(X) + E) + F)/X\}, G\{recX.(\tau.X + E + F)/X\}) \mid \\ G \text{ contains at most } X \text{ free}\} \cup \{(pri(recX.(\tau.(pri(X) + E) + F)) + \\ E\{recX.(\tau.(pri(X) + E) + F)/X\}, recX.(\tau.X + E + F))\}$$

is a weak bisimulation. From this result it straightforwardly follows that  $recX.(\tau.(pri(X) + E) + F) \simeq recX.(\tau.X + E + F)$ .

Again we start the proof that  $\beta$  is a weak bisimulation by considering all the pairs  $(G\{recX.(\tau.(pri(X) + E) + F)/X\}, G\{recX.(\tau.X + E + F)/X\})$  such that  $G$  contains at most  $X$  free. As we did for (Ung2), we prove that  $\beta$  satisfies the following stronger condition

$$\text{if } G\{recX.(\tau.(pri(X) + E) + F)/X\} \xrightarrow{\gamma} H \text{ then, for some } H', \\ G\{recX.(\tau.X + E + F)/X\} \xrightarrow{\gamma} H' \text{ with } (H, H') \in \beta,$$

and symmetrically for a move of  $G\{recX.(\tau.X + E + F)/X\}$ .

This is done, first, by induction on the depth of the inference by which transitions of  $G\{recX.(\tau.(pri(X) + E) + F)/X\} \xrightarrow{\alpha} H$  are inferred. We have the following cases depending on the structure of  $G$ .

In the cases  $G \equiv \underline{0}$ ,  $G \equiv \gamma.G'$ ,  $G \equiv G' + G''$  and  $G \equiv recY.G'$  the proof is analogous to (Ung2).

In the case  $G \equiv X$  we have that  $G\{recX.(\tau.(pri(X) + E) + F)/X\} \equiv recX.(\tau.(pri(X) + E) + F)$  and  $G\{recX.(\tau.X + E + F)/X\} \equiv recX.(\tau.X + E + F)$ . Suppose  $recX.(\tau.(pri(X) + E) + F) \xrightarrow{\alpha} H$ , we have two cases:

- if  $recX.(\tau.(pri(X) + E) + F) \xrightarrow{\tau} H \equiv pri(recX.(\tau.(pri(X) + E) + F)) + E\{recX.(\tau.(pri(X) + E) + F)/X\}$ , then  $recX.(\tau.X + E + F) \xrightarrow{\tau} H' \equiv recX.(\tau.X + E + F)$  with  $(H, H') \in \beta$ ;

- if  $recX.(\tau.(pri(X)+E)+F) \xrightarrow{\alpha} H$  with  $F\{recX.(\tau.(pri(X)+E)+F)/X\} \xrightarrow{\alpha} H$ , then, by induction we derive that, for some  $H'$ ,  $F\{recX.(\tau.X+E+F)/X\} \xrightarrow{\alpha} H'$  with  $(H, H') \in \beta$ . Hence,  $recX.(\tau.X + E + F) \xrightarrow{\alpha} H'$  with  $(H, H') \in \beta$ .

A symmetric inductive proof is performed when we start from a transition of  $G\{recX.(\tau.X + E + F)/X\} \xrightarrow{\alpha} H$  in the condition above. The only exception to symmetry is the case of  $G \equiv X$ . The proof of this case follows.

Suppose  $recX.(\tau.X + E + F) \xrightarrow{\alpha} H$ , we have three cases:

- if  $recX.(\tau.X + E + F) \xrightarrow{\tau} H \equiv recX.(\tau.X + E + F)$ , then  $recX.(\tau.(pri(X) + E) + F) \xrightarrow{\tau} H' \equiv \{(pri(recX.(\tau.(pri(X) + E) + F)) + E\{recX.(\tau.(pri(X) + E) + F)/X\})\}$  with  $(H, H') \in \beta$ ;
- if  $recX.(\tau.X + E + F) \xrightarrow{\alpha} H$ , with  $E\{recX.(\tau.X + E + F)/X\} \xrightarrow{\alpha} H$ , then, by induction we derive that, for some  $H'$ ,  $E\{recX.(\tau.(pri(X) + E) + F)/X\} \xrightarrow{\alpha} H'$  with  $(H, H') \in \beta$ . Hence,  $recX.(\tau.(pri(X) + E) + F) \xrightarrow{\tau} \xrightarrow{\alpha} H'$  with  $(H, H') \in \beta$ .
- if  $recX.(\tau.X + E + F) \xrightarrow{\alpha} H$ , with  $F\{recX.(\tau.X + E + F)/X\} \xrightarrow{\alpha} H$ , then, by induction we derive that, for some  $H'$ ,  $F\{recX.(\tau.(pri(X) + E) + F)/X\} \xrightarrow{\alpha} H'$  with  $(H, H') \in \beta$ . Hence,  $recX.(\tau.(pri(X) + E) + F) \xrightarrow{\alpha} H'$  with  $(H, H') \in \beta$ .

An identical pair of inductive proofs is performed when we consider  $\delta$  transitions in the condition above (we just have to consider  $\delta$  instead of a generic  $\alpha$ ), apart from the cases  $G \equiv X$  and  $G \equiv G' + G''$ . In the case  $G \equiv X$ , due to the priority constraints, no  $\delta$  transitions can be performed, so the condition obviously holds. In the case  $G \equiv G' + G''$  we have to take into account the priority constraints by proceeding exactly as for (Ung2).

We conclude the proof that  $\beta$  is a weak bisimulation by considering the pair:

$(pri(recX.(\tau.(pri(X) + E) + F)) + E\{recX.(\tau.(pri(X) + E) + F)/X\}, recX.(\tau.X + E + F))$ .

Suppose  $pri(recX.(\tau.(pri(X) + E) + F)) + E\{recX.(\tau.(pri(X) + E) + F)/X\} \xrightarrow{\gamma} H$ , first of all we note that  $\gamma \neq \delta$  for the priority constraints. We have the following three cases.

- If  $pri(recX.(\tau.(pri(X)+E)+F))+E\{recX.(\tau.(pri(X)+E)+F)/X\} \xrightarrow{\tau} H \equiv pri(recX.(\tau.(pri(X) + E) + F)) + E\{recX.(\tau.(pri(X) + E) + F)/X\}$ , then  $recX.(\tau.X + E + F)$  just makes no move. Denoting the latter term with  $H'$ , we have  $(H, H') \in \beta$ .
- If  $pri(recX.(\tau.(pri(X)+E)+F))+E\{recX.(\tau.(pri(X)+E)+F)/X\} \xrightarrow{\alpha} H$  with  $F\{recX.(\tau.(pri(X) + E) + F)/X\} \xrightarrow{\alpha} H$ , then, by the first part of the proof we derive that, for some  $H'$ ,  $F\{recX.(\tau.X + E + F)/X\} \xrightarrow{\alpha} H'$  with  $(H, H') \in \beta$ . Hence,  $recX.(\tau.X + E + F) \xrightarrow{\alpha} H'$  with  $(H, H') \in \beta$ .
- If  $pri(recX.(\tau.(pri(X)+E)+F))+E\{recX.(\tau.(pri(X)+E)+F)/X\} \xrightarrow{\alpha} H$  with  $E\{recX.(\tau.(pri(X) + E) + F)/X\} \xrightarrow{\alpha} H$ , then, by the first part of the proof we derive that, for some  $H'$ ,  $E\{recX.(\tau.X + E + F)/X\} \xrightarrow{\alpha} H'$  with  $(H, H') \in \beta$ . Hence,  $recX.(\tau.X + E + F) \xrightarrow{\alpha} H'$  with  $(H, H') \in \beta$ .

Vice-versa if  $recX.(\tau.X + E + F) \xrightarrow{\gamma} H$  (it must be  $\gamma \neq \delta$  for the priority constraints) we have the following three cases.

- If  $recX.(\tau.X + E + F) \xrightarrow{\tau} H \equiv recX.(\tau.X + E + F)$ , then  $pri(recX.(\tau.(pri(X) + E) + F)) + E\{recX.(\tau.(pri(X) + E) + F)/X\}$  just makes no move. Denoting the latter term with  $H'$ , we have  $(H, H') \in \beta$ .
- if  $recX.(\tau.X + E + F) \xrightarrow{\alpha} H$ , with  $E\{recX.(\tau.X + E + F)/X\} \xrightarrow{\alpha} H$ , then, by the first part of the proof we derive that, for some  $H'$ ,  $E\{recX.(\tau.(pri(X) + E) + F)/X\} \xrightarrow{\alpha} H'$

- with  $(H, H') \in \beta$ . Hence,  $\text{pri}(\text{rec}X.(\tau.(\text{pri}(X) + E) + F)) + E\{\text{rec}X.(\tau.(\text{pri}(X) + E) + F)/X\} \xrightarrow{\alpha} H'$  with  $(H, H') \in \beta$ .
- if  $\text{rec}X.(\tau.X + E + F) \xrightarrow{\alpha} H$ , with  $F\{\text{rec}X.(\tau.X + E + F)/X\} \xrightarrow{\alpha} H$ , then, by the first part of the proof we derive that, for some  $H'$ ,  $F\{\text{rec}X.(\tau.(\text{pri}(X) + E) + F)/X\} \xrightarrow{\alpha} H'$  with  $(H, H') \in \beta$ . Hence,  $\text{pri}(\text{rec}X.(\tau.(\text{pri}(X) + E) + F)) + E\{\text{rec}X.(\tau.(\text{pri}(X) + E) + F)/X\} \xrightarrow{\alpha} H'$  with  $(H, H') \in \beta$ .  $\square$

**2.5. Completeness.** We now show that the axiom system  $\mathcal{A}$  is complete over processes of  $\mathcal{P}$ . In order to do this we follow the lines of [29], so we deal with systems of recursive equations.

We start by introducing the machinery necessary for proving completeness over guarded expressions  $E \in \mathcal{E}$ . Afterwards we will show how the axioms (*Ung*) can be used to turn an unguarded processes of  $\mathcal{P}$  into a guarded process of  $\mathcal{P}$ . Notice that the new operator  $\text{pri}(E)$  is used only in the intermediate steps of the second procedure.

**Definition 2.14.** An *equation set* with *formal variables*  $\tilde{X} = \{X_1, \dots, X_n\}$  and *free variables*  $\tilde{W} = \{W_1, \dots, W_m\}$ , where  $\tilde{X}$  and  $\tilde{W}$  are disjoint, is a set  $S = \{X_i = H_i \mid 1 \leq i \leq n \wedge H_i \in \mathcal{E}\}$  of equations such that the expressions  $\tilde{H} = \{H_1, \dots, H_n\}$  have free variables in  $\tilde{X} \cup \tilde{W}$ . In the following we will also represent  $S$  by using the shorthand  $S : \tilde{X} = \tilde{H}$ . We call  $X_1$  the *distinguished* variable of  $S$ .

We say that an expression  $E$  *provably satisfies*  $S$  if there are expressions  $\tilde{E} = \{E_1, \dots, E_n\}$  with free variables in  $\tilde{W}$  such that  $E_1 \equiv E$  and for  $1 \leq i \leq n$  we have  $\mathcal{A} \vdash E_i = H_i\{\tilde{E}/\tilde{X}\}$ . We can also simply write:  $\mathcal{A} \vdash \tilde{E} = \tilde{H}\{\tilde{E}/\tilde{X}\}$ .

Finally, we say that  $S$  is *closed* if  $W = \emptyset$ .

In order to define (similarly as in [29]) guardedness over equation sets, we consider the relation  $\xrightarrow{\text{ung}}_S \subseteq \tilde{X} \times \tilde{X}$ , defined as follows:

$$X_i \xrightarrow{\text{ung}}_S X_j \quad \text{iff} \quad H_i \triangleright X_j$$

where  $E \triangleright X$  denotes that expression  $E$  contains a free unguarded occurrence of  $X$ .

**Definition 2.15.** An equation set  $S$  with formal variables  $\tilde{X} = \{X_1, \dots, X_n\}$  is *guarded* if there is no cycle  $X_i \xrightarrow{\text{ung}}_S^+ X_i$ .

The equation sets that are actually dealt with in the proof of completeness of [29] belong to the subclass of *standard equation sets*. Here we consider the subclass of *prioritized* ones.

**Definition 2.16.** An equation set  $S = \{X_i = H_i \mid 1 \leq i \leq n\}$ , with formal variables  $\tilde{X} = \{X_1, \dots, X_n\}$  and free variables  $\tilde{W} = \{W_1, \dots, W_m\}$ , is *standard* if each expression  $H_i$  ( $1 \leq i \leq n$ ) is of the form:<sup>1</sup>

$$H_i \equiv \sum_{j \in J_i} \gamma_{i,j} \cdot X_{f(i,j)} + \sum_{k \in K_i} W_{g(i,k)}$$

Moreover, if it also holds that:

$$\exists j \in J_i : \gamma_{i,j} = \tau \Rightarrow \nexists j \in J_i : \gamma_{i,j} = \delta$$

<sup>1</sup>We assume  $\sum_{j \in J} E \equiv 0$  if  $J = \emptyset$ .

we say that  $S$  is *prioritized*.

As in [29], for a standard equation set  $S$  we define the relations  $\longrightarrow_S \subseteq \tilde{X} \times Act \times \tilde{X}$  and  $\triangleright_S \subseteq \tilde{X} \times \tilde{W}$  as follows:

$$\begin{array}{ll} X_i \xrightarrow{\gamma}_S X_j & \text{iff } \gamma.X_j \text{ occurs in } H_i \\ X_i \triangleright_S W & \text{iff } W \text{ occurs in } H_i \end{array}$$

Notice that, from Definition 2.15, we have that a standard equation set  $S$  with formal variables  $\tilde{X} = \{X_1, \dots, X_n\}$  is guarded if there is no cycle  $X_i \xrightarrow{\tau}_S^+ X_i$ .

**Definition 2.17.** A free variable  $W$  is guarded in the standard equation set  $S = \{X_i = H_i \mid 1 \leq i \leq n\}$  if it is not the case that  $X_1 \xrightarrow{\tau}_S^* X_i \triangleright_S W$ .

We obviously have that every standard equation set can be reduced to a prioritized one.

**Theorem 2.18.** *For every standard equation set  $S$  there is a prioritized standard equation set  $S'$  such that every expression  $E$  that provably satisfies  $S$ , provably satisfies  $S'$  as well. Moreover, if  $S$  is guarded then  $S'$  is also guarded.*

*Proof.*  $S' = \{X_i = H'_i \mid 1 \leq i \leq n\}$  is obtained from  $S = \{X_i = H_i \mid 1 \leq i \leq n\}$  by eliminating the occurrences of  $\delta.X_i$  (for any  $i$ ) in the equations of  $S$  which include terms  $\tau.X_j$  (for any  $j$ ). The expressions  $\tilde{E} = \{E_1, \dots, E_n\}$  used to show that  $E$  that provably satisfies  $S$  (with  $E_1 \equiv E$ ) can also be used to prove that  $E$  provably satisfies  $S'$ : for every  $i$  such that  $1 \leq i \leq n$  we have that  $\mathcal{A} \vdash E_i = H_i\{\tilde{E}/\tilde{X}\} = H'_i\{\tilde{E}/\tilde{X}\}$  by using laws (Pri6) and (Pri3). If  $S$  is guarded then  $S'$  is easily proven to be guarded by contradiction: if there is a cycle  $X_i \xrightarrow{\tau}_S^+ X_i$  for some  $i$  then the same cycle must be also in  $S$ .  $\square$

The following theorem guarantees that from a guarded expression  $E \in \mathcal{E}$  we can derive a (prioritized) standard guarded equation set which is provably satisfied by  $E$ .

**Theorem 2.19** (representability). *Every guarded expression  $E \in \mathcal{E}$  with free variables  $\tilde{W}$  provably satisfies a standard guarded equation set  $S$  with free variables in  $\tilde{W}$ . Moreover, if a free variable  $W$  is guarded in  $E$  then  $W$  is guarded in  $S$ .*

*Proof.* The proof, by induction on the structure of  $E$ , is exactly as in [29].  $\square$

Once established the structure of prioritized standard guarded equation sets  $S$ , completeness over guarded expressions is a consequence of the uniqueness of the solution of such equation sets and of the possibility to merge them (after saturation, see Definition 2.21) if they have equivalent solutions, as in [29].

The following theorem shows that every guarded equation set (not necessarily standard) has a unique solution up to provable equality.

**Theorem 2.20** (one and only one solution). *If  $S$  is a guarded equation set with free variables in  $\tilde{W}$ , then there is a guarded expression  $E \in \mathcal{E}$  with free variables in  $\tilde{W}$  that provably satisfies  $S$ . Moreover, if  $F$  with free variables in  $\tilde{W}$  provably satisfies  $S$ , then  $\mathcal{A} \vdash E = F$ .*

*Proof.* The proof is exactly as in [29]. We just additionally show that the expression  $E \in \mathcal{E}$  obtained with the inductive procedure in [29] is guarded (not proven in [29]).

The inductive procedure in [29] constructs  $E$  as follows. Given equation set  $S = \{X_i = H_i \mid 1 \leq i \leq n\}$ , the expression  $E$  that provably satisfies  $S$  is obtained by induction on the



number of equations  $n$ . The base case is  $n = 1$ :  $E$  is simply  $recX_1.H_1$ . In the inductive case, given equation set  $S = \{X_i = H_i \mid 1 \leq i \leq n + 1\}$ ,  $E$  is inductively defined as the expression that provably satisfies the equation set  $\{X_i = H'_i \mid 1 \leq i \leq n\}$ , where  $H'_i = H_i\{recX_{n+1}.H_{n+1}/X_{n+1}\}$ .

