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# ACCESSIBILITY, ABNORMAL GEODESICS IN OPTIMAL CONTROL. A GEOMETRIC APPROACH FROM SINGULARITY THEORY USING TWO CASES STUDIES

BERNARD BONNARD, JÉRÉMY ROUOT, AND BORIS WEMBE

ABSTRACT. In this note, we use two case studies to analyze the relation between abnormal geodesics in time minimal control, accessibility and the regularity properties of the (time minimal) value function. The first problem concerns a Zermelo navigation problem in the plane. Using the historical problem of the calculus of variations set by Carathéodory and Zermelo aiming to compute the quickest nautical path of a ship on a river to be transferred from one shore to opposite one, we construct a semi-normal form to analyze the cusp singularity of the abnormal geodesic when meeting the transition between the strong and weak current domains and to evaluate the time minimal value function. The problem is related to reaching the boundary of the two domains in minimum time. The second problem concerns the classification of the time minimal syntheses for chemical reactors near a terminal manifold of codimension one, in particular in relation with the McKeithan network and associated to maximize the production of one chemical species. Both cases studies have the common geometric frame and computations techniques illustrate the role of singularity theory in geometric optimal control, in relation with abnormal geodesics and regularity of the value function solution of Hamilton-Jacobi-Bellman equation.

## INTRODUCTION

The aim of this article is to analyze the role of abnormal geodesics in optimal control in relation with accessibility theory and regularity of the time minimal value function and the techniques from singularity theory that is computation of graded semi-normal forms and determination of time minimal syntheses using the concept of unfolding ([1, 6]) adapted to control theory. We consider two cases studies related to applications. The first case concerns a planar Zermelo navigation problem defined by a couple where  $F_0$  is the current and  $g$  is a Riemannian metric. The problem was first studied by Carathéodory and Zermelo, in the case of a linear current and with the Euclidean metric [4]. The domain can be split into a strong current domain and a weak current domain. In the strong current domain the set of admissible directions forms a cone bounded by two abnormal (limits) directions. When meeting the boundary of the two domains the abnormal geodesics have a cusp singularity and this causes a morphogenesis of the accessibility set. Singularity theory is used to describe this phenomenon in a generic context. The second case concerns the control of a sequence of reactions using the derivative of the temperature as control whose aim is to maximize the production of one species. In particular we consider the case of the McKeithan scheme introduced in [3]. The problem is translated into a time minimal control problem for a single-input affine system with terminal

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manifold  $N$  with codimension one. Again we consider the exceptional case where a bang or singular geodesic is on the zero level of the maximized Hamiltonian. This is related to accessibility properties of the system. Semi-normal forms coming from [7, 2] are presented to describe the optimal syntheses in a neighborhood of the terminal manifold which can be applied to the McKeithan network. In this case the optimal solutions are not smooth and we must evaluate the branches of the switching and cut loci ramifying on the terminal manifold to stratify the time minimal value function.

## 1. ABNORMAL GEODESICS IN PLANAR ZERMELO NAVIGATION PROBLEMS

**1.1. Notations and concepts.** Let  $(M, g, F_0)$  be a smooth navigation problem where  $M$  is a 2d- manifold,  $g$  is a metric and  $F_0$  is a vector field defining the current. The problem under study is local and we denote by  $q = (x, y)$  a coordinate system and let  $F_1, F_2$  be an orthonormal frame for the metric. The problem is set a time minimal control problem for the system

$$(1) \quad \frac{dq(t)}{dt} = F_0(q(t)) + \sum_{i=1,2} u_i(t) F_i(q(t)), \quad u = (u_1, u_2),$$

where the control is  $u$  is such that  $\|u\| \leq 1$ . Let  $H_i(p, q) = p \cdot F_i(q)$  for  $i = 0, 1, 2$  be the Hamiltonian lift of the vector fields  $F_i$ 's,  $p = (p_x, p_y)$  being the adjoint vector and  $H = H_0 + \sum_{i=1,2} H_i$  is the pseudo-(or unmaximized) Hamiltonian. Since  $F_1, F_2$  form a frame, according to the Maximum Principle every minimizer is solution of the smooth Hamiltonian vector field  $\vec{H} = H_0 + (H_1^2 + H_2^2)^{1/2}$ , while the optimal control is given by  $u_i = H_i / (H_1^2 + H_2^2)^{1/2}$  for  $i = 1, 2$ . One can use the Carathéodory-Zermelo-Goh representation introducing the heading angle  $\alpha$  of the ship defined by  $u_1 = \cos \alpha, u_2 = \sin \alpha$ . By setting  $\tilde{q} = (q, \alpha)$ ,  $\tilde{p} = (p, p_\alpha)$ , the system can be extended to the affine system  $\frac{d\tilde{q}(t)}{dt} = X(\tilde{q}(t)) + v(t) Y(\tilde{q}(t))$  where the accessory control is  $v(t) = \frac{d\alpha(t)}{dt}$ , and  $X = F_0 + \cos \alpha F_1 + \sin \alpha F_2$ ,  $Y = \frac{\partial}{\partial \alpha}$ .

The geodesics are the solutions of  $\vec{H}$  and since  $H$  is constant and equal to  $-p_0$ , they can be classified into hyperbolic (resp. elliptic ) geodesics if  $p_0 = -1$  (resp.  $+1$ ) while the case  $p_0 = 0$  corresponds to the abnormal (or exceptional) situation.

In the Carathéodory-Zermelo-Goh extension they can computed as follows. One has

$$\tilde{p} \cdot Y(\tilde{q}) = \tilde{p} \cdot [Y, X](\tilde{q}) = 0,$$

$$\tilde{p} \cdot ([Y, X], X)(\tilde{q}) + v_s [Y, X, Y](\tilde{q}) = 0.$$

Hence  $v_s$  can be computed as the feedback given by

$$v_s = -D'/D, \quad D = \det(Y, [Y, X], [[Y, X], Y]), \quad D' = \det(Y, [Y, X], [[Y, X], X]).$$

Furthermore let  $D'' = \det(Y, [Y, X], [[Y, X], X])$ . The geodesics are projections on the  $q$ -space of the solutions of the vector field defined by  $X_s(\tilde{q}) = X(\tilde{q}) + v_s(\tilde{q})Y(\tilde{q})$  and abnormal geodesics are projections of the solutions contained in  $D'' = 0$ , while hyperbolic (resp. elliptic) geodesics are projections of those contained in  $D D'' > 0$  (resp.  $< 0$ ).

**1.2. Elements of singularity theory and the cusp singularity of the abnormal geodesics.** Let  $\gamma: t \rightarrow (x(t), y(t))$  be a planar smooth curve. A point is a semicubical singular point at  $t = t_{cusp}$  if  $\gamma'(t_{cusp}) = 0$  and  $\gamma^{(2)}(t_{cusp}), \gamma^{(3)}(t_{cusp})$  are independent. Using suitable coordinates the image can be identified to the equation  $x^2 - y^3 = 0$ . This leads to an algebraic model of the singularity on  $\mathbb{R}[x, y]$  as a transition between a  $\mathbb{R}$ -node solution of the equation  $x^3 - x^2 + y^2 = 0$  with two distinct tangents at 0:  $x \pm y = 0$  and a  $\mathbb{C}$ -node solution of  $x^3 + x^2 + y^2 = 0$ , with two complex tangents at 0 given by  $x \pm iy = 0$  and keeping the real branch located near 0 (but not containing 0). See Fig. 1.

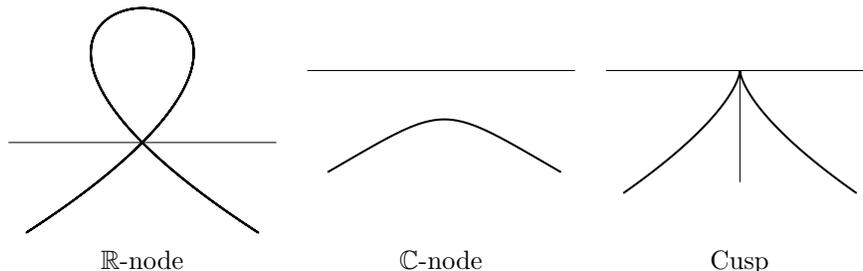


FIGURE 1. A cusp viewed as a transition between a  $\mathbb{R}$ -node and a  $\mathbb{C}$ -node.

Using elementary singularity theory one can construct a versal unfolding of this cusp singularity depending upon one parameter.

In a Zermelo navigation problem one can distinguish between the weak current domain where  $\|F_0(q)\|_g < 1$  and the strong current domain where  $\|F_0(q)\|_g > 1$  and the transition between the two domains is called the moderated current domain and is defined by  $\|F_0(q)\|_g = 1$ .