It is immediate to show that  $E$  obtained from such a procedure is guarded by contradiction. Suppose that  $E$  contains a subterm  $recX.E'$  such that  $E'$  includes a free unguarded occurrence of  $X$ . Now consider the recursion operators that appear in  $recX.E'$  and which include the free unguarded occurrence of  $X$  in their scope (among them there is the recursion operator  $recX.E'$  itself). Based on the inductive procedure above, such recursion operators (considered syntactically from outside to inside) would be in correspondence with a sequence of consecutive  $X_i \xrightarrow{unq}_S X_j$  steps in the equation set  $S$ :  $X_i$  corresponds to the syntactical occurrence of a  $recX_i$  operator and  $X_j$  either to the syntactical occurrence of  $recX_j$  or, if  $X_j \equiv X$ , to the syntactical occurrence of  $X$ . Such a sequence would lead from  $X$  to itself, hence  $S$  would not be guarded.  $\square$

In the following, given a guarded equation set  $S$ , we will refer to the solution  $E$  determined in the proof of the theorem above as the *standard solution* of  $S$ .

**Definition 2.21.** A *saturated* standard equation set  $S$  with formal variables  $\tilde{X}$  is a standard equation set such that for all  $X \in \tilde{X}$ :

$$\begin{aligned} (i) \quad X &\xrightarrow{\tau}_S^* \xrightarrow{\alpha}_S \xrightarrow{\tau}_S^* X' &\Rightarrow & X \xrightarrow{\alpha}_S X' \\ (ii) \quad X &\xrightarrow{\tau}_S^* \xrightarrow{\delta}_S \xrightarrow{\tau}_S^* X' &\Rightarrow & X \xrightarrow{\delta}_S X' \\ (iii) \quad X &\xrightarrow{\tau}_S^* \triangleright_S W &\Rightarrow & X \triangleright_S W \end{aligned}$$

**Lemma 2.22.** *Let  $E \in \mathcal{E}$  provably satisfy  $S$ , standard and guarded. Then there is a saturated, standard and guarded equation set  $S'$  provably satisfied by  $E$ .*

*Proof.* From [29] we know that there is a saturated standard equation set  $S'$  provably satisfied by  $E$ : the proof is based on the axioms (A1) – (A4) and (Tau1) – (Tau3) that are unchanged with respect to the standard machinery.  $\square$

The possibility of saturating standard and guarded equation sets  $S$  leads to the following theorem.

**Theorem 2.23** (mergeability). *Let process  $P \in \mathcal{P}$  provably satisfy  $S$ , and process  $Q \in \mathcal{P}$  provably satisfy  $T$ , where both  $S$  and  $T$  are prioritized, standard, guarded and closed sets of equations, and let  $P \simeq Q$ . Then there is a prioritized, standard, guarded and closed equation set  $U$  provably satisfied by both  $P$  and  $Q$ .*

*Proof.* A standard guarded (and closed) equation set  $U'$  provably satisfied by both  $P$  and  $Q$  can be derived from  $S$  and  $T$  (considered just as standard guarded equation sets) by means of the procedure in the proof of the related theorem in [29]. In particular, as in [29], we can assume standard and guarded equation sets  $S$  and  $T$  to have been preliminarily saturated because of Lemma 2.22. The relation between the variables of saturated  $S$  and  $T$ , which in [29] is deduced from  $P \simeq Q$ , still holds here (in spite of priority). This is because  $S$  and  $T$  are assumed to be prioritized standard guarded equation sets, hence no  $\delta$  prefix that does not actually correspond to a weak transition  $\xrightarrow{\delta}$  occurs in the saturated standard equation sets obtained from  $S$  and  $T$ . From the existence of  $U'$  we derive the existence of the prioritized standard guarded equation set  $U$  by means of Theorem 2.18.  $\square$

Hence we have proved completeness over guarded processes of  $\mathcal{P}$ .

**Theorem 2.24.** *If  $P$  and  $Q$  are guarded processes of  $\mathcal{P}$  and  $P \simeq Q$  then  $\mathcal{A} \vdash P = Q$ .*

Now we show that each unguarded process can be turned into a guarded process of  $\mathcal{P}$ , so that we obtain completeness also over unguarded processes. We start with a technical lemma which, in analogy to [29], will be used in the proof of this result.

**Lemma 2.25.** *If  $X$  occurs free and unguarded in  $E \in \mathcal{E}$ , then  $\mathcal{A} \vdash E = X + E$ .*

*Proof.* The proof is exactly as in [29]. □

**Theorem 2.26.** *For each process  $P \in \mathcal{P}$  there exists a guarded  $P' \in \mathcal{P}$  such that  $\mathcal{A} \vdash P = P'$ .*

*Proof.* We show, by structural induction, that given an expression  $E \in \mathcal{E}$ , it is possible to find an expression  $F \in \mathcal{E}_{pri}$  such that:

- (1) if  $pri(G)$  is a subexpression of  $F$  then  $G \equiv X$  for some free variable  $X$ ;
- (2) for any free variable  $X$ ,  $pri(X)$  is weakly guarded in  $F$ , i.e. each occurrence of  $pri(X)$  is within some subexpression of  $F$  of the form  $\gamma.G$ ;
- (3) a summation cannot have both  $pri(X)$  and  $Y$  as arguments, for any (possibly coincident) variables  $X$  and  $Y$ ;
- (4) for any variable  $X$ , each subterm  $recX.G$  of  $F$  is (strongly) guarded in  $F$ , i.e. each occurrence of  $recX.G$  is within some subexpression of  $F$  of the form  $\gamma.H$ , with  $\gamma \neq \tau$ ;
- (5)  $F$  is guarded;
- (6) all variables occurring free in  $F$  occur free also in  $E$ ;
- (7)  $\mathcal{A} \vdash E = F$ .

Notice that a consequence of property 4 is that each unguarded occurrence of any free variable  $X$  of  $F$  does not lie within the scope of a subexpression  $recY.G$  of  $F$ ;

Showing this result proves the theorem, in that if  $E \in \mathcal{E}$  is a process of  $\mathcal{P}$ , i.e. a closed term, we have (by the properties of  $F$  above) that  $F$  is also a process of  $\mathcal{P}$ , it is guarded, and  $\mathcal{A} \vdash E = F$ .

The result above is derived by structural induction on the syntax of an expression  $E \in \mathcal{E}$  as follows.

- If  $E \equiv 0$ , then  $F \equiv 0$ .
- If  $E \equiv X$ , for some variable  $X$ , then  $F \equiv X$ .
- If  $E \equiv \gamma.E'$  then  $F \equiv \gamma.F'$ , where  $F'$  is the term obtained from  $E'$  via the induction hypothesis.
- If  $E \equiv E' + E''$ , then  $F \equiv F' + F''$ , where  $F'$  and  $F''$  are the terms obtained from  $E'$  and  $E''$ , respectively, via the induction hypothesis.
- If  $E \equiv recX.E'$ , then  $F$  is evaluated as follows.

First of all we derive from the term  $F'$ , obtained from  $E'$  via the induction hypothesis, a term  $F''$  such that:

- $F''$  satisfies the properties 1 – 5 above;
- $X$  is guarded in  $F''$ ;
- all variables occurring free in  $F''$  occur free also in  $E'$ ;
- $\mathcal{A} \vdash recX.E' = recX.F''$ ;

We start by eliminating all fully unguarded occurrences of  $X$  in  $F'$ , via the axiom (*Ung1*), so that we obtain a term  $G$  such that  $X$  is weakly guarded in  $G$  and  $recX.E' = recX.G$ . The axiom (*Ung1*) is sufficient because  $F'$  satisfies the property 2 above, so we do not need to deal with priority. Notice that  $G$  still satisfies properties 1 – 5 above.

Then we need to remove weakly guarded occurrences of  $X$  in  $G$  (if there are not, we just take  $F'' \equiv G$ ). In order to do this we begin by evaluating a term  $G'$  such that:

$$G' \equiv \tau.X + G''$$

where  $X$  is guarded in  $G''$  and  $\text{rec}X.G = \text{rec}X.G'$ .

We do this by employing the following iterative procedure, where initially we let  $H \equiv G$ .

– We start each iteration with  $H$  such that  $X$  is weakly guarded in  $H$  and  $H$  satisfies the properties 1, 3, 4 and 5 above. If there still exists  $H' \not\equiv X$  with  $X$  occurring unguarded in  $H'$  such that  $H$  has the form  $\tau.H' + H''$  we continue with the procedure. Otherwise  $H$  can be turned into a term  $G' \equiv \tau.X + G''$  with the property above, by simply using  $\tau.X + \tau.X = \tau.X$  and we are finished.

– Since  $H$  satisfies property 3, we have the following cases for the structure of  $H'$  (after applying the idempotency law (A3) to  $X$  or  $\text{pri}(X)$  if needed).

(1)  $H'$  has the form  $X + H'''$ , where  $X$  is weakly guarded in  $H'''$ . In this case we consider the term  $H'''' \equiv \tau.X + H''' + H''$  and we have  $\text{rec}X.H = \text{rec}X.H''''$  by applying the axiom (Ung3).

(2)  $H'$  has the form  $\text{pri}(X) + H'''$ , where  $X$  is weakly guarded in  $H'''$ . In this case we consider again the term  $H'''' \equiv \tau.X + H''' + H''$  and we have  $\text{rec}X.H = \text{rec}X.H''''$  by applying the axiom (Ung4).

(3)  $X$  is weakly guarded in  $H'$ . We have two sub-cases.

(i) If  $\text{pri}(X)$  occurs in  $H'$ , then we do the following.

We consider a new variable  $Y'$  for each variable  $Y$  such that  $\text{pri}(Y)$  appears in  $H'$ . Let  $H'''$  be the unique expression of  $\mathcal{E}$  such that if we replace  $\text{pri}(Y)$  for each new variable  $Y'$  inside  $H'''$ , we obtain the term  $H'$ .

Since  $X$  occurs unguarded in  $H'$ , then  $X'$  occurs unguarded in  $H'''$  and from Lemma 2.25 we have that  $H''' = X' + H''''$ . Hence, since substitution preserves equality, we have  $H' = \text{pri}(X) + H'$  and we continue as in the case 2.

(ii) Otherwise we do the following.

We consider a new variable  $Y'$  for each variable  $Y$  such that  $\text{pri}(Y)$  appears in  $H'$ . Let  $H'''$  be the unique expression of  $\mathcal{E}$  such that if we replace  $\text{pri}(Y)$  for each new variable  $Y'$  inside  $H'''$ , we obtain the term  $H'$ .

Since  $X$  occurs unguarded in  $H'$ , then  $X$  occurs unguarded also in  $H'''$  and from Lemma 2.25 we have that  $H''' = X + H''''$ . Hence, since substitution preserves equality, we have  $H' = X + H'$  and we continue as in the case 1.

– Now a new iteration is performed starting from the term  $H''''$  that we have obtained. Since there is at least one unguarded occurrence of  $X$  such that the length of the  $\tau$  path that leads to that occurrence is shorter in  $H''''$  than in  $H$ , we are guaranteed that the iterative procedure will eventually terminate.

Now we are in a position to derive the term  $F''$  with the properties we described above. From  $G' \equiv \tau.X + G''$  we have  $\text{rec}X.G' = \text{rec}X.\tau.\text{pri}(G'')$  by applying the axiom (Ung2).  $X$  is guarded in the term  $\tau.\text{pri}(G'')$  and such term satisfies properties 2, 4 and 5.

Due to the property 4, the term  $G''$  has the following structure:

$$G'' \equiv \sum_i \alpha_i.H_i + \sum_j \delta.H'_j + \sum_k Y_k + \sum_h \text{pri}(Y'_h).$$

where the variables  $Y_k$  and  $Y'_h$  are free and do not coincide with  $X$  because  $X$  is guarded in  $G''$ . By applying the axioms (Pri1), (Pri2), (Pri3), (Pri4) and (Pri5), we obtain a term  $F'' = \tau.\text{pri}(G'')$ , where:

$$F'' \equiv \tau. \left( \sum_i \alpha_i.H_i + \sum_k \text{pri}(Y_k) + \sum_h \text{pri}(Y'_h) \right).$$

We have, thus, obtained an  $F''$  that satisfies also properties 1 and 3.

Finally, once obtained (possibly by applying the procedure above for weak guarded recursion elimination, if needed) a term  $F''$  with the properties listed above, we have to transform the term  $\text{rec}X.F''$  so to obtain a term  $F$  that satisfies the seven properties of the induction statement.

$\text{rec}X.F''$  already satisfies the properties 2 and 3 because  $F''$  satisfies them. Moreover  $\text{rec}X.F''$  already satisfies property 5, i.e. it is guarded, because  $X$  is guarded in  $F''$  and  $F''$  already satisfies that property.

In order to satisfy property 1 we have to remove the occurrences of  $\text{pri}(X)$  because now  $X$  is no longer a free variable.

We unfold  $\text{rec}X.F''$  via the axiom (*Rec1*) and we obtain  $F''' \equiv F''\{\text{rec}X.F''/X\} = \text{rec}X.F''$ . Notice that  $F'''$  may include subterms of the form  $\text{pri}(\text{rec}X.F'')$ .

We deal with such subterms in the following way. We have  $\text{pri}(\text{rec}X.F'') = \text{pri}(F''\{\text{rec}X.F''/X\})$ , by applying law (*Rec1*).

Since  $X$  is guarded in  $F''$  and  $F''$  satisfies the property 4, the term  $F''\{\text{rec}X.F''/X\}$  has the following structure:

$$F''\{\text{rec}X.F''/X\} \equiv \sum_i \alpha_i.H_i + \sum_j \delta.H'_j + \sum_k Y_k + \sum_h \text{pri}(Y'_h).$$

where the variables  $Y_k$  and  $Y'_h$  are free and obviously do not coincide with  $X$ . By applying the axioms (*Pri1*), (*Pri2*), (*Pri3*), (*Pri4*) and (*Pri5*), we obtain a term  $T = \text{pri}(F''\{\text{rec}X.F''/X\})$ , where:

$$T \equiv \sum_i \alpha_i.H_i + \sum_k \text{pri}(Y_k) + \sum_h \text{pri}(Y'_h).$$

Now let us consider a new variable  $X'$ . Let  $T'$  be the unique term, not having  $\text{pri}(\text{rec}X.F'')$  as a subterm, such that  $T'\{\text{pri}(\text{rec}X.F'')/X'\} \equiv T$ . Since  $X'$  is guarded in  $T'$  (because  $X$  is guarded in  $F''$ ) and serial in  $T'$  (thanks to the transformation of  $\text{pri}(F''\{\text{rec}X.F''/X\})$  into  $T$ ), from  $\text{pri}(\text{rec}X.F'') = T'\{\text{pri}(\text{rec}X.F'')/X'\}$  we derive, by applying the law (*Rec2*),  $\text{pri}(\text{rec}X.F'') = \text{rec}X'.T'$ .

Therefore now we can replace all occurrences of  $\text{pri}(\text{rec}X.F'')$  in  $F'''$  with  $\text{rec}X'.T'$ , and we obtain a term  $F'''' = F'''$  not having  $\text{pri}(\text{rec}X.F'')$  as a subterm. In order to satisfy the property 1 we still have to remove the occurrences of  $\text{rec}X.F''$  in  $F''''$ , so that we get rid of the occurrences of  $\text{pri}(X)$  appearing inside  $F''$ .

In order to do this, we consider another new variable  $X''$ . Let  $T''$  be the unique term, not having  $\text{rec}X.F''$  as a subterm, such that  $T''\{\text{rec}X.F''/X''\} \equiv F''''$ . Since  $X''$  is guarded in  $T''$  (because each occurrence of the expression  $\text{rec}X.F''$  is guarded both in  $T'$  and in  $F''''$ , since  $X$  is guarded in  $F''$ ) and serial in  $T''$  (thanks to the substitution of  $\text{pri}(\text{rec}X.F'')$  with  $\text{rec}X'.T'$ ), from  $\text{rec}X.F'' = F'''' \equiv T''\{\text{rec}X.F''/X''\}$  we derive, by applying the law (*Rec2*),  $\text{rec}X.F'' = \text{rec}X''.T''$ .

Now we have that  $\text{rec}X''.T''$  satisfies the property 1, because such term no longer uses the variable  $X$ , and does not include occurrences of  $\text{pri}(X')$  or  $\text{pri}(X'')$ .

Moreover, since  $X''$  is guarded in  $T''$  and each occurrence of  $\text{rec}X'.T''$ , for any  $T''$ , is guarded in  $T''$  (because each occurrence of  $\text{pri}(X)$  is guarded in  $F''$ ), in order to obtain property 4 it is sufficient to unfold  $\text{rec}X''.T''$ , by applying the law (*Rec1*). In this way

we finally obtain  $F \equiv T''\{recX''.T''/X''\} = recX''.T''$  that satisfies the seven properties above. □

**Example 2.27.** We here show an example of transformation of an unguarded process  $P \in \mathcal{P}$  into a guarded  $P' \in \mathcal{P}$ , following the inductive procedure described in the proof of Theorem 2.26. We take  $P$  to be:

$$recY.(recX.(\tau.(X + b.X + \delta.\underline{0}) + Y) + Y)$$

We start from subexpression  $\tau.(X + b.X + \delta.\underline{0}) + Y$ , which is not modified by the inductive transformation.