A simple computation shows that abnormal and elliptic geodesics occur only in the strong current domain. A model of the singularity of the abnormal geodesics when meeting the boundary of the domains can be analyzed using the historical example due to Carathéodory and Zermelo. The geodesics equation is given by the equations

$$(2) \quad \frac{dx}{dt} = y + \cos \alpha, \quad \frac{dy}{dt} = \sin \alpha, \quad \frac{d\alpha}{dt} = -\cos^2 \alpha.$$

In this example the boundary between the two domains can be identified to the set  $y = -1$ . Making the translation  $Y = y + 1$  and expanding at  $\alpha = 0$  up to order 2 one gets the system

$$\frac{dx}{dt} = Y - \alpha^2/2, \quad \frac{dY}{dt} = \alpha, \quad \frac{d\alpha}{dt} = -1 + \alpha^2.$$

Hence at the point  $\tilde{q}_1 = (0, 0, *)$  at the boundary of the two domains, one can observe that  $\alpha$  is of order  $t$ ,  $y$  of order  $t^2$  and  $x$  of order  $t^3$  so that the respective weights of the variables  $(x, y, \alpha)$  are  $(3, 2, 1)$ . Integrating the geodesics near 0, this gives a universal unfolding of the cusp singularity depending upon three parameters  $x_0, y_0, \alpha_0$  corresponding to the initial conditions of the geodesics curves in the neighborhood of 0. In particular one represents on Fig. 2 the versal unfolding associated to  $x_0 = \alpha_0 = 0$ . In this representation the cusp singularity corresponds to the abnormal direction when meeting the boundary of two domains, curves with

self-intersections corresponds to hyperbolic trajectories crossing the boundary while the third other type corresponds to elliptic geodesics.

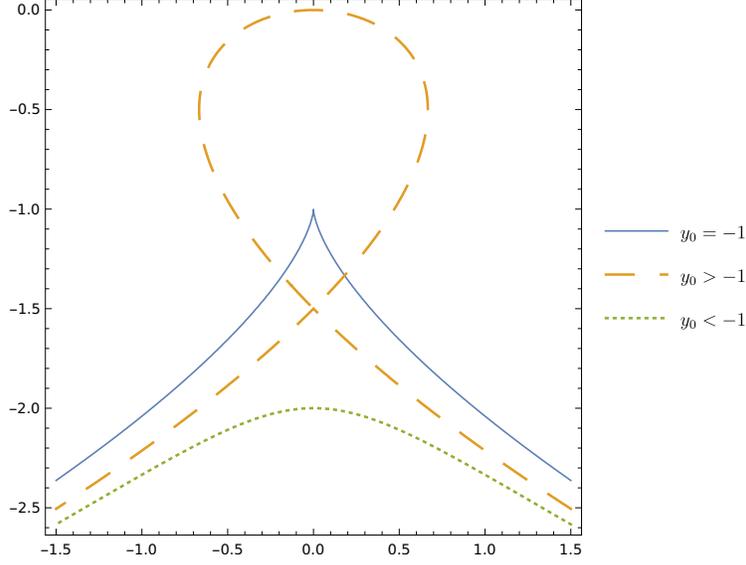


FIGURE 2. Quickest nautical path as a versal unfolding of the generic singularity of the abnormal geodesic.

*Evaluation of the time minimal value function.* Take a pair of points  $q_0, q_1$  we denote by  $T(q_0, q_1)$  the minimum transfer time from  $q_0$  to  $q_1$ . If  $q_1$  is on the boundary of the two domains one can compare the transfer time from geodesics starting from  $q_0$  to an intermediate point  $q_a(t', q_0)$  on the abnormal geodesics with the transfer time along the self-intersecting hyperbolic geodesic. See the corresponding Fig. 3.

This time is clearly strictly greater. Indeed if the limit of the heading angles of the hyperbolic geodesics tends to the heading angle of the abnormal direction hence the final point tends at time  $t'$  tends to  $q_a(t', q_0)$  and the additional loop to intersects exactly adds an additional times. More precisely one has the estimate given by the following proposition.