When  $recX.(\tau.(X + b.X + \delta.\underline{0}) + Y)$  is considered, we first remove fully unguarded occurrences of  $X$  from  $\tau.(X + b.X + \delta.\underline{0}) + Y$  (there are not). Then, we remove weakly unguarded occurrences of  $X$ : one iteration yielding  $\tau.X + b.X + \delta.\underline{0} + Y$  (case (1) for the structure of  $H'$ , which in our example is  $X + b.X + \delta.\underline{0}$ ) is sufficient. Such an iteration is correct because, thanks to (*Ung3*) we have  $recX.(\tau.(X + b.X + \delta.\underline{0}) + Y) = recX.(\tau.X + b.X + \delta.\underline{0} + Y)$ . We can now apply (*Ung2*) to get  $recX.(\tau.pri(b.X + \delta.\underline{0} + Y))$  and (*Pri*) axioms to get  $recX.(\tau.(b.X + pri(Y)))$ . We terminate this level of the structural induction by unfolding recursion via axiom (*Rec1*), thus getting  $\tau.(b.recX.\tau.(b.X + pri(Y)) + pri(Y))$ .<sup>2</sup>

Finally, we consider  $recY.(F' + Y)$  with  $F' \equiv \tau.(b.recX.(\tau.(b.X + pri(Y))) + pri(Y))$ . We first remove fully unguarded occurrences of  $Y$  in  $\tau.(b.recX.\tau.(b.X + pri(Y)) + pri(Y)) + Y$  obtaining  $\tau.(b.recX.\tau.(b.X + pri(Y)) + pri(Y))$ . Such a transformation is correct because, thanks to (*Ung1*), the two expression are equated when used as arguments of  $recY...$ . Then, we remove weakly unguarded occurrences of  $Y$ : one iteration yielding  $\tau.Y + b.recX.\tau.(b.X + pri(Y))$  (case (2) for the structure of  $H'$ , which in our example is  $b.recX.\tau.(b.X + pri(Y)) + pri(Y)$ ) is sufficient. Such an iteration is correct because, thanks to (*Ung4*), the two expression are equated when used as arguments of  $recY...$ . We can now apply (*Ung2*) to get  $recY.\tau.pri(b.recX.\tau.(b.X + pri(Y)))$  and (*Pri*) axioms to get  $recY.F''$ , with  $F'' \equiv \tau.b.recX.\tau.(b.X + pri(Y))$ . In order to get rid of  $pri(Y)$  subterms we now unfold recursion via axiom (*Rec1*), thus getting  $\tau.b.recX.\tau.(b.X + pri(recY.F''))$ . We now aim at replacing  $pri(recY.F'')$  subterms. This is done by using the fact that the same transformation via axiom (*Rec1*) can be done inside a  $pri(\_)$  operator, i.e.  $pri(recY.F'') = pri(\tau.b.recX.\tau.(b.X + pri(recY.F'')))$ . By using (*Pri*) axioms on the latter term we obtain  $\tau.b.recX.\tau.(b.X + pri(recY.F''))$ . Considering a new variable  $Y'$  such a term can be written as  $\tau.b.recX.\tau.(b.X + Y')\{pri(recY.F'')/Y'\}$ . We have therefore that  $pri(recY.F'') = \tau.b.recX.\tau.(b.X + Y')\{pri(recY.F'')/Y'\}$  with  $Y'$  being guarded and serial in  $\tau.b.recX.\tau.(b.X + Y')$ . Hence, by applying axiom (*Rec2*), we have  $pri(recY.F'') = recY'.\tau.b.recX.\tau.(b.X + Y')$ . We can, thus, achieve our goal of replacing such a term inside  $\tau.b.recX.\tau.(b.X + pri(recY.F''))$ , obtaining  $\tau.b.recX.\tau.(b.X + recY'.\tau.b.recX.\tau.(b.X + Y'))$ .

From Theorem 2.24 and Theorem 2.26 we derive the completeness of  $\mathcal{A}$  over processes of  $\mathcal{P}$ .

**Theorem 2.28.** *Let  $P, Q \in \mathcal{P}$ . If  $P \simeq Q$  then  $\mathcal{A} \vdash P = Q$ .*

Note that all the axioms of  $\mathcal{A}$  are actually used in the proof of completeness. In particular in the proof of completeness over guarded expressions (Theorem 2.24) we employ

<sup>2</sup>In the proof of Theorem 2.26 the actual term derived at the end of the induction step is  $\tau.(b.recX''.\tau.(b.X'' + pri(Y)) + pri(Y))$  with variable  $X''$  taking the place of  $X$ .

|   |   |
|---|---|
| $\frac{P \xrightarrow{\alpha} P'}{P \parallel_S Q \xrightarrow{\alpha} P' \parallel_S Q} \quad \alpha \notin S$           | $\frac{Q \xrightarrow{\alpha} Q'}{P \parallel_S Q \xrightarrow{\alpha} P \parallel_S Q'} \quad \alpha \notin S$ |
| $\frac{P \xrightarrow{a} P' \quad Q \xrightarrow{a} Q'}{P \parallel_S Q \xrightarrow{a} P' \parallel_S Q'} \quad a \in S$ |   |
| $\frac{P \xrightarrow{a} P'}{P/L \xrightarrow{\tau} P'/L} \quad a \in L$  | $\frac{P \xrightarrow{\alpha} P'}{P/L \xrightarrow{\alpha} P'/L} \quad \alpha \notin L$                         |

Table 5: Standard Rules for Parallel Composition and Hiding

|  |  |
|--|--|
| $\frac{P \xrightarrow{\delta} P' \quad Q \xrightarrow{\delta} Q'}{P \parallel_S Q \xrightarrow{\delta} P' \parallel_S Q'}$ | $\frac{P \xrightarrow{\delta} P' \quad \exists a \in L. P \xrightarrow{a}}{P/L \xrightarrow{\delta} P'/L}$ |
|--|--|

Table 6: Rules for Discrete Time Transitions

the standard axioms (A1) – (A4), (Tau1) – (Tau3) and (Rec1), (Rec2) as in [29], plus the new axioms (Pri3) and (Pri6). All these axioms are necessary even if we restrict ourselves to consider completeness over guarded processes only. Moreover, proving that a process of  $\mathcal{P}$  can always be turned into a guarded process (Theorem 2.26) requires the use of the remaining axioms (Pri1), (Pri2), (Pri4), (Pri5) and (Ung1) – (Ung4). This supports the claim that our axiomatization is irredundant.

### 3. DISCRETE TIME

We now interpret unprioritized  $\delta$  actions of the basic calculus as representing time delays in the context of *discrete time*, see [22]. We first show that we can extend the basic calculus of Section 2 with *static* operators, like CSP [26] parallel composition and hiding, preserving the congruence property of standard observational congruence. We then consider a full discrete time calculus and provide a complete axiomatization.

In the following we will make use of the CSP [26] parallel composition operator “ $P \parallel_S Q$ ”, where standard actions with type in  $S$  are required to synchronize, while the other standard actions are executed independently from  $P$  and  $Q$ . Such an operator will be used in combination with a hiding operator “ $P/L$ ”, which turns all the standard actions of  $P$  whose type is in  $L$  into  $\tau$  and does not affect the other standard actions. See Table 5 for the standard operational rules of such operators. Along the lines of [22], the behavior of parallel composition concerning “tick” transitions is defined by the operational rule in Table 6, which states that time is allowed to elapse in  $P \parallel_S Q$  only if *both* processes  $P$  and  $Q$  may explicitly make it pass via  $\delta$  transitions. Moreover, due to the maximal progress assumption, the generation of  $\tau$  actions enacted by the hiding operator must cause all alternative  $\delta$  actions to be pre-empted, as formalized by the rule in Table 6.

**Theorem 3.1.**  *$\approx$  and  $\simeq$  are congruences with respect to both  $\_ \parallel_S \_$  and  $\_ /L$  operators.*

|   |   |
|---|---|
| $\alpha^t.P \xrightarrow{\alpha} P$                                 |   |
| $\frac{P \xrightarrow{\alpha} P'}{P +^t Q \xrightarrow{\alpha} P'}$ | $\frac{Q \xrightarrow{\alpha} Q'}{P +^t Q \xrightarrow{\alpha} Q'}$ |

Table 7: Standard Rules for Prioritized Action Transitions

|  |                                     |                                    |
|--|-------------------------------------|------------------------------------|
| $\underline{1} \xrightarrow{\delta} \underline{1}$   | $\delta^t.P \xrightarrow{\delta} P$ | $a^t.P \xrightarrow{\delta} a^t.P$ |
| $\frac{P \xrightarrow{\delta} P' \quad Q \xrightarrow{\delta} Q'}{P +^t Q \xrightarrow{\delta} P' +^t Q'}$ |                                     |                                    |

Table 8: Rules for Discrete Time Transitions

*Proof.* We will show congruence of  $\approx$  for both  $\_||_S \_$  and  $\_/L$  operators and then congruence of  $\simeq$ . In the proof we will often exploit the fact that standard transitions  $\alpha$  are inferred by standard transitions only.

Let  $P \approx Q$ . In the following we will use the reformulation of weak bisimulation introduced in Proposition 2.8.

$P ||_S R \approx Q ||_S R$  is shown by considering the weak bisimulation

$$\beta = \{(P ||_S R, Q ||_S R) \mid P \approx Q \wedge R \in \mathcal{P}_{static}\}$$

with  $\mathcal{P}_{static}$  being the set of basic processes extended with  $\_||_S \_$  and  $\_/L$  operators. The proof for  $\alpha$  transitions is standard. Concerning  $\delta$  transitions, since  $P \approx Q$ , we have that there exists  $Q''$  such that  $Q \xrightarrow{\tau}^* Q''$  and for all  $P'$  we have: if  $P \xrightarrow{\delta} P'$  then, for some  $Q'$ ,  $Q'' \xrightarrow{\delta} \xrightarrow{\tau}^* Q'$  and  $P' \approx Q'$ . Therefore  $Q ||_S R \xrightarrow{\tau}^* Q'' ||_S R$  and for all  $P', R'$  we have: if  $P ||_S R \xrightarrow{\delta} P' ||_S R'$  then (from  $R \xrightarrow{\delta} R'$ ), for the above  $Q', Q'' ||_S R \xrightarrow{\delta} \xrightarrow{\tau}^* Q' ||_S R'$  and  $(P' ||_S R', Q' ||_S R') \in \beta$ .

$P/L \approx Q/L$  is shown by, similarly, considering the weak bisimulation

$$\beta = \{(P/L, Q/L) \mid P \approx Q\}$$

Under the condition that  $P$  does not perform any  $a$  transition with  $a \in L$  and  $P$  performs some  $\delta$  transitions (otherwise there is nothing to prove concerning  $\delta$  transitions of  $P/L$ ), the proof is performed similar to the case of  $||_S$ . We have in addition to rely on the fact that  $Q''$  cannot do  $a$  transitions with  $a \in L$  because otherwise  $P$  (that cannot perform  $\tau$  transitions) should perform a corresponding  $a$  transition.

Concerning congruence of  $\simeq$ , we show that  $P \simeq Q$  implies  $P ||_S R \simeq Q ||_S R$  by resorting to the reformulation of Proposition 2.10. The proof is as that done for the pair  $(P ||_S R, Q ||_S R)$  in the proof of congruence of  $\approx$ , with  $Q''$  being just  $Q$ , and by reaching the pair  $P' ||_S R' \approx Q' ||_S R'$ . A similar reasoning is done for  $P/L \simeq Q/L$ .  $\square$

**3.1. Discrete Time Calculus.** As we mentioned in the introduction, in the discrete time setting, besides the static operators  $-\|_S-$  and  $-/L$ , it is convenient (from a modeling viewpoint, as we detail below) to adopt the specialized “timed” action prefix  $\gamma^t.P$  and “timed” choice operator  $P +^t Q$  used in [22]. Such operators are different from the prefix  $\gamma.P$  and choice  $P + Q$  that we considered in the basic calculus of Section 2 (here we use the “ $t$ ” superscript to distinguish them) in that they allow time to evolve via explicit execution of “ $\delta$ ” transitions as for the semantics of parallel composition:

- The new prefix operator  $\gamma^t.P$  behaves like  $\gamma.P$  plus the additional behaviour defined by “ $a^t.P \xrightarrow{\delta} a^t.P$ ”. That is, if  $\gamma$  is not  $\delta$  or  $\tau$ , it allows time to elapse via the explicit execution of a “tick” transition, see Tables 7 and 8. When specifying systems it is convenient to adopt such a prefix operator, because it allows visible actions to be arbitrarily delayed, so that, e.g., in  $a^t.P \|_{\{a\}} Q$ , the action  $a$  does not cause a time deadlock in the case it is not immediately executable.
- The new choice operator  $P +^t Q$  behaves like  $P + Q$  as far as standard  $\alpha$  transitions are concerned, see Table 7. Concerning “tick” transitions it behaves, instead, according to the rule in Table 8. That is, similarly as for parallel composition operator, it allows one of  $P$  and  $Q$  to let time pass only if the other one may let time pass and is defined in such a way that time passage does not resolve the choice (yielding time determinism, see below). When specifying systems it is convenient to adopt such a choice operator because it allows new prefixes  $a^t.P$  to be used without causing the delays  $\delta$  preceding the execution of the  $a$  to solve the choice.

We also use a terminated  $\underline{1}$  process that allows time to to elapse ( $\underline{0}$  instead deadlocks time).

As explained in the introduction, the idea is that such specialized “timed” prefix  $\gamma^t.P$  and choice  $P +^t Q$  together with  $\underline{0}$ ,  $\underline{1}$ , recursion, parallel composition and hiding form a *specification level* calculus (like that of [22] apart from the action synchronization model) that is used to specify timed systems. On the contrary,  $\gamma.P$  and  $P + Q$  are just used, as auxiliary operators, to express *normal forms* for them, and to produce an axiomatization by using (a variant of) the machinery presented in Section 2.

The set  $\mathcal{E}_{DT}$  of Discrete Time Calculus expressions, ranged over by  $E, F, G, \dots$ , is defined by the following syntax:

$$E ::= \underline{0} \mid \underline{1} \mid X \mid \gamma^t.E \mid E +^t E \mid E/L \mid E \|_S E \mid \text{rec}X.E$$

where  $L, S \subseteq PAct - \{\tau\}$ . The corresponding set of processes, i.e. closed terms, is denoted by  $\mathcal{P}_{DT}$ , ranged over by  $P, Q, R, \dots$ . The operational semantics of the calculus is defined as the least subset of  $\mathcal{P}_{DT} \times Act \times \mathcal{P}_{DT}$  satisfying the standard operational rules of Tables 7 and 5, and the timed ones of Tables 8 and 6. In the following we will assume  $\gamma^t.E$  prefixes to guard variables as for  $\gamma.E$  prefixes in the basic calculus, i.e. we say that a variable is guarded by  $\gamma$  if it occurs in the scope of a  $\gamma^t.E$  prefix.

As far as properties of the generated labeled transition systems are concerned, we first notice that, as for the basic calculus, *maximal progress* (Proposition 2.2) holds also for  $\mathcal{P}_{DT}$  processes (with a similar inductive proof on the inference tree of  $\tau$  transitions, with base cases:  $\tau$  transitions inferred by hiding of a visible transition and  $\tau$  transitions obtained directly by a  $\tau^t$  prefix). Moreover the effect of adopting such a specification level time calculus, is, like in [22], *time determinism*.

**Definition 3.2.** A process  $P$  is called *time-deterministic*, if, for all the states in the semantics of  $P$ , i.e. terms reachable by  $P$ , at most one outgoing  $\delta$  transition can be inferred by the operational rules.



**Proposition 3.3.** *Let  $P \in \mathcal{P}_{DT}$ . Then  $P$  is time-deterministic.*

*Proof.* We show that, for any  $P \in \mathcal{P}_{DT}$ , we have:  $P \xrightarrow{\delta} P_1 \wedge P \xrightarrow{\delta} P_2$  implies  $P_1 = P_2$  and such two transitions are obtained with exactly the same inference (tree). This is proven by induction on the maximum of the inference depth of the two transitions.

We have the following cases depending on the structure of  $P$ .