**Proposition 1.1.** Take  $\tilde{q}_1 = (0, -1, 0)$  on the boundary of the two domains and  $q_a(\cdot, \tilde{q}_1)$  denotes the abnormal geodesic passing through the point  $\tilde{q}_1$ . Define  $q_0 = q_a(t_0, \tilde{q}_1)$  for some  $t_0 < 0$ . In a neighborhood of  $\tilde{q}_1$ , there exists a heading angle defined by  $\tilde{q}'_0 = (q_0, \alpha_0)$  and the hyperbolic geodesic  $q_h(\cdot, \tilde{q}'_0)$  intersects the abnormal geodesic at a single position  $q_2 = q_a(t_2, \tilde{q}_1)$ ,  $t_2 < 0$ . The approximated cost from  $q_0$  to  $q_1$  of the abnormal geodesic is smaller than the cost of the loop trajectory containing the hyperbolic geodesic.

*Proof.* Expanding the geodesics  $q_a(\cdot, \tilde{q}_1)$ ,  $q_h(\cdot, \tilde{q}'_0)$  up to order 3 with respect to the time  $t_2$  and  $\tilde{q}_1, \tilde{q}'_0$  the equation

$$q_2 = q_a(t_2, \tilde{q}_1) = q_h(t_2, \tilde{q}'_0) + O(|t_2, \tilde{q}_1, \tilde{q}'_0|^3),$$

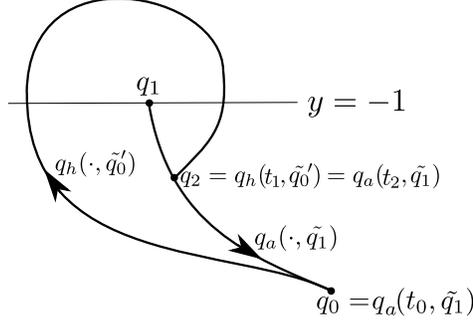


FIGURE 3. Hyperbolic and abnormal geodesics arcs between  $q_1 \in \{\|F_0\|_g = 1\}$  and  $q_0$ . (the abnormal being oriented with negative time)

where

$$(3) \quad \begin{aligned} q_a(t_2, \tilde{q}_1) &= \left( -\frac{t_2^3}{3}, -\frac{t_2^2}{2} - 1 \right), \\ q_h(t_1, \tilde{q}'_0) &= \left( -\frac{t_1^3}{3} + t_1^2 \alpha_0 - \frac{t_1 t_0^2}{2} - \frac{t_1 \alpha_0^2}{2} - \frac{t_0^3}{3}, -\frac{t_1^2}{2} + t_1 \alpha_0 - \frac{t_0^2}{2} - 1 \right), \end{aligned}$$

is satisfied for

$$(4) \quad \begin{aligned} \alpha_0 &= \frac{-2t_2^3 + 6t_2(t_2 + t_0) \sqrt{t_2^2 + t_2 t_0 + t_0^2} - 4(t_2^2 + t_2 t_0 + t_0^2)^{3/2} + 3t_2^2 t_0 - t_0^3}{3(t_2^2 - t_0^2)} \\ t_1 &= 2\sqrt{t_2^2 + t_2 t_0 + t_0^2} - t_2 + t_0. \end{aligned}$$

Hence, the cost of the abnormal geodesic from  $q_0$  to  $q_1$  is  $-t_0 > 0$ , while the cost of the hyperbolic geodesic followed by the abnormal arc  $\widehat{q_2 q_1}$  is

$$(5) \quad \begin{aligned} t_1 - t_2 &= 2 \left( \sqrt{t_2^2 + t_2 t_0 + t_0^2} - t_2 \right) + t_0 = 2 \left( \sqrt{(t_0 - t_2)^2 + 3t_2 t_0} + t_0 - t_2 \right) - t_0 \\ &\geq 2(|t_0 - t_2| + t_0 - t_2) - t_0 \\ &\geq -t_0. \end{aligned}$$

□

*Construction of the semi-normal form.* We proceed to a geometric normalization of the cusp singularity in which the image is a semicubical point. We have to make the following assumptions

- The boundary  $N$  between the two domains is smooth and can be locally identified to the horizontal.
- The geodesic  $t \rightarrow \gamma(t)$  intersects the domain at a semicubical point with a transverse tangent which can be taken as vertical.

From the second condition since since the derivative vanishes, the Hamiltonian is zero and the geodesics is abnormal. To compute the semi-normal form the essential is to normalize the tangent to the cusp to the horizontal. More precisely we make the following normalization.