- If  $P \equiv \underline{0}$ ,  $P \equiv \underline{1}$  or  $P \equiv \gamma^t.P'$ , the condition above obviously holds.
- If  $P \equiv P'/L$ , we have that  $P'/L \xrightarrow{\delta} P_1 \wedge P'/L \xrightarrow{\delta} P_2$  implies:  $P_1 = P'_1/L$ ,  $P_2 = P'_2/L$  and  $P' \xrightarrow{\delta} P'_1 \wedge P' \xrightarrow{\delta} P'_2$ . Therefore, by applying the induction hypothesis to such a pair of transitions (having a strictly smaller maximum inference depth w.r.t. that of the pair of transitions considered for  $P'/L$ ), we have:  $P'_1 = P'_2$  and the two transitions are obtained with exactly the same inference. Hence we have  $P_1 = P_2$  and that the inference of the considered  $P'/L$  transitions, extending the inference determined by induction, is the same.
- If  $P \equiv \text{rec}X.E$ , we have that  $\text{rec}X.E \xrightarrow{\delta} P_1 \wedge \text{rec}X.E \xrightarrow{\delta} P_2$  implies:  $E\{\text{rec}X.E/X\} \xrightarrow{\delta} P_1 \wedge E\{\text{rec}X.E/X\} \xrightarrow{\delta} P_2$ . Therefore, by applying the induction hypothesis to such a pair of transitions (having a strictly smaller maximum inference depth w.r.t. that of the pair of transitions considered for  $\text{rec}X.E$ ), we directly have  $P_1 = P_2$  and that the two transitions are obtained with exactly the same inference.
- If  $P \equiv P' +^t P''$ , we have that  $P' +^t P'' \xrightarrow{\delta} P_1 \wedge P' +^t P'' \xrightarrow{\delta} P_2$  implies:  $P_1 = P'_1 +^t P''_1$ ,  $P_2 = P'_2 +^t P''_2$ ,  $P' \xrightarrow{\delta} P'_1 \wedge P' \xrightarrow{\delta} P'_2$  and  $P'' \xrightarrow{\delta} P''_1 \wedge P'' \xrightarrow{\delta} P''_2$ . Therefore, by applying the induction hypothesis to both such pairs of transitions (both having a strictly smaller maximum inference depth w.r.t. that of the pair of transitions considered for  $P' +^t P''$ ), we have:  $P'_1 = P'_2$ ,  $P''_1 = P''_2$  and, for each pair, it holds that the two transitions are obtained with exactly the same inference. Hence we have  $P_1 = P_2$  and that the inference of the considered  $P' +^t P''$  transitions, extending the inferences (of the  $P'$  and  $P''$  transitions) determined by induction, is the same.
- If  $P \equiv P' \parallel_S P''$ , we have that  $P' \parallel_S P'' \xrightarrow{\delta} P_1 \wedge P' \parallel_S P'' \xrightarrow{\delta} P_2$  implies (with exactly the same reasoning, with  $\parallel_S$  replacing  $+^t$ , as that of the previous item):  $P_1 = P_2$  and such two transitions are obtained with the same inference.  $\square$

Finally, we notice that, for the specification discrete time calculus, fully unguarded recursions cannot perform  $\delta$  transitions.

**Proposition 3.4.** *Let  $\text{rec}X.E \in \mathcal{P}_{DT}$ . If  $X$  occurs fully unguarded in  $E$  then  $\text{rec}X.E \not\xrightarrow{\delta}$ .*

*Proof.* Let us suppose, by contradiction, that  $\text{rec}X.E \xrightarrow{\delta}$ . We now show that, if we remove such a transition from the (timed) transition relation obtained as the semantics of terms, then the obtained transition relation still satisfies the operational rules, thus violating minimality. We just need to observe that, since  $X$  occurs fully unguarded in  $E \in \mathcal{E}_{DT}$ , any possible inference tree of the  $\text{rec}X.E \xrightarrow{\delta}$  transition necessarily includes the transition  $\text{rec}X.E \xrightarrow{\delta}$  itself as a premise. This is because: (i) every operator, apart from  $\underline{1}$  and prefix, generates a  $\delta$  move only if *all* of its arguments perform a  $\delta$  move, and (ii) we know that  $X$  indeed occurs in  $E$  and it occurs, being fully unguarded, not in the scope of prefix. Therefore, if we remove  $\text{rec}X.E \xrightarrow{\delta}$  from the transition relation the operational inference rules are still satisfied.  $\square$

Regarding equivalence, unfortunately, the introduction of the new choice operator  $P +^t Q$  makes standard observational congruence (with  $\delta$  being considered as a standard visible action) no longer a congruence: e.g.,  $\delta.\tau.a.\underline{0} \simeq \delta.a.\underline{0}$  but  $\delta.b.\underline{0} +^t \delta.\tau.a.\underline{0} \not\simeq \delta.b.\underline{0} +^t \delta.a.\underline{0}$ . This is because  $\delta.b.\underline{0} +^t \delta.\tau.a.\underline{0}$  is isomorphic to  $\delta.(b.\underline{0} + \tau.a.\underline{0})$ , while  $\delta.b.\underline{0} +^t \delta.a.\underline{0}$  is isomorphic to  $\delta.(b.\underline{0} + a.\underline{0})$ . The problem is that, since in  $P +^t Q$  the execution of  $\delta$  transitions does not cause the choice to be resolved, in  $P$  (as well as in  $Q$ ) it is incorrect to abstract from  $\tau$  transitions that occur before the execution of a standard  $a$  action (that cause the choice to be resolved). In other words, a finer notion of observational congruence, called *discrete time observational congruence*, must be considered that is defined similarly as in [16]: the “root” condition of the equivalence (where standard transitions are matched as in standard observational congruence) can be “left” only by executing standard transitions (and not by executing  $\delta$  transitions).

**Definition 3.5.** A relation  $\beta \subseteq \mathcal{P}_{DT} \times \mathcal{P}_{DT}$  is a rooted time weak bisimulation if, whenever  $(P, Q) \in \beta$ :

- If  $P \xrightarrow{\alpha} P'$  then, for some  $Q', Q \xrightarrow{\alpha} Q'$  and  $P' \approx Q'$ .
- If  $P \xrightarrow{\delta} P'$  then, for some  $Q', Q \xrightarrow{\delta} Q'$  and  $(P', Q') \in \beta$ .
- If  $Q \xrightarrow{\alpha} Q'$  then, for some  $P', P \xrightarrow{\alpha} P'$  and  $P' \approx Q'$ .
- If  $Q \xrightarrow{\delta} Q'$  then, for some  $P', P \xrightarrow{\delta} P'$  and  $(P', Q') \in \beta$ .

Two processes  $P, Q$  are time observationally congruent, written  $P \simeq_T Q$ , iff  $(P, Q)$  is included in some rooted time weak bisimulation. We consider  $\simeq_T$  as being defined also on open terms by means of free variable substitution, as in Definition 2.5.

Notice that, even if finer than the observational congruence defined in Definition 2.4 (where we just considered time delays as standard actions), time observational congruence is still a *conservative extension* of Milner’s observational congruence: for transition systems without timed transitions it reduces to this equivalence. In particular only the root condition is more restrictive: the notion of weak bisimulation  $\approx$  remains unchanged.

Due to the time determinism of the generated labeled transition systems, it is also possible to simplify the form of weak bisimulation definition: we can provide reformulations as those presented in Proposition 2.8 and 2.9 where *tails of  $\tau$  transitions can be disregarded* when matching weak timed moves.

**Proposition 3.6.** *In the case a relation  $\beta$  over time-deterministic processes is considered, a simplified form of the reformulations of weak bisimulation in Propositions 2.8 and 2.9 holds true, where matching  $\xrightarrow{\delta} \xrightarrow{\tau}^*$  moves are replaced by matching  $\xrightarrow{\delta}$  moves.*

*Proof.* We now show that  $\beta$  satisfying the reformulations in Propositions 2.8 and 2.9 also satisfies the simplified form (the converse is obvious). Assumed  $(P, Q) \in \beta$ , from  $P \xrightarrow{\delta} P'$  and a matching  $Q \xrightarrow{\tau}^* Q''$  such that  $Q'' \xrightarrow{\delta} Q''' \xrightarrow{\tau}^* Q'$  with  $(P', Q') \in \beta$ , we show that we also have  $(P', Q''') \in \beta$ . Since  $Q'' \xrightarrow{\delta} Q'''$  we have that, due to time determinism,  $P \xrightarrow{\delta} P' \xrightarrow{\tau}^* P''$  with  $(P'', Q''') \in \beta$ . Therefore, from  $Q''' \xrightarrow{\tau}^* Q'$  with  $(P', Q') \in \beta$  (showing that  $Q'''$  can match moves made by  $P'$ ) and  $P' \xrightarrow{\tau}^* P''$  with  $(P'', Q''') \in \beta$  (showing that  $P'$  can match moves made by  $Q'''$ ), we can conclude that  $(P', Q''') \in \beta$ .  $\square$

We will show in Theorem 3.7 in the context of a larger signature that  $\simeq_T$  is a congruence with respect to all calculus operators, including recursion.

|   |  |
|---|--|
| $\frac{P \xrightarrow{a} P'}{vis(P) \xrightarrow{a} P'}$  | $\frac{P \xrightarrow{\tau} P'' \wedge vis(P'') \xrightarrow{a} P'}{vis(P) \xrightarrow{a} P'}$                              |
| $\frac{P \xrightarrow{\alpha} P'}{P \parallel_S Q \xrightarrow{\alpha} P' \parallel_S Q} \quad \alpha \notin S$     | $\frac{vis(P) \xrightarrow{a} P' \wedge vis(Q) \xrightarrow{a} Q'}{P  _S Q \xrightarrow{a} P' \parallel_S Q'} \quad a \in S$ |
| $\frac{P \xrightarrow{\delta} P' \wedge Q \xrightarrow{\delta} Q'}{P  _S Q \xrightarrow{\delta} P' \parallel_S Q'}$ |  |

Table 9: Rules for Auxiliary Operators

**3.2. Axiomatizing the Discrete Time Calculus.** Producing a complete axiomatization for the discrete time calculus requires it to be extended so to be able to express normal forms of processes (terms of the basic calculus) and manage and derive them: e.g. the  $pri(P)$  operator and left and synchronization merge operators needed to axiomatize parallel composition, along the lines of [1]. More precisely, we use as normal forms the terms of the basic calculus that are time deterministic (Definition 3.2) and get a complete axiomatization via a variant of the machinery presented in Section 2.

**3.2.1. Extending the Discrete Time Calculus.** We now formally introduce the auxiliary operators needed to build the axiomatization, whose semantics is presented in Table 9. The operators “ $P \parallel_S Q$ ” and “ $P |_S Q$ ” are timed extensions of the left merge and synchronization merge operators of [1], where the definition of the operational rule for “ $P |_S Q$ ” allows actions  $\tau$  to be skipped so to get a congruence. In particular, here this is expressed by using the novel operator “ $vis(P)$ ” that we introduce in this paper: it turns  $\xrightarrow{\tau}^* \xrightarrow{a}$  weak transitions of  $P$  labeled by visible (non- $\tau$ ) standard actions  $a$  into strong transitions. Similarly to the standard setting of [1], the above auxiliary operators will be used for axiomatizing parallel composition. In doing this, “ $vis(P)$ ” will play an important role in that it allows us to check for the absence of executable delays and/or  $\tau$  actions (see axiom (SM7) of Table 11). As we will see, differently from the standard setting of [1], we need here to perform such a check because of priority of  $\tau$  over  $\delta$ .

We define the Extended Discrete Time calculus to be the process algebra obtained by extending the Discrete Time calculus with the auxiliary operators above, the operators of the basic calculus of Section 2 and the  $pri(\_)$  operator. The set  $\mathcal{E}_{EDT}$  of behavior expressions, ranged over by  $E, F, G, \dots$ , is defined by the following syntax:

$$\begin{aligned}
E ::= & \mathbf{0} \mid \mathbf{1} \mid X \mid \gamma^t.E \mid E +^t E \mid E/L \mid E \parallel_S E \mid recX.E \mid \\
& \gamma.E \mid E + E \mid pri(E) \mid \\
& vis(E) \mid E \parallel_S E \mid E |_S E
\end{aligned}$$

where  $L, S \subseteq PAct - \{\tau\}$ . The set of processes, i.e. closed terms, is denoted by  $\mathcal{P}_{EDT}$ , ranged over by  $P, Q, R, \dots$ . The operational semantics of processes is defined as the least subset of  $\mathcal{P}_{EDT} \times Act \times \mathcal{P}_{EDT}$  satisfying all operational rules already presented.

As far as properties of the generated labeled transition system are concerned, we obviously still have that *maximal progress* (Proposition 2.2) holds. On the contrary, differently from

the Discrete Time calculus specification calculus, in general the *time-deterministic* property does not hold (as, e.g., in  $\delta.P + \delta.Q$ ).

**Theorem 3.7.**  $\simeq_T$  is a congruence for the extended discrete time calculus with respect to all of its operators, including recursion.

*Proof.* We show congruence of  $\simeq_T$  with respect to all the operators of the extended discrete time calculus, considering the recursion operator at the end. In the proof we will often exploit the fact that standard transitions  $\alpha$  are inferred by standard transitions only. Let  $P \simeq_T Q$ .

$\gamma.P \simeq_T \gamma.Q$ , in the case  $\gamma = \delta$  (otherwise, the proof is trivial), is shown by considering the rooted time weak bisimulation  $\beta = \{(\delta.P, \delta.Q)\} \cup \simeq_T$ . Similarly,  $P + R \simeq_T Q + R$  is shown by considering the rooted time weak bisimulation  $\beta = \{(P + R, Q + R)\} \cup \simeq_T$ .

$\text{pri}(P) \simeq_T \text{pri}(Q)$  is immediate by just considering the rooted time weak bisimulation  $\{( \text{pri}(P), \text{pri}(Q) )\}$ :  $\text{pri}(P)$  cannot perform  $\delta$  transitions and standard  $\alpha$  transitions are matched by non-zero length standard weak transitions.  $\text{vis}(P) \simeq_T \text{vis}(Q)$  is similarly shown by considering the rooted time weak bisimulation  $\{( \text{vis}(P), \text{vis}(Q) )\}$ :  $\text{vis}(P)$  can perform neither  $\delta$  transitions nor  $\tau$  transitions and each standard  $a$  transition is matched by an  $a$  transition (the transition leading directly to the term  $Q''$  such that  $Q \xrightarrow{\tau} * \xrightarrow{a} Q'' \xrightarrow{\tau} * Q'$  is the weak transition matching the  $\xrightarrow{\tau} * \xrightarrow{a}$  transition of  $P$ ) possibly followed by a sequence of  $\tau$  transitions (the sequence  $Q'' \xrightarrow{\tau} * Q'$ ).

$\gamma^t.P \simeq_T \gamma^t.Q$ , in the case  $\gamma = a$  for some  $a \in PAct - \{\tau\}$  (otherwise, this case reduces to the previous case), is shown by considering the rooted time weak bisimulation  $\{(a^t.P, a^t.Q)\}$ .

$P +^t R \simeq_T Q +^t R$  is shown by considering the rooted time weak bisimulation

$$\beta = \{(P +^t R, Q +^t R) \mid P \simeq_T Q \wedge R \in \mathcal{P}_{EDT}\}$$

The proof for  $\alpha$  transitions is trivial. Concerning  $\delta$  transitions, since  $P \simeq_T Q$ , we have that for all  $P'$ : if  $P \xrightarrow{\delta} P'$  then, for some  $Q'$ ,  $Q \xrightarrow{\delta} Q'$  and  $P' \simeq_T Q'$ . Therefore for all  $P', R'$  we have: if  $P +^t R \xrightarrow{\delta} P' +^t R'$  then (from  $R \xrightarrow{\delta} R'$ ), for the above  $Q'$ ,  $Q +^t R \xrightarrow{\delta} Q' +^t R'$  and  $(P' +^t R', Q' +^t R') \in \beta$ .

$P \parallel_S R \simeq_T Q \parallel_S R$  and  $P/L \simeq_T Q/L$  are shown as for the proof of congruence of  $\approx$  with the simplification that (as for “+<sup>t</sup>” above) the possibility of silently reaching a  $Q''$  state is not considered and by resorting to congruence of  $\approx$  in the case of standard  $\alpha$  moves.

$P \parallel_S R \simeq_T Q \parallel_S R$  is immediate by just considering the rooted time weak bisimulation  $\{(P \parallel_S R, Q \parallel_S R)\}$ :  $P \parallel_S R$  cannot perform  $\delta$  transitions and standard  $\alpha$  transitions not in  $S$  (that must originate in  $P$ ) are matched by non-zero length standard weak transitions not in  $S$  (congruence of  $\approx$  with respect to  $\parallel_S$  is then exploited).

$P \mid_S R \simeq_T Q \mid_S R$  is shown by considering the rooted time weak bisimulation  $\beta = \{(P \mid_S R, Q \mid_S R)\} \cup \simeq_T$ . Concerning standard transitions,  $P \mid_S R$  can just perform (non- $\tau$ )  $a$  transitions with  $a \in S$ : each  $a$  transition originated from  $P$  is matched (as for the  $\text{vis}()$  operator) by an  $a$  transition (the transition leading directly to the term  $Q''$  such that  $Q \xrightarrow{\tau} * \xrightarrow{a} Q'' \xrightarrow{\tau} * Q'$  is the weak transition matching the  $\xrightarrow{\tau} * \xrightarrow{a}$  transition of  $P$ ) possibly followed by a sequence of  $\tau$  transitions (the sequence  $Q'' \xrightarrow{\tau} * Q'$ ); congruence of  $\approx$  with respect to “ $\parallel_S$ ” is then exploited on reached terms. Concerning  $\delta$  transitions, since  $P \simeq_T Q$ , we have that, for all  $P', R'$ : if  $P \mid_S R \xrightarrow{\delta} P' \parallel_S R'$  then (from  $R \xrightarrow{\delta} R'$ ), there exists  $Q'$  such that  $Q \mid_S R \xrightarrow{\delta} Q' \parallel_S R'$  and  $P' \parallel_S R' \simeq_T Q' \parallel_S R'$  because of congruence of  $\simeq_T$  with respect to “ $\parallel_S$ ”.

|           |  |                                    |
|-----------|--|------------------------------------|
| $(Tau1')$ | $\alpha.\tau.E = \alpha.E$                             |                                    |
| $(Tau3')$ | $\alpha.(E + \tau.F) + \alpha.F = \alpha.(E + \tau.F)$ |                                    |
| $(Tau4)$  | $\alpha.F\{\delta.\tau.E/X\} = \alpha.F\{\delta.E/X\}$ | provided that $X$ is serial in $F$ |

Table 10: Axioms of  $\mathcal{A}_{DT}$  for unguarded basic terms

Concerning the recursion operator, from  $E \simeq_T F$  we derive  $recX.E \simeq_T recX.F$  as follows. We show that

$$\beta = \{(G\{recX.E/X\}, G\{recX.F/X\}) \mid G \text{ contains at most } X \text{ free}\}$$

satisfies the conditions:

- If  $G\{recX.E/X\} \xrightarrow{\alpha} H$  then, for some  $H', H''$ ,

$$G\{recX.F/X\} \xrightarrow{\alpha} H'' \text{ with } H'' \approx H' \text{ such that } (H, H') \in \beta,$$

and symmetrically for a move of  $G\{recX.F/X\}$ .