*Semi-normal form of the cusp singularity.* The cusp singularity is identified to 0 and we consider the weighted Taylor expansion of the geodesic vector field. Variables have the weights  $(3, 2, 1)$  and the weights of  $\frac{\partial}{\partial x}$  is  $-3$ , the weight of  $\frac{\partial}{\partial y}$  is  $-2$  and the weight of  $\frac{\partial}{\partial \alpha}$  is  $-1$ . We make the following normalization:

- (C1) The metric  $g$  is written in the isothermal form  $a(x, y) (dx^2 + dy^2)$  so that  $F_1 = \frac{1}{\sqrt{a}} \frac{\partial}{\partial x}$  and  $F_2 = \frac{1}{\sqrt{a}} \frac{\partial}{\partial y}$ .
- (C2) The limit  $N$  between the strong current domain if given by  $y = 0$  so that  $F_0 = b(x, y) \frac{\partial}{\partial x} + c(x, y) \frac{\partial}{\partial y}$  satisfies  $\|F_0\|_g = 1$  on  $N$ .
- (C3) The geodesics control is given by  $v_s = -D'/D$  and the solution at 0 is a semicubical cusp so that  $\alpha$  is of order  $t$ ,  $y$  of order  $t^2$  and  $x$  of order  $t^3$ .

As in the historical example we can expand for small time the geodesics starting at  $q_0$  near 0 at any order and evaluate the time minimal value function. Also this gives the time minimal synthesis to reach from  $\tilde{q}_0$  the barrier  $N$ .

**Lemma 1.2.** *Generically, there exists a local coordinates such that in a neighborhood of  $q_1 = 0$  the system (1) takes the form*

$$(6) \quad \begin{aligned} F_0 &= b(x, y) \frac{\partial}{\partial x} + c(x, y) \frac{\partial}{\partial y}, \\ F_1 &= \frac{1}{\sqrt{a(x, y)}} \frac{\partial}{\partial x}, \\ F_2 &= \frac{1}{\sqrt{a(x, y)}} \frac{\partial}{\partial y}, \end{aligned}$$

with  $a(x, y) = 1 + 2b_{10}x + a_{01}y + \varepsilon_1(x, y)$ ,  $b(x, y) = -1 + b_{10}x + b_{01}y + \varepsilon_2(x, y)$ ,  $c(x, y) = c_{10}x + c_{01}y + \varepsilon_3(x, y)$ ,  $a_{01} - 2b_{01} \neq 0$  and  $\varepsilon_i(x, y)$ ,  $i = 1, 2, 3$  being higher order terms.

*Proof.* We write  $a(x, y) = \sum_{i,j} a_{ij} x^i y^j$ ,  $b(x, y) = \sum_{i,j} b_{ij} x^i y^j$ ,  $c(x, y) = \sum_{i,j} c_{ij} x^i y^j$ . The solution  $q(\cdot)$  to the system (1) from  $q_1$  satisfies

$$\dot{q}(0) = \left( \frac{1}{\sqrt{a_{00}}} + b_{00}, c_{00} \right),$$

and since  $q(0) = q_1$  is a singular point, we must have:  $c_{00} = 0$  and  $b_{00} = -\frac{1}{\sqrt{a_{00}}}$ . Reparameterizing the abnormal geodesic, we can set  $a_{00} = 1$ . For the condition  $\|F_0\|_g = 1$ , we have

$$(7) \quad \begin{aligned} (b(x, y) \quad c(x, y)) \begin{pmatrix} a(x, y) & 0 \\ 0 & a(x, y) \end{pmatrix} \begin{pmatrix} b(x, y) \\ c(x, y) \end{pmatrix} \Big|_{y=0} &= \\ a(x, 0) (b(x, 0)^2 + c(x, 0)^2) &= 1 + (a_{10} - 2b_{10})x + \varepsilon(x, y), \end{aligned}$$

where  $\varepsilon(x, y)$  are higher order terms and it follows  $a_{10} = 2b_{10}$ .

Differentiating, we have:

$$(8) \quad \begin{aligned} \ddot{q}(0) &= \left( 0, \frac{1}{2} (a_{01} - 2b_{01}) \right), \\ q^{(3)}(0) &= \left( -\frac{1}{2} (a_{01} - 2b_{01})^2, 0 \right), \end{aligned}$$

and the condition (C3) is satisfied if  $a_{01} - 2b_{01} \neq 0$  (moreover if  $a_{01} = 2b_{01}$ , then the abnormal geodesic is reduced to  $q_1$ ).  $\square$

Again, the following proposition gives an estimate to the value function.

**Proposition 1.3.**

(a) Considering the semi-normal form (6), to reach in minimum time a point  $q_1 \in N = \{\|F_0\|_g = 1\}$  from  $q_0$  in a neighborhood of  $q_1$ , the cost of the trajectory containing the hyperbolic arc is greater than the cost of the abnormal geodesic from  $q_0$  to  $q_1$ .

(b) The cut locus of a point  $q_0$  can contain a branch formed by an abnormal arc terminating with a cusp on  $N$ .

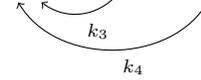
*Proof.* We use the notation of the previous Section (see Fig. 3), and we integrate backward the system  $\dot{q} = X(q) - D'(q)/D(q)Y(q)$  from  $\tilde{q}_1 = (q_1, 0)$  to  $q_0 = q_a(t_0, \tilde{q}_1)$ . Integrating forward along the hyperbolic geodesic from  $\tilde{q}'_0 = (q_0, \alpha_0)$ , we reach the intersecting point  $q_2 = q_h(t_1, \tilde{q}'_0) = q_a(t_2, \tilde{q}_1)$ . The point  $q_2$  is characterized by the equations

$$(9) \quad \begin{aligned} (a_{01} - 2b_{01})^2 (2t_1^3 + 3t_1t_0^2 - 2t_2^3 + 2t_0^3) + 12t_1^2\alpha_0(a_{01} - 2b_{01}) + 12t_1\alpha_0^2 &= 0 \\ (a_{01} - 2b_{01})(t_1^2 - t_2^2 + t_0^2) + 4t_1\alpha_0 &= 0. \end{aligned}$$

Computing, we get  $t_1 = 2\sqrt{t_2^2 + t_2t_0 + t_0^2} - t_2 + t_0$  and  $t_1 - t_2$  is greater than  $-t_0$ .  $\square$

## 2. TIME MINIMAL EXCEPTIONAL GEODESICS IN OPTIMIZATION OF CHEMICAL NETWORKS

**2.1. A brief recap about the optimal control of chemical networks.** In this section, we introduce the concepts for the optimization of chemical networks. In particular we shall consider the McKeithan network:  $T + M \xrightarrow{k_1} A \xrightarrow{k_2} B$ . The



state space is formed by the concentration vector:

$$c = (c_T, c_M, c_A, c_B)$$

of the respective chemical species. We note  $\delta_1 = C + A + B$ ,  $\delta_2 = M + A + B$  the first integrals associated to the dynamics and let  $x = c_A$ ,  $y = c_B$ , then the system is described by the equation:

$$\begin{aligned} \dot{x} &= -\beta_2 x v^{\alpha_2} - \beta_3 x v^{\alpha_3} - \delta_3 v (x + y) + \delta_4 v + v (x + y)^2 \\ \dot{y} &= \beta_2 x v^{\alpha_2} - \beta_4 y v^{\alpha_4} \\ \dot{z} &= u, \quad |u| \leq 1, \end{aligned}$$

with

$$\begin{aligned} 0 \leq x \leq \delta_1, \quad 0 \leq y \leq \delta_2, \quad \delta_3 = \delta_1 + \delta_2, \quad \delta_4 = \delta_1 \delta_2, \\ z = k_1, \quad k_2 = \beta_2 v^{\alpha_2}, \quad k_3 = \beta_3 v^{\alpha_3}, \quad k_4 = \beta_4 v^{\alpha_4} \end{aligned}$$

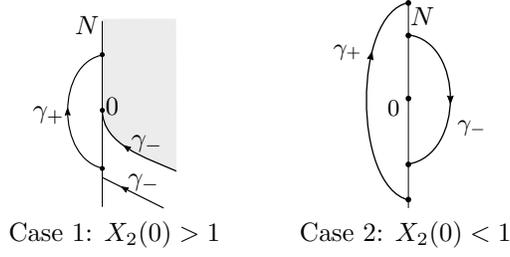
and maximizing the product of the  $A$  species leads to a time minimal control problem with a terminal manifold  $N : x = d$ ,  $d$  being the desired production of  $A$ .

We denote by  $\dot{q} = X + uY$ ,  $|u| \leq 1$  the control system.

The singular geodesics are solutions of the dynamics

$$\dot{q} = X + u_s Y, \quad u_s = -D'/D.$$

Each optimal solution is a concatenation by arcs  $\gamma_+$ ,  $\gamma_-$ , where the control is  $u = \pm 1$  and singular arcs  $\gamma_s$ .