- If  $G\{recX.E/X\} \xrightarrow{\delta} H$  then, for some  $H', H''$ ,

$$G\{recX.F/X\} \xrightarrow{\delta} H'' \text{ with } H'' \simeq_T H' \text{ such that } (H, H') \in \beta,$$

and symmetrically for a move of  $G\{recX.F/X\}$ .

This implies that  $\beta$  is a weak bisimulation up to  $\approx$ , see [33]. As a consequence (by using  $\beta \subseteq \approx$  in the first item) the relation  $\beta \simeq_T$  is a rooted discrete time weak bisimulation, hence  $\beta \subseteq \simeq_T$ . Thus, by taking  $G \equiv X$  we may conclude that  $recX.E \simeq_T recX.F$ .

The proof that  $(G\{recX.E/X\}, G\{recX.F/X\}) \in \beta$  satisfies the conditions above is by induction on the height of the inference tree by which transitions of  $G\{recX.E/X\} \xrightarrow{\gamma} H$  are inferred, first for standard  $\alpha$  transitions (which are inferred from standard transitions only) and then for  $\delta$  transitions. In both cases the induction proof is performed by considering several cases depending on the structure of  $G$ : its topmost operator. For  $\alpha$  transitions the induction hypothesis is exploited in a similar way as we did for the proof of congruence of  $\simeq_T$  for that operator. Thus, here we just detail the cases for  $\delta$  transitions.

$G$  cannot be of the form  $pri(G')$ ,  $vis(G')$  or  $G' \parallel_S G''$  because in these cases it would not be able to perform  $\delta$  transitions. If  $G$  is of the form  $\gamma.G'$ ,  $\gamma^t.G'$ ,  $G' + G''$ , then the proof is trivial.

If  $G \equiv G' \parallel_S G''$  then, from  $(G' \parallel_S G'')\{recX.E/X\} \xrightarrow{\delta} H_1 \parallel_S H_2$ , we have, from the induction hypothesis on  $G'$  and  $G''$ , that  $(G' \parallel_S G'')\{recX.F/X\} \xrightarrow{\delta} H'_1 \parallel_S H''_2$  with  $H'_1 \parallel_S H''_2 \simeq_T H'_1 \parallel_S H'_2$  (due to congruence of  $\simeq_T$  with respect to “ $\parallel_S$ ”) such that  $(H_1 \parallel_S H_2, H'_1 \parallel_S H'_2) \in \beta$ .

If  $G \equiv G'/L$  then we must have that  $G'\{recX.E/X\}$  does not perform any  $a$  transition with  $a \in L$ , otherwise it would not be possible for  $(G'/L)\{recX.E/X\}$  to perform a  $\delta$  transition to  $H/L$ . Since  $G'\{recX.F/X\}$  cannot do  $a$  transitions with  $a \in L$  because otherwise  $G'\{recX.E/X\}$  (that cannot perform  $\tau$  transitions) should perform a corresponding  $a$  transition, we have, from the induction hypothesis on  $G'$ , that  $(G'/L)\{recX.F/X\} \xrightarrow{\delta} H''/L$  with  $H''/L \simeq_T H'/L$  (due to congruence of  $\simeq_T$  with respect to “ $/L$ ”) such that  $(H/L, H'/L) \in \beta$ .

If  $G$  is of the form  $G' +^t G''$  or  $G' \mid_S G''$  then the proof is carried out like in the case of  $\parallel_S$ . The cases  $G \equiv recY.G'$  and  $G \equiv X$  are dealt with as in the standard way.  $\square$

|         |   |                                  |
|---------|---|----------------------------------|
| (Ter)   | $\underline{1} = \text{rec}X.\delta.X$                |                                  |
| (TPre1) | $\alpha^t.E = \text{rec}X.(\delta.X + \alpha.E)$      | provided that X is not free in E |
| (TPre2) | $\delta^t.E = \delta.E$                               |                                  |
| (TCh1)  | $E +^t F = F +^t E$                                   |                                  |
| (TCh2)  | $(E +^t F) +^t G = E +^t (F +^t G)$                   |                                  |
| (TCh3)  | $E +^t \underline{1} = E$                             |                                  |
| (TCh4)  | $\text{pri}(E +^t F) = \text{pri}(E) + \text{pri}(F)$ |                                  |
| (TCh5)  | $\text{pri}(E) +^t F = \text{pri}(E) + \text{pri}(F)$ |                                  |
| (TCh6)  | $(\delta.E) +^t (\delta.F) = \delta.(E +^t F)$        |                                  |
| (TCh7)  | $(E + F) +^t G = (E +^t G) + (F +^t G)$               |                                  |

Table 11: Axioms of  $\mathcal{A}_{DT}$  for timed operators

|         |   |   |
|---------|---|---|
| (Hi1)   | $\underline{0}/L = \underline{0}$   |   |
| (Hi2)   | $(\gamma.E)/L = \gamma.(E/L)$   | $\gamma \notin L$   |
| (Hi3)   | $(a.E)/L = \tau.(E/L)$  | $a \in L$   |
| (Hi4)   | $(E + F)/L = E/L + F/L$   |   |
| (RecHi) | $(\text{rec}X.E)/L = \text{rec}X.(E/L)$   | provided that X is serial in E  |
| (Vis1)  | $\text{vis}(\underline{0}) = \underline{0}$   |   |
| (Vis2)  | $\text{vis}(a.E) = a.E$   |   |
| (Vis3)  | $\text{vis}(E + F) = \text{vis}(E) + \text{vis}(F)$                                 |   |
| (Par)   | $E \parallel_S F = E \parallel_S F + F \parallel_S E + E  _S F$                     |   |
| (LM1)   | $\underline{0} \parallel_S E = \underline{0}$                                       |   |
| (LM2)   | $(\gamma.E) \parallel_S F = \underline{0}$  | $\gamma \in S \cup \{\delta\}$  |
| (LM3)   | $(\alpha.E) \parallel_S F = \alpha.(E \parallel_S F)$                               | $\alpha \notin S$   |
| (LM4)   | $(E + F) \parallel_S G = E \parallel_S G + F \parallel_S G$                         |   |
| (SM1)   | $E  _S F = F  _S E$   |   |
| (SM2)   | $\underline{0}  _S E = \underline{0}$   |   |
| (SM3)   | $(\gamma.E)  _S (\gamma'.F) = \underline{0}$  | $(\gamma \notin S \cup \{\delta\} \vee \gamma \neq \gamma') \wedge \tau \notin \{\gamma, \gamma'\}$ |
| (SM4)   | $(\tau.E)  _S F = \text{pri}(E  _S F)$  |   |
| (SM5)   | $(\gamma.E)  _S (\gamma.F) = \gamma.(E \parallel_S F)$                              | $\gamma \in S \cup \{\delta\}$  |
| (SM6)   | $(\text{pri}(E) + \text{pri}(F))  _S G = \text{pri}(E)  _S G + \text{pri}(F)  _S G$ |   |
| (SM7)   | $(\delta.E + \text{vis}(F))  _S G = \delta.E  _S G + \text{vis}(F)  _S G$           |   |

Table 12: Axioms of  $\mathcal{A}_{DT}$  for parallel composition, hiding and related auxiliary operators

3.2.2. *Axiom System.* We now present the axiom system  $\mathcal{A}_{DT}$ . The idea is that the axiom system must be able: (i) to turn  $\mathcal{P}_{DT}$  terms of the specification calculus into normal form, i.e. basic processes in  $\mathcal{P}$  that are time-deterministic, and (ii) to equate normal forms when they are equivalent according to  $\simeq_T$ .

$\mathcal{A}_{DT}$  is composed of:

- the axioms in Tables 4 and 10, related to axiomatizing basic processes, where the axioms in Table 10 that are primed replace the corresponding axioms in Table 4 (reflecting the modification in the equivalence, i.e. we are axiomatizing  $\simeq_T$  instead of  $\simeq$ );

- the axioms in Table 11 related to axiomatizing timed operators (make it possible to eliminate them, so to obtain normal forms); and
- the axioms in Table 12 related to axiomatizing parallel composition and hiding operators (eliminating such operators via auxiliary operators and dynamically generated unguardedness via the (*RecHi*) axiom taken from [4, 5], so to obtain normal forms).

Concerning Table 10, due to the more restrictive root of equivalence, axioms (*Tau1*) and (*Tau3*) are restricted to  $\gamma = \alpha \in PAct$ , instead of a general  $\gamma$  action including  $\sigma$ , and a specific  $\tau$  elimination axiom (*Tau4*) for time is added, i.e. a restricted version of old axiom (*Tau1*) in the case  $\gamma = \sigma$  that now we have excluded. Notice that variable replacement in axiom (*Tau4*) is needed in order to deal with the case that  $\delta.\tau.E$  occurs inside a recursion in  $F$ . Moreover the “serial” condition in (*Tau4*) is needed because, e.g.,  $\alpha.(\delta.\tau.E +^t G)$  is not equivalent to  $\alpha.(\delta.E +^t G)$ .

Concerning Table 11, we just include axioms for timed operators that are actually needed to prove our completeness result for the discrete time calculus. As a matter of fact, we could have considered other axioms as, e.g.,  $recX.(X +^t E) = recX.pri(E)$ , which allows fully unguarded recursion to be removed in the case of timed choice (such an axiom is sound in that  $recX.(X +^t E)$  cannot perform  $\delta$  transitions, as it can be shown with a proof similar to that of Proposition 3.4).

Concerning Table 12, the axioms are standard, apart from the usage of the auxiliary operator  $vis(\_)$  to deal with distributivity of synchronization merge: we cannot just distribute with the standard axiom  $(E + F) |_S G = E |_S G + F |_S G$  because, e.g.,  $(\tau.E + \delta.F) |_S G$  is not equivalent to  $\tau.E |_S G + \delta.F |_S G$ , due to priority of  $\tau$  over  $\delta$ . Moreover, as explained in the introduction, we make use of the axiom (*RecHi*) taken from [4, 5] for eliminating unguardedness dynamically generated by the hiding operator. Notice that hiding here also has a prioritization effect: this makes it essential to resort to the axiomatization presented in Section 2 that enacts priority of  $\tau$  over  $\delta$ . Also in the case of Table 12 we just include axioms that we need in order to prove our completeness result. For this reason auxiliary operator axioms such as, e.g.,  $vis(\tau.E) = vis(E)$  and  $vis(\delta.E) = \underline{0}$ , are omitted.

The axiom system is sound with respect to  $\simeq_T$  extended to open terms.

**Theorem 3.8.** *Given  $E, F \in \mathcal{E}_{EDT}$ , if  $\mathcal{A}_{DT} \vdash E = F$  then  $E \simeq_T F$ .*

*Proof.* The soundness of the new axiom (*Tau4*) is proved by just showing that

$\beta = \{(G\{\delta.\tau.E/X\}, G\{\delta.E/X\}) \mid G \text{ contains at most } X \text{ free and } X \text{ is serial in } G\} \cup \{(\tau.E, E)\}$  is a weak bisimulation. This holds because, for corresponding transitions,  $G\{\delta.\tau.E/X\}$  and  $G\{\delta.E/X\}$ , with  $X$  serial in  $G$ , either reach related terms ( $G'\{\delta.\tau.E/X\}$  and  $G'\{\delta.E/X\}$ ), respectively, for some  $G'$  such that  $G'$  contains at most  $X$  free and  $X$  is serial in  $G'$ , or reach related terms  $(\tau.E, E)$ .

The soundness of the (*RecHi*) axiom is shown as in [5]. The proof of soundness for the other axioms is standard, see [1]. In particular, concerning axiom (*Par*), notice that the semantics of synchronization merge “ $|_S$ ” follows a standard approach, in that: inferring  $\xrightarrow{a}$  transitions using the new  $vis(\_)$  operator is like directly inferring them from  $\xrightarrow{\tau}^* \xrightarrow{a}$  transitions.  $\square$

**3.2.3. Completeness for Time-Deterministic Basic Processes.** Completeness of the  $\mathcal{A}_{DT}$  axiomatization is proven by resorting to equation sets. In particular, we have to introduce the subclass of *time-deterministic* ones.

**Definition 3.9.** Let  $S = \{X_i = H_i \mid 1 \leq i \leq n\}$ , with formal variables  $\tilde{X} = \{X_1, \dots, X_n\}$ , where  $X_1$  is the distinguished variable of  $S$ , and free variables  $\tilde{W} = \{W_1, \dots, W_m\}$  be a standard equation set and let us suppose each expression  $H_i$  ( $1 \leq i \leq n$ ) to be denoted by

$$H_i \equiv \sum_{j \in J_i} \gamma_{i,j} \cdot X_{f(i,j)} + \sum_{k \in K_i} W_{g(i,k)}.$$

$S$  is *time-deterministic* if it holds that, for all  $i$ , we have

$$\forall j, j' \in J_i. \gamma_{i,j} = \gamma_{i,j'} = \delta \Rightarrow j = j'$$

$S$  is *well-rooted* if it holds that, for all  $i$ , we have

- If  $X_i \in \tilde{X}_R$  then  $\forall j \in J_i. \gamma_{i,j} \neq \delta \Rightarrow X_{f(i,j)} \notin \tilde{X}_R$
- If  $X_i \notin \tilde{X}_R$  then  $\forall j \in J_i. X_{f(i,j)} \notin \tilde{X}_R$

where  $\tilde{X}_R$  is the set of *root variables* of  $S$  defined by  $\tilde{X}_R = \{X_i \mid X_1 \xrightarrow{\delta}_{\mathcal{S}}^* X_i\}$ .

**Proposition 3.10.** *Let expression  $E \in \mathcal{E}$  provably satisfy a standard equation set  $S$ . Then there exists a well-rooted standard equation set  $S'$  provably satisfied by  $E$ . Moreover, if  $S$  is time-deterministic, prioritized, guarded and closed, then  $S'$  is time-deterministic, prioritized, guarded and closed.*

*Proof.* We first consider new variables  $X'_i$ , one for each variable  $X_i \in \tilde{X}_R$ , and new equations  $X'_i = H'_i$  where  $H'_i$  is obtained from the term  $H_i$  such that  $X_i = H_i$  by replacing each occurrence of a variable  $X_j \in \tilde{X}_R$  with  $X'_j$ . The idea is that, once the root is left,  $X'_i$  variables are used to represent the same behavior as that of  $X_i$  variables. Then we modify: the equations  $X_i = H_i$  for the variables  $X_i \in \tilde{X}_R$  by replacing each occurrence of  $\gamma \cdot X_j$  in  $H_i$ , for any  $\gamma \neq \delta$  and  $X_j \in \tilde{X}_R$ , with  $\gamma \cdot X'_j$ ; the equations  $X_i = H_i$  for the variables  $X_i \notin \tilde{X}_R$  by replacing each occurrence of a variable  $X_j \in \tilde{X}_R$  in  $H_i$  with  $X'_j$ . It is immediate to verify that the new equation system has the same root variables  $\tilde{X}_R$  as the original one (because  $\delta$  prefixed variables in the equations of root variables are not modified) and that, by construction, the two statements in the definition of “well-rooted” standard equation systems hold true.  $\square$

Theorems 2.19 and 2.20 concerning representability and one and only one solution still hold with exactly the same proofs (the change from  $\simeq$  to  $\simeq_T$  does not affect the transforming expressions into equation sets and vice-versa). Here we just have to additionally show that the standard solution of a time-deterministic guarded equation set is time-deterministic and that any time-deterministic guarded basic process can be represented by a time-deterministic equation set.

**Theorem 3.11** (time-deterministic solution). *If  $S$  is a time-deterministic, prioritized, standard, guarded and closed equation set, then there is a time-deterministic guarded process  $P \in \mathcal{P}$  which provably satisfies  $S$ .*

*Proof.* The inductive procedure for building  $P$  is that presented in the proof of Theorem 2.20, in turn taken from [29]. In addition here we show that such  $P$  is time-deterministic. We first observe that, since  $S$  is standard and closed, the equation of any of its variables  $X$  is of the form  $X = \sum_{j \in J} \gamma_j \cdot X_j$ , for some variables  $X_j$ , with  $j \in J$ , all having a defining equation in  $S$ .



We now show that every process  $P'$  reachable from  $P$  is an expansion of some variable  $X$  of  $S$ : in short, an  $X$ -expansion. In general an expression  $E$  is an  $X$ -expansion if the following holds: if  $X = \sum_{j \in J} \gamma_j.X_j$  is the defining equation for  $X$  in  $S$ ,  $E \equiv \text{rec}X. \sum_{j \in J} \gamma_j.E_j$  for some expressions  $\{E_j \mid j \in J\}$  such that, for each  $j \in J$ , either  $E_j \equiv X_j$  or  $E_j$  is, itself, an  $X_j$ -expansion. It is immediate to show that the inductive procedure for building  $P$  yields an  $X$ -expansion, with  $X$  being the distinguished (first) variable of  $S$ . Moreover, taken any process  $P'$  that is an  $X$ -expansion for some  $X$  in  $S$ , with  $X = \sum_{j \in J} \gamma_j.X_j \in S$  and  $P' \equiv \text{rec}X. \sum_{j \in J} \gamma_j.E_j$ , we have that process  $P''$  reached from  $P'$  with the  $\gamma_j$  transition is an  $X_j$ -expansion.

Therefore, since  $S$  is time-deterministic, then  $P$  is time-deterministic.  $\square$

**Theorem 3.12** (time-deterministic representability). *Every time-deterministic guarded process  $P \in \mathcal{P}$  provably satisfies a time-deterministic, prioritized, standard, guarded and closed equation set  $S$ .*

*Proof.* Let  $P_1 \dots P_n$  be the states of the transition system of  $P \equiv P_1$ . By applying axiom (Rec1) and the (Pri) axioms, for each  $i \in \{1 \dots n\}$ , there exist  $m_i$ ,  $\{\gamma_j^i\}_{j \leq m_i}$  (denoting actions),  $\{k_j^i\}_{j \leq m_i}$  (denoting natural numbers) s.t. we can derive  $P_i = \sum_{j \leq m_i} \gamma_j^i.P_{k_j^i}$ , where “ $\xrightarrow{\gamma_j^i} P_{k_j^i}$ ”, with  $j \leq m_i$ , are the outgoing transitions of  $P_i$  (no outgoing transitions corresponds to the sum being 0). Hence we can characterize the behavior of  $P_1$  by means of a time-deterministic, prioritized, standard, guarded and closed equation set  $S = \{X_i = H_i \mid 1 \leq i \leq n\}$  where  $H_i \equiv \sum_{j \leq m_i} \gamma_j^i.X_{k_j^i}$  and  $P_1 \dots P_n$  are a solution of the equation set. Such an equation set is guarded because the arguments of the sums in the equations are the outgoing transitions of the states of  $P$  and  $P$  is a guarded basic processes, hence every cycle in its transition system contains at least one non- $\tau$  action.  $\square$

**Definition 3.13.** An  $\alpha$ -saturated standard equation set  $S$  with formal variables  $\tilde{X}$  is a standard equation set such that, for all  $X \in \tilde{X}$ , items (i) and (iii) of Definition 2.21 hold.

**Lemma 3.14.** *Let expression  $E \in \mathcal{E}$  provably satisfy  $S$ , time-deterministic, prioritized, standard and guarded. Then there is an  $\alpha$ -saturated time-deterministic, prioritized, standard and guarded equation set  $S'$  provably satisfied by  $E$ . Moreover  $S'$  is well-rooted and closed if  $S$  is well-rooted and closed.*

*Proof.* Since the saturation involves only standard  $\alpha$  actions, we can derive  $S'$  by following the same procedure as that in [29] to saturate  $X \xrightarrow{\tau}_{S^*} \xrightarrow{\alpha}_S \xrightarrow{\tau}_{S^*} X'$  in standard equation sets. Hence, the procedure involves the axioms (A1) – (A4) and (Tau1'), (Tau2), (Tau3'), i.e. the axioms that correspond to standard axioms when just dealing with (saturation of) standard actions, plus the (Pri) axioms that are needed in order to remove unwanted instances of  $\delta$  prefixes produced by using (Tau2).  $\square$

We now introduce the novel concept of  $\tau$ -saturation of (time-deterministic, prioritized, standard and closed) equation sets  $S$ .  $\tau$ -saturation transforms  $\delta.X$  terms occurring in the definition of a non-root variable of  $S$  into  $\delta.\tau.X$  terms. Such a transformation preserves the equation set solutions due to axiom (Tau4), which makes it possible to apply it inside recursions.  $\tau$ -saturation will play a fundamental role in the equation set mergeability Theorem 3.17 in that it allows us to eliminate, via axiom (Tau1'),  $\tau$  prefixes occurring inside terms used to replace  $X$  in  $\delta.\tau.X$ .

**Definition 3.15.** Let  $S = \{X_i = H_i \mid 1 \leq i \leq n\}$  be a time-deterministic, prioritized, standard and closed equation set with formal variables  $\tilde{X} = \{X_1, \dots, X_n\}$ . Moreover assume expressions  $H_i$  ( $1 \leq i \leq n$ ) to be denoted by:

$$H_i \equiv \sum_{j \in J_i} \gamma_{i,j} \cdot X_{f(i,j)}$$

$\tau$ -saturation of  $S$  yields the equation set  $S' = \{X_i = H'_i \mid 1 \leq i \leq n\}$  with formal variables  $\tilde{X}$  and with

$$H'_i \equiv \sum_{j \in J_i} \gamma_{i,j} \cdot G_{i,j}$$

where:  $G_{i,j} \equiv \tau.X_{f(i,j)}$  if  $X_i \notin \tilde{X}_R$  and  $\gamma_{i,j} = \delta$ ;  $G_{i,j} \equiv X_{f(i,j)}$  otherwise.

**Lemma 3.16.** *Let process  $P \in \mathcal{P}$  provably satisfy  $S$  time-deterministic, prioritized, standard, guarded and closed. Then  $P$  provably satisfies  $S'$  obtained by  $\tau$ -saturating  $S$ .*

*Proof.* We assume, without loss of generality, that the root variables of  $S$  and  $S'$  are the first ones in the index variable ordering (if that is not the case we simply re-order variables in the same way inside both  $S$  and  $S'$  and, obviously, since the initial variable is not involved in such a re-ordering, any term that provably satisfies an equation set still satisfies a re-ordered one and vice-versa).

Let  $Q$  be the standard solution of  $S$  and  $Q'$  be the standard solution of  $S'$ . Since  $S'$  is obtained by  $\tau$ -saturating  $S$ ,  $S$  and  $S'$  just differ for the fact that, in the definition of non-root variables,  $\delta.R$  summands are replaced by  $\delta.\tau.R$  summands. As a consequence, according to the inductive procedure in [28] for deriving the standard solution from a guarded standard equation set (here reported in the proof of Theorem 2.20), we have that  $Q$  and  $Q'$  just differ for the fact that: in subterms  $recX.E$ , where  $X$  is a non-root variable of  $S$ ,  $\delta.R$  summands of the sum  $E$  are replaced by  $\delta.\tau.R$  summands. It is easy to see, by contradiction, that such subterms  $recX.E$  must be in the scope of an  $\alpha$  prefix operator. If that was not the case, i.e. if  $recX.E$  was only in the scope of  $\delta$  prefixes (or not in the scope of any prefix), it would imply that  $X$  is a root variable: since  $Q, Q'$  are built just by variable replacement, in  $S, S'$  variable  $X$  would be reachable from the initial variable by performing  $\delta$  steps only (traversing the variables  $Y$  such that  $recX.E$  is in the scope of  $recY._$ , in the outside-inside order).

Therefore, by multiple applications of the axiom (*Tau4*) we have that  $Q = Q'$  and, since  $P = Q$ , for unique solution of guarded equation sets, we are finished.  $\square$

**Theorem 3.17** (mergeability). *Let process  $P \in \mathcal{P}$  provably satisfy  $S$ , and process  $Q \in \mathcal{P}$  provably satisfy  $T$ , where both  $S$  and  $T$  are time-deterministic, prioritized, standard, guarded and closed sets of equations, and let  $P \simeq Q$ . Then there is a time-deterministic, prioritized, standard, guarded and closed equation set  $U$  provably satisfied by both  $P$  and  $Q$ .*

*Proof.* We may suppose that  $S$  is the time-deterministic prioritized standard guarded and closed equation set  $S : \tilde{X} = \tilde{H}$  and  $T$  is the time-deterministic prioritized standard guarded and closed equation set  $T : \tilde{Y} = \tilde{J}$  where  $\tilde{X} = \{X_1, \dots, X_m\}$  and  $\tilde{Y} = \{Y_1, \dots, Y_n\}$  are disjoint sets of formal variables. We can assume that both  $S$  and  $T$  are both well-rooted and  $\alpha$ -saturated because of Proposition 3.10 and Lemma 3.14.

Since  $P \simeq Q$  and  $S$  and  $T$  are  $\alpha$ -saturated, from Propositions 2.9 and 3.6 we derive that there exists a relation  $\beta \subseteq \tilde{X} \times \tilde{Y}$  such that:

1. Whenever  $(X, Y) \in \beta$ :
  - (i) If  $X \xrightarrow{\alpha}_S X'$  then, either (A)  $\alpha = \tau$  and  $(X', Y) \in \beta$ , or (B) for some  $Y'$ ,  $Y \xrightarrow{\alpha}_T Y'$  and  $(X', Y') \in \beta$ .
  - (ii) If  $X \xrightarrow{\delta}_S X'$  then, either (A)  $Y \xrightarrow{\tau}_T$ , or (B) for some  $Y'$ ,  $Y \xrightarrow{\delta}_T Y'$  and  $(X', Y') \in \beta$ .
  - (iii) If  $Y \xrightarrow{\alpha}_T Y'$  then, either (A)  $\alpha = \tau$  and  $(X, Y') \in \beta$ , or (B) for some  $X'$ ,  $X \xrightarrow{\alpha}_S X'$  and  $(X', Y') \in \beta$ .
  - (iv) If  $Y \xrightarrow{\delta}_T Y'$  then, either (A)  $X \xrightarrow{\tau}_S$ , or (B) for some  $X'$ ,  $X \xrightarrow{\delta}_S X'$  and  $(X', Y') \in \beta$ .
2.  $(X_1, Y_1) \in \beta$ , and when  $(X, Y) \in \tilde{X}_R \times \tilde{Y}_R$  then cases (i)(A), (ii)(A), (iii)(A) and (iv)(A) do not apply.

Notice that, due to condition 2 and to the fact that  $S$  and  $T$  are both well-rooted, we can assume  $\beta$  to be such that  $\beta \subseteq \tilde{X}_R \times \tilde{Y}_R \cup (\tilde{X} - \tilde{X}_R) \times (\tilde{Y} - \tilde{Y}_R)$ .

We now build the new equation set  $U : \tilde{Z} = \tilde{K}$ , where  $\tilde{Z}$  is the set of variables  $\tilde{Z} = \{Z_{i,j} \mid (X_i, Y_j) \in \beta\}$  and  $\tilde{K}$  is the set of expressions  $\tilde{K} = \{K_{i,j} \mid (X_i, Y_j) \in \beta\}$  defined as follows. Each expression  $K_{i,j}$  is a sum containing the terms:

- (i)  $\alpha.Z_{k,l}$ , whenever  $X_i \xrightarrow{\alpha}_S X_k$  and  $Y_j \xrightarrow{\alpha}_T Y_l$  and  $(X_k, Y_l) \in \beta$ .
- (ii)  $\tau.Z_{k,j}$ , whenever  $X_i \xrightarrow{\tau}_S X_k$  and there is no  $l$  such that:  $Y_j \xrightarrow{\tau}_T Y_l$  and  $(X_k, Y_l) \in \beta$ . As a consequence we must have that  $(X_k, Y_j) \in \beta$ .
- (iii)  $\tau.Z_{i,l}$ , whenever  $Y_j \xrightarrow{\tau}_T Y_l$  and there is no  $k$  such that:  $X_i \xrightarrow{\tau}_S X_k$  and  $(X_k, Y_l) \in \beta$ . As a consequence we must have that  $(X_i, Y_l) \in \beta$ .
- (iv)  $\delta.Z_{k,l}$ , whenever  $X_i \xrightarrow{\delta}_S X_k$  and  $Y_j \xrightarrow{\delta}_T Y_l$  and  $(X_k, Y_l) \in \beta$ .

Notice that  $U$  is clearly closed, standard and prioritized and it is also guarded: as in the standard case, by contradiction, any  $\tau$ -cycle  $Z_{i,j} \xrightarrow{\tau}_U Z_{i,j}$  would imply either a  $\tau$ -cycle  $X_i \xrightarrow{\tau}_S X_i$  or a  $\tau$ -cycle  $Y_j \xrightarrow{\tau}_T Y_j$ . Moreover, notice that at most one term of the kind (iv) can occur due to the fact that  $S$  and  $T$  are time-deterministic, hence also  $U$  is time-deterministic. Finally, notice that, due to condition 2 over the relation  $\beta$  and to the fact that  $S$  and  $T$  are both well-rooted, we have that  $\tilde{Z}_R = \{Z_{i,j} \mid (X_i, Y_j) \in \beta \cap (\tilde{X}_R \times \tilde{Y}_R)\}$  and that  $U$  is well-rooted as well.

We now show that  $P$  provably satisfies the equation set  $U$  (with  $Z_{1,1}$  as distinguished variable). To do this, we consider the equation sets  $S' : \tilde{X} = \tilde{H}'$ ,  $T' : \tilde{Y} = \tilde{J}'$  and  $U' : \tilde{Z} = \tilde{K}'$  obtained by  $\tau$ -saturating  $S$ ,  $T$ , and  $U$  respectively. We also suppose that  $\tilde{P} = \{P_1, \dots, P_m\}$ , with  $P_1 \equiv P$ , and  $\tilde{Q} = \{Q_1, \dots, Q_n\}$ , with  $Q_1 \equiv Q$ , both with free variables in  $\tilde{W}$ , are such that  $\mathcal{A} \vdash \tilde{P} = \tilde{H}'\{\tilde{P}/\tilde{X}\}$  and  $\mathcal{A} \vdash \tilde{Q} = \tilde{J}'\{\tilde{Q}/\tilde{Y}\}$ .

We choose expressions  $\tilde{G} = \{G_{i,j} \mid (X_i, Y_j) \in \beta\}$  as

$$G_{i,j} \equiv \begin{cases} \tau.P_i & \text{if } Z_{i,j} \xrightarrow{\tau}_U Z_{i,l} \text{ for some } l \\ P_i & \text{otherwise} \end{cases}$$

and in the following we show that  $\mathcal{A} \vdash \tilde{G} = \tilde{K}'\{\tilde{G}/\tilde{Z}\}$ . Notice that for  $G_{i,j}$  such that  $Z_{i,j} \in \tilde{Z}_R$  we have  $G_{i,j} \equiv P_i$  because the expression  $K_{i,j}$  cannot include terms of the kind (iii). In particular  $G_{1,1} \equiv P_1 \equiv P$ .

Since  $U$  is guarded (hence also  $U'$  is guarded) we can conclude that any solution of  $U$  (being it also a solution for  $U'$  which is guarded) is provably equal to  $P$ . Hence  $P$  is a solution for  $U$ .

For each equation  $G_{i,j} = K'_{i,j}\{\tilde{G}/\tilde{Z}\}$  we have the following two cases:

- $Z_{i,j} \xrightarrow{\tau} U Z_{i,l}$  for any  $l$ , hence  $G_{i,j} \equiv P_i$ . In this case we have the two following subcases:
  - $X_i \xrightarrow{\delta} S$ . In this case we have (since the standard equation set  $S$  is prioritized)  $X_i \xrightarrow{\tau} S$ . Since  $K'_{i,j}$  does not include terms of the kind (iii), this implies  $Y_j \xrightarrow{\tau} T$ . Therefore the structure of  $K'_{i,j}$  is such that it may contain only terms of kind (i) and (iv), where: terms of kind (i) cannot be  $\tau$  prefixes and exactly one term of kind (iv) must be present because (since  $Y_j \xrightarrow{\tau} T$ ) the case (ii)(A) does not apply. Due to the properties of relation  $\beta$  the following holds for the terms included in  $K'_{i,j}\{\tilde{G}/\tilde{Z}\}$ . The terms of kind (i), which become, by possibly using axiom  $(Tau1')$ , terms  $a.P_k$  for some  $k$ , are exactly (with possible repetitions) the terms for which  $X_i \xrightarrow{a} S X_k$ . Moreover, concerning the term of kind (iv), we have the following two cases. If  $Z_{i,j} \in \tilde{Z}_R$  then such a term is  $\delta.P_k$  for some  $k$  (since the  $\delta$  prefixed variable again belongs to  $\tilde{Z}_R$  we are guaranteed it is not replaced by  $\tau.P_k$ ): it is exactly the term for which  $X_i \xrightarrow{\delta} S X_k$ . Otherwise, such a term, due to  $\tau$ -saturation, is a  $\delta$  prefix followed by  $\tau$  and it becomes, by possibly using axiom  $(Tau1')$ , term  $\delta.\tau.P_k$  for some  $k$ : it is exactly the term for which  $X_i \xrightarrow{\delta} S X_k$ . Hence, by using axiom (A3) to deal with repetitions and by applying the hypothesis we derive  $K'_{i,j}\{\tilde{G}/\tilde{Z}\} = H'_i\{\tilde{P}/\tilde{X}\} = P_i$ .
  - $X_i \xrightarrow{\delta} S$ . In this case we have that also  $Z_{i,j} \xrightarrow{\delta} U$ , hence the structure of  $K'_{i,j}$  is such that it may contain only terms of kind (i) and (ii). Therefore this case works exactly as in the untimed case of [29]: due to the properties of relation  $\beta$  the following hold. By using a similar reasoning as in the first subcase, terms of kind (i) and (ii) become, by possibly using axiom  $(Tau1')$ , exactly as prefixes included in the summation  $H'_i\{\tilde{P}/\tilde{X}\}$  (with possible repetitions). Hence, by using axioms (A3) to deal with repetitions and by applying the hypothesis we derive  $K'_{i,j}\{\tilde{G}/\tilde{Z}\} = H'_i\{\tilde{P}/\tilde{X}\} = P_i$ .
- $Z_{i,j} \xrightarrow{\tau} U Z_{i,l}$  for some  $l$ , hence  $G_{i,j} \equiv \tau.P_i$ . In this case we have that the structure of  $K'_{i,j}$  is such that it may contain only terms of kind (i), (ii) and (iii). Due to the properties of relation  $\beta$  the following holds for the terms included in  $K'_{i,j}\{\tilde{G}/\tilde{Z}\}$ . Every term of kind (iii) becomes, by possibly using axiom  $(Tau1')$ ,  $\tau.P_i$ . Moreover (by using a similar reasoning as in the first subcase) the terms of kind (i) and (ii) become, by possibly using axiom  $(Tau1')$ , exactly as the non- $\delta$  prefixes included in the summation  $H_i\{\tilde{P}/\tilde{X}\}$  (with possible repetitions). Hence, by using axioms (A3), (Pri3) and (Pri6) that allow  $\delta$  prefixes to be removed in the presence of a  $\tau$  alternative, we derive  $K'_{i,j}\{\tilde{G}/\tilde{Z}\} = \tau.P_i + H_i\{\tilde{P}/\tilde{X}\}$ . By applying the hypothesis and by using axiom  $(Tau2)$  we have that  $\tau.P_i + H_i\{\tilde{P}/\tilde{X}\} = \tau.P_i + P_i = \tau.P_i$ .