FIGURE 4. Local synthesis near  $\mathcal{E}$  in the generic case.

Our method consists of the construction of *semi-normal forms* for the action of the pseudo-group  $\mathcal{G}$  of local diffeomorphisms  $\varphi$  such that  $\varphi(0) = 0$ ,  $\varphi * Y = Y$  and feedback transformation  $u \rightarrow -u$  (so that  $\gamma_+$  and  $\gamma_-$  can be exchanged).

**2.2. Local syntheses in the exceptional cases.** Take a terminal point  $\tilde{q}_1$  of  $N$ , which can be identified to  $0$ . One wants to describe the time minimal syntheses in a small neighborhood  $U$  of  $0$ . We denote respectively by  $\gamma_{\pm}^0$ ,  $\gamma_s^0$  bang and singular arcs terminating at  $0$  and we consider only the exceptional case where the arc is tangent to  $N$ , which splits into the bang exceptional case or the singular exceptional case. The syntheses are described in details in [2, 7] up to the codimension two situations and we recall the main points to be applied to the McKeithan network.

**2.2.1. The bang exceptional case.** The neighborhood  $U$  of  $0$  can be splitted into two domains denoted by  $U_+$  on which the optimal control is  $u = +1$  and the  $U_-$  where it is given by  $u = -1$ . We have to consider the two cases.

**Generic case (codimension one).** In this case both arcs  $\gamma_+^0$  and  $\gamma_-^0$  are tangent to  $N$  but with a contact of order 2. Using the concept of unfolding, one can define a  $C^0$ -foliation of  $U$  by invariant planes so that in each plane the system takes the semi-normal form:

$$(10) \quad \begin{aligned} \dot{x} &= by + o(|x|, |y|) \\ \dot{y} &= X_2(x, y) + u, \quad |u| \leq 1, \end{aligned}$$

where  $b = n \cdot [Y, X] \neq 0$ , which can be normalized to 1,  $n = (1, 0)$  being the normal to  $N$  identified to  $x = 0$ . Moreover one can assume that  $1 + X_2(0) > 0$  and we have two cases.

**Proposition 2.1.** Using the previous normalizations we have two cases described in Fig.4.

The difference between the two cases is related to different accessibility properties of the system. In the case  $X_2(0) > 1$ , the target  $N$  is not accessible from the points in  $x > 0$  above the arc  $\gamma_-^0$  terminating at  $0$ . In the case  $X_2(0) < 1$ , each point of  $U$  can be steered in minimum time to  $N$ , the domain  $U_+$  where the optimal control is  $+1$  being  $x < 0$  and the domain  $U_-$  with optimal control  $-1$  being  $x > 0$ .

**Codimension two case.** A more complex situation occurs assuming that the arc  $\gamma_-^0$  has a contact of order three with  $N$  while  $\gamma_+^0$  has a contact of order two. The optimal syntheses cannot be described by foliating  $N$  by  $2d$ -planes as in the previous cases.

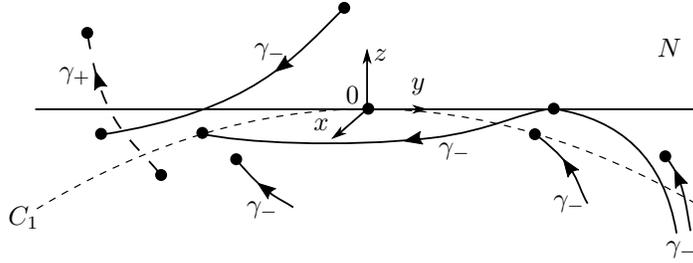


FIGURE 5. Local synthesis near  $\mathcal{E}$  in the codimension 2 case for  $b < 0$ .

One needs to introduce the following assumptions. We assume that  $Y = \frac{\partial}{\partial z}$ ,  $N$  is the plane  $x = 0$  parameterized as the image of:  $(w, s) \mapsto (0, w, s)$ . Denoting  $n = (1, 0, 0)$ , the normal to  $N$  at 0, we assume

- bang exceptional case:  $n \cdot X(0) = 0$ ,  $n \cdot [Y, X] \neq 0$
- $\det(X, Y, [Y, X]) \neq 0$  at 0
- $\{n \cdot X = 0\} \cap N$  is a curve which is neither tangent to  $X$  nor to  $Y$  at 0.