In a completely symmetrical way we can also show that  $Q$  provably satisfies  $U$ . □

Hence we have proved completeness over time-deterministic guarded basic processes.

**Theorem 3.18.** *Let  $P, Q \in \mathcal{P}$  be time-deterministic guarded processes. If  $P \simeq_T Q$  then  $\mathcal{A}_{DT} \vdash P = Q$ .*

Finally, by an analogous of Theorem 2.26, we get also completeness over unguarded time-deterministic basic processes.

**Lemma 3.19.** *For each time-deterministic process  $P \in \mathcal{P}$  there exists a time-deterministic guarded process  $P' \in \mathcal{P}$  such that  $\mathcal{A} \vdash P = P'$ .*

*Proof.* Same proof as that of Theorem 2.26 with the additional observation that the transformations performed preserve time-determinism.  $\square$

**Theorem 3.20.** *Let  $P, Q \in \mathcal{P}$  be time-deterministic processes. If  $P \simeq_T Q$  then  $\mathcal{A}_{DT} \vdash P = Q$ .*

3.2.4. *Completeness for the Discrete Time Calculus.* We first introduce, along the lines of [4, 5], a syntactical characterization that guarantees processes to be finite-state.

**Definition 3.21.**  $\mathcal{E}_{DT}^{fs}$  is the set of expressions  $E \in \mathcal{E}_{DT}$  such that: for any subterm  $E'$  of  $E$ , every free occurrence of a variable  $X$  does not appear in  $E'$  in the scope of a static operator, i.e. “ $\_ \parallel_S \_$ ” or “ $\_/L$ ”, and, if it appears in the scope of a “ $\_ +^t \_$ ” operator, it is guarded inside such an operator by a standard action  $\alpha$ .  $\mathcal{P}_{DT}^{fs}$  is the set of closed  $\mathcal{E}_{DT}^{fs}$  expressions.

**Lemma 3.22.** *Let  $P', P'' \in \mathcal{P}$  be time-deterministic guarded processes.  $P \equiv P' \parallel_S P''$ ,  $P \equiv P' |_S P''$ ,  $P \equiv P' \parallel_S P''$ ,  $P \equiv P'/L$  and  $P \equiv P' +^t P''$  can be turned by the axiom system  $\mathcal{A}_{DT}$  into the form  $\sum_{1 \leq i \leq k} \gamma_i.P_i$ , where  $k \geq 0$  ( $k = 0$  corresponds to the sum being  $\underline{0}$ ) and there exists at most one  $i$ , with  $1 \leq i \leq k$ , such that  $\gamma_i = \delta$ . Moreover,  $\{(\gamma_i, P_i) \mid 1 \leq i \leq k\} = \{(\gamma, Q) \mid P \xrightarrow{\gamma} Q\}$ .<sup>3</sup>*

*Proof.* By means of axiom (Rec1) and (Pri) axioms it is immediate to show that  $P' = P'_{next} \equiv \sum_{1 \leq i \leq n} \gamma'_i.P'_i$  and  $P'' = P''_{next} \equiv \sum_{1 \leq i \leq m} \gamma''_i.P''_i$ , where  $n, m \geq 0$  and there exists at most one  $i$ , with  $1 \leq i \leq n$ , such that  $\gamma_i = \delta$  and at most one  $j$ , with  $1 \leq j \leq m$ , such that  $\gamma_j = \delta$ . Moreover,  $\{(\gamma'_i, P'_i) \mid 1 \leq i \leq n\} = \{(\gamma, Q) \mid P' \xrightarrow{\gamma} Q\}$  and  $\{(\gamma''_i, P''_i) \mid 1 \leq i \leq m\} = \{(\gamma, Q) \mid P'' \xrightarrow{\gamma} Q\}$ .

The cases of left merge, i.e.  $P \equiv P' \parallel_S P''$ , hiding, i.e.  $P \equiv P'/L$ , are proved by using the corresponding axioms (that are standard) to perform the following transformations:<sup>4</sup>

$$P' \parallel_S P'' = P'_{next} \parallel_S P'' = \sum_{i \leq n} (\gamma'_i.P'_i \parallel_S P'') = \sum_{i \leq n, \gamma'_i \notin S \cup \{\delta\}} \gamma'_i.(P'_i \parallel_S P'')$$

$$P'/L = P'_{next}/L = \sum_{i \leq n} (\gamma'_i.P'_i)/L = \sum_{i \leq n, \gamma'_i \in L} \tau.(P'_i/L) + \sum_{i \leq n, \gamma'_i \notin L} \gamma'_i.(P'_i/L)$$

We now consider the case of synchronization merge, i.e.  $P \equiv P' |_S P''$ . We initially have:

$$P' |_S P'' = P'_{next} |_S P''_{next} = \sum_{i \leq n} (\gamma'_i.P'_i |_S P''_{next}) = \sum_{i \leq n} \left( \sum_{j \leq m} (\gamma'_i.P'_i |_S \gamma''_j.P''_j) \right)$$

where the second equality is obtained as follows. First of all we observe that, for any (possibly empty) set of terms  $Q_i$  and visible actions  $a_i$ , with  $i \leq h$ , we have that  $\sum_{i \leq h} a_i.Q_i = \text{vis}(\sum_{i \leq h} a_i.Q_i)$  by applying axioms (Vis1 – 3). Similarly, for any (possibly empty) set of

<sup>3</sup>Since processes  $P_i$  and labels  $\gamma_i$ , with  $1 \leq i \leq k$ , are those such that  $P \xrightarrow{\gamma_i} P_i$  this implies that: if  $\gamma_h = \delta$  for some  $h$ , then  $\gamma_i \neq \tau$  for all  $i$ , with  $1 \leq i \leq k$ .

<sup>4</sup>We assume prefix to take precedence over  $\parallel_S$ ,  $|_S$  and  $\parallel_S$  operators, when writing terms.

terms  $Q_i$  and standard actions  $\alpha_i$ , with  $i \leq h$ , we have that  $\sum_{i \leq h} \alpha_i \cdot Q_i = \text{pri}(\sum_{i \leq h} \alpha_i \cdot Q_i)$  by applying axioms (*Pri1* – 2) and (*Pri4*). We have two cases for the structure of  $P'_{\text{next}}$ . If there exists  $k \leq n$  such that  $\gamma'_k = \delta$  then we have:  $\sum_{i \leq n, i \neq k} \gamma'_i \cdot P'_i = \text{vis}(\sum_{i \leq n, i \neq k} \gamma'_i \cdot P'_i)$ , hence  $P'_{\text{next}} |S P''_{\text{next}} = \delta \cdot P'_k |S P''_{\text{next}} + (\sum_{i \leq n, i \neq k} \gamma'_i \cdot P'_i) |S P''_{\text{next}}$  by axiom (*SM7*); moreover, since, for any  $I \subset \{i | i \leq n, i \neq k\}$ , it holds  $\sum_{i \in I} \gamma'_i \cdot P'_i = \text{pri}(\sum_{i \in I} \gamma'_i \cdot P'_i)$ , we conclude that  $(\sum_{i \leq n, i \neq k} \gamma'_i \cdot P'_i) |S P''_{\text{next}} = \sum_{i \leq n, i \neq k} (\gamma'_i \cdot P'_i |S P''_{\text{next}})$  by repeatedly applying axiom (*SM6*). Otherwise, we directly have: for any  $I \subset \{i | i \leq n\}$ , it holds  $\sum_{i \in I} \gamma'_i \cdot P'_i = \text{pri}(\sum_{i \in I} \gamma'_i \cdot P'_i)$ , hence  $(\sum_{i \leq n} \gamma'_i \cdot P'_i) |S P''_{\text{next}} = \sum_{i \leq n} (\gamma'_i \cdot P'_i |S P''_{\text{next}})$  by repeatedly applying axiom (*SM6*). The third equality is obtained by applying the same procedure above to terms  $P''_{\text{next}}$  for each of the  $n$  summands. Therefore

$$P'_{\text{next}} |S P''_{\text{next}} = \sum_{i \leq n, \gamma'_i = \tau} (\tau \cdot P'_i |S P''_{\text{next}}) + \sum_{j \leq m, \gamma'_j = \tau} (P' |S \tau \cdot P''_j) + \sum_{i \leq n, j \leq m, \gamma'_i = \gamma'_j \in S \cup \{\delta\}} \gamma'_i \cdot (P'_i |S P''_j)$$

In the following we show that, for any closed normal forms  $P', P''$ , we can turn  $P \equiv P' |S P''$  into  $\sum_{1 \leq i \leq n} \gamma_i \cdot P_i$  such that the arguments of the sum correspond to the transitions of  $P$  by inducing on the following measure: the maximal length of the sequences of  $\tau$  transitions performable by  $P'$  plus the maximal length of the sequences of  $\tau$  transitions performable by  $P''$ . From this result we can conclude that any such  $P$  can be turned into the desired form because, since normal forms include only guarded recursion,  $P', P''$  cannot include cycles of  $\tau$  loops (and are finite-state), hence the sequences of  $\tau$  transitions they can perform are bounded.

- The base case of the induction corresponds to such a measure being 0, i.e. both  $P'$  and  $P''$  cannot perform  $\tau$  transitions. This means that, when transforming  $P$  in the sum-form above, the first two sums are not obtained, hence the assertion obviously holds.
- The inductive case is performed by just observing that the summands  $\tau \cdot P'_i |S P''_{\text{next}}$  and  $P' |S \tau \cdot P''_j$  obtained by transforming  $P$  into the sum-form above can be rewritten, by using axiom (*SM4*), into  $\text{pri}(P'_i |S P''_{\text{next}})$  and  $\text{pri}(P' |S P''_j)$ , respectively. For such terms  $P'_i |S P''_{\text{next}}$  and  $P' |S P''_j$  we can apply the induction hypothesis and turn them into the form  $\sum_{1 \leq i \leq n} \gamma'''_i \cdot P'''_i$  such that the arguments of the sum correspond to their transitions. The obtained term  $\text{pri}(\sum_{1 \leq i \leq n} \gamma'''_i \cdot P'''_i)$  can be then turned into  $\sum_{1 \leq i \leq n, \gamma'''_i \neq \delta} \gamma'''_i \cdot P'''_i$  by using axioms (*Pri1* – 4). It is now easy to observe that, once terms  $\text{pri}(P'_i |S P''_{\text{next}})$  and  $\text{pri}(P' |S P''_j)$  have been turned into the form  $\sum_{1 \leq i \leq n, \gamma'''_i \neq \delta} \gamma'''_i \cdot P'''_i$  inside the sum-form above, the arguments of the obtained overall sum correspond to the transitions of  $P' |S P''_{\text{next}}$ . This is because, according to the operational rules for synchronization merge,  $P' |S P''_{\text{next}}$  is endowed, besides visible and  $\delta$  transitions obtained by synchronization of visible and  $\delta$  transitions of  $P'$  and  $P''$ , with the visible transitions of  $P'_i |S P''_{\text{next}}$  ( $P' |S P''_j$ ) whenever  $P' \xrightarrow{\tau} P'_i$  ( $P'' \xrightarrow{\tau} P''_j$ ),

Let us then consider the case of parallel composition operator, i.e.  $P \equiv P' ||_S P''$ . We initially have

$$P' ||_S P'' = P'_{\text{next}} ||_S P''_{\text{next}} + P''_{\text{next}} ||_S P' + P'_{\text{next}} |S P''_{\text{next}}$$

We then apply the transformation for  $P'_{\text{next}} ||_S P''_{\text{next}}$  considered in the proof for the case of left merge (and we also apply it to  $P''_{\text{next}} ||_S P'$ ) and the transformation for  $P'_{\text{next}} |S P''_{\text{next}}$  considered in the proof for the case of synchronization merge. Here, however, instead of dealing with the first and second sums of the sum form obtained from  $P'_{\text{next}} |S P''_{\text{next}}$  by means of an inductive transformation, we just get rid of them as follows. Since, for any  $i$  and

$j$ , we have  $\tau.P'_i|_S P'' = \text{pri}(P'_i|_S P'')$  and  $P'|_S \tau.P''_j = \text{pri}(P'|_S P''_j)$  and terms  $P'_i|_S P''$  and  $P'|_S P''_j$  already occur in the transformation of  $P'_{next} \parallel_S P''$  and  $P''_{next} \parallel_S P'$  (by additionally applying axiom (*Par*) to parallel composition and commutativity via (*SM1*) of the form  $\tau.((P'_i|_S P'') + P''')$  and  $\tau.((P'|_S P''_j) + P''')$ , respectively, by using axioms (*Tau2*) and (*A3*), we derive

$$P' \parallel_S P'' = \sum_{i \leq n, \gamma'_i \notin S \cup \delta} \gamma'_i.(P'_i \parallel_S P'') + \sum_{i \leq m, \gamma''_i \notin S \cup \delta} \gamma''_i.(P' \parallel_S P''_i) + \sum_{i \leq n, j \leq m, \gamma'_i = \gamma''_j \in S \cup \delta} \gamma'_i.(P'_i \parallel_S P''_j)$$

where, in the second sum, we also have exploited the commutativity of “ $\parallel_S$ ” derived by axiom (*SM1*). It is immediate to observe that, being  $\xrightarrow{\gamma'_i} P'_i$ , with  $i \leq n$ , and  $\xrightarrow{\gamma''_i} P''_i$ , with  $i \leq m$ , the outgoing transitions of  $P'$  and  $P''$ , respectively, the arguments of the above sum correspond to the transitions derived for  $P$  from the operational rules of parallel composition.

Finally, let us consider the case  $P \equiv P' +^t P'' = P'_{next} +^t P''_{next}$ . We have two cases. If for both  $P'$  and  $P''$  there exist  $h, k$  such that  $\gamma'_h = \delta$  and  $\gamma''_k = \delta$ , then we have that  $P = (\sum_{1 \leq i \leq n, i \neq h} \gamma'_i.P'_i + \delta.P'_h) +^t (\sum_{1 \leq j \leq m, j \neq k} \gamma''_j.P''_j + \delta.P''_k)$ . By using axioms (*TCh7*) and (*TCh1*) we derive  $P = ((\sum_{1 \leq i \leq n, i \neq h} \gamma'_i.P'_i) +^t (\sum_{1 \leq j \leq m, j \neq k} \gamma''_j.P''_j)) + ((\sum_{1 \leq i \leq n, i \neq h} \gamma'_i.P'_i) +^t \delta.P'_h) + (\delta.P'_h +^t \sum_{1 \leq j \leq m, j \neq k} \gamma''_j.P''_j) + (\delta.P'_h +^t \delta.P''_k)$ . Then, by observing that  $\sum_{1 \leq i \leq n, i \neq h} \gamma'_i.P'_i = \text{pri}(\sum_{1 \leq i \leq n, i \neq h} \gamma'_i.P'_i)$  and  $\sum_{1 \leq j \leq m, j \neq k} \gamma''_j.P''_j = \text{pri}(\sum_{1 \leq j \leq m, j \neq k} \gamma''_j.P''_j)$  can be derived by using axioms (*Pri1* – 2) and (*Pri4*) because, since both  $P'$  and  $P''$  are time deterministic, all involved actions are standard actions, we derive  $P = (\sum_{1 \leq i \leq n, i \neq h} \gamma'_i.P'_i) + (\sum_{1 \leq j \leq m, j \neq k} \gamma''_j.P''_j) + \delta.(P'_h +^t P''_k)$  by using axioms (*TCh5* – 6), (*Pri3*) and (*A3*). Otherwise, if there exists one between  $P'$  and  $P''$  (let us suppose it to be  $P'$ , the other case is symmetric) such that there is no  $h$  yielding  $\gamma'_h = \delta$ , then we directly have:  $P = (\sum_{1 \leq i \leq n} \gamma'_i.P'_i) + (\sum_{1 \leq j \leq m, j \neq k} \gamma''_j.P''_j)$  if there is  $k$  yielding  $\gamma''_k = \delta$ ;  $P = (\sum_{1 \leq i \leq n} \gamma'_i.P'_i) + (\sum_{1 \leq j \leq m} \gamma''_j.P''_j)$  otherwise. Both equalities are derived by preliminarily observing that  $\sum_{1 \leq i \leq n} \gamma'_i.P'_i = \text{pri}(\sum_{1 \leq i \leq n} \gamma'_i.P'_i)$  can be derived by using axioms (*Pri1* – 2) and (*Pri4*) (because all involved actions are standard actions) and by subsequently applying axioms (*TCh5*) and, for the first equality, (*Pri3*).  $\square$

**Theorem 3.23.** *For each process  $P \in \mathcal{P}_{DT}^{fs}$  there exists a time-deterministic process  $P' \in \mathcal{P}$  such that  $\mathcal{A} \vdash P = P'$ .*

*Proof.* We show, by structural induction over the syntax of expressions  $E \in \mathcal{E}_{DT}^{fs}$  that  $E$  can be turned into a basic expression  $F \in \mathcal{E}$  such that  $\mathcal{A} \vdash F = E$  and:

- (1) For any variable  $X$ : if  $X$  occurs free in  $F$  then  $X$  occurs free in  $E$ .
- (2) If all free variable  $X$  occurrences are guarded in  $E$  by a standard action  $\alpha$  then the same holds in  $F$ .
- (3) Every free occurrence of a variable  $X$  in  $F$  that is in the scope of a “ $_+$ ” operator is guarded inside such an operator by a standard action  $\alpha$ .
- (4)  $F\{0/X \mid X \text{ occurs free in } F\}$  is time-deterministic.

The base cases of the induction are  $E \equiv \underline{1}$ ,  $E \equiv \underline{0}$  and  $E \equiv X$ : for the first one we just apply axiom (*Ter*), the second and third ones are trivial because they are of the desired form already. The inductive cases are the following ones:

- If  $E \equiv \gamma^t.E'$  then  $E$  can be turned into the desired form by directly exploiting the inductive argument over  $E'$  and by applying axioms (*TPre1* – 2).

- If  $E \equiv \text{rec}X.E'$  then  $E$  can be turned into the desired form by directly exploiting the inductive argument over  $E'$  and by considering  $F \equiv \text{rec}X.F'$ : since the obtained expression  $F'$  satisfies items 3 and 4, we have that  $G \equiv F\{0/Y \mid Y \text{ occurs free in } F\}$  is time-deterministic because, in states of  $G$ , terms  $\text{rec}X.G'$  (for some  $G'$ ) that are in the scope of a “ $_+ \_$ ” operator are guarded inside such an operator by a standard action  $\alpha$ .
- If  $E \equiv E' \parallel_S E''$  then we can turn  $E$  into the desired form as follows. By exploiting the inductive argument over  $E'$  and  $E''$ , and by observing that  $E$  cannot include free variables, we obtain the two closed basic terms  $F'$  and  $F''$  that, by using Lemma 3.19 can be transformed into a guarded time-deterministic basic processes. As a consequence we have that the closed term  $P_1$  obtained by replacing both  $E'$  and  $E''$  inside  $E$  with such terms has a finite transition system and is time-deterministic. We can therefore turn  $P_1$  into a time-deterministic basic process as follows. Let  $P_1 \dots P_n$  be the (finite) states of the transition system of  $P_1$ . Due to Lemma 3.22, for each  $i \in \{1 \dots n\}$  we can rewrite  $P_i$  into the form  $P_i = \sum_{j \leq m_i} \gamma_j^i \cdot P_{k_j^i}$ , hence we can characterize the behavior of  $P_1$  by means of a time-deterministic prioritized standard guarded equation set exactly as we did for the proof of Theorem 3.12. Such an equation set is guarded because, by Lemma 3.22, the arguments of the sums above are the outgoing transitions of the states of  $P_1$  and  $P_1$  is the parallel composition of two guarded basic processes, hence (since it cannot turn visible actions into  $\tau$  ones) every cycle in its transition system contains at least a non- $\tau$  action. Since guarded equation sets have a unique solution (Theorem 2.20), we have that there exists a time-deterministic (due to Theorem 3.11) basic process  $P'_1$  such that  $P'_1 = P_1$ .
- If  $E \equiv E'/L$  then we can turn  $E$  into the desired form as follows. Similarly as for parallel composition, by exploiting the inductive argument over  $E'$ , and by observing that  $E$  cannot include free variables, we obtain a closed basic term  $F'$  that, by using Lemma 3.19 can be transformed into a guarded time-deterministic basic processes  $P$ . We then proceed exactly as in the proof of Proposition 6.5 of [5] to replace with  $\tau \_$  each  $a \_$  prefix occurring in  $P$  such that  $a \in L$ , thus obtaining a term  $P'$  (we can distribute top-down and bottom-up the hiding inside  $P/L$  thanks to axiom  $(\text{RecHi})$  and to the fact that, due to axiom  $(\text{Hi}2)$ ,  $\delta \_$  prefixes can be dealt with as  $a \_$  prefixes with  $a \notin L$ ). We then turn the weakly guarded basic process  $P'$  into a guarded  $P''$  by applying Lemma 3.19 and we finally turn  $P''/L$  into a basic process by characterizing its behavior by means of a time-deterministic prioritized standard guarded equation set exactly as we did for parallel composition (notice that now the  $\_/L$  has no hiding effect so, being  $P''$  guarded, every cycle in the transition system of  $P''/L$  contains at least a non- $\tau$  action).
- If  $E \equiv E' +^t E''$  then we can turn  $E$  into the desired form as follows. By exploiting the inductive argument over  $E'$  and  $E''$ , and by observing that all free variables in  $E$  are guarded by a standard action  $\alpha$ , we obtain the two basic expressions  $F'$  and  $F''$  for which, due to item 2, the same property holds.  $F'$  and  $F''$ , by using the statement in the proof of Lemma 2.26 (on which Lemma 3.19 is based), can be transformed into guarded expressions  $H'$  and  $H''$ , respectively, such that the same property holds (because weakly unguardedness elimination in Lemma 2.26 introduces a  $\tau$  guard). Moreover the obtained  $H'$  and  $H''$  are both expressions in  $\mathcal{E}$  because both  $F'$  and  $F''$  satisfy item 3, hence, during the induction of Lemma 2.26,  $\text{pri}(\_)$  operators are always generated in front of guards. Finally, notice that, since both  $F'$  and  $F''$  satisfy item 4, the same holds also for  $H'$  and  $H''$ , because the transformations performed in Lemma 2.26 preserve time-determinism. We consider  $G'$  and  $G''$  such that the set of free variables of  $G'$  and  $G''$  is disjoint from the set of free variables of  $H'$  and  $H''$  and  $(G' +^t G'')\{\alpha_X.E_X/X \mid X \text{ occurs free in}$



$G', G''\} \equiv H' +^t H''$  (such  $G', G''$  exist because all free variables in  $H', H''$  are guarded by a standard action  $\alpha$ ). We have that the process  $P_1 \equiv (G' +^t G'')\{\alpha_X.\underline{0}/X \mid X \text{ occurs free in } G', G''\}$  has a finite transition system and is time-deterministic (being a  $+^t$  of two time-deterministic processes). We can therefore turn  $P_1$  into a time-deterministic basic process as follows. Let  $P_1 \dots P_n$  be the (finite) states of the transition system of  $P_1$ . Due to Lemma 3.22, for each  $i \in \{1 \dots n\}$  we can rewrite  $P_i$  into the form  $P_i = \sum_{j \leq m_i} \gamma_j^i \cdot P_{k_j^i}$ , hence we can characterize the behavior of  $P_1$  by means of a time-deterministic prioritized standard guarded equation set exactly as we did for the proof of Theorem 3.12: it is guarded because, by Lemma 3.22, the arguments of the sums above are the outgoing transitions of the states of  $P_1$  and  $P_1$  is the  $+^t$  of two guarded basic processes. Since guarded equation sets have a unique solution (Theorem 2.20), we have that there exists a time-deterministic (due to Theorem 3.11) basic process  $P'_1$  such that  $P'_1 = P_1$ . By applying exactly the same axioms in the same way, we can transform  $(G' +^t G'')\{\alpha_X.E_X/X \mid X \text{ occurs free in } G', G''\} \equiv H' +^t H''$  into  $F \equiv G_1\{\alpha_X.E_X/X \mid X \text{ occurs free in } G', G''\}$ , with  $G_1$  such that  $G_1\{\alpha_X.\underline{0}/X \mid X \text{ occurs free in } G', G''\} \equiv P'_1$ . Moreover, we have that  $F$  satisfies item 2 because all its free variables are inside  $E_X$  terms. Also the other items are satisfied because  $P'_1$  is time-deterministic and every  $E_X$  term satisfies items 3 and 4.  $\square$

From Theorem 3.23 and Theorem 3.20 we derive the completeness of  $\mathcal{A}_{DT}$  over processes of  $\mathcal{P}_{DT}^{fs}$ .

**Theorem 3.24.** *Let  $P, Q \in \mathcal{P}_{DT}^{fs}$ . If  $P \simeq_T Q$  then  $\mathcal{A}_{DT} \vdash P = Q$ .*

#### 4. RELATED WORK

In the following we consider related work on priority mechanisms and on expressing time in process algebra. Here we just comment on the characteristics of our axiomatization using notions introduced in [27]. In [27] the form of Milner's axiomatization [29, 28] is discussed by observing that it is not in pure equational Horn logic: the axioms involve non-equational side-conditions and/or they are schematically infinitary, as, e.g., the folding axiom (*Rec2*). Our axiomatization is an extension/variant of Milner's one that preserves its characteristics.

**4.1. Local and Global Priority.** In classical prioritized calculi the parallel composition operator is usually managed in two ways: either by implementing *local* pre-emption or *global* pre-emption (see [18]).

Assuming local pre-emption means that  $\tau$  actions of a sequential process (i.e. a process not including a parallel composition operator in its immediate behavior) may pre-empt only actions  $\delta$  of the same sequential process. For instance in  $\tau.E \parallel_{\emptyset} \delta.F$  the action  $\delta$  of the righthand process is not pre-empted by the action  $\tau$  of the lefthand process, as instead happens if we assume global pre-emption. It is easy to see that an extension to our basic calculus with a parallel composition operator implementing local pre-emption makes it possible to preserve congruence w.r.t. Milner's observational congruence. However, unfortunately, expressing local pre-emption makes it necessary to introduce location information in the semantics and does not allow to directly produce an axiomatization by means of standard techniques.

If global pre-emption is, instead, assumed, then standard Milner's notion of observational congruence is not a congruence for the parallel composition operator (see [18]). This is because, e.g.,  $\tau.\underline{0}$  is observationally congruent to  $recX.\tau.X$ , but  $\tau.\underline{0} \parallel_{\emptyset} \delta.P$ , whose semantics

is that of  $\tau.\delta.P$ , is not observationally congruent to  $recX.\tau.X \parallel_{\emptyset} \delta.P$ , whose semantics, due to global pre-emption, is that of  $recX.\tau.X$ . In general note that the problem with congruence is related to the behavior of parallel composition for *processes  $Q$  which may initially execute neither “ $\tau$ ” prefixes, nor “ $\delta$ ” prefixes*, among which is  $\underline{0}$  (for any such  $Q$ , the use of  $\tau.Q \simeq recX.(\tau.X + Q)$  with the context “ $\_ \parallel_{\emptyset} \delta.P$ ” provides a counterexample to congruence). In this case a possibility is to resort to a finer notion of observational congruence (which is divergent sensitive in certain cases) similar to those presented in [30] and [24].

An alternative approach is to adopt an explicit priority operator, see e.g. [2]. In this context [2] shows that a finite complete axiomatization is admitted only if the action set of the considered process algebra is finite. With our simpler form of priority such a limitation is not needed: as for the axiomatizations in [29, 28, 4, 5] we can assume a denumerable action set and we have a finite complete axiomatization (with axioms using action variables).

**4.2. Time.** When priority derives from time (maximal progress assumption), i.e. when  $\delta$  actions represent time delays and standard actions are executed in zero time, there is an alternative choice w.r.t. that of just adopting global priority, as, e.g., in the above mentioned approach [24]. Conceptually, under the timed interpretation, the problem with congruence derives from the fact that the parallel composition operator deals with the terminated process  $\underline{0}$  (and in general with processes which may initially execute neither  $\tau$  actions nor  $\delta$  actions) as if it let time pass. For example  $\underline{0} \parallel_{\emptyset} \delta$  may execute  $\delta$  and become  $\underline{0} \parallel_{\emptyset} \underline{0}$ . This is obviously in contrast with the fact that  $\underline{0}$  is weakly bisimilar to  $recX.\tau.X$ , which is clearly a process that does not let time pass (in the context of time it represents a Zeno process which executes infinite  $\tau$  actions in the same time point): it originates a so-called *time deadlock*. The solution adopted in [22] and in this paper is, instead, to consider, as processes which can let time pass, only processes which can actually execute  $\delta$  actions. In this way  $\underline{0}$  is interpreted not as a terminated process which may let time pass, but as a time deadlock. As a consequence the behaviour of parallel composition is defined, as in [22], in such a way that the *absence* of  $\delta$  actions within the actions executable by a process (which means that the process cannot let time pass) pre-empts the other process from executing a timed action  $\delta$ , see Table 6.

As we already explained in the introduction, in order to produce an axiomatization we have to face the problem of standard axiom  $recX.(\tau.X + E) = recX.\tau.E$  unsoundness. In order to overcome this problem, in the above mentioned approach of [24] the distinguished symbol “ $\perp$ ” is introduced, which represents an ill-defined term that can be removed from a summation only if a silent computation is possible. In this way by considering the rule  $recX.(\tau.X + E) = recX.\tau.(E + \perp)$  the resulting term that escapes divergence can be turned into a “normal” term only if  $E$  may execute a silent move. This law is surely sound (over terms without “ $\perp$ ”) also in our language, but is not sufficient to achieve completeness. Since, differently from [24], we do not impose conditions about stability in our definition of observational congruence, we can escape divergence not only when  $E$  includes a silent move but *for all possible* terms  $E$ . For example in our calculus (but not in [24]) the term  $recX.\tau.X$  is equivalent to  $\tau.\underline{0}$  (as in standard CCS), so we can escape divergence in  $F$  even if  $\tau.X$  has not a silent alternative inside  $recX$ . In our case  $\tau$  divergence can always be escaped by turning  $E$  into  $pri(E)$  and the strongly guarded terms we obtain are always “well-defined” terms. Notice that the introduction of this auxiliary operator, representing priority “scope”, is crucial for being able to axiomatize the priority of  $\tau$  actions over  $\delta$  actions when standard

observational congruence is considered. Since we have to remove  $\delta$  actions performable by a term  $E$  even if  $E$  does not include a silent move, we cannot do this by employing a special symbol like “ $\perp$ ” instead of using an operator. This is because  $\perp$  must somehow be removed at the end of the deletion process (in [24]  $\perp$  is eliminated by silent alternatives) in order to obtain a “normal” term.

Concerning the discrete interpretation of time, our Discrete Time Calculus just differs from [22] for the choice of using CSP [26] parallel composition and hiding, instead of CCS [28] parallel composition and restriction, as done in [22]. This allowed us to provide an axiomatization that is complete for finite-state processes by applying the technique of [4, 5] that requires dynamic generation of “ $\tau$ ” actions to be expressed by a dedicated hiding operator (and not by another operator, like CCS [28] parallel, that mixes generation of  $\tau$  actions with other mechanisms). Notice that an analogous complete axiomatization of [22] could have been obtained by expressing the CCS-like parallel composition of [22] in terms of parallel composition and hiding by using the generic process algebra TCP+REC introduced in [5], that was produced during the mentioned collaboration with Prof. Jos Baeten.

Finally, concerning limitations of our approach, we consider the possible extension to discrete-time process algebras with multiple clocks and general clock scoping, as e.g. the calculus in [32]. In this context a complete axiomatization of maximal progress for weakly guarded finite-state processes cannot be achieved by a direct application of our approach. This because a  $\tau$ -circle could involve two or more processes, each of which has an (independent) outgoing  $\delta$  transition: in this situation we could not directly use our axiomatization to remove the  $\tau$ -circle while maintaining time-determinism.

## 5. FUTURE WORK

We just make some remarks concerning future work. We plan to apply the axiomatization techniques used in this paper also in the context of stochastic time, in particular to provide a complete axiomatization of Markovian observational congruence for Revisited Interactive Markov Chains [9], where, due to maximal progress,  $\tau$  actions are prioritized w.r.t. Markovian delays. Differently from original Interactive Markov Chains [23], where the maximal progress assumption is implemented by a global priority mechanism and a  $\tau$ -divergent sensitive equivalence like that in [24] is adopted, Revisited Interactive Markov Chains are based on a parallel composition that requires both processes to let time pass, as for our discrete time calculus. As a consequence the axiomatization of the basic calculus introduced in this paper can also form a basis (once  $\delta$  prefixes are turned into Markovian delays and Markovian observational congruence of [9] is considered) for completely axiomatizing the Markovian calculus of [9].

## ACKNOWLEDGMENTS

We thank the anonymous reviewers for their careful reading, comments and suggestions.

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