We introduce the following normalizations: along the  $y$ -axis,  $n \cdot X = 0$  and  $[X, Y] = \frac{\partial}{\partial x}$ .

Using the concept of semi-normal form the optimal syntheses can be described by the following model:

$$(11) \quad \begin{cases} \dot{x} = z \\ \dot{y} = b, \\ \dot{z} = 1 + u + y \end{cases} \quad N : (w, s) \mapsto (0, w, s) ,$$

and we have two types of time minimal syntheses.

**Proposition 2.2.** Assume  $b < 0$ . Then each point of  $U$  can be steered to  $N$ . Moreover

- (1)  $U^+ \setminus N \subset \{x < 0\}$  and  $U^- \setminus N \subset \{x > 0\}$
- (2) Optimal trajectories  $\gamma_-$  arrive at any point  $(0, w, s < 0)$  or  $(0, w \geq 0, s)$  of  $N \cap U$ .

The optimal synthesis is described by Fig.5.

**Proposition 2.3.** Assume  $b > 0$ . In this case the system is not locally controllable at 0. We represent on Fig.6 the synthesis in this case.

We shall refer to [7] for the full details of the computation and description of the syntheses.

2.2.2. *The singular exceptional case.* In this case we can assume  $Y = \frac{\partial}{\partial z}$ ,  $N$  is the plane  $x = 0$ , the normal to  $N$  at 0 is  $n = (1, 0, 0)$  and moreover:

*Singular exceptional case.*

$$\begin{cases} n \cdot X(0) = 0 \\ n \cdot [Y, X](0) = 0 \end{cases}$$

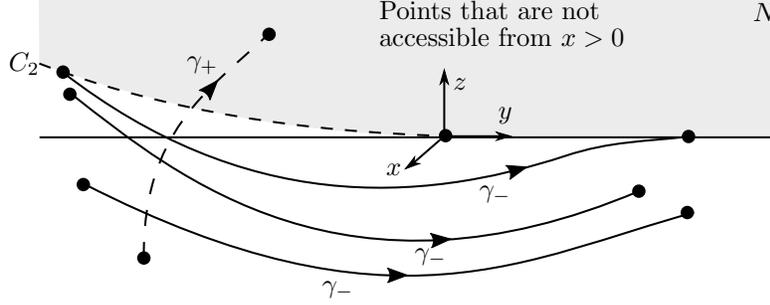
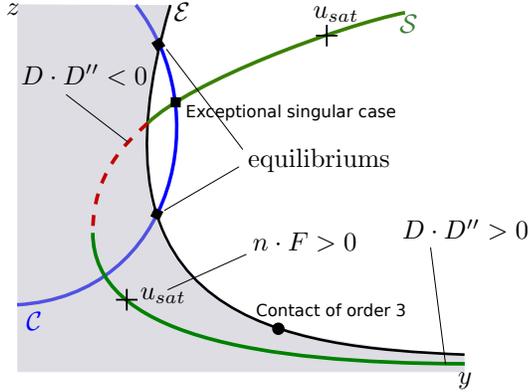
FIGURE 6. Local synthesis near  $\mathcal{E}$  in the codimension 2 case for  $b > 0$ .

FIGURE 7. Description of the codimension 2 cases analyzed in our section for the McKeithan network.

and we add the following generic conditions

$$\begin{cases} X \text{ and } Y \text{ are independent at } 0, \\ \det(Y, [Y, X], \text{ad}^2 Y \cdot X) \neq 0 \text{ at } 0, \\ \{n \cdot X(0) = 0\} \cap N \text{ is a curve which is not tangent to } X \text{ at } 0. \end{cases}$$

Using the concept of semi-normal form the optimal syntheses can be described by the following

$$(12) \quad \begin{cases} \dot{x} = y + z^2 \\ \dot{y} = b + b_1 z \\ \dot{z} = c + u \end{cases},$$

The different syntheses are described in [7].

2.2.3. *The collinearity locus.* Another codimension two case occurs at the collinearity locus

$$\mathcal{C} : X \text{ and } Y \text{ are collinear.}$$

This corresponds to singular points of the  $2d$ -vector  $X$  and it occurs in the McKeithan network.

$\mathcal{C} \cap N : x = d$  leads to isolated singular points and in the generic case such a point is not a singular trajectory. Thus we obtain a model of the form  $\dot{\tilde{q}} = A\tilde{q} + ub$  and the local synthesis can be easily described.

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